Math 51 Notes

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Contents

1 Planes in $\mathbf{R}^3$

1.1 Exercise 3.1

Consider the three different points $(3, -2, 5)$, $(\frac{1}{2}, 0, 4)$, and $(1, -2, 10)$ in $\mathbb{R}^3$.

(a) Use difference vectors to show these points are not on a common line, so there is exactly one plane $\mathcal{P}$ containing all of them. Let

$$P = (3, -2, 5)$$

$$Q = \frac{1}{2}, 0, 4$$

$$R = (1, -2, 10)$$

$$\overrightarrow{PQ} = Q - P = \begin{bmatrix} \frac{5}{2} \\ 2 \\ -1 \end{bmatrix}$$

$$\overrightarrow{PR} = R - P = \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}$$

we need to check whether $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are linearly independent. There should not be any scalar a and b both non-zero such that

$$a\overrightarrow{PQ} + b\overrightarrow{PR} = 0$$

$$a. \begin{bmatrix} \frac{5}{2} \\ 2 \\ -1 \end{bmatrix} + b. \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix} = 0$$

$$\frac{5}{2}a - 2b = 0$$

$$2a = 0$$

$$-a + 5b = 0$$

Substituting

$$a = 0$$

we get

$$0 + 5.b = 0$$

This gives

$$a = 0, b = 0$$

This shows that $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are linearly independent and the points

$$P, Q, R$$

are not on a common line . So there is exactly one plat $\mathcal{P}$ containing all of them.

(b) Give a parametric form for the plane $\mathcal{P}$ from (a).

The plane consists of all vectors of the form

$$P + t\overrightarrow{PQ} + t'\overrightarrow{PR}$$

for scalars t and t'

$$\begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} + t \begin{bmatrix} \frac{5}{2} \\ 2 \\ -1 \end{bmatrix} + t' \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}$$

The parametric form of equation is

$$X = \begin{bmatrix} 3 + \frac{5}{2}t - 2t' \\ -2 + 2t \\ 5 - t + 5t' \end{bmatrix}$$

1.2 Exercise 3.2

Give an equation describing the plane $\mathcal{P}$ in Exercise 3.1

Let non-zero vector n =

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

that is perpendicular to all direction of plane in Exercise 3.1

As the normal vector n is perpendicular we have

$$n.\overrightarrow{PQ} = 0$$

$$n.\overrightarrow{PR} = 0$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} . \begin{bmatrix} \frac{5}{2} \\ 2 \\ -1 \end{bmatrix} = 0$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} . \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix} = 0$$

$$\frac{5}{2}a + 2b - c = 0$$

$$-2a + 5c = 0$$

Substituting c = $\frac{5}{2}$a + 2b we get

$$-2a + 5(\frac{5}{2}a + 2b) = 0$$

$$-4a + 25a + 20b = 0$$

$$\frac{-21}{20}a = b$$

$$c = \frac{5}{2}a + 2 * \frac{-21}{20}a$$

$$c = \frac{9}{20}a$$

It follows that n =

$$\begin{bmatrix} a \\ \frac{-21}{20}a \\ \frac{9}{20}a \end{bmatrix}$$

when n $\neq$ 0 and a $\neq$ 0, let's put a = 1

$$\begin{bmatrix} 1 \\ \frac{-21}{20} \\ \frac{9}{20} \end{bmatrix}$$

Finally let

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

be a point on the plane. The vector from P (3,-2,5) to the above point is the difference vector

$$\begin{bmatrix} x-3 \\ y-2 \\ z-5 \end{bmatrix}$$

The vector from P (3,-2,5) to the above point is perpendicular to the normal vector n, So

$$\begin{bmatrix} x-3 \\ y-2 \\ z-5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \frac{-21}{20} \\ \frac{9}{20} \end{bmatrix} = 0$$

$$x - 3 - (y-3)\frac{21}{20} + (z-5).\frac{9}{20} = 0$$

$$20x - 60 - 21y + 63 + 9z - 45 = 0$$

$$20x - 21y + 9z = 42$$

1.3 Exercise 3.3

Find the parametric form for the plane $\mathbb{R}^3$ given by the equation

$$6x - 6y - z = 7$$

Let

$$x = 1, y = 1$$

Then we from the equation

$$6x - 6y - z = 7$$

we have z = -7 So

$$P = (1, 1, -7)$$

Let

$$x = 0, y = 1$$

Then we from the equation

$$6x - 6y - z = 7$$

we have z = -13 So

$$Q = (0, 1, -13)$$

Let

$$x = 1, y = 0$$

Then we from the equation

$$6x - 6y - z = 7$$

we have z = -1 So

$$R = (1, 0, -1)$$

The parametric form of the for a plane is given by

$$P + t\overrightarrow{PQ} + t'\overrightarrow{PR}$$

$$\overrightarrow{PQ} = \begin{bmatrix} -1 \\ 0 \\ -6 \end{bmatrix}$$

$$\overrightarrow{PR} = \begin{bmatrix} 0 \\ -1 \\ 6 \end{bmatrix}$$

$$P + t\overrightarrow{PQ} + t'\overrightarrow{PR} =$$

$$\begin{bmatrix} -7 \\ 1 \\ 1 \end{bmatrix} + t. \begin{bmatrix} -1 \\ 0 \\ -6 \end{bmatrix} + t' \begin{bmatrix} 0 \\ -1 \\ 6 \end{bmatrix} =$$

