PROBABILITY DENSITIES

Definition: An experiment which can be repeated any number of times under the same identical conditions and even though we know the outcomes of the experiment in advance we could not predict the exact outcome of the experiment is called the random experiment.

Example: (i) Tossing a coin (ii) Rolling a fair die (iii) Taking a card from a deck of 52 cards etc.

Definition: The set of all possible outcomes of a random experiment is called the sample space.

Example: When tossing a coin one time the sample space associated with it is { H, T}

Example: When tossing a fair die one time the sample space associated with it is {1, 2, 3, 4, 5, 6}.

Definition: A random variable is a real valued function defined over the sample space.

Definition: A random variable which takes on finite number of values or countably infinite number of values is called a discrete random variable.

Example: Consider the experiment of tossing a coin n times. Let X denotes the number of heads turns up. Then X is a random variable which takes on the values 0, 1, 2, …, n. So, X is a discrete random variable.

Definition: A random variable which assumes every value in an interval [a, b] is called a continuous random variable.

Example: Let X denotes the life time of a battery. Then X assumes any value in the interval (0, ∞). So, X is a continuous random variable.

Definition:A function $f(x)$ is said to be probability density function of a continuous random variable X if f(x) satisfies:

(i). $f(x) \geq 0$ (ii). $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1$

Definition: The Cumulative distribution function $\mathrm{F}(x)$ of a continuous random variable X is defined as $\mathrm{F}(x) = \mathrm{P}(\mathrm{X} \le x) = \int_{-\infty}^{x} f(x)\,\mathrm{d}x$

Formulae:

1. $\mathrm{P}(\mathrm{X} \le a) = \int_{-\infty}^{a} f(x)\,\mathrm{d}x$

2. $\mathrm{P}(\mathrm{X} \ge a) = \int_{a}^{\infty} f(x)\,\mathrm{d}x$

3. $\mathrm{P}(a \le \mathrm{X} \le b) = \mathrm{P}(a \le \mathrm{X} < b) = \mathrm{P}(a < \mathrm{X} \le b) = \mathrm{P}(a < \mathrm{X} < b) = \int_{a}^{b} f(x)\,\mathrm{d}x$

4. $\mathrm{P}(\mathrm{X} \le a) = F(a)$

5. $\mathrm{P}(\mathrm{X} \ge a) = 1 - F(a)$

6. $\mathrm{P}(a \le X \le b) = F(\mathrm{b}) - F(a)$

7. $F(-\infty) = 0;\ \mathrm{F}(\infty) = 1$

8. $\frac{d}{dx} F(x) = \mathrm{f}(x)$

Definition: The k^th^ moment of the random variable X about the origin is defined by

$$\mu_k^1 = \int_{-\infty}^{\infty} x^k f(x)\,dx$$

The first moment of the random variable X about origin is called mean, and is denoted by μ. The mean of the probability density is $\mu = \int_{-\infty}^{\infty} x\ f(x)\,\mathrm{d}x$

Definition: The k^th^ moment of the random variable X about the mean is defined by

$$\mu_k = \int_{-\infty}^{\infty} (x-\mu)^k\ f(x)\,\mathrm{d}x$$

The second moment of the random variable X about the mean is called variance, denoted by σ^2.

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2\ f(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} x^2\ f(x)\,\mathrm{d}x - \mu^2$$

$= E(X^2) - [E(X)]^2$

The positive square root of variance is called standard deviation and is denoted by σ.

<u>Problem 1</u>: Verify the given function is probability density function or not, If a random variable has the probability density $f(x) = \begin{cases} 2e^{-2x} & \text{for } x > 0 \\ 0 & \text{for } x \le 0 \end{cases}$

Find the probabilities that it will take on a value

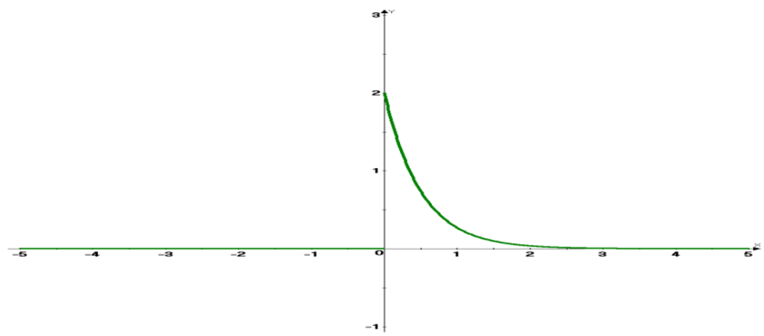
(a). between 1 and 3

(b). greater than 0.5

(c). less than or equal to 1.

Also find mean and variance.

<u>Solution</u>: The graph of the function f(x) is



<u>To verify that f(x) is probability density function :</u>

(i) When $x \le 0,\ f(x) = 0$

When $x > 0,\ f(x) = 2e^{-2x} = \dfrac{2}{e^{2x}} \ge 0$

$\therefore\ f(x) \ge 0\ \ \forall x$

(ii)Consider $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx$

$$= 0 + \int_0^{\infty} 2e^{-2x}\,dx$$

$$= 2\left(\frac{e^{-2x}}{-2}\right)_0^{\infty} = 1$$

$\therefore f(x)$ is a probability density function.

(a). $P(1 < X < 3) = \int_1^3 f(x)\,\mathrm{d}x$

$$= \int_1^3 2e^{-2x}\,dx$$

$$= 2\left(\frac{e^{-2x}}{-2}\right)_1^3$$

$$= e^{-2} - e^{-6} = 0.1328$$

(b). $P(X > 0.5) = \int_{0.5}^{\infty} f(x)\,\mathrm{d}x$

$$= \int_{0.5}^{\infty} 2e^{-2x}\,dx$$

$$= 2\left(\frac{e^{-2x}}{-2}\right)_{0.5}^{\infty}$$

$$= e^{-1} = 0.3678$$

(c). $P(X \le 1) = \int_{-\infty}^{1} f(x)\,\mathrm{d}x$

$$= \int_{-\infty}^{0} f(x)\,\mathrm{d}x + \int_0^1 f(x)\,\mathrm{d}x$$

$$= 0 + \int_0^1 2e^{-2x}\,dx$$

$$= 2\left(\frac{e^{-2x}}{-2}\right)_0^1$$

$$= 1 - e^{-2} = 0.8646$$

<u>To find mean</u>:

We know that Mean, $\mu = \int_{-\infty}^{\infty} x\, f(x)\,\mathrm{d}x$

$$= \int_{-\infty}^{0} f(x)\,\mathrm{d}\,x + \int_{0}^{\infty} f(x)\,\mathrm{d}\,x$$

$$= 0 + \int_{0}^{\infty} x\; 2e^{-2x}\; dx$$

$$= 2\int_{0}^{\infty} xe^{-2x}\; dx$$

$$= 2\left[x\left(\frac{e^{-2x}}{-2}\right) - (1)\left(\frac{e^{-2x}}{(-2)^2}\right)\right]_0^{\infty}$$

$$= 2\left[(0-0) - \left(0 - \frac{1}{4}\right)\right] = 2\left(\frac{1}{4}\right) = \frac{1}{2}$$

<u>To find Variance:</u>

We know that Variance, $\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2\; f(x)\,\mathrm{d}\,x$

$$= \int_{-\infty}^{\infty} x^2\; f(x)\,\mathrm{d}\,x - \mu^2$$

$$= \int_{-\infty}^{0} x^2\; f(x)\,\mathrm{d}\,x + \int_{0}^{\infty} x^2\; f(x)\,\mathrm{d}\,x - \mu^2$$

$$= 0 + \int_{0}^{\infty} x^2\; 2e^{-2x}\; dx - \left(\frac{1}{2}\right)^2$$

$$= 2\int_{0}^{\infty} x^2 e^{-2x}\; dx - \frac{1}{4}$$

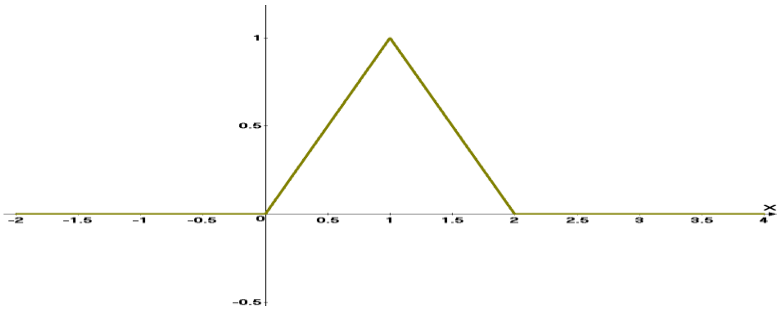
$$= 2\left[x^2\left(\frac{e^{-2x}}{-2}\right) - (2x)\left(\frac{e^{-2x}}{(-2)^2}\right) + (2)\left(\frac{e^{-2x}}{(-2)^3}\right)\right]_0^{\infty} - \frac{1}{4}$$

$$= 2\left[(0-0) - \left(0 - \frac{2}{-8}\right)\right] - \frac{1}{4}$$

$$= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

Problem 2: If the probability density of a random variable is given by $f(x)=\begin{cases} x & \text{if } 0<x<1 \\ 2-x & \text{if } 1\le x<2 \\ 0 & \text{elsewhere} \end{cases}$ Find the probabilities that the the random variable will take on a value (i) between 0.2 and 0.8; (ii) between 0.6 and 1.2; (iii) greater than 1.8.

Solution: The graph of the function f(x) is



Given $f(x)=\begin{cases} x & \text{if } 0<x<1 \\ 2-x & \text{if } 1\le x<2 \\ 0 & \text{elsewhere} \end{cases}$

$$(i)P(0.2<X<0.8)=\int_{0.2}^{0.8} f(x)\,dx$$

$$=\int_{0.2}^{0.8} x\,dx$$

$$=\left(\frac{x^2}{2}\right)_{0.2}^{0.8}$$

$$=\frac{1}{2}\left(0.8^2-0.2^2\right)$$

$$= 0.3$$

$$(ii)P(0.6<X<1.2)=\int_{0.6}^{1.2} f(x)\,dx$$

$$=\int_{0.6}^{1} f(x)\,dx+\int_{1}^{1.2} f(x)\,dx$$

$$=\int_{0.6}^{1} x\,dx+\int_{1}^{1.2} (2-x)\,dx$$

$$=\left(\frac{x^2}{2}\right)_{0.6}^{1}+\left(2x-\frac{x^2}{2}\right)_{1}^{1.2}$$

$$=\frac{1}{2}(1^2-0.6^2)+\left[\left(2\times1.2-\frac{1.2^2}{2}\right)-\left(2\times1-\frac{1^2}{2}\right)\right]$$

$= 0.32 + 0.18$

$= 0.5$

$$(iii) P\left(X>1.8\right)=\int_{1.8}^{\infty} f(x)\, dx$$

$$=\int_{1.8}^{2} f(x)\, dx+\int_{2}^{\infty} f(x)\, dx$$

$$=\int_{1.8}^{2}\left(2-x\right) dx+0$$

$$=\left(2x-\frac{x^2}{2}\right)_{1.8}^{2}$$

$$=\left(2\left[2-1.8\right]-\left[\frac{1}{2}\left(2^2-1.8^2\right)\right]\right) = 0.02$$

To find mean:

We know that Mean, $\mu = \int_{-\infty}^{\infty} x\ f(x)dx$

$$= \int_{-\infty}^{0} x\ f(x)dx+\int_{0}^{1} x\ f(x)dx+\int_{1}^{2} x\ f(x)dx+\int_{2}^{\infty} x\ f(x)dx$$

$$= 0+\int_{0}^{1} x\left(x\right)dx+\int_{1}^{2} x\left(2-x\right)dx+0$$

$$= \left(\frac{x^3}{3}\right)_{0}^{1}+\left(x^2-\frac{x^3}{3}\right)_{1}^{2}$$

$$= \frac{1}{3}+\left[\left(4-\frac{8}{3}\right)-\left(1-\frac{1}{3}\right)\right]$$

$= 1$

To find Variance:

We know that Variance, $\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 \; f(x)\, \mathrm{d}\, x$

$$= \int_{-\infty}^{\infty} x^2 f(x)\, dx - \mu^2$$

$$= \int_{-\infty}^{0} x^2 \; f(x)dx + \int_{0}^{1} x^2 \; f(x)dx + \int_{1}^{2} x^2 \; f(x)dx + \int_{2}^{\infty} x^2 \; f(x)dx - \mu^2$$

$$= 0 + \int_{0}^{1} x^2 (x) dx + \int_{1}^{2} x^2 (2-x) dx + 0 - 1^2$$

$$= \left(\frac{x^4}{4}\right)_0^1 + \left(2\frac{x^3}{3} - \frac{x^4}{4}\right)_1^2 - 1$$

$$= \frac{1}{4} + \left[\left(\frac{16}{3} - 4\right) - \left(\frac{2}{3} - \frac{1}{4}\right)\right] - 1 \; = \; \frac{1}{6}$$

Problem 3: If the distribution function of a random variable is given by

$$F(x) = \begin{cases} 1 - \dfrac{4}{x^2} & \text{for } x > 2 \\ 0 & \text{for } x \le 2 \end{cases}$$

Find the probabilities that this random variable will take on a value (i) less than 3; (ii) between 4 and 5. Also find the probability density function corresponding to F(x).

