Prof: Prabhakar Bhimalapuram

Science - 1

Group - B

Marks :

Endsem : 35 %

Midsem : 20 %

$\pm 5\%$

Quiz : 10 + 10 %

In Class / Tut Quiz : 25 %

Tutorial Quizzes

Science - 1

- Science explains everything on the time-scales, the length scales and the energy scales.

Time Scales : Climate Change (Decade - scale)
Bullet travel (Second - scale)
Molecular movement (Picosecond - scale)

Length Scales: High Energy Physics (Quark / Lepton Scales)
Expansion of Universe (10^{35} m scale)

Energy Scale: Energy to lift a laptop
" " build a building

- There are a variety of things on these scales that we use in our daily life.

→ **What is Science?** :

- Systematic investigation of natural phenomenon.

- Reproducibility is a key feature of Science. Any experiment must be reproducable. Even a single datapoint is sufficient to disprove a theory.

- We can use the data gathered from the experiment, to create a mathematical model, than can be used to predict things. Mathematical modelling has enabled the development of sciences.

- Science will try to explain the behavior / reason behind the parameters present in the model, and also the limitations / inaccur-acies in the model.

  → Newton's first law is not directly provable, since it requires highly ideal conditions. It is only true at asymptotical conditions.

  ⟶ Limitation of Newton's laws.

- Reductionism: Separation of the most important things and neglecting every other factors. ex: The radius of planets while observing their orbital behavior around the sun, through Kepler's planetary laws.

"A System is a sum of its parts" (Superposition-like)

By decomposing a system into its pieces, we can figure out the interactions between the parts and also the most important parts of the system.

- **Microscopics**: The study behind the various interactions between the parts of a system.

- A reductionism model can be used to accurately predict the behavior of the model.

- Reductionism goes from microscopic behavior to macroscopic features.

- Even microscopic interactions follow laws, like conservation of energy.

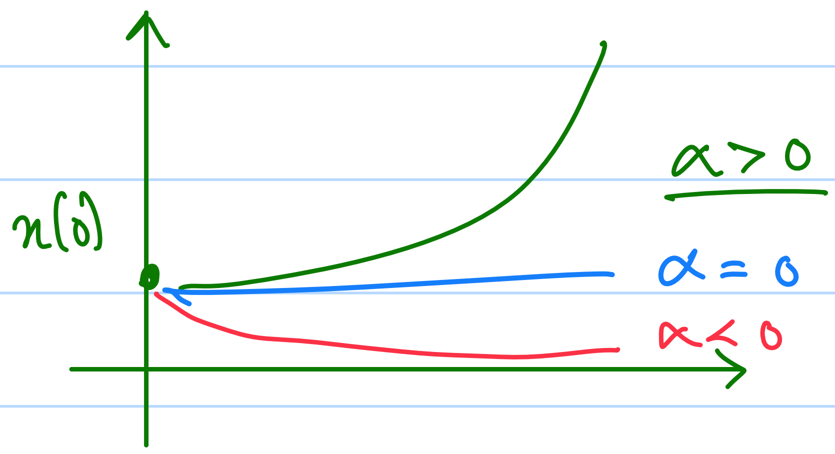
→ **Mathematics in Science:**

- Geometry and Linear Algebra

- Change and Calculus

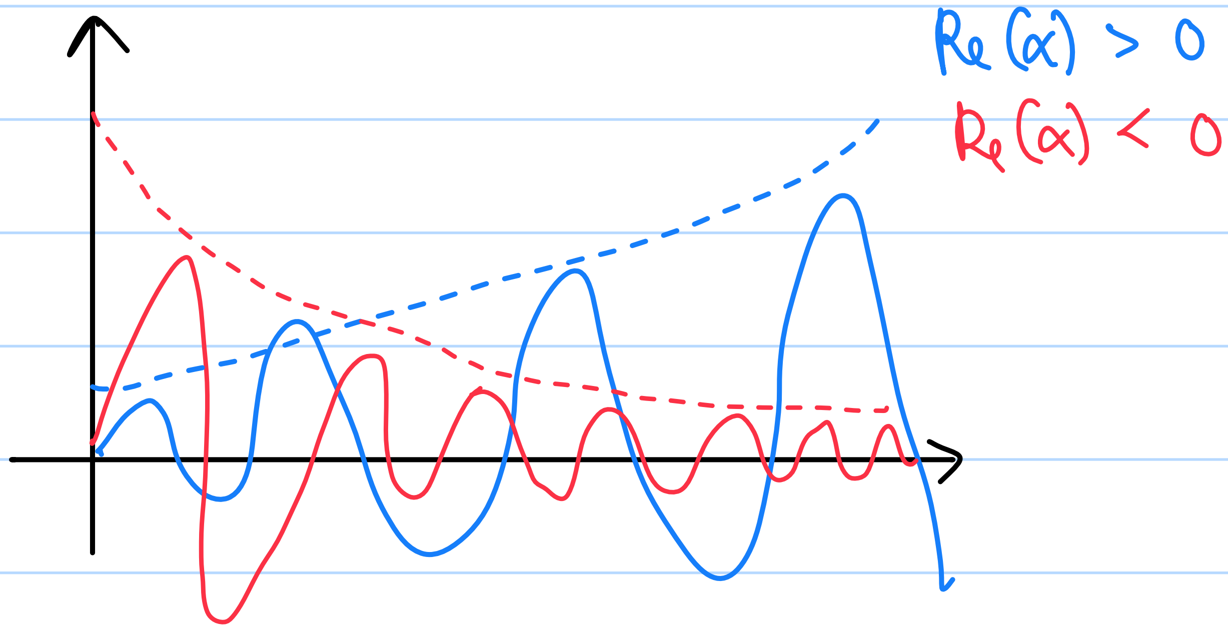
- Chance, probability and statistics. → Uncertainity Principle

ex: The Behavior of a system with $x(t) = x(0)\,e^{\alpha t}$



In the above system, the value of the parameter can fundamentally change the behavior of the system.

