# unit - III
## Multiple Integrals

**Double Integral:** consider a function $f(x,y)$ of the independent variables $x$ and $y$ defined at each point in the finite region $R$ of the xy-plane. Divide the region $R$ into $n$ elementary areas $\delta A_1, \delta A_2, \cdots, \delta A_n$. Let $(x_r, y_r)$ be any point within the $r^{th}$ elementary area $\delta A_r$. consider the sum

$$f(x_1,y_1)\,\delta A_1 + \cdots + f(x_n,y_n)\,\delta A_n = \sum_{r=1}^{n} f(x_r,y_r)\,\delta A_r$$

The limit of this sum, if it exists, as the number of sub-divisions increases indefinitely and area of each subdivision decreases to zero is defined as the double integral of $f(x,y)$ over the region $R$ and is denoted by $\iint_R f(x,y)\,dA$

$$\text{Thus } \iint_R f(x,y)\,dA = \lim_{\substack{n\to\infty \\ \delta A_r \to 0}} \sum_{r=1}^{n} f(x_r,y_r)\,\delta A_r$$

$$\text{Here } \iint_R f(x,y)\,dA = \iint_R f(x,y)\,dx\,dy \text{ (or)}$$

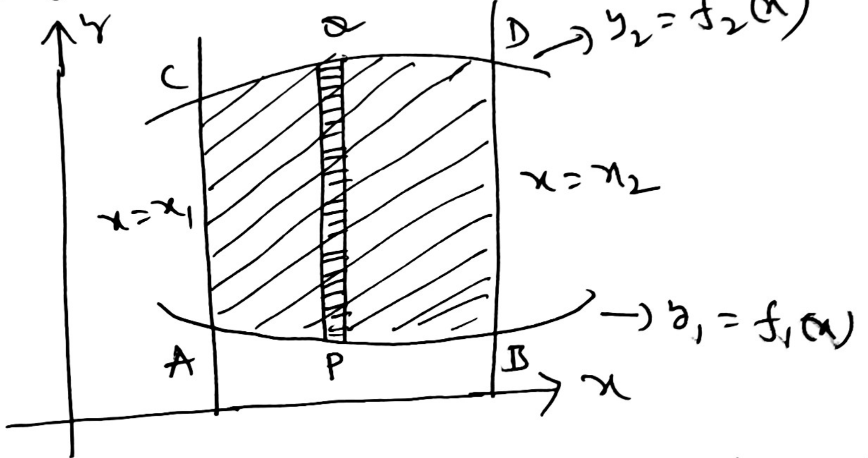
$$\iint_R f(x,y)\,dy\,dx$$

# Evaluation of double integrals

i) when $y$ limits are functions of $x$ and $x$ limits are constants:

In this case $\iint_R f(x,y)\,dA = \int_{x_1}^{x_2} \int_{f_1(x)}^{f_2(x)} f(x,y)\,dy\,dx$

where $x_1, x_2$ are constants and $f_1(x), f_2(x)$ are functions of $x$.

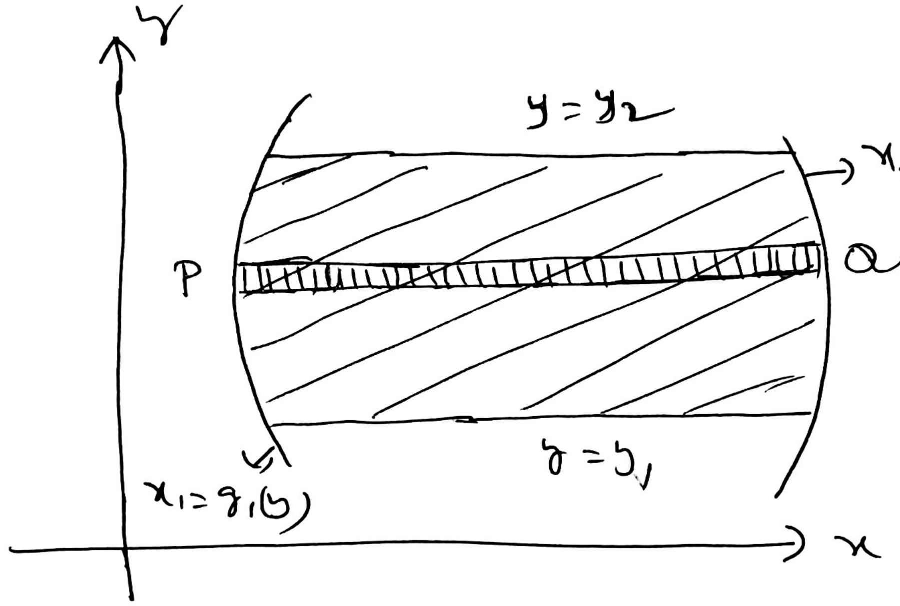



In this case $f(x,y)$ is first integrated w.r.t $y$ keeping $x$ as a constant between the limits $f_1(x)$ and $f_2(x)$. The resultant expression which is in $x$ is then integrated next w.r.t $x$ between the limits $x_1$ and $x_2$.

ii) when $x$ limits are functions of $y$ and $y$ limits are constants:

In this case $\iint_R f(x,y)\,dA = \int_{y_1}^{y_2} \int_{g_1(y)}^{g_2(y)} f(x,y)\,dx\,dy$

where $y_1, y_2$ are constants and $g_1(y), g_2(y)$ are functions of $y$.