Solution: Given $F(x) = \begin{cases} 1 - \dfrac{4}{x^2} & \text{for } x > 2 \\ 0 & \text{for } x \le 2 \end{cases}$

(i) P(X < 3) = F(3) = $1 - \dfrac{4}{3^2} = \dfrac{5}{9}$

(ii) P(4 < X < 5) = F(5) - F(4) = $\left(1 - \dfrac{4}{5^2}\right) - \left(1 - \dfrac{4}{4^2}\right) = \dfrac{9}{100} = 0.09$

And we know that $\dfrac{d}{dx} F(x) = \mathrm{f}(x)$

$$\Rightarrow f(x) = \begin{cases} \dfrac{d}{dx}\left(1 - \dfrac{4}{x^2}\right) & ; x > 2 \\ 0 & ; x \le 2 \end{cases}$$

$$= \begin{cases} \dfrac{8}{x^3}; x > 2 \\ 0 \ ; x \le 2 \end{cases}$$

Problem 4: Find the value k so that the following can serve as the probability density of a random variable $f(x) = \begin{cases} 0 & \text{for } x \le 0 \\ kxe^{-4x^2} & \text{for } x > 0 \end{cases}$

Solution: Given f(x) is a probability density function.

$$\Rightarrow \qquad \int_{-\infty}^{\infty} f(x)\, dx = 1$$

$$\Rightarrow \qquad \int_{-\infty}^{0} f(x)dx + \int_{0}^{\infty} f(x)dx = 1$$

$$\Rightarrow \qquad \int_{0}^{\infty} kxe^{-4x^2} dx = 1$$

$$\Rightarrow \qquad k\int_{0}^{\infty} xe^{-4x^2} dx = 1$$

Let $4x^2 = t$

$\Rightarrow 8x\, dx = dt$

When $x = 0, t = 0$ and when $x \to \infty, t \to \infty$

$$\therefore k\int_{0}^{\infty} e^{-t} \frac{dt}{8} = 1$$

$$\Rightarrow \frac{k}{8}\left(\frac{e^{-t}}{-1}\right)_0^{\infty} = 1$$

$$\Rightarrow \frac{k}{8}(0+1) = 1$$

$$\Rightarrow \quad k = 8$$

Problem 5: If the probability density of a random variable is given by

$$f(x) = \begin{cases} kx^2; 0 < x < 1 \\ 0 \quad \text{elsewhere} \end{cases}$$

Find the value k and the probability that the random variable takes on a value

(i) between 1/4 and 3/4 ; (ii) greater than 2/3. Also find its mean and variance.

Solution: Given f(x) is a probability density function.

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\, dx = 1$$

$$\Rightarrow \int_{-\infty}^{0} f(x)\, dx + \int_{0}^{1} f(x)\, dx + \int_{1}^{\infty} f(x)\, dx = 1$$

$$\Rightarrow 0 + \int_{0}^{1} kx^2\, dx + 0 = 1$$

$$\Rightarrow k\left(\frac{x^3}{3}\right)_0^1 = 1$$

$$\Rightarrow k\left(\frac{1}{3}\right) = 1$$

$$\Rightarrow k = 3$$

$$\therefore f(x) = \begin{cases} 3x^2; 0 < x < 1 \\ 0 \quad \text{elsewhere} \end{cases}$$

(i) The probability that the random variable takes on a value between 1/4 and 3/4 is

$$P\left(\frac{1}{4} < X < \frac{3}{4}\right) = \int_{1/4}^{3/4} f(x)dx$$

$$= \int_{1/4}^{3/4} 3x^2 dx$$

$$= 3\left[\frac{x^3}{3}\right]_{1/4}^{3/4}$$

$$= \left(\frac{3}{4}\right)^3 - \left(\frac{1}{4}\right)^3$$

$$= \frac{13}{32}$$

(ii)The probability that the random variable takes on a value greater than 2/3 is

$$P\left(X > 2/3\right) = \int_{2/3}^{\infty} f(x)dx$$

$$= \int_{2/3}^{1} f(x)dx + \int_{1}^{\infty} f(x)dx$$

$$= \int_{2/3}^{1} 3x^2 dx + 0$$

$$= \left[x^3\right]_{2/3}^{1}$$

$$= 1 - \left(\frac{2}{3}\right)^3$$

$$= \frac{19}{27}$$

Mean, $\mu = \int_{-\infty}^{\infty} x \; f(x) dx$

$= \int_{0}^{1} x\left(3x^2\right) dx + 0$

$= 3\left(\frac{x^4}{4}\right)_0^1$

$= \frac{3}{4}$

Variance, $\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)\, dx - \mu^2 = \int_{0}^{1} x^2 \left(3x^2\right) dx - \left(\frac{3}{4}\right)^2 = 1.1625$

<u>Problem 6</u>: Given the probability density $f(x) = \frac{k}{1+x^2}, -\infty < x < \infty$ find k.

<u>Solution</u>: Given f(x) is a probability density function.

$\Rightarrow \quad \int_{-\infty}^{\infty} f(x)\, dx = 1$

$\Rightarrow \quad \int_{-\infty}^{\infty} \frac{k}{1+x^2}\, dx = 1$

$\Rightarrow \quad k\left(\tan^{-1}(x)\right)_{-\infty}^{\infty} = 1$

$\Rightarrow \quad k\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = 1$

$\Rightarrow \quad k = \frac{1}{\pi}$

<u>Problem 7</u>: In a certain city, the daily consumption of electric power (in millions of kilowatt- hours) is a random variable having the probability density

$f(x) = \begin{cases} \frac{1}{9} x e^{-x/3} & \text{for } x > 0 \\ 0 & \text{for } x \le 0 \end{cases}$.

If the city's power plant has a daily capacity of 12mkwh, what is the probability that this power supply will be inadequate on any given day?

<u>Solution</u>: The daily capacity of the power plant is 12mkwh.

Given the daily consumption of electric power is the random variable X.

The probability that 12mkwh power supply will be inadequate is P(X > 12)

If the daily consumption is greater than the supply then the power supply is inadequate.

$$P(X>12)=\int_{12}^{\infty} f(x)\,dx$$

$$=\int_{12}^{\infty}\frac{1}{9}xe^{-x/3}dx$$

$$=\frac{1}{9}\left[(x)(\frac{e^{-x/3}}{-1/3})-(1)(\frac{e^{-x/3}}{(-1/3)^2})\right]_{12}^{\infty}$$

$$=\frac{1}{9}\left[0-(12(-3)e^{-4}-9e^{-4})\right]$$

$$=\frac{45}{9}e^{-4}$$

$$=5e^{-4}$$

$$= 0.0915$$

Problem 8: The length of satisfactory service (years) provided by a certain model of laptop computer is a random variable having the probability density

$$f(x)=\begin{cases}\frac{1}{4.5}e^{-x/4.5} & \text{for } x>0\\ 0 & \text{for } x\le 0\end{cases}$$

Find the probabilities that one of these laptops will provide satisfactory service for (i) at most 2.5 years; (ii) anywhere from 4 to 6 years; (iii) at least 6.75 years.

Solution: (i) The probability that one of these laptops will provide satisfactory service for at most 2.5 years is

$$P\left(X \le 2.5\right) = \int_{-\infty}^{2.5} f(x)\,\mathrm{d}\,x$$

$$= \int_{0}^{2.5} \frac{1}{4.5} e^{-x/4.5} dx$$

$$= \frac{1}{4.5}\left[\left(-4.5\right)e^{-x/4.5}\right]_{0}^{2.5}$$

$$= -e^{-2.5/4.5} + 1$$

$$= 1 - e^{-5/9}$$

$$= 0.42612$$

(ii)The probability that one of these laptops will provide satisfactory service anywhere from 4 to 6 years is

$$P(4 \le X \le 6) = \int_{4}^{6} f(x)dx$$

$$= \int_{4}^{6} \frac{1}{4.5} e^{-x/4.5} dx$$

$$= \frac{1}{4.5}\left[(-4.5)e^{-x/4.5}\right]_{4}^{6}$$

$$= [e^{-4/4.5} - e^{-6/4.5}]$$

$$= 0.1475$$

(ii)The probability that one of these laptops will provide satisfactory service at least 6.75 years is

$$P\left(X \ge 6.75\right) = \int_{6.75}^{\infty} f(x)\,\mathrm{d}\,x$$

$$= \int_{6.75}^{\infty} \frac{1}{4.5} e^{-x/4.5} dx$$

$$= \frac{1}{4.5}\left[\left(-4.5\right)e^{-x/4.5}\right]_{6.75}^{\infty}$$

$$= 0 + e^{-6.75/4.5}$$

$$= 0.2231$$

Problem 9: Let the phase error in a tracking device have probability density $f(x)=\begin{cases}\cos x\,;0<x<\pi/2\\ 0 \quad \text{elsewhere}\end{cases}$ Find the probability that the phase error is (i) between 0 and $\pi/4$; (ii) greater than $\pi/3$. Also find mean and standard deviation for the distribution of the phase error.

Solution:

(i) $P\left(0<X<\pi/4\right)=\int_0^{\pi/4}\cos x\,dx=[\sin x]_0^{\pi/4}=\frac{1}{\sqrt{2}}$ = 0.7070

(ii) $P\left(X>\pi/3\right)=\int_{\pi/3}^{\infty}f(x)\,dx=\int_{\pi/3}^{\pi/2}\cos x\,dx=[\sin x]_{\pi/3}^{\pi/2}=1-\frac{\sqrt{3}}{2}$ = 0.1339

Mean, $\mu=\int_{-\infty}^{\infty}xf(x)\,dx=\int_0^{\pi/2}x\cos x\,dx=\left[x\sin x+\cos x\right]_0^{\pi/2}=\frac{\pi}{2}-1=0.5714$

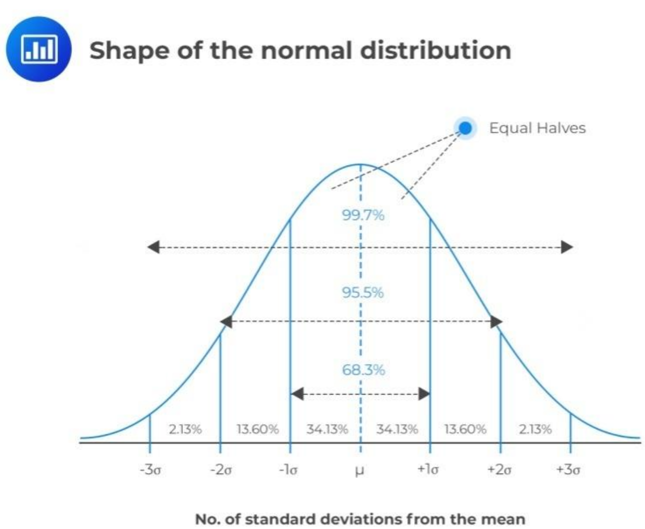
Variance, $\sigma^2=\int_{-\infty}^{\infty}x^2f(x)\,\mathrm{d}x-\mu^2=\int_0^{\pi/2}x^2\cos x\,dx-(0.5714)^2=\left[x^2\sin x+2x\cos x-2\sin x\right]_0^{\pi/2}$ - $(0.5714)^2$ = 0.1429

$\Rightarrow \sigma$ = 0.37802

The Normal distribution:

A random variable X is said to have the Normal distribution if its probability density is given by

$$f\left(x;\mu,\sigma^2\right)=\frac{1}{\sqrt{2\pi}}e^{\frac{-(x-\mu)^2}{2\sigma^2}};-\infty<x<\infty$$

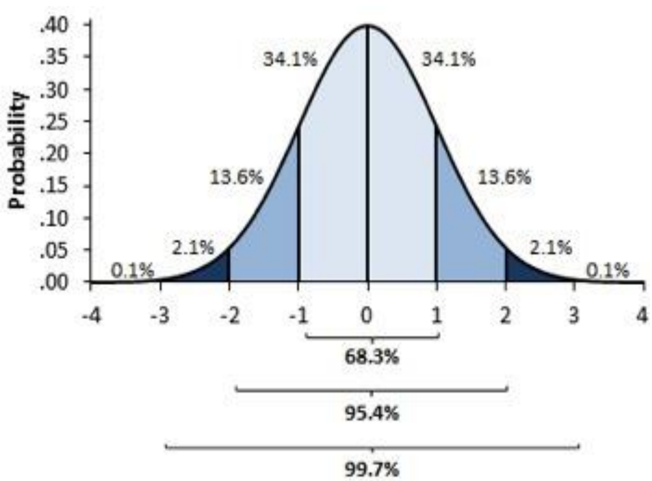


Properties:

- The graph of Normal distribution is a bell shaped curve.
- It is symmetrical about x = μ.
- The area under the normal curve above the x – axis is equal to unity.
- The normal curve meets the x – axis at the points of infinity and so x – axis is the asymptote of the normal curve.

Standard normal distribution:

If we take $Z = \frac{X-\mu}{\sigma}$ then the normal probability density function becomes $f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, -\infty < z < \infty$ which is called the standard normal density function and the random variable Z is called the standard normal random variable. The mean of Z, E(Z) = 0 and the variance of Z, Var(Z) = 1.The graph of the standard normal curve is



The cumulative distribution function of the random variable Z is given by

$$F(z) = P(Z \le z) = \int_{-\infty}^{z} f(t)\,dt$$

$$i.e., F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2} dt$$

Formulae:

1. The probability that a random variable having Standard normal distribution will take on a value between a and b is P(a < Z < b) = F(b) - F(a)

2. $P(Z \le a) = F(a)$

3. $P(Z \ge a) = 1 - F(a)$

4. $F(-z) = 1 - F(z)$

5. Let X be a random variable which has a normal distribution with mean μ, standard deviation σ then the corresponding standardized random variable Z is $Z = \frac{X-\mu}{\sigma}$, which has the standard normal distribution.

(i). $P(X \le a) = P\left(\frac{X-\mu}{\sigma} \le \frac{a-\mu}{\sigma}\right) = P\left(Z \le \frac{a-\mu}{\sigma}\right) = F\left(\frac{a-\mu}{\sigma}\right)$

(ii). $P(a \le X \le b) = P\left(\frac{a-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{b-\mu}{\sigma}\right)$

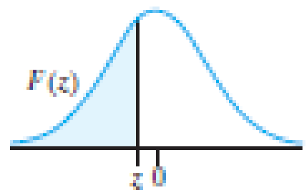
$$= P\left(\frac{a-\mu}{\sigma} \le Z \le \frac{b-\mu}{\sigma}\right)$$

$$= F\left(\frac{b-\mu}{\sigma}\right) - F\left(\frac{a-\mu}{\sigma}\right)$$

(iii). $P(X \ge a) = P\left(\frac{X-\mu}{\sigma} \ge \frac{a-\mu}{\sigma}\right) = P\left(Z \ge \frac{a-\mu}{\sigma}\right) = 1 - F\left(\frac{a-\mu}{\sigma}\right)$

Table 3 Standard Normal Distribution Function

$$F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2}\, dt$$



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
−5.0	0.0000003									
−4.0	0.00003									
−3.5	0.0002									
−3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
−3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0006	0.0003
−3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
−3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
−3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
−1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
−1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
−1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
−1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
−1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
−0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
−0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
−0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
−0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
−0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
−0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
−0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
−0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
−0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
−0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

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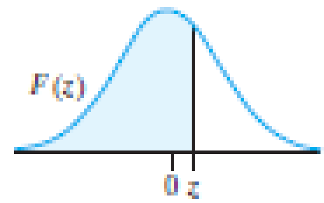
Table 3

$$F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2}\, dt$$



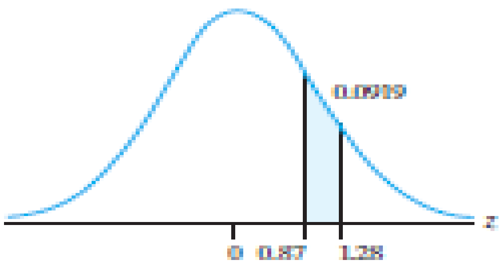
z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5973	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998									
4.0	0.99997									
5.0	0.9999997									

<u>Problem 10</u>: Find the probabilities that a random variable having the standard normal distribution will take on a value (i) between 0.87 and 1.28; (ii) Between -0.34 and 0.62; (iii) Greater than 0.85; (iv) Greater than -0.65.