$$\begin{bmatrix} 1 - t \\ 1 - t' \\ -7 - 6t + 6t' \end{bmatrix}$$

1.4 Exercise 3.4

Find the equation for the plane parallel to

$$4x - 7y + 2z = 1$$

and passing through the point

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

The equation of a plane is given by

$$Ax + By + Cz = D$$

where A,B,C are coefficients of plane's normal vector and D is the constant. The given plane is

$$4x - 7y + 2z = 1$$

So the plane parallel to given plane will also have the same coefficients of plane's normal vector. So the equation of the new plane parallel to given plane is also

$$4x - 7y + 2z = D$$

As the new plane passes through the point

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

, we can substitute this point in above equation:

$$4.1 - 7.2 + 2.3 = D$$

$$D = -4$$

So the equation for the plane parallel to

$$4x - 7y + 2z = 1$$

and passing through the point

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

is

$$4x - 7y + 2z = -4$$

1.5 Exercise 3.5

Let P, Q be different points in $\mathbb{R}^3$, and let $\mathcal{P}$ be the collection of 3-vectors $\mathbf{v}$ with the same distance from P and from Q:

$$\|\mathbf{P} - \mathbf{v}\| = \|\mathbf{Q} - \mathbf{v}\|.$$

(a) By squaring both sides of the distance equality and using that $\|\mathbf{w}\|^2 = \mathbf{w} \cdot \mathbf{w}$, show that $\mathcal{P}$ consists of exactly those 3-vectors $\mathbf{v}$ satisfying

$$\mathbf{v} \cdot (\mathbf{P} - \mathbf{Q}) = \frac{1}{2}\left(\|\mathbf{P}\|^2 - \|\mathbf{Q}\|^2\right).$$

In particular, the plane $\mathcal{P}$ has (nonzero) normal direction $\mathbf{P} - \mathbf{Q}$. Draw a picture to illustrate why it is reasonable that this is a plane with normal direction $\mathbf{P} - \mathbf{Q}$.

Solution:

$$\|\mathbf{P} - \mathbf{v}\| = \|\mathbf{Q} - \mathbf{v}\|.$$

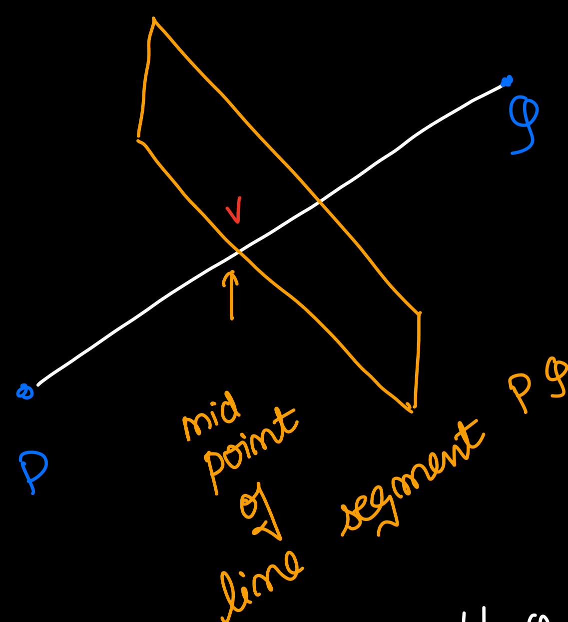
Squaring both sides:

$$\|\mathbf{P}\|^2 + \|\mathbf{v}\|^2 - 2(\mathbf{P} \cdot \mathbf{v}) = \|\mathbf{Q}\|^2 + \|\mathbf{v}\|^2 - 2(\mathbf{Q} \cdot \mathbf{v})$$

Rearranging the terms

$$\mathbf{v} \cdot (\mathbf{P} - \mathbf{Q}) = \frac{1}{2}\left(\|\mathbf{P}\|^2 - \|\mathbf{Q}\|^2\right).$$

See diagram below:



$$\|p - v\| = \|q - v\|$$

(b) For the case $\mathbf{P} = \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}$ and $\mathbf{Q} = \begin{bmatrix} -1 \\ 5 \\ -2 \end{bmatrix}$, give an explicit equation for the plane $\mathcal{P}$.

