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INTRODUCTION

Modern probability theory studies chance processes for which the knowledge of previous outcomes influences predictions for future experiments. In principle, when we observe a sequence of chance experiments, all of the past outcomes could influence our predictions for the next experiment. For example, this should be the case in predicting a student's grades on a sequence of exams in a course. But to allow this much generality would make it very difficult to prove general results.

In 1907, A. A. Markov began the study of an important new type of chance process. In this process, the outcome of a given experiment can affect the outcome of the next experiment. This type of process is called a Markov chain.

Markov Chains are mathematical systems that experience transitions from one state to another according to certain probabilistic rules. A Markov chain is a stochastic process, but it differs from a general stochastic process in that a Markov chain must be "memory-less." In other words, the probability of transitioning to any particular state is dependent solely on the current state, and time elapsed. This is called the Markov property. Markov Chains are a fairly common, and relatively simple way to statistically model random processes.

Markov chain typically has three parts:

- Probability vector: It is a vector whose entries add up to 1.
- Stochastic matrix/Transition matrix: A square matrix whose columns/rows are probability vectors.
- Chain: Sequence of probability vectors (also known as state vectors) along with a transition matrix which will give the next state vector.

Markov chains have many applications as statistical models of real-world processes, such as studying cruise control systems in motor vehicles, queues or lines of passengers arriving at an airport, currency exchange rates, and animal population dynamics. Also, Markov Chains help us to include real-world phenomena in computer simulations. Even the algorithm, which is used by Google, to determine the order of web pages in the search engine's results, is a type of Markov Chain.

The aim and scope of the project report is to provide a brief introduction to Markov Chains and also to present examples of different applications of the same within the financial and political world.

REVIEW OF LITERATURE

Andrey Andreyevich Markov was a Russian mathematician. He is best known for his work on the theory of stochastic Markov processes. His research in the area later became known as the Markov process and Markov chains.

Andrey Andreyevich Markov introduced the Markov chains in 1906 when he produced the first theoretical results for stochastic processes by using the term "chain" for the first time. In 1913 he calculated letter sequences of the Russian language.

The goal of the Markov chain is to statistically model random processes in a relatively simple way compared to decision trees which tend to be confusing and sometimes untraceable. Markov chains is applicable to scenarios which meet the following two conditions :

1. The total population remains fixed.
2. The population of a given state can never become negative.

Definition 1 :

The Markov Property.

For any positive integer n and possible states $i_0, i_1, \ldots, i_n$ of the random variables,

$$P(X_n = i_n \mid X_{n-1} = i_{n-1}) = P(X_n = i_n \mid X_0 = i_0, X_1 = i_1, \ldots, X_{n-1} = i_{n-1}).$$

A generalization to countable infinite state spaces was given by Kolmogorov in 1931. Markov chains are related to Brownian motion and the ergodic hypothesis, two topics in physics that were important in the early years of the twentieth century. But Markov appears to have pursued this out of a mathematical motivation, namely the extension of the law of large numbers to dependent events. Out of this approach grew a general statistical instrument, the so-called stochastic Markov process. In mathematics generally, probability theory and statistics particularly, a Markov process can be considered as a time-varying random phenomenon for which Markov properties are achieved. In a common description, a stochastic process with the Markov property, or memorylessness is one for which conditions on the present state of the system, its future and past are independent.

Markov processes arise in probability and statistics in one of two ways. A stochastic process, defined via a separate argument, may be shown (mathematically) to have the Markov property and as a consequence to have the properties that can be deduced from this for all Markov processes. Of more practical importance is the use of the assumption that the Markov property holds for a certain random process in order to construct a stochastic model for that process. In modeling terms, assuming that the Markov property holds is one of a limited number of simple ways of introducing statistical dependence into a model for a stochastic process in such a way that allows the strength of dependence at different lags to decline as the lag increases.

Often, the term Markov chain is used to mean a Markov process which has a discrete (finite or countable) state-space. Usually a Markov chain would be defined for a discrete set of times (i.e. a discrete-time Markov Chain) although some authors use the same terminology where "time" can take continuous values.

