Triple integrals

Consider a function $f(x,y,z)$ which is defined at every point of the three dimensional finite region V. Divide V into n elementary volumes $\delta V_1, \delta V_2, \cdots, \delta V_n$. Let (x_r, y_r, z_r) be an arbitrary point in the r^{th} sub division δV_r. Consider the sum $\sum_{r=1}^{n} f(x_r, y_r, z_r)\, \delta V_r$

The limit of this sum as $n \to \infty$ and $\delta V_r \to 0$, if it exists, is called the triple integral of $f(x,y,z)$ over the region V and is denoted by

$\iiint f(x,y,z)\, dV$.

Generally, a typical triple integral is given by

$$I = \int_{x_1}^{x_2} \int_{f_1(x)}^{f_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\, dz\, dy\, dx$$

In the above integral x limits are outer limits which are constants, middle limits are y limits which are functions of x and inner limits are z limits which are functions of x and y.

If all the limits are constants then the order of integration is immaterial.

Problem(1): Evaluate $I = \int_0^a \int_0^b \int_0^c (x^2+y^2+z^2)\,dx\,dy\,dz$.

Solution: In the given integral

$x : 0 \to c$

$y : 0 \to b$

$z : 0 \to a$

$$\therefore I = \int_0^a \int_0^b \left(\frac{x^3}{3} + xy^2 + xz^2\right)_0^c dy\,dz$$

$$= \int_0^a \int_0^b \left(\frac{c^3}{3} + cy^2 + cz^2\right) dy\,dz$$

$$= \int_0^a \left(\frac{c^3}{3}y + \frac{cy^3}{3} + cyz^2\right)_0^b dz$$

$$= \int_0^a \left(\frac{bc^3}{3} + \frac{cb^3}{3} + cbz^2\right) dz$$

$$= \left(\frac{bc^3z}{3} + \frac{cb^3z}{3} + \frac{cbz^3}{3}\right)_0^a$$

$$= \frac{abc^3}{3} + \frac{ab^3c}{3} + \frac{a^3bc}{3}$$

$$= \frac{abc}{3}(a^2+b^2+c^2).$$

Problem (2): Evaluate $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)\,dx\,dy\,dz$

Solution:

Let $I = \int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)\,dy\,dx\,dz$

Here inner limits are y limits
middle limits are x limits &
outer limits are z limits

$$\text{so, } I = \int_{-1}^{1}\int_{0}^{z}\left(xy+\frac{y^2}{2}+yz\right)_{x-z}^{x+z} dx\,dz$$

$$= \int_{-1}^{1}\int_{0}^{z}\Big[\Big(x(x+z)+\frac{(x+z)^2}{2}+(x+z)z\Big) - \Big(x(x-z)+\frac{(x-z)^2}{2}+(x-z)z\Big)\Big]dx\,dz$$

$$= \int_{-1}^{1}\int_{0}^{z}\Big[\Big(x^2+xz+\frac{x^2}{2}+\frac{z^2}{2}+xz+xz+z^2\Big) - \Big(x^2-xz+\frac{x^2}{2}+\frac{z^2}{2}-\cancel{xz}+\cancel{xz}-z^2\Big)\Big]dx\,dz$$

$$= \int_{-1}^{1}\int_{0}^{z}\Big[\cancel{x^2}+3xz+\cancel{\frac{x^2}{2}}+\cancel{\frac{z^2}{2}}+z^2-\cancel{x^2}+xz-\cancel{\frac{x^2}{2}} - \cancel{\frac{z^2}{2}}+z^2\Big]dx\,dz$$

$$= \int_{-1}^{1}\int_{0}^{z}\big[4xz+2z^2\big]dx\,dz$$

$$= \int_{-1}^{1}\left(2x^2z+2xz^2\right)_{0}^{z} dz$$

$$= \int_{-1}^{1} (2z^3 + 2z^3)\, dz$$

$$= \int_{-1}^{1} 4z^3\, dz$$

$$= 4\left(\frac{z^4}{4}\right)_{-1}^{1}$$

$$= 4\left(\frac{1}{4} - \frac{1}{4}\right) = 0.$$

Problem(3): Evaluate $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\, dz\, dy\, dx$

solution:

