Completely Randomized Designs

One way ANOVA

Suppose that there are k independent random samples from k different populations and we are interested to test whether the means of these k populations are all equal. Let us denoted the j^{th} observation in the i^{th} sample by y_{ij} and the data for one way classification is as follows:

	Observations					
Sample 1	y_{11}	y_{12}	…	y_{1j}	…	y_{1n_1}
Sample 2	y_{21}	y_{22}	…	y_{2j}	…	y_{2n_2}
…						
Sample i	y_{i1}	y_{i2}	…	y_{ij}	…	y_{in_i}
…						
Sample k	y_{k1}	y_{k2}	…	y_{kj}	…	y_{kn_k}

From the above data compute the following quantities:

$$T_{\bullet} = \sum_{i=1}^{k}\sum_{j=1}^{n_i} y_{ij} \qquad N = \sum_{i=1}^{k} n_i \qquad \bar{y} = \frac{T_{\bullet}}{N}$$

$$C = \frac{T_{\bullet}^2}{N} \qquad T_i = \sum_{j=1}^{n_i} y_{ij}$$

$$SST = \sum_{i=1}^{k}\sum_{j=1}^{n_i} y_{ij}^2 - C$$

$$SS(Tr) = \sum_{i=1}^{k} \frac{T_i^2}{n_i} - C$$

SSE = SST – SS(Tr)

Null Hypothesis, $H_0 : \mu_1 = \mu_2 = \ldots = \mu_k$

Alternative Hypothesis, H_1: The k population means are not all equal.

Level of significance: α

Analysis of variance table:

Source of variation	Degrees of freedom	Sum of squares	Mean square	F
Treatments	K - 1	SS(Tr)	MS(Tr) = SS(Tr)/(k – 1)	MS(Tr)/MSE
Error	N - k	SSE	MSE = SSE/(N – k)	
Total	N - 1	SST		

Decision: Reject the null hypothesis if F exceeds F_α (k – 1, N – k).

Problem 1: Suppose that a sheet of tin plate, sufficiently long and wide, is selected and that the 48 disks are cut as circles. The 12 disks cut from strip 1 are sent to the laboratory 1, The 12 disks cut from strip 2 are sent to the laboratory 2, and so forth. Each laboratory measures the tin-coating weights of 12 disks and that the results are as follows:

Laboratory 1	Laboratory 2	Laboratory 3	Laboratory 4
0.25	0.18	0.19	0.23
0.27	0.28	0.25	0.30
0.22	0.21	0.27	0.28
0.30	0.23	0.24	0.28
0.27	0.25	0.18	0.24
0.28	0.20	0.26	0.34
0.32	0.27	0.28	0.20
0.24	0.19	0.24	0.18
0.31	0.24	0.25	0.24
0.26	0.22	0.20	0.28

0.21	0.29	0.21	0.22
0.28	0.16	0.19	0.21

Perform an analysis of variance to test at the 0.05 level of significance whether the differences among the sample means at the four laboratories are significant.

Solution: Given k = 4 samples and the size of each sample is $n_i = 12$, i = 1,2,3.4

$T_1 = 3.21$ (the sum of all the values under laboratory 1)

$T_2 = 2.72$ (the sum of all the values under laboratory 2)

$T_3 = 2.76$ (the sum of all the values under laboratory 3)

$T_4 = 3.00$ (the sum of all the values under laboratory 4)

The grand total, $T_{\bullet} = T_1 + T_2 + T_3 + T_4 = 11.69$

$N = n_1 + n_2 + n_3 + n_4 = 12 + 12 + 12 + 12 = 48$

$$C = \frac{T_{\bullet}^2}{N} = \frac{11.69^2}{48} = 2.8470$$

$$SST = \sum_{i=1}^{k}\sum_{j=1}^{n_i} y_{ij}^2 - C$$

$$= (0.25)^2 + (0.27)^2 + (0.22)^2 + \ldots\ldots + (0.22)^2 + (0.21)^2 - 2.8470 = 0.0809$$

$$SS(Tr) = \sum_{i=1}^{k} \frac{T_i^2}{n_i} - C$$

$$= \frac{(3.21)^2}{12} + \frac{(2.72)^2}{12} + \frac{(2.76)^2}{12} + \frac{(3.00)^2}{12} - 2.8470 = 0.0130$$