REPORT

★ Analysis:

Markov Chains have numerous applications as statistical models of real-life processes. Like studying cruise control systems in motor vehicles, queues or lines of customers arriving at an airport, currency exchange rates, and so on. As mentioned before, it satisfies Markov property, which means that if the current state of the process is known, then we require no extra information about its past states to obtain the best possible prediction of its future states. This simplicity permits great depletion in the number of parameters. [1]

To define it in mathematical form, we can do it as follows:

Consider a sequence of random variables $X = \{X_t : t \in N\}$ defined on a common underlying probability space (Ω, F, P) with discrete state space (S, S), i.e., technically, X_t is $F-S$-measurable for $t \in N$. It is a Markov Chain only if it holds the condition that:

$P(X_{t+1} = x \mid X_0 = x_0, X_1 = x_1, ... X_{t-1} = x_{t-1}, X_t = y) = P(X_{t+1} = x \mid X_t = y)$, where x,y and all other x_i are elements of the given state space S.

Markov Chains are used to find out the probabilities of events happening by seeing them as states transitioning into other states, or transition into the same state as before.
For example, we use the weather: If it is a rainy day, there is a 10% chance that the next day is a sunny day, and a 20% chance that if it is a sunny day, the next day will be a rainy day. Similarly, if today is a sunny day, there is, therefore, an 80% chance that the next day is another sunny day, and if it is a rainy day, there is a 90% chance that the weather will be rainy next day as well. [2]
This can be condensed into a transition diagram with all possible transitions of states.

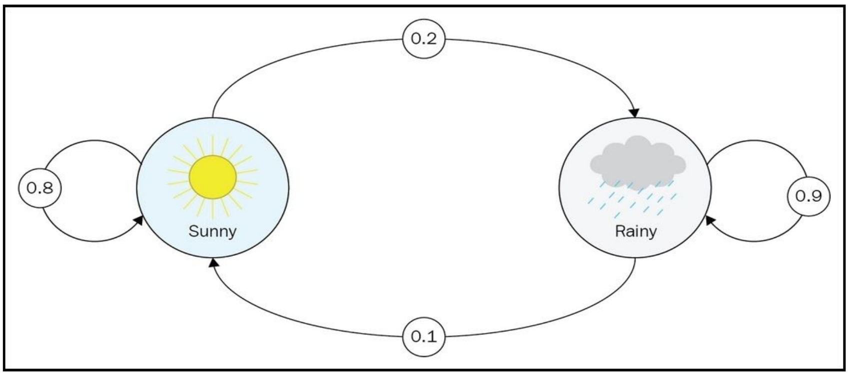

Fig: Transition diagram of the sunny-rainy weather phenomenon.

If we approach this situation mathematically, we view today as the current state, S_0, which is a 1 x m vector. The entries in the vector will be the current state of the process. In this weather example, we establish S = [Sunny Rainy]. Here, S is called our state space, in which all the elements are all feasible states that the process can achieve. If we take today to be a sunny day, then the S_0 vector will be $S_0 = [1 \quad 0]$, since there is a 100% probability of the day being sunny and 0% probability of it being rainy. If we want the next day's information, i.e., the next state, the transition probability matrix (or) stochastic matrix, which is basically the probability of transmission summarized in a matrix. For this case, it will be as such:

$$P = \begin{array}{cc} S & R \\ \end{array}$$
$$P = \begin{bmatrix} 0.8 & 0.2 \\ 0.1 & 0.9 \end{bmatrix} \text{ or more generally in this case: } P = \begin{bmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{bmatrix}$$

To reach the next state, S_1, we can simply find out the matrix product $S_1 = S_0P$. Because the calculations for successive states of S is just of the kind $S_n = S_{n-1}P$, we can compute the general formula for obtaining the probability of the process reaching a particular state is $S_n = S_0P^n$. Hence, this general formula allows huge simplicity while calculating probabilities that are in the distant future.

★ Elucidation of various concepts regarding Markov Chains:

While approaching Markov Chains we see that there are two different types:

1. Discrete-time Markov chains

2. Continuous-time Markov chains

This implies that we have two cases. One where the changes occur at specific states and one where the changes are continuous. For this project report, we will focus solely on discrete-time Markov chains.

To explain the discrete-time Markov chains we take the example of the price of an asset whose value is only taken down at the end of the day. The value of the Markov chain in discrete-time is known as state and in this example, the state corresponds to the closing price. Opposed to this, a continuous Markov chain can change at whatever time. The aforementioned can be explained with an example where the measured events happen at a continuous-time and don't contain "steps" in its process. A famous example of a continuous-time Markov chain is the Poisson process.