We have

$$\mathbf{v} \cdot (\mathbf{P} - \mathbf{Q}) = \frac{1}{2} \left(\|\mathbf{P}\|^2 - \|\mathbf{Q}\|^2 \right).$$

$$P = \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}, Q = \begin{bmatrix} -1 \\ 5 \\ -2 \end{bmatrix}$$

$$P - Q = \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix}$$

$$\|(\mathbf{P}\|)^2 = (3^2 + 4^2 + 3^2) = (9 + 16 + 9) = 34$$

$$\|(\mathbf{Q}\|)^2 = ((-1)^1 + (5)^2 + (-2)^2) = (1 + 25 + 4) = 30$$

Let

$$\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

be any point on the plane.

$$\mathbf{v} \cdot (\mathbf{P} - \mathbf{Q}) = \frac{1}{2} \left(\|\mathbf{P}\|^2 - \|\mathbf{Q}\|^2 \right).$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix} = \frac{1}{2} \left(\|\mathbf{P}\|^2 - \|\mathbf{Q}\|^2 \right)$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix} = \frac{34 - 30}{2} = 2$$

Equation for the plane $\mathcal{P}$:

$$4x - y + 5z = 2$$

2 Span, subspaces, and dimension

2.1 Exercise 4.1

(a)

$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : z = 2x - y \right\}$$

A set of vectors $S \subseteq \mathbb{R}^n$ is a linear subspace if and only if:

(a) The zero vector is in S.

(b) S is closed under vector addition: if $\mathbf{u}, \mathbf{v} \in S$, then $\mathbf{u} + \mathbf{v} \in S$.

(c) S is closed under scalar multiplication: if $\mathbf{u} \in S$ and $c \in \mathbb{R}$, then $c \cdot \mathbf{u} \in S$.

Checking each condition:

(a) **The zero vector is in S.**

$$\mathbf{u} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Check if it satisfies $z = 2x - y$:

$$2 \cdot 0 - 0 = 0.$$

This is true, so the zero vector is in the set.

(b) **S is closed under vector addition.**

Let

$$\mathbf{u} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}.$$

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}.$$

$$z_1 + z_2 = 2 \cdot (x_1 + x_2) - (y_1 + y_2).$$

$$z_1 = 2 \cdot x_1 - y_1, \quad z_2 = 2 \cdot x_2 - y_2.$$

$$z_1 + z_2 = 2 \cdot (x_1 + x_2) - (y_1 + y_2).$$

Thus, the set is closed under addition.

(c) **S is closed under scalar multiplication.**

Let

$$\mathbf{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

$$c \cdot \mathbf{u} = \begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}.$$

$$cz = 2 \cdot (cx) - (cy).$$

But

$$z = 2x - y.$$

So

$$cz = c(2x - y) = 2 \cdot (cx) - (cy).$$

Thus, the set is closed under scalar multiplication. So this is a linear subspace.

(b)

$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : z = 1 + 2x - y \right\}$$

A set of vectors $S \subseteq \mathbb{R}^n$ is a linear subspace if and only if:

(a) The zero vector is in S.

(b) S is closed under vector addition: if $\mathbf{u}, \mathbf{v} \in S$, then $\mathbf{u} + \mathbf{v} \in S$.

(c) S is closed under scalar multiplication: if $\mathbf{u} \in S$ and $c \in \mathbb{R}$, then $c \cdot \mathbf{u} \in S$.

Checking each condition:

(a) **The zero vector is in S.**

$$\mathbf{u} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Check if it satisfies $z = 1 + 2x - y$:

$$2 \cdot 0 - 0 = 1 \neq 0$$

This is false, so the zero vector is not in the set. So this is not a linear subspace.

(c)

$$\left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : y = x^2 \right\}$$

A set of vectors $S \subseteq \mathbb{R}^n$ is a linear subspace if and only if:

(a) The zero vector is in S.

(b) S is closed under vector addition: if $\mathbf{u}, \mathbf{v} \in S$, then $\mathbf{u} + \mathbf{v} \in S$.

(c) S is closed under scalar multiplication: if $\mathbf{u} \in S$ and $c \in \mathbb{R}$, then $c \cdot \mathbf{u} \in S$.

Checking each condition:

(a) **The zero vector is in S.**

$$\mathbf{u} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Check if it satisfies $y = x^2$:

$$0 = 0^2 = 0$$

This is true, so the zero vector is in the set.

(b) S is closed under vector addition: if $\mathbf{u}, \mathbf{v} \in S$, then $\mathbf{u} + \mathbf{v} \in S$.
Let

$$\mathbf{u} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}.$$

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}.$$

$$y_1 + y_2 = (x_1 + x_2)^2$$

Here

$$y1 = x1^2, y2 = x2^2$$

$$y1 + y2 = x1^2 + x2^2 \neq (x1 + x2)^2$$

S is NOT closed under vector addition. So this is not a linear subspace.

