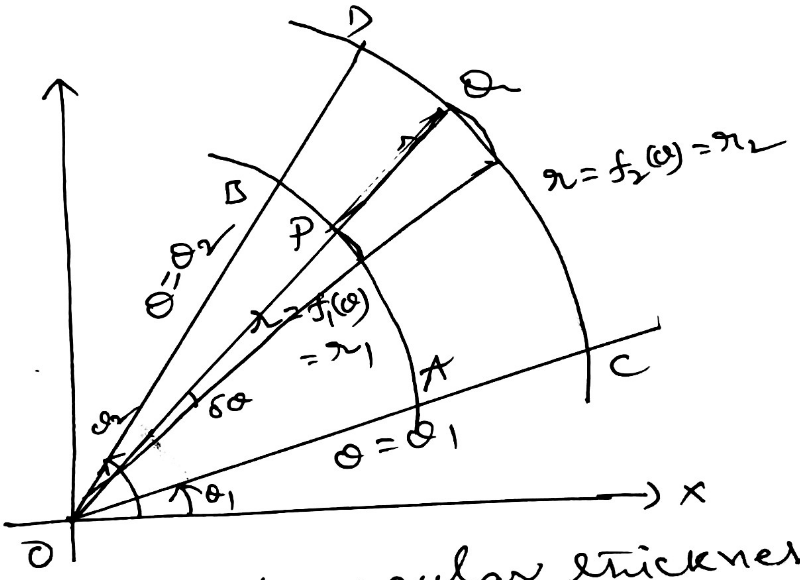
Double integrals in polar coordinates

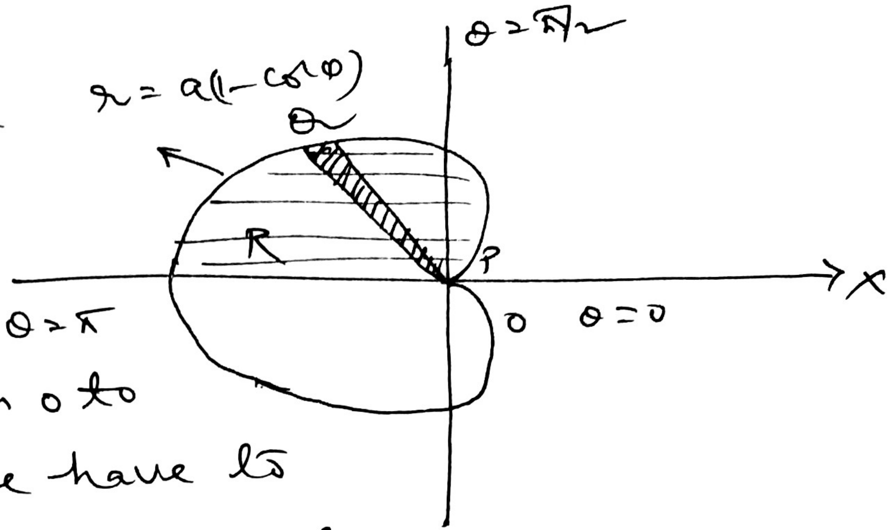
To evaluate $\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r,\theta)\,dr\,d\theta$ over the region bounded by the lines $\theta=\theta_1$ and $\theta=\theta_2$ and the curves $r=r_1$, $r=r_2$ we first integrate w.r.t "r" between the limits $r=r_1$ and $r=r_2$ treating θ as constant. The resulting expression is then integrated w.r.t "θ" between the limits $\theta=\theta_1$ and $\theta=\theta_2$.



Here PQ is wedge of angular thickness $\delta\theta$.
First we will integrate the wedge PQ and later we will rotate the wedge from $\theta=\theta_1$ to $\theta=\theta_2$.

Problem (1): Evaluate $\iint r\sin\theta\,dr\,d\theta$ over the cardioid $r=a(1-\cos\theta)$ above the initial line.

solution:



At P, $r=0$ and at Q, $r=a(1-\cos\theta)$.

Along the wedge PQ, r varies from 0 to $a(1-\cos\theta)$. Later we have to rotate the wedge from $\theta=0$ to $\theta=\pi$.

$$\therefore \iint_R r\sin\theta \, dr\, d\theta = \int_0^{\pi}\int_0^{a(1-\cos\theta)} r\sin\theta\, dr\, d\theta$$

$$= \int_0^{\pi} \sin\theta \left(\frac{r^2}{2}\right)_0^{a(1-\cos\theta)} d\theta$$

