

Quiz 1 - 10
Quiz 2 - 10
Assn - 20
Mid - 25
Final - 35

Theory = 100

Submission - 15
Mid Exam - 10
Project/End Exam - 15

Lab = 40

Textbooks:

Signals & Systems by Oppenheim
DSP, JG Proakis

Addn: Principles of SP, BP Lathi

DSP : Sanjit K Mitra

Signal Processing

Prof. Santosh Nannuru

A > 130

F < 60

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Labs:

MATLAB Based

Github Submission

Tutorial: 10:30 - 11:30 Saturday

Signal Processing

→ Overview of NeSS :-

- Time variant - function of time
- Signals can be a function of any variable, need not be time always.
- Multi-Dimensional signals - images and videos.
- Anything that carries information is a signal.
- The course focuses more on discrete time signals, since most digital devices work in a discrete manner.
- System process signals, $x(t) \rightarrow \text{System}(H) \rightarrow y(t)$.
- LTI system - Linear and Time-Invariant Systems.
- Any circuit with only R,L,C will always form an LTI system, since R,L,C are all linear components.

Diodes and Transistors form non-linear systems.

- In any LTI system, the output $y(t)$ of a signal $x(t)$ is,

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau \rightarrow \underline{\text{Convolution}}$$

$h(t)$ = impulse response of the system.

- Laplace Transform :-

Representation of the signal in the s / complex frequency domain.

- Poles of the system function determine the behaviour / properties of the system.
- ROC - Region in the complex s - plane where the Laplace transform converges. Is always a vertical strip, defined by the location of poles.

- Fourier Series :-

- Any periodic signal can be represented as a sum of sines and cosines.

→ Course Topics:-

- Fourier Analysis, Z transform
- Sampling and Quantization - DSP
- Computational Aspect - fast Fourier Transform
- Manipulating Signals - filter design.

→ Fourier Analysis:-

- Periodic and Continuous-time - Fourier Series Rep.
- Aperiodic and Continuous-time - Fourier Transform
- Aperiodic and Discrete-time - Discrete Time Fourier Transform
- Finite Length and Discrete-time - DFT
 - FFT
Faster
- Fourier Series Representation :
 - Representation of any periodic signal as a sum of sinusoidal harmonics.

$$x(t) = b_0 + \sum_{k \in \mathbb{N}} (a_k \sin(k\omega_0 t) + b_k \cos(k\omega_0 t))$$

ω_0 - Angular frequency of the signal. ($2\pi/\tau$)

$$x(t) = \sum_{k \in \mathbb{Z}} c_k e^{jk\omega_0 t}$$

- Analysis: Computing the coefficients of the given signal

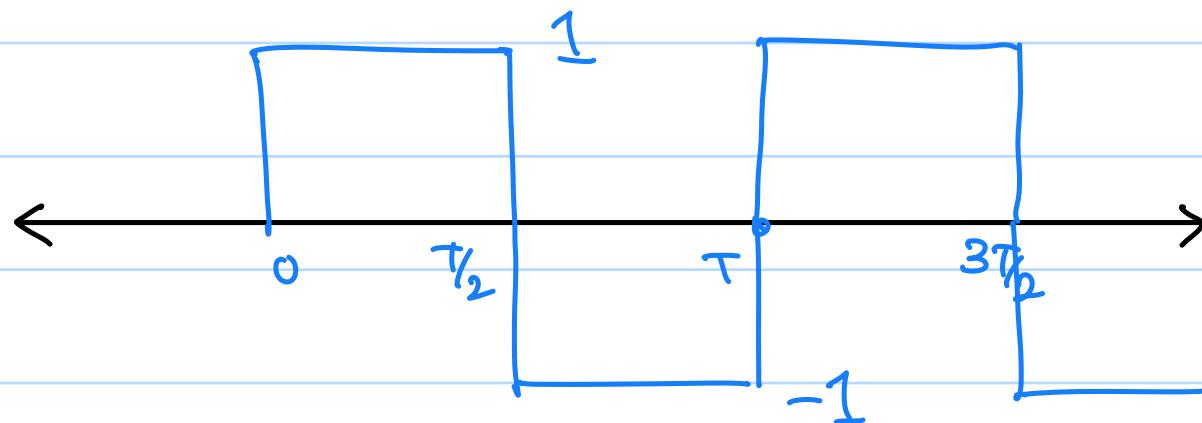
Synthesis: Given the signal, computing the coefficients.

