Gauss-Divergence Theorem:

If $\bar{F}$ is a continuously differentiable vector function in the region E bounded by a closed surface S, then $\int_S \bar{F}\cdot\bar{N}\,dS = \int_E \operatorname{div}\bar{F}\,dv$ where $\bar{N}$ is the external normal vector.

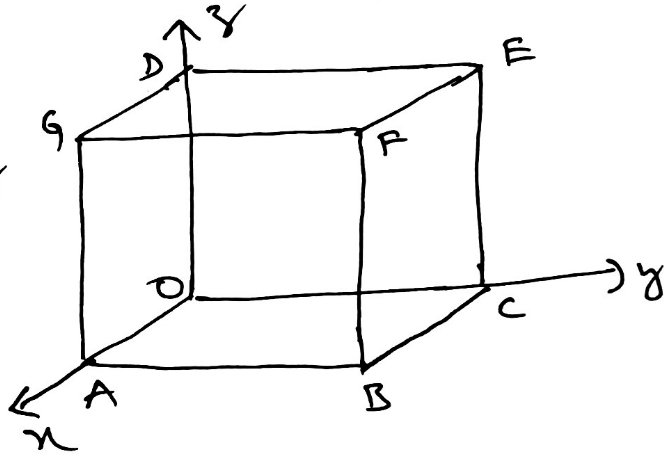
Another form of GDT:

$$\iint_S (f\,dy\,dz + \phi\,dz\,dx + \psi\,dx\,dy) = \iiint_E \left(\frac{\partial f}{\partial x} + \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial z}\right) dx\,dy\,dz$$

Problem (1): Verify Gauss-Divergence Theorem for $\bar{F}$ taken over the cube bounded by $x=0, x=1, y=0, y=1, z=0, z=1$ where $\bar{F} = 4xzI - y^2J + yzK$.

Solution: Given $\bar{F} = 4xzI - y^2J + yzK$

Here S is a closed surface bounded by $x=0, x=1, y=0, y=1, z=0$ and $z=1$ and E is the region enclosed by this closed surface.



Clearly $S = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6$

where S_1: the face OABC $(\bar{N} = -K)$

S_2: the face DEFG $(\bar{N} = K)$

S_3: the face OAGD $(\bar{N} = -J)$

S_4: the face BCEF $(\bar{N} = J)$

S_5: the face ABFG $(\bar{N} = I)$

S_6: the face OCED. $(\bar{N} = -I)$

$$\int_S \bar{F}\cdot\bar{N}\,dS = \int_{S_1} \bar{F}\cdot\bar{N}\,dS_1 + \int_{S_2} \bar{F}\cdot\bar{N}\,dS_2 + \cdots + \int_{S_6} \bar{F}\cdot\bar{N}\,dS_6.$$

$$\int_{S_1} \bar{F}\cdot\bar{N}\,dS_1 = \iint_{S_1} (4xz I - y^2 J + yz K)\cdot(-K)\,dx\,dy$$

$$= \iint_{S_1} -yz\,dx\,dy = 0 \qquad (\because \text{ on } S_1, z=0)$$

$$\int_{S_2} \bar{F}\cdot\bar{N}\,dS_2 = \iint_{S_2} (4xz I - y^2 J + yz K)\cdot K\,dS_2$$

$$= \iint_{S_2} yz K\,dS_2$$

$$= \int_0^1\int_0^1 y\,dx\,dy \qquad (\text{on } S_2, z=1)$$

$$\left(dS_2 = \frac{dx\,dy}{|\bar{N}\cdot K|} = dx\,dy\right)$$

$$= \frac{1}{2}.$$

$$\int_{S_3} \bar{F}\cdot\bar{N}\,dS_3 = \iint_{S_3} (4xz I - y^2 J + yz K)\cdot(-J)\,dS_3$$

$$= \iint_{S_3} y^2\,dS_3 = \int_0^1\int_0^1 y^2\,dx\,dz$$

$$= 0 \qquad (\text{on } S_3, y=0)$$

$$\int_{S_4} \bar{F}\cdot\bar{N}\,dS_4 = \iint_{S_4} (4xz I - y^2 J + yz K)\cdot J\,dS_4$$

$$= \iint_{S_4} -y^2\,dS_4 = \int_0^1\int_0^1 (-1^2)\,\frac{dx\,dz}{|\bar{N}\cdot J|}$$

$$(\text{on } S_4, y=1)$$

$$= \int_0^1\int_0^1 (-1^2)\,\frac{dx\,dz}{|J\cdot J|}$$

$$= -1$$

$$\int_{S_5} \bar{F}\cdot\bar{N}\,dS_5 = \iint_{S_5} (4xz I - y^2 J + yz K)\cdot I\,dS_5$$

