UNIT-IV

Vector Calculus

Scalar point function: If to each point $P(x,y,z)$ of a region E in space there is assigned a real number $u = f(x,y,z)$, Then f is called a scalar point function

Ex: (i) The temperature distribution of an object at any instant

(ii) The density of a body at any instant

Scalar field: A scalar point function defined over some region is called a scalar field.

Vector point function: If to each point $P(x,y,z)$ of a region E in space there is assigned a vector $\bar{a} = \bar{F}(x,y,z)$ then $\bar{F}$ is called a vector point function. Here we can represent $\bar{F}$ as

$$\bar{F} = f_1 \hat{i} + f_2 \hat{j} + f_3 k$$

Ex: (i) The velocity of a moving fluid at any instant.

(ii) The magnetic field generated ~~a~~ by a magnet.

Vector field: A vector point function defined over some region is called a vector field.

If $\bar{F}(x,y,z)$ is a vector point function then

$$\frac{d\bar{F}}{dt} = \frac{\partial \bar{F}}{\partial x}\frac{dx}{dt} + \frac{\partial \bar{F}}{\partial y}\frac{dy}{dt} + \frac{\partial \bar{F}}{\partial z}\frac{dz}{dt}$$

The vector differential operator ∇ is defined as $\nabla = I\frac{\partial}{\partial x} + J\frac{\partial}{\partial y} + K\frac{\partial}{\partial z}$.

Here we shall denote $\bar{R} = xI + yJ + zK$.

Gradient: If $f(x,y,z)$ is a scalar point function then ∇f is a vector point function and is denoted by grad f

ie, $\nabla f = \text{grad } f = I\frac{\partial f}{\partial x} + J\frac{\partial f}{\partial y} + K\frac{\partial f}{\partial z}$.

If $f(x,y,z) = c$ represents a level surface then ∇f is a vector normal to the level surface $f(x,y,z) = c$. and has a magnitude equal to the rate of change of f along this normal. ∇f gives the maximum rate of change of f.

Directional derivative: The directional derivative of a scalar point function f is defined as the resolved part of ∇f in the direction of a given unit vector $\hat{n}$.

ie, directional derivative of $f = \nabla f \cdot \hat{n}$.

Problem(1): Prove that $\nabla r^n = n r^{n-2} \bar{R}$ where $\bar{R} = xI + yJ + zK$ and $r = |\bar{R}| = \sqrt{x^2+y^2+z^2}$.

Solution: Let $f(x,y,z) = r^n = (x^2+y^2+z^2)^{n/2}$

Now $\frac{\partial f}{\partial x} = \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1} 2x = n x r^{n-2}$

$\frac{\partial f}{\partial y} = \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1} 2y = n y r^{n-2}$

$\frac{\partial f}{\partial z} = \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1} 2z = n z r^{n-2}$

$\therefore \nabla f = \nabla r^n = I\frac{\partial f}{\partial x} + J\frac{\partial f}{\partial y} + K\frac{\partial f}{\partial z}$

$= n x r^{n-2} I + n y r^{n-2} J + n z r^{n-2} K$

$= n r^{n-2}(xI + yJ + zK)$

$= n r^{n-2} \bar{R}$

$*\ (x^2+y^2+z^2)^{\frac{n}{2}-1} = (x^2+y^2+z^2)^{\frac{n-2}{2}} = r^{n-2}$

Problem (2): Find $\nabla \phi$, if $\phi = \log(x^2+y^2+z^2)$

Solution: Given $\phi = \log(x^2+y^2+z^2)$

$$\nabla \phi = I \frac{\partial \phi}{\partial x} + J \frac{\partial \phi}{\partial y} + K \frac{\partial \phi}{\partial z}$$

$$= \frac{2x}{x^2+y^2+z^2} I + \frac{2y}{x^2+y^2+z^2} J + \frac{2z}{x^2+y^2+z^2} K$$

$$= \frac{2}{x^2+y^2+z^2}(xI + yJ + zK)$$

$$= \frac{2\bar{R}}{r^2}$$

Problem (3): Find a unit normal vector to the surface $x^3+y^3+ \quad 3xyz = 3$ at the point $(1,2,-1)$.

