— Ch. Murali Krishna

UNIT - IV

DESIGN OF FIR DIGITAL FILTERS & REALIZATIONS :-

Digital filters are classified as finite duration unit impulse response [FIR] filters and Infinite duration unit impulse response [IIR] filters, depending on the form of the unit impulse response of the system.

* In FIR system, impulse response sequence is of finite duration i.e., it has a finite number of non-zero terms. The IIR system has infinite duration of impulse response sequence.
* IIR filters are usually implemented using recursive structures [feedback Poles & Zeros]. FIR filters are usually implemented using non-recursive structures [No feedback - only zeros].
* The response of ~~on~~ the FIR filter depends on the present and past input samples, whereas for the IIR filter, depends on present and past input samples as well as past values of the response.

Advantages of FIR filters :-

①. FIR filters are always stable.

②. FIR filters with exactly linear phase can easily be designed.

③. FIR filters can be realized in both recursive and Non-recursive structures.

④. FIR filters are free of limit cycle oscillations, when implemented on a finite word length digital system.

⑤. Excellent design methods are available for various kinds of FIR filters.

Disadvantages of FIR filters :-

1). The implementation of narrow transition band FIR filters are very costly. It requires more arithmetic operations and hardware components such as multipliers, adders and delay elements.

2). Memory requirement and execution time are very high.

Characteristics of FIR filters with linear phase :-

W.K.T. the difference equation of the LTI System is given by

$$y(n) = -\sum_{K=1}^{N} a_K \cdot y(n-K) + \sum_{K=0}^{M} b_K \cdot x(n-K) \longrightarrow ①$$

[Past output] [Present & past i/p]

Eqn. ① represents IIR filter. For the FIR filter, there are no past outputs. So, Eqn ① becomes

$$y(n) = \sum_{K=0}^{M} b_K \cdot x(n-K) \longrightarrow ②$$

W.K.T, the o/p of LTI system is given by

$$y(n) = \sum_{K=-\alpha}^{\alpha} h(K) \cdot x(n-K) \longrightarrow ③$$

For FIR filter, $h(n) = 0$ for $n<0$ & $n \geq M \longrightarrow ④$

From eqn's ② & ③, for FIR filter, unit impulse response h(K) exists ~~for~~ from '0' to 'M-1' as in eqn ④. Thus, output of FIR filter becomes

$$y(n) = \sum_{K=0}^{M-1} h(K) \cdot x(n-K) \longrightarrow ⑤$$

Therefore, unit impulse response of FIR filter is given by

$$h(K) = b_K \longrightarrow ⑥$$

②

The transfer function of FIR filter is given by

$$H(Z) = \sum_{n=0}^{N-1} h(n) z^{-n} \longrightarrow (7)$$

where h(n) is the impulse response of a filter.

The frequency response of h(n) is given by

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n) e^{-j\omega n} \longrightarrow (8)$$

It can be expressed as $H(e^{j\omega}) = \pm |H(e^{j\omega})| \cdot e^{j\theta(\omega)}$ $\longrightarrow (9)$

where $|H(e^{j\omega})|$ = Magnitude response

$\theta(\omega)$ = phase response

Filters can have a linear or non-linear phase depending upon the delay function namely the phase delay and group delay.

Phase delay of the filter, $\tau_p = \dfrac{-\theta(\omega)}{\omega}$ $\longrightarrow (10)$

Group delay of the filter, $\tau_g = \dfrac{-d\theta(\omega)}{d\omega}$ $\longrightarrow (11)$

For FIR filters with linear phase, we define

$$\theta(\omega) = -\alpha\omega \quad , -\pi \leq \omega \leq \pi \longrightarrow (12)$$

where 'α' is constant phase delay in samples.

from eqn's (10), (11) & (12),

$$\tau_p = \alpha \quad \text{and} \quad \tau_g = \alpha$$

Which means 'α' is independent of frequency from (8) & (9), we have

$$\sum_{n=0}^{N-1} h(n) e^{-j\omega n} = \pm |H(e^{j\omega})| \cdot e^{j\theta(\omega)} \longrightarrow (13)$$

$$\Rightarrow \sum_{n=0}^{N-1} h(n).\cos\omega n - j\sum_{n=0}^{N-1} h(n)\sin\omega n = \pm H(e^{j\omega}).\cos\theta(\omega) \pm |H(e^{j\omega})|.\sin\theta(\omega)$$

which gives,

$$\left.\begin{aligned} \sum_{n=0}^{N-1} h(n).\cos\omega n &= \pm |H(e^{j\omega})|.\cos\theta(\omega) \\ \text{and} \quad -\sum_{n=0}^{N-1} h(n).\sin\omega n &= \pm |H(e^{j\omega})|.\sin\theta(\omega) \end{aligned}\right\} (14)$$

Therefore,

$$\frac{-\sum_{n=0}^{N-1} h(n).\sin\omega n}{\sum_{n=0}^{N-1} h(n).\cos\omega n} = \frac{\cancel{\pm |H(e^{j\omega})|}.\sin(-\alpha\omega)}{\cancel{\pm |H(e^{j\omega})|}.\cos(-\alpha\omega)}$$

$$[\because \theta(\omega) = -\alpha\omega]$$

$$\Rightarrow \sum_{n=0}^{N-1} h(n).\sin\alpha\omega.\cos\omega n - \sum_{n=0}^{N-1} h(n).\cos\alpha\omega.\sin\omega n = 0$$

$$\Rightarrow \sum_{n=0}^{N-1} h(n).\sin(\alpha-n)\omega = 0 \longrightarrow (15)$$

Eqn (15). will satisfies when

$$h(n) = h(N-1-n) \longrightarrow (16)$$

and

$$\alpha = \frac{N-1}{2}, \text{ for } 0 \le n \le N-1 \longrightarrow (17)$$

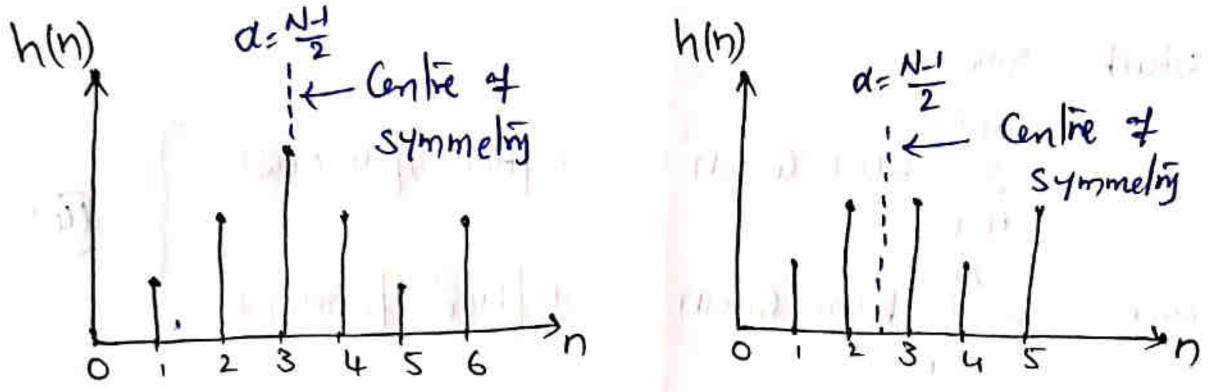
Therefore, FIR filters have constant phase & group delays when the impulse response is symmetrical about $\alpha = \frac{N-1}{2}$.

(3)

The impulse response h(n) satisfying eqns (16) & (17) for odd and even values of 'N' as shown in fig.1.

If $N = 7$ (odd), $\alpha = \frac{N-1}{2} = \frac{7-1}{2} = 3$, Symmetry occurs at third sample

If $N = 6$ (even), $\alpha = \frac{6-1}{2} = 2\frac{1}{2}$, Symmetry occurs at $2\frac{1}{2}$ sample.



(a) N = 7 (odd) (b) N = 6 (even)

Fig.1:- Impulse response Sequences of Symmetric sequences.

If only constant group delay is required and not the phase delay, so, we can write

$$\theta(\omega) = \beta - \alpha\omega \longrightarrow (18)$$

So, Eqn (13) becomes

$$\sum_{n=0}^{N-1} h(n)\, e^{-j\omega n} = \pm|H(e^{j\omega})| \cdot e^{j(\beta-\alpha\omega)} \longrightarrow (19)$$

After simplification,

$$\sum_{n=0}^{N-1} h(n) \cdot \sin\left[\beta - (\alpha - n)\omega\right] = 0 \longrightarrow (20)$$

If $\beta = \pi/2$ then

$$\sum_{n=0}^{N-1} h(n) \cdot \cos(\alpha - n)\omega = 0 \longrightarrow (21)$$

Eqn. (21) will be satisfied when

$$h(n) = -h(N-1-n) \longrightarrow (22)$$

and $$\alpha = \frac{N-1}{2} \longrightarrow (23)$$

Therefore, FIR filters have constant group delay, τ_g and not constant phase delay, τ_p when impulse response is antisymmetrical about $\alpha = \frac{N-1}{2}$.

The impulse response $h(n)$ satisfying eqns (22) & (23) for odd & even values of 'N' as shown in fig.2.

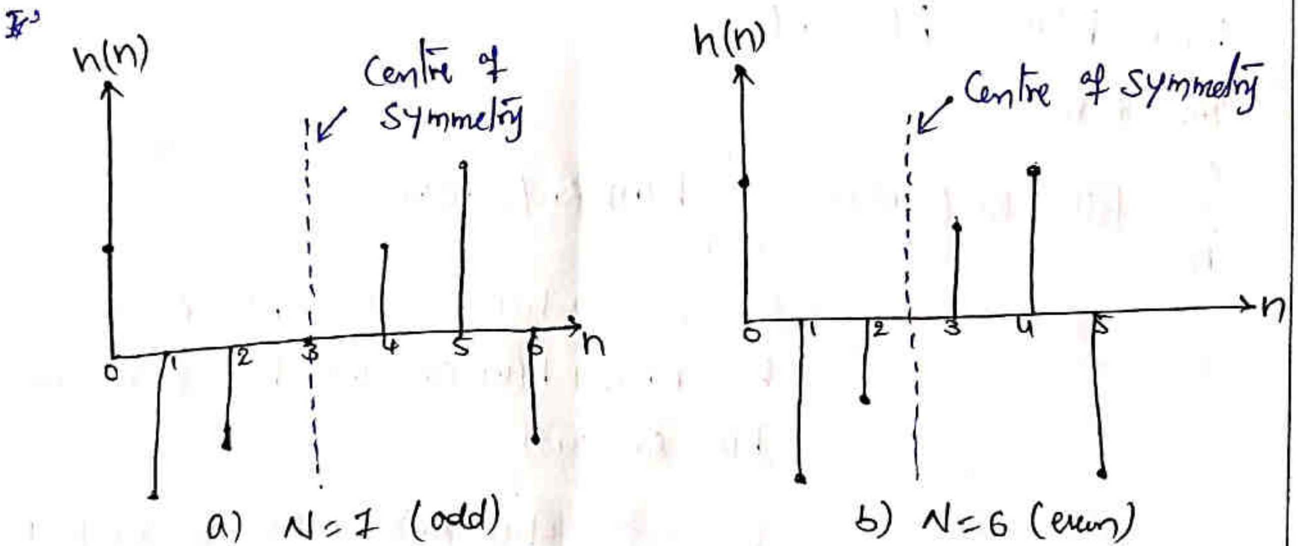


Fig.2:- Impulse response Sequence of antisymmetric sequence

=0=

Eg-1:- The length of an FIR filter is 7. If this filter has a linear phase, show that the equation

$$\sum_{n=0}^{N-1} h(n).\ \sin(\alpha-n)\omega = 0$$ is Satisfied.

Sol:- Given that, length of FIR filter, $N=7$

So, $\alpha = \frac{N-1}{2} = \frac{7-1}{2} = \frac{6}{2} = 3.$

Given that, FIR filter has linear phase. So, the condition for symmetry is $h(n) = h(N-1-n)$

$\Rightarrow h(n) = h(6-n)$

The filter Coefficients are $h(0) = h(6)$, $h(1) = h(5)$, $h(2) = h(4)$, $h(3) = h(3)$.

There fore,

$$\sum_{n=0}^{N-1} h(n).\ \sin(\alpha-n)\omega = \sum_{n=0}^{6} h(n).\sin(3-n)\omega$$

$$= h(0)\cdot\sin 3\omega + h(1)\cdot\sin 2\omega + h(2)\cdot\sin\omega + h(3)\sin 0 + h(4)\sin(-\omega) + h(5)\sin(-2\omega) + h(6)\cdot\sin(-3\omega)$$

$$= \cancel{h(0)\sin 3\omega} + \cancel{h(1)\cdot\sin 2\omega} + \cancel{h(2)\cdot\sin\omega} + 0 - \cancel{h(2)\sin\omega} - \cancel{h(1)\cdot\sin 2\omega} - \cancel{h(0)\sin 3\omega}$$

$$= 0$$

Hence, $\sum_{n=0}^{N-1} h(n).\ \sin(\alpha-n)\omega = 0$ is Satisfied.

=0=

Eg-2:- An FIR filter (N = 9) is characterized by the following transfer function.

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$$

Determine the magnitude response and also prove that the phase and group delays are constant.

Sol:- Given that, $N = 9$

So, $\alpha = \frac{N-1}{2} = \frac{9-1}{2} = \frac{8}{2} = 4.$

For an FIR filter, the symmetric condition is $h(n) = h(N-1-n)$
$\Rightarrow h(n) = h(8-n)$

The filter coefficients are $h(0) = h(8)$, $h(1) = h(7)$, $h(2) = h(6)$, $h(3) = h(5)$ and $h(4)$.

