CS 577: Introduction to Algorithms	Homework 6
Out: 04/14/2022	**Due: 04/26/2022**

Instructions

- Read the lecture notes.
- Read Chapters on Minimum Spanning Trees, Shortest Paths (single source and all pairs).
- You should do ***all*** problems, but hand in only the graded problems.
- Write your name and student ID clearly on your submission.
- Clearly mark the beginning and end of your solution to each problem.

1. Prove that if $G = (V, E, W)$ is a weighted connected (undirected) graph where all edges have distinct weights, then the Minimum Spanning Tree of G is unique.

   Suppose all edges of G have distinct weights except two edges e and e' which have the same weight. Suppose there is a Minimum Spanning Tree of G containing both e and e'. Prove that this G has a unique Minimum Spanning Tree.

2. **Graded Problem (Page limit: 1 sheet; 2 sides)** In our lectures we described an algorithmic approach to build Minimum Spanning Trees by adding lowest cost edges without creating cycles. This problem is an alternative algorithmic approach going the opposite directions, by deleting highest cost edges on a cycle. More specifically:

   - Let $G = (V, E, W)$ be a weighted connected (undirected) graph, where V is the set of vertices, E is the set of edges, $w_e \in W$ for each $e \in E$ is a nonnegative weight of the edge e. This alternative algorithm repeatedly finds a cycle C and if e is an edge on C such that $w_e = \max_{e' \in C}\{w_{e'}\}$, then remove e. Write a pseudocode algorithm implementing this idea.
   - The key lemma in our lecture notes proving the correctness (for both Kruskal's algorithm and Prim's algorithm) is about when a subset of edges $A \subseteq E$ can be "safely" augmented by an edge $e \notin A$, so that if A can be "grown" to a Minimum Spanning Tree, so can $A \cup \{e\}$.

     Prove a similar lemma about breaking a cycle: Suppose $A \subseteq E$ is a subset of edges such that there is a Minimum Spanning Tree $T = (V, E_T)$ such that $E_T \subseteq A$ (i.e., A contains this Minimum Spanning Tree T). Suppose C is a cycle in A (i.e. C only uses edges from A) and $e \in C$ is such that $w_e = \max_{e' \in C}\{w_{e'}\}$, then there is a Minimum Spanning Tree contained in $A - \{e\}$.

     (**Hint**:) To prove this lemma, you are given A and a MST $T = (V, E_T)$ with $E_T \subseteq A$. You consider two cases: Either $e \notin E_T$ or $e \in E_T$. ... The slightly tricky case is when $e \in E_T$ (why?).

     You should provide a counter example to the following fallacious argument: On T, since it is acyclic, some $e' \in C$ does not belong to E_T. Take any such e' and switch e' with e.

     Now that you know this argument is fallacious, give a correct argument proving your lemma.
   - Prove the correctness of your algorithm using the lemma.

     Discuss proper data structures to be used in your algorithm. Suppose the given graph G has size $|E| = n + O(1)$. What algorithm would you use for the MST problem, and why? Justify your answers.

3. This problem is about finding the shortest cycle in an undirected graph G with unit edge lengths.

   Someone proposes the following idea: When a back edge, say (u, v), is discovered during a DFS, it forms a cycle with tree edges from v to u followed by the back edge (u, v). The length of the cycle is easily calculated by the levels of u and v, which we keep track of during the DFS. We then perform a DFS, keeping track of the level of every vertex visited in its DFS tree, and keeping track of the shortest cycle formed by a back edge together with tree edges as described above.

Prove that this idea does not work. Find a counter example to this approach.

Give a correct algorithm for this problem.

4. **Graded Problem (Page limit: 1 sheet; 2 sides)** Often there are more than one shoretest paths from one vertex to another in a weighted (directed or undirected) graph. Among all paths of shoretest distance, we usually prefer a path that also has the minimum number of edges.

   Suppose G is a positive weighted directed graph. s is a vertex of G. Find an efficient algorithm that finds the shoretest path from s to all other vertices v of G that also has the minimum number of edges (among all shoretest paths from s to v).

   Prove your algorithm is correct, and analyse its running time.

5. Suppose in addition to edge weights there are also vertex weights, and a path $v_0, v_1, \ldots, v_k$ with edges $e_1 = (v_0, v_1), \ldots, e_k = (v_{k-1}, v_k)$ has total cost $\sum_{i=0}^{k} w(v_i) + \sum_{i=1}^{k} w(e_i)$, where $w(v_i)$ and $w(e_i)$ are the vertex and edge weights respectively.

   Suppose G is a weighted directed graph. (Some weights could be zero or negative.) Find an efficient algorithm that finds

   - whether there is a negative cycle.
   - Assume there is no negative cycle, for every pair (u, v), find the minimum total cost path from u to v.