$$= \int_0^{\pi} \frac{a^2}{2}\sin\theta(1-\cos\theta)^2\, d\theta \quad *$$

$$= \frac{a^2}{2}\left[\frac{(1-\cos\theta)^3}{3}\right]_0^{\pi}$$

$$= \frac{a^2}{2}\left[\frac{(1-\cos\pi)^3}{3} - \frac{(1-\cos 0)^3}{3}\right]$$

$$= \frac{a^2}{2}\left[\frac{(1-(-1))^3}{3} - 0\right]$$

$$= \frac{a^2}{2}\,\frac{8}{3}$$

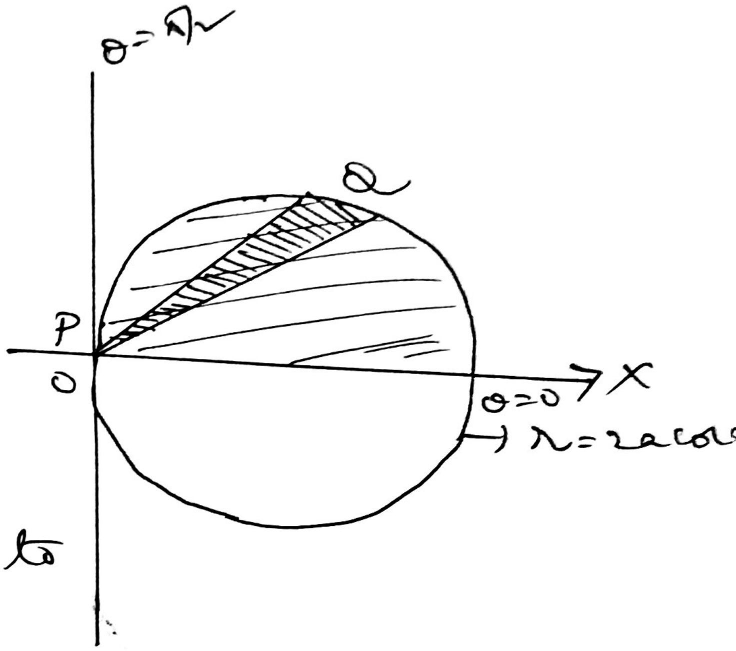
$$= \frac{4a^2}{3}$$

* put $1-\cos\theta = t$

$\Rightarrow +\sin\theta\, d\theta = dt$

Problem (2): Show that $\iint_R r^2 \sin\theta \, dr\, d\theta = \frac{2a^2}{3}$, where R is the semi-circle $r = 2a\cos\theta$ above the initial line.

Solution:



Along the wedge PQ, r varies from 0 to $2a\cos\theta$. Next we have to rotate the wedge PQ from $\theta = 0$ to $\theta = \pi/2$.

$$\therefore \iint_R r^2 \sin\theta \, dr\, d\theta = \int_0^{\pi/2}\int_0^{2a\cos\theta} r^2 \sin\theta \, dr\, d\theta$$

$$= \int_0^{\pi/2} \sin\theta \left(\frac{r^3}{3}\right)_0^{2a\cos\theta} d\theta$$

$$= \int_0^{\pi/2} \frac{8a^3}{3} \cos^3\theta \sin\theta \, d\theta$$

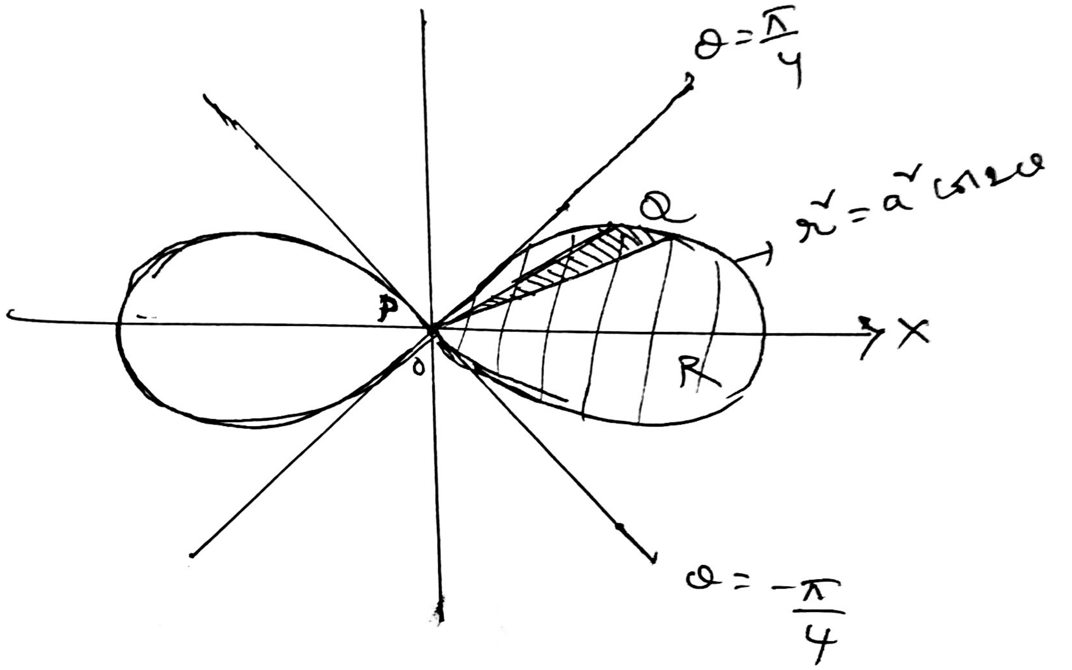
$$= \frac{8a^3}{3} \int_0^{\pi/2} -\cos^3\theta (-\sin\theta)\, d\theta$$