In this case $f(x, y)$ is first integrated w.r.t $x$ keeping $y$ as a constant between the limits $g_1(y)$ and $g_2(y)$. The resultant expression which is in $y$ is then integrated next w.r.t $y$ between the limits $y_1$ and $y_2$.

iii) when both $x$ and $y$ limits are constants:

In this case $\iint_R f(x,y)\, dA = \int_{x_1}^{x_2} \int_{y_2}^{y_2} f(x,y)\, dy\, dx$

$$= \int_{y_1}^{y_2} \int_{x_1}^{x_2} f(x,y)\, dx\, dy.$$

Here $f(x, y)$ is first integrated w.r.t $y$ keeping $x$ as a constant between the limits $y_1$ and $y_2$. The resultant expression which is in $x$ is then integrated next w.r.t $x$ between the limits $x_1$ and $x_2$ (or) $f(x,y)$ is first integrated w.r.t $x$ keeping $y$ as a constant between the limits $x_1$ and $x_2$. The resultant expression which is in $y$ is then integrated next w.r.t $y$ between the limits $y_1$ and $y_2$.

**Problem (1):** Evaluate $\int_1^2 \int_3^4 (xy + e^y)\, dy\, dx$.

**Solution:**

$$\int_1^2 \int_3^4 (xy+e^y)\, dy\, dx = \int_1^2 \left(\frac{xy^2}{2} + e^y\right)_3^4 dx$$

$$= \int_1^2 \left(8x + e^4 - \frac{9}{2}x - e^3\right) dx$$

$$= \int_1^2 \left(\frac{7}{2}x + e^4 - e^3\right) dx$$

$$= \left(\frac{7x^2}{4} + (e^4 - e^3)x\right)_1^2$$

$$= 7 + 2(e^4 - e^3) - \frac{7}{4} - (e^4 - e^3)$$

$$= \frac{21}{4} + e^4 - e^3$$

Problem (2): Evaluate $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\, dy\, dx$.

Solution: In the given problem y limits are functions of x and x limits are constants.

$$\text{Let } I = \int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\, dy\, dx$$

$$= \int_0^1 \left(x^2 y + \frac{y^3}{3}\right)_x^{\sqrt{x}} dx$$

$$= \int_0^1 \left[\left(x^2\sqrt{x} + \frac{(\sqrt{x})^3}{3}\right) - \left(x^2 \cdot x + \frac{x^3}{3}\right)\right] dx$$

$$= \int_0^1 \left(x^{5/2} + \frac{x^{3/2}}{3} - \frac{4x^3}{3}\right) dx$$

$$= \left(\frac{2}{7} x^{7/2} + \frac{2}{15} x^{5/2} - \frac{x^4}{3}\right)_0^1$$

$$= \frac{2}{7} + \frac{2}{15} - \frac{1}{3}$$

$$= \frac{3}{35}.$$

Problem (3): Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\, dx}{1+x^2+y^2}$

Solution:

$$\text{Let } I = \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{1}{(\sqrt{1+x^2})^2+y^2}\, dy\, dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \left( \mathrm{Tan}^{-1} \frac{y}{\sqrt{1+x^2}} \right)_0^{\sqrt{1+x^2}} dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \left( \mathrm{Tan}^{-1} \frac{\sqrt{1+x^2}}{\sqrt{1+x^2}} - \mathrm{Tan}^{-1} 0 \right) dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \mathrm{Tan}^{-1}(1)\, dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \frac{\pi}{4}\, dx$$

$$= \frac{\pi}{4} \int_0^1 \frac{1}{\sqrt{1+x^2}}\, dx$$

$$= \frac{\pi}{4} \left( \log x + \sqrt{1+x^2} \right)_0^1$$

$$= \frac{\pi}{4} \left[ \log(1+\sqrt{2}) - \log 1 \right]$$

$$= \frac{\pi}{4} \log(\sqrt{2}+1)$$

Problem(4): Evaluate $\int_0^1 \int_0^x e^{\cancel{y/x}} \, dy\, dx$

Solution:

Let $I = \int_0^1 \int_0^x e^{\cancel{y/x}}\, dy\, dx$

$$= \int_0^1 x \left( e^{y/x} \right)_0^x dx$$

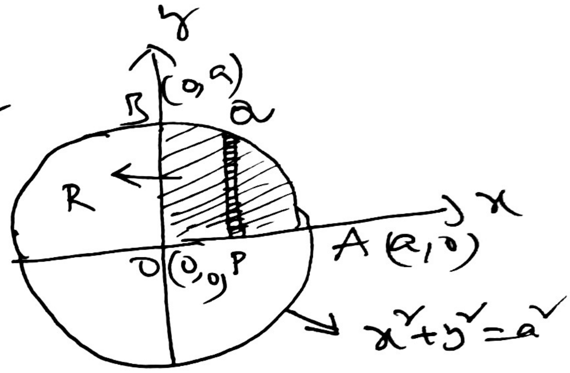
$$= \int_0^1 x \left(e^{x/x} - e^{0}\right) dx$$

$$= \int_0^1 x\,(e-1)\,dx$$

$$= (e-1)\left(\frac{x^2}{2}\right)_0^1$$

$$= \frac{e-1}{2}$$

Problem (5): Evaluate $\iint xy\,dx\,dy$ over the positive quadrant of the circle $x^2+y^2=a^2$.




