LA Assignment - 1 (H2)

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1. To find: Orthogonal basis for the subspace of $\mathbb{R}^3$ spanned by the vectors

$$\bar{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \bar{y} = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}$$

(The inner product is the standard inner product).

Answer:

Let the orthogonal basis be $\{\bar{v}_1, \bar{v}_2\}$ (Since 2 vectors span the subspace, the subspace is 2 dimensional).

By the Gram Schmidt Process,

$$\bar{v}_1 = \bar{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\bar{v}_2 = \bar{y} - \text{proj}_{\bar{v}_1} \bar{y} \Rightarrow \bar{y} - \text{proj}_{\bar{x}} \bar{y}$$

$$= \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} - \frac{\bar{x}.\bar{y}}{\bar{x}.\bar{x}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} - \frac{3+4}{1+1} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} - \frac{7}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1/2 \\ 1/2 \\ 2 \end{pmatrix}$$

$\therefore$ The vectors $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/2 \\ 1/2 \\ 2 \end{pmatrix}$ form the required basis of the subspace.

2. a) <u>To Prove</u>:

If A, B are square matrices and AB is invertible, then both A and B are invertible.

<u>Proof</u>:

If A, B are square matrices and AB is defined, then A and B are of the same order.

If AB is invertible, then $\exists$ square matrix C st,

$$(AB) \cdot C = I \quad , \text{ where } C = (AB)^{-1}$$

Since $(AB)C = I$, by Associativity, $A(BC) = I$

$A(BC) = I \Rightarrow A^{-1} = BC \Rightarrow A$ is invertible

Also,

$$ABC = I$$

$$\Rightarrow A^{-1}ABC = A^{-1}$$

$$\Rightarrow BC = A^{-1}$$

$$\Rightarrow BCA = A^{-1}A$$

$$\Rightarrow B(CA) = I$$

$\Rightarrow B^{-1} = CA \Rightarrow B$ is invertible

$\therefore$ If AB is invertible, then A and B are invertible.

b) <u>To Prove:</u>

If a symmetric matrix is invertible, then its inverse is also symmetric.

<u>Proof:</u>

Let $A_{n \times n}$ be any symmetric, invertible matrix.

$\Rightarrow A^T = A$.

Also, $\exists$ a matrix A^{-1} s.t. $AA^{-1} = I$.

$$AA^{-1} = I$$

$$\Rightarrow (AA^{-1})^T = I^T = I$$

$$\Rightarrow (A^{-1})^T A^T = I$$

$$\Rightarrow (A^{-1})^T A = I$$

$\Rightarrow (A^{-1})^T$ is the inverse of A.

Since the inverse of a matrix is unique, we get $(A^{-1})^T = A^{-1}$

$\Rightarrow A^{-1}$ is symmetric

$\therefore$ A is symmetric and invertible $\Rightarrow A^{-1}$ is symmetric.

3. a) $A = \begin{pmatrix} k & -k & 3 \\ 0 & k+1 & 1 \\ k & -8 & k-1 \end{pmatrix}$

$A^{-1} = \dfrac{1}{\det(A)} \operatorname{adj}(A)$

$\Rightarrow \det(A) \neq 0$

We know that A is invertible iff $\det(A) \neq 0$

$$\det(A) = k\big((k+1)(k-1) - (-8)\big) + k(0-k) + 3\big(0-(k+1)(k)\big)$$
$$= k(k^2 - 1 + 8) + k(-k) - 3(k^2 + k)$$
$$= k^3 + 7k - k^2 - 3k^2 - 3k$$
$$= k^3 - 4k^2 + 4k = k(k^2 - 4k + 4) = k(k-2)^2$$

$\det(A) = 0$

$\Rightarrow k(k-2)^2 = 0 \Rightarrow$ $k = 0$ or $k = 2$

$\Rightarrow \det(A) \neq 0 \ \forall\, k \in \mathbb{R} - \{0, 2\}$.

