Numerical Example :

RS Code $(15, 11, 5)$

$d_{min} = 5 \Longrightarrow t = 2$

transmitted codeword, $c(x) = x^5 + \alpha^6 x^4 + x^3 + \alpha^2 x^2 + \alpha^{12} x + \alpha^{10}$

recieved codeword, $r(x) = x^5 + \alpha^6 x^4 + \underbrace{\alpha x^3}_{Err} + \alpha^2 x^2 + \underbrace{\alpha^{10} x}_{Err} + \alpha^{10}$

PGZ Algorithm :

$S_1 = r(\alpha)$

$= \alpha^5 + \cancel{\alpha^{10}} + \cancel{\alpha^4} + \cancel{\alpha^4} + \alpha^{11} + \cancel{\alpha^{10}} = \alpha^5 + \alpha^{11}$

$= (\alpha^2 + \alpha) + (\alpha^3 + \alpha^2 + \alpha) = \underline{\underline{\alpha^3}}$. syndrome 1

$S_2 = r(\alpha^2)$

$= \cancel{\alpha^{10}} + \alpha^{14} + \alpha^7 + \alpha^6 + \alpha^{12} + \cancel{\alpha^{10}}$

$= (\cancel{\alpha^3} + \cancel{1}) + (\cancel{\alpha^3} + \cancel{\alpha} + \cancel{1}) + (\cancel{\alpha^3} + \cancel{\alpha^2}) + (\cancel{\alpha^3} + \cancel{\alpha^2} + \cancel{\alpha} + 1) = \underline{\underline{1}}$ syndrome 2

$S_3 = r(\alpha^3)$

$= \alpha^{15} + \alpha^{18} + \cancel{\alpha^{10}} + \alpha^8 + \alpha^{13} + \cancel{\alpha^{10}}$

$= (\cancel{1}) + (\cancel{\alpha^3}) + (\cancel{\alpha^2} + \cancel{1}) + (\cancel{\alpha^3} + \cancel{\alpha^2} + 1) = \underline{\underline{1}}$

$S_4 = r(\alpha^4)$

$= \alpha^{20} + \alpha^{22} + \alpha^{13} + \cancel{\alpha^{10}} + \alpha^{14} + \cancel{\alpha^{10}}$

$= \alpha^5 + \alpha^7 + \alpha^{13} + \alpha^{14} = (\cancel{\alpha^2} + \cancel{\alpha}) + (\cancel{\alpha^3} + \cancel{\alpha} + \cancel{1}) + (\cancel{\alpha^3} + \cancel{\alpha^2} + \cancel{1}) + (\alpha^3 + 1)$

$= \alpha^3 + 1 = \underline{\underline{\alpha^{14}}}$

Syndrome matrix, M $= \begin{bmatrix} S_1 & S_2 \\ S_2 & S_3 \end{bmatrix} = \begin{bmatrix} \alpha^3 & 1 \\ 1 & 1 \end{bmatrix}$

Coeff vector $= \begin{bmatrix} \Lambda_2 \\ \Lambda_1 \end{bmatrix}$; syndrome vector $= \begin{bmatrix} S_3 \\ S_4 \end{bmatrix} = \begin{bmatrix} 1 \\ \alpha^{14} \end{bmatrix}$

$$\Longrightarrow \begin{bmatrix} \alpha^3 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \Lambda_2 \\ \Lambda_1 \end{bmatrix} = \begin{bmatrix} 1 \\ \alpha^{14} \end{bmatrix}$$

$$\det(M) = \alpha^3 + 1 = \underline{\alpha^{14}}.$$

$$M^{-1} = \frac{1}{\det(M)} \cdot \operatorname{adj}(M) = \frac{1}{\alpha^{14}} \begin{bmatrix} 1 & 1 \\ 1 & \alpha^3 \end{bmatrix}$$

$$= \alpha \begin{bmatrix} 1 & 1 \\ 1 & \alpha^3 \end{bmatrix} = \begin{bmatrix} \alpha & \alpha \\ \alpha & \alpha^4 \end{bmatrix}$$

$$\begin{bmatrix} \Lambda_1 \\ \Lambda_2 \end{bmatrix} = M^{-1} S = \begin{bmatrix} \alpha & \alpha \\ \alpha & \alpha^4 \end{bmatrix} \begin{bmatrix} 1 \\ \alpha^{14} \end{bmatrix}$$

$$= \begin{bmatrix} \alpha + \alpha^{15} \\ \alpha + \alpha^{18} \end{bmatrix} = \begin{bmatrix} \alpha + 1 \\ \alpha + \alpha^3 \end{bmatrix} = \underline{\underline{\begin{bmatrix} \alpha^4 \\ \alpha^9 \end{bmatrix}}}$$

Error locator polynomial $= \underline{\underline{1 + \alpha^9 x + \alpha^4 x^2}}$

$= \sigma(x)$

we can see $\sigma(\alpha^{14})$ and $\sigma(\alpha^{12}) = 0$.

