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A Concise Method of Pole Placement to Stabilize the Linear Time Invariant MIMO System

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Abstract— **Stabilization of linear time-invariant multi-input-multi-output (LTI-MIMO) systems is presented distinctively and efficiently in this paper. The idea is to decouple the system state matrix depending on different inputs and outputs using the special canonical transformation proposed. Due to the decoupled form of the observer based controller system, it's possible to use separate transformation for observer and controller design. Since the decoupled state matrix resembles the single-input-single-output (SISO) case, the generalized equation for the system is first obtained, and then it's extended to the MIMO system. The computational complexity in getting the controller and observer gain matrix coefficients are further reduced due to the special form of gain matrices taken.**

Index Terms— **controllability, cyclic subspace, Hurwitz, observability, similarity transformation, stability, state feedback.**

I. Introduction

Controlling of a system becomes more simple when the system is in linear form. In most of the problems, a non-linear system is linearised near the operating point. Stabilization of a linear system is straightforward and is more general whereas nonlinear analysis uses numerical methods. Location of the closed-loop poles of the system determines the performance and stability[1]. Generic approach towards the stabilization of single input single output (SISO) linear time-invariant (LTI) systems using pole placement technique are already mentioned in the literature. But when it comes to the multiple input multiple output (MIMO) case, the design is not anymore straight forward. Some of the approaches mentioned in the literature include conversion of the system to Brunovsky canonical form[2], pole placement after decoupling using Luenberger canonical form[3][4][5]. Brasch explained dynamic pole placement of compensator's with an arbitary transformation [6] and Chen described obtaining state feedback matrix using Lyapnov equation without revealing the structure of the resulting feedback system subjected to the condition that state and feedback coefficient matrix has no common eigenvalues[7].

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This paper proposes a generic approach towards the stabilization of LTI MIMO systems. The objective is to obtain a particular form of input and output matrices by using suitable similarity transformation. These transformation matrices are obtained by slightly modifying the controllability and observability matrices. These transformations also decouple the system matrix to a block upper or block lower triangular form corresponding to different inputs or outputs. A generic system consists of a plant, which is to be stabilized with the associated controllers and observers connected to it. The dynamics of the whole system obtained has the decoupled form, corresponding to the controller and observer. The approach allows taking two different transformations so that the controller and the observer have the independent stable dynamics. The stabilization of the MIMO system using controller has been already described by Wonham[8][9]. Here the approach is extended towards the observer design, where the transformation based on the observability matrix is used.

The transformation used decomposes the MIMO system to multiple SISO systems. Hence initially, the approach is applied to the SISO system to get a generalized view of the components later; these generalized equations are used to the diagonal blocks of the augmented system matrix of MIMO case. The calculation of the controller and observer gain matrices are further simplified by taking the special form of these matrices which are related to the mentioned transformations.

The special forms of input and output matrices allow simplifying the complexity in calculating the controller and observer gain matrices.

Definition 1: In the special form of the input and output matrices, each non-zero entity corresponds to the input and output for the corresponding block of the augmented system matrix of controller and observer, respectively.

Special input matrix form

$$\widehat{B} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}$$

Special output matrix form

$$\widetilde{C} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \dots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \end{bmatrix}$$

This form restricts the augmented system matrix in the triangular block structure, where each diagonal block helps to attain the form similar to the SISO system.

A. System Model

Consider the n^{th} order linear dynamic system given by the state and output equation

$$\dot{x} = A\,x + B\,u \tag{1}$$

$$y = Cx \tag{2}$$

where the state vector $x \in$ vector space V which is $\mathbb{R}^{(n\times1)}$, $u \in \mathbb{R}^{(p\times1)}$ is the input vector, $y \in \mathbb{R}^{(q\times1)}$ is the output vector, $A \in \mathbb{R}^{(n\times n)}$ is the state matrix, $B \in \mathbb{R}^{(n\times p)}$ is the input matrix and $C \in \mathbb{R}^{(q\times n)}$ is the output matrix. An observer is used to identify the states of the system from the information of the output of the system. Model of the observer is given as

$$\dot{z} = A\,z + B\,u + L(y - y_m) \tag{3}$$

$$y_m = C\,z \tag{4}$$

where, $z \in \mathbb{R}^{(n\times1)}$ is the observed state vector, $y_m \in \mathbb{R}^{(q\times1)}$ is the observer output vector and $L \in \mathbb{R}^{(n\times q)}$ is the observer gain matrix. The controller uses these states to stabilize the system using state feedback.

$$u = -K\,z \tag{5}$$

where $K \in \mathbb{R}^{(p\times n)}$ controller feedback gain matrix.