(d)

$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : \begin{array}{l} 3x - y + z = 0, \\ x + y - 4z = 0 \end{array} \right\}$$

A set of vectors $S \subseteq \mathbb{R}^n$ is a linear subspace if and only if:

(a) The zero vector is in S.

(b) S is closed under vector addition: if $\mathbf{u}, \mathbf{v} \in S$, then $\mathbf{u} + \mathbf{v} \in S$.

(c) S is closed under scalar multiplication: if $\mathbf{u} \in S$ and $c \in \mathbb{R}$, then $c \cdot \mathbf{u} \in S$.

Checking each condition:

(a) **The zero vector is in S.**

$$\mathbf{u} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Check if it satisfies

$$3x - y + z = 0,$$
$$x + y - 4z = 0$$

$$3 \cdot 0 - 0 + 0 = 0$$

$$0 + 0 - 4.0 = 0$$

This is true, so the zero vector is in the set.

(b)

(c) **S is closed under vector addition.**

Let

$$\mathbf{u} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}.$$

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}.$$

$$3x - y + z = 0,$$
$$x + y - 4z = 0$$

$$z = -3x + y$$

$$z = (x + y)/4$$

$$z_1 + z_2 = -3(x_1 + x_2) + y$$
$$-3(x_1) + y_1 + -3(x_2) + y_2 = -3(x_1 + x_2) + y_1 + y_2$$
$$-3(x_1 + x_2) + y_1 + y_2 = -3(x_1 + x_2) + y_1 + y_2$$

$$z1 + z_2 = (x_1 + x_2 + y_1 + y_2)/4$$
$$(x_1 + y_1)/4 + (x_2 + y_2)/4 = (x_1 + x_2 + y_1 + y_2)/4$$
$$(x_1 + y_1 + x_2 + y_2)/4 = (x_1 + x_2 + y_1 + y_2)/4$$

This is true so S is closed under vector addition.

(d) S **is closed under scalar multiplication.**

Let

$$\mathbf{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

$$c \cdot \mathbf{u} = \begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}.$$

$$cz = -3cx_1 + cy_1$$
$$c(-3x_1 + y_1) = -3cx_1 + cy_1$$
$$-3cx_1 + cy_1 = -3cx_1 + cy_1$$

$$cz_1 = (cx_1 + cy_1)/4$$

$$c(x_1 + y_1)/4 = (cx_1 + cy_1)/4$$
$$(cx_1 + cy_1)/4 = (cx_1 + cy_1)/4$$

This is true so S is closed under scalar multiplication. So this is a linear subspace.

2.2 Exercise 4.2

For

$$v = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad w = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix},$$

find scalars a, b, c so that

$$\operatorname{span}(v, w) = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : ax + by + cz = 0 \right\}.$$

Here ax + by + cz = 0

From

$$v = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

we have

$$a(2) + 0(b) + 1(c) = 0$$

$$c = -2(a)$$

From

$$w = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$$

we have

$$a(-1) + 1(b) + 3(c) = 0$$

$$b = a - 3c$$

Therefore we have scalars a,b,c such that

$$b = a - 3c$$

$$c = -2a$$

Substituting a =1

$$c = -2$$

$$b = 1 - 3(-2.1) = 7$$

Substituting we have

$$1(x) + 7(y) + -2(z) = 0$$

From

$$v = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$1(x) + 7(y) + -2(z) = 0$$

we have

$$1(2) + 7(0) + -2(1) = 2 - 2 = 0$$

From

$$w = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$$

we have

$$1(x) + 7(y) + -2(z) = 0$$

$$1(-1) + 7(1) + -2(3)$$

$$= -1 + 7 + -6 = -7 + 7 = 0$$

The resulting triplets work.

2.3 Exercise 4.3

For

$$v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad w = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix},$$

find scalars a, b, c so that

$$\operatorname{span}(v, w) = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : ax + by + cz = 0 \right\}.$$

For

$$v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad w = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix},$$

find scalars a, b, c so that

$$\operatorname{span}(v, w) = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : ax + by + cz = 0 \right\}.$$

Here ax + by + cz = 0

From

$$v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

we have

$$a(1) + 1(b) + 1(c) = 0$$
$$b = -(a + c)$$

From

$$w = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

we have

$$a(4) + 2(b) + 1(c) = 0$$
$$c = -(4a + 2b)$$

Therefore we have scalars a,b,c such that

$$b = -(a + c)$$

$$c = -(4a + 2b)$$

Substituting a =1 and solving for b and c we get

$$c = 2$$

$$b = -3$$

Substituting we have

$$1(x) - 3(y) + 2(z) = 0$$

From

$$v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$1(x) - 3(y) + 2(z) = 0$$

we have

$$1(1) - 3(1) + 2(1) = 3 - 3 = 0$$

From

$$w = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

we have

$$1(x) - 3(y) + 2(z) = 0$$

$$1(4) + -3(2) + 2(1)$$

$$= 6 - 6 = 0$$

The resulting triplets work.

