Nonlinear PSE derivation with base flow update

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Contents

1 Incompressible Nonlinear PSE derivation

The incompressible Navier-Stokes equations can be written in Einstein notation as Eq. 1 where u_i is the velocity components in all three cartesian coordinate directions and p is the pressure.

$$\begin{aligned} \frac{\partial u_j}{\partial x_j} &= 0 \\ \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} + \frac{1}{\rho}\frac{\partial p}{\partial x_i} - \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} &= 0 \end{aligned} \tag{1}$$

Typical derivations of the Parabalized Stability Equations (PSE) begin by subtracting the base flow ($\bar{u}$) Navier-Stokes equations from the decomposed ($u = \bar{u} + u'$) Navier-Stokes equation resulting in the governing equations for the fluctuating quantities (u'). This is shown in several classical PSE derivations [Herbert, 1997] [Schmid and Henningson, 2001]. Eq. 2 shows the typical fluctuating subtraction.

$$\begin{array}{llllll} & \frac{\partial(\bar{u}+u')_i}{\partial t} & +(\bar{u}+u')_j \frac{\partial(\bar{u}+u')_i}{\partial x_j} & +\frac{1}{\rho}\frac{\partial(\bar{P}+P')}{\partial x_i} & -\nu\frac{\partial^2(\bar{u}+u')_i}{\partial x_j \partial x_j} & \\ - & \left(\frac{\partial(\bar{u})_i}{\partial t}\right. & +(\bar{u})_j \frac{\partial(\bar{u})_i}{\partial x_j} & +\frac{1}{\rho}\frac{\partial(\bar{P})}{\partial x_i} & \left.-\nu\frac{\partial^2(\bar{u})_i}{\partial x_j \partial x_j}\right) & = 0 \end{array} \tag{2}$$

The underlying assumption here is that the base flow will satisfy the Navier-Stokes equation even when nonlinear fluctuation terms begin to get large. To account for this, the mean flow distortion (MFD) (average of fluctuating quantities in the homogeneous directions) will be marched as any other mode, but with slightly different boundary conditions as seen in [Lozano-Durán et al., 2018]. This MFD is essentially thought of as a base flow change. This approach has been shown to not robustly solve once nonlinear terms become large. Additionally, as will be seen later, the approached used in this work uses finite difference derivatives in the spanwise direction. Because of this, this work cannot use separate boundary conditions for the MFD as opposed to other spanwise modes (as is classically done with spectral derivatives in the spanwise direction). The solution that is proposed in this work is derive PSE by subtracting the averaged decomposed Navier-Stokes equations from the decomposed ($\bar{u} + u'$) Navier-Stokes equations. This will yield additional nonlinear terms that will be shown below.

We can then introduce the decomposed solution where $u = \bar{u} + u'$ and $p = \bar{P} + P'$ where $\bar{u}$ and $\bar{P}$ is the guess for the base flow velocity and pressure and u' and P' are the fluctuating quantities. The resulting decomposed Navier-Stokes equation is seen in Eq. 3.

$$\frac{\partial(\bar{u}+u')_i}{\partial t} + (\bar{u}+u')_j \frac{\partial(\bar{u}+u')_i}{\partial x_j} + \frac{1}{\rho}\frac{\partial(\bar{P}+P')}{\partial x_i} - \nu \frac{\partial^2(\bar{u}+u')_i}{\partial x_j \partial x_j} = 0 \tag{3}$$

We can also average this decomposed equation in the homogeneous directions. Subtracting this averaged decomposed equation from Eq. 3 is depicted in 4.

$$\begin{array}{lllll} & \frac{\partial(\bar{u}+u')_i}{\partial t} & +(\bar{u}+u')_j \frac{\partial(\bar{u}+u')_i}{\partial x_j} & +\frac{1}{\rho}\frac{\partial(\bar{P}+P')}{\partial x_i} & -\nu \frac{\partial^2(\bar{u}+u')_i}{\partial x_j \partial x_j} \\ - & \left\langle \frac{\partial(\bar{u}+u')_i}{\partial t} \right. & +(\bar{u}+u')_j \frac{\partial(\bar{u}+u')_i}{\partial x_j} & +\frac{1}{\rho}\frac{\partial(\bar{P}+P')}{\partial x_i} & \left. -\nu \frac{\partial^2(\bar{u}+u')_i}{\partial x_j \partial x_j} \right\rangle = 0 \end{array} \tag{4}$$