Solution: Along the strip PQ $y$ varies from $0$ to $\sqrt{a^2-x^2}$. If we slide the strip PQ from O to A, $x$ varies from $0$ to $a$.

$$\therefore \iint_R xy\,dx\,dy = \int_0^a \int_0^{\sqrt{a^2-x^2}} xy\,dy\,dx$$

$$= \int_0^a x\left(\frac{y^2}{2}\right)_0^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \frac{x}{2}\left(\sqrt{a^2-x^2}\right)^2 dx$$

$$= \int_0^a \frac{x}{2}\left(a^2-x^2\right) dx$$

$$= \int_0^a \left(x\frac{a^2}{2} - \frac{x^3}{2}\right) dx$$

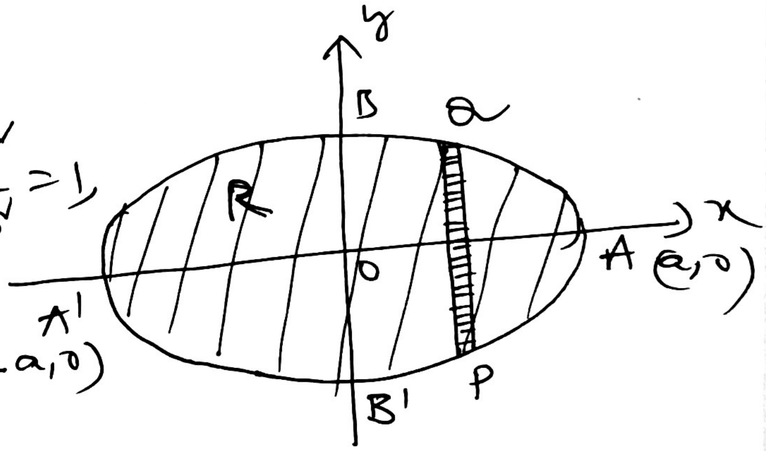
$$= \left(\frac{a^2x^2}{4} - \frac{x^4}{8}\right)_0^a$$

$$= \frac{a^4}{4} - \frac{a^4}{8}$$

$$= \frac{a^4}{8}$$

Problem (6): Evaluate $\iint (x+y)^2\,dx\,dy$ over the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution:




For the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,

$$\frac{y}{b} = \pm\sqrt{1 - \frac{x^2}{a^2}} \quad (\text{or})$$

$$y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$$

Now at P, $y = -\frac{b}{a}\sqrt{a^2-x^2}$ and

at Q, $y = \frac{b}{a}\sqrt{a^2-x^2}$

Also $-a \le x \le a$

$$\text{Now } \iint_R (x+y)^2\,dx\,dy = \int_{-a}^{a} \int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2+y^2+2xy)\,dy\,dx$$

$$= \int_{-a}^{a} \int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2+y^2)\, dy\, dx$$

$$+ \int_{-a}^{a} \int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} 2xy\, dy\, dx$$

$$= \int_{-a}^{a} \int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2+y^2)\, dy\, dx + 0 \qquad \left(\because \int_{-l}^{l} f(x)dx = 0 \text{ if } f(x) \text{ is odd}\right)$$

$$= 4 \int_{0}^{a} \int_{0}^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2+y^2)\, dy\, dx$$

$$= 4 \int_{0}^{a} \left(x^2 y + \frac{y^3}{3}\right)_{0}^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= 4 \int_{0}^{a} \left[\frac{bx^2}{a}\sqrt{a^2-x^2} + \frac{b^3}{3a^3}(a^2-x^2)^{3/2} - 0\right] dx$$

Put $x = a\sin\theta$

$\Rightarrow a^2 - x^2 = a^2\cos^2\theta$

$\Rightarrow \sqrt{a^2-x^2} = a\cos\theta$

$dx = a\cos\theta\, d\theta$

$\theta : 0 \to \pi/2$

$$= 4 \int_{0}^{\pi/2} \left(\frac{b}{a} a^2 \sin^2\theta \cdot a\cos\theta + \frac{b^3}{3a^3} a^3 \cos^3\theta\right) a\cos\theta\, d\theta$$

$$= 4\int_0^{\pi/2} a^3 b \sin^2\theta \cos^2\theta \, d\theta + 4\int_0^{\pi/2} \frac{ab^3}{3} \cos^4\theta \, d\theta$$

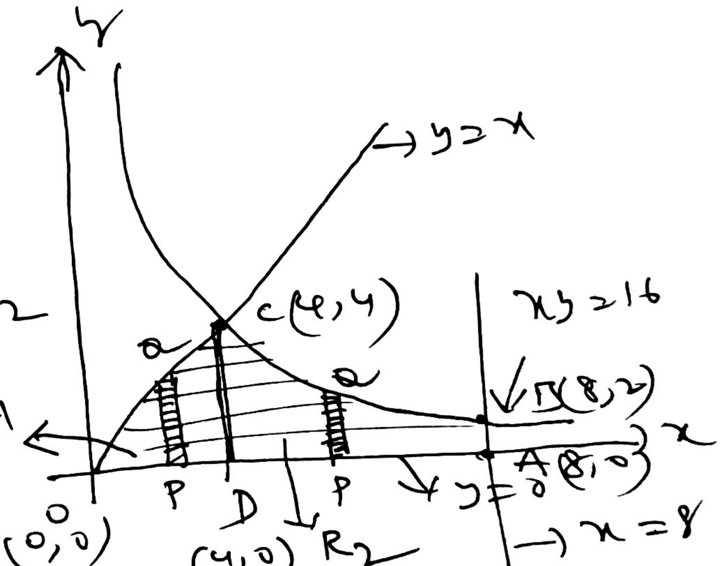
$$= \cancel{4}\, a^3 b \, \frac{1}{\cancel{4}} \cdot \frac{1}{2} \cdot \frac{\pi}{2} + \cancel{4}\, \frac{ab^3}{\cancel{3}} \, \frac{\cancel{3}}{\cancel{4}} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= \frac{a^3 b \pi}{4} + \frac{ab^3 \pi}{4}$$

$$= \frac{\pi ab}{4}(a^2 + b^2)$$

Problem (7): Evaluate $\iint_R x^2 \, dx dy$ where R is the region in the first quadrant bounded by the lines $x = y$, $y = 0$, $x = 8$ and the curves $xy = 16$.