Therefore,

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n} = \sum_{n=0}^{8} h(n) \cdot z^{-n}$$

$$= h(0) + h(1) z^{-1} + h(2) z^{-2} + h(3) z^{-3} + h(4) z^{-4} + h(5) z^{-5} + h(6) z^{-6} + h(7) \cdot z^{-7} + h(8) z^{-8}$$

$$= h(0) + h(1) z^{-1} + h(2) z^{-2} + h(3) z^{-3} + h(4) z^{-4} + h(3) z^{-5} + h(2) z^{-6} + h(1) z^{-7} + h(0) z^{-8}$$

$$= z^{-4} \left[h(0) z^{4} + h(1) z^{3} + h(2) z^{2} + h(3) z + h(4) + h(3) z^{-1} + h(2) z^{-2} + h(1) z^{-3} + h(0) z^{-4} \right]$$

$$\Rightarrow H(z) = z^{-4} \left[h(0)\left[z^{4} + z^{-4}\right] + h(1)\left[z^{3} + z^{-3}\right] + h(2)\left[z^{2} + z^{-2}\right] + h(3)\left[z + z^{-1}\right] + h(4) \right]$$

The frequency response may be obtained by replacing 'z' with '$e^{j\omega}$'

⑤

$$H(e^{j\omega}) = e^{-j4\omega} \cdot \left[h(0) \cdot \left[e^{j4\omega} + e^{-j4\omega}\right] + h(1) \cdot \left[e^{j3\omega} + e^{-j3\omega}\right] + h(2) \cdot \left[e^{j2\omega} + e^{-j2\omega}\right] + h(3) \cdot \left[e^{j\omega} + e^{-j\omega}\right] + h(4) \right].$$

$$= e^{-j4\omega} \cdot \left[2 \cdot h(0) \cdot \cos 4\omega + 2 \cdot h(1) \cdot \cos 3\omega + 2 \cdot h(2) \cdot \cos 2\omega + 2 \cdot h(3) \cdot \cos \omega + h(4) \right]$$

$$\Rightarrow H(e^{j\omega}) = e^{-j4\omega} \cdot \left[h(4) + 2 \cdot \sum_{n=0}^{3} h(n) \cdot \cos(\alpha - n)\omega \right]$$

$$\Rightarrow H(e^{j\omega}) = e^{-j4\omega} \cdot |H(e^{j\omega})|.$$

Where $|H(e^{j\omega})|$ = Magnitude response $= h(4) + 2 \cdot \sum_{n=0}^{3} h(n) \cdot \cos(\alpha - n)\omega$

$\theta(\omega)$ = phase response $= -4\omega$

$\therefore$ phase delay, $\tau_p = \dfrac{-\theta(\omega)}{\omega} = \dfrac{+4\omega}{\omega} = +4$

$\therefore$ Group delay, $\tau_g = \dfrac{-d\theta(\omega)}{d\omega} = \dfrac{-d}{d\omega}(-4\omega) = 4$

Thus, the phase delay and group delay are the same and have same constants.

$= 0 =$

Frequency response of linear phase FIR filters:-

Depending on the value of N (odd (or) even) and the type of symmetry of the filter impulse response sequence (symmetric or antisymmetric), there are four possible types of impulse response for linear phase FIR filters.

1). Symmetrical impulse response when N is odd
2). Symmetrical impulse response when N is even
3). Antisymmetric impulse response when N is odd
4). Antisymmetric impulse response when N is even.

①. Symmetrical impulse response, N odd :-

W.K.T, the frequency response of impulse response can be expressed as

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n)\, e^{-j\omega n} \longrightarrow ①$$

When 'N' is odd, the symmetrical impulse response will at $n = \frac{N-1}{2}$. So eqn ① ~~can~~ is expressed as:

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n).e^{-j\omega n} + h\left(\frac{N-1}{2}\right).e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=\frac{N+1}{2}}^{N-1} h(n).e^{-j\omega n} \longrightarrow ②$$

Let $n = N-1-m \Rightarrow m = N-1-n$

When $n = \frac{N+1}{2}$, $m = (N-1) - \left(\frac{N+1}{2}\right) = \frac{N-3}{2}$

When $n = N-1$, $m = N-1-(N-1) = 0$

$$H(e^{j\omega}) = h\left(\frac{N-1}{2}\right)\cdot e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega n} + \sum_{n=0}^{\frac{N-3}{2}} h(N-1-n)\cdot e^{-j\omega(N-1-n)}$$

For symmetrical response, $h(N-1-n) = h(n)$

$$\Rightarrow H(e^{j\omega}) = h\left(\frac{N-1}{2}\right)\cdot e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=0}^{\frac{N-3}{2}} h(n)\cdot e^{-j\omega n} + \sum_{n=0}^{\frac{N-3}{2}} h(n)\cdot e^{-j\omega(N-1-n)}$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)}\cdot\left[h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} h(n)\cdot e^{-j\omega n + j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=0}^{\frac{N-3}{2}} h(n)\cdot e^{-j\omega(N-1-n) + j\omega\left(\frac{N-1}{2}\right)}\right]$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)}\cdot\left[h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} h(n)\left[e^{j\omega\left(\frac{N-1}{2}-n\right)} + e^{-j\omega\left(\frac{N-1}{2}-n\right)}\right]\right]$$

$$\Rightarrow H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)}\cdot\left[h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} h(n)\cdot 2\cdot \cos\omega\left(\frac{N-1}{2}-n\right)\right] \longrightarrow (3)$$

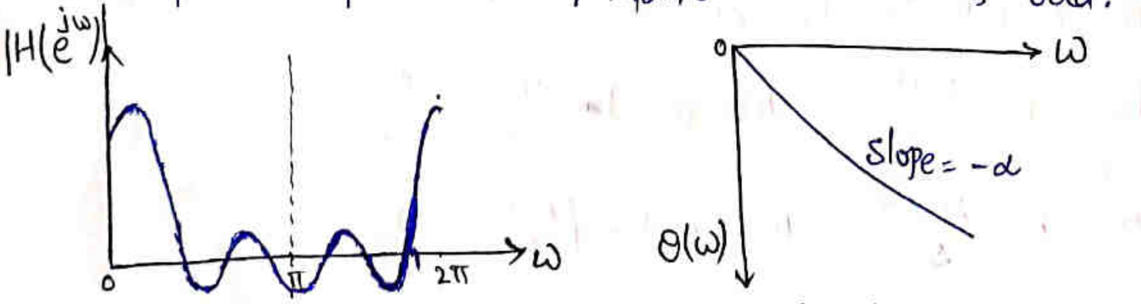
The above eqn. is in the form of

$$H(e^{j\omega}) = |H(e^{j\omega})|\cdot e^{j\theta(\omega)}$$

where $|H(e^{j\omega})| = h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} 2\cdot h(n) \cos\omega\left(\frac{N-1}{2}-n\right)$

and $\theta(\omega) = -\left(\frac{N-1}{2}\right)\omega = -\alpha\omega$, where $\alpha = \frac{N-1}{2}$

Fig.3 shows the Magnitude & phase responses, when the impulse response is symmetric and 'N' is odd.



a) Magnitude response   b). phase response

Fig.3:- Frequency response characteristics

② Symmetrical impulse response, 'N' is even:-

W.K.T, $$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n)\, e^{-j\omega n}$$

For symmetrical impulse response with even number of samples, the centre of symmetry lies between $n = \frac{N}{2}-1$ and $n = \frac{N}{2}$. Hence $H(e^{j\omega})$ is expressed as

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N}{2}-1} h(n) \cdot e^{-j\omega n} + \sum_{n=N/2}^{N-1} h(n)\, e^{-j\omega n} \longrightarrow ①$$

$$= \sum_{n=0}^{\frac{N}{2}-1} h(n)\, e^{-j\omega n} + \sum_{n=0}^{\frac{N}{2}-1} h(N-1-n) \cdot e^{-j\omega(N-1-n)}$$

for symmetrical impulse response, $h(N-1-n) = h(n)$

$$\Rightarrow H(e^{j\omega}) = \sum_{n=0}^{\frac{N}{2}-1} h(n) \cdot e^{-j\omega n} + \sum_{n=0}^{\frac{N}{2}-1} h(n) \cdot e^{-j\omega(N-1-n)}$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \cdot \left[\sum_{n=0}^{\frac{N}{2}-1} h(n) \left[e^{-j\omega n + \frac{N-1}{2} j\omega} + e^{-j\omega(N-1-n) + j\omega\frac{N-1}{2}}\right]\right]$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \cdot \left[\sum_{n=0}^{\frac{N}{2}-1} h(n) \left[e^{j\omega\left(\frac{N-1}{2}-n\right)} + e^{-j\omega\left(\frac{N-1}{2}-n\right)}\right]\right]$$

$$\Rightarrow H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)} \sum_{n=0}^{\frac{N}{2}-1} h(n) \cdot 2 \cdot \cos\omega\left(\frac{N-1}{2}-n\right) \longrightarrow ②$$

where $|H(e^{j\omega})|$ = Magnitude response

$$= \sum_{n=0}^{\frac{N}{2}-1} 2 \cdot h(n) \cos\omega\left(\frac{N-1}{2}-n\right) \longrightarrow ③$$

$$\theta(\omega) = \text{phase response} = -\left(\frac{N-1}{2}\right)\omega = -\alpha\omega.$$

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The magnitude response is shown in fig. 4.

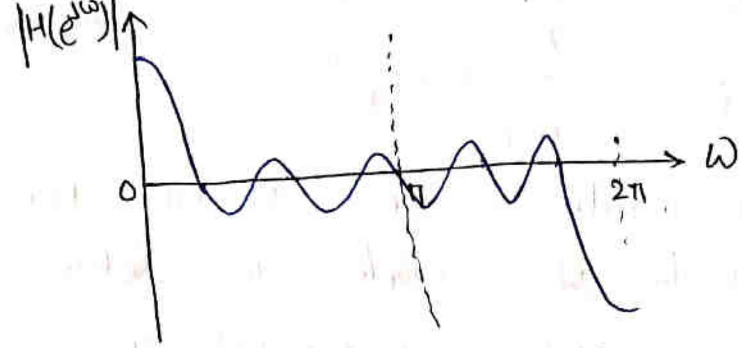

Fig. 4:- Magnitude response characteristics when N = even & symmetric.

③. Antisymmetric impulse response, 'N' is odd :-

The impulse is response is antisymmetric with centre of symmetry at $n = \frac{N-1}{2}$ and its value is zero

i.e., $h\left(\frac{N-1}{2}\right) = 0 \longrightarrow ①$

W.K.T, the frequency response of FIR filter is

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n)\, e^{-j\omega n}$$

$$\Rightarrow H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega n} + h\left(\frac{N-1}{2}\right) e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=\frac{N+1}{2}}^{N-1} h(n).e^{-j\omega n} \longrightarrow ②$$

Substitute eqn ①,

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n).e^{-j\omega n} + \sum_{n=0}^{\frac{N-3}{2}} h(N-1-n).e^{-j\omega(N-1-n)} \longrightarrow ③$$

For antisymmetric, $h(N-1-n) = -h(n)$

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n)\, e^{-j\omega n} - \sum_{n=0}^{\frac{N-3}{2}} h(n)\, e^{-j\omega(N-1-n)}$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \cdot \left[\sum_{n=0}^{\frac{N-3}{2}} h(n) \cdot \left(e^{-j\omega n + j\omega\left(\frac{N-1}{2}\right)} - e^{-j\omega(N-1-n) + j\omega\left(\frac{N-1}{2}\right)}\right)\right]$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \cdot \left[\sum_{n=0}^{\frac{N-3}{2}} h(n) \cdot \left[e^{j\omega\left(\frac{N-1}{2}-n\right)} - e^{-j\omega\left(\frac{N-1}{2}-n\right)}\right]\right]$$

$$\Rightarrow H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)} \sum_{n=0}^{\frac{N-3}{2}} h(n) \cdot 2j \cdot \sin\omega\left(\frac{N-1}{2}-n\right) \longrightarrow (4)$$

W.K.T, $e^{j\pi/2} = j \longrightarrow (5)$

$$\Rightarrow H(e^{j\omega}) = e^{-j\omega\cdot\left(\frac{N-1}{2}\right)} \cdot e^{j\pi/2} \sum_{n=0}^{\frac{N-3}{2}} 2 \cdot h(n) \cdot \sin\omega\left(\frac{N-1}{2}-n\right)$$

$$\Rightarrow H(e^{j\omega}) = e^{j\left(\frac{\pi}{2} - \omega\frac{N-1}{2}\right)} \left[\sum_{n=0}^{\frac{N-3}{2}} 2 \cdot h(n) \cdot \sin\omega\left(\frac{N-1}{2}-n\right)\right] \longrightarrow (6)$$

Let $m = \frac{N-1}{2} - n \Rightarrow n = \frac{N-1}{2} - m$

If $n=0$, $m = \frac{N-1}{2}$ & If $n = \frac{N-3}{2}$, $m = 1$

Eqn (6) becomes

$$H(e^{j\omega}) = e^{j\left(\frac{\pi}{2} - \omega\frac{N-1}{2}\right)} \left[\sum_{n=1}^{\frac{N-1}{2}} 2 \cdot h\left(\frac{N-1}{2}-n\right) \cdot \sin\omega n\right] \longrightarrow (7)$$

The Magnitude function is given by

$$|H(e^{j\omega})| = \sum_{n=1}^{\frac{N-1}{2}} 2.h\left(\frac{N-1}{2}-n\right). \sin\omega n \longrightarrow (8)$$ and

phase function is given by

$$\theta(\omega) = \frac{\pi}{2} - \omega.\frac{N-1}{2} = \beta - \alpha\omega \longrightarrow (9)$$

where $\beta = \pi/2$ and $\alpha = \frac{N-1}{2}$

Fig. 5 shows the Magnitude function of frequency response, when the impulse response is antisymmetric and 'N' is odd.

