

Math 51 Notes

Your Name

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# 1 Chapter 1: Title of Chapter 1

## 1.1 Exercise 4.1

(a)

$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : z = 2x - y \right\}$$

A set of vectors  $S \subseteq \mathbb{R}^n$  is a linear subspace if and only if:

- (a) The zero vector is in  $S$ .
- (b)  $S$  is closed under vector addition: if  $\mathbf{u}, \mathbf{v} \in S$ , then  $\mathbf{u} + \mathbf{v} \in S$ .
- (c)  $S$  is closed under scalar multiplication: if  $\mathbf{u} \in S$  and  $c \in \mathbb{R}$ , then  $c \cdot \mathbf{u} \in S$ .

Checking each condition:

- (a) **The zero vector is in  $S$ .**

$$\mathbf{u} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Check if it satisfies  $z = 2x - y$ :

$$2 \cdot 0 - 0 = 0.$$

This is true, so the zero vector is in the set.

- (b)  **$S$  is closed under vector addition.**

Let

$$\mathbf{u} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}.$$

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}.$$

$$z_1 + z_2 = 2 \cdot (x_1 + x_2) - (y_1 + y_2).$$

$$z_1 = 2 \cdot x_1 - y_1, \quad z_2 = 2 \cdot x_2 - y_2.$$

$$z_1 + z_2 = 2 \cdot (x_1 + x_2) - (y_1 + y_2).$$

Thus, the set is closed under addition.

- (c)  **$S$  is closed under scalar multiplication.**

Let

$$\mathbf{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

$$c \cdot \mathbf{u} = \begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}.$$

$$cz = 2 \cdot (cx) - (cy).$$

But

$$z = 2x - y.$$

So

$$cz = c(2x - y) = 2 \cdot (cx) - (cy).$$

Thus, the set is closed under scalar multiplication. So this is a linear subspace.

(b)

$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : z = 1 + 2x - y \right\}$$

A set of vectors  $S \subseteq \mathbb{R}^n$  is a linear subspace if and only if:

- (a) The zero vector is in  $S$ .
- (b)  $S$  is closed under vector addition: if  $\mathbf{u}, \mathbf{v} \in S$ , then  $\mathbf{u} + \mathbf{v} \in S$ .
- (c)  $S$  is closed under scalar multiplication: if  $\mathbf{u} \in S$  and  $c \in \mathbb{R}$ , then  $c \cdot \mathbf{u} \in S$ .

Checking each condition:

- (a) **The zero vector is in  $S$ .**

$$\mathbf{u} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Check if it satisfies  $z = 1 + 2x - y$ :

$$2 \cdot 0 - 0 = 1 \neq 0$$

This is false, so the zero vector is not in the set. So this is not a linear subspace.

(c)

$$\left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : y = x^2 \right\}$$

A set of vectors  $S \subseteq \mathbb{R}^n$  is a linear subspace if and only if:

- (a) The zero vector is in  $S$ .
- (b)  $S$  is closed under vector addition: if  $\mathbf{u}, \mathbf{v} \in S$ , then  $\mathbf{u} + \mathbf{v} \in S$ .
- (c)  $S$  is closed under scalar multiplication: if  $\mathbf{u} \in S$  and  $c \in \mathbb{R}$ , then  $c \cdot \mathbf{u} \in S$ .

Checking each condition:

(a) **The zero vector is in  $S$ .**

$$\mathbf{u} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Check if it satisfies  $y = x^2$ :

$$0 = 0^2 = 0$$

This is true, so the zero vector is in the set.

(b)  $S$  is closed under vector addition: if  $\mathbf{u}, \mathbf{v} \in S$ , then  $\mathbf{u} + \mathbf{v} \in S$ .

Let

$$\mathbf{u} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}.$$

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}.$$

$$y_1 + y_2 = (x_1 + x_2)^2$$

Here

$$y_1 = x_1^2, y_2 = x_2^2$$

$$y_1 + y_2 = x_1^2 + x_2^2 \neq (x_1 + x_2)^2$$

$S$  is NOT closed under vector addition. So this is not a linear subspace.