Assuming the system is fully state controllable, the controllability matrix M is

$$M = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-2}B & A^{n-1}B \end{bmatrix}, \tag{6}$$

where, input matrix $B = [b_1\; b_2 \dots b_j \dots b_p]$. The column vectors of M span the whole space[10], V. Obtaining the cyclic subspaces S_j corresponding to each input vector b_j's the whole space is written as $S_1 \cup S_2 \cup \cdots \cup S_j \cup \cdots \cup S_p = V$. Where the $dim(S_j)$ gives the controllability index of the corresponding input vector b_j. Assuming fully state observable, the observability matrix N is

$$N = \begin{bmatrix} C \\ C\,A \\ \vdots \\ C\,A^{n-2} \\ C\,A^{n-1} \end{bmatrix}, \tag{7}$$

where, the output matrix $C = \left[c_1^T\, c_2^T \dots c_j^T \dots c_q^T\right]^T$. The row vectors of N span the whole space[10], V. Obtaining the cyclic subspaces R_j corresponding to each output vector c_j's the whole space is written as $R_1 \cup R_2 \cup \cdots \cup R_j \cup \cdots \cup R_q = V$. $dim(R_j)$ gives the observability index of the corresponding output vector c_j.

II. System Stabilization

The total system now consists of the plant, controller and the observer and the state feedback used $u = -Kz$. Now (1) becomes

$$\begin{aligned} \dot{x} =& Ax - BKz \\ =& (A - BK)x + BKe_x \end{aligned} \tag{8}$$

From (1) and (3), the error dynamics $\dot{e}_x = \dot{x} - \dot{z}$ is obtained as

$$\begin{aligned} \dot{e}_x =& (A - LC)\,e_x \\ y - y_m =& C\,e_x \end{aligned} \tag{9}$$

Hence the total system dynamics becomes

$$\begin{bmatrix} \dot{x} \\ \dot{e_x} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ e_x \end{bmatrix} \tag{10}$$

From (10) its evident that the eigenvalues of the total system are the eigenvalues of $A - BK$ and $A - LC$.

Remark 1: **Any system of the form** (10) **is stable if and only if** $A-BK$ **and** $A-LC$ **are individually Hurwitz.**

This form leads to the design of the controller and observer independently. Thus K and L are designed in a way such that $A - BK$ and $A - LC$ have stable eigenvalues. We use different similarity transformations for the design of controller and observer due to the above fact.

III. Single-Input-Single-Output Case

Consider the system, defined in (1) and (2) with the input matrix b and output matrix c, where $b \in \mathbb{R}^{(n\times1)}$ and $c \in \mathbb{R}^{(1\times n)}$. It is possible to control the system if all the states of the system are observable. Hence, first, the observer is designed to estimate the unobservable states of the system.

A. Observer Design

Presuming the system is fully state observable, system can be transformed to observable canonical form where the output matrix c is transformed to $\widetilde{c} = cQ^{-1} = \begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix}_{(1\times n)}$. The transformation also modifies the system matrix to $\widetilde{A} = QAQ^{-1}$. Note the transformation has the form $Q = \begin{bmatrix}c^T & (cA)^T & \cdots & (cA^{n-2})^T & (cA^{n-1})^T\end{bmatrix}^T$ and it is full-ranked. Hence, the observable form of the system matrix is

$$\widetilde{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \cdots & \cdots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 & 1 \\ -a_1 & -a_2 & \cdots & \cdots & -a_{n-1} & -a_n \end{bmatrix}_{(n\times n)} . \tag{11}$$