<u>Solution</u>:(i). $P(0.87 < Z < 1.28) = F(1.28) - F(0.87)$

$= 0.8997 – 0.8078$

$= 0.0919$



(ii). $P(-0.34 < Z < 0.62) = F(0.62) - F(-0.34)$

$= 0.7324 – 0.3669$

$= 0.3655$

(iii). $P(Z > 0.85) = 1 - F(0.85)$

$= 1 - 0.8023$

$= 0.1977$

(iv). $P(Z > -0.65) = 1 - F(-0.65)$

$= 1 – 0.2578$

$= 0.7422$

<u>Problem 11</u>: If a random variable has the standard normal distribution, find the probability that it will take on a value (i) Less than 1.75; (ii) Less than -1.25; (iii) Greater than2.06; (iv) Greater than -1.82.

<u>Solution</u>: (i) $P(Z < 1.75) = F(1.75) = 0.9599$

(ii) $P(Z < -1.25) = F(-1.25) = 0.1056$

(iii) $P(Z > 2.06) = 1 - F(2.06) = 1 - 0.9803 = 0.0197$

(iv) $P(Z > -1.82) = 1 - F(-1.82)$

$= 1 – 0.0344$

$= 0.9656$

Problem 12: If a random variable has the standard normal distribution, find the probability that it will take on a value (i) Between 0 and 2.3; (ii) Between 1.22 and 2.43; (iii) Between -1.45 and -0.45; (iv) Between -1.70 and 1.35.

Solution: (i). $P(0 < Z < 2.3) = F(2.3) - F(0)$

$= 0.9861 – 0.5$

$= 0.4861$

(ii). $P(1.22 < Z < 2.43) = F(2.43) - F(1.22)$

$= 0.9925 – 0.8888$

$= 0.1037$

(iii). $P(-1.45 < Z < -0.45) = F(-0.45) - F(-1.45)$

$= 0.3264 – 0.0735$

$= 0.2529$

(iv). $P(-1.70 < Z < 1.35) = F(1.35) - F(-1.70)$

$= 0.9115 – 0.0446$

$= 0.8669$

Problem 13: Find z if the probability that a random variable having the standard normal distribution will take on a value (i) Less than z is 0.9911; (ii) Greater than z is 0.1093; (iii) Greater than z is 0.6443; (iv) Less than z is 0.0217; (v) Between –z and z is 0.9298.

Solution: (i) $P(Z < z) = 0.9911$

$\Rightarrow F(z) = 0.9911$

$\Rightarrow F(z) = F(2.37)$ (by table-3)

$\Rightarrow z = 2.37$

(ii). $P(Z > z) = 0.1093$

$\Rightarrow 1- F(z) = 0.1093$

$\Rightarrow F(z) = 1-0.1093$

$\Rightarrow F(z) = 0.8907 = F(1.23)$

$\Rightarrow \quad z = 1.23$

(iii) $P(Z > z) = 0.6443$

$\Rightarrow 1- F(z) = 0.6443$

$\Rightarrow F(z) = 1-0.6443$

$\Rightarrow F(z) = 0.3557 = F(-0.37)$ (by table-3)

$\Rightarrow \quad z = -0.37$

(iv) $P(Z < z) = 0.0217$

$\Rightarrow F(z) = 0.0217$

$\Rightarrow F(z) = F(-2.02)$ (by table-3)

$\Rightarrow \quad z = -2.02$

(v) $P(-z < Z < z) = 0.9298$

$\Rightarrow F(z) - F(-z) = 0.9298$

$\Rightarrow 2F(z) = 1.9298$

$\Rightarrow F(z) = 0.9649$

$\Rightarrow F(z) = F(1.81)$

$\Rightarrow \quad z = 1.81$

The Z_α notation for a standard normal distribution:

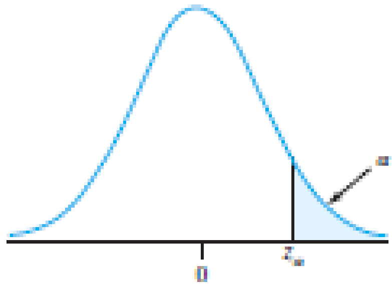
If $\mathbf{Z_\alpha}$ is such that the probability is α that it will be exceeded by a random variable having the standard normal distribution.

i.e, $P(Z > \mathbf{Z_\alpha}) = \alpha$

$\Rightarrow 1- P(Z < Z_\alpha) = \alpha$

$\Rightarrow 1- F(Z_\alpha) = \alpha$

$\Rightarrow F(Z_\alpha) = 1-\alpha$



Problem 14: Find (i) $Z_{0.01}$; (ii) $Z_{0.05}$; (iii) $Z_{0.005}$; (IV) $Z_{0.025}$

Solution:

(i) We know that $F(Z_\alpha) = 1- \alpha$

$\Rightarrow F(Z_{0.01}) = 1- 0.01 = 0.99$

In the standard normal distribution table-3, 0.99 is nearer to 0.9901 corresponding to z = 2.33

i.e, $F(Z_{0.01}) = 0.99 = F(2.33)$

$\Rightarrow Z_{0.01} = 2.33$

(ii) We know that $F(Z_\alpha) = 1- \alpha$

$\Rightarrow F(Z_{0.05}) = 1- 0.05 = 0.95$

In the standard normal distribution table-3, 0.95 is closest to 0.9495 and 0.9505 corresponding to z = 1.64 and z = 1.65

i.e, $F(Z_{0.05}) = 0.95 = F\left(\frac{1.64+1.65}{2}\right) = F(1.645)$

$\Rightarrow Z_{0.05} = 1.645$

(iii) We know that $F(Z_\alpha) = 1-\alpha$

$\Rightarrow F(Z_{0.005}) = 1- 0.005 = 0.995$

In the standard normal distribution table-3, 0.995 is closest to 0.9949 and 0.9951 corresponding to z = 2.57 and z = 2.58

i.e, $F(Z_{0.005}) = 0.995 = F\left(\frac{2.57+2.58}{2}\right) = F(2.575)$

$\Rightarrow Z_{0.005} = 2.575$

(iv) We know that $F(Z_\alpha) = 1-\alpha$

$\Rightarrow F(Z_{0.025}) = 1- 0.025 = 0.975$

In the standard normal distribution table-3, 0.975 is exactly equal corresponding to z = 1.96

i.e, $F(Z_{0.025}) = 0.975 = F(1.96)$

$\Rightarrow Z_{0.025} = 1.96$

<u>Problem 15</u>: Given a random variable having the normal distribution with μ= 16.2 and σ^2=1.5625, find the probabilities that it will take on a value

(i) greater than 16.8

(ii) less than 14.9

(iii) between 13.6 and 18.8

(iv) between 16.5 and 16.7

<u>Solution</u>: Given μ= 16.2 and σ^2=1.5625

$\Rightarrow \sigma = 1.25$

(i) $P(X > 16.8) = P\left(\frac{X-\mu}{\sigma} > \frac{16.8-\mu}{\sigma}\right)$

$$= P\left(Z > \frac{16.8-16.2}{1.25}\right)$$

$$= P(Z > 0.48)$$

$$= 1- F(0.48)$$

$$= 1- 0.6844$$

$$= 0.3156$$

(ii) $P(X < 14.9) = P\left(\frac{X-\mu}{\sigma} > \frac{14.9-\mu}{\sigma}\right)$

$$= P\left(Z > \frac{14.9-16.2}{1.25}\right)$$

$$= P(Z < -1.04)$$

$$= F(-1.04)$$

$$= 1 - F(1.04)$$

$$= 0.1492$$

(iii) P(13.6 < X < 18.8) = $P\left(\frac{13.6-16.2}{1.25} < Z < \frac{18.8-16.2}{1.25}\right)$

$= P(-2.08 < Z < 2.08)$

$= F(2.08) - F(-2.08)$

$= 2F(2.08) - 1$

$= 2(0.9812) - 1 = 0.9624$

(iv) P(16.5 < X < 16.7) = $P\left(\frac{16.5-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{16.7-\mu}{\sigma}\right)$

$= P\left(\frac{16.5-16.2}{1.25} < Z < \frac{16.7-16.2}{1.25}\right)$

$= P(0.24 < Z < 0.40)$

$= F(0.40) - F(0.24)$

$= 0.6554 - 0.5948$

$= 0.0606$

Problem 16: If the amount of cosmic radiation to which a person is exposed while flying by jet across United States is a random variable having the normal distribution with μ=4.35 and σ=0.59. Find the probabilities that the amount of cosmic radiation to which a person will be exposed on such a flight is (i) between 4 and 5; (ii) at least 5.5.

Solution: Here the random variable is the amount of cosmic radiation to which a person is exposed while flying by jet across the United States. Let it be X.

Given X is the random variable having the normal distribution with μ=4.35 and σ = 0.59.

(i) The probability that the amount of cosmic radiation to which a person will be exposed on such a flight between 4 and 5 is

$$P(4 \le X \le 5) = P\left(\frac{4-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{5-\mu}{\sigma}\right)$$

$$= P\left(\frac{4-4.35}{0.59} \le Z \le \frac{5-4.35}{0.59}\right)$$

$$= P(-0.5932 \le Z \le 1.1017)$$

= F(1.10)- F(-0.59)

= F(1.10) - (1- F(0.59)

= 0.8643-0.2776

= 0.5867

(ii) The probability that the amount of cosmic radiation to which a person will be exposed on such a flight at least 5.5 is

$$P(X \geq 5.5) = P\left(\frac{X-\mu}{\sigma} \geq \frac{5.5-\mu}{\sigma}\right)$$

$$= P\left(Z \geq \frac{5.5-4.3}{0.9}\right)$$

$$= P(Z \geq 1.9492)$$

$$= P(Z \geq 1.95)$$

= 1- F(1.95)

= 1- 0.9744

= 0.0256

<u>Problem 17</u>: The time for a super glue to set can be treated as a random variable having a normal distribution with mean 30 seconds. Find its standard deviation if the probability is 0.20 that it will take on a value greater than 39.2 seconds.

Solution: Given that the time for a super glue to set is a random variable having a normal distribution with mean, μ=30 seconds.

Also given the probability that a random variable will take on a value greater than 39.2 seconds is 0.20

i.e, P(X > 39.2) = 0.20

$$\Rightarrow P\left(\frac{X-\mu}{\sigma} > \frac{39.2-\mu}{\sigma}\right) = 0.20$$

$$\Rightarrow \quad P\left(Z > \frac{39.2-30}{\sigma}\right) = 0.20$$

$$\Rightarrow \quad P\left(Z > \frac{39.2-30}{\sigma}\right) = 0.20$$

$$\Rightarrow \quad 1 - F\left(\frac{9.2}{\sigma}\right) = 0.20$$

$$\Rightarrow \quad F\left(\frac{9.2}{\sigma}\right) = 0.80 = F(0.84)$$

$$\Rightarrow \quad \frac{9.2}{\sigma} = 0.84$$

$$\Rightarrow \quad \sigma = 10.9524$$

Problem 18: The time to microwave a bag of popcorn using the automatic setting can be treated as a random variable having a normal distribution with standard deviation 10 seconds. If the probability is 0.8212 that the bag will take less than 282.5 seconds to pop, find the probability that it will take longer than 258.3 seconds to pop.

Solution: Given the time to microwave a bag of popcorn using the automatic setting can be treated as a random variable having a normal distribution. Let it be X.

Also given standard deviation, σ=10 seconds.

And the probability that the bag will take less than 282.5 seconds to pop is 0.8212.

$$i.e,\ P(X \le 282.5) = 0.8212$$

$$\Rightarrow P\left(\frac{X-\mu}{\sigma} \le \frac{282.5-\mu}{\sigma}\right) = 0.8212$$

$$\Rightarrow P\left(Z \le \frac{282.5-\mu}{\sigma}\right) = 0.8212$$

$$\Rightarrow F\left(\frac{282.5-\mu}{\sigma}\right) = 0.8212 = F(0.92)$$

$$\Rightarrow \frac{282.5-\mu}{\sigma} = 0.92$$

$$\Rightarrow \mu = 273.3\,\text{sec}$$

The probability that the bag will take longer than 258.3 seconds to pop is

$$P(X \geq 258.3) = P\left(\frac{X-\mu}{\sigma} \geq \frac{258.3-\mu}{\sigma}\right)$$

$$= P\left(Z \geq \frac{258.3-273.3}{10}\right)$$

$$= P(Z \geq -1.5)$$

= 1-F(-1.5)

= F(1.5) = 0.9332

The Normal Approximation to the Binomial distribution:

If X is a random variable having the binomial distribution with the parameters n and p, the limiting form of the distribution function of the standardized random variable $Z = \frac{X-np}{\sqrt{np(1-p)}}$ as n$\to\infty$, is given by the standard normal distribution

$$F(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt, -\infty < z < \infty.$$

In Binomial distribution, Mean, μ=np and variance, σ^2=npq=np(1-p)

Where n: number of experiments

P: probability of successes

q: probability of failures.