Let $I = \int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\, dz\, dy\, dx$

$$= \int_0^a \int_0^x e^{x+y} \left(e^z\right)_0^{x+y} dy\, dx$$

$$= \int_0^a \int_0^x e^{x+y} \left(e^{x+y} - 1\right) dy\, dx$$

$$= \int_0^a \int_0^x \left(e^{2x+2y} - e^{x+y}\right) dy\, dx$$

$$= \int_0^a \left[\left(\frac{e^{2y}}{2}\right)_0^x e^{2x} - \left(e^y\right)_0^x e^x\right] dx$$

$$= \int_0^a \left[\frac{e^{4x}}{2} \underbrace{- \frac{e^{2x}}{2} - e^{2x}}_{-\frac{3}{2}e^{2x}} + e^x\right] dx$$

$$= \left(\frac{e^{4x}}{8} - \frac{3}{4} e^{2x} + e^{x}\right)_0^a$$

$$= \left(\frac{e^{4a}}{8} - \frac{3}{4} e^{2a} + e^{a}\right) - \left(\frac{1}{8} - \frac{3}{4} + 1\right)$$

$$= \frac{e^{4a}}{8} - \frac{3}{4} e^{2a} + e^{a} - \frac{3}{8}$$

Problem (4): Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz \, dx\, dy\, dz$

solution:

Let $I = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz \, dz\, dy\, dx$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} xy \left(\frac{z^2}{2}\right)_0^{\sqrt{1-x^2-y^2}} dy\, dx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{xy}{2} (1-x^2-y^2)\, dy\, dx$$

$$= \frac{1}{2} \int_0^1 \int_0^{\sqrt{1-x^2}} (xy - x^3 y - xy^3)\, dy\, dx$$

$$= \frac{1}{2} \int_0^1 \left(\frac{xy^2}{2} - \frac{x^3 y^2}{2} - x\frac{y^4}{4}\right)_0^{\sqrt{1-x^2}} dx$$

$$= \frac{1}{2} \int_0^1 \left[\frac{x(1-x^2)}{2} - \frac{x^3(1-x^2)}{2} - \frac{x}{4}(1-x^2)^2\right] dx$$

$$= \frac{1}{2}\int_0^1 \left[\frac{x}{2} - \cancel{\frac{x^3}{2}} - \frac{x^3}{2} + \frac{x^5}{2} - \frac{x}{4} - \frac{x^5}{4} + \cancel{\frac{x^3}{2}}\right] dx$$

$$= \frac{1}{2}\int_0^1 \left(\frac{x}{4} - \frac{x^3}{2} + \frac{x^5}{4}\right) dx$$

$$= \frac{1}{2}\left(\frac{x^2}{8} - \frac{x^4}{8} + \frac{x^6}{24}\right)_0^1$$

$$= \frac{1}{2}\left(\cancel{\frac{1}{8}} - \cancel{\frac{1}{8}} + \frac{1}{24}\right) = \frac{1}{48}.$$

practice problems

1) Evaluate $\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a}(x^2+y^2+z^2)\,dx\,dy\,dz$

2) Evaluate $\int_0^1\int_{y^2}^{1}\int_0^{1-x} x\,dz\,dx\,dy$

3) Evaluate $\int_1^e\int_1^{\log y}\int_1^{e^x} \log z\,dz\,dx\,dy$

Volumes by Triple integration:

i) The volume V of a three dimensional region V is given by $V = \iiint_V dx\,dy\,dz$.

ii) $V = \iiint_V r\,dr\,d\phi\,dz$ (cylindrical coordinates)

iii) $V = \iiint_V r^2 \sin\theta\,dr\,d\theta\,d\phi$ (spherical polar coordinates)

i) Transforming (x, y, z) into (r, θ, ϕ) ie, cartesian coordinates to spherical polar coordinates.

$$x = r \sin\theta \cos\phi$$
$$y = r \sin\theta \sin\phi$$
$$z = r \cos\theta$$

$$dx\,dy\,dz = r^2 \sin\theta \, dr \, d\theta \, d\phi$$

here $0 \leq r \leq a$, $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$ for the sphere $x^2 + y^2 + z^2 = a^2$.

ii) Transforming (x, y, z) into (ρ, ϕ, z) ie, cartesian coordinates to cylindrical co-ordinates.

$$x = \rho \cos\phi$$
$$y = \rho \sin\phi$$
$$z = z$$

$$dx\,dy\,dz = \rho \, d\rho \, d\phi \, dz$$

For the cylinder $x^2 + y^2 = a^2$, $0 \leq z \leq b$ we have

$$0 \leq \rho \leq a$$
$$0 \leq \phi \leq 2\pi$$
$$0 \leq z \leq b$$

Problem (1): Find the volume of the sphere $x^2+y^2+z^2=a^2$.