$\therefore$ A is invertible $\forall\, k \in \mathbb{R} - \{0, 2\}$.

b) $A = \begin{pmatrix} k & k & 0 \\ k^2 & 2 & k \\ 0 & k & k \end{pmatrix}$

$$\det(A) = k(2k - k^2) - k(k^3)$$
$$= 2k^2 - k^3 - k^4$$
$$= -k^2(k^2 + k - 2)$$
$$= -k^2(k^2 - k + 2k - 2)$$
$$= -k^2(k(k-1) + 2(k-1))$$
$$= -k^2(k+2)(k-1)$$

$\det(A) = 0$

$\Rightarrow -k^2(k+2)(k-1) = 0$

$\Rightarrow k = 0$ or $k = 1$ or $k = -2$

$\Rightarrow \det(A) \neq 0 \;\forall\; k \in \mathbb{R} - \{0, 1, -2\}$

$\therefore$ A is invertible $\forall\; k \in \mathbb{R} - \{0, 1, -2\}$

4. <u>Given:</u> $\bar{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\bar{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\bar{v}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

Let $U = [\bar{v}_1 \;\; \bar{v}_2 \;\; \bar{v}_3] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -2 \end{bmatrix}$ (The columns of U are $\bar{v}_1, \bar{v}_2, \bar{v}_3$)

If $\{v_1, v_2, v_3\}$ are to form a basis in $\mathbb{R}^3$, then they must be linearly independent.

If they are linearly independent then

$$UX = 0 \Rightarrow X = \bar{0}, \text{ where } X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, x, y, z \in \mathbb{R}.$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x + y + z \\ x - y + z \\ x - 2z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x - 2z = 0 \Rightarrow z = \frac{x}{2}$$

$$x - y + z = 0$$

$$\Rightarrow 3x/2 - y = 0$$

$$\Rightarrow y = 3x/2$$

$$x + y + z$$

$$= x + 3x/2 + x/2 = 0$$

$$= 3x = 0 \Rightarrow x = 0$$

$$\Rightarrow y = 0 \ \& \ z = 0$$

$\Rightarrow \quad UX = 0 \quad \Rightarrow \quad X = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow$ The cols. of U are linearly independent.

$\Rightarrow$ $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ are linearly independent. Therefore they form a basis in $\mathbb{R}^3$.

To see if they are orthogonal,

$$\bar{v}_1 \cdot \bar{v}_2 = 1(1) + (-1)(1) + (1)(0) = 0$$

$$\bar{v}_1 \cdot \bar{v}_3 = 1(1) + 1(1) + 1(-2) = 0$$

$$\bar{v}_2 \cdot \bar{v}_3 = 1(1) + (-1)(1) + (0)(-2) = 0$$

$\therefore$ $\bar{v}_1, \bar{v}_2, \bar{v}_3$ are orthogonal to each other.

$\therefore$ $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ is an orthogonal basis in $\mathbb{R}_3$

$\bar{w} = \begin{bmatrix} 7 \\ 9 \\ 10 \end{bmatrix}$. Let the coordinates of w be (c_1, c_2, c_3), w.r.t, $(\bar{v}_1, \bar{v}_2, \bar{v}_3)$

$$\Rightarrow \quad \bar{w} = c_1 \cdot \bar{v}_1 + c_2 \cdot \bar{v}_2 + c_3 \bar{v}_3$$

$$\Rightarrow \quad U \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \bar{w}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 9 \\ 10 \end{bmatrix}$$

To convert into RREF,

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 1 & -1 & 1 & 9 \\ 1 & 0 & -2 & 10 \end{array}\right] \xrightarrow{R_2 : R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & -2 & 0 & 2 \\ 1 & 0 & -2 & 10 \end{array}\right]$$

$$\xrightarrow{R_3 : R_3 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & -2 & 0 & 2 \\ 0 & -1 & -3 & 3 \end{array}\right] \xrightarrow{R_2 : \frac{-1}{2} R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & -3 & 3 \end{array}\right]$$

$$\xrightarrow{R_1 : R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 8 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & -3 & 3 \end{array}\right] \xrightarrow{R_3 : R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 8 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -3 & 2 \end{array}\right]$$

$$\xrightarrow{R_3 : \frac{-1}{3} R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 8 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2/3 \end{array}\right] \xrightarrow{R_1 : R_1 - R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 26/3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2/3 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 26/3 \\ -1 \\ -2/3 \end{bmatrix}$$

$$\Rightarrow \underline{c_1 = 26/3, \ c_2 = -1, \ c_3 = -2/3}$$

5. a) For a given matrix to be orthogonal, its columns must form an orthonormal set.