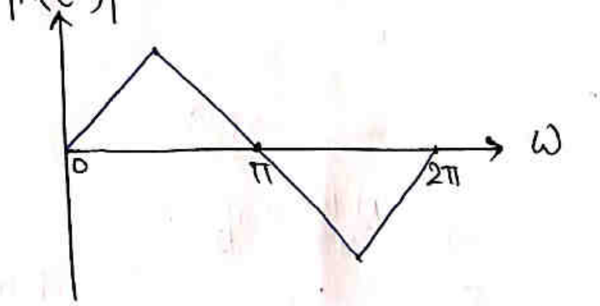

Fig. 5:- Magnitude function of $H(e^{j\omega})$ when anti- -symmetric impulse response, 'N' odd

4). Antisymmetric impulse response, 'N' is even:-

For FIR filter, the frequency response is given by

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n).e^{-j\omega n}$$

$$\Rightarrow H(e^{j\omega}) = \sum_{n=0}^{\frac{N}{2}-1} h(n)e^{-j\omega n} + \sum_{n=N/2}^{N-1} h(n)e^{-j\omega n} \longrightarrow (1)$$

$$= \sum_{n=0}^{\frac{N}{2}-1} h(n)e^{-j\omega n} + \sum_{n=0}^{\frac{N}{2}-1} h(N-1-n)\, e^{-j\omega(N-1-n)}$$

For antisymmetric, $h(N-1-n) = -h(n)$

$$\therefore H(e^{j\omega}) = \sum_{n=0}^{\frac{N}{2}-1} h(n) e^{-j\omega n} - \sum_{n=0}^{N/2-1} h(n) e^{-j\omega(N-1-n)}$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \cdot \left[\sum_{n=0}^{\frac{N}{2}-1} h(n) . \left[e^{-j\omega n + j\omega \frac{N-1}{2}} - e^{-j\omega(N-1-n) + j\omega\left(\frac{N-1}{2}\right)}\right]\right]$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \left[\sum_{n=0}^{\frac{N}{2}-1} h(n) . \left(e^{+j\omega\left(\frac{N-1}{2}-n\right)} - e^{-j\omega\left(\frac{N-1}{2}-n\right)}\right)\right]$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} . \sum_{n=0}^{\frac{N}{2}-1} h(n) . 2j \sin\omega\left(\frac{N-1}{2}-n\right)$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \cdot e^{j\pi/2} \left[\sum_{n=1}^{N/2} 2h\left(\frac{N}{2}-n\right) \cdot \sin\omega\left(n-1/2\right)\right]$$

$$\left[\because m = \frac{N}{2}-n\right]$$

$$\Rightarrow H(e^{j\omega}) = e^{j\left(\frac{\pi}{2} - \omega\frac{N-1}{2}\right)} \cdot \left[\sum_{n=1}^{N/2} 2 . h\left(\frac{N}{2}-n\right) . \sin\omega\left(n-\frac{1}{2}\right)\right] \longrightarrow (2)$$

The Magnitude function is given by

$$|H(e^{j\omega})| = \sum_{n=1}^{N/2} 2 . h\left(\frac{N}{2}-n\right) . \sin\omega\left(n-\frac{1}{2}\right) \longrightarrow (3)$$

Fig. 6 shows the magnitude function of frequency response when antisymmetric impulse response & 'N' is even.

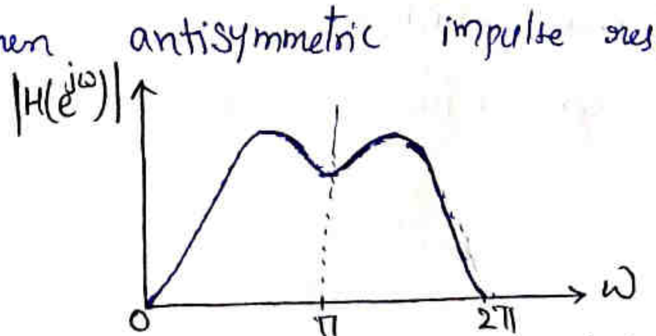

Fig.6:- Magnitude function of $H(e^{j\omega})$, when antisymmetric and 'N' even.

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(9)

Design techniques for FIR filters:-

The well known methods of designing FIR filters are as follows:

1). Fourier series method
2). Window method
3). Frequency sampling method
4). Optimum filter design.

2). Design of FIR filters using windows:-

The desired impulse response is obtained from desired frequency response by inverse fourier transform. This desired impulse response has infinite length. The FIR filters has finite length of impulse response. Hence impulse response is passed through certain window to get the finite length. This is the reason for choosing window techniques.

There are various types of window functions:

1). Rectangular window
2). Bartlett (i.e.., Triangular window)
3). Blackmann window
4). Hamming window
5). Hanning window
6). Kaiser window.

Procedure for designing FIR filters using windows:-

①. Choose the desired frequency response of the filter $H_d(\omega)$

②. Take inverse fourier transform of $H_d(\omega)$ to obtain the desired impulse response $h_d(n)$.

③. Choose a window sequence $\omega(n)$ and multiply $h_d(n)$ by $\omega(n)$ to convert infinite duration impulse response to a finite duration impulse response $h(n)$

④. The transfer function $H(Z)$ of filter is obtained by taking Z-transform of $h(n)$.

1). Rectangular window:-

The weighting function [window function] for an N-point rectangular window is given by

$$W_R(n) = \begin{cases} 1 & : -\left(\frac{N-1}{2}\right) \le n \le \left(\frac{N-1}{2}\right) \\ 0 & ; \text{elsewhere} \end{cases} \longrightarrow ①$$

(or)

$$W_R(n) = \begin{cases} 1 & : 0 \le n \le (N-1) \\ 0 & : \text{elsewhere} \end{cases}$$

Fig. 7 shows the rectangular window.

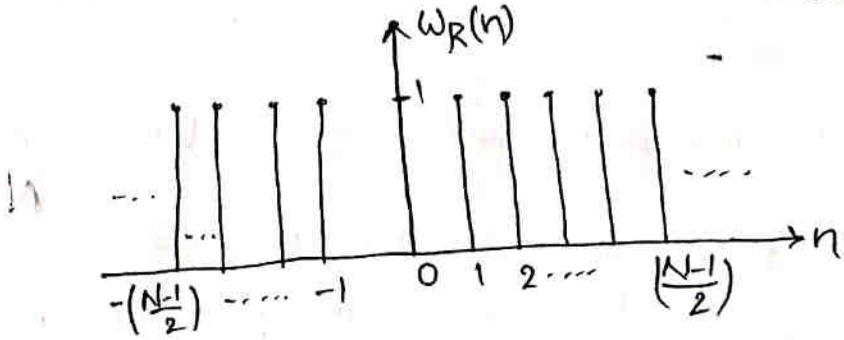


Fig. 7 :- Rectangular window.

Let us consider the fourier transform of rectangular window.

$$W_R(\omega) = \sum_{n=0}^{N-1} w_R(n).e^{-j\omega n} = \sum_{n=0}^{N-1} 1.e^{-j\omega n}$$

$$\Rightarrow W_R(\omega) = \frac{1-e^{-j\omega N}}{1-e^{-j\omega}} = \frac{e^{-j\omega N/2}\left[e^{j\omega N/2} - e^{-j\omega N/2}\right]}{e^{-j\omega/2}\left[e^{j\omega/2} - e^{-j\omega/2}\right]}$$

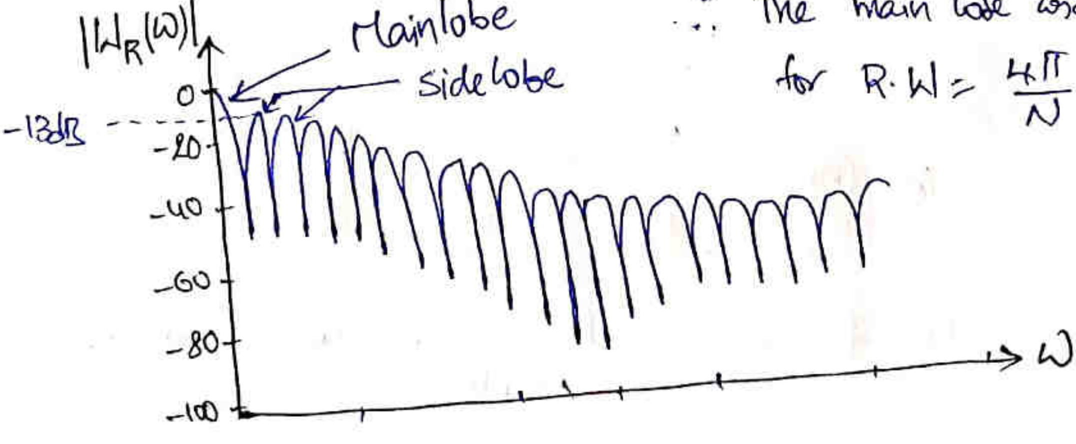
$$\Rightarrow W_R(\omega) = e^{-j\omega\left(\frac{N-1}{2}\right)} \cdot \frac{\cancel{2j}\sin\left(\frac{\omega N}{2}\right)}{\cancel{2j}\sin(\omega/2)}$$

$$\Rightarrow W_R(\omega) = e^{-j\omega\left(\frac{N-1}{2}\right)} \cdot \frac{\sin\left(\frac{\omega N}{2}\right)}{\sin(\omega/2)}$$

The magnitude response of rectangular window is

$$|W_R(\omega)| = \frac{\left|\sin\left(\frac{\omega N}{2}\right)\right|}{\left|\sin(\omega/2)\right|}$$

Fig. 8 shows the magnitude response of rectangular window for N = 50.



$\therefore$ The main lobe width for R.W = $\frac{4\pi}{N}$

Fig. 8 :- Magnitude spectrum of rectangular window for N = 50

From fig, it may be noted that there is one main lobe and many side lobes. As 'N' increases, the main lobe becomes narrower. The area under the side lobes remain same irrespective of changes in the value of 'N'.

W.K.T, the unit sample response of FIR filter is given as

$$h(n) = h_d(n) \cdot w(n)$$

The frequency response of FIR filter can be obtained by taking fourier transform:

$$H(\omega) = F.T\left[h_d(n) \cdot w(n)\right] = H_d(\omega) \circledast W(\omega)$$

This expression shows that the response of FIR filter is equal to the convolution of desired frequency response with that of window function. Because of convolution, $H(\omega)$ has the smoothing effect. The sidelobes of $W(\omega)$ create undesirable ringing effects in $H(\omega)$.

Gibb's phenomenon :-

Let us consider the example of a LPF having desired frequency response $H_d(\omega)$ as depicted in fig. 9. This response has the cut off frequency at ω_c.

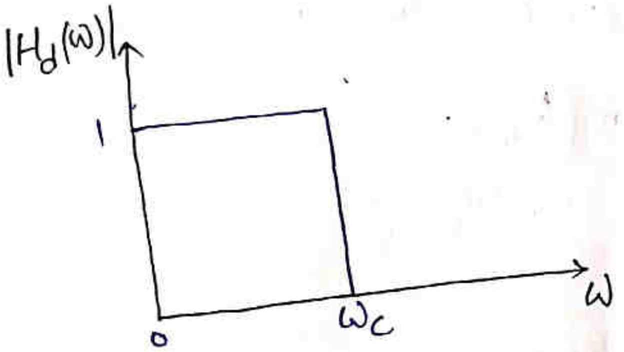


Fig. 9(a); Desired frequency response

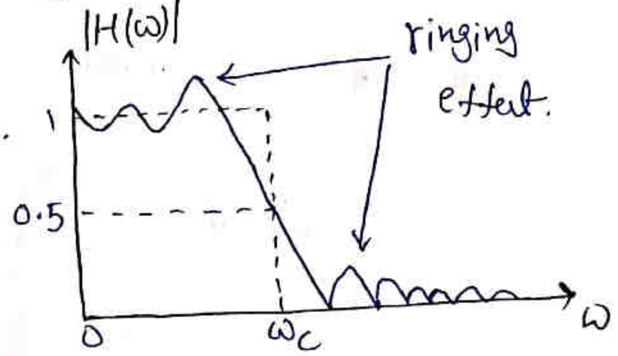


Fig. 9(b):- Frequency response obtained by windowing.

These oscillations (or) ringing is generated because of sidelobes in the frequency response $W(\omega)$ of the window function. This oscillatory behaviour near the band edge (ω_c) of a filter is known as "Gibb's Phenomenon". These sidelobes are generated because of abrupt discontinuity of window function. In case of

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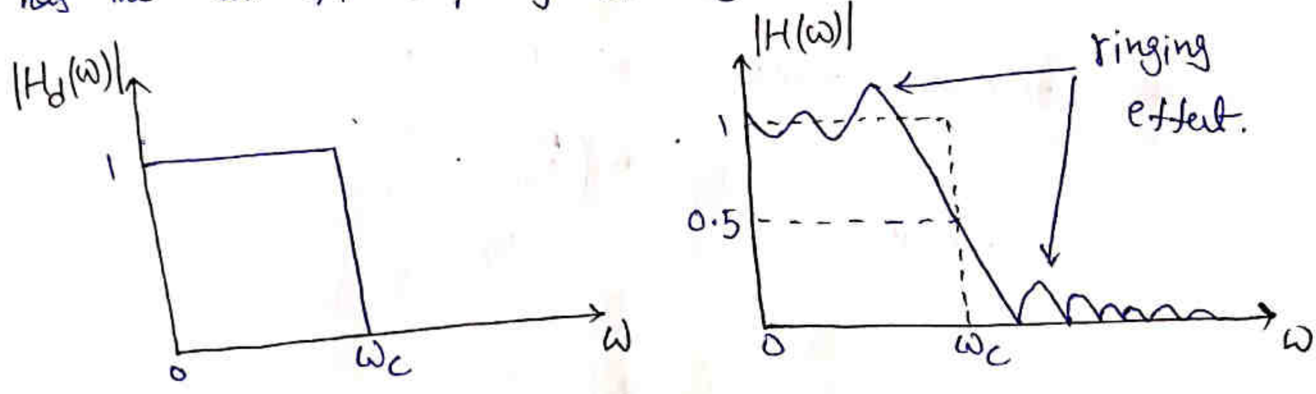

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Fig. 9(b):- Frequency response obtained by windowing.