Computing this subtraction, we obtain the governing equations for the fluctuating quantities in Eq. 5. This equation also shows the main difference in our derivation here, and what is classically done.

$$\underbrace{\frac{\partial u_i'}{\partial t} + \bar{u}_j \frac{\partial u_i'}{\partial x_j} + u_j' \frac{\partial \bar{u}_i}{\partial x_j} + \frac{1}{\rho}\frac{\partial P'}{\partial x_i} - \nu \frac{\partial^2 u_i'}{\partial x_j \partial x_j} = -u_j' \frac{\partial u_i'}{\partial x_j}}_{\text{Classical PSE terms}} + \underbrace{\left\langle u_j' \frac{\partial u_i'}{\partial x_j} \right\rangle}_{\text{Extra terms}} \tag{5}$$

Introducing the ansatz that our fluctuating quantities (velocity and pressure) will take the form of Eq. 6

$$\mathbf{q}(x,y,z,t) = \sum_{m=-M}^{M} \hat{\mathbf{q}}_m(x,y,z) e^{\left(\mathrm{i}\int_{x_0}^{x} \alpha_m(\xi) d\xi - \mathrm{i}m\omega t\right)} \tag{6}$$

where the fluctuating quantities state vector $\mathbf{q} = [u', v', w', P']^T$ and its related slowly varying $\hat{\mathbf{q}} = [\hat{u}, \hat{v}, \hat{w}, \hat{P}]^T$. Introducing this ansatz into Eq. 5 and retaining terms of $\mathcal{O}\left(\frac{1}{Re}\right)$ we can obtain the PSE equations with the nonlinear terms being integrated into the linear operator (left hand side) and the averaged nonlinear terms being treated as a forcing term (right hand side). These equations are shown in matrix form in Eq. 7 with the linear operators defined in Eq. 11 and the nonlinear operators defined in Eq. 12 and the averaged nonlinear operators defined in Eq. 14.

$$(\mathbf{A}_{\mathrm{op}})\hat{\mathbf{q}} = \langle \mathbf{F} \rangle \tag{7}$$

Where Eq. 8 defines the linear operator $\mathbf{A}_{\mathrm{op}}$. $\mathcal{D}_y$ indicates the first derivative in the y direction, $\mathcal{D}_{zz}$ indicates the second derivative with respect to the z axis and so forth. These derivative operators were defined using fourth order finite differences except for the $\mathcal{D}_x$ operator which uses Backward Euler method to march downstream in the slowly varying flow. The subscript F on the operators indicates the non-averaged nonlinear terms that have been linearized by lagging a state vector in a nonlinear iterative loop as is described in [DeBlois, 1997]. It should be noted that the averaged nonlinear terms could also have this lagging, however it may disrupt much of the sparse nature of finite difference operator form, so this was not attempted in this work.

$$(\mathbf{A}_{\text{op}}) = \mathbf{A}+\mathbf{A}_F+(\mathbf{B}+\mathbf{B}_F)\mathcal{D}_y+(\mathbf{C}+\mathbf{C}_F)\mathcal{D}_{yy}+(\mathbf{D}+\mathbf{D}_F)\mathcal{D}_x+(\mathbf{E}+\mathbf{E}_F)\mathcal{D}_z+(\mathbf{G}+\mathbf{G}_F)\mathcal{D}_{zz} \tag{8}$$

Introducing the Backward Euler for the $\mathcal{D}_x$ will give us the PSE linear system as shown in Eq. 9 with operator and right hand side defined in Eqs. 10, 11, 12, and 13. The subscripts i and $i+1$ denotes the spatial marching step of step size h_x where i is the previous known spatial step, and $i+1$ is the unknown next marching step.