$$= -\frac{8a^3}{3} \left(\frac{\cos^4\theta}{4}\right)_0^{\pi/2}$$

$$= -\frac{8a^3}{3}\left(0 - \frac{1}{4}\right) = \frac{2a^3}{3}$$

Problem(3): Evaluate $\iint \frac{r\,dr\,d\theta}{\sqrt{a^2+r^2}}$ over one loop of the lemniscate $r^2 = a^2\cos 2\theta$.

Solution:



Along the wedge PQ, r varies from 0 to $a\sqrt{\cos 2\theta}$. Later we have to rotate the wedge PQ from $\theta = -\frac{\pi}{4}$ to $\theta = \frac{\pi}{4}$.

$$\therefore \iint_R \frac{r\,dr\,d\theta}{\sqrt{a^2+r^2}} = \int_{-\frac{\pi}{4}}^{\pi/4} \int_0^{a\sqrt{\cos 2\theta}} \frac{r}{\sqrt{a^2+r^2}}\,dr\,d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\pi/4} \left(\sqrt{a^2+r^2}\right)_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\pi/4} \left[(a^2 + a^2\cos 2\theta)^{1/2} - a\right] d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[\left(a^2(1+\cos 2\theta)\right)^{1/2} - a \right] d\theta$$

$$= \int_{-\pi/4}^{\pi/4} a\left[(1+\cos 2\theta)^{1/2} - 1\right] d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} a\left[(2\cos^2\theta)^{1/2} - 1\right] d\theta$$

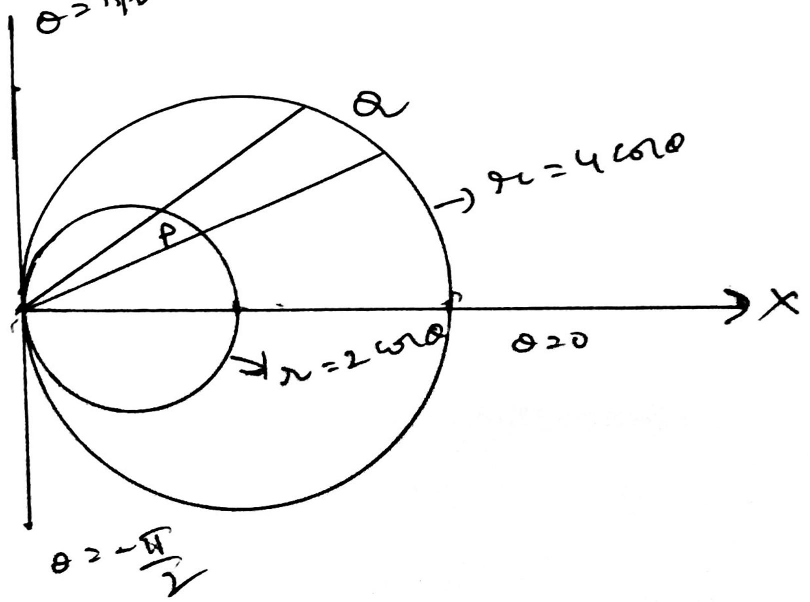
$$= a \int_{-\frac{\pi}{4}}^{\pi/4} (\sqrt{2}\cos\theta - 1)\, d\theta = 2a\int_0^{\pi/4}(\sqrt{2}\cos\theta - 1)\, d\theta$$

$$= 2a\,(\sqrt{2}\sin\theta - \theta)_0^{\pi/4} = 2a\left(\sqrt{2}\cdot\frac{1}{\sqrt{2}} - \frac{\pi}{4}\right)$$

$$= 2a\left(1 - \frac{\pi}{4}\right).$$

Problem(4): Evaluate $\iint r^3\, dr\, d\theta$ over the area bounded between the circles $r = 2\cos\theta$ and $r = 4\cos\theta$.

solution:



Along the wedge PQ, r varies from $2\cos\theta$ to $4\cos\theta$. Later we have to rotate the wedge PQ from $\theta = -\frac{\pi}{2}$ to $\theta = \frac{\pi}{2}$.

$$\therefore \iint r^3\,dr\,d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{2\cos\theta}^{4\cos\theta} r^3\,dr\,d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{r^4}{4}\right)_{2\cos\theta}^{4\cos\theta} d\theta$$

$$= \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (256\cos^4\theta - 16\cos^4\theta)\,d\theta$$

$$= 60 \int_{-\frac{\pi}{2}}^{\pi/2} \cos^4\theta\,d\theta$$

$$= 120 \int_0^{\frac{\pi}{2}} \cos^4\theta\,d\theta$$

$$= 120 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= \frac{45\pi}{2}$$

Practice Problem

1) Calculate $\iint r^3\,dr\,d\theta$ over the area included between the circles $r = 2\sin\theta$ & $r = 4\sin\theta$.