SSE = SST – SS(Tr) = 0.0809 – 0.0130 = 0.0679

$F_{0.05}(4-1, 48-4) = F_{0.05}(3,44) = 2.82$

Analysis of variance table:

Source of variation	Degrees of freedom	Sum of squares	Mean square	F
Treatments	$k - 1$ **3**	SS(Tr) **0.0130**	MS(Tr) = SS(Tr)/(k – 1) **0.0043**	MS(Tr)/MSE **2.87**
Error	$N - k$ **44**	SSE **0.0679**	MSE = SSE/(N – k) **0.0015**	
Total	$N - 1$ **47**	SST **0.0809**		

Decision: Since $F = 2.87$ exceeds the value of $F_{0.05}(3,44) = 2.82$, the null hypothesis can be rejected at the 0.05 level of significance. So, we conclude that the laboratories are not obtaining consistent results.

Problem 2: As a part of the investigation of the collapse of the roof of a building, a testing laboratory is given all the available bolts that connected the steel structure at three 3 different positions on the roof. The forces required to shear each of these bolts are as follows:

Position 1	90	82	79	98	83	91	
Position 2	105	89	93	104	89	95	86
Position 3	83	89	80	94			

Perform an analysis of variance to test at the 0.05 level of significance whether the differences among the sample means at the 3 positions are significant.

Solution: Given $k = 3$ samples and the sizes are $n_1 = 6$, $n_2 = 7$ and $n_3 = 4$

$T_1 = 523$ (the sum of all the values under Position 1)

$T_2 = 661$ (the sum of all the values under Position 2)

$T_3 = 346$ (the sum of all the values under Position 3)

The grand total, $T_{\bullet} = T_1 + T_2 + T_3 = 1530$

$N = n_1 + n_2 + n_3 = 6 + 7 + 4 = 17$

$$C = \frac{T_{\bullet}^2}{N} = \frac{1530^2}{17} = 137700$$

$$SST = \sum_{i=1}^{k} \sum_{j=1}^{n_i} y_{ij}^2 - C$$

$= (90)^2 + (82)^2 + (79)^2 + \ldots + (80)^2 + (94)^2 - 137700$

$= 138638 - 137700 = 938$

$$SS(Tr) = \sum_{i=1}^{k} \frac{T_i^2}{n_i} - C$$

$$= \frac{(523)^2}{6} + \frac{(661)^2}{7} + \frac{(346)^2}{4} - 137700 = 234$$

SSE = SST – SS(Tr) = 938 – 234 = 704

$F_{0.05}(3-1,17-3) = F_{0.05}(2,14) = 3.74$

Analysis of variance table:

Source of variation	Degrees of freedom	Sum of squares	Mean square	F
Treatments	k – 1 **2**	SS(Tr) **234**	MS(Tr) = SS(Tr)/(k – 1) **117**	MS(Tr)/MSE **2.33**
Error	N - k **14**	SSE **704**	MSE = SSE/(N – k) **50.3**	
Total	N – 1 **16**	SST **938**		

Decision: Since F = 2.33 does not exceed the value of $F_{0.05}(2,14) = 3.74$, the null hypothesis cannot be rejected. So, accept the null hypothesis. That is there is no difference in the mean shear strengths of the bolts at the three different positions on the roof.

Problem 3: The following are the number of mistakes made in 5 successive days for 4 technicians working for a photographic laboratory:

Technician-I	Technician-II	Technician-III	Technician-IV
6	14	10	9
14	9	12	12
10	12	7	8
8	10	15	10
11	14	11	11

Test at the level of significance $\alpha = 0.01$ whether the differences among 4 sample means can be attributed to chance.

