PROBABILITY DENSITIES

Definition: An experiment which can be repeated any number of times under the same identical conditions and even though we know the outcomes of the experiment in advance we could not predict the exact outcome of the experiment is called the random experiment.

Example: (i) Tossing a coin (ii) Rolling a fair die (iii) Taking a card from a deck of 52 cards etc.

Definition: The set of all possible outcomes of a random experiment is called the sample space.

Example: When tossing a coin one time the sample space associated with it is { H, T}

Example: When tossing a fair die one time the sample space associated with it is {1, 2, 3, 4, 5, 6}.

Definition: A random variable is a real valued function defined over the sample space.

Definition: A random variable which takes on finite number of values or countably infinite number of values is called a discrete random variable.

Example: Consider the experiment of tossing a coin n times. Let X denotes the number of heads turns up. Then X is a random variable which takes on the values 0, 1, 2, …, n. So, X is a discrete random variable.

Definition: A random variable which assumes every value in an interval [a, b] is called a continuous random variable.

Example: Let X denotes the life time of a battery. Then X assumes any value in the interval (0, ∞). So, X is a continuous random variable.

Definition:A function $f(x)$ is said to be probability density function of a continuous random variable X if f(x) satisfies:

(i). $f(x) \geq 0$ (ii). $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1$

Definition: The Cumulative distribution function $\mathrm{F}(x)$ of a continuous random variable X is defined as $\mathrm{F}(x) = \mathrm{P}(\mathrm{X} \le x) = \int_{-\infty}^{x} f(x)\,\mathrm{d}\,x$

Formulae:

1. $\mathrm{P}(\mathrm{X} \le a) = \int_{-\infty}^{a} f(x)\,\mathrm{d}\,x$
2. $\mathrm{P}(\mathrm{X} \ge a) = \int_{a}^{\infty} f(x)\,\mathrm{d}\,x$
3. $\mathrm{P}(a \le \mathrm{X} \le b) = \mathrm{P}(a \le \mathrm{X} < b) = \mathrm{P}(a < \mathrm{X} \le b) = \mathrm{P}(a < \mathrm{X} < b) = \int_{a}^{b} f(x)\,\mathrm{d}\,x$
4. $\mathrm{P}(\mathrm{X} \le a) = F(a)$
5. $\mathrm{P}(\mathrm{X} \ge a) = 1 - F(a)$
6. $\mathrm{P}(a \le X \le b) = F(\mathrm{b}) - F(a)$
7. $F(-\infty) = 0; \mathrm{F}(\infty) = 1$
8. $\frac{d}{dx} F(x) = \mathrm{f}(x)$

Definition: The kth moment of the random variable X about the origin is defined by

$$\mu_k^1 = \int_{-\infty}^{\infty} x^k f(x)\,dx$$

The first moment of the random variable X about origin is called mean, and is denoted by μ. The mean of the probability density is $\mu = \int_{-\infty}^{\infty} x \; f(x)\,\mathrm{d}\,x$

Definition: The kth moment of the random variable X about the mean is defined by

$$\mu_k = \int_{-\infty}^{\infty} (x - \mu)^k \; f(x)\,\mathrm{d}\,x$$

The second moment of the random variable X about the mean is called variance, denoted by σ^2.

$$\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 \; f(x)\,\mathrm{d}\,x = \int_{-\infty}^{\infty} x^2 \; f(x)\,\mathrm{d}\,x - \mu^2$$

Problem 9: Let the phase error in a tracking device have probability density $f(x)=\begin{cases}\cos x\,; 0<x<\pi/2 \\ 0 \quad \text{elsewhere}\end{cases}$ Find the probability that the phase error is (i) between 0 and $\pi/4$; (ii) greater than $\pi/3$. Also find mean and standard deviation for the distribution of the phase error.