$$a_k = \frac{2}{T} \int_T x(t) \sin(k\omega_0 t) dt, \quad k \neq 0$$

$$b_k = \frac{2}{T} \int_T x(t) \cos(k\omega_0 t) dt$$

$$c_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$$

Compute the FS Coeffs for a Sq. wave.



$$x(t) = \begin{cases} 1, & t \in (0, T/2) \\ -1, & t \in (T/2, T) \end{cases}$$

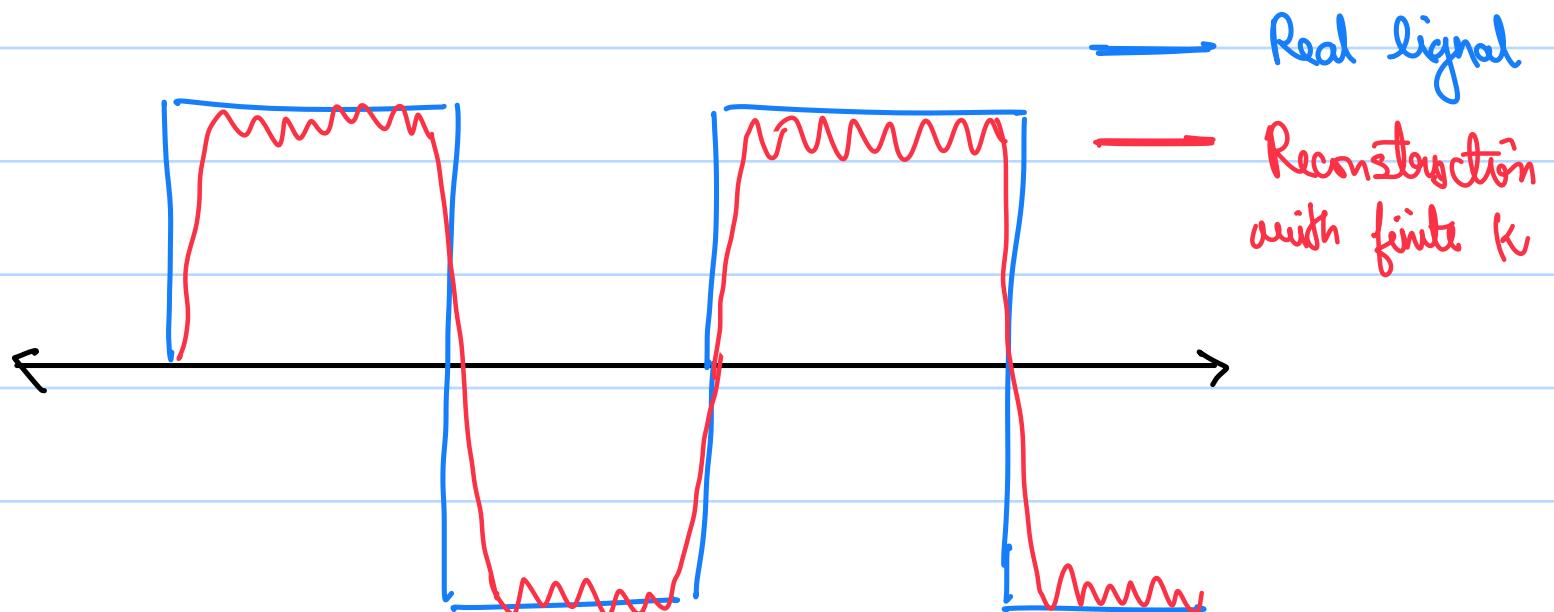
$$\begin{aligned}
c_k &= \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt \\
&= \frac{1}{T} \left[\int_0^{T/2} e^{-jk\omega_0 t} dt - \int_{T/2}^T e^{-jk\omega_0 t} dt \right] \\
&= \frac{1}{T} \left(\frac{-1}{j\omega_0} \right) \left[[e^{-jk\omega_0 t}]_{0}^{T/2} - [e^{-jk\omega_0 t}]_{T/2}^T \right] \\
&= -\frac{1}{k\pi \frac{2\pi}{T}} \left(\left[e^{-jk(\frac{2\pi}{T})(\frac{T}{2})} - e^{-jk\omega_0(0)} \right] - \left[e^{-jk(\frac{2\pi}{T})T} - e^{-jk(\frac{2\pi}{T})(\frac{T}{2})} \right] \right) \\
&= -\frac{1}{k2\pi} \left[\left(e^{-jk\pi} - 1 \right) - \left(e^{-jk2\pi} - e^{-jk\pi} \right) \right] \\
&= -\frac{1}{k2\pi} (2e^{-jk\pi} - 1 - e^{-jk2\pi}) \\
&= -\frac{1}{k2\pi} (2e^{-jk\pi} - 2) \\
&= -\frac{1}{k\pi} (e^{-jk\pi} - 1) \xrightarrow{\text{for even } k} 0 \\
&= \frac{1 - \cos k\pi}{k\pi} \quad \text{if odd } k
\end{aligned}$$