Solution: Let $f = x^3+y^3+ \quad 3xyz = 3$

Then $$\nabla f = I \frac{\partial f}{\partial x} + J \frac{\partial f}{\partial y} + K \frac{\partial f}{\partial z}$$

$$= (3x^2+3yz) I + (3y^2+3xz) J + (\quad 3xy) K$$

At $(1,2,-1)$, $\nabla f = -3I + 9J + 6K$

This is the vector normal to the given surface at $(1,2,-1)$.

Hence a unit vector normal to the given surface at $(1, 2, -1)$ is $\dfrac{\nabla f}{|\nabla f|}$

$$\text{ie, } \frac{\nabla f}{|\nabla f|} = \frac{-3I + 9J + 6K}{\sqrt{9+81+36}}$$

$$= \frac{-3I + 9J + 6K}{\sqrt{126}}$$

$$= \frac{3(-I + 3J + 2K)}{3\sqrt{14}}$$

$$= \frac{-I + 3J + 2K}{\sqrt{14}}$$

Problem(4): Find the directional derivative of $\phi = x^2yz + 4xz^2$ at the point $(1, -2, 1)$ in the direction of the vector $2I - J - 2K$.

solution: Given $\phi = x^2yz + 4xz^2$

we have $\nabla\phi = I\dfrac{\partial\phi}{\partial x} + J\dfrac{\partial\phi}{\partial y} + K\dfrac{\partial\phi}{\partial z}$

$$\Rightarrow \nabla\phi = (2xyz + 4z^2)I + x^2zJ + (x^2y + 8xz)K$$

$\nabla\phi$ at $(1, -2, 1) = 0I + J + 6K$

unit vector in the direction of $2I - J - 2K$ is

$$\frac{2I - J - 2K}{3}$$

The directional derivative of $\phi = x^2yz + 4xz^2$ at $(1,-2,1)$ in the direction of $2I - J - 2K$ is given by

$$(0I + J + 6K) \cdot \frac{2I - J - 2K}{3}$$

$$= \frac{-1 - 12}{3}$$

$$= -\frac{13}{3}$$

Problem (5): what is the directional derivative of $\phi = xy^2 + yz^3$ at the point $(2,-1,1)$ in the direction of the normal to the surface $x \log z - y^2 = -4$ at $(-1,2,1)$.

Solution: Given $\phi = xy^2 + yz^3$

we have $\nabla\phi = I\frac{\partial \phi}{\partial x} + J\frac{\partial \phi}{\partial y} + K\frac{\partial \phi}{\partial z}$

$$\Rightarrow \nabla\phi = y^2 I + (2xy + z^3) J + 3yz^2 K$$

Now $\nabla\phi$ at $(2,-1,1) = I - 3J - 3K$

Let $f = x \log z - y^2 = -4$

vector normal to $f(x,y,z) = x\log z - y^2 = -4$

is $I\frac{\partial f}{\partial x} + J\frac{\partial f}{\partial y} + K\frac{\partial f}{\partial z}$

ie, $\nabla f = \log z \, I - 2y \, J + \frac{x}{z} K$

$\Rightarrow \nabla f$ at $(-1,2,1) = 0I - 4J - K$

Now, the directional derivative of $\phi = xy^2 + yz^3$ at $(2,-1,1)$ in the direction of $0I - 4J - K$ is given by

$$(I - 3J - 3K) \cdot \frac{0I - 4J - K}{\sqrt{0+16+1}}$$

$$= \frac{12+3}{\sqrt{17}} = \frac{15}{\sqrt{17}}$$

Problem (6): Find the directional derivative of $\phi = x^4 + y^4 + z^4$ at the point $A(1,-2,1)$ in the direction AB where B is $(2,6,-1)$. Also find the maximum directional derivative of ϕ at $(1,-2,1)$.

Solution: Given $\phi = x^4 + y^4 + z^4$.