Formulae:

1. $P(X < a) = P(X \leq a - 0.5)$
2. $P(X \leq a) = P(X \leq a + 0.5)$
3. $P(X = a) = P(a - 0.5 \leq X \leq a + 0.5)$
4. $P(X > a) = P(X \geq a + 0.5)$
5. $P(X \geq a) = P(X \geq a - 0.5)$
6. $P(a < X < b) = P(a - 0.5 \leq X \leq b + 0.5)$

Problem 19: If 20% of the memory chips made in a certain plant are defective, what are the probabilities that in a lot of 100 randomly chosen for inspection

(i) At most 15 will be defective;

(ii) Exactly 15 will be defective.

Solution: Given n=100, p=20%=0.2, q=1-p =1-0.2 =0.8

Mean, μ= np =20

Standard deviation, $\sigma = \sqrt{npq} = \sqrt{100(0.2)(0.8)} = \sqrt{16} = 4$

(i) The probability that at most 15 will be defective is

$$P(X \le 15) = P(X \le 15.5)$$
$$= P\left(\frac{X-\mu}{\sigma} \le \frac{15.5-\mu}{\sigma}\right)$$
$$= P\left(Z \le \frac{15.5-20}{4}\right)$$
$$= P(Z \le -1.125)$$

= F(-1.125)

= F(-1.13)

= 0.1292

(ii) The probability that exactly 15 will be defective is

$$P(X = 15) = P(14.5 \le X \le 15.5)$$
$$= P\left(\frac{14.5-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{15.5-\mu}{\sigma}\right)$$
$$= P\left(\frac{14.5-20}{4} \le Z \le \frac{15.5-20}{4}\right)$$
$$= P(-1.375 \le Z \le -1.125)$$
$$= P(-1.38 \le Z \le -1.13)$$

= F(-1.13)- F(-1.38)

= 0.1292 – 0.0838 = 0.0454

Problem 20: If a random variable has the binomial distribution with n=40 and p=0.40, Use the normal approximation to determine the probabilities that it will take on (i) The value 22; (ii) a value less than 8.

Solution: Given n=40, p=0.40, q=1-p =1-0.40 =0.60

Mean, μ=np =16

Standard deviation, $\sigma = \sqrt{npq} = \sqrt{40(0.40)(0.60)} = \sqrt{9.6} = 3.1$

(i) The probability that it will take on a value 22 is

$$P(X = 22) = P(21.5 \le X \le 22.5)$$
$$= P\left(\frac{21.5-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{22.5-\mu}{\sigma}\right)$$
$$= P\left(\frac{21.5-16}{3.1} \le Z \le \frac{22.5-16}{3.1}\right)$$
$$= P(1.7742 \le Z \le 2.0968)$$

= F(2.01)- F(1.77)

= 0.9821 – 0.9616

= 0.0205

(ii)The probability that it will take less than 8 is

$$P(X < 8) = P(X \le 7.5)$$
$$= P\left(\frac{X-\mu}{\sigma} \le \frac{7.5-\mu}{\sigma}\right)$$
$$= P\left(Z \le \frac{7.5-16}{3.1}\right)$$
$$= P(Z \le -2.7419)$$

= F(-2.74)

= 0.0031

<u>Problem 21</u>:A manufacturer knows that, on average, 2% of the electric toasters that he makes will require repairs within 90 days after they are sold. Use the normal approximation to the binomial distribution to determine the probability that among 1200 of these toasters at least 30 will require repairs within the first 90 days after they are sold.

<u>Solution</u>: Given the number of electric toasters that will require repair within 90 days is a random variable.

Here n=1200, p=2%=0.02, q=1-p =1-0.02 =0.98

Mean, μ=np =24

Standard deviation, $\sigma = \sqrt{npq} = \sqrt{1200(0.02)(0.98)} = 4.8$

(i) The probability that at least 30 toasters will require repairs with in the first 90 days is

$$P(X \ge 30) = P(X \ge 29.5)$$
$$= P\left(\frac{X-\mu}{\sigma} \ge \frac{29.5-24}{4.85}\right)$$
$$= P(Z \ge 1.1340)$$
$$= 1\text{-}F(1.13)$$
$$= 1 - 0.8708$$
$$= 0.1292$$

<u>Problem 22</u>: If 62% of all clouds seeded with silver iodide show spectacular growth, what is the probability that among 40 clouds seeded with silver iodide at most 20 will show spectacular growth?

<u>Solution</u>: Given the number of clouds seeded with silver iodide is a random variable.

Here n=40, p=62%=0.62, q=1-p =1-0.62 =0.38

Mean, μ=np =24.8

Standard deviation, $\sigma = \sqrt{npq} = \sqrt{40(0.62)(0.38)} = 3.07$

(i) The probability that 20 clouds will show spectacular growth is

$$P(X \le 20) = P(X \le 20.5)$$
$$= P\left(\frac{X-\mu}{\sigma} \le \frac{20.5-24.8}{3.07}\right)$$
$$= P(Z \le -1.4007)$$
$$= F(-1.40)$$
$$= 0.0808$$

<u>Problem 23</u>: A safety engineer feels that 30% of all industrial accidents in her plant are caused by failure of employees to follow instructions. If this figure is correct, find, approximately, the probability that among 84 industrialized accidents in this plant anywhere from 20 to 30 (inclusive) will be due to failure of employees to follow instructions.

<u>Solution</u>: Treating the number of industrial accidents are done as a random variable.

Given n=84, p=0.30, q=1-p =1-0.30 =0.70

Mean, μ=np =25.2

Standard deviation, $\sigma = \sqrt{npq} = \sqrt{84(0.3)(0.7)} = \sqrt{17.64} = 4.2$

(i) The probability that the industrial accidents between 20 and 30 is

$$
\begin{aligned}
P(20 \leq X \leq 30) &= P(19.5 \leq X \leq 30.5) \\
&= P\left(\frac{19.5-\mu}{\sigma} \leq \frac{X-\mu}{\sigma} \leq \frac{30.5-\mu}{\sigma}\right) \\
&= P\left(\frac{19.5-25.2}{4.2} \leq Z \leq \frac{30.5-25.2}{4.2}\right) \\
&= P(-1.36 \leq Z \leq 1.26)
\end{aligned}
$$

= F(1.26)- F(-1.36)

= 0.8962 – 0.0869

= 0.8093

<u>Problem 24</u>: The probability that an electric component will fail in less than 1000 hours of continuous use is 0.25. Use the normal approximation to find the probability that among 200 such components fewer than 45 will fail in less than 1000 hours of continuous use.

<u>Solution</u>: Here the number of electric components that will fail in less than 1000 hours of continuous use is the random variable.

Given n=200, p=0.25, q=1-p =1-0.25 =0.75

Mean, μ = np =50

Standard deviation, $\sigma = \sqrt{npq} = \sqrt{200(0.2)(0.75)} = \sqrt{37.5} = 6.1237$

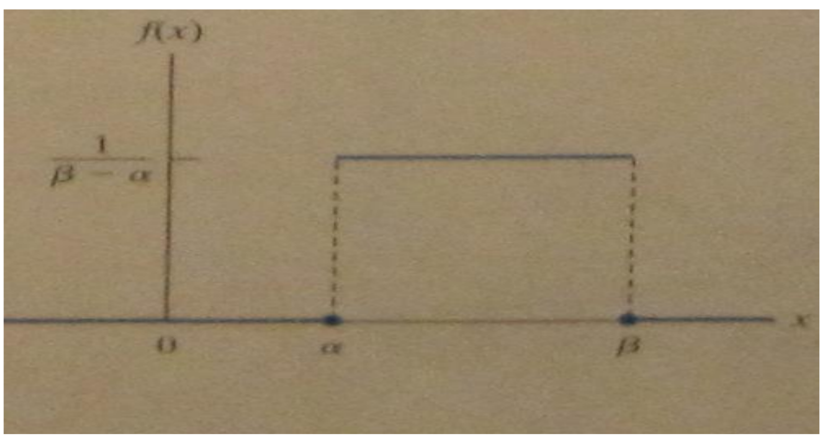
(i) The probability that the component fewer than 45 will fail in less than 1000 hours of continuous use is

$$P(X<45)=P(X\le 44.5)$$
$$=P\left(\frac{X-\mu}{\sigma}\le\frac{44.5-\mu}{\sigma}\right)$$
$$=P\left(Z\le\frac{44.5-50}{6.1237}\right)$$
$$=P(Z\le -0.8981)$$
$$= F(-0.90)$$
$$= 0.1841$$

The Uniform Distribution:

The probability density function of Uniform distribution with the parameters α and β is $f(x)=\begin{cases}\dfrac{1}{\beta-\alpha} & for\ \ \alpha<x<\beta\\ 0 & \text{elsewhere}\end{cases}$ $(OR)\ f(x)=\begin{cases}0 & for\ \ x<\alpha\\ \dfrac{1}{\beta-\alpha} & for\ \ \alpha<x<\beta\\ 0 & x>\beta\end{cases}$

The graph of Uniform Probability density is



Problem 1: Find the mean and variance of the Uniform distribution.

Sol: For the uniform distribution, the probability density function is

$$f(x)=\begin{cases}\dfrac{1}{\beta-\alpha} & for\ \ \alpha<x<\beta\\ 0 & elsewhere\end{cases}$$

$$\text{Mean}, \mu = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_{-\infty}^{\alpha} x\, f(x)\, dx + \int_{\alpha}^{\beta} x\, f(x)\, dx + \int_{\beta}^{\infty} x\, f(x)\, dx$$

$$= \int_{\alpha}^{\beta} x\left(\frac{1}{\beta-\alpha}\right) dx = \left(\frac{1}{\beta-\alpha}\right)\left(\frac{x^2}{2}\right)_{\alpha}^{\beta}$$

$$= \left(\frac{1}{\beta-\alpha}\right)\left(\frac{\beta^2-\alpha^2}{2}\right) = \frac{\alpha+\beta}{2}$$

$$\text{Variance}, \ \sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$$

$$= \int_{-\infty}^{\infty} x^2 f(x)\, dx - \mu^2$$

$$= \int_{\alpha}^{\beta} x^2 \frac{1}{\beta-\alpha}\, dx - \mu^2$$

$$= \frac{1}{\beta-\alpha}\left[\frac{x^3}{3}\right]_{\alpha}^{\beta} - \left(\frac{\beta+\alpha}{2}\right)^2$$

$$= \frac{\beta^3-\alpha^3}{3(\beta-\alpha)} - \left(\frac{\beta+\alpha}{2}\right)^2$$

$$= \frac{(\beta-\alpha)(\beta^2+\alpha^2+\alpha\beta)}{3(\beta-\alpha)} - \left(\frac{\beta+\alpha}{2}\right)^2$$

$$= \frac{(\beta^2+\alpha^2+\alpha\beta)}{3} - \frac{\beta^2+\alpha^2+2\alpha\beta}{4}$$

$$= \frac{\beta^2+\alpha^2-2\alpha\beta}{12} = \frac{(\beta-\alpha)^2}{12}$$

$$\text{Variance}, \sigma^2 = \frac{(\beta-\alpha)^2}{12}$$

Problem 2: In certain experiments, the error made in determining the solubility of a substance is a random variable having the uniform density with $\alpha = -0.025$ and $\beta = 0.025$. What are the probabilities that such an error will be (a) between 0.010 and 0.015; (b) between -0.012 and 0.012?

Also find the mean and variance of this distribution.

Solution: Given $\alpha = -0.025$, $\beta = 0.025$

Now $\frac{1}{\beta-\alpha} == \frac{1}{0.025+0.025} = \frac{1}{0.05} = 20$

Therefore the Uniform density function is

$$f(x) = \begin{cases} 20 & if \ -0.025 < X < 0.025 \\ 0 & \text{otherwise} \end{cases}$$

Here the error made in determining the solubility of substance is the random variable.

(a). The probability that the error will be between 0.010 and 0.015 is

$$P(0.010 \le X \le 0.015) = \int_{0.010}^{0.015} 20\, dx = 20(x)_{0.010}^{0.015} = 20(0.005) = 0.1$$

(b). The probability that the error will be between -0.012 and 0.012 is

$$P(-0.012 \le X \le 0.012) = \int_{-0.012}^{0.012} 20\, dx = 0.48$$

Mean $\mu = \frac{\alpha+\beta}{2} = \frac{-0.025+0.025}{2} = \frac{0}{2} = 0$

Variance $\sigma^2 = \frac{(\beta-\alpha)^2}{12} = \frac{(0.025+0.025)^2}{12} = \frac{(0.05)^2}{12} 0.000208$

The Gamma distribution:

The probability density function for the Gamma distribution is defined as

$$f(x) = \begin{cases} \frac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} & for \ \ x > 0, \ \alpha > 0, \beta > 0 \\ 0 & elsewhere \end{cases}$$

Where the gamma function is defined as $\Gamma(\alpha)=\int_0^\infty x^{\alpha-1}e^{-x}\,dx, \quad \alpha>0$

Note: $\Gamma(\alpha+1)=\alpha\,\Gamma(\alpha),\qquad \Gamma(1)=1,\ \Gamma(n+1)=n!$ for $n\in N$

<u>Problem 3</u>: Show that $\mu=\alpha\ \beta$ and $\sigma^2=\alpha\,\beta^2$ for the gamma distribution.