• Partial Reconstruction :-

- Not all the coefficients are known to us, so only an approximation of the signal is constructed.

- Reconstruction Error $e(t) = x(t) - \hat{x}(t)$

- Discontinuous signals will have an infinite number of non-zero coefficients.



FS Coeffs of Sq. Wave Continuation ,

$$a_k = \frac{2}{2T} \int n(t) \sin(k\omega_0 t) dt$$

$$= \frac{1}{T} \left[\int_0^T -\sin(k\omega_0 t) dt + \int_T^{2T} \sin(k\omega_0 t) dt \right]$$

$$= \frac{1}{T} \left(\frac{1}{k\omega_0} \right) \left[[\cos(k\omega_0 t)]_0^T + [-\cos(k\omega_0 t)]_T^{2T} \right]$$

$$= \frac{1}{k\pi} \left[\left(\cos \left(k \frac{2\pi}{2T} (\pi) \right) - 1 \right) + \left(\cos \left(k \left(\frac{2\pi}{2T} \right) T \right) - \cos \left(k \left(\frac{2\pi}{2T} \right) 2T \right) \right) \right]$$

$$= \frac{1}{k\pi} \left[(\cos k\pi - 1) + (\cos k\pi - 1) \right]$$

$$a_k = \frac{2(\cos k\pi - 1)}{k\pi}$$

$$b_k = \frac{2}{2T} \int_{-T}^T x(t) \cos(k\omega_0 t) dt$$

$$= \frac{1}{T} \left[\int_0^T -\cos(k\omega_0 t) dt + \int_T^{2T} \cos(k\omega_0 t) dt \right] \left(\frac{1}{k\omega_0} \right)$$

$$= \frac{1}{T} \left[(-\sin(k\omega_0 t)) \Big|_0^T + (\sin(k\omega_0 t)) \Big|_T^{2T} \right] \left(\frac{1}{k\omega_0} \right)$$

$$= \frac{1}{T} \left[(0 - \sin(k\pi)) + (\sin k \frac{2\pi}{2T} - \sin k \frac{\pi}{2T}) \right]$$

$$= 0 \neq k$$

- Note: If $a_k \in \mathbb{R} \neq k$, then the signal is Even.

→ Discrete-Time signals :-

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- Represented by $x[n]$, where $n \in \mathbb{Z}$, n is an index where time duration is the sampling interval.