(d)

$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : \begin{array}{l} 3x - y + z = 0, \\ x + y - 4z = 0 \end{array} \right\}$$

A set of vectors  $S \subseteq \mathbb{R}^n$  is a linear subspace if and only if:

(a) The zero vector is in  $S$ .

(b)  $S$  is closed under vector addition: if  $\mathbf{u}, \mathbf{v} \in S$ , then  $\mathbf{u} + \mathbf{v} \in S$ .

(c)  $S$  is closed under scalar multiplication: if  $\mathbf{u} \in S$  and  $c \in \mathbb{R}$ , then  $c \cdot \mathbf{u} \in S$ .

Checking each condition:

(a) **The zero vector is in  $S$ .**

$$\mathbf{u} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Check if it satisfies

$$3x - y + z = 0,$$

$$x + y - 4z = 0$$

$$3 \cdot 0 - 0 + 0 = 0$$

$$0 + 0 - 4 \cdot 0 = 0$$

This is true, so the zero vector is in the set.

(b)

(c)  **$S$  is closed under vector addition.**

Let

$$\mathbf{u} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}.$$

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}.$$

$$3x - y + z = 0,$$

$$x + y - 4z = 0$$

$$z = -3x + y$$

$$z = (x + y)/4$$

$$z_1 + z_2 = -3(x_1 + x_2) + y_1 + y_2$$

$$-3(x_1) + y_1 + -3(x_2) + y_2 = -3(x_1 + x_2) + y_1 + y_2$$

$$-3(x_1 + x_2) + y_1 + y_2 = -3(x_1 + x_2) + y_1 + y_2$$

$$z_1 + z_2 = (x_1 + x_2 + y_1 + y_2)/4$$

$$(x_1 + y_1)/4 + (x_2 + y_2)/4 = (x_1 + x_2 + y_1 + y_2)/4$$

$$(x_1 + y_1 + x_2 + y_2)/4 = (x_1 + x_2 + y_1 + y_2)/4$$

This is true so  $S$  is closed under vector addition.

(d)  **$S$  is closed under scalar multiplication.**

Let

$$\mathbf{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

$$c \cdot \mathbf{u} = \begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}.$$

$$cz = -3cx_1 + cy_1$$

$$c(-3x_1 + y_1) = -3cx_1 + cy_1$$

$$-3cx_1 + cy_1 = -3cx_1 + cy_1$$

$$cz_1 = (cx_1 + cy_1)/4$$

$$c(x_1 + y_1)/4 = (cx_1 + cy_1)/4$$

$$(cx_1 + cy_1)/4 = (cx_1 + cy_1)/4$$

This is true so  $S$  is closed under scalar multiplication. So this is a linear subspace.

## 1.2 Exercise 4.2

For

$$v = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad w = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix},$$

find scalars  $a, b, c$  so that

$$\text{span}(v, w) = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : ax + by + cz = 0 \right\}.$$

Here  $ax + by + cz = 0$

From

$$v = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

we have

$$a(2) + 0(b) + 1(c) = 0$$

$$c = -2(a)$$

From

$$w = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$$

we have

$$a(-1) + 1(b) + 3(c) = 0$$

$$b = a - 3c$$

Therefore we have scalars  $a, b, c$  such that

$$b = a - 3c$$

$$c = -2a$$

Substituting  $a = 1$

$$c = -2$$

$$b = 1 - 3(-2) = 7$$

Substituting we have

$$1(x) + 7(y) + -2(z) = 0$$

From

$$v = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$1(x) + 7(y) + -2(z) = 0$$

we have

$$1(2) + 7(0) + -2(1) = 2 - 2 = 0$$

From

$$w = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$$

we have

$$\begin{aligned} 1(x) + 7(y) + -2(z) &= 0 \\ 1(-1) + 7(1) + -2(3) \\ &= -1 + 7 + -6 = -7 + 7 = 0 \end{aligned}$$

The resulting triplets work.

