LA Assignment - 2 - 19/1/25

1. Prove that row-equivalence is an equivalence relation.

ANS:

A relation R is said to be an equivalence relation if it satisfies the following properties:

1) Reflexivity : $(a,a) \in R$

2) Symmetry : $(a,b) \in R \Rightarrow (b,a) \in R$

3) Transitivity : $(a,b) \in R$ and $(b,c) \in R \Rightarrow (a,c) \in R$

Matrices A and B are said to be row equivalent if A can be represented as the output of a finite number of row operations on B.

To prove that row equivalence is a equivalence relation,

i) Reflexivity,

Clearly, $A = A$

$\therefore$ A is the output of zero row operations on A

$\therefore$ A is row equivalent to itself, ie row equivalence is reflexive.

2) Symmetry,

Let A be row equivalent to B, ie

$$A = e_1(e_2(e_3 \dots e_n(B)))\dots)$$

Where each $e_i(x)$, $i \leq n \in \mathbb{N}$, is a row operation.
We know that for each of these row operation, $\exists$ an inverse row operation e_i' such that,

$$A = e_i'(e(A))$$

Let e_i' be the inverse operation of e_i

$$A = e_1(e_2(e_3 \dots (e_n(B))\dots)$$

$$\Rightarrow \quad e_1'(A) = e_2(e_3(\dots (e_n(B))\dots)$$

$$\Rightarrow \quad e_2'(e_1'(A)) = e_3(\dots (e_n(B))\dots)$$

$$\Rightarrow \quad B = e_n'(e_{n-1}'(e_{n-2}' \dots e_3'(e_2'(e_1'(A)))\dots))$$

$\therefore$ B is the output of n row operations on A

$\therefore$ B is row equivalent to A

$\therefore$ Row equivalence is symmetric

3) Transitive.

Let A be row equivalent to B and B be row equivalent to C, ie,

$$A = e_1(e_2(e_3 \cdots e_n(B))\cdots) \quad — (1)$$

$$B = j_1(j_2(j_3 \cdots j_m(C))\cdots) \quad — (2)$$

$n, m \in \mathbb{N}$

Where $e_i(X)$ and $j_i(X)$ are row operations.

Clearly by applying (2) in (1), we get,

$$A = e_1(e_2(e_3 \cdots e_n(j_1(j_2(j_3 \cdots j_m(C)))\cdots)$$

ie, A is the output of $n+m$ row operations on C.

$\therefore$ A is row equivalent to C.

$\therefore$ Row equivalence is transitive.

As Row equivalence satisfies all the above 3 properties,

<u>Row equivalence is an equivalence relation</u>

2. Prove that every $m \times n$ matrix is row equivalent to a row reduced matrix.

Any $m \times n$ matrix can be converted into a row reduced matrix using the elementary row operations, using the following algorithm. (Let the matrix be $[A_{ij}]_{m \times n}$)

1. Set $j = 1$. Find the minimum possible i s.t. $A_{i1} \neq 0$ and $i \geq j$
2. Swap row i with the jth row.
3. Multiply row j (now row 1) with the multiplicative inverse of A_{jj} (or A_{11}). Now $A_{11} = 1$.
4. For $\forall\, i > 1$, $A_{ij} \longrightarrow A_{ij} - A_{i1}(A_{1j})$ subtraction of a row by the multiple of another row.

   Since $A_{11} = 1$, for $j = 1$

   $$A_{i1} \longrightarrow A_{i1} - A_{i1}(A_{11})$$

   $$\Rightarrow A_{i1} \longrightarrow A_{i1} - A_{i1}$$

   $$\Rightarrow A_{i1} \longrightarrow 0 \quad \forall\, i > j$$

5. Repeat this $\forall\, j$ st $1 < j \leq m$.

Since in step 4, we get that the elements A_{jj} is the topmost non zero element in column j, the resulting matrix is actually a row reduced echelon matrix, which clearly is row-reduced

Therefore, since we have obtained a row reduced matrix after a finite no. of row operations on a generic $m \times n$ matrix, the given statement holds true, ie

"Any matrix is row equivalent to a row reduced matrix."

LA Assignment - 3 - 26/1/25

1. Prove that if $A_{m\times n}$, $m<n$, then $AX=0$ always has a non trivial solution.

Let r be the no. of zero rows. ($r \leq m$ clearly).

Let the row reduced Echelon form of A be R,

Since the matrix is $m\times n$, there are n variables in the system, ie X is $n\times 1$.

There will be $m-r$ leading one's in R, therefore there will be $m-r$ dependent variables / non-free scalars

Since there are n total variables and $m<n$
$\Rightarrow$ $m-r<n$, there are $n-m+r$ free scalars in the system, ie, $\forall$ value of these scalars, $\exists$ a solution to the system.
Using these free scalars, we can form infinite no. of non trivial solutions.

$\therefore$ It is proved that if $m<n$ and $A_{m\times n} \cdot X_{n\times 1}=0$ the system will have non trivial solutions always.

2. Prove that a square matrix A is row equivalent to I, if and only if $AX = 0$ has only trivial solution.

If $A_{n\times n}$ is row equivalent to I, then RREF of A is I itself

Clearly, $IX = 0 \Rightarrow X = 0 \rightarrow$ trivial solution

$\therefore$ If $A_{n\times n}$ is equivalent to I, then $AX = 0$ always has only trivial soln, as $X \neq 0 \Rightarrow IX \neq 0$ which is not true.

Now, if $A_{n\times n}X = 0$ only has trivial solution, there will not be any free scalars in the RREF of A which implies that there are no non-zero rows in the RREF of A, since there has to be n leading ones in the matrix, and there are only n rows.

Since no row in the RREF of A is a zero row, RREF must be Identity matrix.