As stated in Remark 1, it is possible to design $L \in \mathbb{R}^{(n\times 1)}$ independently to stabilize the error dynamics (9). Consider the matrix $\widetilde{L} = \begin{bmatrix}l_n & l_{n-1} & \cdots & l_2 & l_1\end{bmatrix}^T$. The augmented system matrix $G = \widetilde{A} - \widetilde{L}\widetilde{c}$ is calculated where the transformation $\widetilde{L}Q = L$ provides the designed L. Hence,

$$G = \begin{bmatrix} -l_n & 1 & 0 & \cdots & 0 & 0 \\ -l_{n-1} & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \cdots & \cdots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \cdots & \ddots & \vdots \\ -l_2 & 0 & \cdots & \cdots & 0 & 1 \\ -l_1 - a_1 & -a_2 & \cdots & \cdots & -a_{n-1} & -a_n \end{bmatrix} . \tag{12}$$

The design of $\widetilde{L}$ depends upon the desired location of the closed loop poles. Consider the desired characteristic equation to be

$$s^n + r_n s^{n-1} + r_{n-1} s^{n-2} + \cdots + r_1 = 0 \tag{13}$$

The characteristic equation of the augmented system is $|sI - G| = 0$.

Remark 2 (General Characteristic Polynomial): The generalized form of the characteristic equation is obtained by expanding the matrix $|sI - G| = 0$ and has the form given in (14).

$$\begin{aligned} s^n + \sum_{i=0}^{n-1} l_{n-i} s^{n-i-1} + [\sum_{k=0}^{n-2} a_{n-k} \\ (\sum_{j=k+2}^{n} l_j s^{j-k-2} + s^{n-k})] + a_1 = 0 \end{aligned} \tag{14}$$

Equations (13) and (14) are compared to obtain the elements of the matrix $\widetilde{L}$.

Remark 3 (General form of gain matrix elements): The elements of the gain matrix are calculated by the recursive substitution process starting from $i = n, \cdots, 1$. The $(n+1)^{th}$ elements in the calculation is considered as zero.

$$l_i = r_i - a_i - \sum_{j=i+1}^{n} a_j l_{n-j+i+1} \tag{15}$$

B. Feedback Controller Design

The observer provides the information about the unobservable states, now the state z can be used by the controller to stabilize the system. The controllability matrix is rearranged to form $P = \begin{bmatrix}A^{n-1}b & \cdots & A^2 b & Ab & b\end{bmatrix}$. This allows transformations similar to that of observer design. P matrix transforms the system in the controllable form so that, the b matrix takes the form stated in Definition 1 and it has the structure $\widehat{b} = P^{-1}b = \begin{bmatrix}0 & \cdots & 0 & 1\end{bmatrix}^T_{(n\times 1)}$. P also transforms the system matrix, which gives the controllable form of it as $\widehat{A} = P^{-1}AP$. Hence,

$$\widehat{A} = \begin{bmatrix} -a_n & 1 & 0 & \cdots & 0 & 0 \\ -a_{n-1} & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \cdots & \cdots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \cdots & \ddots & \vdots \\ -a_2 & 0 & \cdots & \cdots & 0 & 1 \\ -a_1 & 0 & \cdots & \cdots & 0 & 0 \end{bmatrix}_{(n\times n)} . \tag{16}$$

As stated in Remark 1, the controller can be designed independently and hence the augmented system matrix is obtained as $F = \widehat{A} - \widehat{b}\widehat{K}$ and the transformation $P\widehat{K} = K$ gives the controller gain matrix (K). Consider the matrix $\widehat{K} = \begin{bmatrix}k_1 & k_2 & \cdots & k_n\end{bmatrix}$ which provides

$$F = \begin{bmatrix} -a_n & 1 & 0 & \cdots & 0 & 0 \\ -a_{n-1} & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \cdots & \cdots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \cdots & \ddots & \vdots \\ -a_2 & 0 & \cdots & \cdots & 0 & 1 \\ -a_1 - k_1 & -k_2 & \cdots & \cdots & -k_{n-1} & -k_n \end{bmatrix} \tag{17}$$

The structure of F is similar to G. From Remark 2 and 3, By interchanging l_i with k_i and a_i with l_i we can obtain the elements of the $\widehat{K}$ matrix similarly as that of the observer gain matrix.

IV. Multiple-Input-Multiple-Output Case

Consider the system stated in (1) and (2). The transformation of the system in controllable form and observable form is similar, as described for the SISO system. Here the transformation is chosen in such a way that the input matrix and output matrix have the form stated in Definition 1. The stated transformation gives the structure of the system matrices $\widetilde{A}$, $\widehat{A}$, and augmented system matrices G, F as upper or lower triangular form. Hence, the process of assigning the poles become independent for each of the inputs and outputs.