$$(\mathbf{A}_{\text{solve}})_{i+1}\hat{\mathbf{q}}_{i+1} = \mathbf{b}_i \tag{9}$$

$$\begin{gathered}
\mathbf{b}_i = \langle \mathbf{F}_{i+1} \rangle + \frac{\mathbf{D}_{i+1}+(\mathbf{D}_F)_{i+1}}{h_x}\hat{\mathbf{q}}_i \\
(\mathbf{A}_{\text{solve}}) = \mathbf{A}+\mathbf{A}_F+(\mathbf{B}+\mathbf{B}_F)\mathcal{D}_y+(\mathbf{C}+\mathbf{C}_F)\mathcal{D}_{yy}+\frac{\mathbf{D}+\mathbf{D}_F}{h_x}+(\mathbf{E}+\mathbf{E}_F)\mathcal{D}_z+(\mathbf{G}+\mathbf{G}_F)\mathcal{D}_{zz}
\end{gathered} \tag{10}$$

$$\begin{aligned}
\mathbf{A} &= \begin{bmatrix} r+\bar{u}_x & \bar{u}_y & 0 & i\alpha \\ 0 & r+\bar{v}_y & 0 & 0 \\ 0 & 0 & r & 0 \\ i\alpha & 0 & 0 & 0 \end{bmatrix} &
\mathbf{B} &= \begin{bmatrix} \bar{v} & 0 & 0 & 0 \\ 0 & \bar{v} & 0 & 1 \\ 0 & 0 & \bar{v} & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \\
\mathbf{C} &= \begin{bmatrix} \frac{-1}{Re} & 0 & 0 & 0 \\ 0 & \frac{-1}{Re} & 0 & 0 \\ 0 & 0 & \frac{-1}{Re} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} &
\mathbf{D} &= \begin{bmatrix} \bar{u} & 0 & 0 & 1 \\ 0 & \bar{u} & 0 & 0 \\ 0 & 0 & \bar{u} & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \\
\mathbf{E} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} &
\mathbf{G} &= \begin{bmatrix} \frac{-1}{Re} & 0 & 0 & 0 \\ 0 & \frac{-1}{Re} & 0 & 0 \\ 0 & 0 & \frac{-1}{Re} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
r &= -im\omega + i\alpha\bar{u} + \frac{\alpha^2}{Re}
\end{aligned} \tag{11}$$

$$\begin{aligned}
\mathbf{A}_F &= \begin{bmatrix} a_f & 0 & 0 & 0 \\ 0 & a_f & 0 & 0 \\ 0 & 0 & a_f & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} &
\mathbf{B}_F &= \begin{bmatrix} b_f & 0 & 0 & 0 \\ 0 & b_f & 0 & 0 \\ 0 & 0 & b_f & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
\mathbf{C}_F &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} &
\mathbf{D}_F &= \begin{bmatrix} d_f & 0 & 0 & 0 \\ 0 & d_f & 0 & 0 \\ 0 & 0 & d_f & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
\mathbf{E}_F &= \begin{bmatrix} e_f & 0 & 0 & 0 \\ 0 & e_f & 0 & 0 \\ 0 & 0 & e_f & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} &
\mathbf{G}_F &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned} \tag{12}$$

Where Eq. 13 defines the individual components for the nonlinear operators. The superscript k denotes the previous nonlinear iteration's guess for the current quantities. These quantities are updated as will be shown in section 4

$$\begin{aligned} a_f &= \sum_{m=-M}^{M} \hat{u}_m^k \mathrm{i}\alpha e^{\mathrm{i}\Theta_m} & b_f &= \sum_{m=-M}^{M} \hat{v}_m^k e^{\mathrm{i}\Theta_m} \\ e_f &= \sum_{m=-M}^{M} \hat{w}_m^k e^{\mathrm{i}\Theta_m} & d_f &= \sum_{m=-M}^{M} \hat{u}_m^k e^{\mathrm{i}\Theta_m} \\ \Theta_m &= \int_{x_0}^{x} \alpha_m(\xi) d\xi - \mathrm{i}m\omega t \end{aligned} \tag{13}$$

And the averaged forcing nonlinear terms are defined in Eq. 14. This term is the new additional term to nonlinear PSE when compared to the classical PSE derivation.