2.4 Exercise 4.4

For the 4-vectors

$$w = \begin{bmatrix} -2 \\ 2 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad w' = \begin{bmatrix} 3 \\ 4 \\ 0 \\ 1 \end{bmatrix},$$

show that the collection of vectors

$$V = \left\{ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 : x \cdot w = 0,\ x \cdot w' = 0 \right\}$$

is a linear subspace of $\mathbb{R}^4$ in each of the following ways:

(a) For $x \in V$, solve for each of x_3 and x_4 in terms of x_1 and x_2 to write V as a span of two vectors;

(b) For $x \in V$, solve for each of x_1 and x_4 in terms of x_2 and x_3 to write V as a span of two vectors.

Solution:

(a) For w,

$$w \cdot x = -2x_1 + 2x_2 + x_3 + x_4 = 0 \quad \Rightarrow \quad x_3 + x_4 = 2x_1 - 2x_2 \tag{1}$$

For w',

$$w' \cdot x = 3x_1 + 4x_2 + x_4 = 0 \quad \Rightarrow \quad x_4 = -3x_1 - 4x_2 \tag{2}$$

Substitute x_4 from (2) into (1):

$$x_3 + (-3x_1 - 4x_2) = 2x_1 - 2x_2$$

$$x_3 = 5x_1 + 2x_2 \tag{3}$$

Thus, the components of x are:

$$x_3 = 5x_1 + 2x_2, \quad x_4 = -3x_1 - 4x_2$$

Substitute these into x:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ 5x_1 + 2x_2 \\ -3x_1 - 4x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 5 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 2 \\ -4 \end{bmatrix}$$

The basis vectors are:

$$\begin{bmatrix} 1 \\ 0 \\ 5 \\ -3 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 2 \\ -4 \end{bmatrix}.$$

(b) For w,

$$w \cdot x = -2x_1 + 2x_2 + x_3 + x_4 = 0 \quad \Rightarrow \quad x_4 = -2x_1 + 2x_2 - x_3 \tag{4}$$

For w',

$$w' \cdot x = 3x_1 + 4x_2 + x_4 = 0 \quad \Rightarrow \quad x_4 = -3x_1 - 4x_2 \tag{5}$$

Equating x_4 from (4) and (5):

$$-2x_1 + 2x_2 - x_3 = -3x_1 - 4x_2$$

Simplify:

$$x_1 + 6x_2 - x_3 = 0 \quad \Rightarrow \quad x_1 = -6x_2 + x_3 \tag{6}$$

Substitute x_1 from (6) into (4):

$$x_4 = -2(-6x_2 + x_3) + 2x_2 - x_3$$

$$x_4 = 12x_2 - 2x_3 + 2x_2 - x_3$$

$$x_4 = 14x_2 - 3x_3 \tag{7}$$

Thus, the components of x are:

$$x_1 = -6x_2 + x_3, \quad x_4 = 14x_2 - 3x_3$$

Substitute these into x:

$$x = \begin{bmatrix} -6x_2 + x_3 \\ x_2 \\ x_3 \\ 14x_2 - 3x_3 \end{bmatrix} = x_2 \begin{bmatrix} -6 \\ 1 \\ 0 \\ 14 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \\ -3 \end{bmatrix}$$

The basis vectors are:

$$\begin{bmatrix} -6 \\ 1 \\ 0 \\ 14 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 1 \\ -3 \end{bmatrix}.$$

2.5 Exercise 4.5

Find a nonzero 3-vector $\mathbf{v}$ so that

$$\left\{ \mathbf{x} \in \mathbb{R}^3 : \mathbf{x} \cdot \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = 0, \, \mathbf{x} \cdot \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} = 0 \right\} = \mathrm{span}(\mathbf{v}).$$

Then, using the *geometric* fact that any two different planes through the origin in $\mathbb{R}^3$ meet along a line through the origin, interpret this algebraic outcome that the left side is the span of a single vector.

