Linear Algebra - Assignment 2 (H2)

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1. **Given:** A has SVD $A = USV^T$. Let order of A be $m \times n$.

   To Show: AA^T has the columns of U as its Eigenvectors with associated Eigenvalues S^2.

Let the singular values of A be $\sigma_1, \sigma_2, \sigma_3 \ldots \sigma_n$ such that $\sigma_i = 0 \ \forall\, i > r$, and $\sigma_1 \geq \sigma_2 \geq \sigma_3 \cdots \geq \sigma_r > \sigma_{r+1} = \sigma_{r+2} = \cdots = \sigma_n = 0$

$$\Rightarrow S = \left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right]_{m \times n} \quad \text{where } D = \text{dia}(\sigma_1, \sigma_2, \sigma_3 \ldots, \sigma_r)$$

$A = USV^T$, U, V are orthogonal

$$A^T = (USV^T)^T = VS^TU^T$$

$$AA^T = USV^TVS^TU$$

$$= USS^TU^T \quad (V^{-1} = V^T \text{ by orthogonality})$$

$$SS^T = \left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right]_{m \times n} \left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right]_{n \times m}$$

The submatrix D is unchanged in the transpose since its elements occupy position in the principal diagonal of the matrix.

$$\Rightarrow SS^T = \left[\begin{array}{c|c} D^2 & 0 \\ \hline 0 & 0 \end{array}\right]_{m \times m}$$

Since D is a diagonal matrix, D^2 will also be a diagonal matrix, whose values are the squares of the values of D.

$$\Rightarrow D^2 = dia(\sigma_1^2, \sigma_2^2, \sigma_3^2 \dots \sigma_r^2)$$
$$= dia(\lambda_1, \lambda_2, \lambda_3, \lambda_4 \dots \lambda_r)$$

$\therefore$ The values of D^2 are the Eigenvalues of A^TA

$$\Rightarrow AA^T = U \underbrace{\begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 & \cdots \\ 0 & \lambda_2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & 0 & \cdots & \vdots \\ \vdots & & & & 0 & \vdots \\ & \cdots & & 0 & \lambda_r & \vdots \\ & & & & & 0 \\ \vdots & & \cdots & & & 0 \end{pmatrix}}_{S^2} U^T$$

Also, $\sigma_{r+1}^2 = \sigma_{r+2}^2 \dots = \sigma_m^2 = 0 \Rightarrow \underline{\lambda_{r+1} = \lambda_{r+2} = \dots = \lambda_m = 0}$

$$\Rightarrow \quad AA^T = US^2U^T \longrightarrow \text{Orthogonal Diagonalization of } AA^T$$

Using the above Orthogonally Diagonalized form, the spectral form of AA^T can be written as,

$$AA^T = \lambda_1 \bar{u}_1 \bar{u}_1^T + \lambda_2 \bar{u}_2 u_2^T + \cdots \lambda_m \bar{u}_m \bar{u}_m^T$$

Where $\bar{u}_1, \bar{u}_2, \ldots \bar{u}_n$ are the column vectors of U

$$\Rightarrow \quad AA^T = \sum_{i=1}^{m} \lambda_i \bar{u}_i \bar{u}_i^T$$

Multiplying on both sides by an arbitrary column vector of U, say $\bar{u}_k$,

$$AA^T \bar{u}_k = \sum_{i=1}^{m} \lambda_i \bar{u}_i \bar{u}_i^T \bar{u}_k$$

$$= \sum_{i=1}^{m} \lambda_i \bar{u}_i \left(\bar{u}_i \cdot \bar{u}_k \right)$$

The terms of this summation can be calculated as,

$$\lambda_i \bar{u}_i (\bar{u}_i \cdot \bar{u}_k) = \begin{cases} 0, & i \neq k \ (\text{Orthogonality of } U) \\ \lambda_k \bar{u}_k (\bar{u}_k \cdot \bar{u}_k), & i = k \end{cases}$$

And, $\lambda_k \bar{u}_k (\bar{u}_k \cdot \bar{u}_k) = \lambda_k \bar{u}_k (1) = \lambda_k \bar{u}_k$, due to the columns of U being normalized (Orthogonality of U)

$$\Rightarrow \sum_{i=1}^{m} \lambda_i \bar{v}_i (\bar{v}_i \cdot \bar{v}_k) = \lambda_k \bar{v}_k$$

$$\Rightarrow \underline{AA^T \bar{v}_k = \lambda_k \bar{v}_k}$$

Since $\bar{v}_k$ is any general column vector of V,

$\therefore$ The Eigenvectors of AA^T are the column vectors of V with the corresponding Eigenvalues in S^2.