- Unit Impulse:

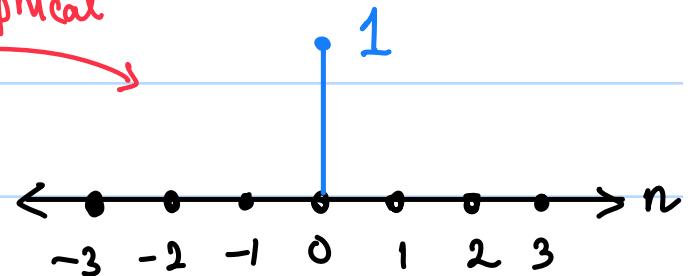
- In continuous time,

$$s(t) = 0 \neq t \neq 0, \text{ s.t. } \int_{-\infty}^{\infty} s(t) dt = 1$$

- In discrete time,

$$s[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

Graphical

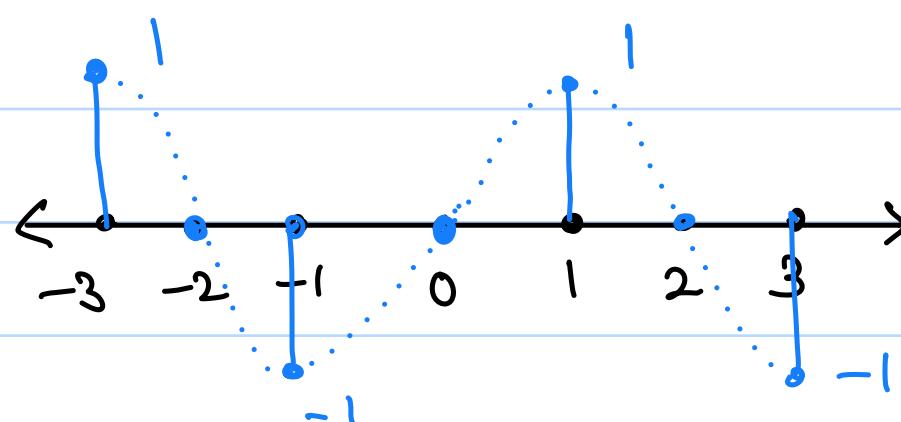


- Unit Step: $u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$

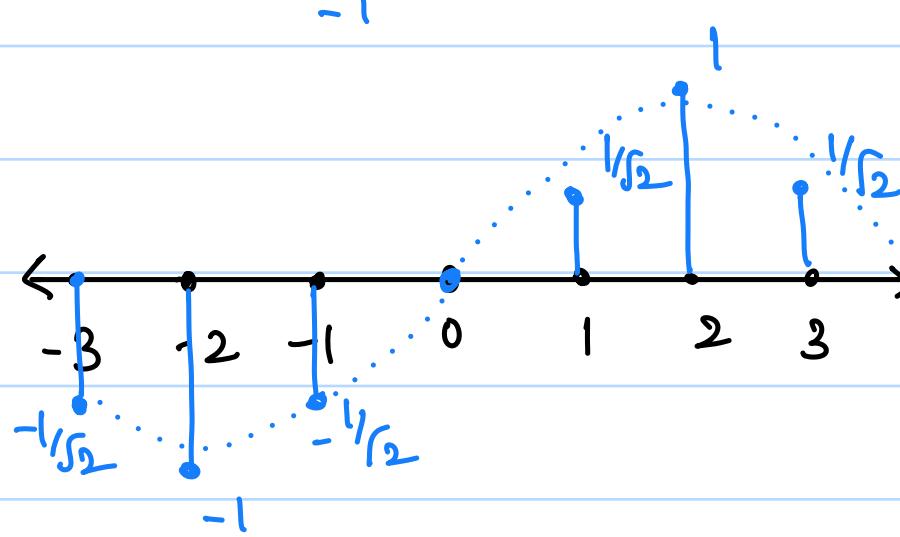
- Exponential signal: $x[n] = a^n, a \in \mathbb{R}$

Example: Plot $x[n] = \sin(\omega_0 n)$ for $\omega_0 = \frac{\pi}{2}, \frac{\pi}{4}$

1)



2)



- A signal that is periodic in continuous time may not be always be periodic in discrete time, due to the sampling frequency. ($x[n] = \sin(n)$)

