# Using the Bondi K-calculus to describe accelerating observers

Sreyam Sengupta

July 13, 2015

**Abstract**

The most common pedagogical approach to special relativity uses the special Lorentz transformations to derive the standard results of special relativity. The Bondi K-calculus is an alternative approach that yields deep physical insights.

Uniformly accelerating observers can be sufficiently explained using special relativity, which fails only when the curvature of spacetime cannot be neglected.

In my project I read about the Bondi K-calculus and accelerating observers in special relativity. The following is a report on what I have read and understood so far.

## 1 Introduction

Special relativity is a simple, straightforward subject. Maybe what it teaches may seem counterintuitive at first, but the mathematics is pretty simple, and when the special Lorentz transformations are brought in, the mathematics is all we have to worry about, and we can forget the physics and get on with solving problems.

The above statement is a common misconception many students have about the special theory. The truth is, not only is the special theory a complicated, conceptually difficult subject, the mathematics itself need not be so simple. And the special Lorentz transformations, though intended to instruct and clarify, sometimes have the opposite effect of concealing more than they reveal. And it is true that once the special Lorentz transformations have been derived, the physics takes a backseat to the calculations. And we often do calculations without even knowing what we're doing or why we're doing them. This is where the K-calculus, an alternative pedagogical tool developed by Hermann Bondi comes in. However, before we begin, it would be helpful to remind ourselves of a few basic facts.

The two postulates of special relativity are:

1. The laws of physics are the same in all inertial reference frames. Here an inertial reference frame is one where Newton's second law holds, i.e.

an observer at rest in an inertial frame would measure the product of a body's mass and its acceleration and find it equal to the sum of all forces acting on it.

2. The speed of light in vacuum is the same as measured by all inertial observers. This value is denoted by $c$.

If these two postulates are true, then the notion of absolute time cannot be true - the two concepts are contradictory. And because the ultimate test of truth is experiment, Einstein's theory stands. At low velocities relativity approximates to Newtonian mechanics.

Now we will discuss the basics of the K-calculus very briefly.

## 2 The K-calculus

One of the main reasons the K-calculus is so insightful is that the role of the observer here is central and essential. No measurement is made and no observation obtained without explicit reference to who the observer is.

In the K-calculus, an *event* is completely specified by its coordinates in spacetime $\left(x^0, x^1, x^2, x^3\right)$, where $x^0 = ct$, $c$ being the velocity of light in vacuum.

Observers measure the distance to a particular object by sending light signals (or any sort of electromagnetic radiation) which are reflected by the object and return to the observer. A key assumption is that the distance travelled by the light ray going to the object is the same as the distance travelled by the light ray returning from the object to the observer, regardless of the relative motion between the observer and the object. This comes from the previous statement that an event is completely specified by its coordinates (relative to some observer of course). There is no ground based reference frame.

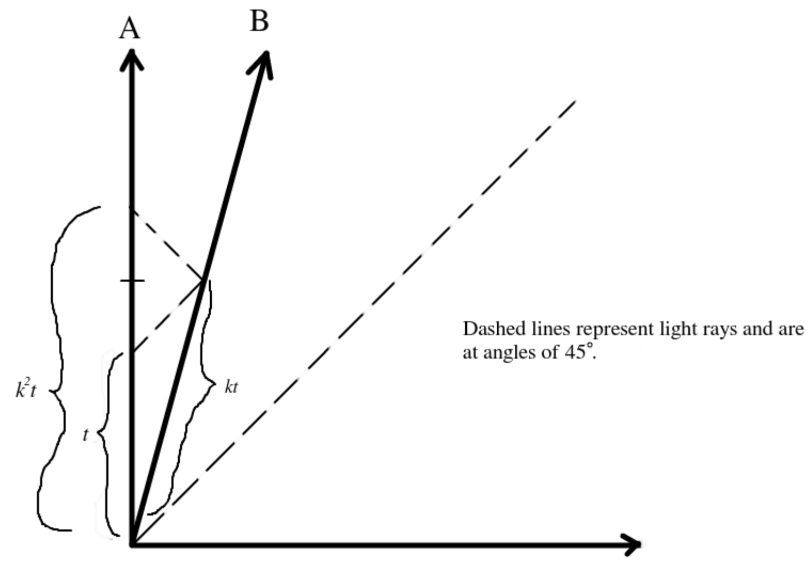
Since the speed of light in vacuum is constant in all inertial frames, all inertial observers measure the same value of $c$. Therefore, the time taken by the ray of light travelling from the observer to the object is the same as the time taken by the reflected ray returning from the object to the observer (the distance is the same, as mentioned earlier, and light travels at $c$ in free space, so the time of journey is the same as well).