2. <u>To Find</u>: SVD and Outer Product Form of $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

To find SVD of A, we need Singular values of A.

$$A^T A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}$$

$$\det(A^T A - \lambda I) = \det \begin{pmatrix} 2-\lambda & 2 & 2 \\ 2 & 2-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{pmatrix}$$

$$= (2-\lambda)\left((2-\lambda)^2 - 4\right) - 2\left(2(2-\lambda) - 4\right) + 2\left(4 - 2(2-\lambda)\right)$$

$$= (2-\lambda)^3 - 4(2-\lambda) - 4(2-\lambda) + 8 + 8 - 4(2-\lambda)$$

$$= (2-\lambda)^3 - 12(2-\lambda) + 16$$

Changing variable $\lambda_0 = 2 - \lambda$,

$$= \lambda_0^3 - 12\lambda_0 + 16$$

$$\begin{array}{r|l} & \lambda_0^2 + 2\lambda_0 - 8 \\ \hline \lambda_0 - 2 & \lambda_0^3 + 0\lambda_0^2 - 12\lambda_0 + 16 \\ & (-)\lambda_0^3 \overset{(+)}{-} 2\lambda_0^2 \\ \hline & 2\lambda_0^2 - 12\lambda_0 \\ & (-)\, 2\lambda_0^2 \overset{(+)}{-} 4\lambda_0 \\ \hline & -8\lambda_0 + 16 \\ & -8\lambda_0 + 16 \\ \hline & 0 \end{array}$$

$$= (\lambda_0 - 2)(\lambda_0^2 + 2\lambda_0 - 8)$$

$$= (\lambda_0 - 2)(\lambda_0 + 4)(\lambda_0 - 2)$$

$$= (\lambda_0 - 2)^2(\lambda_0 + 4)$$

$$\Rightarrow (2 - \lambda - 2)^2(2 - \lambda + 4)$$

$$= \lambda^2(6 - \lambda) = 0 \Rightarrow \text{Eigenvalues: } \lambda_1 = 6\ (AM = 1),\ \lambda_2 = 0\ (AM = 2)$$

$\Rightarrow$ Singular values of A: $\sigma_1 = \sqrt{6}$, $\sigma_2 = 0$

$$\Rightarrow \Sigma = \begin{bmatrix} \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

For matrix V, we need the eigenvectors of A^TA, which are,

Wrt $\lambda_1 = 6$

$$(A^TA - 6I)X = \bar{0}$$

$$\Rightarrow \begin{pmatrix} 2-6 & 2 & 2 \\ 2 & 2-6 & 2 \\ 2 & 2 & 2-6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \bar{0}$$

$$\begin{pmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \bar{0}$$

$$\Rightarrow 2 \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$-2x + y + z = 0$

$y + z = 2x$

$x = \frac{y+z}{2}$

$x - 2y + z = 0$

$\frac{y+z}{2} - 2y + z = 0$

$y + z - 4y + 2z = 0$

$-3y + 3z = 0$

$y = z$

$\Rightarrow \quad x = y = z$

$x + y - 2z = 0$

$z + z - 2z = 0$

$\Rightarrow \quad 0 = 0$

$$\Rightarrow \quad \underline{x = y = z}$$

$$\Rightarrow \quad E_{\lambda_1} = x \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \forall\, x \in \mathbb{R} \Rightarrow \bar{V_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Wrt $\lambda_2 = 0$,

$$(A - 0I)X = \bar{0}$$

$$= \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= 2x + 2y + 2z = 0$$

$$= x + y + z = 0$$

$$= x = -y - z$$

$$\Rightarrow \quad E_{\lambda_2} = y \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad \forall\, y, z \in \mathbb{R}$$

$$\Rightarrow \quad \bar{V_2} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad \bar{V_3} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad \bar{V_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \bar{V_2} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad \bar{V_3} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Vectors $\bar{V_2}$ and $\bar{V_3}$ are not orthogonal to each other, so, to orthogonalize them,