- Discrete Time Periodic Signals:-

$x[n]$ is periodic if,

$$x[n+P] = x[n] \text{ for some } P \in \mathbb{Z}$$

- Complex Exponentials :-

$$x[n] = e^{sn}, s \in \mathbb{C}, \text{ ie. } s = \sigma + j\omega, \sigma, \omega \in \mathbb{R}$$

$$\Rightarrow x[n] = e^{\sigma n} (\cos(\omega n) + j \sin(\omega n))$$

Note: Sinusoids are periodic if $\omega_0 = \frac{2\pi}{P}$ for some $P \in \mathbb{Z}$

- Even and Odd:

$$\text{Even : } x[n] = x[-n] \quad \forall n \in \mathbb{Z}$$

$$\text{Odd : } x[n] = -x[-n] \quad \forall n \in \mathbb{Z}$$

- Energy of a Discrete Time signal :-

$$E = \sum_{n \in \mathbb{Z}} |x[n]|^2$$

It may not relate to any physical quantity (property of the signal).

$$\text{Power} = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N |x[n]|^2$$

Energy signals : signals that carry finite energy, and hence, zero power

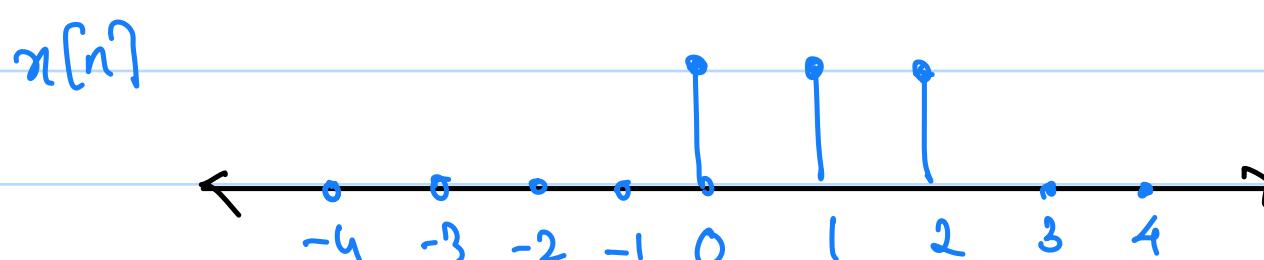
Power signals : signals that carry finite power.

→ Time axis Manipulation for Discrete time signals :-

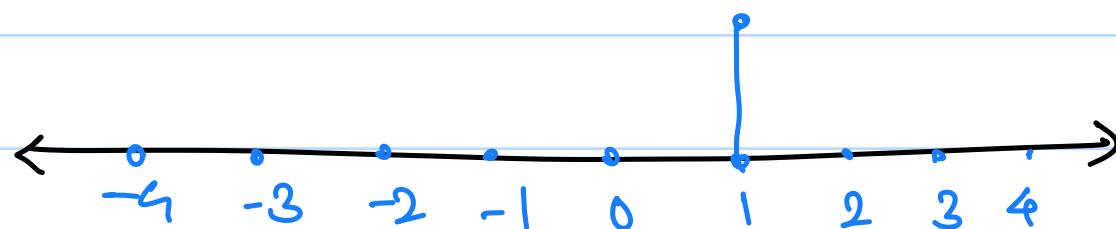
- $x[n - n_0]$ - Shift right by n_0
- $x[n + n_0]$ - Shift left by n_0
- $x[kn]$ - Only every k^{th} sample is taken

Example: $x[n] = u[n] - u[n - 3]$

$$y[n] = x[2n - 1] = ?$$



$$x[2n - 1]$$



- Also, due to the nature of discrete signals, $\omega_0 = \pi$ is the highest possible angular frequency for a sinusoid.

