Math 51 Notes

Your Name

Contents

1 Chapter 1: Title of Chapter 1

1.1 Exercise 4.1

(a)

$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : z = 2x - y \right\}$$

A set of vectors $S \subseteq \mathbb{R}^n$ is a linear subspace if and only if:

(a) The zero vector is in S.

(b) S is closed under vector addition: if $\mathbf{u}, \mathbf{v} \in S$, then $\mathbf{u} + \mathbf{v} \in S$.

(c) S is closed under scalar multiplication: if $\mathbf{u} \in S$ and $c \in \mathbb{R}$, then $c \cdot \mathbf{u} \in S$.

Checking each condition:

(a) **The zero vector is in S.**

$$\mathbf{u} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Check if it satisfies $z = 2x - y$:

$$2 \cdot 0 - 0 = 0.$$

This is true, so the zero vector is in the set.

(b) **S is closed under vector addition.**

Let

$$\mathbf{u} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}.$$

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}.$$

$$z_1 + z_2 = 2 \cdot (x_1 + x_2) - (y_1 + y_2).$$

$$z_1 = 2 \cdot x_1 - y_1, \quad z_2 = 2 \cdot x_2 - y_2.$$

$$z_1 + z_2 = 2 \cdot (x_1 + x_2) - (y_1 + y_2).$$

Thus, the set is closed under addition.

(c) **S is closed under scalar multiplication.**

Let

$$\mathbf{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

$$c \cdot \mathbf{u} = \begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}.$$

$$cz = 2 \cdot (cx) - (cy).$$

But

$$z = 2x - y.$$

So

$$cz = c(2x - y) = 2 \cdot (cx) - (cy).$$

Thus, the set is closed under scalar multiplication. So this is a linear subspace.

(b)

$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : z = 1 + 2x - y \right\}$$

A set of vectors $S \subseteq \mathbb{R}^n$ is a linear subspace if and only if:

(a) The zero vector is in S.

(b) S is closed under vector addition: if $\mathbf{u}, \mathbf{v} \in S$, then $\mathbf{u} + \mathbf{v} \in S$.

(c) S is closed under scalar multiplication: if $\mathbf{u} \in S$ and $c \in \mathbb{R}$, then $c \cdot \mathbf{u} \in S$.

Checking each condition:

(a) **The zero vector is in S.**

$$\mathbf{u} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Check if it satisfies $z = 1 + 2x - y$:

$$2 \cdot 0 - 0 = 1 \neq 0$$

This is false, so the zero vector is not in the set. So this is not a linear subspace.

(c)

$$\left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : y = x^2 \right\}$$

A set of vectors $S \subseteq \mathbb{R}^n$ is a linear subspace if and only if:

(a) The zero vector is in S.

(b) S is closed under vector addition: if $\mathbf{u}, \mathbf{v} \in S$, then $\mathbf{u} + \mathbf{v} \in S$.

(c) S is closed under scalar multiplication: if $\mathbf{u} \in S$ and $c \in \mathbb{R}$, then $c \cdot \mathbf{u} \in S$.

Checking each condition:

(a) **The zero vector is in S.**

$$\mathbf{u} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Check if it satisfies $y = x^2$:

$$0 = 0^2 = 0$$

This is true, so the zero vector is in the set.

(b) S is closed under vector addition: if $\mathbf{u}, \mathbf{v} \in S$, then $\mathbf{u} + \mathbf{v} \in S$.
Let

$$\mathbf{u} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}.$$

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}.$$

$$y_1 + y_2 = (x_1 + x_2)^2$$

Here

$$y1 = x1^2, y2 = x2^2$$

$$y1 + y2 = x1^2 + x2^2 \neq (x1 + x2)^2$$

S is NOT closed under vector addition. So this is not a linear subspace.

(d)

$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : \begin{matrix} 3x - y + z = 0, \\ x + y - 4z = 0 \end{matrix} \right\}$$

A set of vectors $S \subseteq \mathbb{R}^n$ is a linear subspace if and only if:

(a) The zero vector is in S.

(b) S is closed under vector addition: if $\mathbf{u}, \mathbf{v} \in S$, then $\mathbf{u} + \mathbf{v} \in S$.