If $\alpha \in \mathbb{C}$



→ <u>The Prey - Predator Model</u> :-

- x - No. of Rabbits
  y - No. of Tigers

- <u>Parameters</u> :
  α - Growth rate of Rabbits
  β - Effect of foxes on the rabbits (Rabbits are killed)
  γ - Death rate of Foxes
  δ - Effect of rabbits on foxes (Foxes can reproduce with nutrition)

$$\frac{dx}{dt} = \alpha x - \beta xy \qquad \frac{dy}{dt} = -\gamma y + \delta xy$$

$$\Rightarrow \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha x - \beta xy \\ -\gamma y + \delta xy \end{pmatrix} \begin{matrix} \longrightarrow f \\ \longrightarrow g \end{matrix}$$

$$\hookrightarrow \gamma \begin{pmatrix} x \\ y \end{pmatrix} = \nearrow$$

- The equilibrium points of the system,

$$\frac{dx}{dt} = 0 \qquad\qquad \frac{dy}{dt} = 0$$

$$\Rightarrow \alpha x - \beta xy = 0 \qquad \Rightarrow -\gamma y + \delta xy = 0$$

$$\Rightarrow x(\alpha - \beta y) = 0 \qquad \Rightarrow y(-\gamma + \delta x) = 0$$

$$\Rightarrow x = 0, \; y = \frac{\alpha}{\beta} \qquad \Rightarrow y = 0, \; x = \frac{\gamma}{\delta}$$

$\Rightarrow (0,0), \left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$ are the equilibrium points.

$f(x,y) \quad \left[\; \delta f_x \;\; \delta f_y \right.$

Jacobian of the system,

$$\begin{pmatrix} \alpha x - \beta xy \\ -\gamma y + \delta xy \end{pmatrix} \xrightarrow{J} \begin{pmatrix} \alpha - \beta y & -\beta x \\ \delta y & -\gamma + \delta x \end{pmatrix} = J$$

$$J\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right) = \begin{pmatrix} 0 & -\beta\gamma/\delta \\ \delta\alpha/\beta & 0 \end{pmatrix} \longrightarrow \gamma'$$

Let $(x_0, y_0) = \left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$ and define $\varepsilon = x - x_0$, $\eta = y - y_0$

Around the equilibrium point, the function can be approximated as,

$$\frac{d}{dt}\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} = J\Big|_{x_0,y_0} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + \cancel{\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}^2}^{\,0}\Bigg|_{x_0,y_0}$$

$$\Rightarrow \boxed{\frac{d}{dt}\begin{pmatrix} \varepsilon \\ \eta \end{pmatrix} = \begin{pmatrix} 0 & -\beta\gamma/\delta \\ \delta\alpha/\beta & 0 \end{pmatrix}\begin{pmatrix} \varepsilon \\ \eta \end{pmatrix}}$$

$$\approx \quad \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -\beta\gamma/\delta \\ \delta\alpha/\beta & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$\rightarrow$ near the equilibrium of $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

Note: The approximation of the function around (x_0, y_0) is done using the result,

$$h(x,y) = h(x_0, y_0) + \left.\frac{\delta h}{\delta x}\right|_{x_0}(x - x_0) + \left.\frac{\delta h}{\delta y}\right|_{y_0}(y - y_0)$$

$$= h(x_0, y_0) + \left.\begin{bmatrix} \frac{\delta h}{\delta x} & \frac{\delta h}{\delta y} \end{bmatrix}\right|_{x_0, y_0} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$$

is from the Jacobian

Linearization

Ill analysis can be done near $(0,0)$.

Jacobian near $(0,0)$, $J|_{0,0} = \begin{pmatrix} \alpha & 0 \\ 0 & -\delta \end{pmatrix}$

$$\Rightarrow \quad \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & -\delta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$dt = \frac{dx}{\alpha x}$

$$\Rightarrow \quad \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha x \\ -\delta y \end{pmatrix} \rightarrow \quad \frac{dx}{dt} = \alpha x, \quad \frac{dy}{dt} = -\delta y$$

$$\Rightarrow \quad x = c_1 e^{\alpha t}, \quad y = c_2 e^{-\delta t}$$

$$\Rightarrow \quad x = x(0)\, e^{\alpha t} \qquad y = y(0)\, e^{\alpha t}$$

For the other Eq. point, $\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$,

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -\beta\gamma/\delta \\ \delta\alpha/\beta & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \quad \frac{d\bar{r}}{dt} = \bar{J}\,\bar{r}$$

The general solutions of such a system is of the form,

$$\bar{r}(t) = c_1 e^{\lambda_1 t}\,\bar{v}_1 + c_2 e^{\lambda_2 t}\,\bar{v}_2$$

Where, $\bar{v}_1, \bar{v}_2$ are the Eigenvectors of J, λ_1, λ_2 are their respective Eigenvalues.