$$\bar{V_3}' = \bar{V_3} - \frac{\langle \bar{V_2} \cdot \bar{V_3} \rangle}{\langle \bar{V_2} \cdot \bar{V_2} \rangle} \bar{V_2}$$

$$= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad \overline{v_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \overline{v_2} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad \overline{v_3} = \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix}$$

$$\Big\downarrow \text{ Normalizing}$$

$$\overline{v_1} = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} \quad \overline{v_2} = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \quad \overline{v_3} = \begin{pmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ \sqrt{2/3} \end{pmatrix}$$

$$\Rightarrow \quad V = \begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & \sqrt{2/3} \end{pmatrix}$$

For the matrix U, let the column vectors be $\overline{u_1}, \overline{u_2}$

$$\overline{u_1} = \frac{1}{\sigma_1} A\overline{v_1}$$

$$= \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$$

$$= \frac{1}{\sqrt{6}} \begin{pmatrix} \frac{3}{\sqrt{3}} \\ \frac{3}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

To get $\overline{u_2}$, let $X \subseteq \mathbb{R}^2$ be the subspace of vectors orthogonal to $\overline{u_1}$ and let $\overline{x} \in X$, then

$\langle \bar{v}_1 . \bar{x} \rangle = 0$ (Orthogonality)

$\Rightarrow \frac{1}{\sqrt{2}} x_1 + \frac{1}{\sqrt{2}} x_2 = 0$

$\Rightarrow x_1 + x_2 = 0$

$\Rightarrow \underline{x_2 = -x_1}$

$\Rightarrow X = x_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \forall x_1 \in \mathbb{R} \Rightarrow$ Basis of $X = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\Rightarrow \bar{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$ (Normalization)

$\therefore U \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$

$\therefore$ SVD of $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ is,

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\Sigma} \underbrace{\begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & \sqrt{2/3} \end{pmatrix}}_{V^T}$$

Outer Product form,

$$A = \sigma_1 \bar{u}_1 \bar{v}_1^T + \sigma_2 \bar{u}_2 \bar{v}_2^T$$

$$\Rightarrow \quad A = \sqrt{6}\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}\begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix} \qquad (\sigma_2 = 0)$$

3. <u>Given Quadratic Equation</u> :-

$$f(x_1, x_2) = x_1^2 + 8x_1x_2 + x_2^2$$

To find the quadratic form of a 2 variable quadratic equation,

$$f(x_1, x_2) = \begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} a & b \\ b & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= \begin{pmatrix} ax_1 + bx_2 & bx_1 + cx_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= x_1(ax_1 + bx_2) + x_2(bx_1 + cx_2)$$

$$= ax_1^2 + bx_1x_2 + bx_1x_2 + cx_2^2$$

$$= ax_1^2 + 2bx_1x_2 + cx_2^2$$

Comparing coefficients,

$$a = 1, \quad 2b = 8, \quad c = 1$$

$$\Rightarrow b = 4$$

$$\Rightarrow \text{Quadratic form} = \underbrace{\begin{pmatrix} x_1 & x_2 \end{pmatrix}}_{\bar{x}^T}\underbrace{\begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{\bar{x}}$$

To change the Quadratic form into one without any cross product terms, we need to diagonalize the coefficient matrix A.

$A = \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}$. Diagonalization of $A = Q^T D Q$

Eigenvalues of A,

$$\det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 4 \\ 4 & 1-\lambda \end{pmatrix} = 0$$

$$= (1-\lambda)^2 - 16 = 0$$

$$\Rightarrow (1-\lambda)^2 = 16$$

$$\Rightarrow 1-\lambda = 4 \quad \text{or} \quad 1-\lambda = -4$$

$$\Rightarrow \underline{\underline{\lambda = -3}} \qquad \underline{\lambda = 5}$$

$\underline{\lambda_1 = 5, \ \lambda_2 = -3}$ $\Rightarrow D = \begin{pmatrix} 5 & 0 \\ 0 & -3 \end{pmatrix}$