(c) S is closed under scalar multiplication: if $\mathbf{u} \in S$ and $c \in \mathbb{R}$, then $c \cdot \mathbf{u} \in S$.

Checking each condition:

(a) **The zero vector is in S.**

$$\mathbf{u} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Check if it satisfies

$$3x - y + z = 0,$$
$$x + y - 4z = 0$$

$$3 \cdot 0 - 0 + 0 = 0$$

$$0 + 0 - 4.0 = 0$$

This is true, so the zero vector is in the set.

(b)

(c) S **is closed under vector addition.**

Let

$$\mathbf{u} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}.$$

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}.$$

$$3x - y + z = 0,$$
$$x + y - 4z = 0$$

$$z = -3x + y$$
$$z = (x + y)/4$$

$$z_1 + z_2 = -3(x_1 + x_2) + y$$
$$-3(x_1) + y_1 + -3(x_2) + y_2 = -3(x_1 + x_2) + y_1 + y_2$$
$$-3(x_1 + x_2) + y_1 + y_2 = -3(x_1 + x_2) + y_1 + y_2$$

$$z1 + z_2 = (x_1 + x_2 + y_1 + y_2)/4$$
$$(x_1 + y_1)/4 + (x_2 + y_2)/4 = (x_1 + x_2 + y_1 + y_2)/4$$
$$(x_1 + y_1 + x_2 + y_2)/4 = (x_1 + x_2 + y_1 + y_2)/4$$

This is true so S is closed under vector addition.

(d) S **is closed under scalar multiplication.**

Let

$$\mathbf{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

$$c \cdot \mathbf{u} = \begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}.$$

$$cz = -3cx_1 + cy_1$$
$$c(-3x_1 + y_1) = -3cx_1 + cy_1$$
$$-3cx_1 + cy_1 = -3cx_1 + cy_1$$

$$cz_1 = (cx_1 + cy_1)/4$$

$$c(x_1 + y_1)/4 = (cx_1 + cy_1)/4$$
$$(cx_1 + cy_1)/4 = (cx_1 + cy_1)/4$$

This is true so S is closed under scalar multiplication. So this is a linear subspace.

1.2 Exercise 4.2

For

$$v = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad w = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix},$$

find scalars a, b, c so that

$$\operatorname{span}(v, w) = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : ax + by + cz = 0 \right\}.$$

Here ax + by + cz = 0

From

$$v = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

we have

$$a(2) + 0(b) + 1(c) = 0$$

$$c = -2(a)$$

From

$$w = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$$

we have

$$a(-1) + 1(b) + 3(c) = 0$$

$$b = a - 3c$$

Therefore we have scalars a,b,c such that

$$b = a - 3c$$

$$c = -2a$$

Substituting a =1

$$c = -2$$

$$b = 1 - 3(-2.1) = 7$$

Substituting we have

$$1(x) + 7(y) + -2(z) = 0$$

From

$$v = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$1(x) + 7(y) + -2(z) = 0$$

we have

$$1(2) + 7(0) + -2(1) = 2 - 2 = 0$$

From

$$w = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$$

we have

$$1(x) + 7(y) + -2(z) = 0$$

$$1(-1) + 7(1) + -2(3)$$

$$= -1 + 7 + -6 = -7 + 7 = 0$$

The resulting triplets work.

1.3 Exercise 4.3

For

$$v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad w = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix},$$

find scalars a, b, c so that

$$\operatorname{span}(v, w) = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : ax + by + cz = 0 \right\}.$$

For

$$v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad w = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix},$$

find scalars a, b, c so that

$$\operatorname{span}(v, w) = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : ax + by + cz = 0 \right\}.$$

Here ax + by + cz = 0

From

$$v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

we have

$$a(1) + 1(b) + 1(c) = 0$$

$$b = -(a + c)$$

From

$$w = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

we have

$$a(4) + 2(b) + 1(c) = 0$$

$$c = -(4a + 2b)$$

Therefore we have scalars a,b,c such that

$$b = -(a + c)$$

$$c = -(4a + 2b)$$

Substituting a =1 and solving for b and c we get

$$c = 2$$

$$b = -3$$

Substituting we have

$$1(x) - 3(y) + 2(z) = 0$$

From

$$v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$1(x) - 3(y) + 2(z) = 0$$

we have

$$1(1) - 3(1) + 2(1) = 3 - 3 = 0$$

From

$$w = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

we have

$$1(x) - 3(y) + 2(z) = 0$$

$$1(4) + -3(2) + 2(1)$$

$$= 6 - 6 = 0$$

The resulting triplets work.

