

# Math 51 Notes

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# 1 Span, subspaces, and dimension

## 1.1 Exercise 4.1

(a)

$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : z = 2x - y \right\}$$

A set of vectors  $S \subseteq \mathbb{R}^n$  is a linear subspace if and only if:

- (a) The zero vector is in  $S$ .
- (b)  $S$  is closed under vector addition: if  $\mathbf{u}, \mathbf{v} \in S$ , then  $\mathbf{u} + \mathbf{v} \in S$ .
- (c)  $S$  is closed under scalar multiplication: if  $\mathbf{u} \in S$  and  $c \in \mathbb{R}$ , then  $c \cdot \mathbf{u} \in S$ .

Checking each condition:

- (a) **The zero vector is in  $S$ .**

$$\mathbf{u} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Check if it satisfies  $z = 2x - y$ :

$$2 \cdot 0 - 0 = 0.$$

This is true, so the zero vector is in the set.

- (b)  **$S$  is closed under vector addition.**

Let

$$\mathbf{u} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}.$$

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}.$$

$$z_1 + z_2 = 2 \cdot (x_1 + x_2) - (y_1 + y_2).$$

$$z_1 = 2 \cdot x_1 - y_1, \quad z_2 = 2 \cdot x_2 - y_2.$$

$$z_1 + z_2 = 2 \cdot (x_1 + x_2) - (y_1 + y_2).$$

Thus, the set is closed under addition.

- (c)  **$S$  is closed under scalar multiplication.**

Let

$$\mathbf{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

$$c \cdot \mathbf{u} = \begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}.$$

$$cz = 2 \cdot (cx) - (cy).$$

But

$$z = 2x - y.$$

So

$$cz = c(2x - y) = 2 \cdot (cx) - (cy).$$

Thus, the set is closed under scalar multiplication. So this is a linear subspace.

(b)

$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : z = 1 + 2x - y \right\}$$

A set of vectors  $S \subseteq \mathbb{R}^n$  is a linear subspace if and only if:

- (a) The zero vector is in  $S$ .
- (b)  $S$  is closed under vector addition: if  $\mathbf{u}, \mathbf{v} \in S$ , then  $\mathbf{u} + \mathbf{v} \in S$ .
- (c)  $S$  is closed under scalar multiplication: if  $\mathbf{u} \in S$  and  $c \in \mathbb{R}$ , then  $c \cdot \mathbf{u} \in S$ .

Checking each condition:

- (a) **The zero vector is in  $S$ .**

$$\mathbf{u} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Check if it satisfies  $z = 1 + 2x - y$ :

$$2 \cdot 0 - 0 = 1 \neq 0$$

This is false, so the zero vector is not in the set. So this is not a linear subspace.

(c)

$$\left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : y = x^2 \right\}$$

A set of vectors  $S \subseteq \mathbb{R}^n$  is a linear subspace if and only if:

- (a) The zero vector is in  $S$ .
- (b)  $S$  is closed under vector addition: if  $\mathbf{u}, \mathbf{v} \in S$ , then  $\mathbf{u} + \mathbf{v} \in S$ .
- (c)  $S$  is closed under scalar multiplication: if  $\mathbf{u} \in S$  and  $c \in \mathbb{R}$ , then  $c \cdot \mathbf{u} \in S$ .

Checking each condition:

(a) **The zero vector is in  $S$ .**

$$\mathbf{u} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Check if it satisfies  $y = x^2$ :

$$0 = 0^2 = 0$$

This is true, so the zero vector is in the set.

(b)  $S$  is closed under vector addition: if  $\mathbf{u}, \mathbf{v} \in S$ , then  $\mathbf{u} + \mathbf{v} \in S$ .

Let

$$\mathbf{u} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}.$$

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}.$$

$$y_1 + y_2 = (x_1 + x_2)^2$$

Here

$$y_1 = x_1^2, y_2 = x_2^2$$

$$y_1 + y_2 = x_1^2 + x_2^2 \neq (x_1 + x_2)^2$$

$S$  is NOT closed under vector addition. So this is not a linear subspace.

(d)

$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : \begin{array}{l} 3x - y + z = 0, \\ x + y - 4z = 0 \end{array} \right\}$$

A set of vectors  $S \subseteq \mathbb{R}^n$  is a linear subspace if and only if:

(a) The zero vector is in  $S$ .

(b)  $S$  is closed under vector addition: if  $\mathbf{u}, \mathbf{v} \in S$ , then  $\mathbf{u} + \mathbf{v} \in S$ .