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(11)

rectangular window, the side lobes are larger in size because the discontinuity is abrupt. Therefore, the ringing effect is maximum in rectangular window.

The Gibb's phenomenon can be reduced by using a less abrupt truncation of filter coefficients. This can be achieved using a window function that tapers smoothly towards zero at both ends. One such type of window is triangular window.

② Triangular (or) Bartlett window:-

The N-point triangular window is given by

$$W_T(n) = \begin{cases} 1 - \dfrac{2|n|}{N-1} & \text{for } -(N-1)/2 \le n \le (N-1)/2 \\ 0 & : \text{otherwise} \end{cases}$$

The fourier transform of the triangular window is

$$W_T(e^{j\omega}) = \left[\frac{\sin\left(\frac{N-1}{4}\right)\omega}{\sin \omega/2}\right]^2$$

(or)

It can also be defined as

$$W_T(n) = \begin{cases} 1 - \dfrac{2\left|n - \frac{N-1}{2}\right|}{N-1} & : \text{for } n = 0, 1, 2, \ldots N-1 \\ 0 & : \text{otherwise} \end{cases}$$

Fig. 10(a) shows the triangular window and fig. 10(b) shows the frequency spectrum (or) Magnitude response.

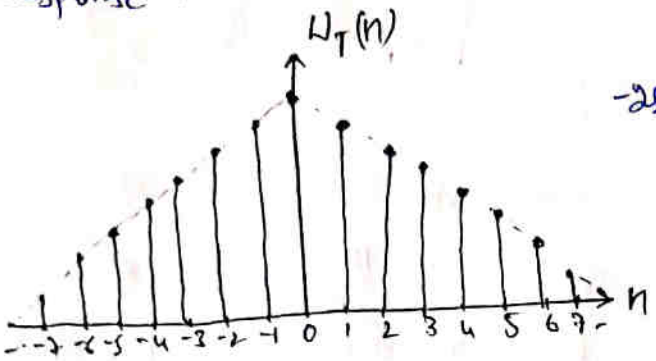

Fig. 10(a): Triangular window

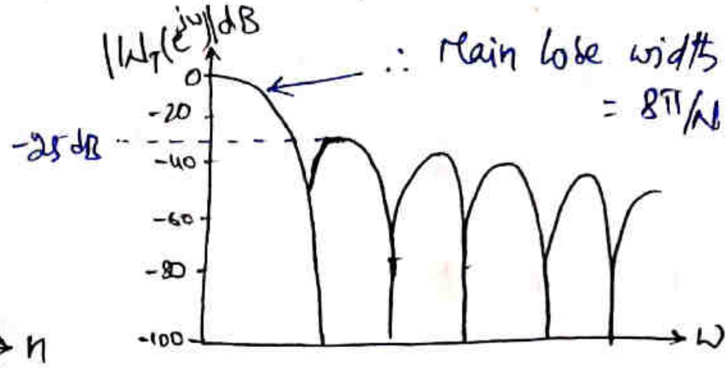

Fig. 10(b): Magnitude response

Fig. 11 shows that the triangular window produces a smooth magnitude response in both pass band and stopband. But it has following disadvantages:

1). The transition region is more
2). The attenuation in stop band is less.

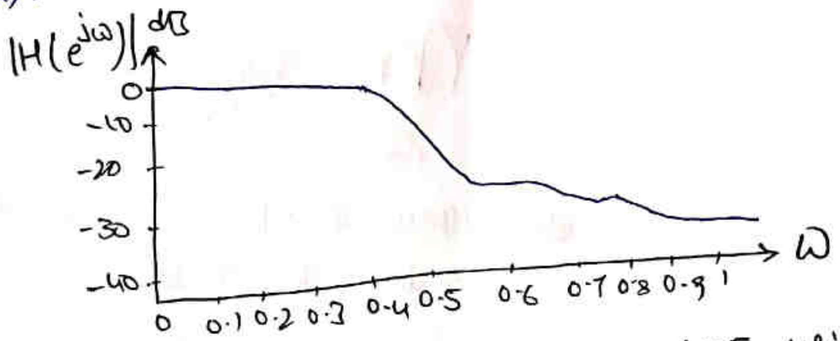

Fig. 11 : Magnitude response of LPF using triangular window for N=25.

③. Blackman window:-

It is defined as

$$\omega_B(n) = \begin{cases} 0.42 + 0.5\cos\dfrac{2\pi n}{N-1} + 0.08\cos\dfrac{4\pi n}{N-1} & ; -\dfrac{(N-1)}{2} \le n \le \dfrac{(N-1)}{2} \\ 0 & ; \text{otherwise} \end{cases}$$

(12)

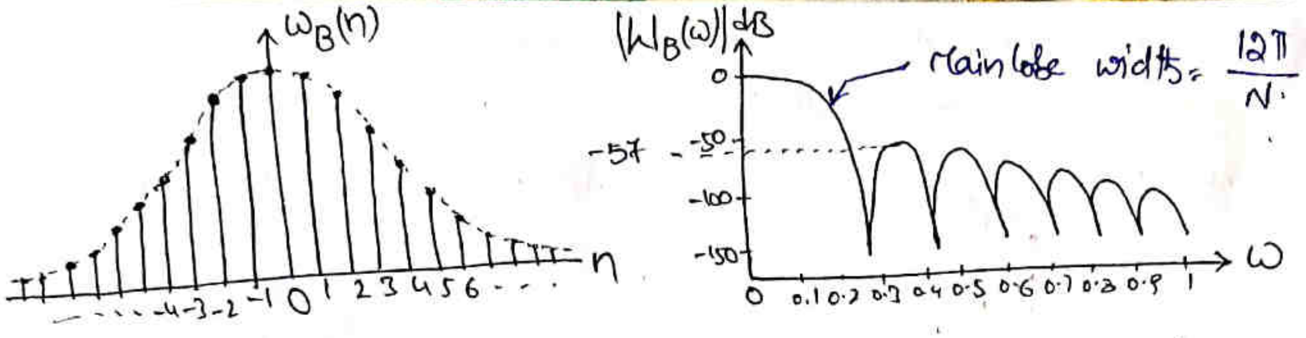

Fig. 12 :- a) window sequence b). Magnitude response in dB

The main lobe width increases to $\frac{12\pi}{N}$. The frequency response of the blackmann window as shown in fig. 12(b) the peak side lobe level is down about 57dB from the main lobe peak.

Fig. 13 shows the side lobe attenuation of a LPF using blackmann window is -74 dB.

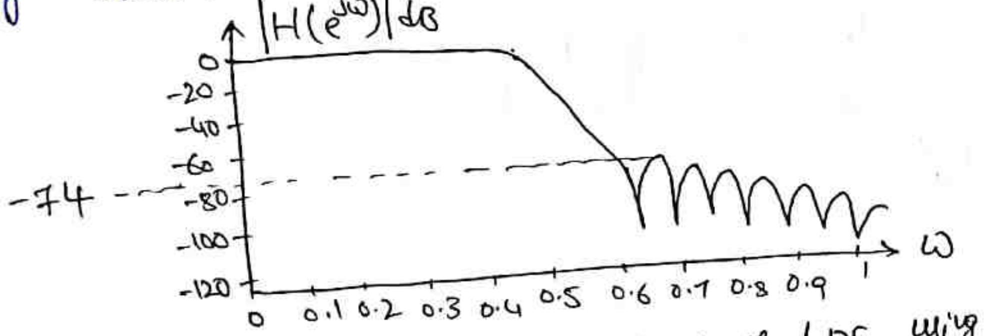

Fig. 13 :- Magnitude response of LPF using Blackmann window for N=25.

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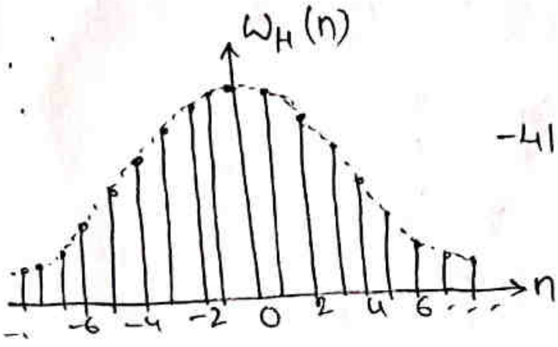
④. Hamming window :-

The equation for hamming window is

$$\omega_H(n) = \begin{cases} 0.54 + 0.46\cos\left(\frac{2\pi n}{N-1}\right) & ; -\frac{(N-1)}{2} \le n \le \frac{(N-1)}{2} \\ 0 & ; \text{otherwise} \end{cases}$$

The frequency response of hamming window is

$$W_H(e^{j\omega}) = 0.54\frac{\sin \omega N/2}{\sin \omega/2} + 0.23\frac{\sin\left(\omega N/2 - \pi N/(N-1)\right)}{\sin\left(\omega/2 - \pi/(N-1)\right)} + 0.23\frac{\sin\left(\omega N/2 + \pi N/(N-1)\right)}{\sin\left(\omega/2 + \pi/(N-1)\right)}$$



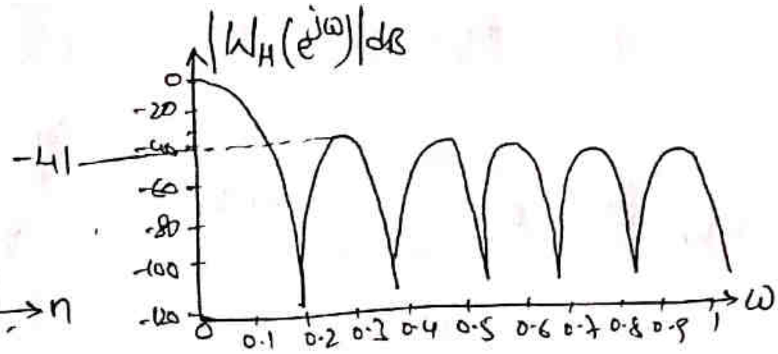


Fig.14:- a) window Sequence b). Magnitude response

The peak side lobe level is down about 41dB from the main lobe peak. The magnitude response of LPF designed using Hamming window is shown in fig.15. The first side lobe peak is -53dB. However at higher frequencies the stop band attenuation is low because this window generates lesser oscillation in the side lobes than hanning window for same main lobe width.

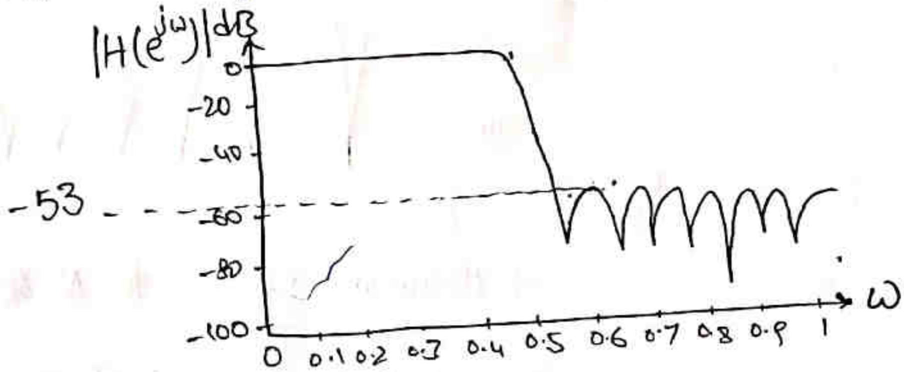


Fig. 15:- Magnitude response of LPF using Hamming window for N=25.

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(5). Hanning window:-

The Hanning window can be expressed as

$$W_{Hn}(n) = \begin{cases} 0.5+0.5\cos\left(\frac{2\pi n}{N-1}\right) & ; -\left(\frac{N-1}{2}\right) \le n \le \left(\frac{N-1}{2}\right) \\ 0 & ; \text{otherwise} \end{cases}$$

(13)

The frequency response of Hanning window is

$$W_{Hn}(e^{j\omega}) = 0.5 \frac{\sin \frac{\omega N}{2}}{\sin \omega/2} + 0.25 \frac{\sin\left(\frac{\omega N}{2} - \frac{\pi N}{N-1}\right)}{\sin\left(\omega/2 - \pi/N-1\right)} + 0.25 \frac{\sin\left(\frac{\omega N}{2} + \frac{\pi N}{N-1}\right)}{\sin\left(\frac{\omega}{2} + \pi/N-1\right)}$$

The window sequence and frequency response are shown in fig.16. The main lobe width of Hanning window is twice that of the rectangular window. The magnitude of side ~~lobe~~ lobe level is -31dB. The first side lobe of hanning window is approximately one tenth that of of rectangular window.

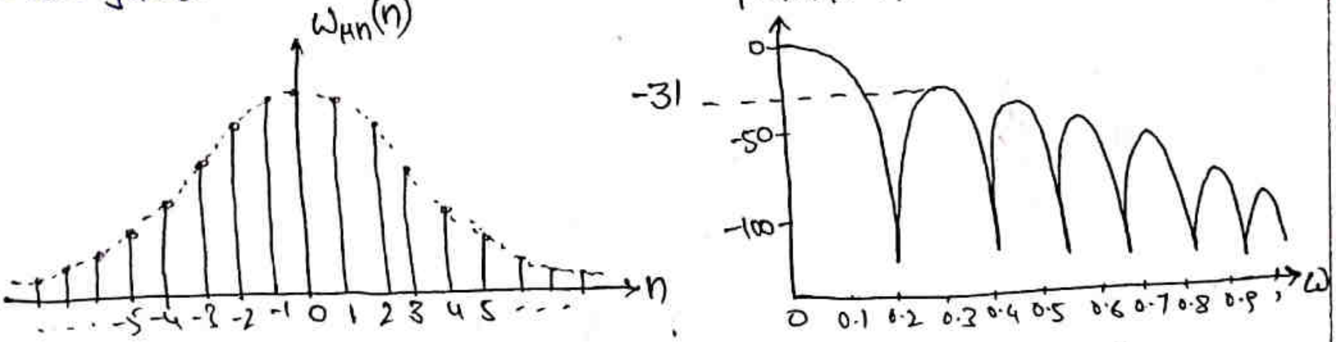

Fig. 16 : a) window sequence b). Magnitude response for N=25.