Solution:

(i) $P\left(0<X<\pi/4\right)=\int_0^{\pi/4}\cos x\,dx=[\sin x]_0^{\pi/4}=\dfrac{1}{\sqrt{2}}$ = 0.7070

(ii) $P\left(X>\pi/3\right)=\int_{\pi/3}^{\infty} f(x)\,dx=\int_{\pi/3}^{\pi/2}\cos x\,dx=[\sin x]_{\pi/3}^{\pi/2}=1-\dfrac{\sqrt{3}}{2}$ = 0.1339

Mean, $\mu=\int_{-\infty}^{\infty} xf(x)\,dx=\int_0^{\pi/2} x\cos x\,dx=\left[x\sin x+\cos x\right]_0^{\pi/2}=\dfrac{\pi}{2}-1=0.5714$

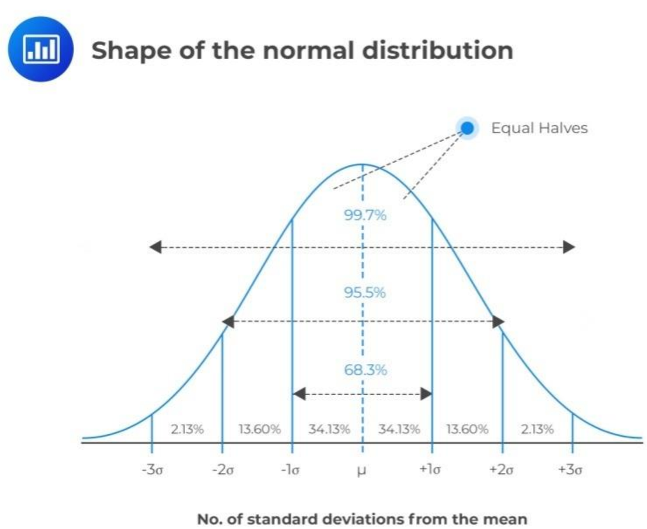
Variance, $\sigma^2=\int_{-\infty}^{\infty} x^2 f(x)\,\mathrm{d}x-\mu^2=\int_0^{\pi/2} x^2\cos x\,dx-(0.5714)^2=\left[x^2\sin x+2x\cos x-2\sin x\right]_0^{\pi/2}$ - $(0.5714)^2$ = 0.1429

$\Rightarrow \sigma$ = 0.37802

The Normal distribution:

A random variable X is said to have the Normal distribution if its probability density is given by

$$f\left(x;\mu,\sigma^2\right)=\frac{1}{\sqrt{2\pi}}e^{\frac{-(x-\mu)^2}{2\sigma^2}};-\infty<x<\infty$$

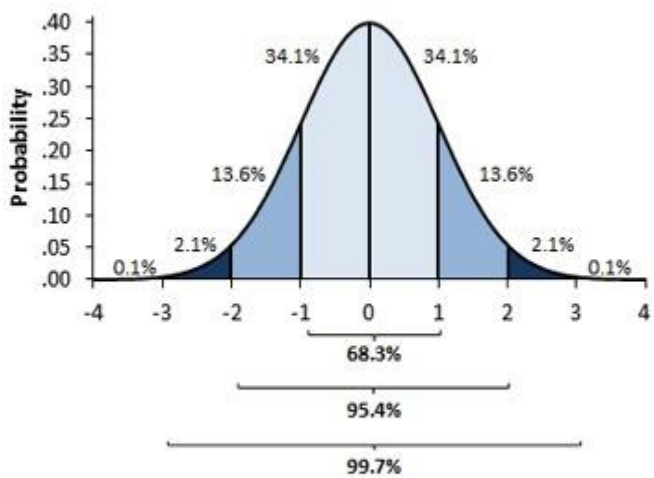


Properties:

- The graph of Normal distribution is a bell shaped curve.
- It is symmetrical about x = μ.
- The area under the normal curve above the x – axis is equal to unity.
- The normal curve meets the x – axis at the points of infinity and so x – axis is the asymptote of the normal curve.