A. Observer Design

It is stated in (10) the observer design and the controller design is independent with each other. Hence, the observer design is demonstrated first.

1) Observability matrix: The output matrix $C \in \mathbb{R}^{(q\times n)} = \begin{bmatrix} c_1^T & \cdots & c_j^T & \cdots & c_q^T \end{bmatrix}^T$ where $c_j \in \mathbb{R}^{(n\times 1)}$. The transformation is taken such that system matrix takes block upper triangular form and each diagonal block is of observable form corresponding to the SISO case.

Definition 2: Transformation matrix Q is obtained from the corresponding basis of the constrained cyclic subspaces, U_j generated by c_j.

$$Q = [\, {c_1}^T \cdots (c_1 A^{m_1-1})^T | \cdots | \, {c_j}^T \cdots (c_j A^{m_j-1})^T \, | \cdots | {c_k}^T \cdots (c_k A^{m_k-1})^T \,]^T \tag{18}$$

Here the subspaces are subjected to the condition where the total space $V = U_1 \oplus U_2 \oplus \cdots \oplus U_j \oplus \cdots \oplus U_k$, $k \le q$ where $dim(U_j) = m_j$ and $\sum_{j=1}^{k} m_j = dim(V) = n$.

Q transforms the system matrix to block upper triangular matrix and the output matrix to the special canonical form. Hence, the choice of this transformation gives

$$\widetilde{A} = \begin{bmatrix} A_1 & \times & \times & \cdots & \cdots & \times \\ 0 & A_2 & \times & \cdots & \cdots & \times \\ 0 & 0 & A_i & \cdots & \cdots & \times \\ \vdots & \vdots & \vdots & \ddots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & A_k \end{bmatrix} \tag{19}$$

where,

$$A_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & \cdots & \ddots & 1 \\ -\alpha_1^i & -\alpha_2^i & \cdots & -\alpha_k^i & \cdots & -\alpha_{m_j}^i \end{bmatrix}_{(m_j \times m_j)} \tag{20}$$

The matrix $\widetilde{L}$ is chosen in such a way that the non-zero elements of $\widetilde{L}$ in each column depends upon the order of the A_i matrix. In $\widetilde{L}$, the first m_j elements of the j^{th} column starting from the $(\sum_{i=1}^{j-1} m_i + 1)^{th}$ position has non-zero values and the rest $(n - m_j)$ elements are zero. Where $j = 1, 2, \ldots, k$. Hence, the generalised form of the column of $\widetilde{L}$ matrix is

$$L_j^T = \begin{bmatrix} 0 & \cdots & l_{m_j}^j & \cdots & l_1^j & \cdots & 0 & \cdots & 0 \end{bmatrix}. \tag{21}$$

The choice of the $\widetilde{L}$ matrix gives the observable form of the augmented system matrix G. Hence,

$$G = \begin{bmatrix} G_1 & \times & \cdots & \times \\ 0 & G_j & \cdots & \vdots \\ \vdots & \vdots & \ddots & \times \\ 0 & \cdots & 0 & G_k \end{bmatrix} \tag{22}$$

where,

$$G_j = \begin{bmatrix} l_{m_j}^1 & 1 & 0 & \cdots & \cdots & 0 \\ l_{m_j-1}^1 & 0 & 1 & 0 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ l_2^1 & 0 & \cdots & \cdots & \cdots & 1 \\ (l_1^1 - \alpha_1) & -\alpha_2 & \cdots & \cdots & -\alpha_{n-1} & -\alpha_n \end{bmatrix} \tag{23}$$

In (23) G_j resembles the form in (12). As (22) has the form of upper block triangular matrix, only G_j's constitute the characteristic equation of the augmented system. $\left|(sI_{m_j} - G_j)\right| = 0$ helps to place m_j number of poles independently with the help of (15).

B. Feedback Controller Design

The feedback controller design is similar to the observer design. In this case, the augmented system matrix (F) takes the structure of the upper triangular block matrix. The controllability matrix transforms the system matrix to get the special structure of the augmented form.