Solution:

Let a,b,c be scalar so that

$$\begin{aligned} 3a + 2b + c &= 0 \\ b &= \frac{-c - 3a}{2} \\ -2a - b + c &= 0 \\ -2a - \frac{-c - 3a}{2} + c &= 0 \\ c &= \frac{a}{3} \\ 2b &= -\frac{a}{3} - 3a \\ b &= -\frac{5a}{3} \end{aligned} \tag{1}$$

writing v in terms of a,

$$\begin{bmatrix} a \\ -\frac{5a}{3} \\ \frac{a}{3} \end{bmatrix}$$

$$v = a. \begin{bmatrix} 1 \\ -\frac{5}{3} \\ \frac{1}{3} \end{bmatrix}$$

The two planes defined by the equations

$$3a + 2b + c = 0 \tag{2}$$

$$-2a - b + c = 0 \tag{3}$$

intersect along a line through the origin in $\mathbb{R}^3$ This line is spanned by the vector v. Hence, the solution set is

$$a. \begin{bmatrix} 1 \\ -\frac{5}{3} \\ \frac{1}{3} \end{bmatrix}$$

Here a $\neq$ 0 so the span is

$$\begin{bmatrix} 1 \\ -\frac{5}{3} \\ \frac{1}{3} \end{bmatrix}$$

2.6 Exercise 4.6

Find a pair of 3-vectors $\mathbf{v}, \mathbf{w}$ so that

$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : 2x - 3y + 2z = 0 \right\} = \operatorname{span}(\mathbf{v}, \mathbf{w}).$$

we have

$$2x - 3y + 2z = 0$$

Therefore

$$x = \frac{3y - 2z}{2}$$

we can write x=

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

as x=

$$\begin{bmatrix} \frac{3y-2z}{2} \\ y \\ z \end{bmatrix}$$

x=

$$y \begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Two linearly independent vectors spanning the subspace are:

$$v1 = \begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \end{bmatrix}$$

and

$$v2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

2.7 Exercise 4.7

part (a)

$$v = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad w = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$v' = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \quad w' = \begin{bmatrix} -2 \\ -3 \\ 4 \end{bmatrix}$$

Find scalars a, b, c, d such that

$$v' = av + bw$$
$$w' = cv + dw$$

$$v' = a\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

$$a + b = 3$$
$$\boxed{b = 1}$$
$$a - b = 1 \implies a - 1 = 1$$
$$\implies \boxed{a = 2}$$

$$v' = 2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

$$w' = a\begin{bmatrix}1\\0\\1\end{bmatrix} + b\begin{bmatrix}1\\1\\-1\end{bmatrix} = \begin{bmatrix}-2\\-3\\4\end{bmatrix}$$

$$= a + b = -2$$

$$\boxed{b = -3}$$

$$a - b = 4$$

$$a - (-3) = 4$$

$$\boxed{a = 1}$$

$$w' = 1\begin{bmatrix}1\\0\\1\end{bmatrix} - 3\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}-2\\-3\\4\end{bmatrix}$$

part (b)

plane spanned by v and w

$ax + by + cz = 0$ where

$\begin{bmatrix}a\\b\\c\end{bmatrix}$ is normal vector perpendicular to both v and w

$$v = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad w = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

The normal vector perpendicular to v and w is got by the cross product of v and w

$$v \times w = \begin{vmatrix} i & j & k \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$v \times w = i \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} - j \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} + k \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$$

$$i = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} \Rightarrow (0 * -1) - (1 * 1) = -1$$

$$j = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = 1 * (-1) - (1 * 1) = -2$$

$$k = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1 \cdot 1 - 0 \cdot 1 = 1$$

$$v * w = (-1) i - (-2) j + (1) k = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$v * w = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$v' = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \text{ and } w' = \begin{bmatrix} -2 \\ -3 \\ 4 \end{bmatrix}$$

$$v * w = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$-1x + 2y + z = 0$$

Substituting $v' = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$

$$v' = -1(3) + 2(1) + (1) =$$

$$-3+3 = 0$$

Substituting $w' = \begin{bmatrix} -2 \\ -3 \\ 4 \end{bmatrix}$

$$w' = -1(-2) + 2(-3) + 4$$

$$= 2 - 6 + 4$$

$$= 0$$

2.8 Exercise 4.8

Use Theorem 4.2.5 to determine the dimension (1, 2, or 3) of each of the following linear subspaces in $\mathbb{R}^3$. Show the work to justify your answer.

(a) span($\mathbf{v}, \mathbf{w}$) for

$$\mathbf{v} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

(b) span($\mathbf{v}', \mathbf{w}'$) for

$$\mathbf{v}' = \begin{bmatrix} 3 \\ 6 \\ -2 \end{bmatrix}, \quad \mathbf{w}' = \begin{bmatrix} -2 \\ -4 \\ 2 \end{bmatrix}.$$

(c) span($\mathbf{v}'', \mathbf{w}'', \mathbf{u}''$) for

$$\mathbf{v}'' = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \quad \mathbf{w}'' = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{u}'' = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}.$$

Solution:

