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CS577 Homework #1
Problem #5

#5a) By the constraints of the problem, at any given legal configuration, the plates on any pillar are ordered such that the largest plate is at the bottom of the tower and their sizes strictly decrease to the top. This means that regardless of the configuration there is exactly one tower that contains the smallest disk. Additionally this disk must be at the top of the tower, otherwise the tower is not legal. Since this disk is the smallest, it can be placed on top of any other disk in the configuration. Therefore, regardless of the number of plates on the tower adjacent to the tower with the smallest disk, there is a legal move that consists of moving the smallest disk to the next tower.

#5b) *Statement:* There are a finite number of moves to solve the tower of hanoi problem with the constraint of only moving one step counterclockwise.

Base Case: n = 0

All the disks are correctly ordered on tower(2).

Inductive Step: n >= 1

To move the kth disk from tower(1) to tower(2). Using the induction hypothesis we move k-1 disks from tower(0) to tower(2). Then we can move the kth disk one step to tower(1). Then using the induction hypothesis we move the k-1 disks back to tower(1) as they are all strictly smaller than the k+1 disk. Following this step, we move the kth disk to tower(2). Once again we can use the inductive hypothesis to assume we can move the k-1 disks to tower(2). We have successfully moved the kth disk from tower(0) to tower(2). This series of steps can be repeated to move all n disks from tower(0) to tower(2).

5c)

T(n): moving n disks two steps, for example from tower(0) to tower(2)
S(n): moving n disks one step, for example from tower(0) to tower(1)

In order to move a given disk one tower. The n-1 disks above it are moved two towers followed by the nth disk moving one tower and moving the n-1 disks back by moving them forward two towers. The recurrence relation for moving one tower would then be the following:

$$S(n) = T(n-1) + 1 + T(n-1)$$
$$= 2T(n-1) + 1$$

In order to move a disk two towers, we need to move the n-1 disks above it two towers to tower(2). Then we can move the nth disk one tower to tower(1). Follow this with a one tower move for the n-1 disks to tower(0). Now we can finally move the nth disk to tower(2). This will be its second move. In order to restore the n-1 disks we need to move them two more towers so they are also on tower(2). The recurrence relation for moving a disk two towers would be the following:

$$T(n) = T(n-1) + 1 + S(n-1) + 1 + T(n-1)$$
$$= 2T(n-1) + S(n-1) + 2$$

Substituting the recurrence relation for a one step move into the above recurrence relation gives:

$$T(n) = 2T(n-1) + 2T(n-2) + 3$$

5d)

Base Case: n = 1:

It takes two moves to move the single peg from tower(0) to tower(2)

$2 > (\sqrt{3} + 1)^{1-1} = 1$

Base Case: n = 2:

Two steps to move the smallest peg from tower(0) to tower(2).

One step to move the larger peg from tower(0) to tower(1)

One step to move the smaller peg from tower(2) to tower(0)

One step to move the larger peg from tower(1) to tower(2).

Two steps to move the smaller peg from tower(0) to tower(2).

Overall it takes 7 steps. $7 > (\sqrt{3} + 1)^{2-1} \approx 2.7$

Inductive Step: $k \Rightarrow k + 1$

Inductive Hypothesis: Assume T(j) holds: $T(j) \geq (\sqrt{3} + 1)^{j-1}$ for $1 \leq j \leq k$

$T(k+1) = 2T(k) + 2T(k-1) + 3$

By the inductive hypothesis 2T(k) and 2T(k-1) are both larger than $(\sqrt{3} + 1)^{k-1}$ and $(\sqrt{3} + 1)^{k-2}$.

Since T(k+1) can be represented by T(k) and T(k-1). It follows that $T(k+1) \geq (\sqrt{3} + 1)^{k}$.