Solution: Here the ~~region~~ region of integration R is divided into two sub-regions $R_1$ and $R_2$ where $R_1$ is the area of OCD and $R_2$ is the area ACD.




$$\therefore \iint_R x^2 \, dx dy = \iint_{R_1} x^2 \, dx dy + \iint_{R_2} x^2 \, dx dy$$

$$= \int_0^4 \int_0^x x^2 \, dx dy + \int_4^8 \int_0^{16/x} x^2 \, dx dy$$

$$= \int_0^4 \int_0^x x^2 \, dy dx + \int_4^8 \int_0^{16/x} x^2 \, dy dx$$

$$= \int_0^4 (x^2 y)_0^x \, dx + \int_4^8 (x^2 y)_0^{16/x} \, dx$$

$$= \int_0^4 x^3 \, dx + \int_4^8 16x \, dx$$

$$= \left(\frac{x^4}{4}\right)_0^4 + \left(16\frac{x^2}{2}\right)_4^8$$

$$= 64 + \frac{8}{\cancel{8}}(64-16)$$

$$= 64 + \frac{8 \times 48}{\cancel{8}}$$

$$= 64 + \cancel{8(\times 16)}$$

$$= 64 + 384$$

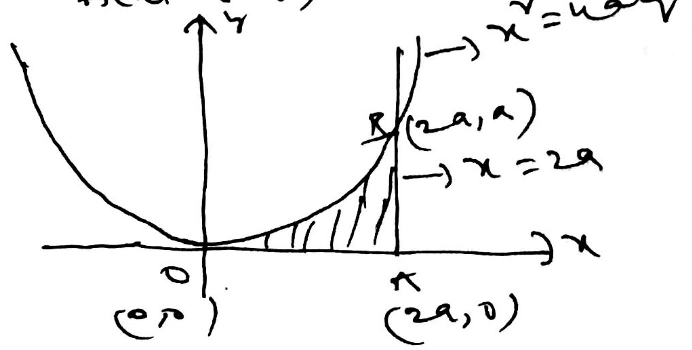
$$= 448.$$

## Practice problems

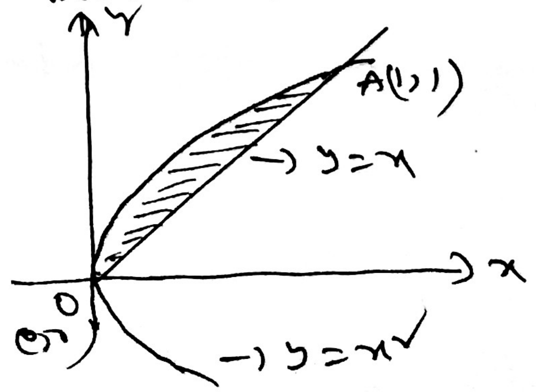
1) Evaluate $\iint_A xy \, dx\, dy$ where A is the domain bounded by x-axis, ordinate $x = 2a$ and the curve $x^2 = 4ay$.

2) Evaluate $\iint xy(x+y) \, dx\, dy$ over the area bounded by $y = x^2$ and $y = x$.

Hint for (1)




Hint for (2)



## Change of order of Integration

In a double integral if the limits of integration are constant then order of integration is immaterial. That is

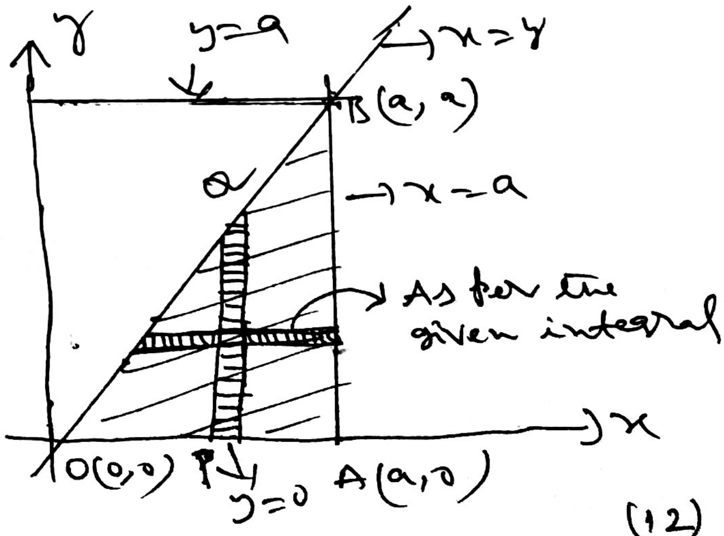
$$\int_{x_1}^{x_2} \int_{y_2}^{y_2} f(x,y)\, dy\, dx = \int_{y_2}^{y_2} \int_{x_1}^{x_2} f(x,y)\, dx\, dy.$$

But when the limits of integration are variable, a change of order of integration changes the limits of integration. Here finding the region of integration is very essential.

**Problem (1):** change the order and hence evaluate $\int_0^a \int_y^a \frac{x}{x^2+y^2}\, dx\, dy$

**Solution:** In the given integral

$x: y \to a$ and $y: 0 \to a$ and strip is horizontal.




clearly $x = y$, $x = a$

$y = 0$, $y = a$

At P, $y = 0$ and at Q, $y = x$. Slide PQ from O(0,0) to A(a, 0).