(a) span($\mathbf{v}, \mathbf{w}$) for

$$\mathbf{v} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Here check if we can combine u and v such that they cancel out each other. This can be checked by finding out a , b such that

$$a.v + b.w = 0$$

a.v means we are stretching (or shrinking) v by a. similar for b.w by combining a.v+b.w, we get a new vector if this new vector is 0 then they cancel each other. If we find non-zero values of a and b such a.v+b.w is 0 then it means v, w are linearly dependent. If the only solution is a=0 and b=0 then then v, w are linearly independent and v and we point to different directions.

$$a.v + b.w = 0$$

$$a. \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} + b. \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0$$

This gives

$$-2a + b = 0$$

$$a + b = 0$$

$$a + b = 0$$

Substituting

$$a = -b$$

$$-2(-b) + b = 0$$

Therefore

$$b = 0$$

$$a = 0$$

This means v and w are independent. Together v and w span a plane which has dimension 2.

(b) span$(\mathbf{v}', \mathbf{w}')$ for

$$\mathbf{v}' = \begin{bmatrix} 3 \\ 6 \\ -2 \end{bmatrix}, \quad \mathbf{w}' = \begin{bmatrix} -2 \\ -4 \\ 2 \end{bmatrix}.$$

Similar to (a) we have a.v + b.w = 0

$$a. \begin{bmatrix} 3 \\ 6 \\ -2 \end{bmatrix} + b. \begin{bmatrix} -2 \\ -4 \\ 2 \end{bmatrix} = 0$$

Now we have

$$3a - 2b = 0$$

$$6a - 4b = 0$$

$$-2a + 2b = 0$$

Substituting

$$a = b$$

we have

$$6(b) - 4b = 0$$

From this we have

$$b = 0$$

$$a = 0$$

This means v' and w' are independent. Together v' and w' span a plane which has dimension 2.

(c) span$(\mathbf{v}'', \mathbf{w}'', \mathbf{u}'')$ for

$$\mathbf{v}'' = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \quad \mathbf{w}'' = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{u}'' = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}.$$

Similar to (b) we have

a.v" + b.w" + c.u" = 0

$$a. \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + b. \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} + c. \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} = 0$$

$$a + 3b + 3c = 0$$

$$-2a - 2c = 0$$

$$3a + 2b + 4c = 0$$

Substituting

$$a = c$$

$$c + 3b + 3c = 0$$

$$b = \frac{-4}{3}c$$

$$3c + 2.\frac{-4}{3}c + 4c = 0$$

$$a = 0$$

$$b = 0$$

$$c = 0$$

This means v”, w” and u” are independent and dimension is 3

2.9 Exercise 4.9

Consider the three nonzero vectors

$$v_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}.$$

(a) Show that v_1 does not belong to the span of v_2 and v_3. (Hint: if $v_1 = av_2 + bv_3$ for some scalars a and b, express this as a system of 3 equations on a and b and show that these equations have no simultaneous solution.)

(b) Similarly show v_2 does not belong to the span of v_1 and v_3, and that v_3 does not belong to the span of v_1 and v_2.

(c) Using (a) and (b), apply Theorem 4.2.5 to conclude that the linear subspace $V = \text{span}(v_1, v_2, v_3)$ in $\mathbb{R}^3$ has dimension equal to 3 (so it coincides with $\mathbb{R}^3$, by Theorem 4.2.8).

Solution:

(a) Show that v_1 does not belong to the span of v_2 and v_3. (Hint: if $v_1 = av_2 + bv_3$ for some scalars a and b, express this as a system of 3 equations on a and b and show that these equations have no simultaneous solution.)

As the hint suggested, we want to find 2 scalar a and b and express

$$v_1 = av_2 + cv_3$$

$$\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = a. \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + b. \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

$$1 = 2a + 3b$$
$$-2 = -2b$$
$$3 = a + b$$

Substituting

$$b = 1$$
$$3 = a + 1$$

Solving we get

$$a = 2$$
$$b = 1$$

But if we put a=2 and b=1 in the first equation we get

$$1 = 2(2) + 3(1)$$
$$1 \neq 7$$

This shows that there is no simultaneous solution to these system of equations. This indicates that v_1 does not belong to the span of v_2 and v_3.

(b) Similarly show v_2 does not belong to the span of v_1 and v_3, and that v_3 does not belong to the span of v_1 and v_2.

(i) v_2 does not belong to the span of v_1 and v_3

$$\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = a \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + b. \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$
$$2 = a + 3b$$
$$0 = -2a - 2b$$
$$1 = 3a + b$$

Substituting

$$a = -b$$

we get

$$1 = 3(-b) + b$$
$$b = \frac{-1}{2}$$

Substituting

$$a = \frac{1}{2}$$
$$b = \frac{-1}{2}$$

in first equation we get

$$2 = \frac{1}{2} + 3\frac{-1}{2}$$
$$2 \neq -1$$

This shows that there is no simultaneous solution to these system of equations. This indicates that v_2 does not belong to the span of v_1 and v_3.