$$\langle \mathbf{F} \rangle = \begin{bmatrix} \langle \sum_{m=-M}^{M} (\hat{u}_m^k(\hat{u}_x^k + \hat{u}^k \mathrm{i}\alpha) + \hat{v}_m^k \hat{u}_y^k + \hat{w}_m^k \hat{u}_z^k) e^{\mathrm{i}\Theta_m} \rangle \\ \langle \sum_{m=-M}^{M} (\hat{u}_m^k(\hat{v}_x^k + \hat{v}^k \mathrm{i}\alpha) + \hat{v}_m^k \hat{v}_y^k + \hat{w}_m^k \hat{v}_z^k) e^{\mathrm{i}\Theta_m} \rangle \\ \langle \sum_{m=-M}^{M} (\hat{u}_m^k(\hat{w}_x^k + \hat{w}^k \mathrm{i}\alpha) + \hat{v}_m^k \hat{w}_y^k + \hat{w}_m^k \hat{w}_z^k) e^{\mathrm{i}\Theta_m} \rangle \\ 0 \end{bmatrix} \tag{14}$$

2 Incompressible Base flow with forcing update

Starting from the averaged decomposed Navier-Stokes equation shown in Eq. 15

$$\left\langle \frac{\partial(\bar{u}+u')_i}{\partial t} + (\bar{u}+u')_j \frac{\partial(\bar{u}+u')_i}{\partial x_j} + \frac{1}{\rho}\frac{\partial(\bar{P}+P')}{\partial x_i} - \nu \frac{\partial^2(\bar{u}+u')_i}{\partial x_j \partial x_j} \right\rangle = 0 \tag{15}$$

This equation can be simplified and we obtain Eq. 16. Where the classical terms retained in Boundary Layer (BL) equations and extra terms have been identified.

$$\underbrace{\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j}}_{\text{Classic terms}} + \underbrace{\left\langle u'_j \frac{\partial u'_i}{\partial x_j} \right\rangle}_{\text{Extra terms}} + \underbrace{\frac{1}{\rho}\frac{\partial P'}{\partial x_i} - \nu \frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j} = 0}_{\text{Classic terms}} \tag{16}$$

We can follow many of the same assumptions to derive a new BL equation with some forcing. We will assume here a steady, two dimensional, and incompressible flow. The Navier-Stokes equations are then reduced to the x and y momentum equations shown in Eq. 17

$$\begin{aligned} \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} + \left\langle u'_j \frac{\partial u'}{\partial x_j} \right\rangle + \frac{1}{\rho}\frac{\partial \bar{P}}{\partial x} - \nu\left(\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2}\right) = 0 \\ \bar{u}\frac{\partial \bar{v}}{\partial x} + \bar{v}\frac{\partial \bar{v}}{\partial y} + \left\langle u'_j \frac{\partial v'}{\partial x_j} \right\rangle + \frac{1}{\rho}\frac{\partial \bar{P}}{\partial y} - \nu\left(\frac{\partial^2 \bar{v}}{\partial x^2} + \frac{\partial^2 \bar{v}}{\partial y^2}\right) = 0 \end{aligned} \tag{17}$$

We can then introduce the nondimensional scaling as shown in Eq. 18 [White, 1974].

$$
\begin{aligned}
x^* &= \frac{x}{L} & y^* &= \frac{y}{L}\sqrt{Re} \\
\bar{u}^* &= \frac{\bar{u}}{U_\infty} & \bar{v}^* &= \frac{\bar{v}}{U_\infty}\sqrt{Re} \\
\bar{P}^* &= \frac{\bar{P} - P_0}{\rho U_\infty^2}
\end{aligned} \tag{18}
$$

Where the reference values U_∞ is the constant freestream velocity, L is the length scale of the problem, and P_0 is the reference pressure and the * indicates nondimensional quantities. We can take the terms in the limit as $Re \to \infty$ and we obtain the modified BL equations shown in Eq. 19 where the classic BL equation terms and the extra terms are identified.

$$
\begin{aligned}
\underbrace{\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y}}_{\text{Classic BL terms}} &= 0 \\
\underbrace{\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} + \frac{1}{\rho}\frac{\partial \bar{P}}{\partial x} - \nu\left(\frac{\partial^2 \bar{u}}{\partial y^2}\right)}_{\text{Classic BL terms}} &= \underbrace{-\left\langle u_j' \frac{\partial u'}{\partial x_j}\right\rangle}_{\text{Extra terms}} \\
\underbrace{\left\langle u_j' \frac{\partial v'}{\partial x_j}\right\rangle}_{\text{Extra terms}} + \underbrace{\frac{1}{\rho}\frac{\partial \bar{P}}{\partial y}}_{\text{Classic BL terms}} &= 0
\end{aligned} \tag{19}
$$