Solution: The sphere is symmetrical about xy, yz and xz planes. So, the required volume is equal to 8 times volume in the first octant of the sphere

ie, $$V = 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx$$

changing into spherical polar coordinates by taking

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$

$r: 0 \to a$

$\theta: 0 \to \frac{\pi}{2}$

$\phi: 0 \to \frac{\pi}{2}$

Now $$V = 8\int_0^a \int_0^{\pi/2} \int_0^{\pi/2} r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= 8\int_0^a r^2\,dr \int_0^{\pi/2} \sin\theta\,d\theta \int_0^{\pi/2} d\phi$$

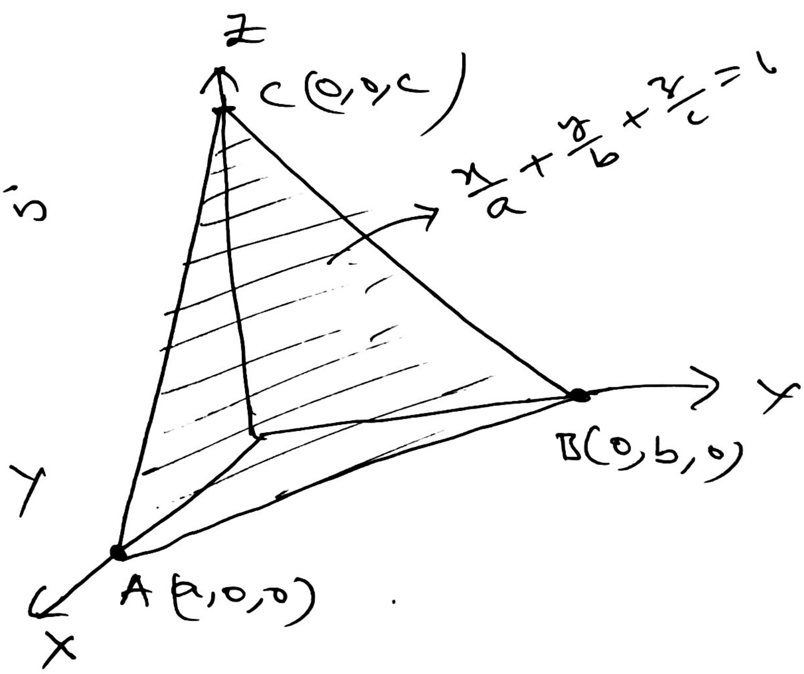
$$= 8\left(\frac{r^3}{3}\right)_0^a \left(-\cos\theta\right)_0^{\pi/2} \left(\phi\right)_0^{\pi/2}$$

$$= \frac{8a^3}{3}(1)\frac{\pi}{2}$$

$$= \frac{4}{3}\pi a^3 \text{ cubic units.}$$

Problem(2): Find the volume of the tetrahedran bounded by the coordinate planes and the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

Solution:



Given plane is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

Put $\frac{x}{a}=X$, $\frac{y}{b}=Y$

and $\frac{z}{c}=Z$

Then $x=aX$, $y=bY$, $z=cZ$, and

$$dx\,dy\,dz = abc\,dX\,dY\,dZ$$

Now $V = \iiint\limits_{\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1} dx\,dy\,dz$

$$= \iiint_{x+y+z=1} abc \, dx\, dy\, dz$$

$$= abc \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\, dy\, dx$$

$$= abc \int_0^1 \int_0^{1-x} (z)_0^{1-x-y} dy\, dx$$

$$= abc \int_0^1 \int_0^{1-x} (1-x-y)\, dy\, dx$$

$$= abc \int_0^1 \left(y - xy - \frac{y^2}{2}\right)_0^{1-x} dx$$

$$= abc \int_0^1 \left(1-x - x(1-x) - \frac{(1-x)^2}{2}\right) dx$$

$$= abc \int_0^1 \left(1 - \cancel{x} - x + x^2 - \frac{1}{2} - \frac{x^2}{2} + \cancel{x}\right) dx$$

$$= abc \int_0^1 \left(\frac{1}{2} - x + \frac{x^2}{2}\right) dx$$

$$= abc \left(\frac{x}{2} - \frac{x^2}{2} + \frac{x^3}{6}\right)_0^1$$

$$= abc \left[\left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{2}} + \frac{1}{6}\right) - 0\right]$$

$$= \frac{abc}{6} \text{ cu. units.}$$

Problem (3): Find the volume common to the cylinders $x^2+y^2=a^2$ and $x^2+z^2=a^2$.