For a finite Markov chain, the state space S is usually given by S = {1, 2, 3,.., M}, and the infinite but countable Markov chain state space is recorded as S = {0, 1, 2, ….}. [3]

A Markov chain can be stationary in nature and thus be independent of the initial state in the process. This type of Markov chain is also called a steady-state Markov chain. Its outcome can be perceived in the example of market trends later on, where the probabilities of distinct outcomes converge to a particular value. An infinite Markov chain need not be a steady-state Markov chain, however, a steady-state Markov chain must be time-homogeneous which means that the stochastic matrix $P_{i,j}(n,n + 1)$ does not depend on the value of n.

★ Applications of Markov Chains:

Seeing that Markov chains can be designed to model many real-world processes, they are applied in a wide range of situations. From frequently being used to describe consumer behavior to analyzing brand loyalty to sales forecasting and even in the composition of music. It has a deep impact in economics where it is used to predict events like market crashes and cycles between recession and expansion and also to predict credit risk. When in a financial market, continuous-time Markov chains are used. The price of an asset is set using a random factor, a stochastic discount factor, which can be defined using a Markov chain. [4]

★ Estimating vote count using Markov chains

We consider here one more application of Markov chains in voting behavior. A population of voters is distributed between the Congress (C), BJP (B), and the rest (R) parties. The state space becomes p = [C B R]. [5]

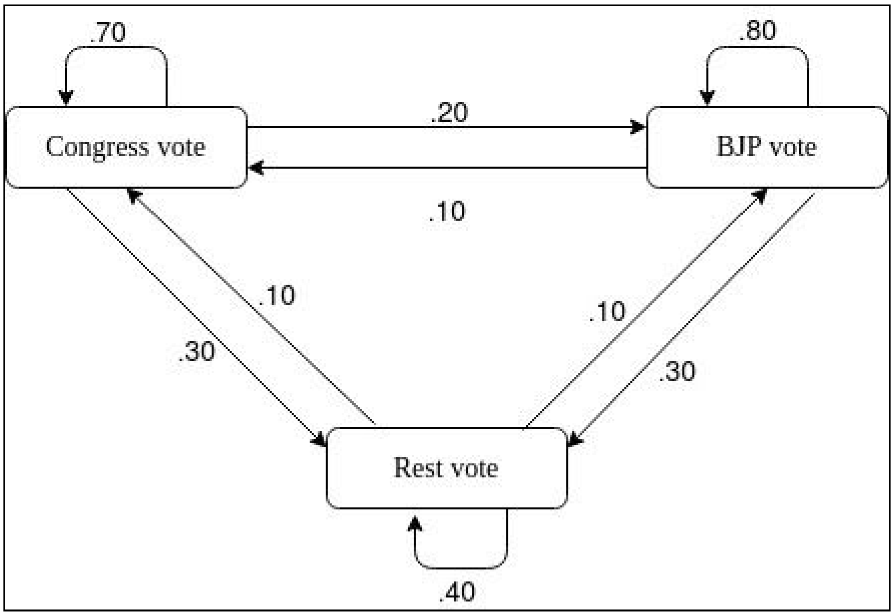

Fig: Voter shift between two elections

Let us take the example that, in an upcoming election, of those who voted for BJP in the previous election, 10% will switch to Congress, 10% to the rest, and the remaining 80% will stick to BJP. We describe this tendency by writing a transition matrix. The column labels are the initial party a portion of the voters have sided with, and the row labels are the final party.

$$T = \begin{array}{c} \begin{array}{ccc} C & B & R \end{array} \\ \begin{bmatrix} 0.7 & 0.1 & 0.3 \\ 0.2 & 0.8 & 0.3 \\ 0.1 & 0.1 & 0.4 \end{bmatrix} \end{array}$$

Suppose after an election campaign, the voters were distributed as such:

$$p_1 = \begin{bmatrix} .48 \\ .51 \\ .01 \end{bmatrix}$$

Therefore, using the transition matrix, we can expect the outcome of the next elections as follows:

$$P_2 = Tp_1 = \begin{bmatrix} 0.7 & 0.1 & 0.3 \\ 0.2 & 0.8 & 0.3 \\ 0.1 & 0.1 & 0.4 \end{bmatrix} \begin{bmatrix} .48 \\ .51 \\ .01 \end{bmatrix} = \begin{bmatrix} .390 \\ .507 \\ .103 \end{bmatrix}$$

To state it definitely, 70% of the original Congress voters (.48) voted for Congress itself, 10% of the BJP voters (.51) voted for Congress and 30% of the people who voted for rest (.01) also voted for Congress:

$$.70(0.48) + .10(.51) + .30(.01) = .336 + .051 + .03 = .390$$

Similarly, the shift of votes can be calculated for BJP and the rest.