Splitting these equations into operator form using finite differences for the y and z axis and Backward Euler Method we obtain Eq. 20 with the operators defined in 21

$$
[\mathbf{A} + (\mathbf{B} + \mathbf{B}_F)\mathcal{D}_y + \mathbf{C}\mathcal{D}_{yy} + (\mathbf{D} + \mathbf{D}_F)\mathcal{D}_x]\mathbf{Q} = \langle \mathbf{F} \rangle \tag{20}
$$

Where $\mathbf{Q}$ is the base flow state vector $\mathbf{Q}(x, y, z) = [\bar{u}, \bar{v}, \bar{P}]^T$.

$$
\begin{aligned}
\mathbf{A} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad
\mathbf{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad
\mathbf{B}_F = \begin{bmatrix} \bar{v}^k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
\mathbf{C} &= \begin{bmatrix} -\nu & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad
\mathbf{D} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad
\mathbf{D}_F = \begin{bmatrix} \bar{u}^k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
\langle \mathbf{F} \rangle &= \begin{bmatrix} -\langle u' \frac{\partial u'}{\partial x} + v' \frac{\partial u'}{\partial y} + w' \frac{\partial u'}{\partial z} \rangle \\ -\langle u' \frac{\partial v'}{\partial x} + v' \frac{\partial v'}{\partial y} + w' \frac{\partial v'}{\partial z} \rangle \\ 0 \end{bmatrix}
\end{aligned} \tag{21}
$$

Introducing the Backward Euler method and integrating, we achieve a parabolic equation that can be solved for each step as seen in Eq. 22 with the operators defined in Eq. 21.

$$
\begin{gathered}
(\mathbf{A}_{\text{solve}})_{i+1}\mathbf{Q}_{i+1} = \mathbf{b}_i \\
\text{or} \\
\left[\mathbf{A} + (\mathbf{B} + \mathbf{B}_F)\mathcal{D}_y + \mathbf{C}\mathcal{D}_{yy} + \frac{\mathbf{D} + \mathbf{D}_F}{h_x}\right]_{i+1} \mathbf{Q}_{i+1} = \frac{(\mathbf{D} + \mathbf{D}_F)_{i+1}}{h_x}\mathbf{Q}_i + \langle \mathbf{F}_{i+1} \rangle
\end{gathered} \tag{22}
$$

This base flow is updated at each nonlinear step as depicted in Fig. 1. It is noted that the boundary conditions will contain the no slip condition at the wall, the free stream velocity U_∞ at the freestream and an initial known inlet condition (perhaps the Blasius Solution).

3 Ambiguity for PSE and constraints

It should be mentioned that there arises an ambiguity concerning the streamwise variation of both $\hat{\mathbf{q}}$ and α. This variation is resolved via the constraint and update in α shown in Eq. 23 where n denotes an iterative procedure to resolve the constraint [Herbert T., 1994]. This update is then used to update the linear PSE equations shown in Eq. 9

$$\begin{aligned} &\int_\Omega \hat{\mathbf{q}}^{\mathsf{H}} \hat{\mathbf{q}}_x d\Omega = 0 \\ &\alpha_{i+1}^{n+1} = \alpha_{i+1}^{n} - \frac{\mathrm{i}}{h_x} \frac{\int_\Omega \hat{\mathbf{q}}_{i+1}^{\mathsf{H}} (\hat{\mathbf{q}}_{i+1} - \hat{\mathbf{q}}_j) d\Omega}{\int_\Omega |\hat{\mathbf{q}}_{i+1}|^2 d\Omega} \end{aligned} \tag{23}$$

Where H denotes the conjugate transpose.

4 Solution Procedure

This section outlines the solution procedure for solving the nonlinear PSE equations with the base flow forcing terms in the boundary layer equations. Fig. 1 outlines the procedure for the simulation. Here, superscript n indicates the constraint iteration, superscript k indicates the nonlinear iterations, and subscript i indicates the spatial marching location (marching of constant step size h_x) in the x streamwise direction.