(i) v_3 does not belong to the span of v_1 and v_2

$$\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = a \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + b. \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$3 = a + 2b$$

$$-2 = -2a$$

$$1 = 3a + b$$

Substituting

$$a = 1$$

we get

$$1 = 3(1) + b$$

$$b = -2$$

Substituting

$$a = 1$$

$$b = -2$$

in first equation

$$3 = 1 + 2(-2)$$

$$3 \neq -3$$

This shows that there is no simultaneous solution to these system of equations. This indicates that v_3 does not belong to the span of v_1 and v_2.

(c) Using (a) and (b), apply Theorem 4.2.5 to conclude that the linear subspace $V = \text{span}(v_1, v_2, v_3)$ in $\mathbb{R}^3$ has dimension equal to 3 (so it coincides with $\mathbb{R}^3$, by Theorem 4.2.8).

Finding 3 scalar such that

$$a.v_1 + b.v_2 + c.v_3 = 0$$

$$a. \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + b. \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + c. \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = 0$$

$$a + 2b + 3c = 0$$

$$-2a - 2c = 0$$

$$3a + b + c = 0$$

Substituting

$$a = -c$$

$$3(-c) + b + c = 0$$

$$b = 2c$$

$$-c + 2(2c) + 3c = 0$$

$$c = 0$$

$$b = 0$$

$$a = 0$$

This means v_1, v_2, v_3 are linearly independent vectors and they have a dimension of 3

2.10 Exercise 4.10

Let V be the span of the collection of three nonzero 3-vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}.$$

Here is an approach based on orthogonality to show $\dim V = 3$ (so $V = \mathbb{R}^3$, by Theorem 4.2.8).

(a) Explain either geometrically or algebraically why if the dimension were 1 or 2 then there would be a *nonzero* 3-vector $\mathbf{n}$ orthogonal to the span (hint: show that for *any* linear subspace of $\mathbb{R}^3$ with dimension 1 or 2 there is a nonzero 3-vector orthogonal to it).

Solution

These vectors v_1, v_2, v_3 could

(i) all point along the same line. This means they have dimension of 1.

(ii) lie on a flat plane. This means they have dimension of 2.

(iii) fill the 3D space. This means they have dimension of 3.

Algebraically this means,

$$n.v_i = 0$$

This means

$$n.v_1 = 0$$

$$n.v_2 = 0$$

$$n.v_3 = 0$$

where n is a non-zero vector

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Solving the system of equations we get

$$a - 2b + 3c = 0$$

$$2a + c = 0$$

$$3a - 2b + c = 0$$

Substituting

$$c = -2a$$

we get

$$3a - 2b + -2a = 0$$

$$a = 2b$$

$$(-2b) - 2b + 3(-2)(-2b) = 0$$

$$b = 0$$

$$a = 0$$

$$c = 0$$

This means there is no nonzero vector n that is orthogonal to all three vectors v_1, v_2, v_3 As there is no non-zero vector n satisfying

$$n.v_1 = 0$$

$$n.v_2 = 0$$

$$n.v_3 = 0$$

The span of v_1, v_2, v_3 must have dimension 3.

(a) (a line): There would exist a plane of vectors (dimension 2) perpendicular to the line, meaning you could always find a nonzero vector n orthogonal to v_1

(b) (a plane): There would exist a line of vectors (dimension 1) perpendicular to the plane, meaning you could always find a nonzero vector n orthogonal to both $v_1 and v_2$

(c) (all of $\mathbb{R}^3$) There is no "room" left for a nonzero vector n to be perpendicular to the entire space. The only solution is the trivial vector n=0.

(b) Check directly that the simultaneous conditions

$$\mathbf{n} \cdot \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = 0, \quad \mathbf{n} \cdot \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = 0, \quad \mathbf{n} \cdot \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = 0,$$

on the entries of $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ have no solution $(a, b, c) \neq (0, 0, 0)$.

Solution: This is already shown above

(c) Use the conclusion of (b) to rule out the possibilities of the dimension being 1 or 2 with the aid of (a) (so the dimension must be 3).

Solution

From part(b), we know that no non-zero vector n exists such that $n.v_1 = 0, n.v_2 = 0, n.v_3 = 0$ If the span had dimension 1, then a plane of 2d vector would have exists orthogonal to span. But this contradicts the dimension = 3 concluded in part(b)

If the span had dimension 2, then there would be atleast one non-zero vector orthogonal to the it. But the only orthogonal vector n is the zero vector.

This means the only possibility is span having dimension 3.This aligns with the conclusion of part (b), which showed that the only n satisfying the orthogonality conditions is the zero vector.