<u>Solution</u>: For the gamma distribution, the probability density function is

$$f(x)=\begin{cases}\dfrac{1}{\beta^\alpha\ \Gamma(\alpha)}x^{\alpha-1}e^{-x/\beta} & for\ x>0,\alpha>0,\beta>0\\ 0 & elsewhere\end{cases}$$

Mean, μ= $\int_{-\infty}^{\infty} x\ f(x)dx$

$$=\int_0^\infty x\frac{1}{\beta^\alpha\ \Gamma(\alpha)}x^{\alpha-1}e^{-x/\beta}dx$$

$$=\frac{1}{\beta^\alpha\ \Gamma(\alpha)}\int_0^\infty x^\alpha e^{-x/\beta}dx$$

$$=\frac{1}{\beta^\alpha\ \Gamma(\alpha)}\int_0^\infty (\beta t)^\alpha\ e^{-t}\beta dt\ ;\ let\ \frac{x}{\beta}=t\Rightarrow dx=\beta\ dt$$

$$=\frac{1}{\beta^\alpha\ \Gamma(\alpha)}\int_0^\infty \beta^{\alpha+1}t^\alpha e^{-t}dt$$

$$=\frac{\beta}{\Gamma(\alpha)}\int_0^\infty t^{\alpha+1-1}e^{-t}dt$$

$$=\frac{\beta}{\Gamma(\alpha)}\Gamma(\alpha+1)$$

$$=\frac{\beta}{\Gamma(\alpha)}\alpha\,\Gamma(\alpha)$$

$$=\alpha\beta$$

Variance, $\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$

$$= \int_{-\infty}^{\infty} x^2 \; f(x) dx - \mu^2$$

$$= \int_{0}^{\infty} x^2 \frac{1}{\beta^{\alpha}\,\Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} dx - \mu^2$$

$$= \frac{1}{\beta^{\alpha}\,\Gamma(\alpha)} \int_{0}^{\infty} x^{\alpha+1} e^{-x/\beta} dx - (\alpha\beta)^2$$

$$= \frac{1}{\beta^{\alpha}\,\Gamma(\alpha)} \int_{0}^{\infty} (\beta t)^{\alpha+1} e^{-t} \beta dt - (\alpha\beta)^2 \;\; ; \; let \; \frac{x}{\beta} = t \Rightarrow dx = \beta \; dt$$

$$= \frac{1}{\beta^{\alpha}\,\Gamma(\alpha)} \int_{0}^{\infty} \beta^{\alpha+2} t^{\alpha+1} e^{-t} dt - (\alpha\beta)^2$$

$$= \frac{\beta^2}{\Gamma(\alpha)} \int_{0}^{\infty} t^{\alpha+2-1} e^{-t} dt - (\alpha\beta)^2$$

$$= \frac{\beta^2}{\Gamma(\alpha)} \Gamma(\alpha+2) - (\alpha\beta)^2$$

$$= \frac{\beta^2}{\Gamma(\alpha)} (\alpha+1)\alpha\,\Gamma(\alpha) - (\alpha\beta)^2 \;\; \left(\text{Since } \Gamma(\alpha+2) = (\alpha+1)\Gamma(\alpha+1)\right)$$

$$= \alpha\beta^2\,(\alpha+1) - (\alpha\beta)^2$$

$$= \alpha\beta^2$$

__Note:__ If $\alpha=1$ in the gamma distribution then we get **__exponential distribution__**.

The probability density function for the exponential distribution is

$$f(x) = \begin{cases} \frac{1}{\beta} e^{-x/\beta} & for \; x > 0, \beta > 0 \\ 0 & elsewhere \end{cases}$$

For the exponential distribution $\mu = \beta$, $\sigma^2 = \beta^2$

<u>Problem 4</u>: If a random variable has the **<u>gamma distribution</u>** with $\alpha = 2$ and $\beta = 3$, find the mean and standard deviation of this distribution. Also find the probability that the random variable will take on a value less than 5.

<u>Solution</u>: Let X be a random variable having the gamma distribution with $\alpha = 2$ and $\beta = 3$

Mean, $\mu = \alpha\beta = 6$

Variance, $\sigma^2 = \alpha\beta^2 = 18$

Standard deviation, $\sigma = 4.2426$

The probability that the random variable will take on a value **less than 5** is

$$P(X < 5) = \int_{-\infty}^{5} f(x)dx$$

$$= \int_{0}^{5} \frac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} dx$$

$$= \int_{0}^{5} \frac{1}{3^2\Gamma(2)} x^{2-1} e^{-x/3} dx$$

$$= \frac{1}{9}\int_{0}^{5} x\, e^{-x/3} dx$$

$$= \frac{1}{9}\left[-3x\, e^{-x/3} - 9e^{-x/3}\right]_0^5$$

$$= \frac{1}{9}\left[-15e^{-5/3} - 9e^{-5/3} + 9\right]$$

$$= \frac{1}{9}\left[9 - 24e^{-5/3}\right]$$

$= \mathbf{0.4963}$

<u>Problem 5</u>: In a certain city, the daily consumption of electric power (in millions of kilowatt- hours) can be treated as a random variable having a gamma distribution with α = 3 and β = 2. If the city's power plant has a daily capacity of 12MKWH, what is the probability that this power supply will be inadequate on any given day?

<u>Solution</u>: The daily capacity of the power plant is 12MKWH.

Given the daily consumption of electric power is the random variable having a gamma distribution with α = 3 and β = 2. Let it be X.

The probability that 12MKWH power supply will be inadequate is

$$P(X > 12) = \int_{12}^{\infty} f(x)dx$$

$$= \int_{12}^{\infty} \frac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} dx$$

$$= \int_{12}^{\infty} \frac{1}{2^{3}\Gamma(3)} x^{3-1} e^{-x/2} dx$$

$$= \frac{1}{16}\int_{12}^{\infty} x^{2} e^{-x/2} dx$$

$$= \frac{1}{16}\left[-2x^{2}e^{-x/2} - 8xe^{-x/2} - 16e^{-x/2}\right]_{12}^{\infty}$$

$= 25e^{-6}$

= 0.06197

<u>The Beta distribution:</u>

For the beta distribution, the probability density is defined as

$$f(x)=\begin{cases}\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1} & \text{for } 0<x<1, \alpha>0, \beta>0\\ 0 & \text{elsewhere}\end{cases}$$

Note: $\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx=\beta(n,m)$

<u>Problem 6:</u> Find mean and variance for Beta distribution.

<u>Solution:</u> For beta distribution, the probability density function is

$$f(x)=\begin{cases}\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1} & \text{for } 0<x<1,\ \alpha>0,\ \beta>0\\ 0 & \text{elsewhere}\end{cases}$$

Mean, $\mu=\int_{-\infty}^{\infty} x\ f(x)dx$

$$=\int_{-\infty}^{0} x\ f(x)dx+\int_0^1 x\ f(x)dx+\int_1^{\infty} x\ f(x)dx$$

$$=\int_0^1 x\ \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}\ dx$$

$$=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 x^{\alpha}(1-x)^{\beta-1}\ dx$$

$$=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\beta(\alpha+1,\beta)\quad\left(\text{Since } \beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\ dx\right)$$

$$=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\ \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)}\quad\left(\text{Since } \beta(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\right)$$

$$\mu=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\ \frac{\alpha\,\Gamma(\alpha)\Gamma(\beta)}{(\alpha+\beta)\,\Gamma(\alpha+\beta)}\quad\left(\text{Since } \Gamma(n+1)=n\,\Gamma(n)\right)$$

$$\mu=\frac{\alpha}{\alpha+\beta}$$

Variance, $\sigma^2 = \mu_2^1 - \mu^2$

Consider $\mu_2^1 = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

$$= \int_{-\infty}^{0} x^2 f(x)\,dx + \int_{0}^{1} x^2 f(x)\,dx + \int_{1}^{\infty} x^2 f(x)\,dx$$

$$= \int_0^1 x^2 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}\;dx$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 x^{\alpha+1}(1-x)^{\beta-1}\;dx$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\,\beta(\alpha+2,\beta)$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\;\frac{\Gamma(\alpha+2)\Gamma(\beta)}{\Gamma(\alpha+\beta+2)}$$

$$\mu_2^1 = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\;\frac{(\alpha+1)\,\Gamma(\alpha+1)\Gamma(\beta)}{(\alpha+\beta+1)\,\Gamma(\alpha+\beta+1)}$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\;\frac{(\alpha+1)\,\alpha\,\Gamma(\alpha)}{(\alpha+\beta+1)\,(\alpha+\beta)\,\Gamma(\alpha+\beta)}$$

$$= \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$$

$$\sigma^2 = \mu_2^1 - \mu^2$$

$$= \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} - \frac{\alpha^2}{(\alpha+\beta)^2}$$

$$= \frac{(\alpha+\beta)\alpha(\alpha+1) - \alpha^2(\alpha+\beta+1)}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$$= \frac{\alpha^3+\alpha^2\beta+\alpha^2+\alpha\beta-\alpha^3-\alpha^2\beta-\alpha^2}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$$= \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Problem 7: In a certain country, the proportion of highway sections requiring repairs in any given year is a random variable having the beta distribution with $\alpha = 3$ and $\beta = 2$. Find

(a) On the average, what percentage of the highway sections require repairs in any given year?

(b) Find the probability that at most half of the highway sections will require repairs in any given year.

Solution: let X be a random variable having beta distribution with $\alpha = 3$ and $\beta = 2$.

(a) Average (Mean), $\mu = \dfrac{\alpha}{\alpha+\beta} = \dfrac{3}{5} = 0.6.$ Therefore on average 60% of the highway sections require repairs in any given year.

(b) The probability that at most half of the highway sections will require repairs in any given year is

$$P\left(X \le \frac{1}{2}\right) = \int_{-\infty}^{\frac{1}{2}} f(x)\,dx \quad = \int_{0}^{\frac{1}{2}} f(x)\,dx$$

(c)

$$= \int_{0}^{\frac{1}{2}} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}\,dx$$

$$= \int_{0}^{\frac{1}{2}} \frac{\Gamma(5)}{\Gamma(3)\Gamma(2)} x^{2}(1-x)\,dx$$

$$= \frac{4!}{2!\ 1!} \int_{0}^{\frac{1}{2}} x^{2}(1-x)\,dx$$

$$= 12\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^{\frac{1}{2}}$$

$$= 12\left[\frac{1}{24} - \frac{1}{64}\right]$$

$$= \frac{5}{16} = 0.3125$$

<u>Problem 8:</u> Verify for α = 3 and β = 2 that the integral of the beta density, from 0 to 1, is equal to 1.

<u>Solution</u>: Let f(x) be the probability density function with α = 3 and β = 3.

$$\text{Consider} \quad \int_0^1 f(x)\,dx = \int_0^1 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}\ dx$$

$$= \int_0^1 \frac{\Gamma(6)}{\Gamma(3)\Gamma(3)} x^2(1-x)^2\ dx$$

$$= \frac{5!}{2!\ 2!} \int_0^1 x^2(1+x^2-2x)\,dx$$

$$= \frac{120}{(2)(2)} \int_0^1 x^2+x^4-2x^3\ dx$$

$$= 30\left[\frac{x^3}{3}+\frac{x^5}{5} - \frac{2x^4}{4}\right]_0^1$$

$$= 30\left[\frac{1}{3}+\frac{1}{5}-\frac{1}{2}\right] = 1$$

<u>Problem 9</u>: Suppose that the proportion of defectives shipped by a vendor, which varies somewhat from shipment to shipment, may be looked upon as a random variable having the beta distribution with α = 1 and β = 4.

(a). Find the mean of this beta distribution, namely, the average proportion of defectives in a shipment from this vendor.

(b). Find the probability that a shipment from this vendor will contain 25% or more defectives.

<u>Solution</u>: Let X be a random variable having the beta distribution with α = 1 and β = 4.

(a) Average (mean), $\mu = \frac{\alpha}{\alpha+\beta} = \frac{1}{5} = 0.2.$

(b) The probability that a shipment from this vendor will contain 25% or more defectives is

$$P\left(X \geq 0.25\right) = 1 - P\left(X < 0.25\right)$$

$$= 1 - \int_{-\infty}^{0.25} f(x)\, dx$$

$$= 1 - \int_{0}^{0.25} f(x)\, dx$$

$$= 1 - \int_{0}^{0.25} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}\ dx$$

$$= 1 - \int_{0}^{0.25} \frac{\Gamma(5)}{\Gamma(1)\Gamma(4)} x^{0}(1-x)^{3}\ dx$$

$$= 1 - \frac{4!}{3!} \int_{0}^{0.25} (1 - x^3 + 3x^2 - 3x)\, dx$$

$$= 1 - 4\left[x - \frac{x^4}{4} + x^3 \quad - \frac{3x^2}{2} \right]_0^{0.25}$$

$$= 1 - 4\left[0.25 - \frac{(0.25)^4}{4} + (0.25)^3 - \frac{3(0.25)^2}{2} \right]$$

$$= 1 - 0.3164 = 0.6836$$

Problem 10: If the annual proportion of erroneous income tax returns filed with the IRS can be looked upon as a random variable having a beta distribution with α = 2 and β = 9, what is the probability that in any given year there will be fewer than 10% erroneous returns?

Solution: Let X be the random variable having beta distribution with α = 2 and β = 9.

$$P\left(X < 10\%\right) = P\left(X < 0.1\right)$$

$$= \int_{-\infty}^{0.1} f(x)\, dx$$

$$= \int_{0}^{0.1} f(x)\, dx$$

$$= \int_{0}^{0.1} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}\ dx$$

$$= \int_0^{0.1} \frac{\Gamma(11)}{\Gamma(2)\Gamma(9)} x(1-x)^8 \; dx$$

$$= \frac{10!}{1!\; 8!} \int_0^{0.1} x(1-\mathrm{x})^8 \; dx$$

$$= 90 \int_0^{0.1} (x-1+1)(1-x)^8 \; dx$$

$$= 90 \int_0^{0.1} \left[(x-1)(1-x)^8 + (1-x)^8 \right] dx$$

$$= 90 \int_0^{0.1} \left[-(1-x)^9 + (1-x)^8 \right] dx$$

$$= 90 \left[\frac{-(1-x)^{10}}{-10} + \frac{(1-x)^9}{-9} \right]_0^{0.1}$$

$$= 0.2639$$

The Weibull Distribution

The probability density function of the Weibull distribution is given by

$$f(x) = \begin{cases} \alpha\beta x^{\beta-1} e^{-\alpha x^{\beta}} & \text{for } x > 0, \alpha > 0, \beta > 0 \\ 0 & \text{elsewhere} \end{cases}$$

The mean of the Weibull distribution is given by $\mu = \alpha^{-1/\beta} \Gamma(1 + \frac{1}{\beta})$ and the variance of the Weibull distribution is given by $\sigma^2 = \alpha^{-2/\beta} \left\{ \Gamma(1+\frac{2}{\beta}) - \left[\Gamma(1+\frac{1}{\beta}) \right]^2 \right\}$

The Weibull distribution is widely used in reliability and life data analysis.

Problem 11: Suppose that the lifetime of a certain kind of an emergency backup battery (in hours) is a random variable X having Weibull distribution with α = 0.1 and β = 0.5. Find

(a) The mean life of these batteries.

(b) The probability that such a battery will last more than 300 hours.

Solution : Given X is a random variable having Weibull distribution that denotes the lifetime of a certain kind of an emergency backup battery (in hours) with α = 0.1 and β = 0.5.

(a) We know that the mean life of X is $\mu = \alpha^{-1/\beta}\Gamma(1+\frac{1}{\beta})$

$so\ \mu = (0.1)^{-1/0.5}\Gamma(3)$ = 200 hours.

