part (a)

$$v = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad w = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$v' = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \quad w' = \begin{bmatrix} -2 \\ -3 \\ 4 \end{bmatrix}$$

Find scalars $a, b, c, d$ such that

$$v' = av + bw$$
$$w' = cv + dw$$

$$v' = a \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

$$a + b = 3$$
$$\boxed{b = 1}$$
$$a - b = 1 \Rightarrow a - 1 = 1$$
$$\Rightarrow \boxed{a = 2}$$

$$v' = 2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

$$w' = a\begin{bmatrix}1\\0\\1\end{bmatrix} + b\begin{bmatrix}1\\1\\-1\end{bmatrix} = \begin{bmatrix}-2\\-3\\4\end{bmatrix}$$

$$= a+b=-2$$

$$\boxed{b=-3}$$

$$a-b=4$$

$$a-(-3)=4$$

$$\boxed{a=1}$$

$$w' = 1\begin{bmatrix}1\\0\\1\end{bmatrix} - 3\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}-2\\-3\\4\end{bmatrix}$$

part (b)

plane spanned by $v$ and $w$

$ax+by+cz=0$ where

$\begin{bmatrix}a\\b\\c\end{bmatrix}$ is normal vector perpendicular to both $v$ and $w$

$$v = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad w = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

The normal vector perpendicular to v and w is got by the cross product of v and w

$$v \times w = \begin{vmatrix} i & j & k \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$v \times w = i \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} - j \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} + k \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$$

$$i = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} \Rightarrow (0 * -1) - (1 * 1) = -1$$

$$j = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = 1 * (-1) - (1 * 1) = -2$$

$$k = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1 \cdot 1 - 0 \cdot 1 = 1$$

$$v * w = (-1) i - (-2) j + (1) k = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$v * w = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$v' = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \text{ and } w' = \begin{bmatrix} -2 \\ -3 \\ 4 \end{bmatrix}$$

$$v * w = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$-1x + 2y + z = 0$$

Substituting $v' = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$

$$v' = -1(3) + 2(1) + (1) =$$

$$-3 + 3 = 0$$

Substituting $w' = \begin{bmatrix} -2 \\ -3 \\ 4 \end{bmatrix}$

$$w' = -1(-2) + 2(-3) + 4$$

$$= 2 - 6 + 4$$

$$= 0$$