Computer Assignment 1

Date Assigned:
September 10, 2011

Date Due:
September 21, 2011

Gauss' theorem is important for understanding the equations of fluid dynamics. In unstructured grid CFD, it is often necessary to compute gradients of the flow variables (density, velocity, energy, etc.). Gauss' theorem often provides the basis for these computations.

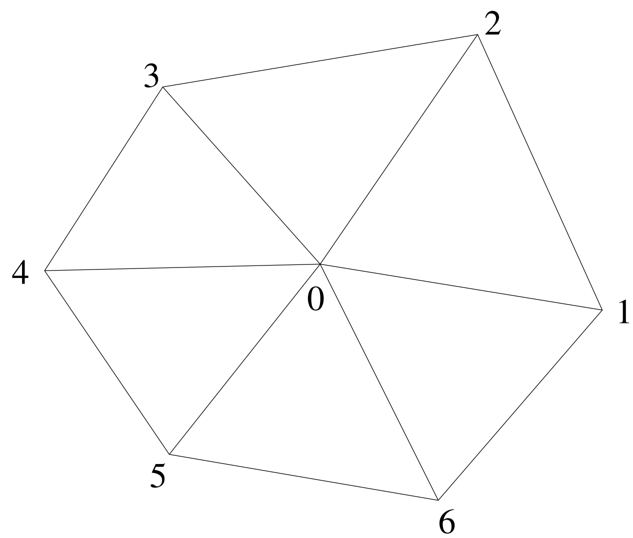

Figure 1: Unstructured stencil needed for the gradient at node 0.

Consider node 0 surrounded by nodes i in an unstructured grid, shown in Figure 1. We wish to compute the gradient of a function, $f(x, y)$, at node 0. While f can be arbitrary, we assume f is linear for the time being. Part of this assignment is to quantify the error by making this linear assumption. For linear f, the gradient is constant, and Gauss' Theorem becomes

$$\int_V \boldsymbol{\nabla} f dV = \int_A f \boldsymbol{n} dA \longrightarrow \boldsymbol{\nabla} f = \frac{1}{V} \int_A f \boldsymbol{n} dA. \tag{1}$$

Thus, we can use Gauss' Theorem directly to compute the gradient. All we need to know is the volume of the "super cell" surrounding node 0, and how to evaluate the area integral around the perimeter of the super cell. The area integral may be evaluated exactly for a linear function by using the trapezoidal rule on each edge surrounding node 0 (which has constant $\boldsymbol{n}$):

$$\boldsymbol{\nabla} f_0 = \frac{1}{V_0} \sum_i \frac{1}{2}(f_i + f_{i+1}) \boldsymbol{n}_{i,i+1} A_{i,i+1}. \tag{2}$$

Here, $\boldsymbol{n}_{i,i+1}$ is the unit normal of the edge connecting nodes i and $i+1$, and $A_{i,i+1}$ is the area (length in 2D) of the same edge. Also, $f_i = f(x_i, y_i)$. It is easy to verify that the normal times the area has components:

$$\boldsymbol{n}_{i,i+1} A_{i,i+1} = \begin{Bmatrix} y_{i+1} - y_i \\ -(x_{i+1} - x_i) \end{Bmatrix}. \tag{3}$$

Therefore the gradient formulas become

$$\boldsymbol{\nabla} f_0 = \begin{Bmatrix} \frac{1}{V_0} \sum_i \frac{1}{2}(f_i + f_{i+1})(y_{i+1} - y_i) \\ -\frac{1}{V_0} \sum_i \frac{1}{2}(f_i + f_{i+1})(x_{i+1} - x_i) \end{Bmatrix} = \begin{Bmatrix} \frac{1}{2V_0} \sum_i f_i (y_{i+1} - y_{i-1}) \\ -\frac{1}{2V_0} \sum_i f_i (x_{i+1} - x_{i-1}) \end{Bmatrix}, \tag{4}$$

where the second form may be slightly easier to work with. The volume of the super cell surrounding node 0 is easily computed exactly by setting $f = x$ in Equation 4.

For this assignment, verify that the above node-centered gradient formulas in Equation 4 for functions expressed on 2D triangular grids we derived in class are exact for arbitrary linear functions, and first-order accurate for any function. Do this in two ways:

1. First, verify this theoretically by computing the truncation error of the discrete gradient formulas.
2. Next, verify this computationally by computing the error between the computed gradient (using the formula derived in class) and the exact gradient of a known function for a series of increasingly refined triangles.

Part two is known as a grid refinement study. The truncation error may be computed as

$$\boldsymbol{E}_t = |\boldsymbol{\nabla} f^h - \boldsymbol{\nabla} f^e| = \boldsymbol{K} h^p + \text{H.O.T.S} \tag{5}$$

Where $\boldsymbol{K}$ is a constant, p is the order of accuracy, and H.O.T.S stands for "higher-order terms." Taking the log of both sides of this equation yields (neglecting H.O.T.S)

$$\log \boldsymbol{E}_t = \log \boldsymbol{K} + p \log h, \tag{6}$$

which is a straight line on a log-log plot, with p the slope of the line.

Turn in a short write-up describing your procedure and results, written using AIAA format. (See `https://www.aiaa.org/Secondary.aspx?id=4597` for the AIAA format requirements.) It is highly encouraged to use the AIAA Latex template provided on the AIAA website.