1) Controllability matrix: Let the input matrix $B \in \mathbb{R}^{(n\times p)} = \begin{bmatrix} b_1 & \cdots & b_j & \cdots & b_p \end{bmatrix}$ where $b_j \in \mathbb{R}^{(1\times n)}$. The transformation is taken such that system matrix takes block lower triangular form and each diagonal block is of controllable form corresponding to the SISO case.

Definition 3: Transformation matrix P is obtained from the corresponding basis of the constrained cyclic subspaces (W_j) generated by b_j.

$$P = [(A^{n_k-1} b_k) \cdots b_k | \cdots | \, (A^{n_j-1} b_j) \cdots b_j \, | \cdots | \, (A^{n_1-1} b_1) \cdots b_1] \tag{24}$$

Here the subspaces are subjected to the condition where the total space $V = W_1 \oplus W_2 \oplus \cdots \oplus W_j \oplus \cdots \oplus W_k$, $k \le p$ where $dim(W_j) = n_j$ and $\sum_{j=1}^{k} n_j = dim(V) = n$.

$$\widehat{A} = \begin{bmatrix} A_k & 0 & 0 & \cdots & \cdots & 0 \\ \times & A_{k-1} & 0 & \cdots & \cdots & 0 \\ \times & \times & A_i & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \ddots & \vdots \\ \times & \times & \times & \cdots & \times & A_1 \end{bmatrix} \tag{25}$$

where,

$$A_i = \begin{bmatrix} -\alpha_{n_j}^i & 1 & 0 & \cdots & \cdots & 0 \\ -\alpha_{n_{j-1}}^i & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 1 & \vdots & 0 \\ -\alpha_m^i & 0 & \cdots & \ddots & \ddots & 0 \\ \vdots & \vdots & \cdots & \cdots & \ddots & 1 \\ -\alpha_1^i & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}_{(n_j \times n_j)}. \tag{26}$$

The next step is the choice of the $\widehat{K}$ matrix. Just like the observer design the non-zero elements in each row of $\widehat{K}$

depends upon the order (n_j) of each of the diagonal block matrices of $\widehat{A}$. The last n_j elements of the j^{th} row starting from the $(\sum_{i=j+1}^{k} n_i+1)^{th}$ position, has non-zero values and the rest $(n-n_j)$ elements are zero. Where $j = 1, 2, \ldots, k$. Hence, the generalised form of a row of $\widehat{K}$ is

$$K_j = \begin{bmatrix} 0 & \cdots & k_1^j & \cdots & k_{n_j}^j & \cdots & 0 & \cdots & 0 \end{bmatrix}. \quad (27)$$

Hence the controllable form of the augmented system matrix is

$$F = \begin{bmatrix} F_k & 0 & \cdots & 0 \\ \times & F_{k-1} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \times & \times & \times & F_1 \end{bmatrix} \quad (28)$$

where,

$$F_j = \begin{bmatrix} -\alpha_{n_j}^j & 1 & 0 & \cdots & \cdots & 0 \\ -\alpha_{n_{j-1}}^j & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 1 & \vdots & 0 \\ -\alpha_m^j & 0 & \cdots & \ddots & \cdots & 0 \\ \vdots & \vdots & \cdots & \cdots & \ddots & 1 \\ (-\alpha_1^j - k_1^j) & k_2^j & \cdots & 0 & \cdots & k_n^j \end{bmatrix}. \quad (29)$$

The rest of the calculation for assigning the poles for state feedback controller is similar to the observer design.With the help of (15), n_j number of poles can be placed independently from $\left|(sI_{n_j} - F_j)\right| = 0$.

V. Conclusion

Reducing the computational complexity and effective time management is always the key factor in designing any system. This paper provides a simple and efficacious way to stabilize an LTI-MIMO system. The special transformations proposed for MIMO system helps to obtain the required input and output matrices and helps to stabilize the system similar to SISO case. The intricacy in computing the gain matrices for controller and observer is further reduced by the form of the taken gain matrices. The attainment is, a single equation format can be used for both controller and observer design. It may be noted that a change in the order of input or output can correspond to changes in input and output matrices. The change, in turn, modifies the transformations and the number of inputs and outputs required for the pole placement can be adjusted in this way.

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