Area enclosed by plane curves

(i) cartesian coordinates:

The area enclosed by the curves $y = f_1(x)$, $y = f_2(x)$, $x = x_1$ and $x = x_2$ is given by

$$A = \int_{x_1}^{x_2} \int_{f_1(x)}^{f_2(x)} dy\, dx$$

Similarly, the area enclosed by the curves $x = g_1(y)$, $x = g_2(y)$, $y = y_1$ and $y = y_2$ is given by

$$A = \int_{y_1}^{y_2} \int_{g_1(y)}^{g_2(y)} dx\, dy$$

(ii) Polar coordinates:

The area enclosed by the curves $r = f_1(\theta)$, $r = f_2(\theta)$, $\theta = \theta_1$ and $\theta = \theta_2$ is given by

$$A = \int_{\theta_1}^{\theta_2} \int_{f_1(\theta)}^{f_2(\theta)} r\, dr\, d\theta.$$

Note: i) In cartesian coordinates

$$A = \iint dx\, dy$$

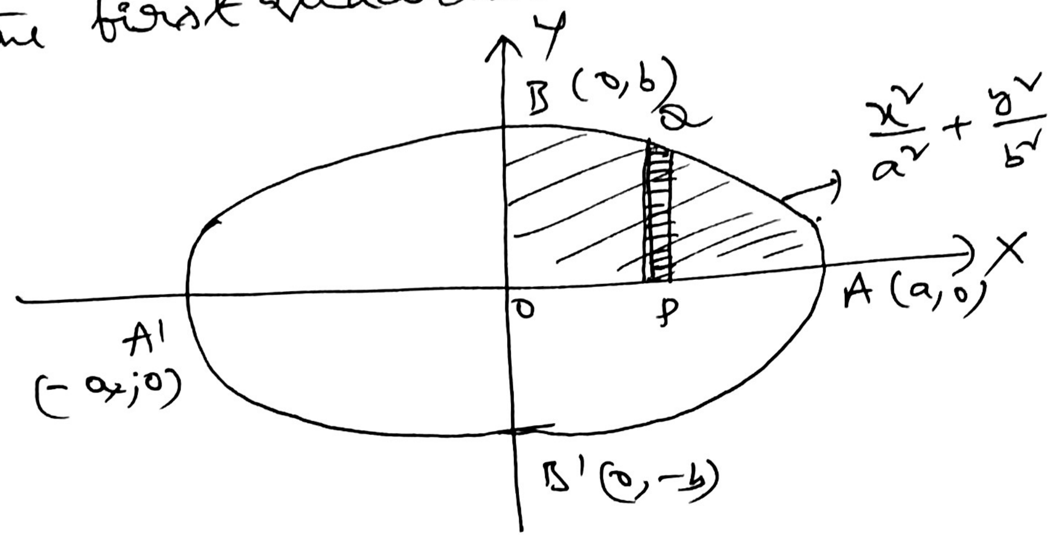
ii) In polar coordinates

$$A = \iint r\, dr\, d\theta.$$

Problem (1): Find the area of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Solution: The area of the ellipse is symmetrical about the coordinate axes. So, area of the ellipse is equal to $4\times$ area of the ellipse in the first quadrant.



i.e., $A = 4\times$ area of OAB

$$\therefore \text{Area of the ellipse} = 4\int_0^a\int_0^{\frac{b}{a}\sqrt{a^2-x^2}} dy\,dx$$

$$=4\int_0^a (y)_0^{\frac{b}{a}\sqrt{a^2-x^2}}\,dx$$

$$=4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$$

$$=\frac{4b}{a}\left(\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right)_0^a$$