Beyond $\omega_0 = \pi$, the signal will be the same as that of a lower frequency, due to the properties of sinusoids.

$$\cos(\pi n) = (-1)^n \longrightarrow \begin{array}{l} \text{Oscillation b/w 1 and -1} \\ \text{at the freq of sampling} \end{array}$$

$$\begin{aligned}
 \sin((\pi + \phi)n) &= \sin(\pi n + \phi n) \\
 &= \sin \pi n \cos \phi n + \cos \pi n \sin \phi n \\
 &= (-1)^n \sin \phi n \quad \rightarrow \text{freq} < \pi
 \end{aligned}$$

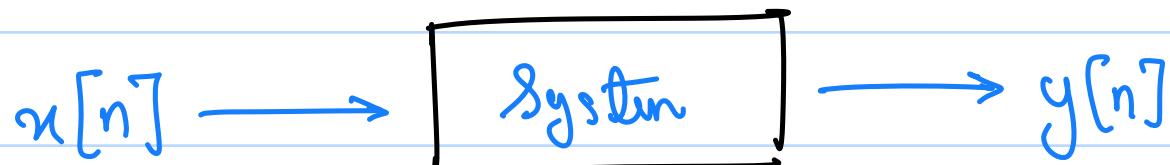
$$\begin{aligned}
 \cos((\pi + \phi)n) &= \cos(\pi n) \cos(\phi n) - \sin(\pi n) \sin(\phi n) \\
 &= (-1)^n \cos \phi n \quad \rightarrow \text{freq} < \pi
 \end{aligned}$$

- Signal Representation By Unit Impulses :-

$$x[n] = \sum_{m \in \mathbb{Z}} x[m] \delta[n-m] \quad \forall n \in \mathbb{Z}$$

Since $\delta[n-m] = 0 \quad \forall n \neq m$, all the other values of the signal are nullified at each n .

- System :-



- Delay :

$$y[n] = x[n-\Delta], \quad \Delta \in \mathbb{Z}$$

- Amplitude Scaling :

$$y[n] = \alpha x[n], \quad \alpha \in \mathbb{C}$$

- Integrator (Summation) :

$$y[n] = \sum_{m=-\infty}^n x[m]$$

o Difference:

$$y[n] = x[n] - x[n-1]$$

Q. $x(t) = \cos(4\pi t)$ is sampled with sampling frequency below.

Identify the frequency of the obtained DT signal.

$$f_s: 1\text{Hz}, 2\text{Hz}, 4\text{Hz}, 8\text{Hz}$$

$$x(t) = \cos(4\pi t)$$

In Sampling, $t \rightarrow nT$, $T = \frac{1}{f_s}$

$$\Rightarrow x[n] = \cos\left(\frac{4\pi n}{f_s}\right), \quad \text{freq} f = \frac{4\pi}{f_s}$$

$$\Rightarrow f = \frac{2}{f_s}$$

→ Properties of System :-

o Linearity: If x_1 gives o/p y_1 , x_2 gives o/p y_2 then,

$$\alpha x_1 + \beta x_2 \longrightarrow \alpha y_1 + \beta y_2. \quad \alpha, \beta \in \mathbb{C} \text{ constants}$$

o Time Invariant: If $x[n]$ gives o/p $y[n]$, then,

$$x[n-n_0] \longrightarrow y[n-n_0] \quad \forall n_0 \in \mathbb{Z}$$

Also termed as shift invariant.

Example: Analyze time invariance of

1) $y[n] = x[n - \Delta]$, $\Delta \in \mathbb{Z}$

2) $y[n] = x^2[n]$

3) $y[n] = x[n] + x[n - 1]$

4) $y[n] = n x[n]$

1) $y[n] = x[n - \Delta] \rightarrow y[n - n_0] = x[n - n_0 - \Delta]$

$$x[n - n_0] \xrightarrow{\text{Sys}} x[n - n_0 - \Delta] \quad \therefore \text{Time Invariant}$$

2) $y[n] = x^2[n] \rightarrow y[n - n_0] = (x[n - n_0])^2$

$$y_2[n] = (x[n - n_0])^2 \quad \therefore \text{Time Invariant}$$

3) $y[n] = x[n] + x[n - 1] \rightarrow y[n - n_0] = x[n - n_0] + x[n - n_0 - 1]$

$$y_2[n] = x[n - n_0] + x[n - n_0 - 1] \quad \therefore \text{Time Invariant}$$

4) $y[n] = n x[n] \rightarrow y[n - n_0] = (n - n_0) x[n - n_0]$

$$y_2[n] = n x[n - n_0] \quad \therefore \text{Time Variant}$$

• Causality: A system is said to be causal if the output $y[n] \forall n \in \mathbb{Z}$ should only depend on $x[m]$ for $m \leq n$.