The Eigenvalues of J are,

$$\det\begin{pmatrix} -\lambda & -\beta\gamma/\delta \\ \delta\alpha/\beta & -\lambda \end{pmatrix} = 0$$

$$= \lambda^2 - \left(\frac{-\cancel{\beta}\gamma}{\cancel{\delta}}\right)\left(\frac{\cancel{\delta}\alpha}{\cancel{\beta}}\right) = 0$$

$$= \lambda^2 - (-\alpha\gamma) = 0$$

$$= \lambda^2 = -\alpha\gamma$$

$$= \lambda = \pm i\sqrt{\alpha\gamma}$$

For $\lambda_1 = i\sqrt{\alpha\gamma}$,

$$\begin{pmatrix} -i\sqrt{\alpha\gamma} & \frac{-\beta\gamma}{\delta} \\ \frac{\delta\alpha}{\beta} & -i\sqrt{\alpha\gamma} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad -x_1 i\sqrt{\alpha\gamma} - \frac{\beta\gamma}{\delta} x_2 = 0$$

$$\Rightarrow \quad x_1 i\sqrt{\alpha\gamma} = \frac{-\beta\gamma}{\delta} x_2$$

$$x_1 = \frac{-\beta\gamma}{i\delta\sqrt{\alpha\gamma}} x_2 = -\frac{\beta}{i\delta}\sqrt{\frac{\gamma}{\alpha}}\, x_2$$

$$\frac{\delta\alpha}{\beta} x_1 - i\sqrt{\alpha\gamma}\, x_2 = 0$$

$$\Rightarrow \quad x_1 = \frac{i\sqrt{\alpha\gamma}\,\beta}{\delta\alpha} x_2 = \frac{i\beta}{\delta}\sqrt{\frac{\gamma}{\alpha}}\, x_2$$

$$\Rightarrow \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{i\beta}{\delta}\sqrt{\frac{\gamma}{\alpha}} \\ 1 \end{pmatrix} = \bar{v}_1$$

For $\lambda_2 = -i\sqrt{\alpha\gamma}$

$$\begin{pmatrix} i\sqrt{\alpha\gamma} & -\beta\gamma/\delta \\ \delta\alpha/\beta & i\sqrt{\alpha\gamma} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad i\sqrt{\alpha\gamma}\, x_1 - \frac{\beta\gamma}{\delta} x_2 \Rightarrow x_1 = \frac{\beta}{i\delta}\sqrt{\frac{\gamma}{\alpha}} = \frac{-i\beta}{\delta}\sqrt{\frac{\gamma}{\alpha}}$$

$$\Rightarrow \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -i\beta/\delta\sqrt{\gamma/\alpha} \\ 1 \end{pmatrix} = \bar{v}_2$$

$$\Rightarrow \quad \bar{\gamma}(t) = c_1 e^{i\sqrt{\alpha\gamma}t} \begin{pmatrix} i\beta/\delta\sqrt{\gamma/\alpha} \\ 1 \end{pmatrix} + c_2 e^{-i\sqrt{\alpha\gamma}t} \begin{pmatrix} -i\beta/\delta\sqrt{\gamma/\alpha} \\ 1 \end{pmatrix}$$

$$= \quad \bar{r}(t) = c_1\left(\cos\left(\sqrt{\alpha\gamma}\,t\right) + i\sin\left(\sqrt{\alpha\gamma}\,t\right)\right)\begin{pmatrix} \frac{i\beta\sqrt{\gamma}}{\delta\sqrt{\alpha}} \\ 1 \end{pmatrix}$$

$$+ \; c_2\left(\cos\left(\sqrt{\alpha\gamma}\,t\right) - i\sin\left(\sqrt{\alpha\gamma}\,t\right)\right)\begin{pmatrix} -\frac{i\beta\sqrt{\gamma}}{\delta\sqrt{\alpha}} \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad \bar{r}(t) = \left[c_1\begin{pmatrix} \frac{i\beta\sqrt{\gamma}}{\delta\sqrt{\alpha}} \\ 1 \end{pmatrix} + c_2\begin{pmatrix} \frac{-i\beta\sqrt{\gamma}}{\delta\sqrt{\alpha}} \\ 1 \end{pmatrix}\right]\cos\left(\sqrt{\alpha\gamma}\,t\right)$$

$$+ \left[c_1\begin{pmatrix} \frac{i\beta\sqrt{\gamma}}{\delta\sqrt{\alpha}} \\ 1 \end{pmatrix} + c_2\begin{pmatrix} \frac{-i\beta\sqrt{\gamma}}{\delta\sqrt{\alpha}} \\ 1 \end{pmatrix}\right] i\sin\left(\sqrt{\alpha\gamma}\,t\right)$$

→ **Common Differential Equations :-**

- $\dfrac{dx}{dt} = \alpha x \Rightarrow x = c_1 e^{\alpha t}$

- $\dfrac{\delta^2 f}{\delta t^2} = c^2 \dfrac{\delta^2 f}{\delta x^2} \longrightarrow$ Diffusion Equation