Provided this voter behavior is valid for a long period of time, we can predict the outcome of any future election by applying the transition matrix appropriate number of times to the initial vector. Also, since the transition matrix is regular, i.e., it doesn't have any value equal to zero, a steady-state solution can be expected. Using the eigenvalue method, we find $|T - \lambda I| = 0$

Hide Steps

$$\det\left(\begin{pmatrix}0.7 & 0.1 & 0.3\\ 0.2 & 0.8 & 0.3\\ 0.1 & 0.1 & 0.4\end{pmatrix} - \lambda\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right): \quad -\lambda^3 + 1.9\lambda^2 - 1.08\lambda + 0.18$$

$$\det\left(\begin{pmatrix}0.7 & 0.1 & 0.3\\ 0.2 & 0.8 & 0.3\\ 0.1 & 0.1 & 0.4\end{pmatrix} - \lambda\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right)$$

Show Steps

$$\begin{pmatrix}0.7 & 0.1 & 0.3\\ 0.2 & 0.8 & 0.3\\ 0.1 & 0.1 & 0.4\end{pmatrix} - \lambda\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}0.7-\lambda & 0.1 & 0.3\\ 0.2 & 0.8-\lambda & 0.3\\ 0.1 & 0.1 & 0.4-\lambda\end{pmatrix}$$

$$= \det\begin{pmatrix}0.7-\lambda & 0.1 & 0.3\\ 0.2 & 0.8-\lambda & 0.3\\ 0.1 & 0.1 & 0.4-\lambda\end{pmatrix}$$

Show Steps

$$\det\begin{pmatrix}0.7-\lambda & 0.1 & 0.3\\ 0.2 & 0.8-\lambda & 0.3\\ 0.1 & 0.1 & 0.4-\lambda\end{pmatrix} = -\lambda^3 + 1.9\lambda^2 - 1.08\lambda + 0.18$$

$$= -\lambda^3 + 1.9\lambda^2 - 1.08\lambda + 0.18$$

Show Steps

Solve $-\lambda^3 + 1.9\lambda^2 - 1.08\lambda + 0.18 = 0$: $\lambda = 1, \lambda = \frac{3}{10}, \lambda = \frac{3}{5}$

The eigenvalues are:

$$\lambda = 1, \lambda = \frac{3}{10}, \lambda = \frac{3}{5}$$

Fig: Calculations of eigenvalues for the given problem.

The eigenvector corresponding to the eigenvalue $\lambda = 1$ therefore, has components $x_1 = 9/4$, $x_2 = 15/4$, x3 = free. To transform this into a valid population vector, we can add the entries which should account to 1. Therefore, the entries are summed and divided by the result, hence we get:

$$p_n = \begin{bmatrix}.32\\ .54\\ .14\end{bmatrix}$$

The vector p_n shows that the population of voters will gradually settle into a steady-state in which 54% of the votes will be cast for the BJP candidate.

```
-> T = [ 0.7 0.1 0.3 ; 0.2 0.8 0.3 ; 0.1 0.1 0.4 ]
T  =

  0.7   0.1   0.3
  0.2   0.8   0.3
  0.1   0.1   0.4

->

-> [c,d] = spec(T);

-> disp("The Eigen-values are:",spec(T));

 "The Eigen-values are:"

  1.  + 0.i
  0.6 + 0.i
  0.3 + 0.i
```

Fig: Calculations of Eigenvalues in Scilab.

★ Utilizing Markov chains to predict market trends

Markov chains can be used to model the probabilities of certain financial market conditions and therefore, predicting the prospect of future market conditions. [6]

The market conditions/trends are:

1. Bull market: A bull market is the condition of a financial market in which prices are rising or are expected to rise.
2. Bear market: A bear market is when a market experiences prolonged price declines. It typically describes a condition in which securities prices fall 20% or more from recent highs amid widespread pessimism and negative investor sentiment.
3. Stagnant market: Stagnation is a prolonged period of little or no growth in the economy. An absence of stock market booms or highs. [7]

In just markets, it is supposed that the market information is shared equally among its participants and the prices fluctuate erratically. Hence, every participant has an equal share and no one has an upper hand by having inside information. Through analysis of historic data, patterns can be found as well as probabilities can be estimated.