Standard normal distribution:

If we take $Z = \frac{X-\mu}{\sigma}$ then the normal probability density function becomes $f(z)=\frac{1}{\sqrt{2\pi}}e^{-z^2/2}, -\infty<z<\infty$ which is called the standard normal density function and the random variable Z is called the standard normal random variable. The mean of Z, E(Z) = 0 and the variance of Z, Var(Z) = 1.The graph of the standard normal curve is



The cumulative distribution function of the random variable Z is given by

$$F(z)=P(Z\le z)=\int_{-\infty}^{z} f(t)\,dt$$

$$i.e., F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2} dt$$

Formulae:

1. The probability that a random variable having Standard normal distribution will take on a value between a and b is P(a < Z < b) = F(b) - F(a)

2. $P(Z \le a) = F(a)$

3. $P(Z \ge a) = 1 - F(a)$

4. $F(-z) = 1 - F(z)$

5. Let X be a random variable which has a normal distribution with mean μ, standard deviation σ then the corresponding standardized random variable Z is $Z = \frac{X - \mu}{\sigma}$, which has the standard normal distribution.

(i). $P(X \le a) = P\left(\frac{X-\mu}{\sigma} \le \frac{a-\mu}{\sigma}\right) = P\left(Z \le \frac{a-\mu}{\sigma}\right) = F\left(\frac{a-\mu}{\sigma}\right)$

(ii). $P(a \le X \le b) = P\left(\frac{a-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{b-\mu}{\sigma}\right)$

$$= P\left(\frac{a-\mu}{\sigma} \le Z \le \frac{b-\mu}{\sigma}\right)$$
$$= F\left(\frac{b-\mu}{\sigma}\right) - F\left(\frac{a-\mu}{\sigma}\right)$$

(iii). $P(X \ge a) = P\left(\frac{X-\mu}{\sigma} \ge \frac{a-\mu}{\sigma}\right) = P\left(Z \ge \frac{a-\mu}{\sigma}\right) = 1 - F\left(\frac{a-\mu}{\sigma}\right)$

$$P(X \geq 258.3) = P\left(\frac{X-\mu}{\sigma} \geq \frac{258.3-\mu}{\sigma}\right)$$
$$= P\left(Z \geq \frac{258.3-273.3}{10}\right)$$
$$= P(Z \geq -1.5)$$
$$= 1-F(-1.5)$$
$$= F(1.5) = 0.9332$$

The Normal Approximation to the Binomial distribution:

If X is a random variable having the binomial distribution with the parameters n and p, the limiting form of the distribution function of the standardized random variable $Z = \frac{X-np}{\sqrt{np(1-p)}}$ as n$\to\infty$, is given by the standard normal distribution

$$F(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt, -\infty < z < \infty .$$

In Binomial distribution, Mean, μ=np and variance, σ^2=npq=np(1-p)

Where n: number of experiments

P: probability of successes

q: probability of failures.

Formulae:

1. $P(X < a) = P(X \leq a - 0.5)$
2. $P(X \leq a) = P(X \leq a + 0.5)$
3. $P(X = a) = P(a - 0.5 \leq X \leq a + 0.5)$
4. $P(X > a) = P(X \geq a + 0.5)$
5. $P(X \geq a) = P(X \geq a - 0.5)$
6. $P(a < X < b) = P(a - 0.5 \leq X \leq b + 0.5)$

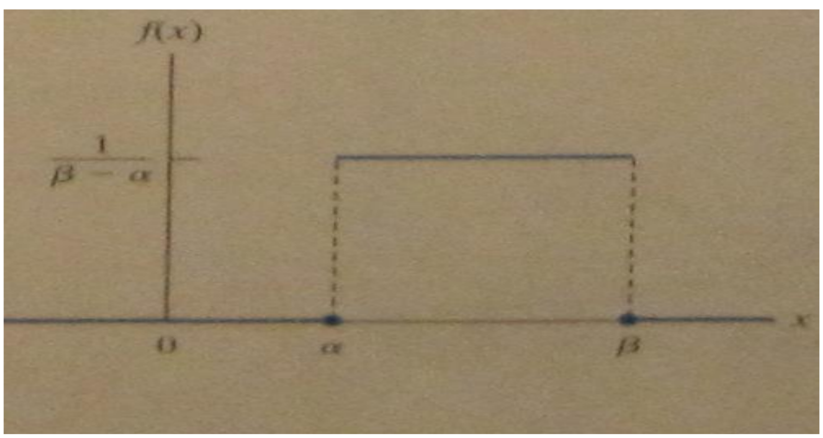
Problem 19: If 20% of the memory chips made in a certain plant are defective, what are the probabilities that in a lot of 100 randomly chosen for inspection

$$P(X<45) = P(X \le 44.5)$$
$$= P\left(\frac{X-\mu}{\sigma} \le \frac{44.5-\mu}{\sigma}\right)$$
$$= P\left(Z \le \frac{44.5-50}{6.1237}\right)$$
$$= P(Z \le -0.8981)$$
$$= F(-0.90)$$
$$= 0.1841$$