$$= \iint_{S_5} 4xz \, dS_5 = \int_0^1 \int_0^1 4z \frac{dy\,dz}{|\bar{N}\cdot I|} \quad (\text{on } S_5, x=1)$$

$$= \int_0^1 \int_0^1 4z \frac{dy\,dz}{|I\cdot I|}$$

$$= 4\left(\frac{z^2}{2}\right)_0^1 (y)_0^1$$

$$= 4 \times \frac{1}{2} \times 1 = 2$$

$$\int_{S_6} \bar{F}\cdot\bar{N}\, dS_6 = \iint_{S_6} (4xz\, I - y^2 J + yz K)\cdot(-I)\, dS_6$$

$$= \iint_{S_6} -4xz \, dS_6$$

$$= \int_0^1 \int_0^1 -4xz \frac{dy\,dz}{|\bar{N}\cdot I|}$$

$$= \int_0^1 \int_0^1 -4xz \frac{dy\,dz}{|-I\cdot I|}$$

$$= \int_0^1 \int_0^1 -4xz \, dy\,dz = 0 \quad (\text{on } S_6, x=0)$$

$$\therefore \int_S \bar{F}\cdot\bar{N}\, dS = \frac{1}{2} - 1 + 2 = \frac{5}{2} - 1 = \frac{3}{2}.$$

$$\text{Div } \bar{F} = \frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)$$

$$= 4z - 2y + y$$

$$= 4z - y$$

$$\int_E \text{Div } \bar{F}\, dv = \int_0^1 \int_0^1 \int_0^1 (4z - y)\, dz\, dy\, dx$$

$$= \int_0^1 \int_0^1 \left(4\frac{z^2}{2} - yz\right)_0^1 dy\, dx$$

$$= \int_0^1 \int_0^1 (2-y)\, dy\, dx$$

$$= \int_0^1 \left(2y - \frac{y^2}{2}\right)_0^1 dx$$

$$= \int_0^1 \left(2 - \frac{1}{2}\right) dx$$

$$= \int_0^1 \frac{3}{2}\, dx$$

$$= \frac{3}{2}$$

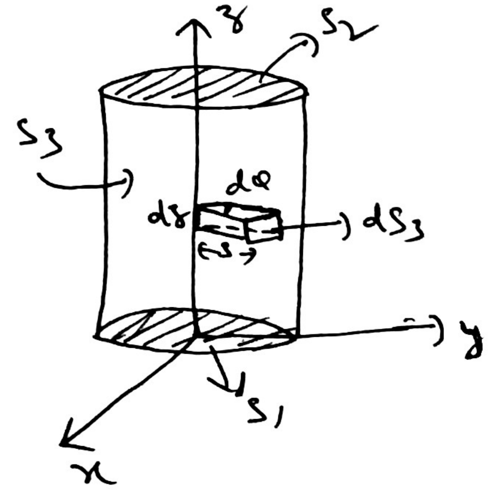
$\therefore$ Divergence theorem is verified.

Problem (2): verify Gauss-Divergence theorem for $\bar{F} = yI + xJ + z^2K$ over the cylindrical region bounded by $x^2+y^2=9$, $z=0$ and $z=2$.

Solution: Given $\bar{F} = yI + xJ + z^2K$

Here S is a closed surface bounded by S_1 $(z=0)$, S_2 $(z=2)$ and S_3 (the cylindrical surface $x^2+y^2=9$).

$$\therefore \int_S \bar{F}\cdot\bar{N}\, dS = \int_{S_1} \bar{F}\cdot\bar{N}\, dS_1 + \int_{S_2} \bar{F}\cdot\bar{N}\, dS_2 + \int_{S_3} \bar{F}\cdot\bar{N}\, dS_3$$



on S_1, $\bar{N} = -K$

on S_2, $\bar{N} = K$

on S_3, $\bar{N} = \dfrac{\nabla(x^2+y^2-9)}{|\nabla(x^2+y^2-9)|}$

$$\Rightarrow \bar{N} = \frac{2x I + 2y J}{|2x I + 2y J|}$$

$$= \frac{2x I + 2y J}{\sqrt{4x^2 + 4y^2}}$$

$$= \frac{2x I + 2y J}{\sqrt{4(x^2 + y^2)}}$$

$$= \frac{2(x I + y J)}{2\sqrt{x^2 + y^2}}$$

$$= \frac{x I + y J}{\sqrt{9}} \qquad (\text{on } S_3,\ x^2 + y^2 = 9)$$

$$= \frac{x I + y J}{3}$$

$$\int_{S_1} F \cdot \bar{N}\, dS_1 = \iint_{S_1} (y I + x J + z^2 K) \cdot (-K)\, dS_1$$