(c)  $S$  is closed under scalar multiplication: if  $\mathbf{u} \in S$  and  $c \in \mathbb{R}$ , then  $c \cdot \mathbf{u} \in S$ .

Checking each condition:

(a) **The zero vector is in  $S$ .**

$$\mathbf{u} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Check if it satisfies

$$3x - y + z = 0,$$

$$x + y - 4z = 0$$

$$3 \cdot 0 - 0 + 0 = 0$$

$$0 + 0 - 4 \cdot 0 = 0$$

This is true, so the zero vector is in the set.

(b)

(c)  **$S$  is closed under vector addition.**

Let

$$\mathbf{u} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}.$$

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}.$$

$$3x - y + z = 0,$$

$$x + y - 4z = 0$$

$$z = -3x + y$$

$$z = (x + y)/4$$

$$z_1 + z_2 = -3(x_1 + x_2) + y_1 + y_2$$

$$-3(x_1) + y_1 + -3(x_2) + y_2 = -3(x_1 + x_2) + y_1 + y_2$$

$$-3(x_1 + x_2) + y_1 + y_2 = -3(x_1 + x_2) + y_1 + y_2$$

$$z_1 + z_2 = (x_1 + x_2 + y_1 + y_2)/4$$

$$(x_1 + y_1)/4 + (x_2 + y_2)/4 = (x_1 + x_2 + y_1 + y_2)/4$$

$$(x_1 + y_1 + x_2 + y_2)/4 = (x_1 + x_2 + y_1 + y_2)/4$$

This is true so  $S$  is closed under vector addition.

(d)  **$S$  is closed under scalar multiplication.**

Let

$$\mathbf{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

$$c \cdot \mathbf{u} = \begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}.$$

$$cz = -3cx_1 + cy_1$$

$$c(-3x_1 + y_1) = -3cx_1 + cy_1$$

$$-3cx_1 + cy_1 = -3cx_1 + cy_1$$

$$cz_1 = (cx_1 + cy_1)/4$$

$$c(x_1 + y_1)/4 = (cx_1 + cy_1)/4$$

$$(cx_1 + cy_1)/4 = (cx_1 + cy_1)/4$$

This is true so  $S$  is closed under scalar multiplication. So this is a linear subspace.

## 1.2 Exercise 4.2

For

$$v = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad w = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix},$$

find scalars  $a, b, c$  so that

$$\text{span}(v, w) = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : ax + by + cz = 0 \right\}.$$

Here  $ax + by + cz = 0$

From

$$v = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

we have

$$\begin{aligned} a(2) + 0(b) + 1(c) &= 0 \\ c &= -2(a) \end{aligned}$$

From

$$w = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$$

we have

$$\begin{aligned} a(-1) + 1(b) + 3(c) &= 0 \\ b &= a - 3c \end{aligned}$$

Therefore we have scalars  $a, b, c$  such that

$$\begin{aligned} b &= a - 3c \\ c &= -2a \end{aligned}$$

Substituting  $a = 1$

$$\begin{aligned} c &= -2 \\ b &= 1 - 3(-2) = 7 \end{aligned}$$

Substituting we have

$$1(x) + 7(y) + -2(z) = 0$$

From

$$v = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$1(x) + 7(y) + -2(z) = 0$$

we have

$$1(2) + 7(0) + -2(1) = 2 - 2 = 0$$

From

$$w = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$$

we have

$$\begin{aligned} 1(x) + 7(y) + -2(z) &= 0 \\ 1(-1) + 7(1) + -2(3) \\ &= -1 + 7 + -6 = -7 + 7 = 0 \end{aligned}$$

The resulting triplets work.

### 1.3 Exercise 4.3

For

$$v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad w = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix},$$

find scalars  $a, b, c$  so that

$$\text{span}(v, w) = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : ax + by + cz = 0 \right\}.$$

For

$$v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad w = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix},$$

find scalars  $a, b, c$  so that

$$\text{span}(v, w) = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : ax + by + cz = 0 \right\}.$$