After changing the order of integration

$$\int_0^a \int_y^a \frac{x}{x^2+y^2}\,dx\,dy = \int_0^a \int_0^x \frac{x}{x^2+y^2}\,dy\,dx$$

$$= \int_0^a \cancel{x}\cdot\left(\frac{1}{\cancel{x}} \tan^{-1} \frac{y}{x}\right)_0^x dx$$

$$= \int_0^a \left(\tan^{-1} \frac{x}{x} - \tan^{-1} 0\right)$$

$$= \int_0^a (\tan^{-1} 1 - 0)\,dx$$

$$= \int_0^a \frac{\pi}{4}\,dx$$

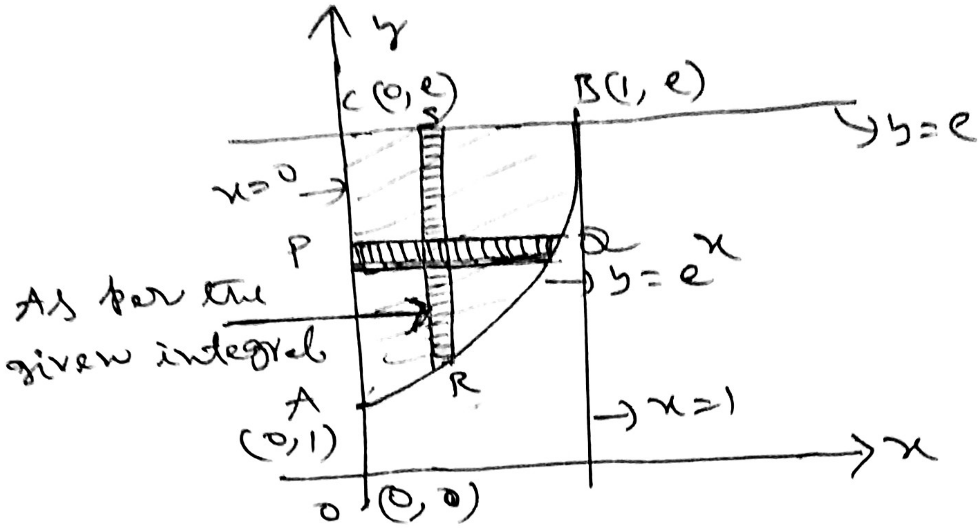
$$= \frac{\pi}{4}(x)_0^a$$

$$= \frac{\pi a}{4}.$$

Problem (2): Change the order of integration and hence evaluate $\int_0^1 \int_{e^x}^{e} \frac{dy\,dx}{\log y}$.

Solution:

As per the given integral $y: e^x \to e$, $x: 0 \to 1$ and strip is vertical. The region of integration is bounded by $y = e^x$, $y = e$, $x = 0$ and $x = 1$.




At P, $x = 0$ and at Q, $x = \log y$. The strip PQ slides from A(0,1) to C(0, e). Then after changing the order we have

$$\int_0^1 \int_{e^x}^{e} \frac{dy\,dx}{\log y} = \int_1^e \int_0^{\log y} \frac{dx\,dy}{\log y}$$

$$= \int_1^e \frac{1}{\log y} (x)_0^{\log y} dy$$

$$= \int_1^e \frac{1}{\log y} (\log y - 0)\, dy$$

$$= \int_1^e \frac{1}{\cancel{\log y}} \cancel{\log y}\, dy$$

$$= \int_1^e dy$$

$$= (y)_1^e$$

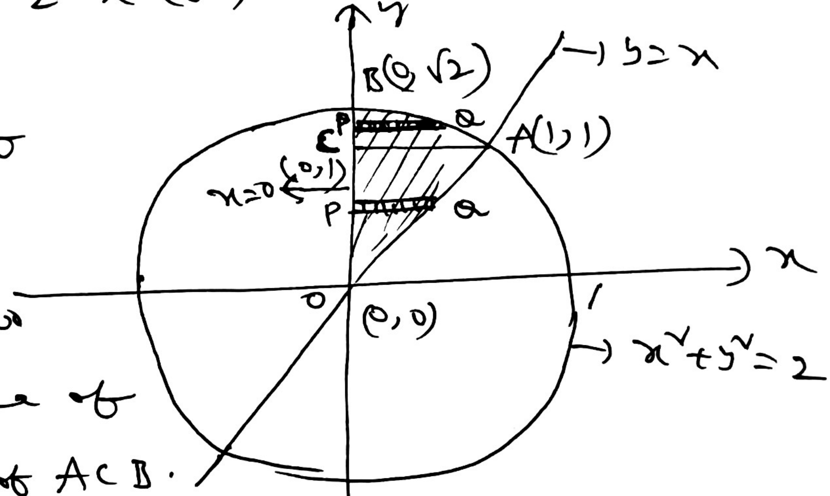
$$= e - 1.$$

Problem (3): Change the order of integration and hence evaluate $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}}$.

Solution:

In the given integral, $y: x \to \sqrt{2-x^2}$ and $x: 0 \to 1$ and the strip is vertical. The region of integration is bounded by $y=x$, $y=\sqrt{2-x^2}$ (or) $y^2 = 2-x^2$ (or) $x^2+y^2=2$, $x=0$ & $x=1$.

Here the region of integration is divided into two sub regions, area of OAC and area of ACB.




$$\therefore \int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}} = \iint_{OAC} \frac{x\,dx\,dy}{\sqrt{x^2+y^2}} + \iint_{ACB} \frac{x\,dx\,dy}{\sqrt{x^2+y^2}}$$

In the region OAC, $x: 0 \to y$ and $y: 0 \to 1$

In the region ACB, $x: 0 \to \sqrt{2-y^2}$ and $y: 1 \to \sqrt{2}$

$$\therefore \int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}} = \int_0^1 \int_0^y \frac{x\,dx\,dy}{\sqrt{x^2+y^2}} + \int_1^{\sqrt{2}} \int_0^{\sqrt{2-y^2}} \frac{x\,dx\,dy}{\sqrt{x^2+y^2}}$$