<u>The Uniform Distribution:</u>

The probability density function of Uniform distribution with the parameters α and β is $f(x) = \begin{cases} \frac{1}{\beta-\alpha} & for\ \alpha < x < \beta \\ 0 & \text{elsewhere} \end{cases}$ $(OR)\ f(x) = \begin{cases} 0 & for\ x < \alpha \\ \frac{1}{\beta-\alpha} & for\ \alpha < x < \beta \\ 0 & x>\beta \end{cases}$

The graph of Uniform Probability density is



<u>Problem 1</u>: Find the mean and variance of the Uniform distribution.

<u>Sol:</u> For the uniform distribution, the probability density function is

$$f(x) = \begin{cases} \frac{1}{\beta-\alpha} & for\ \alpha < x < \beta \\ 0 & elsewhere \end{cases}$$

Problem 2: In certain experiments, the error made in determining the solubility of a substance is a random variable having the uniform density with $\alpha = -0.025$ and $\beta = 0.025$. What are the probabilities that such an error will be (a) between 0.010 and 0.015; (b) between -0.012 and 0.012?

Also find the mean and variance of this distribution.

Solution: Given $\alpha = -0.025$, $\beta = 0.025$

Now $\dfrac{1}{\beta - \alpha} == \dfrac{1}{0.025 + 0.025} = \dfrac{1}{0.05} = 20$

Therefore the Uniform density function is

$$f(x) = \begin{cases} 20 & \text{if } -0.025 < X < 0.025 \\ 0 & \text{otherwise} \end{cases}$$

Here the error made in determining the solubility of substance is the random variable.

(a). The probability that the error will be between 0.010 and 0.015 is

$$P(0.010 \le X \le 0.015) = \int_{0.010}^{0.015} 20\, dx = 20(x)_{0.010}^{0.015} = 20(0.005) = 0.1$$

(b). The probability that the error will be between -0.012 and 0.012 is

$$P(-0.012 \le X \le 0.012) = \int_{-0.012}^{0.012} 20\, dx = 0.48$$

Mean $\mu = \dfrac{\alpha + \beta}{2} = \dfrac{-0.025 + 0.025}{2} = \dfrac{0}{2} = 0$

Variance $\sigma^2 = \dfrac{(\beta - \alpha)^2}{12} = \dfrac{(0.025 + 0.025)^2}{12} = \dfrac{(0.05)^2}{12} 0.000208$

The Gamma distribution:

The probability density function for the Gamma distribution is defined as

$$f(x) = \begin{cases} \dfrac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} & \text{for } x > 0,\ \alpha > 0, \beta > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Where the gamma function is defined as $\Gamma(\alpha)=\int_0^\infty x^{\alpha-1}e^{-x}\,dx, \quad \alpha>0$

Note: $\Gamma(\alpha+1)=\alpha\,\Gamma(\alpha)$, $\qquad \Gamma(1)=1,\ \Gamma(n+1)=n!$ for $n\in N$

<u>Problem 3</u>: Show that $\mu=\alpha\ \beta$ and $\sigma^2=\alpha\,\beta^2$ for the gamma distribution.