$\Longrightarrow$ $\frac{1}{\alpha^{14}}$ and $\frac{1}{\alpha^{12}}$ are roots.

ie α, α^3

Error vector is of structure ; $y_1 x + y_2 x^3$

$r(x) = c(x) + e(x)$

$r(\alpha) = \cancel{c(\alpha)}^{0} + e(\alpha)$

$\Longrightarrow$ $S_1 = r(\alpha) = e(\alpha)$

$S_2 = r(\alpha^2) = e(\alpha^2)$

$S_1 = y_1\alpha + y_2\alpha^3$

$S_2 = y_1\alpha^2 + y_2\alpha^6$

$$\begin{bmatrix} S_1 \\ S_2 \end{bmatrix} = \begin{bmatrix} \alpha & \alpha^3 \\ \alpha^2 & \alpha^6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\begin{bmatrix} S_1 \\ S_2 \end{bmatrix} = \begin{bmatrix} X_1 & X_2 \\ X_1^2 & X_2^2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \alpha & \alpha^3 \\ \alpha^2 & \alpha^6 \end{bmatrix}^{-1} \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}$$

$$= \frac{1}{(\alpha^7+\alpha^5)} \cdot \begin{bmatrix} \alpha^6 & \alpha^3 \\ \alpha^2 & \alpha \end{bmatrix} \begin{bmatrix} \alpha^3 \\ 1 \end{bmatrix}$$

$$= \frac{1}{\alpha^{13}} \begin{bmatrix} \alpha^6 & \alpha^3 \\ \alpha^2 & \alpha \end{bmatrix} \begin{bmatrix} \alpha^3 \\ 1 \end{bmatrix}$$

$$= \alpha^2 \begin{bmatrix} \alpha^9 + \alpha^3 \\ \alpha^5 + \alpha \end{bmatrix} = \alpha^2 \begin{bmatrix} \alpha \\ \alpha^2 \end{bmatrix} = \begin{bmatrix} \alpha^3 \\ \alpha^4 \end{bmatrix}$$

Error polynomial $e(x)$ $= \alpha^3 x + \alpha^4 x^3$

Corrected polynomial, $\hat{c}(x) = r(x) + e(x)$

$$= x^5 + \alpha^6 x^4 + \alpha x^3 + \alpha^2 x^2 + \alpha^{10} x + \alpha^{10}$$

$$+ \alpha^3 x + \alpha^4 x^3$$

$$= x^5 + \alpha^6 x^4 + (\alpha + \alpha^4) x^3 + \alpha^2 x^2 + (\alpha^{10} + \alpha^3) x + \alpha^{10}$$

$$\alpha + \alpha^4 = 1$$

$$\alpha^{10} + \alpha^3 = \cancel{\alpha^2 + \alpha + 1 + \alpha^3 \ldots = \alpha^{14}}$$

$$= \alpha^2 + \alpha + 1 + \alpha^3 = \alpha^{12}$$

$$\hat{c}(x) = x^5 + \alpha^6 x^4 + x^3 + \alpha^2 x^2 + \alpha^{12} x + \alpha^{10}$$ $\Longrightarrow$ original transmitted codeword.

Massey's + Forney's Algorithm :

$S_1 = \alpha^3$; $S_2 = 1$; $S_3 = 1$; $S_4 = \alpha^{14}$

① find error locator polynomial, $\sigma(x)$

$\sigma(x) = 1$; $P(x) = x$; $l = 0$

Step 1) $i = 1$; $l = 0$

$d = S_1 = \alpha^3$

$\sigma_n(x) = 1 + \alpha^3 x$

$2l < i$? yes

$l = i - l = 1$; $P(x) = \dfrac{\sigma(x)}{d} = \dfrac{1}{\alpha^3} = \alpha^{12}$

$P(x) = \alpha^{12} x$

Step 2: $i = 2$; $l = 1$

$$d = S_i + \sum_{j=1}^{l} \sigma_j S_{i-j} \qquad \sigma_1 = \alpha^3$$

$$= S_2 + \sigma_1 S_1 = 1 + \alpha^3 . \alpha^3 = 1 + \alpha^6 = \underline{\underline{\alpha^{13}}}$$

$$\sigma_D(x) = \sigma(x) + d\, P(x)$$

$$= (1 + \alpha^3 x) + \alpha^{13} . \alpha^{12} x$$

$$= 1 + (\alpha^3 + \alpha^{10}) x = \underline{\underline{1 + \alpha^{12} x}}.$$