Here  $ax + by + cz = 0$

From

$$v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

we have

$$\begin{aligned} a(1) + 1(b) + 1(c) &= 0 \\ b &= -(a + c) \end{aligned}$$

From

$$w = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

we have

$$\begin{aligned} a(4) + 2(b) + 1(c) &= 0 \\ c &= -(4a + 2b) \end{aligned}$$

Therefore we have scalars a,b,c such that

$$\begin{aligned} b &= -(a + c) \\ c &= -(4a + 2b) \end{aligned}$$

Substituting a =1 and solving for b and c we get

$$\begin{aligned} c &= 2 \\ b &= -3 \end{aligned}$$

Substituting we have

$$1(x) - 3(y) + 2(z) = 0$$

From

$$v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$1(x) - 3(y) + 2(z) = 0$$

we have

$$1(1) - 3(1) + 2(1) = 3 - 3 = 0$$

From

$$w = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

we have

$$\begin{aligned} 1(x) - 3(y) + 2(z) &= 0 \\ 1(4) + -3(2) + 2(1) \\ &= 6 - 6 = 0 \end{aligned}$$

The resulting triplets work.



### 1.4 Exercise 4.4

For the 4-vectors

$$w = \begin{bmatrix} -2 \\ 2 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad w' = \begin{bmatrix} 3 \\ 4 \\ 0 \\ 1 \end{bmatrix},$$

show that the collection of vectors

$$V = \left\{ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 : x \cdot w = 0, x \cdot w' = 0 \right\}$$

is a linear subspace of  $\mathbb{R}^4$  in each of the following ways:

- (a) For  $x \in V$ , solve for each of  $x_3$  and  $x_4$  in terms of  $x_1$  and  $x_2$  to write  $V$  as a span of two vectors;
- (b) For  $x \in V$ , solve for each of  $x_1$  and  $x_4$  in terms of  $x_2$  and  $x_3$  to write  $V$  as a span of two vectors.

Solution:

- (a) For  $w$ ,

$$w \cdot x = -2x_1 + 2x_2 + x_3 + x_4 = 0 \quad \Rightarrow \quad x_3 + x_4 = 2x_1 - 2x_2 \quad (1)$$

For  $w'$ ,

$$w' \cdot x = 3x_1 + 4x_2 + x_4 = 0 \quad \Rightarrow \quad x_4 = -3x_1 - 4x_2 \quad (2)$$

Substitute  $x_4$  from (2) into (1):

$$\begin{aligned} x_3 + (-3x_1 - 4x_2) &= 2x_1 - 2x_2 \\ x_3 &= 5x_1 + 2x_2 \end{aligned} \quad (3)$$

Thus, the components of  $x$  are:

$$x_3 = 5x_1 + 2x_2, \quad x_4 = -3x_1 - 4x_2$$

Substitute these into  $x$ :

$$x = \begin{bmatrix} x_1 \\ x_2 \\ 5x_1 + 2x_2 \\ -3x_1 - 4x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 5 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 2 \\ -4 \end{bmatrix}$$

The basis vectors are:

$$\begin{bmatrix} 1 \\ 0 \\ 5 \\ -3 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 2 \\ -4 \end{bmatrix}.$$

(b) For  $w$ ,

$$w \cdot x = -2x_1 + 2x_2 + x_3 + x_4 = 0 \quad \Rightarrow \quad x_4 = -2x_1 + 2x_2 - x_3 \quad (4)$$

For  $w'$ ,

$$w' \cdot x = 3x_1 + 4x_2 + x_4 = 0 \quad \Rightarrow \quad x_4 = -3x_1 - 4x_2 \quad (5)$$

Equating  $x_4$  from (4) and (5):

$$-2x_1 + 2x_2 - x_3 = -3x_1 - 4x_2$$

Simplify:

$$x_1 + 6x_2 - x_3 = 0 \quad \Rightarrow \quad x_1 = -6x_2 + x_3 \quad (6)$$

Substitute  $x_1$  from (6) into (4):

$$x_4 = -2(-6x_2 + x_3) + 2x_2 - x_3$$

$$x_4 = 12x_2 - 2x_3 + 2x_2 - x_3$$

$$x_4 = 14x_2 - 3x_3 \quad (7)$$

Thus, the components of  $x$  are:

$$x_1 = -6x_2 + x_3, \quad x_4 = 14x_2 - 3x_3$$

Substitute these into  $x$ :

$$x = \begin{bmatrix} -6x_2 + x_3 \\ x_2 \\ x_3 \\ 14x_2 - 3x_3 \end{bmatrix} = x_2 \begin{bmatrix} -6 \\ 1 \\ 0 \\ 14 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \\ -3 \end{bmatrix}$$

The basis vectors are:

$$\begin{bmatrix} -6 \\ 1 \\ 0 \\ 14 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 1 \\ -3 \end{bmatrix}.$$

## 1.5 Exercise 4.5

Find a nonzero 3-vector  $\mathbf{v}$  so that

$$\left\{ \mathbf{x} \in \mathbb{R}^3 : \mathbf{x} \cdot \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = 0, \mathbf{x} \cdot \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} = 0 \right\} = \text{span}(\mathbf{v}).$$

Then, using the *geometric* fact that any two different planes through the origin in  $\mathbb{R}^3$  meet along a line through the origin, interpret this algebraic outcome that the left side is the span of a single vector.