We will work in $(1+1)$ dimensions. K-calculus does not translate very effectively into $(3+1)$ dimensions, but this is not a major problem as most physical insights in special relativity can be obtained by working with one time and one space dimension. Indeed, the special Lorentz transformations are also restricted to $(1+1)$ dimensions, i.e. a boost along the $x$ axis.

As usual, let us use units in which the value of $c$ is 1. The speed we will use is the dimensionless speed $\beta \equiv \frac{v}{c}$, where $v$ is the velocity of the object or frame with respect to our observer.

An effective way to grasp the basics of K-calculus is to consider a simple picture. Our observer is A, or equivalently we are in A's rest frame. There is another observer B who is moving at a speed $v$ along the positive $x$ axis.

The spacetime diagram (where the observer is A) would be something like the following:




A uses the unprimed coordinate system $(t, x)$ while B uses $(t', x')$ - remember we are only working in one time and one space dimension. Their origins coincide, i.e at $(t = 0, x = 0)$, $(t' = 0, x' = 0)$. At proper time $t$ after their coincidence, A sends out a light signal that bounces off B and returns to A. Now it is reasonable to assume that the time B receives the light signal (as measured after the coincidence by B's watch) is proportional to $t$ and linear in $t$. The linearity is an important assumption, but a valid one as we know all inertial observers are equivalent, so there should be a one-to-one correspondence between the coordinates of A and B, and that is only possible if the relationship between their coordinates is linear.

We denote the time B receives the signal (as measured by B's watch) as $kt$. It bounces off and returns to A at $k^2t$ (as their situation is symmetric - relative to B, A is moving away at a speed $-v$.

Now, let us see what A sees. According to A, the light signal left him at $t$ and returned to him at $k^2t$. It travelled to and from B, so the same distance both ways. It is simple to conclude the bounce occurred exactly midway between those two times, i.e. at $\frac{(k^2+1)t}{2}$. And the light travelled for a total time of $(k^2 - 1)t$, for a two-way trip, so the time taken for one leg of the journey is $\frac{(k^2-1)tc}{2} = \frac{(k^2-1)t}{2}$ as $c = 1$.

So, by A's watch, B has travelled a distance of $\frac{(k^2-1)t}{2}$ in $\frac{(k^2+1)t}{2}$, putting B's speed at $\frac{(k^2-1)}{(k^2+1)} = \beta$. The value of $k$ is therefore $\sqrt{\frac{1+\beta}{1-\beta}}$. We realise this is just the inverse of the Doppler factor between two observers receding from each other at $\beta$ speed.

### 2.1 Some basic properties of the K-factor

Because the mapping from $v \to k$ is a bijection from $(-c, c)$ to $(0, \infty)$, the speed of an object can be parametrised with $k$ just as well as with $v$. In fact, all that is needed for two observers to determine their mutual K-factor is to measure the frequency of the same source of light.

The first advantage of the K-factor is that it is easier to use than the corresponding velocity, especially when we calculate the relative velocities of two objects with respect to a third object. The relativistic velocity addition law is rather complicated. Let's say $v_{AB}$ is the velocity of A with respect to B and so on. In the simple Galilean velocity transformation, $v_{AC} = v_{AB} + v_{BC}$. On the other hand, the relativistic rule is $v_{AC} = \frac{v_{AB}+v_{BC}}{1+v_{AB}v_{BC}}$. However, the corresponding rule for K-factors cannot possibly be any simpler: $k_{AC} = k_{AB}k_{BC}$.

The special Lorentz transformations can be derived using K-calculus as well. In terms of $k$, $t' = \frac{1}{2}\{(k + \frac{1}{k})t - (k - \frac{1}{k})x\}$ and $x' = \frac{1}{2}\{-(k - \frac{1}{k})t + (k + \frac{1}{k})x\}$. $\gamma = \frac{1}{2}(k + \frac{1}{k})$ and $\beta\gamma = \frac{1}{2}(k - \frac{1}{k})$, where $\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$ is the Lorentz factor or time dilation factor.

All these are relatively basic changes, more cosmetic than anything else. The real physical insights, which we have spoken about come when momentum and energy are addressed in terms of the K-factor.

### 2.2 Momentum and energy in terms of the K-factor

Special relativity tells us the momentum is a 4-vector whose components are ($\gamma mc$,$\gamma mv_x$,$\gamma mv_y$, $\gamma mv_z$). In one dimension, the momentum of a body as seen by a particular observer is just $\gamma mv$, where $v$ is its velocity relative to said observer and $m$ is the invariant mass. In terms of the K-factor, the momentum of a body is just $(\frac{k^2-1}{2k})mc$.

The kinetic energy of a body is $(\frac{k^2+1}{2k} - 1)mc^2$, and the total energy is just the rest energy of $mc^2$ added to the kinetic energy to give a total of $(\frac{k^2+1}{2k})mc^2$.