<u>Solution</u>: For the gamma distribution, the probability density function is

$$f(x)=\begin{cases}\dfrac{1}{\beta^\alpha\ \Gamma(\alpha)}x^{\alpha-1}e^{-x/\beta} & for\ x>0,\alpha>0,\beta>0\\ 0 & elsewhere\end{cases}$$

Mean, μ= $\int_{-\infty}^{\infty} x\ f(x)dx$

$$=\int_0^\infty x\frac{1}{\beta^\alpha\ \Gamma(\alpha)}x^{\alpha-1}e^{-x/\beta}dx$$

$$=\frac{1}{\beta^\alpha\ \Gamma(\alpha)}\int_0^\infty x^\alpha e^{-x/\beta}dx$$

$$=\frac{1}{\beta^\alpha\ \Gamma(\alpha)}\int_0^\infty (\beta t)^\alpha\ e^{-t}\beta dt\ ;\ let\ \frac{x}{\beta}=t\Rightarrow dx=\beta\ dt$$

$$=\frac{1}{\beta^\alpha\ \Gamma(\alpha)}\int_0^\infty \beta^{\alpha+1}t^\alpha e^{-t}dt$$

$$=\frac{\beta}{\Gamma(\alpha)}\int_0^\infty t^{\alpha+1-1}e^{-t}dt$$

$$=\frac{\beta}{\Gamma(\alpha)}\Gamma(\alpha+1)$$

$$=\frac{\beta}{\Gamma(\alpha)}\alpha\,\Gamma(\alpha)$$

$$=\alpha\beta$$

Variance, $\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$

$$= \int_{-\infty}^{\infty} x^2 \; f(x) dx - \mu^2$$

$$= \int_0^{\infty} x^2 \frac{1}{\beta^\alpha \; \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} dx - \mu^2$$

$$= \frac{1}{\beta^\alpha \; \Gamma(\alpha)} \int_0^{\infty} x^{\alpha+1} e^{-x/\beta} dx - (\alpha\beta)^2$$

$$= \frac{1}{\beta^\alpha \; \Gamma(\alpha)} \int_0^{\infty} (\beta t)^{\alpha+1} e^{-t} \beta dt - (\alpha\beta)^2 \;\; ; \; let \; \frac{x}{\beta} = t \Rightarrow dx = \beta \; dt$$

$$= \frac{1}{\beta^\alpha \; \Gamma(\alpha)} \int_0^{\infty} \beta^{\alpha+2} t^{\alpha+1} e^{-t} dt - (\alpha\beta)^2$$

$$= \frac{\beta^2}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha+2-1} e^{-t} dt - (\alpha\beta)^2$$

$$= \frac{\beta^2}{\Gamma(\alpha)} \Gamma(\alpha+2) - (\alpha\beta)^2$$

$$= \frac{\beta^2}{\Gamma(\alpha)} (\alpha+1)\alpha \; \Gamma(\alpha) - (\alpha\beta)^2 \;\; \left(\text{Since } \Gamma(\alpha+2) = (\alpha+1)\Gamma(\alpha+1)\right)$$

$$= \alpha\beta^2 \, (\alpha+1) - (\alpha\beta)^2$$

$$= \alpha\beta^2$$

Note: If $\alpha=1$ in the gamma distribution then we get **exponential distribution**.

The probability density function for the exponential distribution is

$$f(x) = \begin{cases} \frac{1}{\beta} e^{-x/\beta} & for \; x > 0, \beta > 0 \\ 0 & elsewhere \end{cases}$$

The Beta distribution:

For the beta distribution, the probability density is defined as

$$f(x)=\begin{cases}\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1} & for\ 0<x<1,\alpha>0,\beta>0\\ 0 & elsewhere\end{cases}$$

Note: $\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx=\beta(n,m)$

Problem 6: Find mean and variance for Beta distribution.

Solution: For beta distribution, the probability density function is

$$f(x)=\begin{cases}\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\ x^{\alpha-1}(1-x)^{\beta-1} & for\ 0<x<1,\ \alpha>0,\ \beta>0\\ 0 & elsewhere\end{cases}$$