We have $\nabla\phi = I\frac{\partial\phi}{\partial x} + J\frac{\partial\phi}{\partial y} + K\frac{\partial\phi}{\partial z}$

$\Rightarrow \nabla\phi = 4x^3 I + 4y^3 J + 4z^3 K$

At $A(1,-2,1)$, $\nabla\phi = 4I - 32J + 4K$

vector in the direction of $\overline{AB} = \overline{OB} - \overline{OA}$

ie $\overline{AB} = (2I + 6J - K) - (I - 2J + K)$

$= I + 8J - 2K$

unit vector in the direction of $\bar{AB}$ $= \dfrac{\bar{AB}}{|\bar{AB}|}$

$$= \frac{I + 8J - 2K}{\sqrt{1+64+4}}$$

$$= \frac{I + 8J - 2K}{\sqrt{69}}.$$

now the directional derivative of $\phi = x^4 + y^4 + z^4$

at $(1,-2,1)$ in the direction of $I + 8J - 2K$ is

given by

$$(4I - 32J + 4K) \cdot \frac{I + 8J - 2K}{\sqrt{69}}$$

$$= \frac{4 - 256 - 8}{\sqrt{69}}$$

$$= \frac{-260}{\sqrt{69}}$$

The directional derivative is maximum in the direction of normal to the surface

ie, in the direction of $\nabla \phi$

The maximum value of this directional derivative $= |\nabla \phi|$

$$= |4I - 32J + 4K|$$

$$= \sqrt{16 + 1024 + 16}$$

$$= \sqrt{1056}.$$

Problem (7): Find the directional derivative of $\phi = 5x^2y - 5y^2z + 2.5z^2x$ at the point $P(1,1,1)$ in the direction of the line

$$\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z-0}{1}.$$

Solution: Given $\phi = 5x^2y - 5y^2z + 2.5z^2x$

we have $\nabla\phi = I\frac{\partial\phi}{\partial x} + J\frac{\partial\phi}{\partial y} + K\frac{\partial\phi}{\partial z}$

$$\Rightarrow \nabla\phi = (10xy + 2.5z^2)I + (-10yz + 5x^2)J + (5zx - 5y^2)K$$

$\nabla\phi$ at $P(1,1,1) = 12.5I - 5J + 0K$

vector in the direction of the line

$\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z-0}{1}$ is $\bar{A} = 2I - 2J + 1K$

unit vector in the direction of $\bar{AB} = \frac{2I - 2J + K}{3}$

Hence the required directional derivative is given by

$$(12.5I - 5J + 0K)\cdot\frac{2I - 2J + K}{3}$$

$$= \frac{25+10}{3} = \frac{35}{3}$$

Problem (8): Find the angle between the tangent planes to the surfaces $x \log z = y^2 - 1$, $x^2 y = 2 - z$ at the point $(1,1,1)$.

Solution: Given surfaces are

$$f_1(x,y,z) = x \log z - y^2 + 1 \longrightarrow (1)$$

$$\& \; f_2(x,y,z) = x^2 y + z - 2 \longrightarrow (2)$$

Given point is $P(1,1,1)$

Now, angle between tangent planes to the surfaces (1) & (2) at $P(1,1,1)$ is same as the angle between their normals at $P(1,1,1)$

So, normal vector to (1) at $P(1,1,1)$ is $(\nabla f_1)_P$

we have $\nabla f_1 = I \dfrac{\partial f_1}{\partial x} + J \dfrac{\partial f_1}{\partial y} + K \dfrac{\partial f_1}{\partial z}$

$$= \log z \, I - 2y J + \frac{x}{z} K$$

$$\nabla f_1 \text{ at } P(1,1,1) = 0I - 2J + K = \overline{N_1}$$

Also normal vector to (2) at $P(1,1,1)$ is $(\nabla f_2)_P$

we have $\nabla f_2 = I \dfrac{\partial f_2}{\partial x} + J \dfrac{\partial f_2}{\partial y} + K \dfrac{\partial f_2}{\partial z}$

$$= 2xy I + x^2 J + K$$

$$\nabla f_2 \text{ at } P(1,1,1) = 2I + J + K = \overline{N_2}$$

Let θ be the angle between $\bar{N}_1$ and $\bar{N}_2$.

Then $$\cos\theta = \frac{\bar{N}_1 \cdot \bar{N}_2}{|\bar{N}_1|\,|\bar{N}_2|}$$

$$= \frac{(0I-2J+K)\cdot(2I+J+K)}{\sqrt{4+1}\;\sqrt{4+1+1}}$$

$$= \frac{-1}{\sqrt{30}}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{-1}{\sqrt{30}}\right)$$

Practice Problems

(1) Find a unit vector normal to the surface $xy^3z^2 = 4$ at $(-1,-1,2)$.

