MATH 446/546 Homework 2 Solution

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For problems 1-3, you must compute the mean and covariance of the process to argue whether it's stationary or not.

Problem 1

First compute it's expectation:

$$\mathbb{E}(X_t) = \cos(\omega t)\mathbb{E}(A) + \sin(\omega t)\mathbb{E}(B) = 0.$$

Then the covariance, for any $h \in \mathbb{Z}$:

$$\begin{aligned}
cov(X_t, X_{t+h}) &= cov(A\cos(\omega t) + B\sin(\omega t), A\cos(\omega t + \omega h) + B\sin(\omega t + \omega h)) \\
&= cov(A\cos(\omega t) + B\cos(\omega t), A\cos(\omega t)\cos(\omega h) \\
&+ B\sin(\omega t)\cos(\omega h) + B\cos(\omega t)\sin(\omega h))) \\
&= cov(A\cos(\omega t), A\cos(\omega t)\cos(\omega h)) - cov(A\cos(\omega t), A\sin(\omega t)\sin(\omega h)) \\
&+ cov(B\sin(\omega t), B\sin(\omega t)\cos(\omega h)) + cov(B\sin(\omega t), B\cos(\omega t)\sin(\omega h)) \\
&= \cos(\omega h)\cos^2(\omega t)Var(A) - \frac{1}{2}\sin(2\omega t)\sin(\omega h)Var(A) \\
&+ \sin^2(\omega t)\cos(\omega h)Var(B) + \frac{1}{2}\sin(2\omega t)\sin(\omega h)Var(B) \\
&= \cos(\omega h).
\end{aligned}$$

Since the mean and covariance are constants independent of time t, indicating it's *weak stationary*.

Problem 2

First re-write the model:

$$X_n = X_{n-1} + \epsilon_n = X_{n-2} + \epsilon_{n-1} + \epsilon_n = \cdots = X_0 + \sum_{i=1}^{n} \epsilon_i = \sum_{i=1}^{n} \epsilon_i.$$

Then compute the expectation:

$$\mathbb{E}(X_n) = 0$$

The covariance:

$$\begin{aligned} cov(X_m, X_n) &= cov(\sum_{i=1}^{m} \epsilon_i, \sum_{j=1}^{n} \epsilon_j) \\ &= Var(\sum_{i=1}^{m \wedge n} \epsilon_i) = (m \wedge n)\sigma^2. \end{aligned}$$

Since the covariance is dependent of time (m and n), it's varying as time changes, thus it's *weak non-stationary*.

Problem 3

First compute the expectation and variance of binary white noise process:

$$\begin{aligned} \mathbb{E}(\epsilon_t) &= p \cdot \frac{1-p}{p} + (1-p) \cdot (-1) = 0. \\ Var(\epsilon_t) &= \mathbb{E}(\epsilon_t^2) = \frac{(1-p)^2}{p^2} \cdot p + (-1)^2 \cdot (1-p) = \frac{1-p}{p}. \end{aligned}$$

Similar as problem 2 above, $Y_t = Y_0 + \sum_{k=1}^{t} \epsilon_k$. Then we compute the expectation and covariance of Y_t:

$$\begin{aligned} \mathbb{E}(Y_t) &= Y_0 \\ cov(Y_t, Y_{t+h}) &= cov(Y_0 + \sum_{p=1}^{t} \epsilon_p, Y_0 + \sum_{q=1}^{t+h} \epsilon_q) \\ &= cov(\sum_{p=1}^{t} \epsilon_p, \sum_{q=1}^{t+h} \epsilon_q) \end{aligned}$$

$$= Var(\sum_{p=1}^{t} \epsilon_p) = t\frac{1-p}{p}.$$

Since the covariance is dependent on time t, it's *weak non-stationary.*

Problem 4&5

Outline of procedure:

1. Compare the realization of your model with the original data: list some of their commonalities and differences,
2. Evaluate your model: which features does the model capture but others don't?
3. Also make assumptions of the model, for example in the random walk (with drift) model, you are assuming that the trend of data does exist, and the difference of the total death data are i.i.d white noise.

See TA or instructor for more detailed feedback.