Mean, $\mu=\int_{-\infty}^{\infty} x\ f(x)dx$

$$=\int_{-\infty}^{0} x\ f(x)dx+\int_0^1 x\ f(x)dx+\int_1^{\infty} x\ f(x)dx$$

$$=\int_0^1 x\ \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}\ dx$$

$$=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 x^{\alpha}(1-x)^{\beta-1}\ dx$$

$$=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\beta(\alpha+1,\beta)\quad\left(\text{Since }\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\ dx\right)$$

$$=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\ \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)}\qquad\left(\text{Since }\beta(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\right)$$

$$\mu=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\ \frac{\alpha\,\Gamma(\alpha)\Gamma(\beta)}{(\alpha+\beta)\,\Gamma(\alpha+\beta)}\qquad\left(\text{Since }\Gamma(n+1)=n\,\Gamma(n)\right)$$

$$\mu=\frac{\alpha}{\alpha+\beta}$$

Variance, $\sigma^2 = \mu_2^1 - \mu^2$

Consider $\mu_2^1 = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

$$= \int_{-\infty}^{0} x^2 f(x)\,dx + \int_{0}^{1} x^2 f(x)\,dx + \int_{1}^{\infty} x^2 f(x)\,dx$$

$$= \int_0^1 x^2 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}\,dx$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 x^{\alpha+1}(1-x)^{\beta-1}\,dx$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \beta(\alpha+2,\beta)$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+2)\Gamma(\beta)}{\Gamma(\alpha+\beta+2)}$$

$$\mu_2^1 = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{(\alpha+1)\,\Gamma(\alpha+1)\Gamma(\beta)}{(\alpha+\beta+1)\,\Gamma(\alpha+\beta+1)}$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} \frac{(\alpha+1)\,\alpha\,\Gamma(\alpha)}{(\alpha+\beta+1)\,(\alpha+\beta)\,\Gamma(\alpha+\beta)}$$

$$= \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$$

$$\sigma^2 = \mu_2^1 - \mu^2$$

$$= \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} - \frac{\alpha^2}{(\alpha+\beta)^2}$$

$$= \frac{(\alpha+\beta)\alpha(\alpha+1) - \alpha^2(\alpha+\beta+1)}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$$= \frac{\alpha^3+\alpha^2\beta+\alpha^2+\alpha\beta-\alpha^3-\alpha^2\beta-\alpha^2}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$$= \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$$= \int_0^{0.1} \frac{\Gamma(11)}{\Gamma(2)\Gamma(9)} x(1-x)^8 \; dx$$

$$= \frac{10!}{1!\ 8!} \int_0^{0.1} x(1-\text{x})^8 \; dx$$

$$= 90 \int_0^{0.1} (x-1+1)(1-x)^8 \; dx$$

$$= 90 \int_0^{0.1} \left[(x-1)(1-x)^8 + (1-x)^8\right] dx$$

$$= 90 \int_0^{0.1} \left[-(1-x)^9 + (1-x)^8\right] dx$$

$$= 90 \left[\frac{-(1-x)^{10}}{-10} + \frac{(1-x)^9}{-9} \right]_0^{0.1}$$

$$= 0.2639$$

<u>The Weibull Distribution</u>

The probability density function of the Weibull distribution is given by

$$f(x) = \begin{cases} \alpha\beta x^{\beta-1} e^{-\alpha x^{\beta}} & for\ x > 0, \alpha > 0, \beta > 0 \\ 0 & elsewhere \end{cases}$$

The mean of the Weibull distribution is given by $\mu = \alpha^{-1/\beta}\Gamma(1+\frac{1}{\beta})$ and the variance of the Weibull distribution is given by $\sigma^2 = \alpha^{-2/\beta}\left\{\Gamma(1+\frac{2}{\beta}) - \left[\Gamma(1+\frac{1}{\beta})\right]^2\right\}$

The Weibull distribution is widely used in reliability and life data analysis.

<u>Problem 11</u>: Suppose that the lifetime of a certain kind of an emergency backup battery (in hours) is a random variable X having Weibull distribution with α = 0.1 and β = 0.5. Find

(a) The mean life of these batteries.

(b) The probability that such a battery will last more than 300 hours.

<u>Solution</u> : Given X is a random variable having Weibull distribution that denotes the lifetime of a certain kind of an emergency backup battery (in hours) with α = 0.1 and β = 0.5.