Put $\sqrt{x^2+y^2} = t$

$$\Rightarrow \frac{2x}{2\sqrt{x^2+y^2}}\, dx = dt$$

$$\Rightarrow \frac{x\, dx}{\sqrt{x^2+y^2}} = dt$$

$$\Rightarrow \int \frac{x\, dx}{\sqrt{x^2+y^2}} = \int dt = t = \sqrt{x^2+y^2}$$

$$\therefore \int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x\, dy\, dx}{\sqrt{x^2+y^2}} = \int_0^1 \left(\sqrt{x^2+y^2}\right)_0^y dy + \int_1^{\sqrt{2}} \left(\sqrt{x^2+y^2}\right)_0^{\sqrt{2-y^2}} dy$$

$$= \int_0^1 \left(\sqrt{2y^2} - \sqrt{y^2}\right) dy + \int_1^{\sqrt{2}} \left(\sqrt{2} - \sqrt{y^2}\right) dy$$

$$= \int_0^1 (\sqrt{2} - 1)\, y\, dy + \int_1^{\sqrt{2}} (\sqrt{2} - y)\, dy$$

$$= (\sqrt{2}-1)\left(\frac{y^2}{2}\right)_0^1 + \left(\sqrt{2}y - \frac{y^2}{2}\right)_1^{\sqrt{2}}$$

$$= \frac{\sqrt{2}-1}{2} + \left(2 - \frac{2}{2}\right) - \left(\sqrt{2} - \frac{1}{2}\right)$$

$$= \frac{1}{\sqrt{2}} - \cancel{\frac{1}{2}} + 1 - \sqrt{2} + \cancel{\frac{1}{2}}$$

$$= \frac{1}{\sqrt{2}} + 1 - \frac{2}{\sqrt{2}}$$

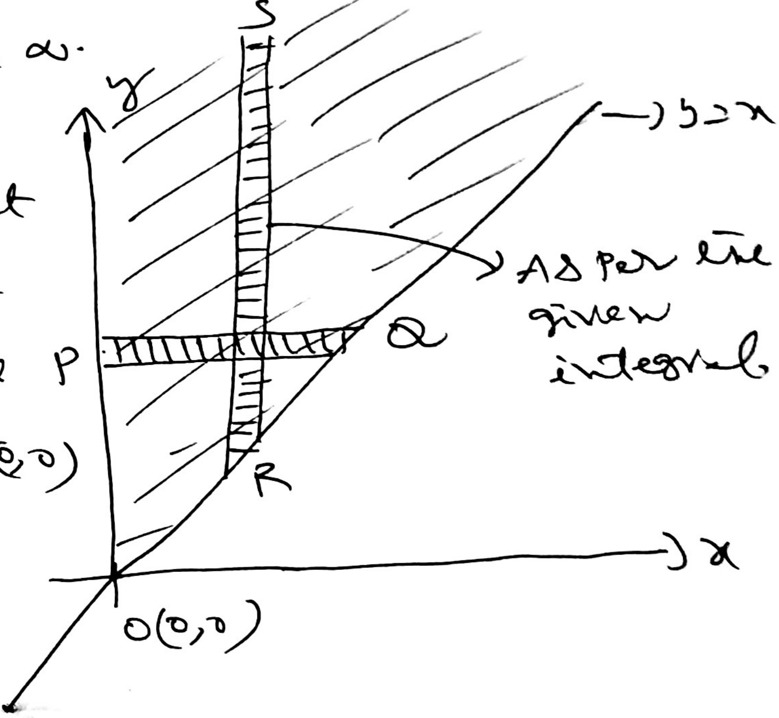
$$= 1 - \frac{1}{\sqrt{2}}.$$

(15')

Problem (4): change the order and hence evaluate $\int_0^{\infty}\int_x^{\infty} \frac{e^{-y}}{y}\, dy\, dx$

Solution:

In the given integral $y : x \to \infty$ and $x : 0 \to \infty$ and the strip is a vertical strip. The region of integration is bounded by $y = x$, $y = \infty$, $x = 0$ and $x = \infty$.




At P, $x = 0$ and at Q, $x = y$. Next we have to slide the strip PQ from O(0,0) to infinity.

$$\therefore \int_0^{\infty}\int_x^{\infty} \frac{e^{-y}}{y}\, dy\, dx = \int_0^{\infty}\int_0^{y} \frac{e^{-y}}{y}\, dx\, dy$$

$$= \int_0^{\infty} \frac{e^{-y}}{y} (x)_0^{y}\, dy$$

$$= \int_0^{\infty} \frac{e^{-y}}{\cancel{y}} \cancel{y}\, dy$$

$$= \int_0^\infty e^{-y}\, dy$$

$$= \left(-e^{-y}\right)_0^\infty$$

$$= \left(-e^{-\infty} - (-e^{0})\right)$$

$$= 0 + 1$$

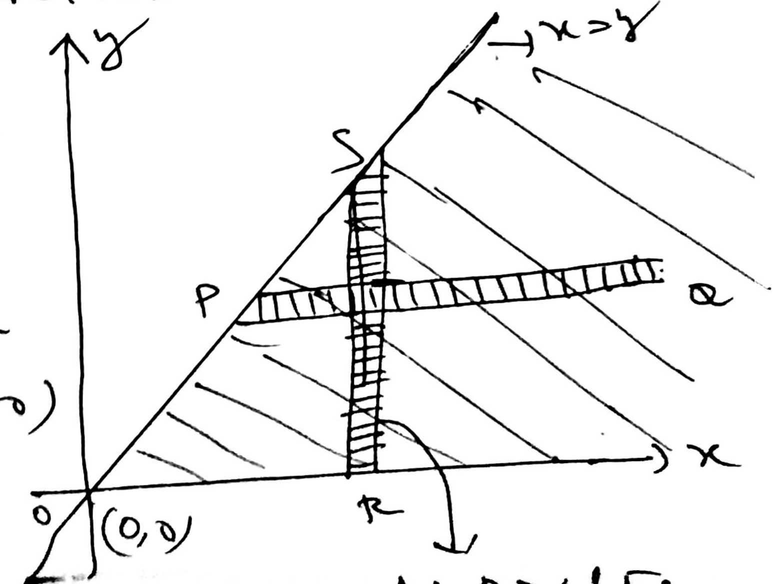
$$= 1.$$

**Problem (5):** change the order of integration and hence evaluate $\int_0^\infty \int_0^x x\, e^{-x^2/y}\, dy\, dx$.

**Solution:**

In the given integral $y: 0 \to x$ & $x: 0 \to \infty$ and the strip is a vertical strip. The region of integration is bounded by $y=0$, $y=x$, $x=0$ and $x=\infty$.