(2) Find the angle between the surfaces $x^2+y^2+z^2 = 9$ and $z = x^2+y^2-3$ at $(2,-1,2)$.

(3) In what direction from $(3,1,-2)$ is the directional derivative of $\phi = x^2y^2z^4$ maximum. Find also the magnitude of this maximum.

(4) What is the greatest rate of increase of $u = xyz^2$ at $(1,0,3)$.

(5) Find the directional derivative of $f = x^2-y^2+2z^2$ at the point $P(1,2,3)$ in the direction of the line PQ where Q is the point $(5,0,4)$. Also calculate the ~~maximum~~ magnitude of the maximum directional derivative.

Divergence: Let $\bar{F} = f_1 I + f_2 J + f_3 K$ be a continuously differentiable vector point function. Then the divergence of $\bar{F}$ denoted by $\nabla \cdot \bar{F}$ (or) div $\bar{F}$ is defined as

$$\text{div}\,\bar{F} = \nabla \cdot \bar{F} = I \cdot \frac{\partial \bar{F}}{\partial x} + J \cdot \frac{\partial \bar{F}}{\partial y} + K \cdot \frac{\partial \bar{F}}{\partial z}$$

$$= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

clearly, div $\bar{F}$ is a scalar point function.

curl: Let $\bar{F} = f_1 I + f_2 J + f_3 K$ be a continuously differentiable vector point function. Then the curl of $\bar{F}$ denoted by $\nabla \times \bar{F}$ (or) curl $\bar{F}$ is defined as

$$\text{curl}\,\bar{F} = \nabla \times \bar{F}$$

$$= I \times \frac{\partial \bar{F}}{\partial x} + J \times \frac{\partial \bar{F}}{\partial y} + K \times \frac{\partial \bar{F}}{\partial z}$$

$$= \begin{vmatrix} I & J & K \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) I + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \right) J + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) K$$

clearly, curl of a vector point function is again a vector point function.

Problem (1): If $\bar{R} = xI + yJ + zK$, show that

(i) $\nabla \cdot \bar{R} = 3$ (ii) $\nabla \times \bar{R} = \bar{0}$

Solution: Given $\bar{R} = xI + yJ + zK$

(i) $$\nabla \cdot \bar{R} = \left(I\frac{\partial}{\partial x} + J\frac{\partial}{\partial y} + K\frac{\partial}{\partial z}\right)\cdot\left(xI + yJ + zK\right)$$

$$= \frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y + \frac{\partial}{\partial z}z$$

$$= 1 + 1 + 1$$

$$= 3.$$

(ii) $$\nabla \times \bar{R} = \begin{vmatrix} I & J & K \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}z - \frac{\partial}{\partial z}y\right)I + \left(\frac{\partial}{\partial z}x - \frac{\partial}{\partial x}z\right)J$$

$$+ \left(\frac{\partial}{\partial x}y - \frac{\partial}{\partial y}x\right)K$$

$$= 0I + 0J + 0K$$

$$= \bar{0}$$

Problem (2): Evaluate div $\bar{F}$ and curl $\bar{F}$ at the point $(1, 2, 3)$ given $\bar{F} = \text{grad}(x^3 y + y^3 z + z^3 x - x^2 y^2 z^2)$.

solution: Given $\bar{F} = \text{grad}\left(x^3 y + y^3 z + z^3 x - x^2 y^2 z^2\right)$

$$\Rightarrow \bar{F} = I \frac{\partial}{\partial x}\left(x^3 y + y^3 z + z^3 x - x^2 y^2 z^2\right)$$

$$+ J \frac{\partial}{\partial y}\left(x^3 y + y^3 z + z^3 x - x^2 y^2 z^2\right)$$

$$+ K \frac{\partial}{\partial z}\left(x^3 y + y^3 z + z^3 x - x^2 y^2 z^2\right)$$

$$= \left(3x^2 y + z^3 - 2x y^2 z^2\right) I$$

$$+ \left(x^3 + 3y^2 z - 2x^2 y z^2\right) J$$

$$+ \left(y^3 + 3z^2 x - 2x^2 y^2 z\right) K$$

Now $\text{div}\,\bar{F} = \frac{\partial}{\partial x}\left(3x^2 y + z^3 - 2x y^2 z^2\right)$