$= 125 * 6 = 750$

(b) $P(X < 300) = \int_0^{300} \alpha\beta x^{\beta-1} e^{-\alpha x^\beta}\, dx$ (5 hours means 300 minutes)

$$= \int_0^{300} \left(\frac{1}{5}\right)\left(\frac{1}{3}\right) x^{\frac{1}{3}-1} e^{-\frac{1}{5}x^{1/3}}\, dx \qquad \text{put } x^{1/3} = t \text{ then } (1/3)\, x^{(1/3)-1} = dt$$

$$= \frac{1}{5} \int_0^{\sqrt[3]{300}} e^{-\frac{1}{5}t}\, dt$$

$$= \frac{1}{5} \left[\frac{e^{-\frac{1}{5}t}}{-\frac{1}{5}} \right]_0^{\sqrt[3]{300}}$$

$$= \left[\frac{e^{-\frac{1}{5}t}}{-1} \right]_0^{\sqrt[3]{300}}$$

$$= \left[1 - e^{-1/5(300)^{1/3}} \right]$$

$$= 0.7379$$

<u>Problem 13</u>: Suppose that the service life (in hours) of a semiconductor device is a random variable having Weibull distribution with α = 0.025 and β = 0.500. What is the probability that such a device will still be operating condition after 4,000 hours?

HINT: Find P(X > 4000).

<u>Joint distributions:</u>

Let X_1 and X_2 be two **discrete** random variables defined on a sample space **S** of an experiment. The probability of the intersection of events X_1 will take a values x_1 and X_2 will take on a value x_2 is denoted by $P\left(X_1 = x_1,\, X_2 = x_2\right) = f\left(x_1, x_2\right)$.

<u>Joint Probability distribution:</u>

A function $f\left(x_1, x_2\right)$ is said to be a **joint probability distribution** function of X_1 and X_2 if it satisfies $(i)\ f\left(x_1, x_2\right) \geq 0 \qquad (ii)\ \sum_{x_1}\sum_{x_2} f\left(x_1, x_2\right) = 1$

Marginal probability distribution:

Let $f\left(x_1, x_2\right)$ be a joint probability distribution of two random variables X_1 and X_2. The marginal probability distribution of the random variable X_1 is denoted by $f_1\left(x_1\right)$ and defined as $f_1\left(x_1\right) = \sum_{x_2} f\left(x_1, x_2\right)$.

Similarly, the marginal probability distribution of the random variable X_2 is denoted by $f_2\left(x_2\right)$ and defined as $f_2\left(x_2\right) = \sum_{x_1} f\left(x_1, x_2\right)$

Conditional probability distribution: Let $f\left(x_1, x_2\right)$ be a joint probability distribution of two random variables X_1 and X_2, then the conditional probability distribution of X_1 given $X_2 = x_2$ is denoted by $f_1\left(x_1 \mid x_2\right)$ and is defined as

$f_1\left(x_1 \mid x_2\right) = \dfrac{f\left(x_1, x_2\right)}{f_2\left(x_2\right)}$ for all x_1, x_2 provided $f_2\left(x_2\right) \neq 0$.

Similarly, the conditional probability distribution of X_2 given $X_1 = x_1$ is denoted by $f_2\left(x_2 \mid x_1\right)$ and is defined as

$f_2\left(x_2 \mid x_1\right) = \dfrac{f\left(x_1, x_2\right)}{f_1\left(x_1\right)}$ for all x_1, x_2 provided $f_1\left(x_1\right) \neq 0$.

Independent random variables: Let $f\left(x_1, x_2\right)$ be a joint probability distribution of two random variables X_1 and X_2. The two random variables X_1 and X_2 are said to be **independent** if $f\left(x_1, x_2\right) = f_1\left(x_1\right) f_2\left(x_2\right)$ for all x_1, x_2.