### 1.3 Exercise 4.3

For

$$v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad w = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix},$$

find scalars  $a, b, c$  so that

$$\text{span}(v, w) = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : ax + by + cz = 0 \right\}.$$

For

$$v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad w = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix},$$

find scalars  $a, b, c$  so that

$$\text{span}(v, w) = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : ax + by + cz = 0 \right\}.$$

Here  $ax + by + cz = 0$

From

$$v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

we have

$$\begin{aligned} a(1) + 1(b) + 1(c) &= 0 \\ b &= -(a + c) \end{aligned}$$

From

$$w = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

we have

$$\begin{aligned} a(4) + 2(b) + 1(c) &= 0 \\ c &= -(4a + 2b) \end{aligned}$$

Therefore we have scalars a,b,c such that

$$\begin{aligned} b &= -(a + c) \\ c &= -(4a + 2b) \end{aligned}$$

Substituting a =1 and solving for b and c we get

$$\begin{aligned} c &= 2 \\ b &= -3 \end{aligned}$$

Substituting we have

$$1(x) - 3(y) + 2(z) = 0$$

From

$$v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$1(x) - 3(y) + 2(z) = 0$$

we have

$$1(1) - 3(1) + 2(1) = 3 - 3 = 0$$

From

$$w = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

we have

$$\begin{aligned} 1(x) - 3(y) + 2(z) &= 0 \\ 1(4) + -3(2) + 2(1) \\ &= 6 - 6 = 0 \end{aligned}$$

The resulting triplets work.



### 1.4 Exercise 4.4

For the 4-vectors

$$w = \begin{bmatrix} -2 \\ 2 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad w' = \begin{bmatrix} 3 \\ 4 \\ 0 \\ 1 \end{bmatrix},$$

show that the collection of vectors

$$V = \left\{ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 : x \cdot w = 0, x \cdot w' = 0 \right\}$$

is a linear subspace of  $\mathbb{R}^4$  in each of the following ways:

- (a) For  $x \in V$ , solve for each of  $x_3$  and  $x_4$  in terms of  $x_1$  and  $x_2$  to write  $V$  as a span of two vectors;
- (b) For  $x \in V$ , solve for each of  $x_1$  and  $x_4$  in terms of  $x_2$  and  $x_3$  to write  $V$  as a span of two vectors.

Solution:

- (a) For  $w$ ,

$$w \cdot x = -2x_1 + 2x_2 + x_3 + x_4 = 0 \quad \Rightarrow \quad x_3 + x_4 = 2x_1 - 2x_2 \quad (1)$$

For  $w'$ ,

$$w' \cdot x = 3x_1 + 4x_2 + x_4 = 0 \quad \Rightarrow \quad x_4 = -3x_1 - 4x_2 \quad (2)$$

Substitute  $x_4$  from (2) into (1):

$$\begin{aligned} x_3 + (-3x_1 - 4x_2) &= 2x_1 - 2x_2 \\ x_3 &= 5x_1 + 2x_2 \end{aligned} \quad (3)$$

Thus, the components of  $x$  are:

$$x_3 = 5x_1 + 2x_2, \quad x_4 = -3x_1 - 4x_2$$

Substitute these into  $x$ :

$$x = \begin{bmatrix} x_1 \\ x_2 \\ 5x_1 + 2x_2 \\ -3x_1 - 4x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 5 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 2 \\ -4 \end{bmatrix}$$

The basis vectors are:

$$\begin{bmatrix} 1 \\ 0 \\ 5 \\ -3 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 2 \\ -4 \end{bmatrix}.$$

(b) For  $w$ ,

$$w \cdot x = -2x_1 + 2x_2 + x_3 + x_4 = 0 \quad \Rightarrow \quad x_4 = -2x_1 + 2x_2 - x_3 \quad (4)$$

For  $w'$ ,

$$w' \cdot x = 3x_1 + 4x_2 + x_4 = 0 \quad \Rightarrow \quad x_4 = -3x_1 - 4x_2 \quad (5)$$

