MATH 446/546 Homework 1 Solution

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Problem 1

(a) We have

$$\begin{aligned}
\gamma_X(t+h,t) &= Cov(X_{t+h}, X_t) = Cov(Z_{t+h} + \theta Z_{t+h-2}, Z_t + \theta Z_{t-2}) \\
&= Cov(Z_{t+h}, Z_t) + \theta Cov(Z_{t+h}, Z_{t-2}) + \theta Cov(Z_{t+h-2}, Z_t) \\
&+ \theta^2 Cov(Z_{t+h-2}, Z_{t-2}) \\
&= \mathbb{1}_{\{0\}}(h) + \theta \mathbb{1}_{\{-2\}}(h) + \theta \mathbb{1}_{\{2\}}(h) + \theta^2 \mathbb{1}_{\{0\}}(h) \\
&= \begin{cases} 1+\theta^2, & \text{if } h = 0, \\ \theta, & \text{if } |h| = 2, \end{cases} \begin{cases} 1.64, & \text{if } h = 0, \\ 0.8, & \text{if } |h| = 2, \end{cases}
\end{aligned}$$

Hence the ACVF depends only on h and we write $\gamma_X(h) = \gamma_X(t+h,h)$. The ACF is then

$$\rho(h) = \frac{\gamma_X(h)}{\gamma_X(0)} = \begin{cases} 1, & \text{if } h = 0 \\ 0.8/1.64 \approx 0.49, & \text{if } |h| = 2 \end{cases}$$

(b) We have

$$\begin{aligned}
&Var(\frac{1}{4}(X_1 + X_2 + X_3 + X_4)) = \frac{1}{16} Var(X_1 + X_2 + X_3 + X_4) \\
&= \frac{1}{16}(Var(X_1) + Var(X_2) + Var(X_3) + Var(X_4) + 2Cov(X_1, X_3) + 2Cov(X_2, X_4)) \\
&= \frac{1}{16}(4\gamma_X(0) + 4\gamma_X(2)) = \frac{1}{4}(\gamma_X(0) + \gamma_X(2)) = 0.61.
\end{aligned}$$

(c) $\theta = -0.8$ implies $\gamma_X(h) = -0.8$ for $|h| = 2$ so

$$Var(\frac{1}{4}(X_1 + X_2 + X_3 + X_4)) = 0.21.$$

Because of the negative covariance at lag 2 the variance in (c) is considerably smaller.

Problem 2

(a) By definition, $\bar{x} = \frac{\sum_{t=1}^{n} x_t}{n} = a + b(n+1)/2$. Thus,

$$\begin{aligned}
\hat{\gamma}(h) &= \frac{1}{n}\sum_{t=1}^{n-h}(a + bh + bt - a - b(n+1)/2)(a + bt - a - b(n+1)/2) \\
&= \frac{b^2}{h}\sum_{t=1}^{n-h}(h + t - (n+1)/2)(t - (n+1)/2) \\
&= \frac{b^2}{n}\sum_{t=1}^{n-h}(h(t - (n+1)/2) + (t - (n+1)/2)^2) \\
&= \frac{b^2}{n}\sum_{t=1}^{n-h}(t - (n+1)/2)^2 + \frac{b^2}{n}\sum_{t=1}^{n-h} h(t - (n+1)/2).
\end{aligned}$$

Now look at

$$\hat{\gamma}(0) = \frac{1}{n}\sum_{t=1}^{n}(bt - b(n+1)/2)^2 = \frac{b^2}{n}\sum_{t=1}^{n}(t - (n+1)/2)^2.$$

Compare the two equations above, we have

$$\lim_{n\to\infty}\hat{\rho}(h) = \lim_{n\to\infty}\frac{\hat{\gamma}(h)}{\hat{\gamma}(0)} = 1 + \lim_{n\to\infty}\frac{\sum_{t=1}^{n-h} h(t - (n+1)/2)}{\sum_{t=1}^{n}(t - (n+1)/2)^2} = 1.$$

And the second term converges to zero is apparent since h is fixed, or it can be checked explicitly.

(b) Using the fact that $\lim_{n\to\infty}\bar{x} = 0$. Then

$$\begin{aligned}
\hat{\gamma}(h) &= \frac{c^2}{n}\sum_{t=1}^{n-h}\cos(\omega t + \omega h)\cos\omega t \\
&= \frac{c^2}{n}\sum_{t=1}^{n-h}((\cos\omega t)^2\cos\omega h - \cos\omega t\sin\omega t\sin\omega h).
\end{aligned}$$

and

$$\hat{\gamma}(0) = \frac{c^2}{n}\sum_{t=1}^{n}(\cos\omega t)^2$$

Thus,

$$\lim_{n\to\infty}\hat{\rho}(h) = \cos\omega h - \lim_{n\to\infty}\frac{\frac{c^2}{n}\sum_{t=1}^{n-h}(\cos\omega t\sin\omega t\sin\omega h)}{\hat{\gamma}(0)}$$
$$= \cos\omega h - \lim_{n\to\infty}\frac{\frac{c^2}{n}(\sum_{t=1}^{n-h}\sin 2\omega t)\sin(\omega h)/2}{\hat{\gamma}(0)} = \cos\omega h,$$

using the fact that $\mathbb{E}(\sin 2\omega t) = 0$ as $t \to \infty$.

Problem 3

1. preprocessing the data: either de-mean the data(model the mean and residual separately), or take the difference of the data [5]

2. choose appropriate set of parameters of p,q(for example, p,q=1,2,3,4,5), then compute aic for each pair of p,q. Choose the model with lowest aic. [5]

if straight fit the data into ARIMA model but explain why the result of p,q=0, you will get part of the marks.