CS 577: Introduction to Algorithms	Homework 0
Out: 01/25/2022	**Due: 02/03/2022**

Ground Rules

- Graded problems in this set are 8 and 9.
- You should do all problems, but hand in only the graded problems.
- Hand in two separate pdf files, one for each problem, on canvas.
- Write your name and student ID clearly on your submission.
- Clearly mark the beginning and end of your solution to each problem.

Problems

1. Review the topics from CS 240 curriculum, especially: induction, solving recurrences, asymptotic notation, and graphs.

2. Prove by induction that $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$.

3. Prove by induction that the number of leaves in a full binary tree of depth d is 2^d, and is one more than the number of internal nodes. (A full binary tree of depth 0 consists of a single vertex, and the full binary tree of depth $d > 0$ consists of a root vertex r and two full binary trees of depth $d - 1$ called its left and right subtrees; their roots are children of r.)

4. Prove that every integer (positive, negative, or zero) can be written in the form $\sum_i \pm 3^i$, where the sum is over a finite subset of distinct non-negative integers i (zero is the empty sum). For example,

$$44 = 3^4 - 3^3 - 3^2 - 3^0$$
$$23 = 3^3 - 3^1 - 3^0$$
$$19 = 3^3 - 3^2 + 3^0$$

   Given an arbitrary integer n, is such an expression for n unique?

5. You are given a $2^n \times 2^n$ chessboard with one square missing (that we will call a hole). Prove using induction on n that regardless of the position of the hole, you can tile the chessboard with L-shaped pieces containing three squares each. That is, you can find an arrangement of the L-shaped tiles such that every square of the chessboard is covered by exactly one tile and the hole is left uncovered.

6. You are given an $n \times n$ chessboard, where $n \geq 2$. Suppose two antipodal squares at the corners are missing. For concreteness, you can think of the Northeast corner and Southwest corner squares are missing. A *domino* is a 1×2 piece (you can think of it as consisting of two adjacent squares). Prove that there is no way to tile the chessboard (with the two missing holes) by dominos. (Hint: You should separately consider n is even or n is odd.)

7. Order the following functions according to the asymptotically smallest to the asymptotically largest.

$$\begin{array}{ccc} \log n & \sqrt{n} & 5^n \\ n^{\log n} & 5^{\sqrt{\log n}} & (\log n)^n \\ 3^{n+10} & \log(5^n) & \sqrt{5^{\log n}} \end{array}$$

8. Consider the following sorting algorithm. We assume A is an array of n integers, where $n \geq 1$. The initial call is SuperSort$(A, 1, n)$.

```
SuperSort(A,i,j){  \\sorts the subarray A[i..j]
    if (j = i) then return;
    if (j = i+1)                        \\when there are only 2 elements
      if (A[i] > A[j]) swap(A,i,j)      \\swaps A[i] and A[j]
    else {
      k = floor of ( (j-i+1)/3 );       \\integer part of the length (j-i+1)/3
      SuperSort(A,i,j-k);               \\sort first two thirds
      SuperSort(A,i+k,j);               \\sort second two thirds
      SuperSort(A,i,j-k);               \\sort first two thirds again
      }
}
```

   (a) Prove using induction that this algorithm is correct, that is, it always produces a sorted array.

   (b) Determine the asymptotic number of comparisons this algorithm makes.

9. Let Σ be a finite alphabet set, e.g., $\Sigma = \{A, B, \ldots, Z\}$ is the set of 26 English alphabet symbols. A string x over Σ is a concatenation of symbols from Σ. E.g., $x =$ NEVERODDOREVEN. We denote by xy the concatenation of x followed by y, and x^m a concatenation of x by itself m times. E.g., for the above example, $x^2 =$ NEVERODDOREVENNEVERODDOREVEN.

   We want to solve the following computational question: Given two strings x and y over Σ, are there positive integers m and n such that $x^m = y^n$?

   Prove the following:

   (a) Such positive integers m and n exist iff it is true for $m = |y|$ and $n = |x|$, the lengths of y and x respectively. From this derive a quadratic-time algorithm for this problem.

   (b) Suppose $|y| \leq |x|$. If the answer to the problem is yes, then y is a prefix as well as a suffix of x.

   (c) The answer to the problem is yes iff the concatenations in two different orders are the same: $xy = yx$. Devise a linear-time algorithm from this.

   (Hint: To prove this condition is necessary, consider $d = \gcd(|x|, |y|)$.)

10. (challenge problem) You are given 12 coins, and you are told that exactly 11 of which are real and one is fake. The only way to tell which is real and which is fake is by weighing them. All real coins weigh exactly the same, but the fake one weighs either heavier or lighter than the real one.

    You are given a scale on which you can weigh any two disjoint subsets of the 12 coins. This is the basic operation: To weigh any chosen subset against another disjoint subset. For example, you can decide to weigh any 2 coins against another 2 coins. (There is no point to weigh 2 subsets of different cardinality. Do you see why?) Upon given the result (either the first subset weighs more or equal or less than the second one), your subsequent choice as to what to weigh can depend on this result.

    Your task is to devise a strategy that identifies the fake coin in no more than 3 sequential weighings and at the end identify the fake coin and determine whether the fake coin weighs more or less than the real one.