Fig. 17 shows the frequency response of LPF design using Hanning window. The minimum stopband attenuation of the filter is 44dB. At higher frequencies the stopband attenuation is even greater.

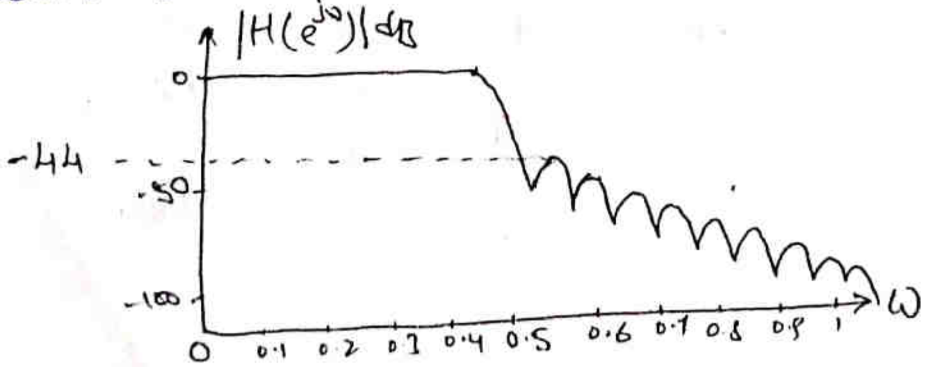

Fig. 17 : Freq. response of LPF using Hanning window for N=25

Comparison between various windows:-

S.No	Name of the window	width of the main lobe	Transition Stop band Attenuation	Minimum amplitude of side lobe	Relative passband ripple
1.	Rectangular	$4\pi/N+1$	$-21\,dB$	$-13\,dB$	$0.7416\,dB$
2.	Triangular	$8\pi/N$	$-25\,dB$	$-25\,dB$	$0.1428\,dB$
3.	Hanning	$8\pi/N$	$-44\,dB$	$-31\,dB$	$0.0546\,dB$
4.	Hamming	$8\pi/N$	$-53\,dB$	$-41\,dB$	$0.0194\,dB$
5.	Backmann	$12\pi/N$	$-74\,dB$	$-57\,dB$	$0.0017\,dB$

Eg-3:- Design the symmetric FIR LPF for which desired frequency response is expressed as

$$H_d(\omega) = \begin{cases} e^{-j\omega\tau}, & \text{for } |\omega| \le \omega_c \\ 0, & \text{elsewhere} \end{cases}$$

The length of the filter should be 7 and $\omega_c = 1$ rad/sample. Make use of rectangular window.

Sol:- Given that, length of filter, $M = 7$ and

$$\omega_c = 1 \text{ rad/sample}$$

$$H_d(\omega) = \begin{cases} e^{-j\omega\tau}, & |\omega| \le \omega_c \\ 0, & \text{elsewhere} \end{cases} = \begin{cases} e^{-j\omega\tau}, & -1 \le \omega \le 1 \\ 0, & \text{elsewhere} \end{cases} \longrightarrow (1)$$

We know that, $h(n) = h_d(n).\,\omega(n) \longrightarrow (2)$

where $\omega(n) = \begin{cases} 1, & 0 \le n \le 6 \\ 0, & \text{elsewhere} \end{cases} \longrightarrow (3)$

W.K.T, the desired unit sample response is

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\, e^{+j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-1}^{1} e^{-j\omega\tau} \cdot e^{j\omega n}\, d\omega \longrightarrow (4)$$

$$\Rightarrow h_d(n) = \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\tau)}}{j(n-\tau)}\right]_{-1}^{1} = \frac{\sin(n-\tau)}{\pi(n-\tau)} \quad \text{for } n \neq \tau$$

If $n = \tau$ then

$$h_d(n) = \frac{1}{2\pi}\int_{-1}^{1} 1 \cdot d\omega = \frac{1}{\pi}$$

$$\therefore h_d(n) = \begin{cases} \frac{\sin(n-\tau)}{\pi(n-\tau)} & \text{for } n \neq \tau \\ \frac{1}{\pi} & \text{for } n = \tau \end{cases} \longrightarrow (5)$$

To determine the value of 'τ' :-

Given that, filter is symmetric. So,

$$h_d(n) = h_d(M-1-n)$$

$$\Rightarrow h_d(n) = h_d(6-n)$$

using eqn (5),

$$\frac{\sin(n-\tau)}{\pi(n-\tau)} = \frac{\sin(6-n-\tau)}{\pi(6-n-\tau)} \longrightarrow (6)$$

This equation will be satisfied, when

$$-(n-\tau) = 6-n-\tau$$

$$\Rightarrow \cancel{-n}+\tau = 6\cancel{-n}-\tau$$

$$\Rightarrow 2\tau = 6$$

$$\Rightarrow \tau = 3$$

$$\therefore h_d(n) = \begin{cases} \frac{\sin(n-3)}{\pi(n-3)} & \text{for } n \neq 3 \\ 1/\pi & \text{for } n = 3 \end{cases} \longrightarrow (7)$$

using ②, ③ & ⑦,

$$h(n) = \begin{cases} \dfrac{\sin(n-3)}{\pi(n-3)} & ; \ 0 \le n \le 6 \quad \text{and} \quad n \ne 3 \\ \dfrac{1}{\pi} & ; \ n=3 \end{cases}$$

If $n=0$:- $h(0) = \dfrac{\sin(-3)}{\pi(-3)} = 0.01497$

$n=1$:- $h(1) = \dfrac{\sin(-2)}{\pi(-2)} = 0.14472$

$n=2$:- $h(2) = \dfrac{\sin(-1)}{\pi(-1)} = 0.26785$

$n=3$:- $h(3) = \dfrac{1}{\pi} = 0.31831$

$n=4$:- $h(4) = \dfrac{\sin(1)}{\pi(1)} = 0.26785$

$n=5$:- $h(5) = \dfrac{\sin(2)}{\pi(2)} = 0.14472$

$n=6$:- $h(6) = \dfrac{\sin(3)}{\pi(3)} = 0.01497$

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Eg:4:- Design a filter with $H_d(e^{j\omega}) = \begin{cases} e^{-j3\omega} & ; -\frac{\pi}{4} \le \omega \le \frac{\pi}{4} \\ 0 & , \frac{\pi}{4} \le |\omega| \le \pi \end{cases}$

using a Hamming window with $N=7$.

Sol:- Given that,

$$H_d(e^{j\omega}) = \begin{cases} e^{-j3\omega} & , -\frac{\pi}{4} \le \omega \le \frac{\pi}{4} \\ 0 & , \frac{\pi}{4} \le |\omega| \le \pi \end{cases}$$

The filter coefficients are given by

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega) \cdot e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi/4}^{\pi/4} e^{-j3\omega} \cdot e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi/4}^{\pi/4} e^{j\omega(n-3)} \, d\omega$$

$$= \frac{1}{2\pi} \left[\frac{e^{j\omega(n-3)}}{j(n-3)} \right]_{-\pi/4}^{\pi/4}$$

$$= \frac{1}{2\pi} \left[\frac{e^{j\pi/4(n-3)} - e^{-j\pi/4(n-3)}}{j(n-3)} \right]$$

$$h_d(n) = \frac{\sin \pi(n-3)/4}{\pi(n-3)}$$

$$\cancel{\frac{1}{4} \times \frac{\sin \pi(n-3)/4}{\pi(n-3)/4}}$$

According to L-Hospital rule, $\lim_{\theta \to 0} \frac{\sin A\theta}{\theta} = A$

$h_d(n) = \frac{1}{4}$, when $n=3$

$$\therefore h_d(n) = \begin{cases} \frac{\sin \pi(n-3)/4}{\pi(n-3)} & ; n=0 \text{ to } 6 \text{ but } n \neq 3 \\ \frac{1}{4} & ; n=3 \end{cases}$$

The filter Coefficients are given by

If $n=0$, $h_d(0) = \frac{\sin \pi(-3)/4}{\pi(-3)} = 0.075$

$n=1$, $h_d(1) = \frac{\sin \pi(-2)/4}{\pi(-2)} = 0.159$

$n=2$, $h_d(2) = \frac{\sin \pi(-1)/4}{\pi(-1)} = 0.225$; $n=3$, $h_d(3) = 1/4$

$n=4$, $h_d(4) = \frac{\sin \pi/4}{\pi} = 0.225$

$n=5$, $h_d(5) = \frac{\sin 2\pi/4}{\pi(2)} = 0.159$,

$n=6$, $h_d(6) = \frac{\sin 3\pi/4}{\pi(3)} = 0.075$

The hamming window function of a Causal filter is:

$$\omega(n) = \begin{cases} 0.54 - 0.46 \cos \frac{2\pi n}{N-1} & ; 0 \le n \le N-1 \\ 0 & ; \text{otherwise} \end{cases}$$

Here $N=7$,

If $n=0$, $\omega(0) = 0.54 - 0.46 = 0.08$

$n=1$, $\omega(1) = 0.54 - 0.46 \cos\left(\frac{2\pi}{6}\right) = 0.31$

$n=2$, $\omega(2) = 0.54 - 0.46 \cos\left(\frac{4\pi}{6}\right) = 0.77$

$n=3$, $\omega(3) = 0.54 - 0.46 \cos\left(6\pi/6\right) = 1$

$n=4$, $\omega(4) = 0.54 - 0.46 \cos\left(\frac{8\pi}{6}\right) = 0.77$

$n=5$, $\omega(5) = 0.54 - 0.46 \cos\left(\frac{10\pi}{6}\right) = 0.31$

$n=6$, $\omega(6) = 0.54 - 0.46 \cos\left(\frac{12\pi}{6}\right) = 0.08$

The filter Coefficients of resultant filter are

$$h(n) = h_d(n) . \omega(n)$$

$h(0) = h_d(0) . \omega(0) = 0.006$; $h(4) = h_d(4) . \omega(4) = 0.173$

$h(1) = h_d(1) . \omega(1) = 0.049$; $h(5) = h_d(5) . \omega(5) = 0.049$

$h(2) = h_d(2) . \omega(2) = 0.173$; $h(6) = h_d(6) . \omega(6) = 0.006$

$h(3) = h_d(3) . \omega(3) = 0.25$;

The frequency response of Causal filter is given by

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n) . e^{-j\omega n}$$

$$= h(0) + h(1) e^{-j\omega} + h(2) e^{-j2\omega} + h(3) e^{-j3\omega} + h(4) e^{-j4\omega} + h(5) e^{-j5\omega} + h(6) e^{-j6\omega}$$

$$= e^{-j3\omega} \left[h(0) e^{j3\omega} + h(1) e^{j2\omega} + h(2) e^{j\omega} + h(3) + h(4) e^{-j\omega} + h(5) e^{-j2\omega} + h(6) e^{-j3\omega} \right]$$

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$$= e^{-j3\omega}\left[0.006e^{j3\omega} + 0.049e^{j2\omega} + 0.173e^{j\omega} + 0.25 + 0.173e^{-j\omega} + 0.049e^{-j2\omega} + 0.006e^{-j3\omega}\right]$$

$$= e^{-j3\omega}\cdot\left[0.25 + 0.173\left[e^{j\omega}+e^{-j\omega}\right] + 0.049\left[e^{j2\omega}+e^{-j2\omega}\right] + 0.006\left[e^{j3\omega}+e^{-j3\omega}\right]\right]$$

$$\Rightarrow H_d(e^{j\omega}) = e^{-j3\omega}\cdot\left[0.25 + 0.346\cos\omega + 0.094\cos 2\omega + 0.012\cos 3\omega\right]$$

The transfer function of the digital FIR filter is

$$H(Z) = \sum_{n=0}^{N-1} h(n)Z^{-n}$$

$$= h(0) + h(1)Z^{-1} + h(2)Z^{-2} + h(3)Z^{-3} + h(4)Z^{-4} + h(5)Z^{-5} + h(6)Z^{-6}$$

$$\Rightarrow H(Z) = 0.006 + 0.049Z^{-1} + 0.173Z^{-2} + 0.25Z^{-3} + 0.173Z^{-4} + 0.049Z^{-5} + 0.006Z^{-6}$$

=o=

Eg.5:- Design a HPF using Hamming window, with a cutoff frequency of 1.2 rad/sec and $N=9$.