$$= \iint_{S_1} -z^2\, dx\, dy$$

$$= 0 \qquad (\text{on } S_1,\ z = 0)$$

$$\int_{S_2} F \cdot \bar{N}\, dS_2 = \iint_{S_2} (y I + x J + z^2 K) \cdot K\, dS_2$$

$$= \iint_{S_2} z^2\, dS_2$$

$$= \iint_{S_2} 2^2\, dS_2 \qquad (\text{on } S_2,\ z = 2)$$

$$= \iint_{x^2+y^2=9} 4 \frac{dx\,dy}{|\bar{N}\cdot K|}$$

$$= 4\iint_{x^2+y^2=9} dx\,dy \qquad (\bar{N} = K,\ K\cdot K = 1)$$

$= 4 \times$ Area of the circle $x^2+y^2=9$

$$= 4\pi 3^2$$

$$= 36\pi$$

$$\int_{S_3} \bar{F}\cdot\bar{N}\,dS_3 = \iint_{S_3} (yI + xJ + z^2K)\cdot\left(\frac{xI+yJ}{3}\right) dS_3$$

$$= \iint_{S_3} \frac{xy + xy}{3}\, dS_3$$

$$= \iint_{S_3} \frac{2xy}{3}\, dS_3 \qquad (dS_3 = 3\,d\theta\,dz)$$

$$= \int_0^{2\pi}\int_0 \frac{2(3\cos\theta)(3\sin\theta)}{\cancel{3}}\,\cancel{3}\,d\theta\,dz \quad (*)$$

$$= 18\int_0^{2\pi} \sin\theta\cos\theta\,d\theta \int_0 dz$$

$$= 18\left(\frac{\sin^2\theta}{2}\right)_0^{2\pi} (2)$$

$$= 9(\sin^2 2\pi - \sin^2 0)(2)$$

$$= 18 \times 0$$

$$= 0$$

$*$ (on S_3, $x = 3\cos\theta$, $y = 3\sin\theta$)

$$\text{Div } \bar{F} = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(z^2)$$

$$= 2z$$

$$\therefore \int_E \text{div } \bar{F} \, dV = \iiint_E 2z \, dx \, dy \, dz$$

Put $x = r\cos\theta$ (cylindrical co-ordinates)

$y = r\sin\theta$

$z = z$

$dx\,dy\,dz = r\,dr\,d\theta\,dz$

$r: 0$ to 3

$\theta: 0$ to 2π

$z: 0$ to 2

$$\therefore \iiint_E 2z \, dx\,dy\,dz = \int_{\theta=0}^{2\pi} \int_{r=0}^{3} \int_{z=0}^{2} 2z \, r\,dr\,d\theta\,dz$$

$$= 2\int_0^{2\pi} d\theta \int_0^3 r\,dr \int_0^2 z\,dz$$

$$= 2(2\pi)\left(\frac{r^2}{2}\right)_0^3 \left(\frac{z^2}{2}\right)_0^2$$

$$= 4\pi \times \frac{9}{\cancel{2}} \times \frac{4}{\cancel{2}}$$

$$= 36\pi$$

$\therefore$ Gauss-Divergence theorem is verified.

Problem (3): using divergence theorem evaluate

$\int_S \bar{R}\cdot\bar{N}\, dS$ where S is the surface of the sphere $x^2+y^2+z^2=9$

Solution: By the divergence theorem, we have

$$\int_S \bar{R}\cdot\bar{N}\, dS = \int_E \operatorname{div}\bar{R}\, dV \quad \text{where } \bar{R} = xI + yJ + zK$$

$$\operatorname{div}\bar{R} = \frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y + \frac{\partial}{\partial z}z = 3$$

$$\therefore \int_E \operatorname{div}\bar{R}\, dV = \iiint_E 3\, dV$$

$$= 3\iiint_E dV$$

$= 3 \times$ volume of the sphere $x^2+y^2+z^2=9$

$$= \cancel{3} \times \frac{4}{\cancel{3}}\pi 3^3 = 108\pi$$

Problem (4): For any closed surface S, prove that

$$\int_S [x(y-z)I + y(z-x)J + z(x-y)K]\cdot d\bar{S} = 0.$$

Solution: By the divergence theorem, we have

$$\int_S \bar{F}\cdot d\bar{S} = \int_E \operatorname{div} F\, dV \quad \text{where } \bar{F} = x(y-z)I$$

$$+ y(z-x)J + z(x-y)K$$

$$\operatorname{div} \bar{F} = \frac{\partial}{\partial x} x(y-z) + \frac{\partial}{\partial y} y(z-x) + \frac{\partial}{\partial z} z(x-y)$$

$$= \cancel{y} - \cancel{z} + \cancel{z} - \cancel{x} + \cancel{x} - \cancel{y}$$

$$= 0$$

$$\therefore \int_E \operatorname{div} \bar{F}\, dV = 0$$

So, $\int_S \bar{F} \cdot d\bar{S} = 0$ for any closed surface S.