1.4 Exercise 4.4

For the 4-vectors

$$w = \begin{bmatrix} -2 \\ 2 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad w' = \begin{bmatrix} 3 \\ 4 \\ 0 \\ 1 \end{bmatrix},$$

show that the collection of vectors

$$V = \left\{ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 : x \cdot w = 0,\ x \cdot w' = 0 \right\}$$

is a linear subspace of $\mathbb{R}^4$ in each of the following ways:

(a) For $x \in V$, solve for each of x_3 and x_4 in terms of x_1 and x_2 to write V as a span of two vectors;

(b) For $x \in V$, solve for each of x_1 and x_4 in terms of x_2 and x_3 to write V as a span of two vectors.

Solution:

(a) For w,

$$w \cdot x = -2x_1 + 2x_2 + x_3 + x_4 = 0 \quad \Rightarrow \quad x_3 + x_4 = 2x_1 - 2x_2 \tag{1}$$

For w',

$$w' \cdot x = 3x_1 + 4x_2 + x_4 = 0 \quad \Rightarrow \quad x_4 = -3x_1 - 4x_2 \tag{2}$$

Substitute x_4 from (2) into (1):

$$x_3 + (-3x_1 - 4x_2) = 2x_1 - 2x_2$$

$$x_3 = 5x_1 + 2x_2 \tag{3}$$

Thus, the components of x are:

$$x_3 = 5x_1 + 2x_2, \quad x_4 = -3x_1 - 4x_2$$

Substitute these into x:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ 5x_1 + 2x_2 \\ -3x_1 - 4x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 5 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 2 \\ -4 \end{bmatrix}$$

The basis vectors are:

$$\begin{bmatrix} 1 \\ 0 \\ 5 \\ -3 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 2 \\ -4 \end{bmatrix}.$$

(b) For w,

$$w \cdot x = -2x_1 + 2x_2 + x_3 + x_4 = 0 \quad \Rightarrow \quad x_4 = -2x_1 + 2x_2 - x_3 \tag{4}$$

For w',

$$w' \cdot x = 3x_1 + 4x_2 + x_4 = 0 \quad \Rightarrow \quad x_4 = -3x_1 - 4x_2 \tag{5}$$

Equating x_4 from (4) and (5):

$$-2x_1 + 2x_2 - x_3 = -3x_1 - 4x_2$$

Simplify:

$$x_1 + 6x_2 - x_3 = 0 \quad \Rightarrow \quad x_1 = -6x_2 + x_3 \tag{6}$$

Substitute x_1 from (6) into (4):

$$x_4 = -2(-6x_2 + x_3) + 2x_2 - x_3$$

$$x_4 = 12x_2 - 2x_3 + 2x_2 - x_3$$

$$x_4 = 14x_2 - 3x_3 \tag{7}$$

Thus, the components of x are:

$$x_1 = -6x_2 + x_3, \quad x_4 = 14x_2 - 3x_3$$

Substitute these into x:

$$x = \begin{bmatrix} -6x_2 + x_3 \\ x_2 \\ x_3 \\ 14x_2 - 3x_3 \end{bmatrix} = x_2 \begin{bmatrix} -6 \\ 1 \\ 0 \\ 14 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \\ -3 \end{bmatrix}$$

The basis vectors are:

$$\begin{bmatrix} -6 \\ 1 \\ 0 \\ 14 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 1 \\ -3 \end{bmatrix}.$$

1.5 Exercise 4.5

Find a nonzero 3-vector $\mathbf{v}$ so that

$$\left\{ \mathbf{x} \in \mathbb{R}^3 : \mathbf{x} \cdot \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = 0, \, \mathbf{x} \cdot \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} = 0 \right\} = \text{span}(\mathbf{v}).$$

Then, using the *geometric* fact that any two different planes through the origin in $\mathbb{R}^3$ meet along a line through the origin, interpret this algebraic outcome that the left side is the span of a single vector.