Sol:- The desired frequency response $H_d(e^{j\omega})$ for a HPF is

$$H_d(e^{j\omega}) = \begin{cases} 0 & ; -\omega_c \le \omega \le \omega_c \\ e^{-j\omega\alpha} & ; \omega_c \le |\omega| \le \pi \end{cases}$$

The desired impulse response is $h_d(n)$ is obtained by

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)e^{+j\omega n}d\omega$$

$$\Rightarrow h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{-\omega_c} e^{-j\omega\alpha}\cdot e^{j\omega n}d\omega + \frac{1}{2\pi}\int_{\omega_c}^{\pi} e^{-j\omega\alpha}\cdot e^{j\omega n}d\omega$$

$$\Rightarrow h_d(n) = \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{-\pi}^{-\omega_c} + \frac{1}{2\pi}\cdot\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{\omega_c}^{\pi}$$

$$\therefore h_d(n) = \frac{1}{2\pi}\left[\frac{e^{-j\omega_c(n-\alpha)} - e^{-j\pi(n-\alpha)}}{j(n-\alpha)}\right] + \frac{1}{2\pi}\left[\frac{e^{j\pi(n-\alpha)} - e^{j\omega_c(n-\alpha)}}{j(n-\alpha)}\right]$$

$$= \frac{1}{2\pi}\cdot\frac{1}{(n-\alpha)}\cdot\left[\frac{e^{j\pi(n-\alpha)} - e^{-j\pi(n-\alpha)}}{j} - \frac{e^{j\omega_c(n-\alpha)} - e^{-j\omega_c(n-\alpha)}}{j}\right]$$

$$\Rightarrow h_d(n) = \frac{1}{\pi(n-\alpha)}\cdot\left[\sin(n-\alpha)\pi - \sin(n-\alpha)\omega_c\right]$$

If $n=\alpha$, then by using L-Hospital's rule.

$$h_d(n) = 1 - \frac{\omega_c}{\pi}$$

$$\therefore h_d(n) = \begin{cases} \frac{1}{\pi(n-\alpha)}\cdot\left[\sin(n-\alpha)\pi - \sin(n-\alpha)\omega_c\right] & ; n \neq \alpha \\ 1 - \frac{\omega_c}{\pi} & ; n = \alpha \end{cases}$$

W.K.T, $\alpha = \frac{N-1}{2} = \frac{9-1}{2} = 4$ and $\omega_c = 1.2$ rad/sec

Here 'n' & 'α' are integers, So $(n-\alpha)$ is also integer hence

$$\sin(n-\alpha)\pi = 0$$

$$\therefore h_d(n) = \begin{cases} \frac{1}{\pi(n-4)}\cdot(-\sin 1.2(n-4)) & ; 0 \le n \le 8 \text{ \& } n \neq 4 \\ 1 - \frac{1.2}{\pi} = 0.618 & ; n = 4 \end{cases}$$

If $n=0$, $h_d(0) = 0.0792$; $n=5$, $h_d(5) = -0.2966$

$n=1$, $h_d(1) = 0.0469$; $n=6$, $h_d(6) = -0.1075$

$n=2$, $h_d(2) = -0.1075$; $n=7$, $h_d(7) = -0.0469$

$n=3$, $h_d(3) = -0.2966$; $n=8$, $h_d(8) = -0.0792$

$n=4$, $h_d(4) = 0.618$

17

The window sequence for Hamming window is given by

$$w(n) = \begin{cases} 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right) & , 0 \le n \le N-1 \\ 0 & , \text{otherwise} \end{cases}$$

$n=0, \quad w(0) = 0.08$	$n=5, \quad w(5) = 0.8652$
$n=1, \quad w(1) = 0.2147$	$n=6, \quad w(6) = 0.54$
$n=2, \quad w(2) = 0.54$	$n=7, \quad w(7) = 0.2147$
$n=3, \quad w(3) = 0.8652$	$n=8, \quad w(8) = 0.08$
$n=4, \quad w(4) = 1$	

The filter coefficients are $h(n) = h_d(n).\,w(n)$

If $n=0, \quad h(0) = 0.0063$	$n=5, \quad h(5) = -0.2566$
$n=1, \quad h(1) = 0.01$	$n=6, \quad h(6) = -0.058$
$n=2, \quad h(2) = -0.058$	$n=7, \quad h(7) = 0.01$
$n=3, \quad h(3) = -0.2566$	$n=8, \quad h(8) = 0.0063.$
$n=4, \quad h(4) = 0.618$	

The frequency response of the filter is

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n)\, e^{-j\omega n}$$

$$= h(0) + h(1)e^{-j\omega} + h(2)e^{-j2\omega} + h(3)e^{-j3\omega} + h(4)e^{-j4\omega} + h(5).e^{-j5\omega} + h(6)e^{-j6\omega} + h(7)e^{-j7\omega} + h(8)e^{-j8\omega}$$

$$= e^{-j4\omega}.\Big[h(0).e^{j4\omega} + h(1)e^{j3\omega} + h(2)e^{j2\omega} + h(3)e^{j\omega} + h(4) + h(5)e^{-j\omega} + h(6)e^{-j2\omega} + h(7)e^{-j3\omega} + h(8)e^{-j4\omega}\Big]$$

$$= e^{-j4\omega}.\begin{bmatrix} 0.0063\left[e^{j4\omega} + e^{-j4\omega}\right] + 0.01\left[e^{j3\omega} + e^{-j3\omega}\right] \\ - 0.058\left[e^{j2\omega} + e^{-j2\omega}\right] - 0.2566\left[e^{j\omega} + e^{-j\omega}\right] + 0.618 \end{bmatrix}$$

$$\Rightarrow H(e^{j\omega}) = e^{-j4\omega}\begin{bmatrix} 0.618 - 0.5132\cos\omega - 0.116\cos 2\omega + 0.02\cos 3\omega \\ + 0.0126\cos 4\omega \end{bmatrix}$$

The transfer function of filter is given by

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n} = \sum_{n=0}^{8} h(n) z^{-n}$$

$$= \cancel{h(4) + \sum_{n=0}^{3} h(n)\left[z^{n} + z^{-n}\right]}$$

$$= h(0) + h(1)z^{-1} + h(2)z^{-2} + \dots + h(8)z^{-8}$$

$$= z^{-4}\left[h(4) + h(0)z^{4} + h(8)z^{-4} + h(1)z^{3} + h(7)z^{-3} + h(2)z^{2} + h(6)z^{-2} + h(3)z^{1} + h(5)z^{-1}\right]$$

$$\Rightarrow H(z) = z^{-4}\left[0.618 - 0.2566(z+z^{-1}) - 0.058(z^{2}+z^{-2}) + 0.01(z^{3}+z^{-3}) + 0.0063(z^{4}+z^{-4})\right]$$

=0=

Eg-6:- Design a band pass filter to pass frequencies in the range 1 to 2 rad/sec using Hanning window with $N=5$.

Sol:- The desired frequency response $H_d(e^{j\omega})$ for band pass filter is

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha} & ; -\omega_{c_2} \le \omega \le -\omega_{c_1} \ \& \ \omega_{c_1} \le \omega \le \omega_{c_2} \\ 0 & ; \text{otherwise} \end{cases}$$

Given that, the band pass filter has to pass frequencies in the range 1 to 2 rad/sec. Therefore,

$\omega_{c_1} = 1$ rad/sec & $\omega_{c_2} = 2$ rad/sec.

The desired impulse response is

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_{c2}}^{-\omega_{c1}} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega + \frac{1}{2\pi}\int_{\omega_{c1}}^{\omega_{c2}} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-2}^{-1} e^{-j\omega\alpha}\cdot e^{j\omega n}\, d\omega + \frac{1}{2\pi}\int_{1}^{2} e^{-j\omega\alpha}\cdot e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\cdot\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{-2}^{-1} + \frac{1}{2\pi}\cdot\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{1}^{2}$$

$$= \frac{1}{2\pi(n-\alpha)j}\left[e^{-j(n-\alpha)} - e^{-2j(n-\alpha)} + e^{j2(n-\alpha)} - e^{j(n-\alpha)}\right]$$

$$= \frac{1}{\pi(n-\alpha)}\cdot\left[\frac{e^{j2(n-\alpha)} - e^{-j2(n-\alpha)}}{2j} - \frac{e^{j(n-\alpha)} - e^{-j(n-\alpha)}}{2j}\right]$$

$$\Rightarrow h_d(n) = \frac{1}{\pi(n-\alpha)}\cdot\left[\sin 2(n-\alpha) - \sin(n-\alpha)\right] \quad \text{for } n \neq \alpha,$$

If $n=\alpha$, according to L-Hospital's rule,

$$h_d(n) = \lim_{(n-\alpha)\to 0} \frac{1}{\pi(n-\alpha)}\left[\sin 2(n-\alpha) - \sin(n-\alpha)\right]$$

$$= \frac{1}{\pi}\left[2-1\right]$$

$$= \frac{1}{\pi}$$

$$\therefore h_d(n) = \begin{cases} \frac{1}{\pi(n-\alpha)}\cdot\left[\sin 2(n-\alpha) - \sin(n-\alpha)\right] & ; n \neq \alpha \\ \frac{1}{\pi} & ; n = \alpha \end{cases}$$

W.K.T, $\alpha = \frac{N-1}{2} = \frac{5-1}{2} = 2$

$$H(e^{j\omega}) = e^{-j2\omega} \cdot \left[0.3183 + 0.0216 \cos\omega\right]$$

The transfer function of Digital FIR BPF is

$$H(Z) = \sum_{n=0}^{N-1} h(n) Z^{-n} = \sum_{n=0}^{4} h(n) Z^{-n}$$

$$= h(0) + h(1) Z^{-1} + h(2) Z^{-2} + h(3) Z^{-3} + h(4) Z^{-4}$$

$$\therefore H(Z) = 0.0108 Z^{-1} + 0.3183 Z^{-2} + 0.0108 Z^{-3}$$

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⑥. Kaiser window :-

From the Comparison table, the main lobe width is inversly ~~to~~ proportional to 'N'. As 'N' increases, the transition band becomes decreases. However, the minimum stop band attenuation is independent of 'N' and is a function of selected window. Thus, in order to achieve prescribed minimum stopband attenuation and pass band ripple, the designer must find a window with an appropriate side lobe level and then choose 'N' to ~~at~~ achieve the prescribed transition width.

In this process, the designer may have to settle for a new window with undesirable design specifications. To overcome this problem, Kaiser has choosen a class of windows based on the prolate spheroidal functions.

The Kaiser window function is given by

$$w_K(n) = \begin{cases} \dfrac{I_0(\beta)}{I_0(\alpha)} & \text{for } |n| \le \dfrac{N-1}{2} \\ 0 & ; \text{otherwise} \end{cases}$$

Where α = adjustable parameter i.e., side lobe level can be controlled w.r.t. the main lobe peak

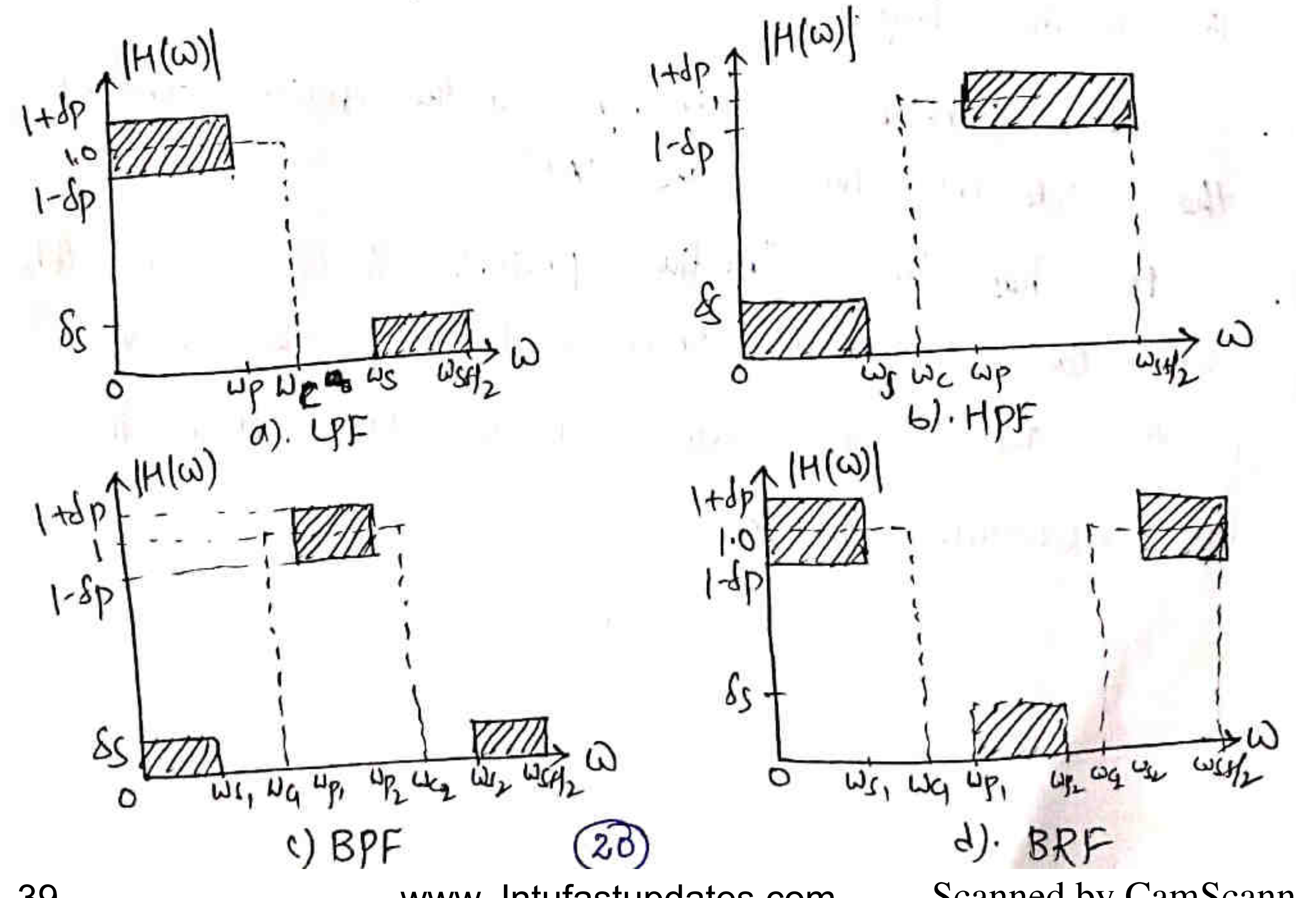
$$\beta = \alpha \left[1 - \left(\frac{2n}{N-1}\right)^2\right]^{1/2}$$

$I_0(x)$ = Modified zeroth order bessel function

$$= 1 + \sum_{K=1}^{\alpha} \left[\frac{1}{K!}\left(\frac{x}{2}\right)^K\right]^2$$

$$= 1 + \frac{0.25x^2}{(1!)^2} + \frac{(0.25x^2)^2}{(2!)^2} + \frac{(0.25x^2)^3}{(3!)^2} + \cdots$$

Fig. shows the idealized frequency responses of different filters with their pass band and stop band specifications.