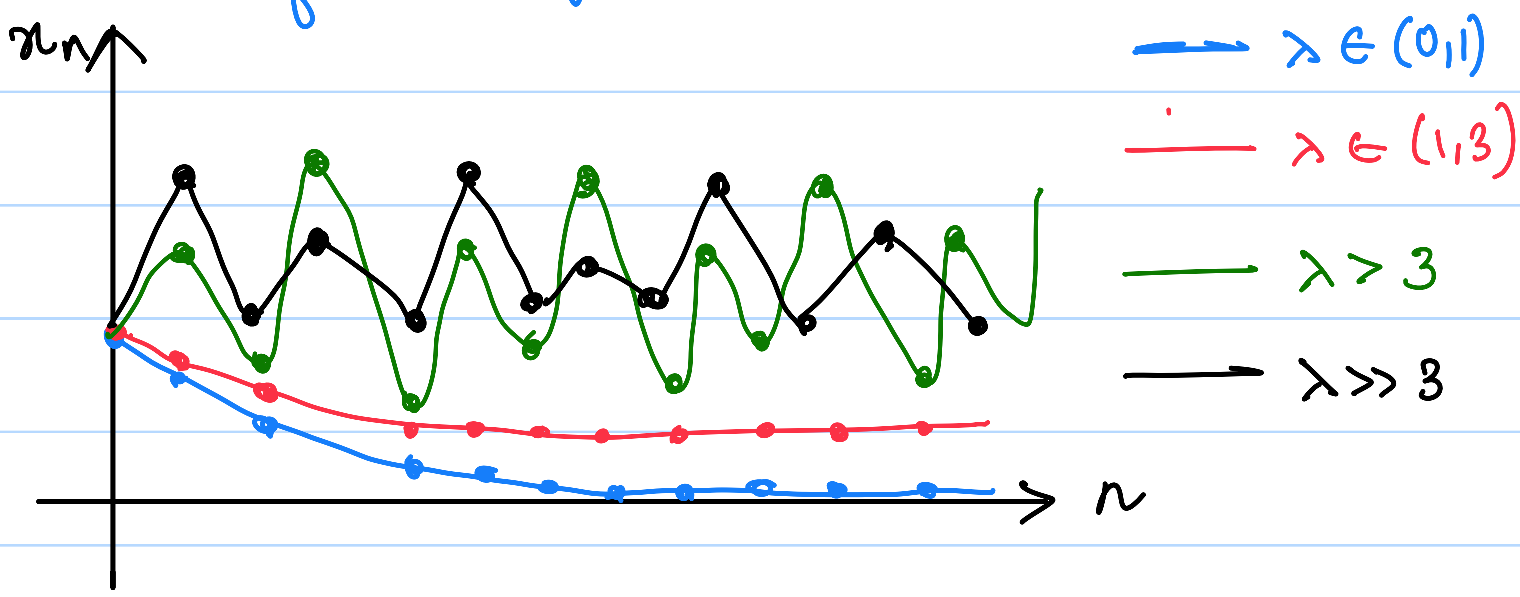
  $\dfrac{\delta f}{\delta t} = c^2 \dfrac{\delta^2 f}{\delta x^2} \longrightarrow$ Quantum Mechanics

→ **Logistics Mapping :-**

- A sequence such that,

$$x_{n+1} = \lambda x_n (1 - x_n), \quad \forall n \in \mathbb{N}$$

- Analyzing the terms of the sequence,



Therefore, depending on the value of λ, the same initial value can result in very different sequences, and for some λ, small differences in initial value can result in large differences in the final value.

- Therefore, this sequence is said to exhibit chaotic behavior.

→ Classical Mechanics :-

- **Scalar Field** : Each point in space is associated with a scalar quantity. ex: Temperature

- **Vector Field** : Each point in space is associated with a vector of some direction and magnitude. ex: Air Speeds.

- **Gradient** : The change of a quantity within its field.

$$\nabla f(x_0, y_0, z_0) = \left[\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y} \quad \frac{\partial f}{\partial z} \right]$$

- **Divergence** : Scalar that measures the outflowness of a vector field at a point.

- **Laws Of Motion** :

1) In an inertial frame, an object undergoes uniform rectilinear motion in the absence of external forces.

2) Rate of change of momentum is equal to the net force on the object.

3) Reactive force is equal and opposite to the action force.

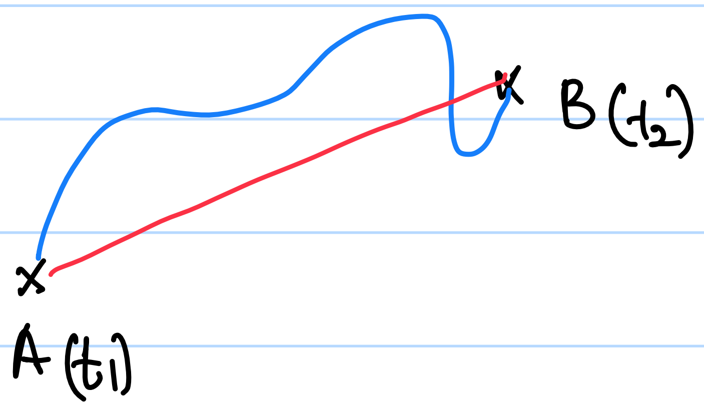
◦ **Work:** The work done by the application of an external force is given by,

$$W = \int_{t_1}^{t_2} \bar{F}(t)\, d\bar{r}(t)$$ infinitesimal F_r

Using the 2nd Law,

$$W = \int_{t_1}^{t_2} \frac{d\bar{p}}{dt}\, v(t)\, dt = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^{1}$$

$$\Rightarrow \quad W = KE_2 - KE_1 \longrightarrow$$ Work is just change in kinetic energy.



Different paths may have different amounts of work done.
(No difference in conservative fields)

In vectorial form, $KE = \frac{1}{2}m(\bar{v}\cdot\bar{v})$

◦ **Conservative Forces:** Forces that can be modelled as a gradient of a scalar field, ie,

$$\bar{F}(\bar{r}) = -\nabla U(\bar{r}).$$

$$\Rightarrow \quad W = \int_1^2 (-\nabla U(\bar{r}))\, d\bar{r} = -(U_2 - U_1)$$

• The scalar quantity / field $U(\bar{r})$ is the potential energy due to the force.

- Since Work now only depends on PE at 1 and 2, it does not depend on the exact path travelled between 1 and 2.

$$W = K_2 - K_1 = -(U_2 - U_1)$$

$$\Rightarrow \boxed{K_2 + U_2 = K_1 + U_1} \rightarrow \text{Law of Conservation of Mechanical Energy}$$

- Conservation of Linear Momentum :-

- For an object, if $\bar{F} \cdot \bar{s} = 0$ ($\bar{F}$ is zero or $\perp$ lar to $\bar{s}$)

$$\bar{F} \cdot \bar{s} = 0 \Rightarrow \frac{d\bar{p}}{dt} \cdot \bar{s} = 0 \Rightarrow \frac{d(\bar{p} \cdot \bar{s})}{dt} = 0$$

(Assuming $\bar{s}$ is const)

$\Rightarrow$ $\bar{p} \cdot \bar{s}$ is constant $\Rightarrow$ Momentum in the direction of s is conserved

$\therefore$ If force in a direction is zero, momentum is conserved in that direction.