Solution:

Let  $a, b, c$  be scalar so that

$$\begin{aligned}
3a + 2b + c &= 0 \\
b &= \frac{-c - 3a}{2} \\
-2a - b + c &= 0 \\
-2a - \frac{-c - 3a}{2} + c &= 0 \\
c &= \frac{a}{3} \\
2b &= -\frac{a}{3} - 3a \\
b &= -\frac{5a}{3}
\end{aligned} \tag{1}$$

writing  $\mathbf{v}$  in terms of  $a$ ,

$$\begin{aligned}
&\begin{bmatrix} a \\ -\frac{5a}{3} \\ \frac{a}{3} \end{bmatrix} \\
\mathbf{v} &= a \cdot \begin{bmatrix} 1 \\ -\frac{5}{3} \\ \frac{1}{3} \end{bmatrix}
\end{aligned}$$

The two planes defined by the equations

$$3a + 2b + c = 0 \tag{2}$$

$$-2a - b + c = 0 \tag{3}$$

intersect along a line through the origin in  $\mathbb{R}^3$ . This line is spanned by the vector  $\mathbf{v}$ . Hence, the solution set is

$$a \cdot \begin{bmatrix} 1 \\ -\frac{5}{3} \\ \frac{1}{3} \end{bmatrix}$$

Here  $a \neq 0$  so the span is

$$\begin{bmatrix} 1 \\ -\frac{5}{3} \\ \frac{1}{3} \end{bmatrix}$$

## 1.6 Exercise 4.6

Find a pair of 3-vectors  $\mathbf{v}, \mathbf{w}$  so that

$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : 2x - 3y + 2z = 0 \right\} = \text{span}(\mathbf{v}, \mathbf{w}).$$

we have

$$2x - 3y + 2z = 0$$

Therefore

$$x = \frac{3y - 2z}{2}$$

we can write  $x =$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

as  $x =$

$$\begin{bmatrix} \frac{3y-2z}{2} \\ y \\ z \end{bmatrix}$$

$x =$

$$y \begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Two linearly independent vectors spanning the subspace are:

$$v_1 = \begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \end{bmatrix}$$

and

$$v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

## 1.7 Exercise 4.7

part (a)

$$v = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, w = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$v' = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, w' = \begin{bmatrix} -2 \\ -3 \\ 4 \end{bmatrix}$$

Find scalars  $a, b, c, d$  such that

$$v' = av + bw$$

$$w' = cv + dw$$

$$v' = a \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

$$a + b = 3$$

$$\boxed{b = 1}$$

$$a - b = 1$$

$$\Rightarrow a - 1 = 1$$

$$\Rightarrow \boxed{a = 2}$$

$$v' = 2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

$$w' = a \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ 4 \end{bmatrix}$$

$$= a + b = -2$$

$$\boxed{b = -3}$$

$$a - b = 4$$

$$a - (-3) = 4$$

$$\boxed{a = 1}$$

$$w' = 1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - 3 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ 4 \end{bmatrix}$$

part (b)

plane spanned by  $v$  and  $w$

$$ax + by + cz = 0 \quad \text{where}$$

$\begin{bmatrix} a \\ b \\ c \end{bmatrix}$  is normal vector perpendicular to both  $v$  and  $w$

$$v = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad w = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

The normal vector perpendicular to  $v$  and  $w$  is got by the cross product of  $v$  and  $w$

$$v \times w = \begin{vmatrix} i & j & k \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$v \times w = i \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} - j \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} + k \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$$

$$i = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} \Rightarrow (0 \times -1) - (1 \times 1) = -1$$

$$j = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = 1 \times (-1) - (1 \times 1) = -2$$

$$k = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1 \cdot 1 - 0 \cdot 1 = 1$$

$$v \times w = (-1)i - (-2)j + (1)k = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$v * w = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$v' = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad w' = \begin{bmatrix} -2 \\ -3 \\ 4 \end{bmatrix}$$

$$v * w = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$-1x + 2y + z = 0$$

substituting  $v' = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$

$$v' = -1(3) + 2(1) + (1) = -3 + 3 = 0$$

substituting  $w' = \begin{bmatrix} -2 \\ -3 \\ 4 \end{bmatrix}$

$$\begin{aligned} w' &= -1(-2) + 2(-3) + 4 \\ &= 2 - 6 + 4 \\ &= 0 \end{aligned}$$



## 1.8 Exercise 4.8

Use Theorem 4.2.5 to determine the dimension (1, 2, or 3) of each of the following linear subspaces in  $\mathbb{R}^3$ . Show the work to justify your answer.