$$+ \frac{\partial}{\partial y}\left(x^3 + 3y^2 z - 2x^2 y z^2\right)$$

$$+ \frac{\partial}{\partial z}\left(y^3 + 3z^2 x - 2x^2 y^2 z\right)$$

$$= 6xy - 2y^2 z^2 + 6yz - 2x^2 z^2$$

$$+ 6zx - 2x^2 y^2$$

$\text{div}\,\bar{F}$ at $(1,2,3) = 12 - 72 + 36 - \cancel{18} + \cancel{18} - 8$

$$= 48 - 80$$

$$= -32$$

$$\operatorname{curl}\bar{F} = \begin{vmatrix} I & J & K \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2y+z^3 & x^3+3y^2z & y^3+3z^2x \\ -2xy^2z^2 & -2x^2yz^2 & -2x^2y^2z \end{vmatrix}$$

$$= (\cancel{3y^2} - \cancel{4x^2yz} - \cancel{3y^2} + \cancel{4x^2yz})\, I$$
$$+ (\cancel{3z^2} - \cancel{4xy^2z} - \cancel{3z^2} + \cancel{4xy^2z})\, J$$
$$+ (\cancel{3x^2} - \cancel{4xyz^2} - \cancel{3x^2} + \cancel{4xyz^2})\, K$$

$$= 0I + 0J + 0K$$

$$= \bar{0}$$

Problem (3): If $\bar{F} = (x+y+1)I + J - (x+y)K$, show that $\bar{F} \cdot \operatorname{curl}\bar{F} = 0$

Solution: Given $\bar{F} = (x+y+1)I + J - (x+y)K$

$$\operatorname{curl}\bar{F} = \begin{vmatrix} I & J & K \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y+1 & 1 & -x-y \end{vmatrix}$$

$$= (-1-0)I + (0-(-1))J + (0-1)K$$

$$= -I + J - K$$

Now $\bar{F}\cdot \text{curl}\,\bar{F} = \cancel{-x}\cancel{-z}\cancel{-y}+\cancel{y}+\cancel{x}+\cancel{z}$

$= 0$

$\therefore \bar{F}\cdot \text{curl}\,\bar{F} = 0$

Problem(4): Find the value of a if the vector $(ax^2y+yz)I+(xy^2-xz^2)J+(2xyz-2x^2y^2)K$ has zero divergence. Also find the curl $\bar{F}$.

Solution:

Given $\bar{F} = (ax^2y+yz)I+(xy^2-xz^2)J$
$\quad +(2xyz-2x^2y^2)K$

$\text{div}\,\bar{F} = \frac{\partial}{\partial x}(ax^2y+yz)+\frac{\partial}{\partial y}(xy^2-xz^2)$
$\quad + \frac{\partial}{\partial z}(2xyz-2x^2y^2)$

$= 2axy+2xy+2xy$

$= 2axy+4xy$

Suppose $\text{div}\,\bar{F}=0$

$\Rightarrow 2axy+4xy=0$

$\Rightarrow (2a+4)xy=0$

$\Rightarrow 2a+4=0 \quad (\because xy\neq 0)$

$\Rightarrow \boxed{a=-2}$

$$\text{curl } \bar{F} = \begin{vmatrix} I & J & K \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -2x^2y+yz & xy^2-xz^2 & 2xyz-2x^2y^2 \end{vmatrix}$$

$$= (2xz-4x^2y+2xz)\,I$$

$$+ (y-2yz+4xy^2)\,J$$

$$+ (y^2-z^2+2x^2-z)\,K$$

$$= (4xz-4x^2y)\,I + (y-2yz+4xy^2)\,J$$

$$+ (y^2-z^2+2x^2-z)\,K$$

Practice Problems

(1) Find div $\bar{F}$ and curl $\bar{F}$, where $\bar{F} = \text{grad}(x^3+y^3+z^3-3xyz)$

(2) If $\bar{V} = (xI+yJ+zK)/\sqrt{x^2+y^2+z^2}$, show that $\nabla\cdot\bar{V} = 2/\sqrt{x^2+y^2+z^2}$ and $\nabla\times\bar{V} = \bar{0}$.

(3) Find div $\bar{F}$ and curl $\bar{F}$, where $\bar{F} = x^2yzI + xy^2zJ + xyz^2K$ at $(1,2,3)$.