is $2l < i$ NO

$$P(x) = x P(x) = \underline{\underline{\alpha^{12} x^2}}$$

Step 3: $i = 3$; $l = 1$

$$d = S_3 + \sigma_1 S_2 \qquad \sigma_1 = \alpha^{12}$$

$$= 1 + \alpha^{12} . 1 = 1 + \alpha^{12} = \underline{\underline{\alpha^{11}}}.$$

$$\sigma_D(x) = \sigma(x) + d\, P(x)$$

$$= (1 + \alpha^{12} x) + \alpha^{11} \alpha^{12} x^2$$

$$= \underline{\underline{1 + \alpha^{12} x + \alpha^8 x^2}}.$$

is $2l < i$ YES

$$l = 2 \; ; \; P = \frac{1 + \alpha^{12} x}{\alpha^{11}} = \alpha^4 + \alpha^{16} x$$

$$= \underline{\underline{\alpha^4 + \alpha x}}.$$

$$P(x) = \underline{\underline{\alpha^4 x + \alpha x^2}}$$

Step 4 : $i = 4$; $l = 2$

$$d = S_4 + \sigma_1 S_3 + \sigma_2 S_2$$

$$= \alpha^{14} + \alpha^{12} \cdot 1 + \alpha^{8} \cdot 1$$

$$= (\cancel{\alpha^3} + 1) + (\cancel{\alpha^3} + \cancel{\alpha^2} + \alpha + \cancel{1}) + (\cancel{\alpha^2} + \cancel{1}) = \alpha + 1 = \underline{\alpha^4}$$

$$\sigma_n(x) = \sigma(x) + d\,P(x)$$

$$= (1 + \alpha^{12}x + \alpha^{8}x^2) + \alpha^4 (\alpha^4 x + \alpha x^2)$$

$$= 1 + \alpha^{12}x + \alpha^{8}x^2 + \alpha^{8}x + \alpha^{5}x^2$$

$$= 1 + (\alpha^{12} + \alpha^{8})x + (\alpha^{8} + \alpha^{5})x^2$$

$$= \underline{\underline{1 + \alpha^{9}x + \alpha^{4}x^2}}$$

" $i = 2t \Rightarrow$ stop Itration "

(2) roots of $\sigma(x)$

α^{14} and $\alpha^{12} \Longrightarrow$ error locations $\frac{1}{\alpha^{14}}, \frac{1}{\alpha^{12}}$

$$= \underline{\underline{\alpha, \alpha^3}} \; !$$

(3) find error magnitude polynomial, $\Lambda(x)$

$$\Lambda(x) = \sigma(x) \cdot (1 + S(x)) \bmod x^{2t+1}$$

$$S(x) = \alpha^3 x + x^2 + x^3 + \alpha^{14} x^4$$

$$\Lambda(x) = (1 + \alpha^9 x + \alpha^4 x^2) * (1 + \alpha^3 x + x^2 + x^3 + \alpha^{14} x^4)$$

$$\Lambda(x) = 1 + \alpha^3 x + x^2 + x^3 + \alpha^{14} x^4$$
$$\alpha^9 x + \alpha^{12} x^2 + \alpha^9 x^3 + \alpha^9 x^4 + \alpha^{23} x^5$$
$$+ \alpha^4 x^2 + \alpha^7 x^3 + \alpha^4 x^4 + \alpha^4 x^5 + \alpha^{18} x^6 \quad . \bmod x^5$$

$$= 1 + (\alpha^3 + \alpha^9) x + (1 + \alpha^{12} + \alpha^4) x^2 + (1 + \alpha^9 + \alpha^7) x^3$$
$$+ (\alpha^{14} + \alpha^9 + \alpha^4) x^4 + (\alpha^{23} + \alpha^4) x^5 + \alpha^{18} x^6 \quad . \bmod x^5$$

$$= 1 + \alpha x + \alpha^{13} x^2 + 0 x^3 + 0 x^4 + \alpha^5 x^5 + \alpha^3 x^6$$

$\Lambda(x) . \bmod 5 \Longrightarrow$

$$\begin{array}{r|l} & \alpha^3 x + \alpha^5 \\ \hline x^5 & \alpha^3 x^6 + \alpha^5 x^5 + \alpha^{13} x^2 + \alpha x + 1 \\ & \alpha^3 x^6 + \\ \hline & \alpha^5 x^5 + \alpha^{13} x^2 + \alpha x + 1 \\ & \alpha^5 x^5 + \\ \hline & \alpha^{13} x^2 + \alpha x + 1 \end{array}$$

$$\Lambda(x) = \alpha^{13} x^2 + \alpha x + 1$$

$$\sigma(x) = 1 + \alpha^9 x + \alpha^4 x^2$$

$$\sigma'(x) = 0 + \alpha^9 + 2\alpha^4 . x \qquad (2\alpha^4 = \alpha^4 + \alpha^4 = 0)$$