- Torque and Conservation of Angular Momentum :-

- Angular momentum $\bar{L} = \bar{r} \times \bar{p}$

- Torque $\bar{\tau} = \frac{d}{dt}\bar{L} = \frac{d}{dt}\bar{r} \times \bar{p} = \frac{d\bar{r}}{dt} \times \bar{p} + \bar{r} \times \frac{d\bar{p}}{dt}$

$$\bar{\tau} = \cancelto{0}{\bar{v} \times \bar{p}} + \bar{r} \times \bar{F}$$

$$\Rightarrow \boxed{\bar{\tau} = \bar{r} \times \vec{F}}$$

• Conservation of Angular Momentum,

If $\bar{\tau} \cdot \bar{s} = 0$, then $\bar{L} \cdot \bar{s}$ is constant.

$\therefore$ Angular momentum is conserved along the axis of zero torque.

∘ $$E = T(\bar{v}(t)) + U(\bar{r}(t), t)$$

$$\frac{dT}{dt} = m\bar{v}\, d\bar{v}/dt = m\bar{v} \cdot \bar{a} = \bar{v} \cdot \bar{F}$$

$$\frac{dU(\bar{r}, t)}{dt} = \frac{\partial U}{\partial \bar{r}} \frac{d\bar{r}}{dt} + \frac{\partial U}{\partial t} = \bar{v} \cdot \Delta U + \frac{\partial U}{\partial t}$$

$\bar{F} = -\Delta U$

$$\Rightarrow \frac{dE}{dt} = \bar{v}\,(\cancel{\bar{F} + \Delta U})^{0} + \frac{\partial U}{\partial t}$$

$$\Rightarrow \frac{dE}{dt} = \frac{\partial U}{\partial t}$$ $\Rightarrow$ If PE is constant, E is conserved

∘ Galilean Relativity :-

Let there be 2 inertial frames K and K' and K' is at relative velocity $\bar{V}$ wrt K. Let an object be moving with velocity $\bar{v}$ and $\bar{v}_1$, ie

$$\bar{v}_1 = \bar{v} - \bar{V}$$

$$\Rightarrow \frac{d\bar{v}_1}{dt} = \frac{d(\bar{v} - \bar{V})}{dt} = \frac{d\bar{v}}{dt}$$ $\bar{V}$ is const

$$\Rightarrow \quad \boxed{\bar{F_1} = \bar{F}}$$

$\therefore$ All mechanical laws are the same in any inertial frame.

→ <u>Simple Harmonic Motion</u> :-

- Hooke's law : $F(x) = -kx \Rightarrow m\dfrac{d^2x}{dt^2} = -kx$

To solve the DE,

$$m\frac{d^2x}{dt^2} = -kx$$

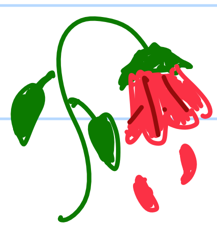
Take $p = m\dfrac{dx}{dt} \Rightarrow m\dfrac{d^2x}{dt^2} = \dfrac{dp}{dt}$

$$\Rightarrow \quad \frac{dp}{dt} = -kx \quad \Rightarrow \quad dp = -kx\,dt$$

$$= \quad p = [-kxt]_0^t = -kxt$$

$$\frac{dx}{dt} = \frac{p}{m}$$

$\Rightarrow$



ts too tuff :(

→ **Limitations of Newtonian Mechanics :-**

- Galilean relativity violated by EM waves.
- Stability of atoms cannot be explained by Newtonian mechanics.

→ **Special Theory of Relativity :-**

The postulates of STR are as follows,

i) The laws of physical phenomena are the same in all inertial reference frames.

ii) The velocity of light (in free space) is a universal constant, independent of any relative motion.

- **Galilean Invariance :-**

Consider a frame K and another frame K' moving at a speed of v_r along the x-axis.

- Let (x, y, z) be coordinates in K and (x', y', z') in K'. The transformation of coordinates is given by,

$$x' = x - v_r t$$

$$y' = y$$

$$z' = z$$

$$t' = t$$

- The above transformation is termed as a Galilean transformation.
- Length is conserved in a Galilean transformation,

$$ds^2 = \Sigma dx_i^2 = \Sigma dx_i'^2 = ds'^2$$

Newton's laws are also conserved,

$$F = ma = ma' = F'$$

- Velocity in K' is given by,

$$v' = v - v_r$$

That means the velocity of light in K' is $c - v_r \neq c$, in violation of STR.

- <u>Lorentz Transformation</u> :-

- If a light pulse is emitted from the common origin of the systems K and K_1 when they are coincident, then,

$$x = ct$$
$$x_1 = ct_1$$

Let $x_1 = \gamma(x - v_r t) \Rightarrow x = \gamma'(x_1 + v_r t_1)$

Since the laws of physics must be the same in K and K',
$\gamma = \gamma'$.

