4033/5033 Assignment: MLE

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Let X be a random variable following a Gaussian distribution $N(\mu, \sigma^2)$. This means the pdf of X is

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp(-\frac{(x-\mu)^2}{2\sigma^2}). \tag{1}$$

Let $x_1, \ldots, x_n$ be an i.i.d. sample of X.

Task 1. Derive the MLE estimate of μ based on the sample. Treat σ as a known constant.

Let X be a random variable following a Gaussian distribution $\mathcal{N}(\mu, \sigma^2)$. This means the probability density function (pdf) of X is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

Let $x_1, \ldots, x_n$ be an i.i.d. sample of X. We want to derive the Maximum Likelihood Estimator (MLE) of μ based on this sample, treating σ as a known constant.

The likelihood function is given by

$$L(\mu \,|\, x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x_i-\mu)^2}{2\sigma^2}\right)$$

Taking the natural logarithm (log-likelihood) to simplify,

$$\ln(L(\mu \,|\, x_1, x_2, \ldots, x_n)) = -n\ln(\sqrt{2\pi\sigma^2}) - \sum_{i=1}^{n} \frac{(x_i-\mu)^2}{2\sigma^2}$$

To find the MLE for μ, we take the derivative of the log-likelihood with respect to μ and set it equal to zero:

$$0 = -\sum_{i=1}^{n} \frac{-2(x_i-\mu)}{2\sigma^2}$$

Simplifying and solving for μ,

$$0 = \sum_{i=1}^{n} \frac{x_i}{\sigma^2} - \sum_{i=1}^{n} \frac{\mu}{\sigma^2}$$

This leads to the MLE for μ:

$$\mu^* = \frac{1}{n}\sum_{i=1}^{n}\frac{x_i}{\sigma^2}$$

So, the Maximum Likelihood Estimator (MLE) for μ based on the sample $x_1, x_2, \ldots, x_n$ when σ is treated as a known constant is:

$$\mu^* = \frac{1}{n}\sum_{i=1}^{n}\frac{x_i}{\sigma^2}$$

Task 2. Derive the MLE estimate of σ based on the sample. Treat μ as a known constant.

Let X be a random variable following a Gaussian distribution $\mathcal{N}(\mu, \sigma^2)$. This means the probability density function (pdf) of X is given by:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

Let $x_1, \ldots, x_n$ be an independent and identically distributed (i.i.d.) sample of X. We want to derive the Maximum Likelihood Estimator (MLE) of σ^2 based on this sample, with μ treated as a known constant.

The likelihood function $L(\sigma^2)$ for the sample $x_1, x_2, \ldots, x_n$ is given by the product of the pdfs for each observation, due to their independence:

$$L(\sigma^2) \propto \exp\left(-\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{2\sigma^2}\right)$$

To find the MLE of σ^2, we maximize the log-likelihood function, which simplifies the calculations:

$$\ln(L(\sigma^2)) \propto -\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{2\sigma^2}$$

Now, we take the derivative of the log-likelihood with respect to σ^2 and set it equal to zero to find the maximum:

$$\frac{d}{d(\sigma^2)}\left[\ln(L(\sigma^2))\right] = -\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2(\sigma^2)^2} + \frac{n}{2\sigma^2} = 0$$

Solving for σ^2:

$$-\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2(\sigma^2)^2} + \frac{n}{2\sigma^2} = 0$$

$$\sum_{i=1}^{n}(x_i-\mu)^2 = n\sigma^2$$

$$\sigma^2 = \frac{\sum_{i=1}^{n}(x_i - \mu)^2}{n}$$

So, the Maximum Likelihood Estimator (MLE) of σ^2 is:

$$\sigma^2_{\mathrm{MLE}} = \frac{\sum_{i=1}^{n}(x_i - \mu)^2}{n}$$

This is the MLE estimate of the population variance σ^2 based on the given i.i.d. sample $x_1, x_2, \ldots, x_n$, with μ treated as a known constant.