(b) P(X > 300) = $\int_{300}^{\infty} \alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}\, dx$

$$= \int_{300}^{\infty} (0.1)(0.5)x^{0.5-1}e^{-0.1x^{0.5}}\, dx$$

$$= \int_{300}^{\infty} (0.1)(0.5)x^{-0.5}e^{-0.1x^{0.5}}\, dx$$

$$= \int_{300}^{\infty} (0.05)x^{-0.5}e^{-0.1x^{0.5}}\, dx$$

put $x^{0.5}$ = t then 0.5 $x^{-0.5}$dx = dt

$$= \int_{\sqrt{300}}^{\infty} (0.1)e^{-0.1t}\, dt$$

$$= \left[(0.1)\frac{e^{-0.1t}}{-0.1}\right]_{\sqrt{300}}^{\infty}$$

$$= e^{-0.1\sqrt{(300)}}$$

= 0.177

<u>Problem 12</u>: Suppose that the time to failure (in minutes) of a certain electronic components subjected to continuous vibrations may be looked upon as a random variable is having Weibull distribution with α = 1/5 and β = 1/3.

(a) How long can such a component be expected to last?

(b) What is the probability that such a component will fail in less than 5 hours?

<u>Solution</u> : Given that X is a random variable having Weibull distribution with α = 1/5 and β = 1/3 that denotes the time to failure (in minutes) of a certain electronic components.

(a) The mean life of such components is given by $\mu = \alpha^{-1/\beta}\Gamma(1+\frac{1}{\beta})$

$$\mu = \left(\frac{1}{5}\right)^{-3}\Gamma(4)$$

$= 125 * 6 = 750$

(b) $P(X < 300) = \int_0^{300} \alpha\beta x^{\beta-1} e^{-\alpha x^{\beta}}\, dx$ (5 hours means 300 minutes)

$$= \int_0^{300} \left(\frac{1}{5}\right)\left(\frac{1}{3}\right) x^{\frac{1}{3}-1} e^{-\frac{1}{5}x^{1/3}}\, dx$$ put $x^{1/3} = t$ then $(1/3)\, x^{(1/3)-1} = dt$

$$= \frac{1}{5} \int_0^{\sqrt[3]{300}} e^{-\frac{1}{5}t}\, dt$$

$$= \frac{1}{5} \left[\frac{e^{-\frac{1}{5}t}}{-\frac{1}{5}} \right]_0^{\sqrt[3]{300}}$$

$$= \left[\frac{e^{-\frac{1}{5}t}}{-1} \right]_0^{\sqrt[3]{300}}$$

$$= \left[1 - e^{-1/5(300)^{1/3}} \right]$$

$$= 0.7379$$

Problem 13: Suppose that the service life (in hours) of a semiconductor device is a random variable having Weibull distribution with $\alpha = 0.025$ and $\beta = 0.500$. What is the probability that such a device will still be operating condition after 4,000 hours?

HINT: Find P(X > 4000).

Joint distributions:

Let X_1 and X_2 be two **discrete** random variables defined on a sample space **S** of an experiment. The probability of the intersection of events X_1 will take a values x_1 and X_2 will take on a value x_2 is denoted by $P\left(X_1 = x_1,\, X_2 = x_2\right) = f\left(x_1, x_2\right)$.

Joint Probability distribution:

A function $f(x_1, x_2)$ is said to be a **joint probability distribution** function of X_1 and X_2 if it satisfies $(i)\ f(x_1, x_2) \geq 0 \qquad (ii)\ \sum_{x_1} \sum_{x_2} f(x_1, x_2) = 1$

Marginal probability distribution:

Let $f(x_1, x_2)$ be a joint probability distribution of two random variables X_1 and X_2. The marginal probability distribution of the random variable X_1 is denoted by $f_1(x_1)$ and defined as $f_1(x_1) = \sum_{x_2} f(x_1, x_2)$.

Similarly, the marginal probability distribution of the random variable X_2 is denoted by $f_2(x_2)$ and defined as $f_2(x_2) = \sum_{x_1} f(x_1, x_2)$

Conditional probability distribution: Let $f(x_1, x_2)$ be a joint probability distribution of two random variables X_1 and X_2, then the conditional probability distribution of X_1 given $X_2 = x_2$ is denoted by $f_1(x_1 | x_2)$ and is defined as

$f_1(x_1 | x_2) = \dfrac{f(x_1, x_2)}{f_2(x_2)}$ for all x_1, x_2 provided $f_2(x_2) \neq 0$.

Similarly, the conditional probability distribution of X_2 given $X_1 = x_1$ is denoted by $f_2(x_2 | x_1)$ and is defined as

$f_2(x_2 | x_1) = \dfrac{f(x_1, x_2)}{f_1(x_1)}$ for all x_1, x_2 provided $f_1(x_1) \neq 0$.

Independent random variables: Let $f(x_1, x_2)$ be a joint probability distribution of two random variables X_1 and X_2. The two random variables X_1 and X_2 are said to be **independent** if $f(x_1, x_2) = f_1(x_1)\, f_2(x_2)$ for all x_1, x_2.

Problem 14: Let X_1 and X_2 have the joint probability distribution in the table given below

Joint Probability Distribution $f(x_1, x_2)$ of X_1 and X_2				
		x_1		
		0	1	2
x_2	0	0.1	0.4	0.1
	1	0.2	0.2	0

(a) Find $P(X_1 + X_2 > 1)$.

(b) Find the probability distribution $f_1(x_1) = P(X_1 = x_1)$ of the individual random variable X_1.

(c) Find the conditional probability distribution of X_1 at $X_2 = 1$.

(d) Are X_1 and X_2 independent?

Solution: From the given table

$$f(0,0) = 0.1, \qquad f(0,1) = 0.2$$
$$f(1,0) = 0.4 \qquad f(1,1) = 0.2$$
$$f(2,0) = 0.1 \qquad f(2,1) = 0$$

(a) The pair of values corresponding to $X_1 + X_2 > 1$ are $(1,1),(2,0)$ and $(2,1)$. Then by adding the corresponding probabilities,

$$P(X_1 + X_2 > 1) = f(1,1) + f(2,0) + f(2,1)$$
$$= 0.2 + 0.1 + 0 = 0.3$$

(b) Since $X_1 = 0,1 \text{ or } 2$, corresponding to the event $X_1 = 0$ the set of all possible pairs of values are (0,0), (0,1). Then by adding the corresponding probabilities

$$f_1(0) = f(0,0) + f(0,1)$$
$$= 0.1 + 0.2 = 0.3$$

Similarly, corresponding to the event $X_1 = 1$ the set of all possible pairs of values are (1,0), (1,1). Then by adding the corresponding probabilities

$$f_1(1) = f(1,0) + f(1,1)$$
$$= 0.4 + 0.2 = 0.6$$

Similarly, corresponding to the event $X_1 = 2$ the set of all possible pairs of values are (2,0), (2,1). Then by adding the corresponding probabilities

$$f_1(2) = f(2,0) + f(2,1)$$
$$= 0.1 + 0 = 0.1$$

Corresponding to the event $X_2 = 0$ the set of all possible pairs of values are (0,0), (1,0), (2,0). Then by adding the corresponding probabilities

$$f_2(0) = f(0,0) + f(1,0) + f(2,0)$$
$$= 0.1 + 0.4 + 0.1 = 0.6$$

Corresponding to the event $X_2 = 1$ the set of all possible pairs of values are (0,1), (1,1), (2,1). Then by adding the corresponding probabilities

$$f_2(1) = f(0,1) + f(1,1) + f(2,1)$$
$$= 0.2 + 0.2 + 0 = 0.4$$

Then the joint distribution table together with marginal distributions are

Joint Probability Distribution $f(x_1, x_2)$ of X_1 and X_2 with Marginal Distributions

		x_1: 0	1	2	Total $f_2(x_2)$
x_2	0	0.1	0.4	0.1	0.6
	1	0.2	0.2	0	0.4
Total	$f_1(x_1)$	0.3	0.6	0.1	1.0

(c) The conditional probability of X_1 at $X_2 = 1$ is $f(x_1 \mid X_2 = 1) = \dfrac{f(x_1,1)}{f_2(1)}$

Since X_1 can take the values 0, 1 and 2 and $f_2(1) = 0.4$

$$f(0 \mid 1) = \frac{f(0,1)}{f_2(1)} = \frac{0.2}{0.4} = 0.5$$

$$f(1|1)=\frac{f(1,1)}{f_2(1)}=\frac{0.2}{0.4}=0.5$$

$$f(2|1)=\frac{f(2,1)}{f_2(1)}=\frac{0}{0.4}=0$$

(d) Since at the point (0,0), from the table f(0,0) = 0.1, $f_1(0)=0.3$, $f_2(0)=0.6$

Then $f(x_1,x_2)\neq f_1(x_1)\,f(x_2)$ at $x_1=0, x_2=0$

Hence the two random variables X_1 and X_2 are not independent.

Problem 15: Two scanners are needed for an experiment of the five available, two have electronic defects, another one has a defect in the memory and two are in good working order. Two units are selected at random.

(a) Find the joint probability distribution of X_1 = the number with electronic defects, and X_2 = the number with a defect in memory.

(b) Find the probability of 0 or 1 total defects among the two selected.

(c) Find the marginal probability distribution of X_1.

(d) Find the conditional probability distribution of X_1 given X_2 = 0.

Solution: Given that the total number of scanners = 5,

Number of electronic defect scanners = 2,

Number of defects in the memory scanners = 1,

Number of good conditioned scanners = 2.

Let X_1 be the random variable corresponds to the number with electronic defects, then $X_1=\{0,1,2\}$ and X_2 be the random variable corresponds to the number of scanners with defect in the memory, then $X_2=\{0,1\}$

(a) $P(X_1=x_1, X_2=x_2)=f(x_1,x_2)=\dfrac{{}^2C_{x_1}\,{}^1C_{x_2}\,{}^2C_{2-(x_1+x_2)}}{{}^5C_2}\quad X_1=\{0,1,2\}, X_2=\{0,1\}$

Then $f(0,0)=\frac{{}^2C_0\,{}^1C_0\,{}^2C_2}{{}^5C_2}=\frac{1}{10}=0.1$

$f(0,1)=\frac{{}^2C_0\,{}^1C_1\,{}^2C_1}{{}^5c_2}=\frac{2}{10}=0.2$

$f(1,0)=\frac{{}^2C_1\,{}^1C_0\,{}^2C_1}{{}^5C_2}=\frac{4}{10}=0.4$

$f(1,1)=\frac{{}^2C_1\,{}^1C_1\,{}^2C_0}{{}^5C_2}=\frac{1}{10}=0.1$

$f(2,0)=\frac{{}^2C_2\,{}^1C_0\,{}^2C_0}{{}^5C_2}=\frac{1}{10}=0.1$

$f(2,1)=0$

(b) The probability of 0 or 1 total defects among the two selected is

$$f(0,0)+f(0,1)+f(1,0)=$$

$$\frac{{}^2C_0\,{}^1C_0\,{}^2C_2}{{}^5C_2}+\frac{{}^2C_1\,{}^1C_0\,{}^2C_1}{{}^5C_2}+\frac{{}^2C_0\,{}^1C_1\,{}^2C_1}{{}^5C_2}=\frac{1}{10}+\frac{4}{10}+\frac{2}{10}=0.7$$

The joint probability distribution table is as follows:

Joint probability distribution of the random variables X_1 and X_2				
	X_1			
X_2		0	1	2
	0	**0.1**	**0.4**	**0.1**
	1	**0.2**	**0.2**	**0**

(c) The marginal distribution of X_1 is $f_1(x_1)=\sum_{x_2} f(x_1,x_2)$

Then $f_1(0)=f(0,0)+f(0,1)=0.1+0.2=0.3$

$f_1(1) = f(1,0) + f(1,1) = 0.4 + 0.2 = 0.6$

$f_1(2) = f(2,0) + f(2,1) = 0.1 + 0 = 0.1$

<table>
<tr><td colspan="5">Joint probability distribution of the random variables X_1 and X_2 with marginal distributions</td><td></td></tr>
<tr><td></td><td colspan="4">X_1</td><td></td></tr>
<tr><td rowspan="4">X_2</td><td></td><td>0</td><td>1</td><td>2</td><td></td></tr>
<tr><td>0</td><td>0.1</td><td>0.4</td><td>0.1</td><td>$f_2(0) = 0.6$</td></tr>
<tr><td>1</td><td>0.2</td><td>0.2</td><td>0</td><td>$f_2(1) = 0.4$</td></tr>
<tr><td></td><td>$f_1(0) = 0.3$</td><td>$f_1(1) = 0.6$</td><td>$f_1(2) = 0.1$</td><td>1(grand total)</td></tr>
</table>

(d) The conditional probability of X_1 given $X_2 = 0$ is

$$f_1(x_1 | 0) = \frac{f(x_1, 0)}{f_2(0)}$$

Then $f_1(0|0) = \frac{f(0,0)}{f_2(0)} = \frac{0.1}{0.6} = 0.17$

$$f_1(1|0) = \frac{f(1,0)}{f_2(0)} = \frac{0.4}{0.6} = 0.67$$

$$f_1(2|0) = \frac{f(2,0)}{f_2(0)} = \frac{0.1}{0.6} = 0.17$$

Problem 16: Two random variables are independent and each has a binomial distribution with success probability 0.3 and 2 trials.

(a) Find the joint probability distribution.

(b) Find the probability that the second random variable is greater than the first.

Solution: Let X_1 and X_2 be two independent random variables each having the binomial distribution with success probability p = 0.3 and sample size n = 2

Then each of the random variables X_1 and X_2 takes the values 0, 1 or 2.