Eigenvectors of A

E_{λ_1}, $(A - \lambda_1 I) X = \overline{0}$

$$\Rightarrow (A - 5I) X = \overline{0}$$

$$\Rightarrow \begin{pmatrix} -4 & 4 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow -4x + 4y = 0 \quad \& \quad 4x - 4y = 0$$

$$\Rightarrow \underline{x = y} \qquad \Rightarrow \underline{x = y}$$

$$\Rightarrow E_{\lambda_1} = x \begin{pmatrix} 1 \\ 1 \end{pmatrix} \ \forall x \in \mathbb{R} \Rightarrow \overline{v_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

E_{λ_2}, $(A - \lambda_2 I)X = 0$

$\Rightarrow (A + 3I)X = 0$

$$\Rightarrow \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow 4x + 4y = 0 \Rightarrow \underline{\underline{y = -x}}$

$$\Rightarrow E_{\lambda_2} = x \begin{pmatrix} 1 \\ -1 \end{pmatrix} \; \forall x \in \mathbb{R} \Rightarrow \bar{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\bar{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \bar{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\bar{v}_1 \cdot \bar{v}_2 = 1(1) + 1(-1) = 0 \Rightarrow \bar{v}_1, \bar{v}_2$ are orthogonal,

Normalizing $\bar{v}_1, \bar{v}_2$, we get,

$$\bar{q}_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \qquad \bar{q}_2 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

$$\Rightarrow Q = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

$$\Rightarrow A = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

$$\Rightarrow f(x_1, x_2) = \underbrace{\begin{pmatrix} x_1 & x_2 \end{pmatrix}}_{x^T} \underbrace{\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}}_{Q^T} \underbrace{\begin{pmatrix} 5 & 0 \\ 0 & -3 \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}}_{Q} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{x}$$

$$\Rightarrow f(x_1, x_2) = \underbrace{\begin{pmatrix} \frac{x_1}{\sqrt{2}} + \frac{x_2}{\sqrt{2}} & \frac{x_1}{\sqrt{2}} - \frac{x_2}{\sqrt{2}} \end{pmatrix}}_{y^T} \underbrace{\begin{pmatrix} 5 & 0 \\ 0 & -3 \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} \frac{x_1}{\sqrt{2}} + \frac{x_2}{\sqrt{2}} \\ \frac{x_1}{\sqrt{2}} - \frac{x_2}{\sqrt{2}} \end{pmatrix}}_{y}$$

$\therefore$ The change of variable is $\bar{y} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \bar{x}$

$$\Rightarrow \underline{\bar{x} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \bar{y}}$$ ($Q^{-1} = Q^T$ due to Orthogonality)

4. <u>To find</u> : Orthogonal Diagonalization of $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$

The Orthogonal diagonalization of a Real Symmetric matrix A is given by,

$$A = Q^T D Q,$$ where Q is orthogonal and D is diagonal.

Eigenvalues of A,

$$\det(A - \lambda I) = 0$$

$$= \det \begin{pmatrix} 1-\lambda & 2 & 2 \\ 2 & 1-\lambda & 2 \\ 2 & 2 & 1-\lambda \end{pmatrix} = 0$$

$$= (1-\lambda)\left((1-\lambda)^2 - 4\right) - 2\left(2(1-\lambda) - 4\right) + 2\left(4 - 2(1-\lambda)\right) = 0$$

$$= (1-\lambda)^3 - 4(1-\lambda) - 4(1-\lambda) + 8 + 8 - 4(1-\lambda) = 0$$

$$= (1-\lambda)^3 - 12(1-\lambda) + 16 = 0$$

Change of variable $\lambda_0 = 1-\lambda$,

$$\Rightarrow \lambda_0^3 - 12\lambda_0 + 16 = 0$$

$$= (\lambda_0 - 2)(\lambda_0^2 + 2\lambda_0 - 8) = 0$$

$$\begin{array}{r|l} & \lambda_0^2 + 2\lambda_0 - 8 \\ \hline \lambda_0 - 2 & \lambda_0^3 + 0\lambda_0^2 - 12\lambda_0 + 16 \\ & (-)\ \lambda_0^3 \overset{(+)}{-} 2\lambda_0^2 \\ \hline & 2\lambda_0^2 - 12\lambda_0 \\ & (-)\ 2\lambda_0^2 \overset{(+)}{-} 4\lambda_0 \\ \hline & -8\lambda_0 + 16 \\ & -8\lambda_0 + 16 \\ \hline & 0 \end{array}$$