(a)  $\text{span}(\mathbf{v}, \mathbf{w})$  for

$$\mathbf{v} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

(b)  $\text{span}(\mathbf{v}', \mathbf{w}')$  for

$$\mathbf{v}' = \begin{bmatrix} 3 \\ 6 \\ -2 \end{bmatrix}, \quad \mathbf{w}' = \begin{bmatrix} -2 \\ -4 \\ 2 \end{bmatrix}.$$

(c)  $\text{span}(\mathbf{v}'', \mathbf{w}'', \mathbf{u}'')$  for

$$\mathbf{v}'' = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \quad \mathbf{w}'' = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{u}'' = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}.$$

Solution:

(a)  $\text{span}(\mathbf{v}, \mathbf{w})$  for

$$\mathbf{v} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Here check if we can combine  $\mathbf{v}$  and  $\mathbf{w}$  such that they cancel out each other. This can be checked by finding out  $a, b$  such that

$$a.v + b.w = 0$$

$a.v$  means we are stretching (or shrinking)  $\mathbf{v}$  by  $a$ . similar for  $b.w$  by combining  $a.v+b.w$ , we get a new vector if this new vector is 0 then they cancel each other. If we find non-zero values of  $a$  and  $b$  such  $a.v+b.w$  is 0 then it means  $\mathbf{v}, \mathbf{w}$  are linearly dependent. If the only solution is  $a=0$  and  $b=0$  then  $\mathbf{v}, \mathbf{w}$  are linearly independent and  $\mathbf{v}$  and  $\mathbf{w}$  point to different directions.

$$a.v + b.w = 0$$

$$a. \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} + b. \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0$$

This gives

$$-2a + b = 0$$

$$a + b = 0$$

$$a + b = 0$$

Substituting

$$\begin{aligned}a &= -b \\ -2(-b) + b &= 0\end{aligned}$$

Therefore

$$\begin{aligned}b &= 0 \\ a &= 0\end{aligned}$$

This means  $\mathbf{v}$  and  $\mathbf{w}$  are independent. Together  $\mathbf{v}$  and  $\mathbf{w}$  span a plane which has dimension 2.

(b)  $\text{span}(\mathbf{v}', \mathbf{w}')$  for

$$\mathbf{v}' = \begin{bmatrix} 3 \\ 6 \\ -2 \end{bmatrix}, \quad \mathbf{w}' = \begin{bmatrix} -2 \\ -4 \\ 2 \end{bmatrix}.$$

Similar to (a) we have  $a.\mathbf{v} + b.\mathbf{w} = 0$

$$a. \begin{bmatrix} 3 \\ 6 \\ -2 \end{bmatrix} + b. \begin{bmatrix} -2 \\ -4 \\ 2 \end{bmatrix} = 0$$

Now we have

$$\begin{aligned}3a - 2b &= 0 \\ 6a - 4b &= 0 \\ -2a + 2b &= 0\end{aligned}$$

Substituting

$$a = b$$

we have

$$6(b) - 4b = 0$$

From this we have

$$\begin{aligned}b &= 0 \\ a &= 0\end{aligned}$$

This means  $\mathbf{v}'$  and  $\mathbf{w}'$  are independent. Together  $\mathbf{v}'$  and  $\mathbf{w}'$  span a plane which has dimension 2.

(c)  $\text{span}(\mathbf{v}'', \mathbf{w}'', \mathbf{u}'')$  for

$$\mathbf{v}'' = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \quad \mathbf{w}'' = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{u}'' = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}.$$