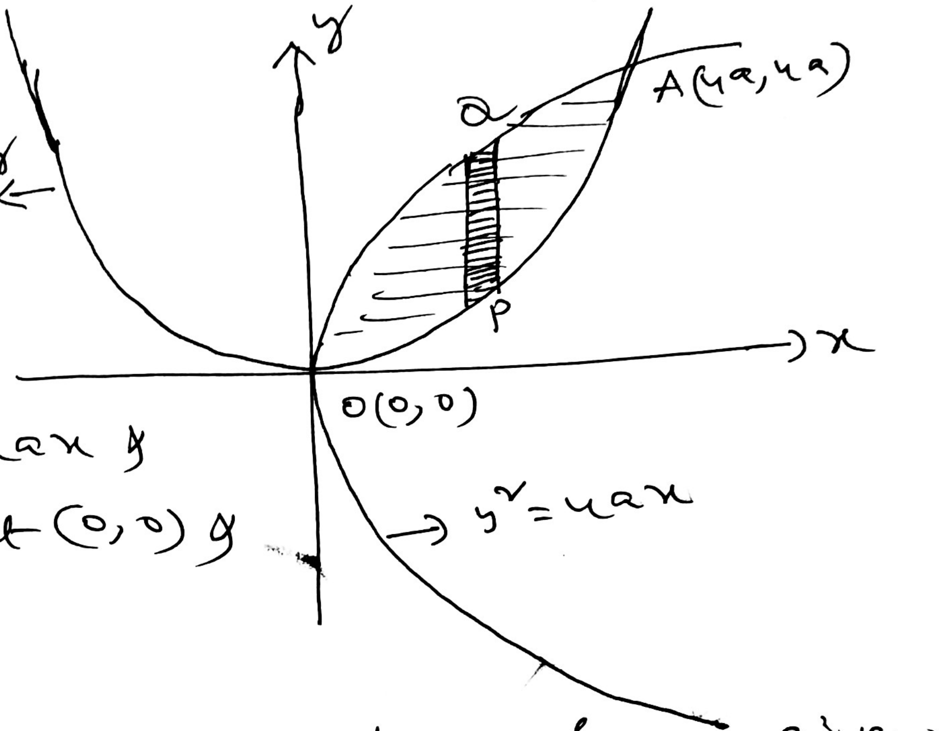
$$=\frac{4b}{a}\left[\left(0+\frac{a^2}{2}\sin^{-1}(1)\right)-0\right]$$

$$= \frac{4b}{a} \cdot \frac{a^2}{2} \cdot \frac{\pi}{2}$$

$$= \frac{\pi ab}{4} \text{ sq. units}$$

Problem(2): Show that the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ is $\frac{16}{3} a^2$.

Solution:



Solving the parabolas $y^2 = 4ax$ & $x^2 = 4ay$ we get $(0,0)$ & $(4a, 4a)$

$\therefore$ Area bounded by the parabolas is given by

$$A = \int_0^{4a} \int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy\, dx$$

$$= \int_0^{4a} (y)_{\frac{x^2}{4a}}^{2\sqrt{ax}} dx$$

$$= \int_0^{4a} \left(2\sqrt{ax} - \frac{x^2}{4a}\right) dx$$

$$= \int_0^{4a} \left(2\sqrt{a}\, x^{1/2} - \frac{1}{4a} x^2\right) dx$$

$$= \left(2\sqrt{a}\, \frac{2}{3} x^{3/2} - \frac{1}{12a} x^3\right)_0^{4a}$$

$$= \frac{4}{3}\sqrt{a}\,(4a)^{3/2} - \frac{1}{12a}(4a)^3$$

$$= \frac{4}{3}\sqrt{a}\; 8a\sqrt{a} - \frac{1}{12a}\, 64a^3$$

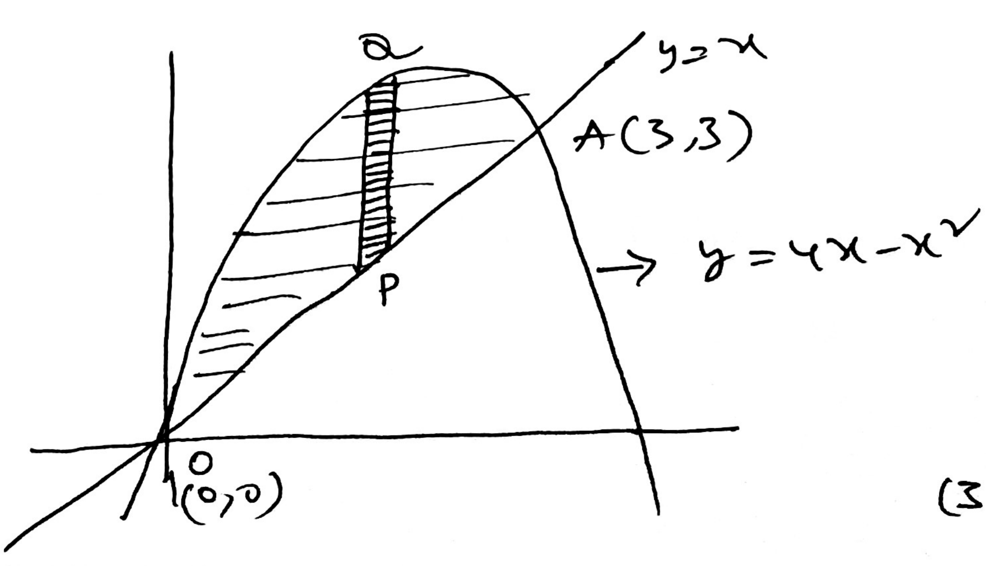
$$= \frac{4}{3}\, 8a^2 - \frac{16}{3} a^2$$

$$= \frac{32-16}{3} a^2$$

$$= \frac{16}{3} a^2 \text{ sq. units}$$

Problem (3): Find, by double integration, the area lying between the parabola $y = 4x - x^2$ and the line $y = x$.