We consider a hypothetical example here, where through historical data analysis, we obtain the following points:

- After a week of bull market trend, there is a 90% chance that another week of bull market conditions will follow, 7.5% chance that bear market trend will follow, and 2.5% chance that it will be a stagnant week.
- After a week of bear market trend, there is an 80% chance that the trend will remain, 15% chance that the trend will change to bear market, and the remaining 5% chance that it will be followed by a stagnant week.
- After a stagnant trend, there is a 50% chance that the market will remain stagnant, and a 25% chance that it will be either a bullish or bearish week.

Collating all the probabilities into a table and then converting into a transition matrix M,

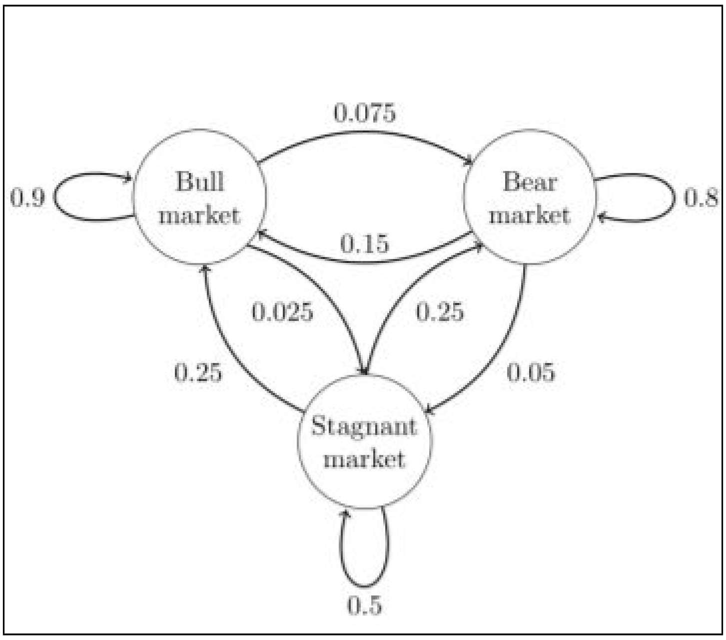

Fig: Transition diagram of the market trend process.

	Bull	Bear	Stagnant
Bull	0.9	0.075	0.025
Bear	0.15	0.8	0.05
Stagnant	0.25	0.25	0.5

Table: Tabular form of the probabilities

$$M = \begin{bmatrix} 0.9 & 0.075 & 0.025 \\ 0.15 & 0.8 & 0.05 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}$$

Now, we can make a state vector H which helps us to tell which of the three different conditions the current week is in. Column 1 represents a bull market trend, column 2 a bear market trend, and column 3 a stagnant market trend respectively. For this example, we choose the current week to be bearish. Hence the vector H = [0 1 0].

Provided the state of the current week, we can now calculate the possibilities of there being a bull trend, bear trend, or stagnant trend for any number of weeks in the future. This is done by matrix multiplying the state vector H with transition matrix M.

1 week from now: $H * M^1 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0.9 & 0.075 & 0.025 \\ 0.15 & 0.8 & 0.05 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}^1 = \begin{bmatrix} 0.15 & 0.8 & 0.05 \end{bmatrix}$

5 weeks from now: $H * M^5 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0.9 & 0.075 & 0.025 \\ 0.15 & 0.8 & 0.05 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}^5 = \begin{bmatrix} 0.48 & 0.45 & 0.07 \end{bmatrix}$

52 weeks from now: $H * M^{52} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0.9 & 0.075 & 0.025 \\ 0.15 & 0.8 & 0.05 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}^{52} = \begin{bmatrix} 0.63 & 0.31 & 0.06 \end{bmatrix}$

99 weeks from now: $H * M^{99} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0.9 & 0.075 & 0.025 \\ 0.15 & 0.8 & 0.05 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}^{99} = \begin{bmatrix} 0.63 & 0.31 & 0.06 \end{bmatrix}$