At P, $x=0$ and at Q, $x=\infty$. Next we have to slide the strip PQ from O(0,0) to $\infty$




As per the given integral

$$\therefore \int_0^\infty \int_0^x x\, e^{-x^2/y}\, dy\, dx = \int_0^\infty \int_y^\infty x\, e^{-x^2/y}\, dx\, dy$$

$$= \int_0^\infty \int_y^\infty -\frac{y}{2}\left(-\frac{2x}{y} e^{-x^2/y}\right) dx\, dy \quad (*)$$

$$= \int_0^\infty -\frac{y}{2}\left(e^{-x^2/y}\right)_y^\infty dy$$

$$= \int_0^\infty -\frac{y}{2}\left(0 - e^{-y}\right) dy$$

$$= \int_0^\infty \frac{y e^{-y}}{2}\, dy$$

$$= \frac{1}{2}\left[(y)(-e^{-y}) - (1)(e^{-y})\right]_0^\infty$$

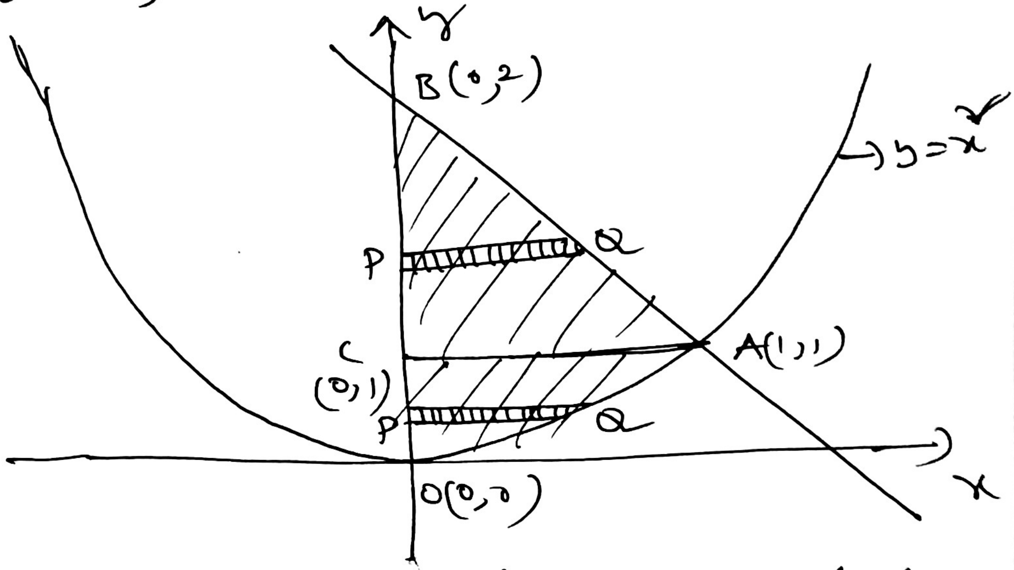
$$= \frac{1}{2}\left[(0-0) - (0-1)\right]$$

$$= \frac{1}{2}.$$

(*) when we are integrating first w.r.t $x$, $y$ is a constant. So, we can multiply and divide with $\left(-\frac{2}{y}\right)$.

Problem (b): change the order and hence evaluate $\int_0^1 \int_{x^2}^{2-x} xy\, dy\, dx$.

Solution: As per the given integral $y: x^2 - 2-x$, $x: 0 \to 1$ and the strip is a vertical strip. The region of integration is bounded by the curves $y = x^2$, $y = 2-x$, $x=0$ and $x=1$.




Here the region of integration is divided into two sub regions namely area of OAC and area of ABC. In the sub region OAC, $x: 0 \to \sqrt{y}$ and $y: 0 \to 1$. Also in the sub region ABC, $x: 0 \to 2-y$ and $y: 1 \to 2$.

$$\therefore \int_0^1 \int_{x^2}^{2-x} xy\, dy\, dx = \int_0^1 \int_0^{\sqrt{y}} xy\, dx\, dy + \int_1^2 \int_0^{2-y} xy\, dx\, dy$$

$$= \int_0^1 y\left(\frac{x^2}{2}\right)_0^{\sqrt{y}} dy + \int_1^2 y\left(\frac{x^2}{2}\right)_0^{2-y} dy$$

$$= \int_0^1 \frac{y}{2}\, y\, dy + \int_1^2 \frac{y}{2}(2-y)^2\, dy$$

$$= \int_0^1 \frac{y^2}{2}\, dy + \int_1^2 \frac{1}{2}(4y + y^3 - 4y^2)\, dy$$

$$= \left(\frac{y^3}{6}\right)_0^1 + \frac{1}{2}\left(2y^2 + \frac{y^4}{4} - \frac{4}{3}y^3\right)_1^2$$

$$= \frac{1}{6} + \frac{1}{2}\left[\left(8 + 4 - \frac{32}{3}\right) - \left(2 + \frac{1}{4} - \frac{4}{3}\right)\right]$$