$$\Rightarrow \quad x = \gamma\left(\gamma\left(x - v_\gamma t\right) + v_\gamma t_1\right)$$

$$\Rightarrow \quad x = \gamma^2\left(x - v_\gamma t\right) + \gamma v_\gamma t_1$$

$$\Rightarrow \quad x = x\gamma^2 - \gamma^2 v_\gamma t + \gamma v_\gamma t_1$$

$$\Rightarrow \quad \gamma v_\gamma t_1 = x - x\gamma^2 + \gamma^2 v_\gamma t$$

$$= x\left(1 - \gamma^2\right) + \gamma^2 v_\gamma t$$

$$\Rightarrow \quad t_1 = \frac{x}{\gamma v_\gamma}\left(1 - \gamma^2\right) + \gamma t$$

$$= \quad t_1 = \frac{x}{v_\gamma}\left(\frac{1-\gamma^2}{\gamma}\right) + \gamma t \quad \text{——— } (1)$$

$$x = ct \quad \Rightarrow \quad x' = \gamma\left(ct - v_\gamma t\right) = \gamma\left(c - v_\gamma\right)t$$

$$\Rightarrow \quad t_1 = \frac{ct}{v_\gamma}\left(\frac{1-\gamma^2}{\gamma}\right) + \gamma t$$

$$x_1 = ct_1 \quad \Rightarrow \quad c = \frac{x_1}{t_1} = \frac{\gamma\left(c - v_\gamma\right)t}{\frac{ct}{v_\gamma}\left(\frac{1-\gamma^2}{\gamma}\right) + \gamma t}$$

$$\Rightarrow \quad c = \frac{\gamma\left(c - v_\gamma\right)}{\frac{c}{v_\gamma}\left(\frac{1-\gamma^2}{\gamma}\right) + \gamma}$$

$$\Rightarrow \quad \gamma\left(c - v_\gamma\right) = \frac{c^2}{v_\gamma}\left(\frac{1-\gamma^2}{\gamma}\right) + c\gamma$$

$$\Rightarrow \cancel{\gamma c} - \gamma v_r = \frac{c^2}{v_\gamma}\left(\frac{1-\gamma^2}{\gamma}\right) + \cancel{c\gamma}$$

$$= \quad \gamma v_\gamma = \frac{c^2}{v_\gamma}\left(\frac{\gamma^2 - 1}{\gamma}\right)$$

$$= \quad \frac{\gamma v_\gamma}{c^2} = \frac{1}{v_r}\left(\frac{\gamma^2 - 1}{\gamma}\right) \quad \text{———} \quad (2)$$

Substitute (2) in (1),

$$\Rightarrow t_1 = x\left(\frac{-\gamma v_\gamma}{c^2}\right) + \gamma t$$

$$= \quad t_1 = \gamma\left(t - \frac{v_r}{c^2} x\right)$$

$$\Rightarrow \gamma = \frac{1}{\sqrt{1 - v_r^2/c^2}}$$

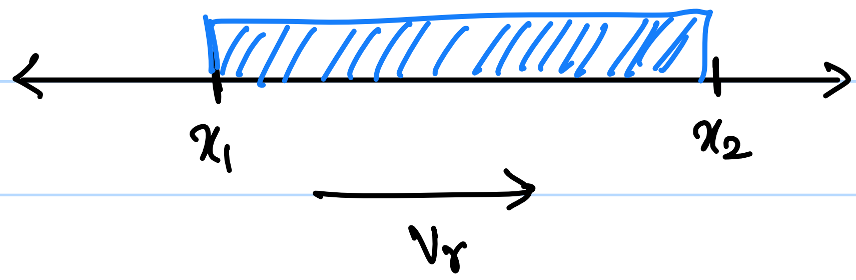
With this, the transformation can be described as,

$$x' = \frac{x - v_r t}{\sqrt{1 - v_r^2/c^2}}$$

$$t' = \frac{t - v_r x/c^2}{\sqrt{1 - v_r^2/c^2}} = \gamma\left(t - \frac{v_r x}{c^2}\right)$$

The above transformation is described as the Lorentz transformation.

- **Length Contraction:**



- An observer is moving velocity v_r towards x_2, from x_1. He measures the length of the above rod $(x_2 - x_1)$ at a time instant t.

The length of the rod in the observer's frame is,

$$l' = x_2' - x_1'$$

$$= \gamma\left(x_2 - v_r t_2\right) - \gamma\left(x_1 - v_r t_1\right)$$

$$= \gamma\left(x_2 - x_1 - v_r\left(t_2 - t_1\right)\right)$$

Since $t_2' = t_1'$

$$\Rightarrow \gamma\left(t_2 - \frac{v_r x_2}{c^2}\right) = \gamma\left(t_1 - \frac{v_r x_1}{c^2}\right)$$

$$= \quad t_2 - t_1 = \frac{v_r}{c^2}\left(x_2 - x_1\right)$$

$$\Rightarrow l' = \gamma\left(x_2 - x_1 - \frac{v_r^2}{c^2}\left(x_2 - x_1\right)\right)$$

$$= \frac{1}{\sqrt{1 - \frac{v_r^2}{c^2}}}\left(x_2 - x_1\right)\left(1 - \frac{v_r^2}{c^2}\right)$$

$$= \left(x_2 - x_1\right)\sqrt{1 - \frac{v_r^2}{c^2}}$$

$$\Rightarrow \quad \underline{l' = \gamma l} \quad \Rightarrow \quad l' < l$$

This is termed as length contraction.

- **Time Dilation :-**

— A clock, at a fixed position x, ticks at 2 time instants t_1 and t_2. An observer moving in frame K', with relative velocity v_r measures the time interval.

$$\Delta t' = t_2' - t_1'$$

$$\Delta t' = \gamma\left(t_2 - \frac{v_r}{c^2} x_2\right) - \gamma\left(t_1 - \frac{v_r}{c^2} x_1\right)$$

$$= \gamma\left(t_2 - t_1 - \frac{v_r}{c^2}\cancelto{0}{(x_2 - x_1)}\right)$$

$$= \gamma(t_2 - t_1)$$

$$\Rightarrow \quad \underline{\Delta t' = \gamma \Delta t} \quad \Rightarrow \quad \Delta t' > \Delta t$$

This phenomenon is termed as time dilation.