$$= \underline{\underline{\alpha^9}}$$

$P_1 = 1 \Rightarrow \alpha$

$P_2 = 3 \Rightarrow \alpha^3$

$$m_1 = \frac{(\alpha^{P_1})\,(\Lambda(\alpha^{-P_1}))}{\sigma'(\alpha^{-P_1})}$$

$$= \frac{\alpha}{\alpha^9}\left[\alpha^{13}\cdot(\alpha^{-2}) + \alpha(\alpha^{-1}) + 1\right] = \alpha^{-8}\left[\alpha^{11} + 1 + 1\right]$$

$$= \underline{\underline{\alpha 3}}$$

$$m_2 = \frac{\alpha^{3}}{\alpha^{9}}\left[\alpha^{13}\cdot\alpha^{-6} + \alpha\,\alpha^{-3} + 1\right]$$

$$= \alpha^{-6}\left[\alpha^{7} + \alpha^{-2} + 1\right]$$

$$= \alpha + \alpha^{-8} + \alpha^{-6} = \alpha + \alpha^{7} + \alpha^{9}$$

$$= (\alpha) + (\cancel{\alpha^{3}} + \cancel{\alpha} + 1) + (\cancel{\alpha^{3}} + \cancel{\alpha}) = \alpha + 1 = \underline{\alpha^{4}}$$

Error polynomial :

$$e(x) = \underline{\alpha^{3}x + \alpha^{4}x^{3}}$$

Same as PGZ Algorithm.

Corrected polynomial $\hat{c}(x) = r(x) + e(x)$

$$= x^5 + \alpha^6 x^4 + \alpha x^3 + \alpha^2 x^2 + \alpha^{10} x + \alpha^{10}$$
$$+ \alpha^3 x + \alpha^4 x^3$$

$$= x^5 + \alpha^6 x^4 + (\alpha + \alpha^4) x^3 + \alpha^2 x^2 + (\alpha^{10} + \alpha^3) x + \alpha^{10}$$

$$\alpha + \alpha^4 = \alpha + (\alpha + 1) = 1$$

$$\alpha^{10} + \alpha^3 = (\alpha^2 + \alpha + 1) + (\alpha^3) = \alpha^{12}$$

$\Longrightarrow$

$$\hat{c}(x) = \underline{x^5 + \alpha^6 x^4 + x^3 + \alpha^2 x^2 + \alpha^{12} x + \alpha^{10}}$$

Same as original transmitted codeword.

GF (2^4)

Primitive poly $\Rightarrow \alpha^4 + \alpha + 1$

$0 \longrightarrow$	0	(0000)	(0)
$1 \longrightarrow$	1	(0001)	(1)
$\alpha \longrightarrow$	α	(0010)	(2)
$\alpha^2 \longrightarrow$	α^2	(0100)	(4)
$\alpha^3 \longrightarrow$	α^3	(1000)	(8)
$\alpha^4 \longrightarrow$	$\alpha+1$	(0011)	(3)
$\alpha^5 \longrightarrow$	$\alpha^2+\alpha$	(0110)	(6)
$\alpha^6 \longrightarrow$	$\alpha^3+\alpha^2$	(1100)	(12)
$\alpha^7 \longrightarrow$	$\alpha^3+\alpha+1$	(1011)	(11)
$\alpha^8 \longrightarrow$	α^2+1	(0101)	(5)
$\alpha^9 \longrightarrow$	$\alpha^3+\alpha$	(1010)	(10)
$\alpha^{10} \longrightarrow$	$\alpha^2+\alpha+1$	(0111)	(7)
$\alpha^{11} \longrightarrow$	$\alpha^3+\alpha^2+\alpha$	(1110)	(14)
$\alpha^{12} \longrightarrow$	$\alpha^3+\alpha^2+\alpha+1$	(1111)	(15)
$\alpha^{13} \longrightarrow$	$\alpha^3+\alpha^2+1$	(1101)	(13)
$\alpha^{14} \longrightarrow$	α^3+1	(1001)	(9)

GF (2^3)

Primitive poly $= \alpha^3 + \alpha + 1$

0	$\longrightarrow$	0	(0000)	(0)
1	$\longrightarrow$	1	(001)	(1)
α	$\longrightarrow$	α	(010)	(2)
α^2	$\longrightarrow$	α^2	(100)	(4)
α^3	$\longrightarrow$	$\alpha+1$	(011)	(3)
α^4	$\longrightarrow$	$\alpha^2+\alpha$	(110)	(6)
α^5	$\longrightarrow$	$\alpha^2+\alpha+1$	(111)	(7)
α^6	$\longrightarrow$	α^2+1	(101)	(5)