The values of these quantities are not important. What is important is that these familiar expressions can actually be derived using the K-calculus from principles as basic as the conservation of momentum and the conservation of energy. There is no need to define an energy-momentum 4-vector.

In the next section we will look at the other part of the project, accelerating observers using special relativity.

## 3 Accelerating observers in special relativity

Contrary to popular belief, there is nothing in special relativity that forbids the treatment of accelerated observers. Special relativity assumes that spacetime is Minkowskian, i.e. it is flat, and the special theory does not have the mathematical machinery to deal with curved spacetime. But when we have assumed spacetime is flat, there is nothing to prevent us from using special relativity.

However, like many other things, acceleration, and especially uniform acceleration have rigorous definitions in special relativity, and it is these we studied in this part of the project.

## 3.1 Defining acceleration in special relativity

In special relativity all physical observables are defined with explicit reference to the observer. It is the same with acceleration.

An accelerated body's velocity changes with time. This requires us to define a co-moving frame. A co-moving frame is defined as an inertial frame that is, at that particular instant, at rest with respect to the accelerating body. An accelerating body, it can be said, hops from one inertial frame to another. We can always find an inertial frame with respect to which the accelerating body is momentarily at rest. This frame is the co-moving frame and the acceleration of the accelerating body with respect to its co-moving frame is defined as its proper acceleration.

(In special relativity, all 'proper' quantities are those quantities that are measured in the object's rest frame. It is the same with proper acceleration.)

A body is said to be uniformly accelerating when its proper acceleration remains the same, that is, when the value of its acceleration as measured in each co-moving frame at each instant remains the same.

## 3.2 The invariant quantity in uniform acceleration

In uniform acceleration, the accelerating body's proper acceleration is invariant. However, this is the acceleration as measured by each co-moving observer. For any arbitrary inertial observer, we shall have to express that same quantity in terms of observables that he (the arbitrary inertial observer) can measure.

Let us take two inertial frames S and S'. S is our rest frame. S' moves at velocity $v$ with respect to us.

There is an accelerating body, let's call it a rocket. Its acceleration is not necessarily uniform. We wish to find its acceleration with respect to S' in terms of quantities directly measureable by S.

Let's say the instantaneous velocity of the rocket with respect to S is $u$, and with respect to S' is $u'$. All we are interested in is what happens at that particular instant.

We know, $u' = \dfrac{u-v}{1-uv/c^2}$. Talking differentials on both sides,

$\mathrm{d}u' = \dfrac{\mathrm{d}u}{1-uv/c^2} + \dfrac{(u-v)v\,\mathrm{d}u}{c^2(1-uv/c^2)^2}$, i.e. $\dfrac{1}{D^2}[D\,\mathrm{d}u + \frac{v}{c^2}(u-v)\,\mathrm{d}u]$ where $D \equiv 1 - uv/c^2$.

Also, $t' = \gamma(t - vx/c^2)$ so $\mathrm{d}t' = \gamma\,\mathrm{d}t(1 - uv/c^2) = \gamma D\,\mathrm{d}t$. Here $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$, or the Lorentz factor between S and S'.

So, dividing $\mathrm{d}u'$ by $\mathrm{d}t'$, we obtain (after some simplification),

$a' = \dfrac{a}{\gamma^3 D^3}$, where $a$ is the acceleration of the rocket as seen by S and $a'$ is the corresponding quantity as seen by S'. Note that $a$, $\gamma$ and $D$ are all quantities measureable by S.

Now, we consider the specific case where S' is a co-moving frame of the rocket. This means $u = v$. So, the quantity $\gamma^3 D^3$ reduces to $(1 - \frac{v^2}{c^2})^{3/2}$. So, we have

$$a' = \frac{a}{(1 - v^2/c^2)^{3/2}}.$$

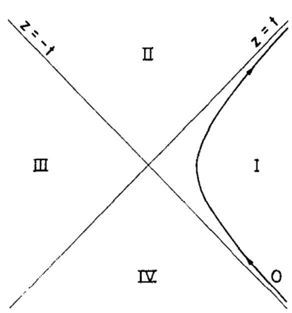
We have thus expressed the proper acceleration of a rocket in terms of quantities measureable by any arbitrary observer, namely $a$, the instantaneous acceleration, and $v$, the instantaneous velocity of the rocket, both measured by S.

If the rocket is uniformly accelerating, the proper acceleration is constant. So the quantity $\dfrac{a}{(1 - v^2/c^2)^{3/2}}$ is an invariant quantity in the case of uniform acceleration.

## 3.3 Some properties of uniformly accelerating bodies

So, the quantity $\alpha = \dfrac{a}{(1 - v^2/c^2)^{3/2}}$ is an invariant quantity or constant, the proper acceleration. We may write $a = \dfrac{\mathrm{d}v}{\mathrm{d}t}$ (since we are dealing with proper quantities, t and $\tau$ are equivalent). Now $\alpha = \gamma^3 \dfrac{\mathrm{d}v}{\mathrm{d}t}$ can be twice integrated, and upon some simplification, we get the equation $x^2 - c^2t^2 = \dfrac{c^4}{\alpha^2}$. This means the worldlines of uniformly accelerating bodies appear as hyperbolas in the spacetime diagram.