Problem(5): By the divergence theorem, evaluate $\iint (x\, dy\, dz + y\, dz\, dx + z\, dx\, dy)$ over the surface of a sphere of radius a.

Solution: By the divergence theorem, we have

$$\iint_S (f\, dy\, dz + \phi\, dz\, dx + \psi\, dx\, dy) =$$

$$\iiint_E \left(\frac{\partial f}{\partial x} + \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial z} \right) dx\, dy\, dz$$

Here $f = x$, $\phi = y$, $\psi = z$ so, $\frac{\partial f}{\partial x} = 1$, $\frac{\partial \phi}{\partial y} = 1$, $\frac{\partial \psi}{\partial z} = 1$

Now $\iint_S (x\, dy\, dz + y\, dz\, dx + z\, dx\, dy) = \iiint_E 3\, dx\, dy\, dz$

$$= 3 \iiint_E dx\, dy\, dz$$

$$= \cancel{3} \times \frac{4}{\cancel{3}} \pi a^3$$

$$= 4\pi a^3$$

Problem(6): Use divergence theorem to evaluate $\int_S \bar{F}\cdot\bar{n}\,dS$ where $\bar{F} = e^x I + e^y J + e^z K$ and S is the surface of the cube $|x|\le 1, |y|\le 1, |z|\le 1$.

Solution: By the divergence theorem, we have

$$\int_S \bar{F}\cdot\bar{n}\,dS = \int_E \operatorname{div}\bar{F}\,dV.$$

$$\int_E \operatorname{div}\bar{F}\,dV = \iiint_E (e^x + e^y + e^z)\,dx\,dy\,dz$$

$$= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}(e^x + e^y + e^z)\,dx\,dy\,dz$$

now $$\int_{-1}^{1}(e^x + e^y + e^z)\,dx = \left[e^x + xe^y + xe^z\right]_{-1}^{1}$$

$$= (e + e^y + e^z) - (e^{-1} - e^y - e^z)$$

$$= (e - e^{-1}) + 2e^y + 2e^z$$

$$\therefore \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}(e^x + e^y + e^z)\,dx\,dy\,dz = \int_{-1}^{1}\int_{-1}^{1}\left[(e-e^{-1}) + 2(e^y + e^z)\right]dy\,dz$$

Now $$\int_{-1}^{1}\left[(e-e^{-1}) + 2(e^y + e^z)\right]dy = \left[(e-e^{-1})y + 2e^y + 2ye^z\right]_{-1}^{1}$$

$$= ((e-e^{-1}) + 2e + 2e^z) - (-(e-e^{-1}) + 2e^{-1} - 2e^z)$$

$$= 2(e-e^{-1}) + 2(e-e^{-1}) + 4e^z$$

$$= 4\left[(e-e^{-1}) + e^z\right]$$

$$\therefore \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}(e^x+e^y+e^z)\,dx\,dy\,dz = 4\int_{-1}^{1}[(e-e^{-1})+e^z]\,dz$$

$$= 4\left[(e-e^{-1})z + e^z\right]_{-1}^{1}$$

$$= 4\left[((e-e^{-1})+e) - (-(e-e^{-1})+e^{-1})\right]$$

$$= 4\left[2(e-e^{-1}) + (e-e^{-1})\right]$$

$$= 12(e-e^{-1}).$$

Practice problems

(1) Verify GDT for $\bar{F} = (x^2-yz)I + (y^2-xz)J + (z^2-xy)K$ taken over the rectangular parallelpiped $0 \le x \le a,\ 0 \le y \le b,\ 0 \le z \le c.$

(2) Using divergence theorem evaluate $\int_S \bar{F}\cdot\bar{n}\,ds$ where $\bar{F} = 4xI - 2y^2J + z^2K$ and S is the surface $x^2+y^2=4,\ z=0$ and $z=3.$

(3) Evaluate $\int_S \bar{F}\cdot\bar{n}\,ds$ where $\bar{F} = y^2z^2I + z^2x^2J + x^2y^2K$ and S is the upper part of the sphere $x^2+y^2+z^2=a^2$ above XOY plane.

(4) If S is any closed surface enclosing a volume V and $\bar{F} = axI + byJ + czK$, prove that $\int_S \bar{F}\cdot\bar{n}\,ds = (a+b+c)V.$