$$\Rightarrow (\lambda_0 - 2)(\lambda_0 + 4)(\lambda_0 - 2) = 0$$

$$\Rightarrow (\lambda_0 - 2)^2(\lambda_0 + 4) = 0$$

$$\Rightarrow \lambda_0 = 2, \quad \lambda_0 = -4$$

$$= 1-\lambda = 2, \quad 1-\lambda = -4$$

$$= \lambda = -1, \quad \lambda = 5$$

$$\text{AM} = 2 \qquad \text{AM} = 1$$

$\therefore$ The Eigenvalues of A are $\lambda_1 = -1$, $\lambda_2 = 5$

$$\Rightarrow D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

The Eigenvectors of A are,

$$E_{\lambda_1} = (A - (-1)I)X = \bar{0}$$

$$= \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad 2x + 2y + 2z = 0$$

$$= \quad x + y + z = 0$$

$$\Rightarrow \quad x = -y - z$$

$$\Rightarrow \quad E_{\lambda_1} = y \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad \forall\ y, z \in \mathbb{R}$$

$$\Rightarrow \quad \overline{v_1} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \qquad \overline{v_2} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$E_{\lambda_2}, \quad (A - 5\mathbb{I})X = \overline{0}$$

$$\Rightarrow \quad \begin{pmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-4x + 2y + 2z = 0$$

$$= \quad 2y + 2z = 4x$$

$$= \quad x = \frac{y+z}{2}$$

$$2x + 2y - 4z = 0$$

$$2z + 2z - 4z = 0$$

$$= \quad 0 = 0$$

$$2x - 4y + 2z = 0$$

$$y + z - 4y + 2z = 0$$

$$-3y + 3z = 0$$

$$\Rightarrow \quad y = z$$

$$\Rightarrow \quad x = y = z$$

$$\Rightarrow \quad E_{\lambda_2} = x\begin{pmatrix}1\\1\\1\end{pmatrix} \quad \forall\, x \in \mathbb{R}$$

$$\Rightarrow \quad \overline{v_3} = \begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\therefore \quad \overline{v_1} = \begin{pmatrix}-1\\1\\0\end{pmatrix} \quad \overline{v_2} = \begin{pmatrix}-1\\0\\1\end{pmatrix} \quad \overline{v_3} = \begin{pmatrix}1\\1\\1\end{pmatrix}$$

$\overline{v_1} \cdot \overline{v_3} = -1(1) + 1(1) + 0(1) = 0 \quad \therefore \overline{v_1}, \overline{v_3}$ are orthogonal

$\overline{v_2} \cdot \overline{v_3} = -1(1) + 0(1) + 1(1) = 0 \quad \therefore \overline{v_2}, \overline{v_3}$ are orthogonal

$\overline{v_1} \cdot \overline{v_2} = -1(-1) + 0(1) + 0(1) = 1 \neq 0$

Orthogonalizing $\overline{v_2}$ wrt $\overline{v_1}$,

$$\overline{v_2}' = \overline{v_2} - \frac{\overline{v_2} \cdot \overline{v_1}}{\overline{v_1} \cdot \overline{v_1}}\,\overline{v_1}$$

$$= \begin{pmatrix}-1\\0\\1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}-1\\1\\0\end{pmatrix}$$

$$= \begin{pmatrix}-1/2\\-1/2\\1\end{pmatrix}$$

Normalizing the vectors we get,

$$\overline{q_1} = \begin{pmatrix}-1/\sqrt{2}\\1/\sqrt{2}\\0\end{pmatrix} \quad \overline{q_2} = \begin{pmatrix}-1/\sqrt{6}\\-1/\sqrt{6}\\\sqrt{2/3}\end{pmatrix} \quad \overline{q_3} = \begin{pmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix}$$

$$\Rightarrow \quad Q = \begin{pmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & \sqrt{2/3} & 1/\sqrt{3} \end{pmatrix}$$

$$A = Q^T D Q$$

$$\Rightarrow \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & \sqrt{2/3} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & \sqrt{2/3} & 1/\sqrt{3} \end{pmatrix}$$