Solution:



Given parabola is $y = 4x - x^2$

$$= -(x^2 - 4x)$$

$$= -(x^2 - 4x + 4 - 4)$$

$$= -(x-2)^2 + 4$$

$\Rightarrow y - 4 = -(x-2)^2$

This is a parabola whose vertex is at $(2,4)$ and it passes through $(0,0), (4,0)$

Next, solving $y = 4x - x^2$ and $y = x$, we get

$$x = 4x - x^2$$

$$\Rightarrow x^2 - 3x = 0$$

$$\Rightarrow x(x-3) = 0$$

$$\Rightarrow x = 0, 3$$

when $x = 0$, $y = 0$

when $x = 3$, $y = 3$

so, the parabola meets the line $y = x$ at $(0,0)$ & $(3,3)$.

Required area $= \int_0^3 \int_x^{4x-x^2} dy\, dx$

$$= \int_0^3 (y)_x^{4x-x^2} dx$$

$$= \int_0^3 \left[(4x - x^2) - x\right] dx$$

$$= \int_0^3 (3x - x^2)\, dx$$

$$= \left(\frac{3}{2}x^2 - \frac{x^3}{3}\right)_0^3$$

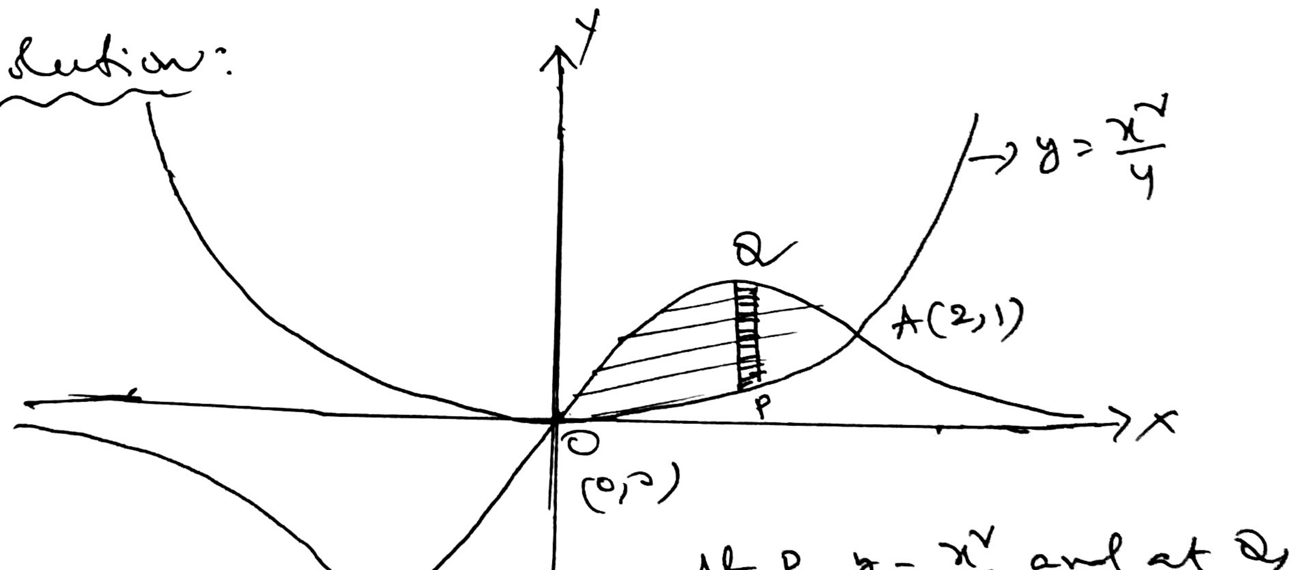
$$= \frac{27}{2} - \frac{27}{3}$$

$$= \frac{27}{2} - 9$$

$$= \frac{9}{2} \text{ sq. units}$$

Problem (4): Find the area enclosed by the curves $y = \dfrac{3x}{x^2+2}$ and $4y = x^2$.

Solution:



At P, $y = \dfrac{x^2}{4}$ and at Q $y = \dfrac{3x}{x^2+2}$. And x varies from 0 to 2