We know that from Binomial distribution, the probability of x successes in n trails

is $b(x;n,p) = {}^{n}C_x\ p^x (1-p)^{n-x}$

Consider $b(0;2,0.3) = {}^{2}C_0\ (0.3)^0 (1-0.3)^{2-0} = 0.49$

$b(1;2,0.3) = {}^{2}C_1\ (0.3)^1 (1-0.3)^{2-1} = 0.42$

$b(2;2,0.3) = {}^{2}C_2\ (0.3)^2 (1-0.3)^{2-2} = 0.09$

(a) Since X_1 and X_2 are independent

Then the joint probability distribution function is

$f(x_1,x_2) = b(x_1;n,p)\, b(x_2;n,p) \quad x_1,x_2 \in \{0,1,2\}$

Then $f(x_1,x_2) = b(x_1;2,0.3)\, b(x_2;2,0.3) \quad x_1,x_2 \in \{0,1,2\}$

The corresponding joint probability distribution table is as follows:

		X₁			
		0	1	**2**	
X_2	0	**0.2401**	**0.2058**	**0.0441**	0.49
	1	**0.2058**	**0.1764**	**0.0378**	0.42
	2	**0.0441**	**0.0378**	**0.0081**	0.09
		0.49	0.42	0.09	

(b) The possible points for second random variable greater than the first $X_2 > X_1$ are (0,1) , (0,2) & (1,2). By adding all the probabilities,

$P(X_2 > X_1) = f(0,1) + f(0,2) + f(1,2)$

$= 0.2058 + 0.0441 + 0.0378 = 0.2877$

<u>Continuous Variables</u>:

Let $X_1, X_2, ..., X_k$ be k continuous random variables then the probability of $X_1 = x_1, X_2 = x_2, ..., X_k = x_k$ is denoted by

$P(X_1 = x_1, X_2 = x_2, ..., X_k = x_k) = f(x_1, x_2, ..., x_k)$

A function in k-variables $f(x_1, x_2, ..., x_k)$ is said to be **joint probability density function** if

(i) $f(x_1, x_2, ..., x_k) \geq 0$

(ii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}...\int_{-\infty}^{\infty} f(x_1, x_2, ..., x_k)\,dx_1 dx_2 ... dx_k = 1.$

Let $f(x_1, x_2, ..., x_k)$ be a joint probability density of $X_1, X_2, ..., X_k$ then

$$P(a_1 \leq X_1 \leq b_1, a_2 \leq X_2 \leq b_2, ..., a_k \leq X_k \leq b_k)$$

$$= \int_{a_1}^{b_1}\int_{a_2}^{b_2}...\int_{a_k}^{b_k} f(x_1, x_2, ..., x_k)\,dx_k dx_{k-1} ... dx_1$$

Let $f(x_1, x_2, ..., x_k)$ be a joint probability density of $X_1, X_2, ..., X_k$ then the **joint cumulative distribution** of k random variables is

$$F(x_1, x_2, ..., x_k) = P(X_1 \leq x_1, X_2 \leq x_2, ..., X_k \leq x_k)$$

$$= \int_{-\infty}^{x_1}\int_{-\infty}^{x_2}........\int_{-\infty}^{x_k} f(x_1, x_2, ..., x_k)\,dx_1 dx_2 ... dx_k$$

Let $f(x_1, x_2, ..., x_k)$ be a joint probability density of $X_1, X_2, ..., X_k$ then the **individual density** for X_i (or) **marginal probability density** for X_i is defined as

$$f_i(x_i) = \int_{x_1=-\infty}^{\infty}\int_{x_2=-\infty}^{\infty}........\int_{x_{i-1}=-\infty}^{\infty}\int_{x_{i+1}=-\infty}^{\infty}.....\int_{x_k=-\infty}^{\infty} f(x_1, x_2, ..., x_k)\,dx_1 dx_2 ... dx_{i-1} dx_{i+1} ... dx_k$$

Let $f(x_1, x_2, ..., x_k)$ be a joint probability density of $X_1, X_2, ..., X_k$. The random variables $X_1, X_2, ..., X_k$ are said to be **independent** if

$$f(x_1, x_2, ..., x_k) = f_1(x_1) f_2(x_2) ... f_k(x_k) \text{ for all } x_1, x_2, ..., x_k$$

Problem 17: If the joint probability density of two random variables is given by

$$f(x_1, x_2) = \begin{cases} 6e^{-2x_1 - 3x_2} & \text{for } x_1 > 0, x_2 > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Find the probabilities that

(a) the first random variable will take on a value between 1 and 2 and the second random variable will take on a value between 2 and 3;

(b) the first random variable will take on a value less than 2 and the second random variable will take on a value greater than 2.

(c) Find the joint distribution of X_1, X_2 and use it to find the probability that both X_1, X_2 will take on a value less than 1.

(d) Find the marginal density of X_1.

(e) Are X_1 and X_2 independent?

<u>Solution</u>: The joint probability density function is

$$f\left(x_1,x_2\right)=\begin{cases}6e^{-2x_1-3x_2} & \text{for } x_1>0, x_2>0\\ 0 & \text{elsewhere}\end{cases}$$

(a) The probability of the first random variable will take on a value between 1 and 2 and the second random variable will take on a value between 2 and 3 is, $P\left(1\le X_1\le 2, 2\le X_2\le 3\right)=\int\limits_{x_1=1}^{2}\int\limits_{x_2=2}^{3} f\left(x_1,x_2\right)dx_1\,dx_2$

$$=\int\limits_{x_1=1}^{2}\int\limits_{x_2=2}^{3} 6e^{-2x_1-3x_2}\ dx_2\ dx_1$$

$$=\int\limits_{x_1=1}^{2} 6e^{-2x_1}\left(\frac{e^{-3x_2}}{-3}\right)_2^3 dx_1$$

$$=-2\int\limits_{x_1=1}^{2} e^{-2x_1}\left(e^{-9}-e^{-6}\right)dx_1$$

$$=-2\left(\frac{e^{-2x_1}}{-2}\right)_1^2\left(e^{-9}-e^{-6}\right)$$

$$=\left(e^{-4}-e^{-2}\right)\left(e^{-9}-e^{-6}\right)$$

(b) The probability of the first random variable will take on a value less than 2 and the second random variable will take on a value greater than 2 is

$$P\left(X_1 < 2, X_2 > 2\right) = \int_{x_1=-\infty}^{2} \int_{x_2=2}^{\infty} f\left(x_1, x_2\right) dx_1\, dx_2$$

$$= \int_{x_1=0}^{2} \int_{x_2=2}^{\infty} 6e^{-2x_1-3x_2}\, dx_2\, dx_1$$

$$= \int_{x_1=0}^{2} 6\, e^{-2x_1} \left(\frac{e^{-3x_2}}{-3}\right)_2^{\infty} dx_1$$

$$= -2 \int_{x_1=0}^{2} e^{-2x_1}\left(0 - e^{-6}\right) dx_1$$

$$= -2\left(\frac{e^{-2x_1}}{-2}\right)_0^2 \left(-e^{-6}\right)$$

$$= \left(e^{-4} - 1\right)\left(-e^{-6}\right)$$

$$= \left(e^{-6} - e^{-10}\right)$$

(c) The joint distribution function of x_1 and x_2 is $F\left(x_1, x_2\right) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} f\left(x_1, x_2\right) dx_1\, dx_2$

$$= \int_0^{x_1} \int_0^{x_2} 6e^{-2x_1-3x_2}\, dx_2 dx_1$$

$$= \int_0^{x_1} 6e^{-2x_1} \left(\frac{e^{-3x_2}}{-3}\right)_0^{x_2} dx_1$$

$$= 2\int_0^{x_1} e^{-2x_1}\left(1 - e^{-3x_2}\right) dx_1$$

$$= 2\left(1 - e^{-3x_2}\right)\left(\frac{e^{-2x_1}}{-2}\right)_0^{x_1}$$

$$= \left(1 - e^{-2x_1}\right)\left(1 - e^{-3x_2}\right)$$

Hence $F\left(x_1, x_2\right) = \begin{cases} \left(1 - e^{-2x_1}\right)\left(1 - e^{-3x_2}\right) & for\ x_1 > 0, x_2 > 0 \\ 0 & \text{elsewhere} \end{cases}$

Now the probability of both the random variables less than 1 is

$P\left(X_1 < 1, X_2 < 1\right) = F\left(1,1\right) = \left(1 - e^{-2}\right)\left(1 - e^{-3}\right)$

d) Marginal density of x_1 is $f_1(x_1) = \int\limits_{x_2=-\infty}^{\infty} f(x_1, x_2)\,dx_2$

$$= \int\limits_{x_2=0}^{\infty} 6e^{-2x_1-3x_2}\ dx_2$$

$$= 6e^{-2x_1}\left(\frac{e^{-3x_2}}{-3}\right)_0^{\infty}$$

$$= 2e^{-2x_1}$$

$$\therefore\ f_1(x_1) = \begin{cases} 2e^{-2x_1} & , for\ x_1 > 0 \\ 0 & ,\ elsewhere \end{cases}$$

Marginal density for x_2 is $f_2(x_2) = \int\limits_{x_1=-\infty}^{\infty} f(x_1, x_2)\,dx_1$

$$= \int\limits_{x_1=0}^{\infty} 6e^{-2x_1-3x_2}\ dx_1$$

$$= 6e^{-3x_2}\left(\frac{e^{-2x_1}}{-2}\right)_0^{\infty}$$

$$= 3e^{-3x_2}$$

$$\therefore\ f_2(x_2) = \begin{cases} 3e^{-3x_2} & , for\ x_2 > 0 \\ 0 & ,\ elsewhere \end{cases}$$

(c) Now $f_1(x_1)\, f_2(x_2) = \begin{cases} 2e^{-2x_1}\ 3e^{-3x_2} & , for\, x_1 > 0, x_2 > 0 \\ 0, & elsewhere \end{cases}$

$$= \begin{cases} 6e^{-2x_1-3x_2}, & for\ x_1 > 0, x_2 > 0 \\ 0 & ,\text{elsewhere} \end{cases}$$

$$= f(x_1, x_2) \quad \text{for all } x_1, x_2$$

$\therefore$ The two random variables X_1 and X_2 are independent.

<u>Problem 18:</u> The joint density of two continuous random variables X and Y is

$$f(x, y) = \begin{cases} c\,xy & , 0 < x < 4,\ 1 < y < 5 \\ 0 & ,\text{elsewhere} \end{cases}$$

(i) Find the value of c.

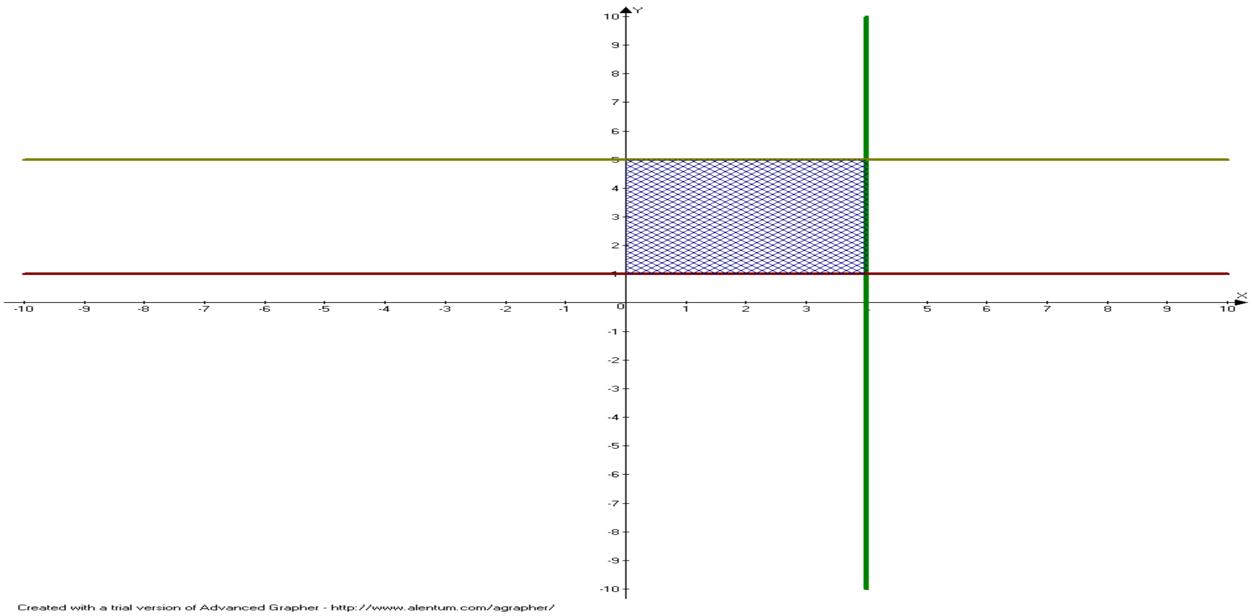
(ii) Find $P(X \geq 3, Y < 2)$

(iii)Find $P(X + Y < 3)$

Solution:

(i) Since the given function f(x,y) is joint density function we know that

$$f(x,y) \geq 0, \text{ and } \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1.$$



Hence $c \geq 0$, and $\int_{x=0}^{4}\int_{y=1}^{5} c\,x\,y\,dy\,dx = 1$

$$\Rightarrow \int_{x=0}^{4} cx\left(\frac{y^2}{2}\right)_1^5 dx = 1$$

$$\Rightarrow 12c\int_{x=0}^{4} x\,dx = 1$$

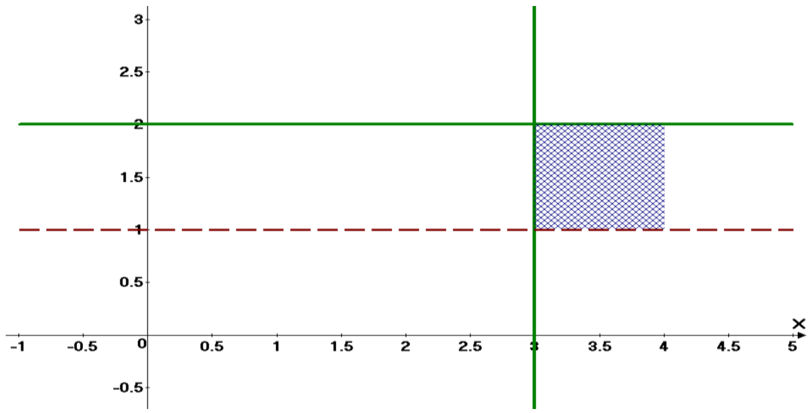
$$\Rightarrow 12c\left(\frac{x^2}{2}\right)_0^4 = 1$$

$$\Rightarrow 96c = 1$$

$$\Rightarrow c = \frac{1}{96}$$

$$\therefore\ f(x,y) = \begin{cases} \frac{1}{96}xy, & 0 < x < 4,\ 1 < y < 5 \\ 0, & elsewhere \end{cases}$$