- Stability: If $x[n] \leq B + n \in \mathbb{Z}$ for some $B \in \mathbb{C}$, then $y[n] \leq B_0 + n \in \mathbb{Z}$, for some $B_0 \in \mathbb{C}$. Bounded Input, Bounded Output.

→ Linear - Time Systems:-

- LTI Systems are linear and time invariant.
- Impulse Response: Response of the system to the impulse signal $\delta[n]$. Denoted by $h[n]$.
- Step Response: Response of the system to the step signal $v[n]$. Denoted by $s[n]$

$$\alpha \delta[n - n_0] \xrightarrow{H} \alpha h[n - n_0]$$

- Since we can denote any signal in the form,

$$x[n] = \sum_m x[m] \delta[n - m] \quad \begin{matrix} \nearrow \\ \text{scaled and delayed} \end{matrix} \quad \begin{matrix} \searrow \\ \text{impulse.} \end{matrix}$$

$$\Rightarrow y[n] = \sum_m x[m] h[n - m]$$

Applying the above 2 results.

$$\Rightarrow y[n] = x[n] * h[n]$$

$x[n]$ is a constant w.r.t n .

Properties of Convolution :

III to multiplication

1) Commutativity : $x[n] * y[n] = y[n] * x[n]$

Proved using
change of variable

2) Distributive of Addition : $x[n] * (y[n] + z[n]) = x[n] * y[n] + x[n] * z[n]$

3) Associative : $(x[n] * y[n]) * z[n] = x[n] * (y[n] * z[n])$

- Properties of $\delta[n]$ in Convolution :

1) $x[n] * \delta[n] = x[n]$

2) $x[n] * \delta[n-\Delta] = x[n-\Delta] \quad \forall \Delta \in \mathbb{Z}$

3) $x[n] * S[n-\Delta] = \underbrace{x[\Delta] \delta[n-\Delta]}_{\text{Scaled impulse}} + \Delta \in \mathbb{Z}$

Description of a System :-

1) Working description : Describe the working of the system.

2) Mathematical description : Describe the mathematical function relating input and output.

3) Impulse Response (LTI) : Describe the impulse response of the LTI system. Output can be calculated through convolution.

o Properties of a system through Impulse Response :-

- Causal Systems: $h[n] = 0 \quad \forall n \in \mathbb{Z}^-$ for a Causal system.

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \sum_m x[m] h[n-m] \\ &= \sum_{m=-\infty}^n x[m] h[n-m] + \sum_{m=n+1}^{\infty} x[m] h[n-m] \end{aligned}$$

should be zero

$$\Rightarrow h[n-m] = 0 \quad \forall m \geq n+1$$

$$m-n \geq 1$$

$$n-m \leq -1 \Rightarrow n-m < 0$$

$$\Rightarrow h[r] = 0 \quad \forall r < 0$$

- Stable Systems: $\sum_n |h[n]|$ should be bounded for stability

$$|y[n]| = \left| \sum_m x[m] h[n-m] \right|$$

$$\Rightarrow |y[n]| \leq \sum_m |x[m]| |h[n-m]| \quad \Delta \text{le inequality}$$

$$\Rightarrow |y[n]| \leq \sum_m B |h[n-m]| = B \sum_k |h[k]|$$

\therefore If $\sum |h[k]|$ is bounded, $|y[n]|$ becomes bounded. (Sufficient Condition)

Note:

If $A \Rightarrow B$, then A is sufficient for B

If $A \Leftarrow B$, then A is Necessary for B

$A \Leftrightarrow B$, A is Necessary and sufficient (iff)