Required area bounded by the two curves is

$$A = \int_0^2 \int_{\frac{x^2}{4}}^{\frac{3x}{x^2+2}} dy\, dx$$

$$= \int_0^2 (y)_{\frac{x^2}{4}}^{\frac{3x}{x^2+2}} dx$$

$$= \int_0^2 \left(\frac{3x}{x^2+2} - \frac{x^2}{4}\right) dx$$

$$= \left[\frac{3}{2}\log(x^2+2) - \frac{x^3}{12}\right]_0^2$$

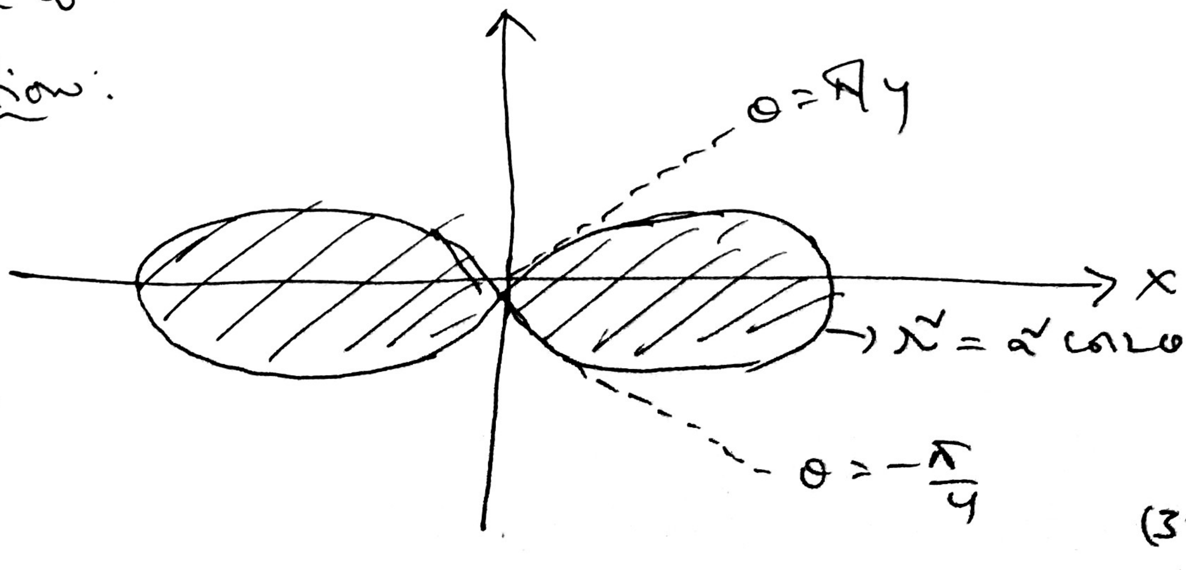
$$= \left(\frac{3}{2}\log 6 - \frac{8}{12}\right) - \left(\frac{3}{2}\log 2 - 0\right)$$

$$= \frac{3}{2}(\log 6 - \log 2) - \frac{2}{3}$$

$$= \frac{3}{2}\log 3 - \frac{2}{3} \text{ sq. units}$$

Problem (5): Find, by double integration, the area of the lemniscate $r^2 = a^2\cos 2\theta$.

Solution:



The lemniscate is symmetrical about the perpendicular line, $\theta = \frac{\pi}{2}$ as well as initial line $\theta = 0$.

So, Required area = 4 × Area of one loop above the initial line

$$= 4\int_0^{\pi/4}\int_0^{a\sqrt{\cos 2\theta}} r\,dr\,d\theta$$

$$= 4\int_0^{\pi/4} \left(\frac{r^2}{2}\right)_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$= \frac{4}{2}\int_0^{\pi/4} a^2 \cos 2\theta\,d\theta$$

$$= 4\frac{a^2}{2}\left(\frac{\sin 2\theta}{2}\right)_0^{\pi/4}$$

$$= 4\frac{a^2}{4}\left(\sin 2\times\frac{\pi}{4} - \sin 0\right)$$

$$= 4\frac{a^2}{4}(1)$$

$$= 4\frac{a^2}{4}$$

$$= a^2 \text{ sq. units}$$

Thus the area of the lemniscate is a^2 sq. units

Problem (6): Find the area lying inside the circle $r = a\sin\theta$ and outside the cardioid $r = a(1-\cos\theta)$.

Solution: Solving the two curves, we have

$$\cancel{a}\sin\theta = \cancel{a}(1-\cos\theta)$$

$$\Rightarrow \sin\theta = 1-\cos\theta$$

$$\Rightarrow \cancel{2}\sin\theta/_2 \cos\theta/_2 = \cancel{2}\sin^2\theta/_2$$