Solution:

Let a,b,c be scalar so that

$$
\begin{aligned}
3a + 2b + c &= 0 \\
b &= \frac{-c - 3a}{2} \\
-2a - b + c &= 0 \\
-2a - \frac{-c - 3a}{2} + c &= 0 \\
c &= \frac{a}{3} \\
2b &= -\frac{a}{3} - 3a \\
b &= -\frac{5a}{3}
\end{aligned} \tag{1}
$$

writing v in terms of a,

$$\begin{bmatrix} a \\ -\frac{5a}{3} \\ \frac{a}{3} \end{bmatrix}$$

$$v = a. \begin{bmatrix} 1 \\ -\frac{5}{3} \\ \frac{1}{3} \end{bmatrix}$$

The two planes defined by the equations

$$3a + 2b + c = 0 \tag{2}$$

$$-2a - b + c = 0 \tag{3}$$

intersect along a line through the origin in $\mathbb{R}^3$ This line is spanned by the vector v. Hence, the solution set is

$$a. \begin{bmatrix} 1 \\ -\frac{5}{3} \\ \frac{1}{3} \end{bmatrix}$$

Here a $\neq 0$ so the span is

$$\begin{bmatrix} 1 \\ -\frac{5}{3} \\ \frac{1}{3} \end{bmatrix}$$

1.6 Exercise 4.6

Find a pair of 3-vectors $\mathbf{v}, \mathbf{w}$ so that

$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : 2x - 3y + 2z = 0 \right\} = \mathrm{span}(\mathbf{v}, \mathbf{w}).$$

we have

$$2x - 3y + 2z = 0$$

Therefore

$$x = \frac{3y - 2z}{2}$$

we can write x=

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

as x=

$$\begin{bmatrix} \frac{3y-2z}{2} \\ y \\ z \end{bmatrix}$$

x=

$$y \begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Two linearly independent vectors spanning the subspace are:

$$v1 = \begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \end{bmatrix}$$

and

$$v2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

1.7 Exercise 4.7

part (a)

$$v = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad w = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$v' = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \quad w' = \begin{bmatrix} -2 \\ -3 \\ 4 \end{bmatrix}$$

Find scalars a, b, c, d such that

$$v' = av + bw$$
$$w' = cv + dw$$

$$v' = a \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

$$a + b = 3$$
$$\boxed{b = 1}$$
$$a - b = 1 \Rightarrow a - 1 = 1$$
$$\Rightarrow \boxed{a = 2}$$

$$v' = 2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

$$w' = a\begin{bmatrix}1\\0\\1\end{bmatrix} + b\begin{bmatrix}1\\1\\-1\end{bmatrix} = \begin{bmatrix}-2\\-3\\4\end{bmatrix}$$

$$= a + b = -2$$

$$\boxed{b = -3}$$

$$a - b = 4$$

$$a - (-3) = 4$$

$$\boxed{a = 1}$$

$$w' = 1\begin{bmatrix}1\\0\\1\end{bmatrix} - 3\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}-2\\-3\\4\end{bmatrix}$$

part (b)

plane spanned by v and w

$ax + by + cz = 0$ where

$\begin{bmatrix}a\\b\\c\end{bmatrix}$ is normal vector perpendicular

to both v and w

$$v = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad w = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

The normal vector perpendicular to v and w is got by the cross product of v and w

$$v \times w = \begin{vmatrix} i & j & k \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$v \times w = i \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} - j \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} + k \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$$

$$i = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} \Rightarrow (0 * -1) - (1 * 1) = -1$$

$$j = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = 1 * (-1) - (1 * 1) = -2$$

$$k = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1 \cdot 1 - 0 \cdot 1 = 1$$

$$v * w = (-1)i - (-2)j + (1)k = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$v * w = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$v' = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad w' = \begin{bmatrix} -2 \\ -3 \\ 4 \end{bmatrix}$$

$$v * w = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$-1x + 2y + z = 0$$

Substituting $v' = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$

$$v' = -1(3) + 2(1) + (1) =$$

$$-3 + 3 = 0$$

Substituting $w' = \begin{bmatrix} -2 \\ -3 \\ 4 \end{bmatrix}$

$$w' = -1(-2) + 2(-3) + 4$$

$$= 2 - 6 + 4$$

$$= 0$$