tanh rule for every check node in parity check matrix of LDPC :

An example 3 bit case is taken here and later an expansion to general n bits is also shown.

$$c_1 = c_2 \oplus c_3$$

input from channel = $[r_1 \; r_2 \; r_3]$; each corresponding to $[c_1 \; c_2 \; c_3]$

Intrinsic llr for $l_2 = \log \left\{ \frac{P(c_2=0 \mid r_2)}{P(c_2=1 \mid r_2)} \right\}$

let $P(c_2 = 0 \mid r_2) = P_2$

$$\Rightarrow P(c_2 = 1 \mid r_2) = 1 - P_2$$

$$\Rightarrow l_2 = \log \left\{ \frac{P_2}{1-P_2} \right\}$$

Similarly $l_3 = \log \left\{ \frac{P_3}{1-P_3} \right\}$

Extrinsic llr on c_1 from c_2 and c_3 ?

ie $P(c_1 = 0 \mid r_2, r_3)$ and $P(c_1 = 1 \mid r_2, r_3)$

$$l_{ext,1} = \log \left\{ \frac{P(c_1 = 0 \mid r_2, r_3)}{P(c_1 = 1 \mid r_2, r_3)} \right\}$$

let $P_1 = P(c_1 = 0 | r_2, r_3)$

$$\Rightarrow (1-P_1) = P(c_1 = 1 | r_2, r_3)$$

$$\Rightarrow l_{ext,1} = \log \left\{ \frac{P_1}{1-P_1} \right\}$$

Truth Table : $c_1 = c_2 \oplus c_3$

c_1	c_2	c_3
0	0	0
0	1	1
1	0	1
1	1	0

first two rows are cases of $c_1 = 0$

$$P_1 = P(c_2 = 0) \times P(c_3 = 0) + P(c_2 = 1) \times P(c_3 = 1)$$

$$= P_2 \cdot P_3 + (1-P_2)(1-P_3) \quad \text{——— (1)}$$

Similarly from row 3 and 4

$$(1-P_1) = P(c_2 = 0) \cdot P(c_3 = 1) + P(c_2 = 1) \cdot P(c_3 = 0)$$

$$= P_2 (1-P_3) + (1-P_2) P_3 \quad \text{——— (2)}$$

(1) − (2) $\Rightarrow$

$$[P_1 - (1-P_1)] = [P_2 - (1-P_2)] \cdot [P_3 - (1-P_3)]$$

dividing by $(p_1 + (1-p_1)) = 1$ on LHS

and $(p_2 + (1-p_2))$ and $(p_3 + (1-p_3))$ on RHS

$$\frac{[p_1 - (1-p_1)]}{p_1 + (1-p_1)} = \frac{[p_2 - (1-p_2)]}{p_2 + (1-p_2)} \cdot \frac{[p_3 - (1-p_3)]}{p_3 + (1-p_3)}$$

$$\frac{1 - \frac{(1-p_1)}{p_1}}{1 + \frac{(1-p_1)}{p_1}} = \frac{\left(1 - \frac{(1-p_2)}{p_2}\right)}{1 + \left(\frac{1-p_2}{p_2}\right)} \cdot \frac{\left(1 - (1-p_3)/p_3\right)}{1 + (1-p_3)/p_3} \quad \text{--- (3)}$$

Note : $l_{ext,1} = \log \left\{ \frac{p_1}{1-p_1} \right\}$

$$\Longrightarrow \left(\frac{p_1}{1-p_1}\right) = e^{l_{ext,1}}$$

Similarly $e^{l_2} = \frac{p_2}{1-p_2}$

$$e^{l_3} = \frac{p_3}{1-p_3}$$

(3) becomes :

$$\frac{1 - e^{-l_{ext,1}}}{1 + e^{-l_{ext,1}}} = \left(\frac{1 - e^{-l_2}}{1 + e^{-l_2}}\right) \cdot \left(\frac{1 - e^{-l_3}}{1 + e^{-l_3}}\right) \quad \text{--- (4)}$$

note : $\tanh(x) = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$

$$= \frac{1 - e^{-2x}}{1 + e^{-2x}}$$

④ becomes

$$\boxed{\tanh\left(\frac{l_{ext,1}}{2}\right) = \tanh\left(\frac{l_2}{2}\right) \cdot \tanh\left(\frac{l_3}{2}\right)}$$

MINSUM approximation on tanh rule.

tanh is an odd function.

$$\text{Sign}(l_{ext,1}) = \text{Sign}(l_2) \cdot \text{Sign}(l_3)$$

$$\tanh\left(\frac{|l_{ext,1}|}{2}\right) = \tanh\left(\frac{|l_2|}{2}\right) \cdot \tanh\left(\frac{|l_3|}{2}\right)$$

taking log on both sides

$$\log \tanh\left(\frac{|l_{ext,1}|}{2}\right) = \log \tanh\left(\frac{|l_2|}{2}\right) + \log \tanh\left(\frac{|l_3|}{2}\right)$$

define a function, $f(x) = \left|\log \tanh\left(\frac{|x|}{2}\right)\right|$

Note: tanh is less than 1, so log will give -ve values;

A property of $f(x)$:

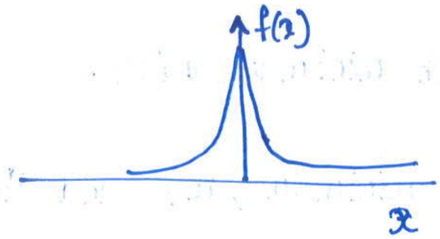
$$f^{-1}(x) = f(x)$$

$$f(|l_{ext,1}|) = f(|l_2|) + f(|l_3|)$$

$$f^{-1}\left(f(|l_{ext,1}|)\right) = f^{-1}\left[f(|l_2|) + f(|l_3|)\right]$$

$$|l_{ext,1}| = f^{-1}\left[f(|l_2|) + f(|l_3|)\right] \quad \text{——— (5)}$$

Plot of $f(x)$ =



for small x, $f(x)$ is larger.

$$\Longrightarrow \quad f(|l_2|) + f(|l_3|) \approx f(\min(|l_2|, |l_3|))$$

(5) becomes.

$$|l_{ext,1}| \approx f^{-1}\left[f(\min(|l_2|, |l_3|))\right]$$

$$\boxed{|l_{ext,1}| \approx \min(|l_2|, |l_3|)}$$

min-sum approximation of tanh rule.

Generalization to n bit code :

$$\mathrm{Sign}(l_{ext,1}) = \mathrm{Sign}(l_2) \cdot \mathrm{Sign}(l_3) \cdots \times \mathrm{Sign}(l_n)$$

$$|l_{ext,1}| = \min(|l_2|, |l_3| \ldots |l_n|)$$

Note 1: dealing with sign

define $S = \text{Sign}(l_1) \cdot \text{Sign}(l_2) \cdot \cdots \times \text{Sign}(l_n)$

$\text{Sign}(l_{ext,k}) = S \cdot \text{Sign}(l_k)$

Note 2: dealing with minimum values.

find overall minimum, m_1 and its position, P_{os}

$$m_1 = \min \{ |l_1|, |l_2| \cdots |l_n| \}$$

find second minimum m_2.

$$m_2 = \min \{ |l_1|, |l_2| \cdots |l_{pos-1}|, |l_{pos+1}| \cdots |l_n| \}$$

$\Longrightarrow |l_{ext,i}| = m_1$ for all $i \neq pos$

$|l_{ext,pos}| = m_2$