The given matrix is $M = \begin{pmatrix} 1/3 & 1/2 & 1/5 \\ 1/3 & -1/2 & 1/5 \\ -1/3 & 0 & 2/5 \end{pmatrix}$

Define the column vectors as,

$$\bar{c_1} = \begin{bmatrix} 1/3 \\ 1/3 \\ -1/3 \end{bmatrix}, \quad \bar{c_2} = \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \end{bmatrix}, \quad \bar{c_3} = \begin{bmatrix} 1/5 \\ 1/5 \\ 2/5 \end{bmatrix}$$

The magnitude of a vector $\bar{a}$ is defined as $\sqrt{|\langle \bar{a}, \bar{a} \rangle|}$

If a set of vectors is orthonormal, their magnitude must be 1

$$|\bar{c_1}| = \sqrt{\left(\frac{1}{9} + \frac{1}{9} + \frac{1}{9}\right)} = \sqrt{1/3} = 1/\sqrt{3} \neq 1$$

Since $\bar{c_1}$ does not have unit magnitude, the set $\{\bar{c_1}, \bar{c_2}, \bar{c_3}\}$ cannot be orthonormal.

$\therefore$ The given matrix cannot be Orthogonal.

b) <u>To Prove</u> :

If Q is an orthogonal 2×2 matrix, then Q must have the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ or $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$.

<u>Proof</u>:

Let $Q = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be any 2×2 orthogonal matrix.

Any orthogonal matrix must satisfy 2 properties,

i) Orthogonality of Column Vectors,

Let the column vectors of Q be $\bar{c_1}, \bar{c_2}$

$$\Rightarrow \bar{c_1} = \begin{bmatrix} a \\ b \end{bmatrix} \quad \bar{c_2} = \begin{bmatrix} c \\ d \end{bmatrix}$$

If Q is orthogonal, then $\langle \bar{c_1}, \bar{c_2} \rangle = 0$.

$$\langle \bar{c_1} \cdot \bar{c_2} \rangle = ac + bd = 0$$

$$\Rightarrow ac = -bd$$

$$\Rightarrow \frac{a}{b} = -\frac{d}{c} \quad \text{———} \quad (1)$$

ii) Normality of Column Vectors,

The column vectors $\bar{c_1}, \bar{c_2}$ must have unit magnitude.

$$|\bar{c_1}| = \sqrt{\langle \bar{c_1}, \bar{c_1} \rangle} = \sqrt{a^2+b^2} = 1$$
$$|\bar{c_2}| = \sqrt{\langle \bar{c_2}, \bar{c_2} \rangle} = \sqrt{c^2+d^2} = 1$$

$$\Rightarrow \quad a^2+b^2=1 \ , \quad c^2+d^2=1$$

Also we know that for any orthogonal matrix $M_{n\times n}$,

$$MM^T = \mathbb{I}_n$$

$\Rightarrow$ If Q is orthogonal, then,

$$QQ^T = \mathbb{I}_2$$
$$\Rightarrow \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} a^2+c^2 & ab+cd \\ ab+cd & b^2+d^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$\Rightarrow$
$$a^2+c^2 = 1 = b^2+d^2$$
$$ab+cd = 0$$
$$\Rightarrow \quad ab = -cd$$
$$\Rightarrow \quad \frac{a}{c} = \frac{-d}{b} \quad \text{——— (2)}$$

From (1) and (2),

$$\frac{a}{b} = -\frac{d}{c} \ , \quad \frac{a}{c} = -\frac{d}{b}$$
$$\Rightarrow \quad a = \frac{-db}{c} \quad \text{(3)} \qquad a = \frac{-dc}{b}$$

$$\Rightarrow \frac{\cancel{d}b}{c} = \frac{\cancel{d}c}{b}$$

$$= \frac{b}{c} = \frac{c}{b}$$

$$= b^2 = c^2 \Rightarrow \underline{\underline{b \pm c}} \text{ --- } (4)$$

Using (4) in (3)

$$a = \frac{-db}{c}$$

If $b = -c$,

$$a = \frac{-d(-\cancel{c})}{\cancel{c}}$$

$$a = d$$

$$\Rightarrow \underline{d = a}$$

If $b = c$

$$a = \frac{-d\cancel{c}}{\cancel{c}}$$

$$a = -d$$

$$\Rightarrow \underline{d = -a}$$

$$\Rightarrow \left.\begin{array}{l} \text{If } c = b, \ d = -a \\ \text{If } c = -b, \ d = a \end{array}\right\} \text{ --- } (6)$$

Using (6) we get Q as,

$$\begin{pmatrix} a & b \\ b & -a \end{pmatrix} \text{ or } \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

Assuming that $a^2 + b^2 = 1$