- **Relativistic Doppler Effect :-**

Consider a source of light and a reciever, with approach velocity v.

Let the reciever be in a frame K and the source in K'.

In time Δt, the source emits n waves. The distance between the front and back is,

$$l = c\Delta t - v\Delta t$$

$$\Rightarrow \quad \lambda = \frac{c\Delta t - v\Delta t}{n}$$

$$\Rightarrow \quad f = \frac{nc}{c\Delta t - v\Delta t}$$

Since the source emits with frequency f_0,

$$n = f_0 \Delta t' = f_0 \frac{\Delta t}{\gamma} \quad \text{[time dilation]}$$

$$\Rightarrow \quad f = \frac{f_0 c\Delta t}{\gamma(c\Delta t - v\Delta t)}$$

$$= \frac{1}{\gamma\left(1 - \frac{v}{c}\right)} f_0$$

$$\underline{f = \frac{\sqrt{1 - v^2/c^2}}{1 - v/c} f_0}$$

If the source and receiver are separating with velocity v,

$$d = c\Delta t + v\Delta t$$

$$\Rightarrow \lambda = \frac{c\Delta t + v\Delta t}{n}$$

$$\Rightarrow f = \frac{nc}{c\Delta t + v\Delta t}$$

$$n = f_0 \Delta t' = f_0 \frac{\Delta t}{\gamma}$$

$$\Rightarrow f = \frac{c\Delta t}{\gamma(c\Delta t + v\Delta t)} f_0$$

$$\Rightarrow f = \frac{\sqrt{1 - v^2/c^2}}{1 + v/c} f_0$$

Note:

Lorentz transformation for velocities,

Let K and K' be 2 inertial frames with relative velocity v_r, with an object moving at velocities v and v' in the frames respectively.

$$v' = \frac{dx'}{dt'} = \frac{dx'}{dt} \Big/ \frac{dt'}{dt}$$

Wkt,

$$x' = \gamma(x - v_r t) \qquad t' = \gamma\left(t - \frac{v_r x}{c^2}\right)$$

γ is a constant w.r.t t.

$$\Rightarrow \quad \frac{dx'}{dt} = \gamma\left(\frac{dx}{dt} - v_r\right) \qquad \frac{dt'}{dt} = \gamma\left(1 - \frac{v_r}{c^2}\frac{dx}{dt}\right)$$

$$= \gamma(v - v_r) \qquad = \gamma\left(1 - \frac{v_r v}{c^2}\right)$$

$$\Rightarrow \quad v' = \frac{\cancel{\gamma}(v - v_r)}{\cancel{\gamma}\left(1 - \frac{v_r v}{c^2}\right)} = \frac{v - v_r}{1 - \frac{v_r v}{c^2}}$$

Verification by taking $v = c$

$$v' = \frac{c - v_r}{1 - \frac{v_r \cancel{c}}{c^{\cancel{2}}}} = \frac{c - v_r}{\frac{(c - v_r)}{c}} = \underline{\underline{c}}$$

$\therefore$ $v = v'$ if $v = c$. STR is not violated.

- **Proper Time:-**

Time as measured in the observer's frame of reference, with respect to events that occur at the same time.

- **Proper Length :-**

Length as measured in the observer's frame of reference, with respect to an object that is at rest in his frame.

- Events that happen at different times will undergo length contraction between them.

Note: An event is defined with both space and time.

- **The Twin Paradox :-**

- If there are 2 twins, one on Earth and one on an interstellar voyage travelling near the speed of light, the space travelling twin will age much slower than the twin on Earth.

- The paradox is that, since with respect to the traveller, the twin on Earth is the one moving, the Earth twin must be the younger one, due to symmetry.

- The solution, is that since the traveller is going away, and then coming back to Earth, he is actually in 2 different reference frames. (One while separation, one while approach)

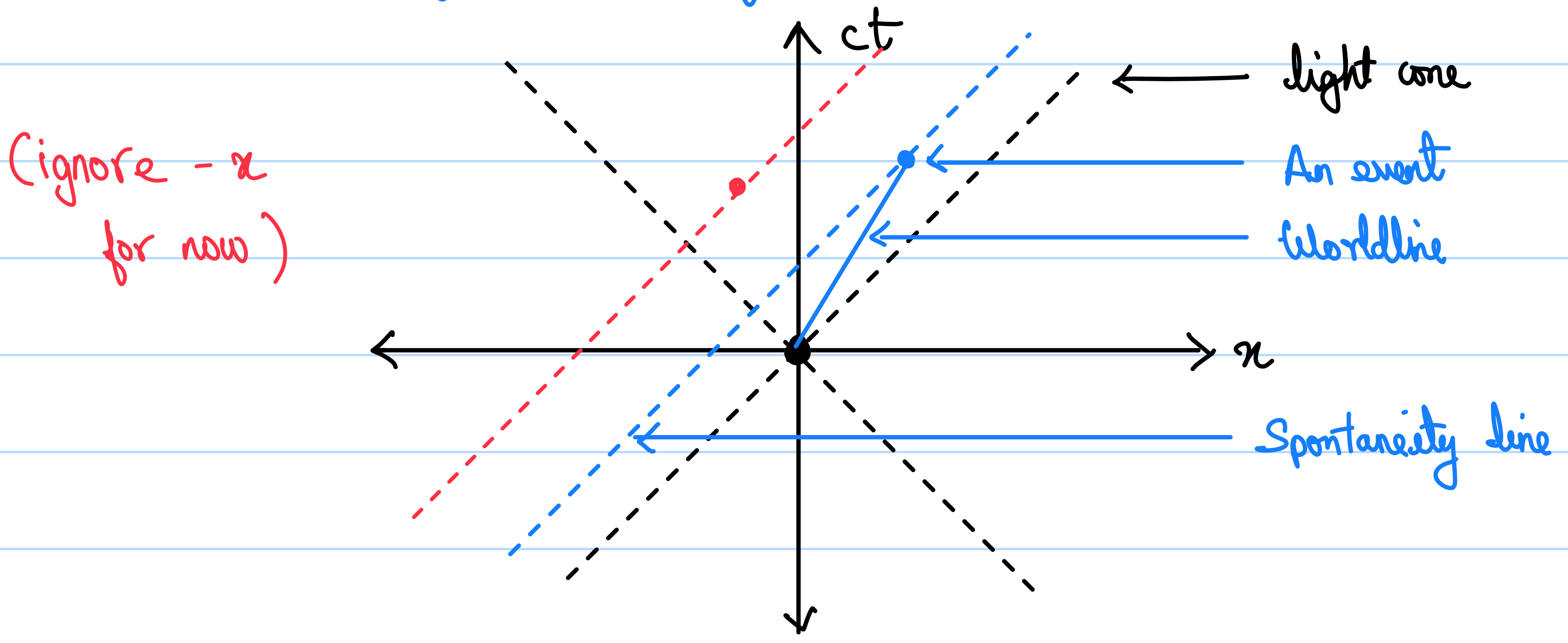
- To explain this further, we can use space time diagrams.