From the above example, we can infer that as the number of weeks tends to infinity, the probabilities will converge to an equilibrium state, meaning that the probability of next week being bullish, bearish or stagnant from that state will be the same, i.e., 63% probability for bullish, 31% probability for bearish, and 6% for stagnant. These results can be used in different ways.

```
In [6]: import numpy as np
        import pandas as pd
        from random import seed
        from random import random
        import matplotlib.pyplot as plt
        P = np.array([[0.9, 0.075, 0.025],
                      [0.15, 0.8, 0.05],
                      [0.25, 0.25, 0.5]])
        state=np.array([[0.0, 1.0, 0.0]])
        stateHist=state
        dfStateHist=pd.DataFrame(state)
        distr_hist = [[0,0,0]]
        for x in range(99):
          state=np.dot(state,P)
          print(state)
          stateHist=np.append(stateHist,state,axis=0)
          dfDistrHist = pd.DataFrame(stateHist)
        dfDistrHist.plot()

        plt.show()
```

Fig: Python code for finding stationary distribution π (steady-state).

```
[[0.625  0.3125 0.0625]]
[[0.625  0.3125 0.0625]]
[[0.625  0.3125 0.0625]]
[[0.625  0.3125 0.0625]]
```

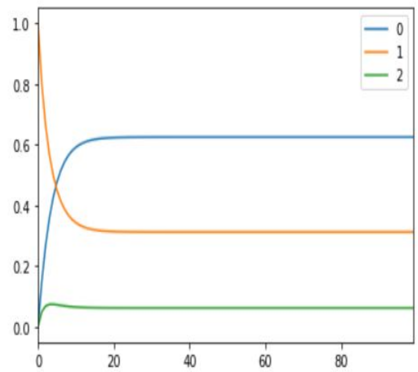

Fig: Graph showing where the probabilities eventually converge to.

```
M  =

   0.9    0.075   0.025
   0.15   0.8     0.05
   0.25   0.25    0.5

--> H = [0 1 0]
 H  =

   0.   1.   0.

--> // CALCULATING PROBABILITIES AFTER WEEK 1

--> H*M
 ans  =

   0.15   0.8   0.05

--> // CALCULATING PROBABILITIES AFTER WEEK 5

--> H*(M^5)
 ans  =

   0.47661   0.450515   0.072875

--> // CALCULATING PROBABILITIES AFTER WEEK 52

--> H*(M^52)
 ans  =

   0.6249999   0.3125001   0.0625

--> // CALCULATING PROBABILITIES AFTER WEEK 99

--> H*(M^99)
 ans  =

   0.625   0.3125   0.0625
```

Fig: calculation of vectors in Scilab.

★ Finding steady-state vector (Example of linear algebra application)

Consider this example. Suppose there are a total of 10 lakh people and 7 lakh people reside in the city whereas 3 lakh people reside in the suburban areas. Every year, 10% of the city's population moves to the suburbs and 2% of the suburb's population moves to the city. Will there be any year where the net movement is zero? We can find out using the Markov chain. [8]

Let C be the state vector. $C_0 = \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix}$

The transition matrix will be $P = \begin{bmatrix} 0.9 & 0.02 \\ 0.1 & 0.98 \end{bmatrix}$

We can calculate the population after the first year.

$C_1 = P * C_0 = \begin{bmatrix} 0.636 \\ 0.364 \end{bmatrix}$

This means the population of the city after one year is 6.36 lakhs and the population of suburbs is 3.64 lakhs.

Now, if the net movement needs to be zero, then $P * C_n = C_n$

$\Rightarrow P * C_n - C_n = 0$

$\Rightarrow (P - I) * C_n = 0$

where I is the identity matrix.