$$= \frac{1}{6} + \frac{1}{2}\left[\frac{4}{3} - \frac{11}{12}\right]$$

$$= \frac{1}{6} + \frac{1}{2}\left[\frac{16 - 11}{12}\right]$$

$$= \frac{1}{6} + \frac{1}{2} \times \frac{5}{12}$$

$$= \frac{1}{6} + \frac{5}{24}$$

$$= \frac{4 + 5}{24}$$

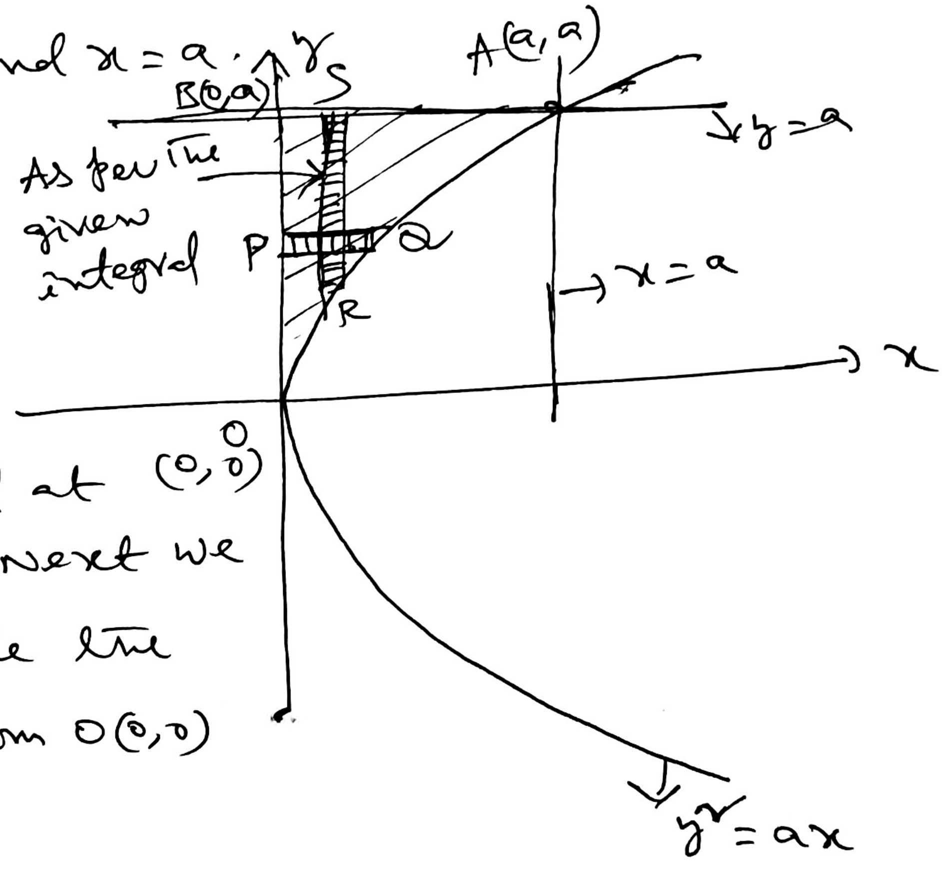
$$= \frac{9}{24}$$

$$= \frac{3}{8}.$$

Problem (7): change the order of integration and hence evaluate $\int_0^a \int_{\sqrt{ax}}^a \frac{y^2\,dy\,dx}{\sqrt{y^4 - a^2x^2}}$.

Solution:

In the given integral $y: \sqrt{ax} \to a$ & $x: 0 \to a$ and the strip is a vertical strip. The region of integration is bounded by $y = \sqrt{ax}$ (or) $y^2 = ax$, $y = a$, $x = 0$ and $x = a$.




At P, $x = 0$ and at Q, $x = y^2/a$. Next we have to slide the strip PQ from O(0,0) to B(0,a).

$$\therefore \int_0^a \int_{\sqrt{ax}}^a \frac{y^2\,dy\,dx}{\sqrt{y^4 - a^2x^2}} = \int_0^a \int_0^{y^2/a} \frac{y^2\,dx\,dy}{\sqrt{y^4 - a^2x^2}}$$

$$= \int_0^a \int_0^{y^2/a} \frac{y^2\,dx\,dy}{\sqrt{\left[a^2\left(\left(\frac{y^2}{a}\right)^2 - x^2\right)\right]}}$$

$$= \frac{1}{a}\int_0^a \int_0^{y^2/a} \frac{y^2}{\sqrt{\left(\frac{y^2}{a}\right)^2 - x^2}}\, dx\, dy$$

$$= \frac{1}{a}\int_0^a y^2 \left(\sin^{-1}\frac{x}{\frac{y^2}{a}}\right)_0^{y^2/a} dy$$

$$= \frac{1}{a}\int_0^a y^2 \left[\sin^{-1}(1) - \sin^{-1}(0)\right] dy$$

$$= \frac{1}{a}\int_0^a y^2 \frac{\pi}{2}\, dy$$

$$= \frac{\pi}{2a}\left(\frac{y^3}{3}\right)_0^a$$

$$= \frac{\pi}{2a}\,\frac{a^3}{3}$$

$$= \frac{\pi a^2}{6}.$$

Practice problems

Change the order of integration and hence evaluate

1) $\displaystyle\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\, dx$ 2) $\displaystyle\int_0^{3}\int_1^{\sqrt{4-y}} (x+y)\, dx\, dy$

3) $\displaystyle\int_0^{1}\int_x^{\sqrt{x}} xy\, dy\, dx$ 4) $\displaystyle\int_0^{a}\int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} xy\, dx\, dy$