Example: Derive the impulse response of $y[n] = x[n] + \alpha y[n-1]$

$$y[n] = x[n] + \alpha y[n-1]$$

$$= x[n] + \alpha(x[n-1] + \alpha y[n-2])$$

$$= x[n] + \alpha(x[n-1] + \alpha(x[n-2] + \dots))$$

$$= x[n] + \alpha x[n-1] + \alpha^2 x[n-2] + \alpha^3 x[n-3] + \dots$$

$$= \sum_{i=0}^n \alpha^i x[n-i]$$

[Assuming initial rest, i.e
 $y[n] = 0 \forall n < 0$]

$$h[n] = \sum_{i=0}^n \alpha^i \delta[n-i]$$

$$h[n] = \alpha^n$$

→ Linear Constant Coefficient Difference Equations :- (LCCDE)

• First order difference equation : $y[n] = x[n] - x[n-1]$

• Generic Difference Eqn : $y[n] = \sum_{p=0}^n a_p x[n-p] + \sum_{q=1}^{n_2} b_q y[n-q]$

If $x[n] = e^{j\omega n}$ is the input signal for an LTI system H ,

$$y[n] = h[n] * e^{j\omega n}$$

$$= \sum_{m=-\infty}^{\infty} h[m] e^{j\omega(n-m)}$$

$$= e^{j\omega n} \sum_m h[m] e^{-j\omega m}$$

$$y[n] = C e^{j\omega n} = \underline{C x[n]}$$

For any LTI system H , $e^{j\omega n}$ is an Eigensignal of H . The output waveform will be a scaled and phase-shifted input waveform.

→ Discrete Time Fourier Transform :-

The Discrete Time Fourier Transform of a discrete signal $x[n]$ is defined as,

$$X(\omega) = \sum_{-\infty}^{\infty} x[n] e^{-j\omega n}$$

→ Analysis Equation

Applying this in the Eigensignal concept,

$$y[n] = e^{j\omega_0 n} \sum_{m=-\infty}^{\infty} h[m] e^{-j\omega_0 m}$$

$$\Rightarrow y[n] = e^{j\omega_0 n} H(\omega_0)$$

$H(\omega) \rightarrow$ FT of $h[n]$,
ie, system function

- For a discrete signal,

$$X(\omega + 2\pi) = X(\omega)$$

$X(\omega)$ is continuous in ω .

- Inverse DTFT :-

$$x[n] \xleftarrow{\text{DTFT}} X(\omega)$$

- We only need the values of $X(\omega)$ for $\omega \in [-\pi, \pi]$ or $[0, 2\pi]$, due to its periodicity.
- Inverse DTFT is done as follows,

$$\begin{aligned} X(\omega) e^{j\omega m} &= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} e^{j\omega m} \\ &= \int_0^{2\pi} X(\omega) e^{j\omega m} d\omega = \int_0^{2\pi} \sum_{n=-\infty}^{\infty} x[n] e^{j\omega(m-n)} d\omega \\ &= \sum_{n=-\infty}^{\infty} x[n] \int_0^{2\pi} e^{j\omega(m-n)} d\omega \end{aligned}$$

$$\Rightarrow x[m] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega m} d\omega \rightarrow \text{Synthesis Equation}$$

Example: Compute DTFT of

$$i) x[n] = a^n u[n] \quad (|a| < 1)$$

$$X(\omega) = \sum x[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \frac{1}{1 - ae^{-j\omega}} \quad \begin{bmatrix} \text{Sum of } \infty \\ \text{GP} \end{bmatrix}$$

$$2) x[n] = \delta[n]$$

$$\begin{aligned} X(\omega) &= \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\omega n} \\ &= e^{-j\omega(0)} = 1 + \omega \end{aligned}$$

$$3) x[n] = \begin{cases} 1, & -M \leq n \leq M \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned} X(\omega) &= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = \sum_{n=-M}^{M} e^{-j\omega n} \\ &= \frac{e^{j\omega M} (1 - e^{-j\omega 2M})}{1 - e^{-j\omega}} \\ X(\omega) &= \frac{e^{j\omega M} - e^{-j\omega M}}{1 - e^{-j\omega}} \end{aligned}$$

