Strassen Matrix Multilication

Matrix Multiplication

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} * \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

$$c_{11} = a_{11}b_{11} + a_{12}b_{21}$$

$$c_{12} = a_{11}b_{12} + a_{12}b_{22}$$

$$c_{21} = a_{21}b_{11} + a_{22}b_{21}$$

$$c_{22} = a_{21}b_{12} + a_{22}b_{22}$$

Matrix Multiplication

A_{11} A_{12}

$$\begin{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} & \begin{bmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{bmatrix} \\ \begin{bmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{bmatrix} & \begin{bmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{bmatrix} \end{bmatrix} * \begin{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} & \begin{bmatrix} b_{13} & b_{14} \\ b_{23} & b_{24} \end{bmatrix} \\ \begin{bmatrix} b_{31} & b_{32} \\ b_{41} & b_{42} \end{bmatrix} & \begin{bmatrix} b_{33} & b_{34} \\ b_{43} & b_{44} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} & \begin{bmatrix} c_{13} & c_{14} \\ c_{23} & c_{24} \end{bmatrix} \\ \begin{bmatrix} c_{31} & c_{32} \\ c_{41} & c_{42} \end{bmatrix} & \begin{bmatrix} c_{33} & c_{34} \\ c_{43} & c_{44} \end{bmatrix} \end{bmatrix}$$

A_{21} A_{22}

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} * \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

$$C_{11} = A_{11}B_{11} + A_{12}B_{21}$$

$$C_{12} = A_{11}B_{12} + A_{12}B_{22}$$

$$C_{21} = A_{21}B_{11} + A_{22}B_{21}$$

$$C_{22} = A_{21}B_{12} + A_{22}B_{22}$$

Matrix Multiplication

- To compute AB of nxn matrices, we need 8 multiplication of $\frac{n}{2}$x$\frac{n}{2}$ matrices and 4 sums of $\frac{n}{2}$x$\frac{n}{2}$ matrices.
- Sum of $\frac{n}{2}$x$\frac{n}{2}$ matrices takes $\frac{n^2}{4}$ additions, 4 sums take n^2 operations and hence is $\theta(n^2)$
- $T(n) = 8T\left(\frac{n}{2}\right) + \theta(n^2) = \theta(n^3)$

Matrix Multiplication

$$\begin{bmatrix} 1 & 2 & 2 & 1 \\ 3 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 0 & 1 & 3 & 1 \\ 1 & 0 & 2 & 0 \\ 2 & 1 & 1 & 2 \\ 0 & 1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 4 & 12 & 6 \\ 3 & 4 & 12 & 5 \\ 5 & 3 & 7 & 5 \\ 0 & 3 & 6 & 2 \end{bmatrix}$$

Matrix Multiplication

$$\begin{bmatrix} \overset{A_{11}}{\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}} & \overset{A_{12}}{\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}} \\ \underset{A_{21}}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} & \underset{A_{22}}{\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}} \end{bmatrix} * \begin{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix} \\ \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 6 & 4 \\ 3 & 4 \end{bmatrix} & \begin{bmatrix} 12 & 6 \\ 12 & 5 \end{bmatrix} \\ \begin{bmatrix} 5 & 3 \\ 0 & 2 \end{bmatrix} & \begin{bmatrix} 7 & 5 \\ 6 & 2 \end{bmatrix} \end{bmatrix}$$

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} * \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

$$C_{11} = A_{11}B_{11} + A_{12}B_{21}$$

$$= \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} * \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} * \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 3 & 4 \end{bmatrix}$$

$$C_{12} = A_{11}B_{12} + A_{12}B_{22}$$

$$C_{21} = A_{21}B_{11} + A_{22}B_{21}$$

$$C_{22} = A_{21}B_{12} + A_{22}B_{22}$$

Matrix Multiplication

- To compute AB of nxn matrices, we need 8 multiplication of $\frac{n}{2}$x$\frac{n}{2}$ matrices and 4 sums of $\frac{n}{2}$x$\frac{n}{2}$ matrices.
- Sum of $\frac{n}{2}$x$\frac{n}{2}$ matrices takes $\frac{n^2}{4}$ additions, 4 sums take n^2 operations and hence is $\theta(n^2)$
- $T(n) = 8T\left(\frac{n}{2}\right) + \theta(n^2) = \theta(n^3)$