Equating  $x_4$  from (4) and (5):

$$-2x_1 + 2x_2 - x_3 = -3x_1 - 4x_2$$

Simplify:

$$x_1 + 6x_2 - x_3 = 0 \quad \Rightarrow \quad x_1 = -6x_2 + x_3 \quad (6)$$

Substitute  $x_1$  from (6) into (4):

$$\begin{aligned} x_4 &= -2(-6x_2 + x_3) + 2x_2 - x_3 \\ x_4 &= 12x_2 - 2x_3 + 2x_2 - x_3 \\ x_4 &= 14x_2 - 3x_3 \end{aligned} \quad (7)$$

Thus, the components of  $x$  are:

$$x_1 = -6x_2 + x_3, \quad x_4 = 14x_2 - 3x_3$$

Substitute these into  $x$ :

$$x = \begin{bmatrix} -6x_2 + x_3 \\ x_2 \\ x_3 \\ 14x_2 - 3x_3 \end{bmatrix} = x_2 \begin{bmatrix} -6 \\ 1 \\ 0 \\ 14 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \\ -3 \end{bmatrix}$$

The basis vectors are:

$$\begin{bmatrix} -6 \\ 1 \\ 0 \\ 14 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 1 \\ -3 \end{bmatrix}.$$

## 1.5 Exercise 4.5

Find a nonzero 3-vector  $\mathbf{v}$  so that

$$\left\{ \mathbf{x} \in \mathbb{R}^3 : \mathbf{x} \cdot \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = 0, \mathbf{x} \cdot \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} = 0 \right\} = \text{span}(\mathbf{v}).$$

Then, using the *geometric* fact that any two different planes through the origin in  $\mathbb{R}^3$  meet along a line through the origin, interpret this algebraic outcome that the left side is the span of a single vector.

Solution:

Let  $a, b, c$  be scalar so that

$$\begin{aligned}
3a + 2b + c &= 0 \\
b &= \frac{-c - 3a}{2} \\
-2a - b + c &= 0 \\
-2a - \frac{-c - 3a}{2} + c &= 0 \\
c &= \frac{a}{3} \\
2b &= -\frac{a}{3} - 3a \\
b &= -\frac{5a}{3}
\end{aligned} \tag{1}$$

writing  $\mathbf{v}$  in terms of  $a$ ,

$$\begin{aligned}
&\begin{bmatrix} a \\ -\frac{5a}{3} \\ \frac{a}{3} \end{bmatrix} \\
\mathbf{v} &= a \cdot \begin{bmatrix} 1 \\ -\frac{5}{3} \\ \frac{1}{3} \end{bmatrix}
\end{aligned}$$

The two planes defined by the equations

$$3a + 2b + c = 0 \tag{2}$$

$$-2a - b + c = 0 \tag{3}$$

intersect along a line through the origin in  $\mathbb{R}^3$ . This line is spanned by the vector  $\mathbf{v}$ . Hence, the solution set is

$$a \cdot \begin{bmatrix} 1 \\ -\frac{5}{3} \\ \frac{1}{3} \end{bmatrix}$$

Here  $a \neq 0$  so the span is

$$\begin{bmatrix} 1 \\ -\frac{5}{3} \\ \frac{1}{3} \end{bmatrix}$$

## 1.6 Exercise 4.6

Find a pair of 3-vectors  $\mathbf{v}, \mathbf{w}$  so that

$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : 2x - 3y + 2z = 0 \right\} = \text{span}(\mathbf{v}, \mathbf{w}).$$

we have

$$2x - 3y + 2z = 0$$

Therefore

$$x = \frac{3y - 2z}{2}$$

we can write  $\mathbf{x} =$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

as  $\mathbf{x} =$

$$\begin{bmatrix} \frac{3y-2z}{2} \\ y \\ z \end{bmatrix}$$

$\mathbf{x} =$

$$y \begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Two linearly independent vectors spanning the subspace are:

$$v_1 = \begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \end{bmatrix}$$

and

$$v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$