From the specifications, the actual pass band ripple (α_p) and stop band ripple (α_s) are given by

$$\alpha_p = 20 \cdot \log_{10} \frac{1+\delta_p}{1-\delta_p} \text{ dB} \quad \& \quad \alpha_s = -20 \cdot \log_{10} \delta_s \text{ dB}$$

The transition bandwidth is

$$\Delta F = f_s - f_p = \omega_s - \omega_p$$

Let α_p' & α_s' be the specified pass band & stop band attenuation respectively and

$$\alpha_p \leq \alpha_p' \quad \text{and} \quad \alpha_s \leq \alpha_s'$$

where 'α_p' & 'α_s' are the actual pass band peak to peak ripple & minimum stop band attenuation respectively.

Advantages of Kaiser window :-

1). In Kaiser window, the pass band & stopband ripple size are not affected much by the change in the window length.

2). It provides flexibility for the designer to select the side lobe level and 'N'.

3). It has the attractive property that the side lobe level can be varied continuously from the low value in the Blackmann window to the high value in the rectangular window.

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Design of Linear phase FIR Filters using Frequency Sampling Method :-

Let us consider the desired frequency response of FIR filter is $H_d(\omega)$.

The frequency response is sampled uniformly at 'N' points. Such frequency response samples are given at

$$\omega_K = \frac{2\pi}{N}K \quad , K=0,1,2,\dots N-1 \longrightarrow (a)$$

Such sampled frequency response is "Discrete Fourier Transform (DFT)".

It may be denoted by $H(K)$ i.e.,

$$H(K) = H_d(\omega)\Big|_{\omega=\omega_K} = H_d\left(\frac{2\pi}{N}K\right) \quad , K=0,1,2,\dots N-1 \longrightarrow (b)$$

Hence, $H(K)$ is called N-Point DFT.

The unit sample response $h(n)$ of FIR filter is

$$h(n) = \frac{1}{N}\sum_{K=0}^{N-1} H(K)\cdot e^{\frac{j2\pi Kn}{N}} \quad , n=0,1,\dots N-1 \longrightarrow (c)$$

Hence, the unit sample response of FIR filter of length 'N' is obtained using frequency sampling technique.

For the FIR filter to be realizable, the coefficients $h(n)$ should be real.

Let us consider, $H(N-K)\cdot e^{j2\pi n(N-K)/N} = H(N-K)\cdot e^{\frac{-j2\pi nK}{N}}$

usually, $|H(N-K)| = |H(K)| \longrightarrow (d)$

Hence, $H(N-K)\cdot e^{j2\pi n(N-K)/N} = H(K)\cdot e^{\frac{-j2\pi nK}{N}} \longrightarrow (e)$

using the relation of complex conjugate terms, the equation (c) is simplified to

$$h(n) = \frac{1}{N}\left[H(0) + 2.\sum_{K=1}^{P} Re\left\{H(K).e^{\frac{j2\pi Kn}{N}}\right\}\right] \longrightarrow (f)$$

where $P = \begin{cases} \frac{N-1}{2} & ; \text{if } N \text{ is odd} \\ \frac{N}{2}-1 & ; \text{if } N \text{ is even.} \end{cases}$

The above expression used to compute coefficients of FIR filter.

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Eg-7:- Design a low pass FIR filter using frequency sampling technique having cut-off frequency of $\pi/2$ rad/sample. The filter should have linear phase & and length of 17.

Sol:- Given that, $\omega_c = \pi/2$ rad/sample & $N=17$.

For linear phase ~~filter~~ FIR filter, the desired frequency response is

$$H_d(\omega) = \begin{cases} e^{-j\omega\left(\frac{N-1}{2}\right)} & ; 0 \le \omega \le \omega_c \\ 0 & ; \omega_c \le \omega \le \pi \end{cases} = \begin{cases} e^{-j\omega 8} & ; 0 \le \omega \le \pi/2 \\ 0 & ; \pi/2 \le \omega \le \pi \end{cases} \longrightarrow (1)$$

Now to sample $H_d(\omega)$, substitute $\omega = \frac{2\pi K}{N}$, $K=0,1,\ldots N-1$

For $N=17$, $\omega = \frac{2\pi K}{17}$, $K=0,1,2,\ldots 16$ $\longrightarrow (2)$

we obtain,

$$H(K) = H_d(\omega)\Big|_{\omega=\frac{2\pi K}{17}} = \begin{cases} e^{-j\frac{2\pi K}{17}8} & ; 0 \le \frac{2\pi K}{17} \le \pi/2 \\ 0 & ; \frac{\pi}{2} \le \frac{2\pi K}{17} \le \pi \end{cases} = \begin{cases} e^{-j\frac{16\pi K}{17}} & ; 0 \le K \le 4 \\ 0 & ; 5 \le K \le 8 \end{cases}$$

$\longrightarrow (3)$

Here $N=17$, using eqn ①.

$$h(n) = \frac{1}{17}\left[H(0) + 2.\sum_{K=1}^{8} Re\left\{H(K).e^{j\frac{2\pi Kn}{17}}\right\}\right]$$

using eqn ③,

$$h(n) = \frac{1}{17}\left[1 + 2.\sum_{K=1}^{4} Re\left\{e^{\frac{-j16\pi K}{17}}.e^{j\frac{2\pi Kn}{17}}\right\}\right]$$

$$\Rightarrow h(n) = \frac{1}{17}\left[1+2.\sum_{K=1}^{4} Re\left\{e^{j2\pi K(n-8)/17}\right\}\right]$$

∴ This is the unit sample response of the FIR filter obtained using frequency sampling technique.

W.K.T, $Re\left[e^{j\theta}\right] = Re\left[\cos\theta + j\sin\theta\right] = \cos\theta$

$$\therefore h(n) = \frac{1}{17}\left[1+2.\sum_{K=1}^{4}\cos\left(\frac{2\pi K(n-8)}{17}\right)\right], \quad n=0,1,2,\dots 16$$

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Eg-8:- Find the impulse response $h(n)$ of a filter having desired frequency response which is given by

$$H_d(e^{j\omega}) = \begin{cases} e^{-j(N-1)\omega/2} & \text{for } 0 \le |\omega| \le \pi/2 \\ 0 & \text{for } \frac{\pi}{2} \le |\omega| \le \pi \end{cases}$$

It is given that $N=7$. use Frequency sampling approach.

Sol:- Given that, $N=7$ and

$$H_d(e^{j\omega}) = \begin{cases} e^{-j(N-1)\omega/2} \\ 0 \end{cases} = \begin{cases} e^{-j3\omega} & ; 0 \le |\omega| \le \pi/2 \\ 0 & ; \frac{\pi}{2} \le |\omega| \le \pi \end{cases}$$

To sample $H_d(e^{j\omega})$. Let us substitute $\omega = \frac{2\pi K}{7}$, $K=0,1,\dots 6$

(22)

$$\therefore H(K) = \begin{cases} e^{-j3 \times \frac{2\pi K}{7}} & ; \quad 0 \le \frac{2\pi K}{7} \le \pi/2 \\ 0 & ; \quad \frac{\pi}{2} \le \frac{2\pi K}{7} \le \pi \end{cases}$$

$$= \begin{cases} e^{-j\frac{6\pi K}{7}} & : 0 \le K \le 7/4 \\ 0 & : \frac{7}{4} \le K \le 7/2 \end{cases} = \begin{cases} e^{-j\frac{6\pi K}{7}} & ; 0 \le K \le 2 \\ 0 & ; 2 \le K \le 4 \end{cases}$$

Let us obtain $h(n)$,

$$h(n) = \frac{1}{N}\left\{ H(0) + 2 \cdot \sum_{K=1}^{3} Re\left[H(K) \cdot e^{j\frac{2\pi K n}{7}} \right] \right\}$$

$$\Rightarrow h(n) = \frac{1}{7}\left\{ 1 + 2 \cdot \sum_{K=1}^{2} Re\left[e^{-j\frac{6\pi K}{7}} \cdot e^{j\frac{2\pi K n}{7}} \right] \right\}$$

$$= \frac{1}{7} \cdot \left\{ 1 + 2 \cdot \sum_{K=1}^{2} Re\left[e^{j2\pi K(n-3)/7} \right] \right\}$$

$$\Rightarrow h(n) = \frac{1}{7}\left\{ 1 + 2 \cdot \sum_{K=1}^{2} \cos\left[\frac{2\pi K(n-3)}{7} \right] \right\} \quad , n = 0, 1, 2, \cdots 6$$

$\therefore$ This is the required unit sample response of the desired filter

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Comparison between IIR & FIR filters:-

IIR filters	FIR filters
①. All the infinite samples of impulse response are considered.	①. only a finite number of samples of impulse response are considered.
②. IIR filters are easily realized recursively	②. FIR filters are easily realized recursively and Non-recursively.
③. Linear phase characteristics Can't be achieved.	③. Linear phase characteristics Can be achieved.
④. The impulse response Cann't be directly converted to digital filter transfer function.	④. The impulse response Can be directly converted to digital filter transfer function.
⑤. The specifications include the desired characteristics for magnitude response only.	⑤. The specifications include the desired characteristics for both magnitude & phase responses.
⑥. The design involves design of analog filter and then transforming analog filter to digital filter.	⑥. The digital filter can be directly designed to achieve the desired specifications.
⑦. The round off noise in IIR filters is more	⑦. The round off noise in FIR filters is less mainly because feedback is not used.

(23).

Basic structures of FIR systems :-

The direct W.K.T, FIR stands for finite impulse response. An FIR filter doesn't have feedback. The difference equation of an FIR filter is given by

$$y(n) = \sum_{K=0}^{M-1} b_K \cdot x(n-K). \longrightarrow (1)$$

$$\Rightarrow Y(Z) = \sum_{K=0}^{M-1} b_K \cdot Z^{-K} \cdot X(Z)$$

$$\Rightarrow H(Z) = \frac{Y(Z)}{X(Z)} = \sum_{K=0}^{M-1} b_K \cdot Z^{-K}$$

by taking inverse Z-transform, we get sample response of FIR filter. i.e.,

$$h(n) = \begin{cases} b_n & ; \text{ for } 0 \le n \le M-1 \\ 0 & ; \text{ otherwise} \end{cases} \longrightarrow (2)$$

* Direct form structure (or) Transversal structure :-

The linear convolution sum is defined as

$$y(n) = \sum_{K=0}^{M-1} h(K) \cdot x(n-K)$$

$$\Rightarrow y(n) = h(0) \cdot x(n) + h(1) \cdot x(n-1) + h(2) \cdot x(n-2) + \ldots + \cancel{h(K) \cdot x(n-K)} + h(M-1) \cdot x(n-M+1).$$

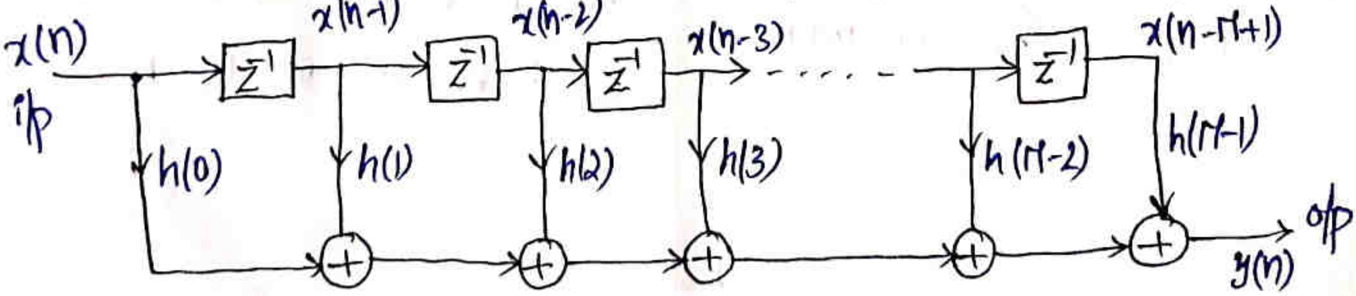

Fig:- Direct form realization of FIR filter

Cascade form structure for FIR filters:-

W.K.T., $$H(Z) = \sum_{K=0}^{M-1} b_K Z^{-K} \longrightarrow (1)$$

Basically, the higher order FIR filter is realized by using a series connection of different FIR sections. Here each section is characterised by second order transfer function. Then due to cascade connection, the transfer function $H(Z)$ will be multiplication of all second order transfer functions i.e.,

$$H(Z) = H_1(Z).\ H_2(Z)\ldots\ H_K(Z). \longrightarrow (2)$$

Here $H_K(Z)$ = second order transfer function

$$= b_{K0} + b_{K1} Z^{-1} + b_{K2} \cdot Z^{-2} \longrightarrow (3)$$

$$\Rightarrow Y_K(Z) = b_{K0} \cdot X_K(Z) + b_{K_1} \cdot Z^{-1} \cdot X_K(Z) + b_{K_2} \cdot Z^{-2} \cdot X_K(Z)$$

taking inverse Z-transform,

$$y_K(n) = b_{K0} \cdot x_K(n) + b_{K_1} \cdot x_K(n-1) + b_{K_2} \cdot x_K(n-2) \longrightarrow (4)$$

From eqn (4), Direct form realization of second order section as shown in fig.

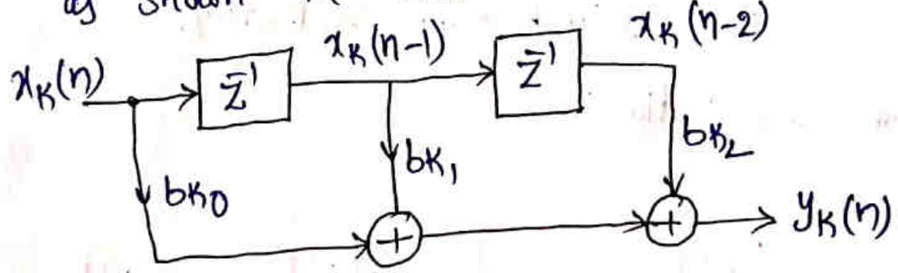

Fig:- Direct form realization of second order section.