This is of the form Ax = 0

Where $A = P - I = \begin{bmatrix} -0.1 & 0.02 \\ 0.1 & -0.02 \end{bmatrix}$ and $x = C_n$

$$\textit{Augmenting matrices A and O we get :}$$

$$D = \begin{bmatrix} -0.1 & 0.02 & 0 \\ 0.1 & -0.02 & 0 \end{bmatrix}$$

$$\textit{Applying row operation } R_2 = R_2 - (-R_1), \textit{we get}$$

$$D = \begin{bmatrix} -0.1 & 0.02 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\textit{From this we can state that} : -0.1x_1 + 0.02x_2 = 0,$$
$$\textit{where } x_1 = \textit{population of city and } x_2 = \textit{population of suburbs.}$$
$$\textit{Simplifying, we get} : x_2 = 5x_1, \textit{but sum of elements of vector} = 1$$
$$\textit{Therefore}, x_1 + x_2 = 1 \Rightarrow 6x_1 = 1 \Rightarrow x_1 = 0.166$$
$$\textit{and } x2 = 5x_1 = 5(0.166) = 0.833$$

Using Gaussian elimination we get Cn = $\begin{bmatrix} 0.166 \\ 0.833 \end{bmatrix}$

We can conclude that even after the movement of people takes place at the end of the year, there won't be any change in the city's and suburb's population when the populations of the city and the suburbs are 1.66 lakhs and 8.33 lakhs respectively.

SUMMARY

Markov chains are a fundamental concept in stochastic processes. They can be used for processes that satisfy Markov property. This means that even after knowing about the history of the process there cannot be any effect on future predictions. This helps to significantly reduce the data which has to be considered while predicting.

To put it in mathematical terms, Markov chains basically consist of a state space, which is a vector that contains every possible state which can be attained by the variable, the current state of the variable, and the stochastic/transition matrix. The transition matrix contains all possible probabilities of the variable changing from one state to another or remain at the same state. To find out the state the variable will be in after n subdivisions of time, simply multiply that state vector with the transition matrix raised to the power of n.

There exists different types and concepts of Markov chains which can be selected depending upon the parameters of the processes. They can either be calculated over discrete or continuous time. The state spaces can be finite or infinite. A countably infinite state space contained by a Markov chain can be stationary, i.e., the process converges to steady space.

Markov chains can be broadly used in a range of academic fields, from economics to biology.

Application of linear algebra and matrix methods to Markov chains provides an efficient means of monitoring the progress of a dynamical system over discrete time intervals. Such systems exist in many fields. The model affords us with good insight and perhaps /serves as a helpful starting point from which more complicated and inclusive models can be developed.

By using empirical data, it is very much likely to obtain certain patterns. From these patterns, Markov chains can be used to predict future trends in the market and the risk associated with them.

BIBLIOGRAPHY

[1] Sericola, Bruno. (2013). Markov Chains. Theory, Algorithms, and Applications.

Author: Bruno Sericola

Publication: Wiley

Year: 2013

No. of pages: 397

Avialable:https://onlinelibrary.wiley.com/doi/book/10.1002/9781118731543

[2] Predicting the weather with Markov Chains.

Avialable:https://towardsdatascience.com/predicting-the-weather-with-markov-chains-a34735f0c4df

[3] Chapter 3, Finite State Markov Chains.

Available:https://ocw.mit.edu/courses/electrical-engineering-and-computer-science/6-262-discrete-stochastic-processes-spring-2011/course-notes/MIT6_262S11_chap03.pdf

[4] Van der Hoek, John & Elliott, Robert. (2012). Asset Pricing Using Finite State Markov Chain Stochastic Discount Functions. Stochastic Analysis and Applications - STOCHASTIC ANAL APPL. 30. 865-894. 10.1080/07362994.2012.704852.

Available:https://www.researchgate.net/publication/254354459_Asset_Pricing_Using_Finite_State_Markov_Chain_Stochastic_Discount_Functions

[5] Andrew Berger, "Linear Algebra applications: Markov chains".

Available:https://www2.kenyon.edu/Depts/Math/hartlaub/Math224%20Fall2008/Markov-Sample2.pdf

[6] Deju Zhang & Xiaomin Zhang, "Study on forecasting the stock market based on the stochastic analysis method", *International Journal of Business and Management.*

Available:https://pdfs.semanticscholar.org/baf1/f9b83c2707c8a8382cfd84eb3d83027b2c5f.pdf?_ga=2.129521797.841601305.1589520807-392796746.1589520807

[7] What is the bull market, bear market, and stagnant market?
Available:https://www.investopedia.com/investing-essentials-4689754

[8] Lecture 10: Markov chains.
Available:https://www.math.uh.edu/~schiu/2017summerLinearAlgebraLab/Lecture_note/lecture10_Markov_chain.pdf

References that are not cited in this paper were used to gain a deeper insight of the respective topic. The information provided by these references have been used in the document as a whole and cannot be cited at specific locations.