Problem 14: Let X_1 and X_2 have the joint probability distribution in the table given below

We know that from Binomial distribution, the probability of x successes in n trails

is $b(x;n,p) = {}^nC_x \; p^x (1-p)^{n-x}$

Consider $b(0;2,0.3) = {}^2C_0 \; (0.3)^0 (1-0.3)^{2-0} = 0.49$

$b(1;2,0.3) = {}^2C_1 \; (0.3)^1 (1-0.3)^{2-1} = 0.42$

$b(2;2,0.3) = {}^2C_2 \; (0.3)^2 (1-0.3)^{2-2} = 0.09$

(a) Since X_1 and X_2 are independent

Then the joint probability distribution function is

$f(x_1,x_2) = b(x_1;n,p)\, b(x_2;n,p) \quad x_1,x_2 \in \{0,1,2\}$

Then $f(x_1,x_2) = b(x_1;2,0.3)\, b(x_2;2,0.3) \quad x_1,x_2 \in \{0,1,2\}$

The corresponding joint probability distribution table is as follows:

		X1			
		0	1	**2**	
X_2	0	**0.2401**	**0.2058**	**0.0441**	0.49
	1	**0.2058**	**0.1764**	**0.0378**	0.42
	2	**0.0441**	**0.0378**	**0.0081**	0.09
		0.49	0.42	0.09	

(b) The possible points for second random variable greater than the first $X_2 > X_1$ are (0,1) , (0,2) & (1,2). By adding all the probabilities,

$P(X_2 > X_1) = f(0,1) + f(0,2) + f(1,2)$

$=$ 0.2058 + 0.0441 + 0.0378 = 0.2877

<u>Continuous Variables</u>:

Let $X_1, X_2, ..., X_k$ be k continuous random variables then the probability of $X_1 = x_1, X_2 = x_2, ..., X_k = x_k$ is denoted by

$P(X_1 = x_1, X_2 = x_2, ..., X_k = x_k) = f(x_1, x_2, ..., x_k)$

A function in k-variables $f\left(x_1,x_2,...,x_k\right)$ is said to be **joint probability density function** if

(i) $f\left(x_1,x_2,...,x_k\right)\geq 0$

(ii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}...\int_{-\infty}^{\infty} f\left(x_1,x_2,...,x_k\right)dx_1dx_2...dx_k = 1.$

Let $f\left(x_1,x_2,...,x_k\right)$ be a joint probability density of $X_1,X_2,...,X_k$ then

$$P\left(a_1\leq X_1\leq b_1, a_2\leq X_2\leq b_2, ..., a_k\leq X_k\leq b_k\right)$$

$$=\int_{a_1}^{b_1}\int_{a_2}^{b_2}...\int_{a_k}^{b_k} f\left(x_1,x_2,...,x_k\right)dx_kdx_{k-1}...dx_1$$

Let $f\left(x_1,x_2,...,x_k\right)$ be a joint probability density of $X_1,X_2,...,X_k$ then the **joint cumulative distribution** of k random variables is

$$F\left(x_1,x_2,...,x_k\right)=P\left(X_1\leq x_1, X_2\leq x_2, ..., X_k\leq x_k\right)$$

$$=\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}........\int_{-\infty}^{x_k} f\left(x_1,x_2,...,x_k\right)dx_1dx_2...dx_k$$

Let $f\left(x_1,x_2,...,x_k\right)$ be a joint probability density of $X_1,X_2,...,X_k$ then the **individual density** for X_i (or) **marginal probability density** for X_i is defined as

$$f_i\left(x_i\right)=\int_{x_1=-\infty}^{\infty}\int_{x_2=-\infty}^{\infty}........\int_{x_{i-1}=-\infty}^{\infty}\int_{x_{i+1}=-\infty}^{\infty}.....\int_{x_k=-\infty}^{\infty} f\left(x_1,x_2,...,x_k\right)dx_1dx_2...dx_{i-1}dx_{i+1}...dx_k$$

Let $f\left(x_1,x_2,...,x_k\right)$ be a joint probability density of $X_1,X_2,...,X_k$. The random variables $X_1,X_2,...,X_k$ are said to be **independent** if

$f\left(x_1,x_2,...,x_k\right)=f_1\left(x_1\right)f_2\left(x_2\right)...f_k\left(x_k\right)$ for all $x_1,x_2,...,x_k$

Problem 17: If the joint probability density of two random variables is given by

$$f\left(x_1,x_2\right)=\begin{cases}6e^{-2x_1-3x_2} & \text{for } x_1>0, x_2>0\\ 0 & \text{elsewhere}\end{cases}$$

Find the probabilities that