The complete cascade realization structure for eqn. (2) is shown in fig.

(24)

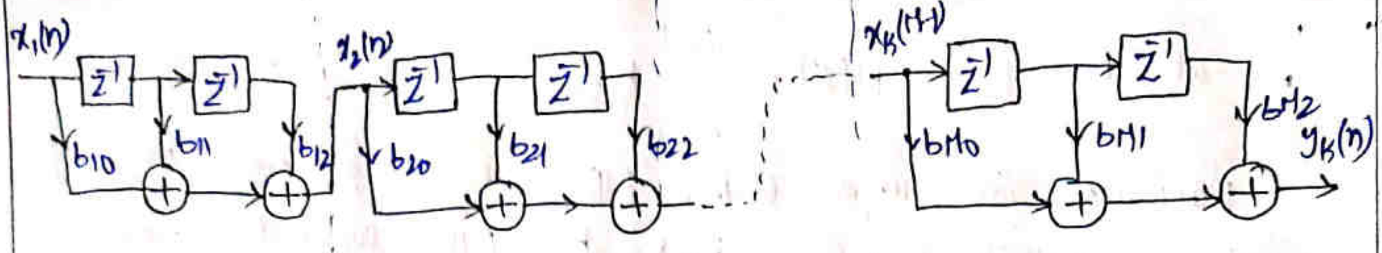

Fig :- Cascade realization of FIR filter.

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Eg :- Determine the Direct form and Cascade form realizations for the transfer function of an FIR filter which is given by

$$H(z) = \underbrace{\left(1 - \frac{1}{4}z^{-1} + \frac{3}{8}z^{-2}\right)}_{H_1(z)} \underbrace{\left(1 - \frac{1}{8}z^{-1} - \frac{1}{2}z^{-2}\right)}_{H_2(z)}$$

Sol :-

Direct form realization :-

$$H(z) = 1 - \frac{3}{8}z^{-1} - \frac{3}{32}z^{-2} + \frac{5}{64}z^{-3} - \frac{3}{16}z^{-4}$$

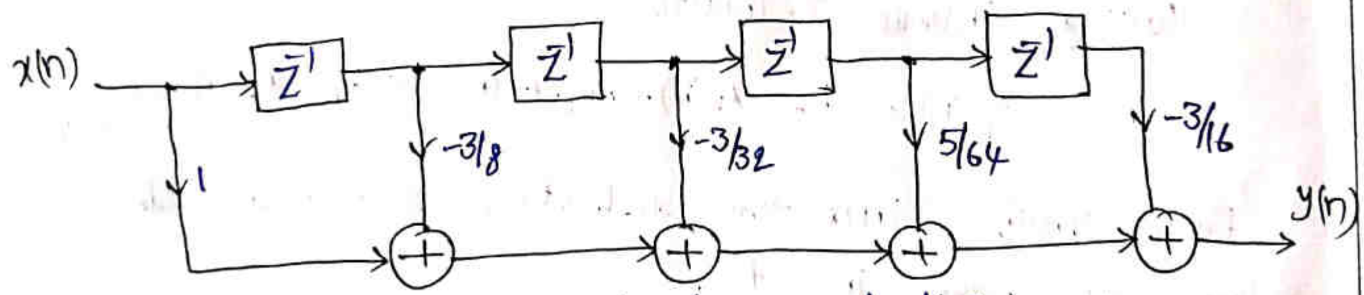

Fig :- Direct form realization.

Cascade form realization :-

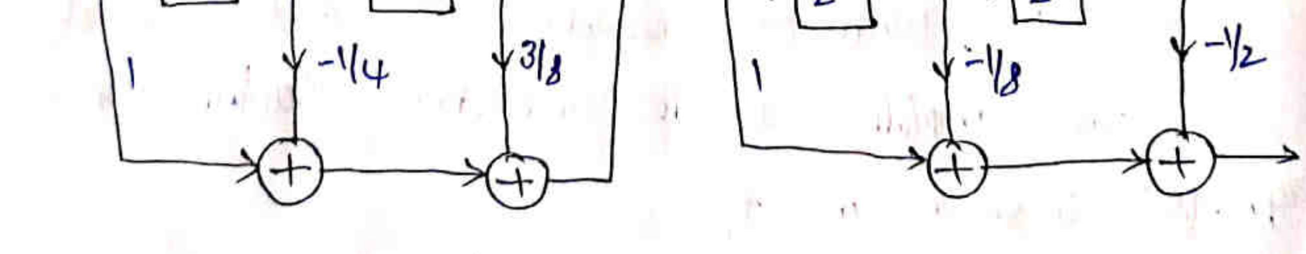

Fig :- Cascade form realization

Lattice structure of an FIR filter :-

- Let us consider an FIR filter with system function

$$H(Z) = A_m(Z) = 1 + \sum_{K=1}^{m} \alpha_m(K) \cdot Z^{-K} \quad ; m \geq 1 \longrightarrow (1)$$

$$\Rightarrow \quad Y(Z) = X(Z) \cdot \left[1 + \sum_{K=1}^{m} \alpha_m(K) Z^{-K}\right].$$

Applying Inverse Z-T,

$$\Rightarrow \quad y(n) = x(n) + \sum_{K=1}^{m} \alpha_m(K) \cdot x(n-K) \longrightarrow (2)$$

Interchange the role of input & o/p.

$$x(n) = y(n) + \sum_{K=1}^{m} \alpha_m(K) \cdot y(n-K) \longrightarrow (3)$$

* Eqn (2) represents an FIR system having system function $H(Z) = A_m(Z)$

* Eqn (3) represents an IIR system having system function $H(Z) = \frac{1}{A_m(Z)}$

For an all zero FIR system of order 'M-1',

Let the input $x(n) = f_0(n)$ and
output $y(n) = f_{M-1}(n)$

For m=1, eqn (2) becomes

$$y(n) = x(n) + \alpha_1(1) \cdot x(n-1) \longrightarrow (4)$$

The basic single stage lattice filter is as shown in fig.

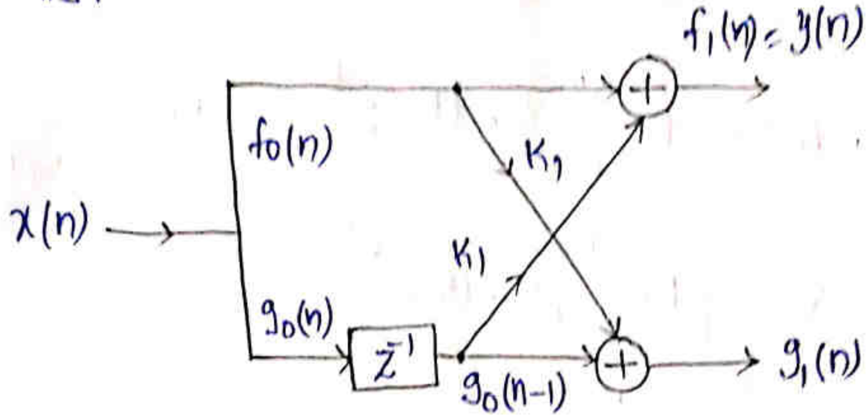

Fig:- Single stage all zero lattice filter.

From fig., $x(n) = f_0(n) = g_0(n)$

$$y(n) = f_1(n) = f_0(n) + K_1 . g_0(n-1)$$

$$\Rightarrow \quad y(n) = x(n) + K_1 . x(n-1) \longrightarrow (5)$$

and

$$g_1(n) = K_1 . f_0(n) + g_0(n-1)$$

$$\Rightarrow g_1(n) = K_1 . x(n) + x(n-1) \longrightarrow (6)$$

Comparing eqn (4) & (5),

$$\alpha_1(0) = 1, \quad \alpha_1(1) = K_1$$

<u>for m=2:-</u>

$$y(n) = x(n) + \alpha_2(1) . x(n-1) + \alpha_2(2) . x(n-2) \longrightarrow (7)$$

By cascading two lattice stages as shown in fig. & is possible to obtain the output $y(n)$.

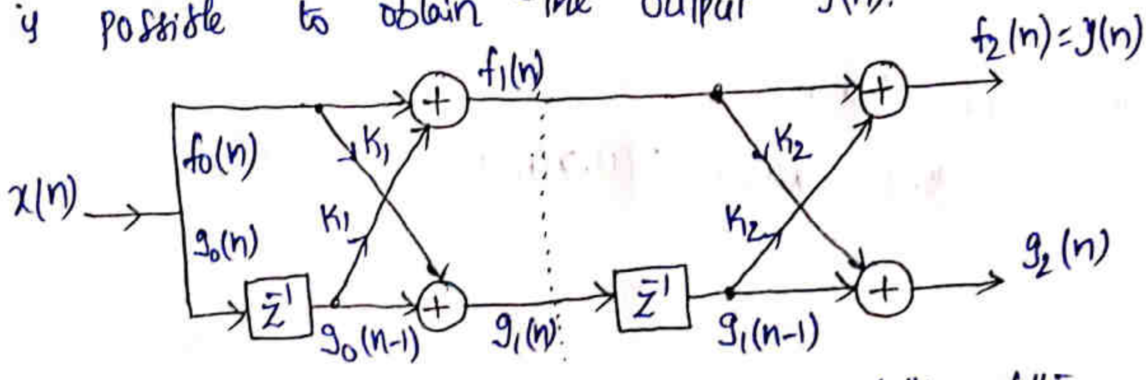

Fig:- Two stage all zero lattice filter.

From fig.,

$$y(n) = f_2(n) = f_1(n) + K_2 \cdot g_1(n-1)$$

using eqn's ⑤ & ⑥.

$$y(n) = f_2(n) = f_0(n) + K_1 \cdot g_0(n-1) + K_2 \cdot \left[K_1 \cdot f_0(n-1) + g_0((n-1)-1)\right]$$

$$= f_0(n) + K_1 \cdot g_0(n-1) + K_1 \cdot K_2 \cdot f_0(n-1) + K_2 \cdot g_0(n-2)$$

$$\Rightarrow y(n) = x(n) + K_1 \cdot x(n-1) + K_1 K_2 \, x(n-1) + K_2 \cdot x(n-2)$$

$$\Rightarrow y(n) = x(n) + K_1 \left[1+K_2\right] \cdot x(n-1) + K_2 \cdot x(n-2) \longrightarrow ⑧$$

Comparing eqns. ⑦ & ⑧,

$$\alpha_2(0) = 1; \quad \alpha_2(1) = K_1(1+K_2) \quad ; \quad \alpha_2(2) = K_2$$

$$= \alpha_1(1)\left[1+\alpha_2(2)\right]$$

Similarly,

$$g_2(n) = K_2 \cdot f_1(n) + g_1(n-1)$$

$$= K_2 \left[f_0(n) + K_1 \cdot g_0(n-1)\right] + K_1 \cdot f_0(n-1) + g_0(n-2)$$

$$= K_2 \cdot f_0(n) + K_1 K_2 \cdot g_0(n-1) + K_1 \cdot f_0(n-1) + g_0(n-2)$$

$$= K_2 \cdot x(n) + K_1 K_2 \, x(n-1) + K_1 \cdot x(n-1) + x(n-2)$$

$$\left.\begin{aligned} &\Rightarrow g_2(n) = K_2 \cdot x(n) + K_1 \cdot (1+K_2) \, x(n-1) + x(n-2) \\ &\Rightarrow g_2(n) = \alpha_2(2) \, x(n) + \alpha_2(1) \cdot x(n-1) + \alpha_2(0) \cdot x(n-2) \end{aligned}\right\} \longrightarrow ⑨$$

From ⑧ & ⑨, observe that the filter coefficients for $f_2(n)$ are $\{1, \alpha_2(1), \alpha_2(2)\}$. While the coefficients for the filter with output $g_2(n)$ are $\{\alpha_2(2), \alpha_2(1), \alpha_2(0)\}$. We note that, these two sets of coefficients are in reverse order.

For $(M-1)^{th}$ stage filter,

$$f_m(n) = f_{m-1}(n) + K_m \cdot g_{m-1}(n-1) \quad , \quad m = 1, 2, \ldots M-1$$

$$g_m(n) = K_m \cdot f_{m-1}(n) + g_{m-1}(n-1) \quad , \quad m = 1, 2, \ldots M-1 \longrightarrow (10)$$

The o/p of $(M-1)^{th}$ stage filter $y(n) = f_{M-1}(n)$

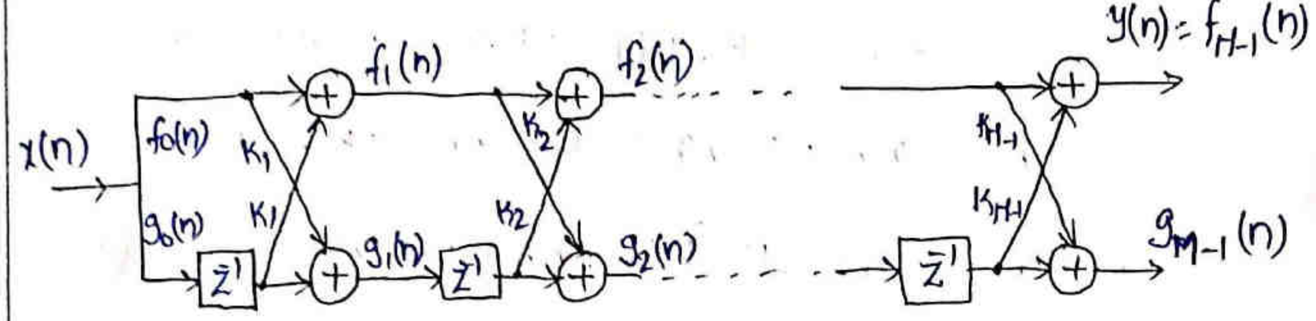

Fig: All zero lattice structure

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