$$\Rightarrow \sin\theta/_2 \cos\theta/_2 - \sin^2\theta/_2 = 0$$

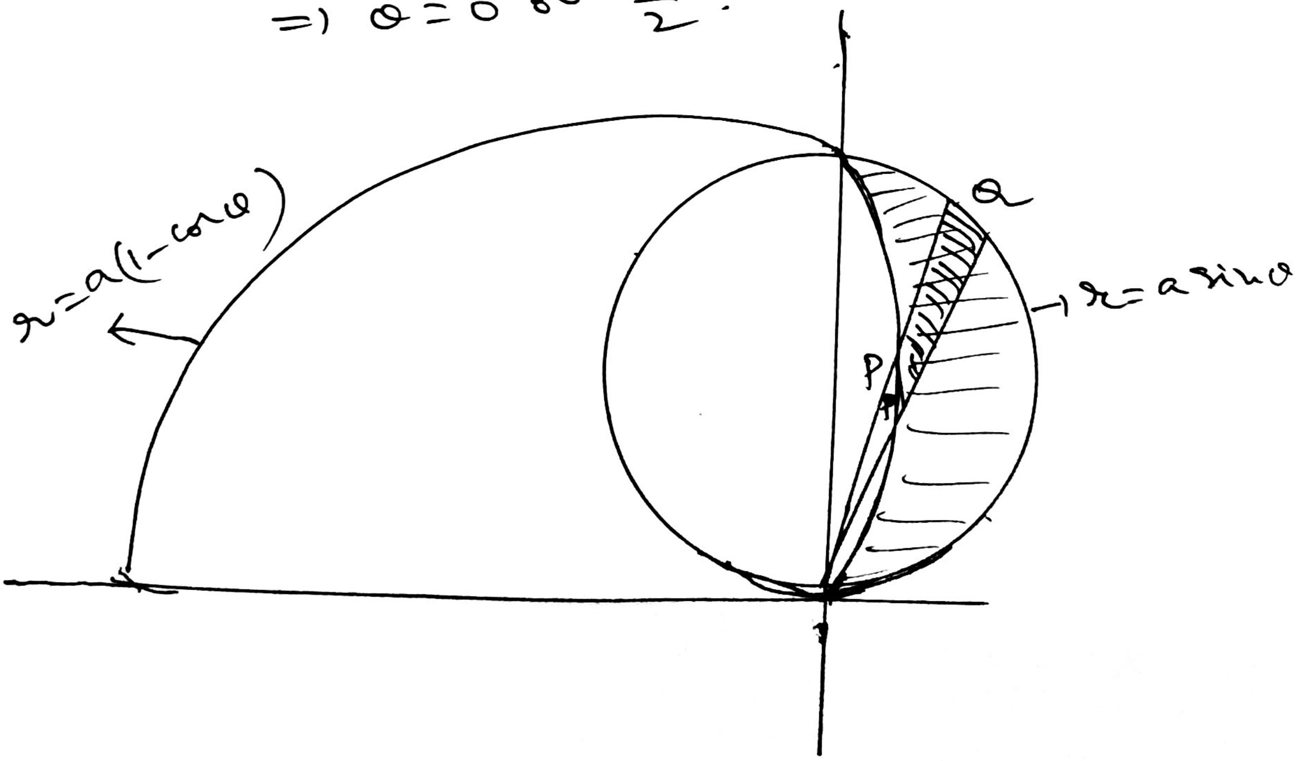
$$\Rightarrow \sin\theta/_2 (\cos\theta/_2 - \sin\theta/_2) = 0$$

$$\Rightarrow \sin\theta/_2 = 0 \text{ (or) } \cos\theta/_2 - \sin\theta/_2 = 0$$

$$\Rightarrow \sin\frac{\theta}{2} = 0 \text{ (or) } \tan\theta/_2 = 1$$

$$\Rightarrow \frac{\theta}{2} = 0 \text{ (or) } \frac{\pi}{4}$$

$$\Rightarrow \theta = 0 \text{ or } \frac{\pi}{2}.$$



In the region of integration r varies from $a(1-\cos\theta)$ to $a\sin\theta$ and θ varies from 0 to $\pi/2$

$$\text{Area} = \int_0^{\pi/2} \int_{a(1-\cos\theta)}^{a\sin\theta} r\, dr\, d\theta$$

$$= \int_0^{\pi/2} \left(\frac{r^2}{2}\right)_{a(1-\cos\theta)}^{a\sin\theta} d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} \left(a^2 \sin^2\theta - a^2(1-\cos\theta)^2\right) d\theta$$

$$= \frac{a^2}{2} \int_0^{\pi/2} \left(\sin^2\theta - 1 - \cos^2\theta + 2\cos\theta\right) d\theta$$

$$= \frac{a^2}{2} \int_0^{\pi/2} \left(1 - \cos^2\theta - 1 - \cos^2\theta + 2\cos\theta\right) d\theta$$

$$= \frac{a^2}{2} \int_0^{\pi/2} \left(-2\cos^2\theta + 2\cos\theta\right) d\theta$$

$$= -2 \cdot \frac{a^2}{2} \int_0^{\pi/2} \cos^2\theta\, d\theta + a^2 \left(\sin\theta\right)_0^{\pi/2}$$

$$= -a^2 \cdot \frac{1}{2} \cdot \frac{\pi}{2} + a^2$$

$$= a^2\left(1 - \frac{\pi}{4}\right) \text{ sq. units.}$$

Practice problems

1) Find the area lying between the parabola $y = x^2$ and the line $x + y - 2 = 0$.

2) Find the area lying inside the cardioid $r = a(1+\cos\theta)$ and outside the circle $r = a$.

3) Find the area common to the circles $r = a\cos\theta$, $r = a\sin\theta$.