$\therefore$ If $A_{n\times n}X = 0$ only has trivial solutions, $A_{n\times n}$ is row equivalent to Identity matrix.

$\therefore$ Square matrix A is row equivalent to I iff $AX = 0$ has only trivial solutions.

8. Prove that for any row operation e and $m \times m$ elementary matrix E st $E = e(I)$, then $\forall\ A_{m \times n}$, $e(A) = E \times A$.

Case I: e is the row operation of multiplication of a row r by a scalar c.

$$\Rightarrow \quad E_{ij} = \begin{cases} 1, & i = j \ \& \ i \neq r \\ c, & i = j = r \\ 0, & \text{otherwise} \end{cases}$$

$$e(A) = \begin{cases} A_{ij}, & i \neq r \\ cA_{ij}, & i = r \end{cases}$$

Let $E_{m \times m} \times A_{m \times n} = P_{m \times n}$

$$P_{ij} = \sum_{s=1}^{n} E_{is} A_{sj}$$

$$E_{is} A_{sj} = \begin{cases} 0, & i \neq s \\ A_{sj}, & i = s \end{cases}$$

$$\Rightarrow \quad P_{ij} = E_{ii} A_{ij} = \begin{cases} A_{ij}, & i \neq r \\ cA_{ij}, & i = r \end{cases} = e(A)_{ij}$$

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Since $P_{ij} = e(A)_{ij}$

$$\Rightarrow \quad \underline{\underline{E \times A = e(A)}}$$

<u>Case 2:</u> e is the row operation of adding the scalar multiple c of row r to row s.

$$\Rightarrow E_{ij} = \begin{cases} 1, & i=j \\ c, & i\neq j \ \& \ i=s, j=r \\ 0, & i\neq j \ \& \ i\neq s \end{cases}$$

$$e(A_{ij}) = \begin{cases} A_{ij}, & i\neq s \\ A_{ij} + cA_{rj}, & i=s \end{cases}$$

Let $P_{m\times n} = E_{m\times m} \times A_{m\times n}$.

$$P_{ij} = \sum_{t=1}^{n} E_{it} A_{tj}$$

$$E_{it} A_{tj} = \begin{cases} A_{ij}, & t=i \\ cA_{rj}, & i=s, t=r \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow P_{ij} = \sum E_{it} A_{tj} = \begin{cases} A_{ij} + cA_{rj}, & i=s \\ A_{ij}, & \text{otherwise} \end{cases}$$

$$= e(A)_{ij}$$

$$\Rightarrow \underline{E \times A = e(A)}$$

Case 3: e is the row operation of swapping 2 rows r, s.

$$E_{ij} = \begin{cases} 1, & i=j \ \& \ i \notin \{r,s\} \\ 1, & i=r, j=s \\ 1, & i=s, j=r \\ 0, & \text{otherwise} \end{cases}$$

$$e(A_{ij}) = \begin{cases} A_{ij}, & i \notin \{r,s\} \\ A_{sj}, & i=r \\ A_{rj}, & i=s \end{cases}$$

Let $P = E \times A$,

$$P_{ij} = \sum_{t=1}^{n} E_{it} A_{tj}$$

$$E_{it} A_{tj} = \begin{cases} A_{ij}, & i \notin \{r,s\} \\ A_{sj}, & i=r, t=s \\ A_{rj}, & i=s, t=r \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow P_{ij} = \begin{cases} A_{ij}, & i \notin \{r,s\} \\ A_{sj}, & i=r \\ A_{rj}, & i=s \end{cases}$$

$$= e(A)_{ij}$$

$$\Rightarrow \underline{E \times A = e(A)}$$

Therefore, for any row operation e,

$$e(A) = E \times A, \text{ where } E = e(I)$$

4. Prove that the following statements are equivalent for a square matrix $A_{n\times n}$.
   1) A is invertible
   2) $AX = 0$ has only trivial solution
   3) $AX = Y$ has a solution for each $Y_{m\times 1}$

Starting with statement 2,

$AX = 0$ has only trivial solutions $\Rightarrow$ A is row equivalent to I (Theorem form Q2).

$$\Rightarrow A = e_1(e_2(e_3(e_4 \cdots\cdots (e_n(I)) \cdots)$$

$$\Rightarrow A = E_1 \times E_2 \times E_3 \cdots \times E_n \times I \quad \text{where } E_i = e_i(I)$$
$$\forall i \in \mathbb{N} \;\&\; i \leq n$$

We know that E_i is invertible $\forall i$, since they are all elementary matrices, and elementary matrices are invertible.

$$\Rightarrow E_1^{-1} \times E_2^{-1} \cdots \times E_n^{-1} \times A = I$$

Let $\prod_{i=1}^{n} E_i^{-1} = B$

$$\Rightarrow B \times A = I \Rightarrow B \text{ is the inverse of } A$$

$\therefore$ $AX = 0$ has only trivial solutions $\Rightarrow$ A is invertible

For the backward proof,

Let $A_{n\times n}$ be any invertible $n \times n$ matrix, ie, $\exists A^{-1}$ st,

$$A \cdot A^{-1} = I$$

For, $AX = 0$

$$\Rightarrow A^{-1} \cdot A \cdot X = 0$$

$$\Rightarrow I \cdot X = 0$$

$$\Rightarrow X = 0$$

$\Rightarrow$ $AX = 0$ has only trivial solution

$\therefore$ Statements 1 and 2 are equivalent.

For statement 3,

$AX = Y$ has a solution for each Y.

$\Leftrightarrow$ There are no free scalars in the system.

$\Leftrightarrow$ $AX = 0$ has only trivial solutions (from Q2)

↳ Statement 2

Statement 3 is equivalent to statement 2.

$\therefore$ The given 3 statements are equal to each other.