Strassen Matrix Multiplication

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} * \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$c_{11} = P + S - T + V$$

$$c_{12} = R + T$$

$$c_{21} = Q + S$$

$$c_{22} = P + R - Q + U$$

$$P = (a_{11} + a_{22})(b_{11} + b_{22})$$

$$Q = (a_{21} + a_{22})b_{11}$$

$$R = a_{11}(b_{12} - b_{22})$$

$$S = a_{22}(b_{21} - b_{11})$$

$$T = (a_{11} + a_{12})b_{22}$$

$$U = (a_{21} - a_{11})(b_{11} + b_{12})$$

$$V = (a_{12} - a_{22})(b_{21} + b_{22})$$

Strassen Matrix Multiplication

$$\begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 17 & 24 \\ 6 & 10 \end{bmatrix}$$

$$c_{11} = P + S - T + V = 17$$

$$c_{12} = R + T = 24$$

$$c_{21} = Q + S = 6$$

$$c_{22} = P + R - Q + U = 10$$

$$P = 15$$

$$Q = 4$$

$$R = -4$$

$$S = 2$$

$$T = 28$$

$$U = 3$$

$$V = 28$$

Strassen Matrix Multiplication

$$\begin{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} & \begin{bmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{bmatrix} \\ \begin{bmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{bmatrix} & \begin{bmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{bmatrix} \end{bmatrix} * \begin{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} & \begin{bmatrix} b_{13} & b_{14} \\ b_{23} & b_{24} \end{bmatrix} \\ \begin{bmatrix} b_{31} & b_{32} \\ b_{41} & b_{42} \end{bmatrix} & \begin{bmatrix} b_{33} & b_{34} \\ b_{43} & b_{44} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} & \begin{bmatrix} c_{13} & c_{14} \\ c_{23} & c_{24} \end{bmatrix} \\ \begin{bmatrix} c_{31} & c_{32} \\ c_{41} & c_{42} \end{bmatrix} & \begin{bmatrix} c_{33} & c_{34} \\ c_{43} & c_{44} \end{bmatrix} \end{bmatrix}$$

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} * \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

Strassen Matrix Multiplication

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} * \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

$$P = (A_{11} + A_{22})(B_{11} + B_{22})$$

$$C_{11} = P + S - T + V$$

$$Q = (A_{21} + A_{22})A_{11}$$

$$C_{12} = R + T$$

$$R = A_{11}(B_{12} - B_{22})$$

$$C_{21} = Q + S$$

$$S = A_{22}(B_{21} - B_{11})$$

$$C_{22} = P + R - Q + U$$

$$T = (A_{11} + A_{12})B_{22}$$

$$U = (A_{21} - A_{11})(B_{11} + B_{12})$$

$$V = (A_{12} - B_{22})(B_{21} + B_{22})$$

Strassen Matrix Multiplication

- To compute AB of nxn matrices, we need 7 multiplication of $\frac{n}{2}$x$\frac{n}{2}$ matrices and 18 sums of $\frac{n}{2}$x$\frac{n}{2}$ matrices.
- Sum of $\frac{n}{2}$x$\frac{n}{2}$ matrices takes $\frac{n^2}{4}$ additions, 18 sums take $18n^2/4$ operations and hence is $\theta(n^2)$
- $T(n) = 7T\left(\frac{n}{2}\right) + \theta(n^2) = \theta\left(n^{\log_2 7}\right) = \theta(n^{2.8})$