Something very interesting happens to uniformly accelerating observers, which can be better explained using the following diagram.



In the above diagram, the straight lines intersect at the origin and divide spacetime into four distinct regions or zones. The accelerating object, let's call it a rocket as usual, is denoted by O and its worldline is a hyperbola. The asymptotes of the hyperbola are precisely the straight lines. We may say the lines are the worldlines of two rays of light, one travelling in the positive $z$ direction (denoted by $z = t$) and the other in the negative $z$ direction (denoted by $z = -t$). Here we denote the one space dimension by $z$ rather than $x$.

We notice the rocket's worldline is restricted to region I. The rocket can both send signals to and receive signals from any point in region I.

The rocket can send signals to, but cannot receive from any point in region II. This means observers in region II can see the rocket but not vice versa.

The rocket can receive signals from, but cannot send to region IV. This means the rocket can see observers in region IV but they cannot see the rocket.

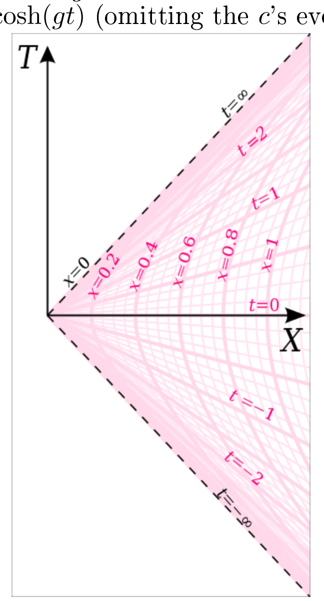
Finally, the rocket can neither send to nor receive from region III. The rocket cannot see, and is invisible to, all observers residing in region III.

## 3.4 Rindler coordinates

Since the worldlines for uniformly accelerating observers are hyperbolas, using hyperbolic coordinates makes calculations easier.

Our rocket is accelerating with a proper acceleration of $g$, a constant. Let us say $(T, X)$ are our standard inertial coordinates. We apply the following coordinate transformation:

$t = \frac{1}{g}\tanh^{-1}(\frac{T}{X})$ and $x = \sqrt{X^2 - T^2}$. Now, $T = x\sinh(gt)$ and $X = x\cosh(gt)$ (omitting the $c$'s everywhere).




The above diagram indicates the transformation between standard inertial coordinates and Rindler coordinates. The hyperbolas are lines of constant $x$, and the $x$ coordinate is actually a measure of the hyperbola's closest approach to the origin (in fact, $x$ is the distance of the hyperbola from the origin measured along the line $T = 0$). The horizon, denoted by $x = 0$, can be thought of as the worldline of a photon initially travelling along the $-X$ axis, then bouncing off the observer at $(T = 0, X = 0)$ and thereafter travelling along the $+X$ axis.

The straight lines starting from the origin are lines of constant $t$, and the $t$ coordinate is actually a measure of the line's slope, with $t = 0$ coinciding with the line $T = 0$ and the line $t = \infty$ coincides with the line $T = X$ (or to be more precise the part of the line in the quadrant where both $T$ and $X$ are positive).

### 3.5 The next step

The next step of the project will be to apply the K-calculus to accelerated observers and see if any new physical insights arise, and if so, what. To do this we will use a variation of the approach used by inertial observers.

## 4 Conclusion

When I began this project I had no idea special relativity held anything more than the special Lorentz transformations and the 4-vector formulations, and problems based on these.

Special relativity is a very interesting subject, and one that holds enormous surprises - even today, a hundred and ten years after it was first proposed and a hundred years after a more complete theory of relativity, one that had the mathematical machinery to handle curved spacetime was introduced by the same genius.

I realised while working on this project that there is much to be learned. Luckily, the process of learning is extremely enjoyable, and in most cases learning is its own reward.

Physics is difficult, and not just because of the long calculations. Some concepts in relativity are not very easy to understand, and require dedicated thought. However, once understood, these concepts can be used to take our understanding one step ahead, showing us new truths.

It seems apt to end with a quote by Einstein:

> *Subtle is the Lord, but malicious he is not.*

In my humble opinion, the same can be said about physics. Perhaps that is what Einstein meant, though will will never know. Perhaps the two are not so different.

## 5 Acknowledgements

I am very grateful to all my teachers, especially those who have taught me physics. However, I owe an enormous debt of gratitude to Dr. Ananda Das-

gupta, not only for being my project mentor, but also for being a wonderful teacher and an inspiration beyond the classroom.

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