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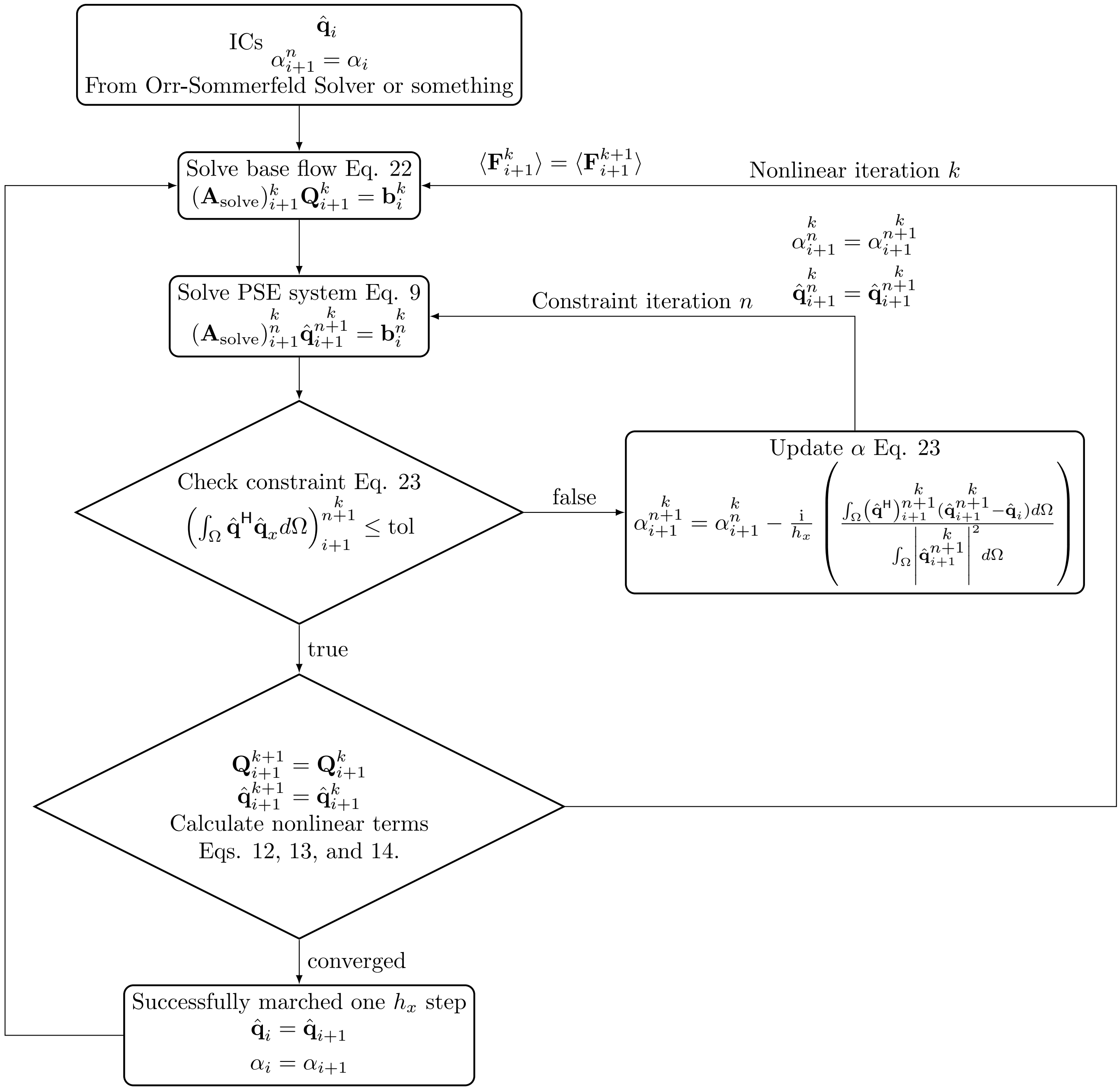

Figure 1: Solution Procedure for nonlinear PSE and modified boundary layer base flow