(ii) Consider $P\left(X \geq 3, Y < 2\right) = \int_{x=3}^{\infty} \int_{y=-\infty}^{2} f\left(x,y\right) dx\, dy$

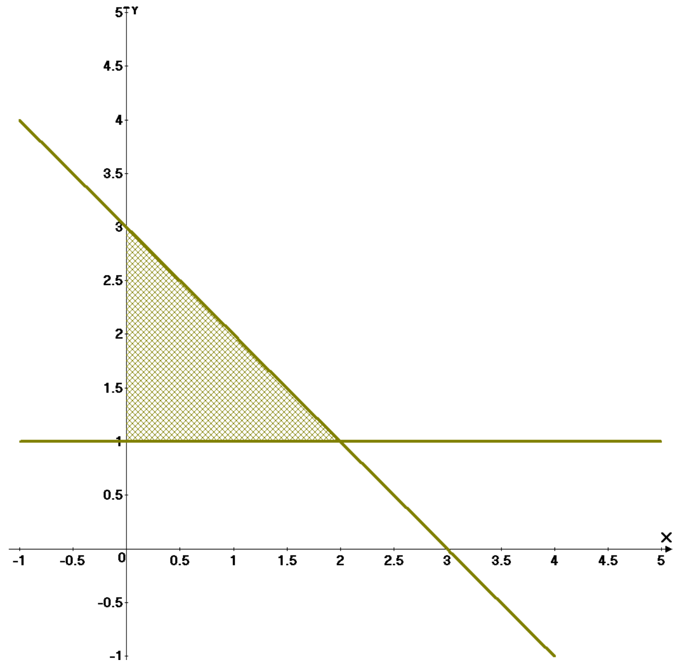


$$= \int_{x=3}^{4} \int_{y=1}^{2} \frac{1}{96} xy\ dy\ dx$$

$$= \int_{x=3}^{4} \frac{x}{96}\left(\frac{y^2}{2}\right)_1^2 dx$$

$$= \frac{3}{192}\left(\frac{x^2}{2}\right)_3^4 = \frac{7}{128}$$

(iii) $P\left(X + Y < 3\right) = \int_{x=0}^{2} \int_{y=1}^{3-x} \frac{1}{96} xy\ dy\ dx$



$$= \int_{x=0}^{2} \frac{x}{96}\left(\frac{y^2}{2}\right)_1^{3-x} dx$$

$$= \int_{x=0}^{2} \frac{x^3 + 8x - 6x^2}{192} dx$$

$$= \frac{1}{192}\left(\frac{x^4}{4} + \frac{8x^2}{2} - \frac{6x^3}{3}\right)_0^2$$

$$= \frac{1}{48}$$

Problem 19: If three random variables have the joint density

$$f(x,y,z) = \begin{cases} k(x+y)e^{-z} & ,0 < x < 1,\ 0 < y < 2,\ z > 0 \\ 0\ , & \text{elsewhere} \end{cases}$$

(i) Find the value of k.

(ii) Find $P(X < Y, Z > 1)$

(iii) Are X, Y, Z independent?

Solution: **(i)** Since f(x,y,z) is joint density function, we know that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,z)\,dx\,dy\,dz = 1$$

$$\Rightarrow \int_{x=0}^{1}\int_{y=0}^{2}\int_{z=0}^{\infty} k(x+y)e^{-z}dz\,dy\,dx = 1$$

$$\Rightarrow \int_{x=0}^{1}\int_{y=0}^{2} k(x+y)\left(\frac{e^{-z}}{-1}\right)_0^{\infty} dy\,dx = 1$$

$$\Rightarrow \int_{x=0}^{1}\int_{y=0}^{2} k(x+y)\,dy\,dx = 1$$

$$\Rightarrow \int_{x=0}^{1} k\left(xy+\frac{y^2}{2}\right)_0^2 dx = 1$$

$$\Rightarrow \int_{x=0}^{1} k(2x+2)\,dx = 1$$

$$\Rightarrow k\left(x^2+2x\right)_0^1 = 1$$

$$\Rightarrow 3k = 1$$

$$\Rightarrow k = \frac{1}{3}$$

$$\therefore\ f(x,y,z) = \begin{cases} \frac{1}{3}(x+y)e^{-z} & ,0<x<1, 0<y<2,\ z>0 \\ 0\ , & \text{elsewhere} \end{cases}$$

(ii) $P(X<Y,\ Z>1) = \iiint f(x,y,z)\,dx\,dy\,dz$

$$= \iint_R \int_{z=1}^{\infty} \frac{1}{3}(x+y)e^{-z}\,dz\,dy\,dx \qquad \left(\text{Here R is the region X<Y}\right)$$

$$= \iint_R \frac{1}{3}(x+y)\left(\frac{e^{-z}}{-1}\right)_1^{\infty} dy\,dx$$

$$= \iint_R \frac{e^{-1}}{3}(x+y)\ dy\,dx$$

$$= \frac{e^{-1}}{3}\int_{x=0}^{1}\int_{y=x}^{2}(x+y)\ dy\,dx$$

$$= \frac{e^{-1}}{3}\int_{x=0}^{1}\left(xy+\frac{y^2}{2}\right)_x^2 dx$$

$$= \frac{e^{-1}}{3}\int_{x=0}^{1}\left(2x+2-\frac{3x^2}{2}\right) dx$$

$$= \frac{e^{-1}}{3}\left(x^2+2x-\frac{x^3}{2}\right)_0^1 = \frac{5}{6e}$$

(iii) Consider marginal distribution of X, $f_1(x) = \int_{y=0}^{2}\int_{z=0}^{\infty}\frac{1}{3}(x+y)e^{-z}\,dz\,dy$

$$= \int_{y=0}^{2}\frac{1}{3}(x+y)\left(\frac{e^{-z}}{-1}\right)_0^{\infty} dy$$

$$= \frac{1}{3}\int_{y=0}^{2}(x+y)\ dy$$

$$= \frac{1}{3}\left(xy+\frac{y^2}{2}\right)_0^2$$

$$= \frac{1}{3}(2x+2)$$

$$\therefore\ f_1(x) = \begin{cases}\frac{1}{3}(2x+2) & \text{for } 0<x<1\\ 0 & \text{elsewhere}\end{cases}$$

Marginal distribution of Y**,** $f_2(y) = \int_{x=0}^{1}\int_{z=0}^{\infty}\frac{1}{3}(x+y)e^{-z}\,dz\ dx$

$$= \frac{1}{3}\int_{x=0}^{1}(x+y)\left(\frac{e^{-z}}{-1}\right)_0^{\infty} dx$$

$$= \frac{1}{3}\int_{x=0}^{1}(x+y)\left(\frac{e^{-z}}{-1}\right)_0^{\infty} dx$$

$$= \frac{1}{3}\left(xy+\frac{x^2}{2}\right)_0^1$$

$$= \frac{1}{3}\left(y+\frac{1}{2}\right)$$

$$\therefore f_2(y) = \begin{cases} \frac{1}{6}(2y+1)\ for 0<y<2 \\ 0 \quad elsewhere \end{cases}$$

And marginal distribution of Z, $f_3(z) = \int_{x=0}^{1}\int_{y=0}^{2}\frac{1}{3}(x+y)e^{-z}\,dy\,dx$

$$= \frac{1}{3}\int_{x=0}^{1} e^{-z}\left(xy+\frac{y^2}{2}\right)_0^2 dx$$

$$= \frac{1}{3}\int_{x=0}^{1} e^{-z}(2x+2)\,dx$$

$$= \frac{1}{3}e^{-z}\left(x^2+2x\right)_0^1$$

$$= e^{-z}$$

$$\therefore f_3(z) = \begin{cases} e^{-z} & \text{for } z>0 \\ 0 & \text{elsewhere} \end{cases}$$

Consider $f_1(x)\,f_2(y)\,f_3(z) = \frac{1}{3}(2x+2)\frac{1}{6}(2y+1)e^{-z} = \frac{1}{9}(x+1)(2y+1)e^{-z}$

$$\neq \frac{1}{3}(x+y)e^{-z} = f(x,y,z)$$

$\therefore$ The three random variables X, Y, Z are **not independent.**

Problem 20: If two random variables have the joint density

$$f(x_1,x_2) = \begin{cases} x_1 x_2 \text{ for } 0<x_1<2,\ 0<x_2<1 \\ 0 \quad \text{elsewhere} \end{cases}$$

Find the probabilities that

(i) Both random variables will take on values less than 1.

(ii) The sum of the values taken on by the two random variables will be less than 1.

(iii) Find the marginal densities of the two random variables.

(iv) Are the two random variables independent?

(v) Find the joint distribution function of the two random variables.

Sol: Given joint probability density function is

$$f(x_1, x_2) = \begin{cases} x_1 x_2 & \text{for } \ 0 < x_1 < 2,\ 0 < x_2 < 1 \\ 0 & \text{elsewhere} \end{cases}$$

(i) The probability that both random variables will take on a values less than 1 is

$$p\ (x_1 < 1,\ x_2 < 1) = \int_{-\infty}^{1}\int_{-\infty}^{1} f(x_1, x_2)\, dx_1\, dx_2$$

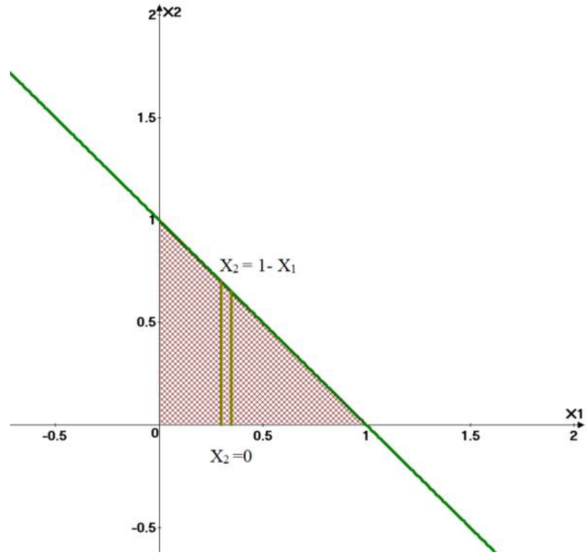
$$= \int_0^1\int_0^1 x_1\, x_2\, dx_2\, dx_1$$

$$= \int_0^1 x_1 \left(\frac{x_2^{\ 2}}{2}\right)_0^1 dx_1$$

$$= \frac{1}{2}\left(\frac{x_1^{\ 2}}{2}\right)_0^1 = \frac{1}{4}$$

(ii) Probability that the sum of the values taken on by the two random variables will be less than 1 is $X_2 = 1 - X_1$

$$p\ (x_1 + x_2 < 1) = \int_{x_1=0}^{1}\int_{x_2=0}^{1-x_1} x_1 x_2\, dx_2\, dx_1$$



$$= \int_{x_1=0}^{1} x_1 \left(\frac{x_2^{\,2}}{2} \right)_0^{1-x_1} dx_1$$

$$= \frac{1}{2} \int_{x_1=0}^{1} x_1 (1-x_1)^2 \, dx_1$$

$$= \frac{1}{24}$$

(iii) Marginal density for the random variable X_1 is

$$f_1(x_1) = \int_{x_2=-\infty}^{\infty} f(x_1, x_2) \, dx_2$$

$$= \int_{x_2=0}^{1} x_1 \, x_2 \, dx_2 = \frac{x_1}{2}$$

$$\therefore \; f_1(x_1) = \begin{cases} \dfrac{x_1}{2} & \text{for } \; 0 < x_1 < 2 \\ 0 & \text{elsewhere} \end{cases}$$

And marginal density for the random variable X_2 is

$$f_2(x_2) = \int_{x_1=-\infty}^{\infty} f(x_1, x_2) \, dx_1$$

$$= \int_{x_1=0}^{2} x_1 x_2 \, dx_1$$

$$= 2x_2$$

$$\therefore\ f_2(x_2)=\begin{cases}2x_2 & \text{for } 0<x_2<1\\ 0 & \text{elsewhere}\end{cases}$$

(iv)From the above two marginal densities

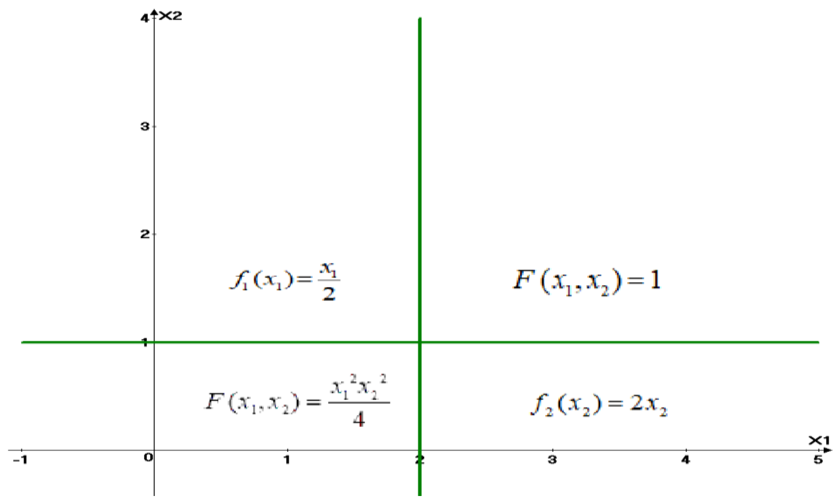
$$f_1(x_1)\,f_2(x_2)=\ \frac{x_1}{2}\times 2x_2$$

$$=\ \begin{cases}x_1\,x_2 & \text{for}\quad 0<x_1<2, 0<x_2<1\\ 0 & \text{elsewhere}\end{cases}=\ f(x_1,x_2)$$

Hence the two random variables X_1, X_2 are independent.

(v) The joint distribution of X_1, X_2 is F (x_1, x_2)

$$=P(X_1\le x_1,\ X_2\le x_2)$$



$$\therefore\ F(x_1,x_2)=\begin{cases}\dfrac{x_1^{\,2}x_2^{\,2}}{4} & \text{for } 0<x_1<2,\ 0<x_2<1\\ 1 & \text{if } x_1>2, x_2>1\\ f_2(x_2) & \text{if } x_1>2\\ f_1(x_1) & \text{if } x_2>1\\ 0 & \text{elsewhere}\end{cases}$$