Spacetime Diagrams (Minkowski Geometry) :-

- In Minkowski Geometry, the spacetime interval between 2 events is defined as,

$$ds^2 = dx^2 + dy^2 + dz^2 - (c\,dt)^2$$

- A spacetime diagram is as follows, (on one spatial dimension x)



Event : A point on the spacetime diagram

Worldline: The path travelled by an event, ie the history of an object through space and time.

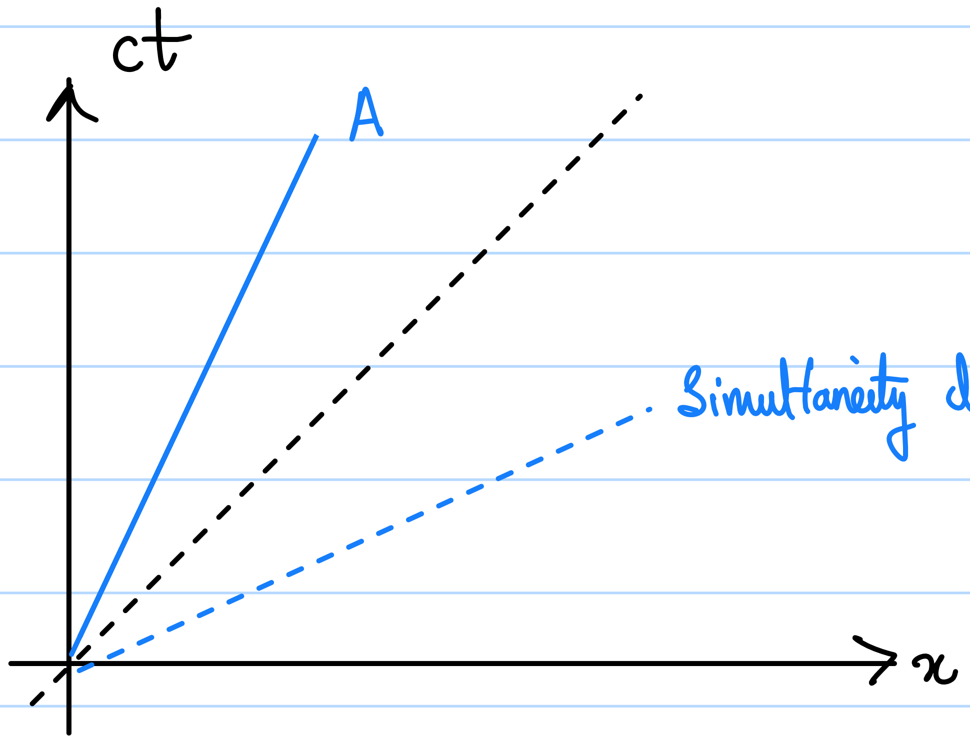
Simultaneity line : Set of all events that are spontaneous with the event, in the observer's frame (if observer is at rest ∥ to light cone)

Light cone: Speed of light representation (Worldlines cannot cross light cone, as nothing can cross the speed of light). Also, the set of events that are simultaneous to origin.

- The slope of the simultaneity line depends on the velocity of the object being analysed in the observer frame. Higher the speed, higher the slope, but cannot cross $m = 1$ or -1 (speed of light)
  -x direction

- Length contraction and time dilation distort the object's spacetime diagram, messing up the spontaneity line.

Let A represent a moving observer wrt ground, the ground's space time diagram is as follows.



- To determine the simultaneity line of A, let velocity of A be v.

Applying Lorentz Transformation,

$$x' = \gamma(x - vt)$$
$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

Simultaneity means $t' =$ const (let it be k).

The slope of a worldline is given by $\frac{d(ct)}{dx}$.

$$K = \gamma\left(t - \frac{vx}{c^2}\right)$$

$$\Rightarrow \quad t - \frac{v}{c^2}x = \frac{K}{\gamma}$$

$$\Rightarrow \quad ct = \frac{v}{c}x + \frac{cK}{\gamma}$$

$\therefore$ Slope of ct vs x = $\underline{\frac{v}{c}}$.

$\therefore$ The simultaneity line of an object at (ct_0, x_0) moving at velocity v with respect to the observer is

$$\underline{(t - t_0) = \frac{v}{c}(x - x_0)}$$

Verification: If $v = 0$, slope $= 0$, ie, set of all points (x, t) such that $t = t_0$. This is clearly correct.