Solution: The required volume is symmetrical about all the coordinate planes. So, the required volume bounded by the cylinders $x^2+y^2=a^2$ and $x^2+z^2=a^2$ is

$$V = 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2}} dz\, dy\, dx$$

$$= 8\int_0^a \int_0^{\sqrt{a^2-x^2}} (z)_0^{\sqrt{a^2-x^2}}\, dy\, dx$$

$$= 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2}\, dy\, dx$$

$$= 8\int_0^a \sqrt{a^2-x^2}\, (y)_0^{\sqrt{a^2-x^2}}\, dx$$

$$= 8\int_0^a (a^2-x^2)\, dx$$

$$= 8\left(a^2x - \frac{x^3}{3}\right)_0^a$$

$$= 8\left(a^3 - \frac{a^3}{3}\right)$$

$$= \frac{16a^3}{3} \text{ cu. units}$$

Problem (4): Find the volume bounded by the xy-plane, the cylinder $x^2+y^2=1$ and the plane $x+y+z=3$.

Solution: Given ~~curves~~ $x^2+y^2=1 \rightarrow (1)$

$$x+y+z=3 \rightarrow (2)$$

on the xy-plane, $z=0$ and on the plane $x+y+z=3$, $z=3-x-y$. Then $z: 0 \rightarrow 3-x-y$

Next x, y varie over the circle $x^2+y^2=1$ on the xy-plane

$$\text{Required volume} = \iiint_V dx\,dy\,dz$$

$$= \underbrace{\iint}_{x^2+y^2=1} \int_0^{3-x-y} dz\,dy\,dx$$

$$= \iint_{x^2+y^2=1} (3-x-y)\,dy\,dx$$

Put $x = r\cos\theta,\ y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

$r: 0 \rightarrow 1$ & $\theta: 0 \rightarrow 2\pi$

$$\text{volume} = \int_0^{2\pi}\int_0^1 (3 - r\cos\theta - r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 (3r - r^2\cos\theta - r^2\sin\theta)\,dr\,d\theta$$

$$= \int_0^{2\pi} \left(\frac{3r^2}{2} - \frac{r^3}{3}\cos\theta - \frac{r^3}{3}\sin\theta\right)_0^1 d\theta$$

$$= \int_0^{2\pi} \left(\frac{3}{2} - \frac{1}{3}\cos\theta - \frac{1}{3}\sin\theta\right) d\theta$$

$$= \left(\frac{3\theta}{2} - \frac{\sin\theta}{3} + \frac{\cos\theta}{3}\right)_0^{2\pi}$$

$$= \left(\frac{6\pi}{2} - 0 + \frac{1}{3}\right) - \left(0 - 0 + \frac{1}{3}\right)$$

$= 3\pi$ cu units

Problem (5): Find the volume cut from the sphere $x^2+y^2+z^2=a^2$ by the cone $x^2+y^2=z^2$.

Solution: Given $x^2+y^2+z^2=a^2 \rightarrow (1)$

& $x^2+y^2=z^2 \rightarrow (2)$

Transforming into spherical polar coordinates, we have

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

(1) becomes $x^2+y^2+z^2 = r^2 = a^2$

$\Rightarrow \boxed{r=a}$

& $x^2+y^2=z^2 \Rightarrow r^2\sin^2\theta = r^2\cos^2\theta$

$\Rightarrow \tan^2\theta = 1$

$\Rightarrow \boxed{\theta = \pm\frac{\pi}{4}}$

Required volume = 8 × volume in the first octant

$$= 8 \int_0^a \int_0^{\pi/4} \int_0^{\pi/2} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= 8 \int_0^a r^2 \, dr \int_0^{\pi/4} \sin\theta \, d\theta \int_0^{\pi/2} d\phi$$

$$= 8 \left(\frac{r^3}{3}\right)_0^a \left(-\cos\theta\right)_0^{\pi/4} \left(\phi\right)_0^{\pi/2}$$

$$= \frac{8}{3} a^3 \left(1 - \frac{1}{\sqrt{2}}\right) \frac{\pi}{2}$$

$$= \frac{4}{3} \pi a^3 \left(1 - \frac{1}{\sqrt{2}}\right) \text{ cu. units}$$