Solution: Given k = 4 samples and the size of each sample is $n_i = 5$, i = 1,2,3.4

$T_1 = 49$ (the sum of all the values under laboratory 1)

$T_2 = 59$ (the sum of all the values under laboratory 2)

$T_3 = 55$ (the sum of all the values under laboratory 3)

$T_4 = 50$ (the sum of all the values under laboratory 4)

The grand total, $T_{\bullet} = T_1 + T_2 + T_3 + T_4 = 213$

$N = n_1 + n_2 + n_3 + n_4 = 5 + 5 + 5 + 5 = 20$

$$C = \frac{T_{\bullet}^2}{N} = \frac{213^2}{20} = 2268.45$$

$$SST = \sum_{i=1}^{k}\sum_{j=1}^{n_i} y_{ij}^2 - C$$

$$= (6)^2 + (14)^2 + (10)^2 + \ldots\ldots + (10)^2 + (11)^2 - 2268.45 = 114.55$$

$$SS(Tr) = \sum_{i=1}^{k} \frac{T_i^2}{n_i} - C$$

$$= \frac{(49)^2}{5} + \frac{(59)^2}{5} + \frac{(55)^2}{5} + \frac{(50)^2}{5} - 2.268.45 = 12.95$$

SSE = SST – SS(Tr) = 101.6

$F_{0.01}(4-1,20-4) = F_{0.01}(3,16) = 5.29$

Analysis of variance table:

Source of variation	Degrees of freedom	Sum of squares	Mean square	F
Treatments	k – 1 **3**	SS(Tr) **12.95**	MS(Tr) = SS(Tr)/(k – 1) **4.3167**	MS(Tr)/MSE **0.6798**
Error	N - k **16**	SSE **101.6**	MSE = SSE/(N – k) **6.35**	
Total	N – 1 **19**	SST **114.55**		

Decision: Since F = 0.6798 does not exceed the value of $F_{0.01}(3,16) = 3.2389$, the null hypothesis can't be rejected at the 0.01 level of significance. So, we conclude that the 4 sample means are all equal.

Problem 4: The following are the weight losses of certain machine parts (in milligrams) due to friction when three different lubricants were used under controlled conditions:

Lub A	12.2	11.8	13.1	11.0	3.9	4.1	10.9	8.4
Lub B	10.9	5.7	13.5	9.4	11.4	15.7	10.8	14.0
Lub C	12.7	19.9	13.6	11.7	18.3	14.3	22.8	20.4

Test at the 0.01 level of significance whether the differences among the means can be attributed to chance. Also estimate the parameters of the model used in the analysis of experiment.

Solution: Given k = 3 samples and the sizes are $n_1 = 8$, $n_2 = 8$ and $n_3 = 8$

$T_1 = 74.8$ (the sum of all the values under Lub A)

$T_2 = 91.4$ (the sum of all the values under Lub B)

$T_3 = 133.7$ (the sum of all the values under Lub C)

The grand total, $T\bullet = T_1 + T_2 + T_3 = 299.9$

$N = n_1 + n_2 + n_3 = 8 + 8 + 8 = 24$

$$C = \frac{T\bullet^2}{N} = \frac{(299.9)^2}{24} = 3747.5$$

$$SST = \sum_{i=1}^{k}\sum_{j=1}^{n_i} y_{ij}^2 - C$$

$$= (12.2)^2 + (11.8)^2 + (13.1)^2 + \ldots + (22.8)^2 + (20.4)^2 - 3747.5$$

$$= 507.3896$$

$$SS(Tr) = \sum_{i=1}^{k}\frac{T_i^2}{n_i} - C$$

$$= 230.5858$$

SSE = SST – SS(Tr) = 276.8038

$F_{0.01}(3-1, 24-3) = F_{0.01}(2,21) = 5.78$

Analysis of variance table:

Source of variation	Degrees of freedom	Sum of squares	Mean square	F
Treatments	k – 1 **2**	SS(Tr) **230.5858**	MS(Tr) = SS(Tr)/(k – 1) **115.2929**	MS(Tr)/MSE **8.7468**
Error	N - k **21**	SSE **276.8038**	MSE = SSE/(N – k) **13.1811**	
Total	N – 1 **23**	SST **507.3896**		

Decision: Since F = 8.7468 exceed the value of $F_{0.01}(2,21) = 5.78$, the null hypothesis must be rejected. So, accept the alternative hypothesis. That is the weight losses are not the same for the three lubricants.