Similar to (b) we have

$$a.\mathbf{v}'' + b.\mathbf{w}'' + c.\mathbf{u}'' = 0$$

$$\begin{aligned}a. \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + b. \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} + c. \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} &= 0 \\ a + 3b + 3c &= 0\end{aligned}$$

$$-2a - 2c = 0$$

$$3a + 2b + 4c = 0$$

Substituting

$$a = c$$

$$c + 3b + 3c = 0$$

$$b = \frac{-4}{3}c$$

$$3c + 2 \cdot \frac{-4}{3}c + 4c = 0$$

$$a = 0$$

$$b = 0$$

$$c = 0$$

This means  $v''$ ,  $w''$  and  $u''$  are independent and dimension is 3

## 1.9 Exercise 4.9

Consider the three nonzero vectors

$$v_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}.$$

- Show that  $v_1$  does not belong to the span of  $v_2$  and  $v_3$ . (Hint: if  $v_1 = av_2 + bv_3$  for some scalars  $a$  and  $b$ , express this as a system of 3 equations on  $a$  and  $b$  and show that these equations have no simultaneous solution.)
- Similarly show  $v_2$  does not belong to the span of  $v_1$  and  $v_3$ , and that  $v_3$  does not belong to the span of  $v_1$  and  $v_2$ .
- Using (a) and (b), apply Theorem 4.2.5 to conclude that the linear subspace  $V = \text{span}(v_1, v_2, v_3)$  in  $\mathbb{R}^3$  has dimension equal to 3 (so it coincides with  $\mathbb{R}^3$ , by Theorem 4.2.8).

Solution:

- Show that  $v_1$  does not belong to the span of  $v_2$  and  $v_3$ . (Hint: if  $v_1 = av_2 + bv_3$  for some scalars  $a$  and  $b$ , express this as a system of 3 equations on  $a$  and  $b$  and show that these equations have no simultaneous solution.)

As the hint suggested, we want to find 2 scalar  $a$  and  $b$  and express

$$v_1 = av_2 + bv_3$$

$$\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = a \cdot \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + b \cdot \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

$$1 = 2a + 3b$$

$$-2 = -2b$$

$$3 = a + b$$

Substituting

$$b = 1$$

$$3 = a + 1$$

Solving we get

$$a = 2$$

$$b = 1$$

But if we put  $a=2$  and  $b=1$  in the first equation we get

$$1 = 2(2) + 3(1)$$

$$1 \neq 7$$

This shows that there is no simultaneous solution to these system of equations. This indicates that  $v_1$  does not belong to the span of  $v_2$  and  $v_3$ .

(b) Similarly show  $v_2$  does not belong to the span of  $v_1$  and  $v_3$ , and that  $v_3$  does not belong to the span of  $v_1$  and  $v_2$ .

(i)  $v_2$  does not belong to the span of  $v_1$  and  $v_3$

$$\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = a \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + b \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

$$2 = a + 3b$$

$$0 = -2a - 2b$$

$$1 = 3a + b$$

Substituting

$$a = -b$$

we get

$$1 = 3(-b) + b$$

$$b = \frac{-1}{2}$$

Substituting

$$a = \frac{1}{2}$$

$$b = \frac{-1}{2}$$

in first equation we get

$$2 = \frac{1}{2} + 3\frac{-1}{2}$$

$$2 \neq -1$$

This shows that there is no simultaneous solution to these system of equations. This indicates that  $v_2$  does not belong to the span of  $v_1$  and  $v_3$ .

(i)  $v_3$  does not belong to the span of  $v_1$  and  $v_2$

$$\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = a \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + b \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$3 = a + 2b$$

$$-2 = -2a$$

$$1 = 3a + b$$

Substituting

$$a = 1$$

we get

$$1 = 3(1) + b$$

$$b = -2$$

Substituting

$$a = 1$$

$$b = -2$$

in first equation

$$3 = 1 + 2(-2)$$

$$3 \neq -3$$

This shows that there is no simultaneous solution to these system of equations. This indicates that  $v_3$  does not belong to the span of  $v_1$  and  $v_2$ .

(c) Using (a) and (b), apply Theorem 4.2.5 to conclude that the linear subspace  $V = \text{span}(v_1, v_2, v_3)$  in  $\mathbb{R}^3$  has dimension equal to 3 (so it coincides with  $\mathbb{R}^3$ , by Theorem 4.2.8).

Finding 3 scalar such that

$$a.v_1 + b.v_2 + c.v_3 = 0$$

$$a. \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + b. \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + c. \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = 0$$

$$a + 2b + 3c = 0$$

$$-2a - 2c = 0$$

$$3a + b + c = 0$$

Substituting

$$a = -c$$

$$3(-c) + b + c = 0$$

$$b = 2c$$

$$-c + 2(2c) + 3c = 0$$

$$c = 0$$

$$b = 0$$

$$a = 0$$

This means  $v_1, v_2, v_3$  are linearly independent vectors and they have a dimension of 3