MATH 446/546 Homework 3 Solution

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Problem 1

Discretize the SHM equation, we know that the second derivative of Y_n can be expressed as:

$$\begin{aligned}\frac{\frac{\Delta Y_n}{\Delta n} - \frac{\Delta Y_{n-1}}{\Delta n}}{\Delta n} &= \Delta Y_n - \Delta Y_{n-1} \\ &= Y_n - Y_{n-1} - (Y_{n-1} - Y_{n-2}) = Y_n - 2Y_{n-1} + Y_{n-2}\end{aligned}$$

since $\Delta n = n - (n-1) = 1$. Then the discrete version of SHM will be

$$Y_n - 2Y_{n-1} + Y_{n-2} = -\omega^2 Y_n.$$

which is equivalent to(adding white noise):

$$Y_n = \frac{2}{1+\omega^2} Y_{n-1} - \frac{1}{1+\omega^2} Y_{n-2} + \epsilon_n$$

To find AR(2) casual is equivalent to find the condition such that the roots for the function of lag operator is outside the unit circle on the complex plane. In other words, the AR(2) model is equivalent to:

$$\phi(L) Y_n = \epsilon_n$$

where $\phi(L) := 1 - \frac{2}{1+\omega^2} L + \frac{1}{1+\omega^2} L^2$, and L is the lag operator. To find the roots for function ϕ, we can solve it as a quadratic equation and we know the roots for equation $ax^2 + bx + c = 0$ are:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

and thus we can compute that the roots for ϕ are:

$$1 \pm |\omega| i$$

that means the coordinates for the roots are $(1, \pm\omega)$ which means for $\forall \omega \in \mathbb{R}$, the root is outside the unit circle.

Problem 2

(1) Find the ARMA model fits the data best by comparing AIC. Conclude that if AR(2) is the best choice indicating if it satisfies SHM equation.

(2) Find the confidence interval of the parameters in AR(2) model by either computing manually with the mean and standard deviation of your MLE estimator, or the embedded function "confint" in r. But notice that you need to finally find the CI for $\hat{\omega}$, not just the parameters.

(3)&(4) Any reasonable explanation is reasonable. You can explain that although the frequency in the model is constant, there might be measurement error and sampling error so that the MLE estimator has a distribution, and these uncertainty may come from the problems with the sensor, mechanical anomalies, spring getting hot or there being friction in air and so on. Also collecting more data will help with reducing such uncertainty.