RA Assignment 1

Sricharan Vinoth Kumar

October 30, 2024

Question 1

It is given that

$$a_n - a_{n-2} \to 0 \text{ as } n \to \infty \tag{1}$$

Therefore, it is implied that the odd-numbered terms and the even-numbered terms of the sequence are very close to each other after a certain number of terms. That is, for any $\epsilon > 0$,

$$|a_n - a_{n-2}| < \epsilon \; \forall \; n \geq N_0, N_0 \in \mathbb{N} \tag{2}$$

If we take $\{e_n\}$ and $\{o_n\}$ to be the sequence of even numbered terms and odd numbered terms of $\{a_n\}$, using the above result, we can state that $\{e_n\}$ and $\{o_n\}$ are Cauchy sequences.
Therefore, assuming that the matrix space to be closed, these sequences are convergent.
Since these sequences are convergent, they must be bounded Let the bounds of $\{e_n\}$ and $\{o_n\}$ be $|E|$ and $|O|$. E and O will be finite numbers.
So,

$$b_n = \frac{a_n - a_{n-1}}{n} \leq \frac{|E| + |O|}{n} \tag{3}$$

Since $\frac{|E|+|O|}{n} \to 0$ (as the numerator is finite), by the inequality,

$$b_n \to 0 \tag{4}$$

Question 2

Since n is large, $\frac{k}{n^2}$ will be a small number. Therefore, using the binomial approximation, we get,

$$\begin{aligned}
\lim_{n\to\infty}\sum_{k=1}^{n}\left(\sqrt{1+\frac{k}{n^2}}-1\right) &= \lim_{n\to\infty}\sum_{k=1}^{n}\left(1+\frac{k}{2n^2}-1\right)\\
&= \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{k}{2n^2}\right)\\
&= \lim_{n\to\infty}\left(\frac{1}{2n^2}\right)(\sum_{k=1}^{n}k)\\
&= \lim_{n\to\infty}\left(\frac{1}{2n^2}\right)\left(\frac{n(n-1)}{2}\right)\\
&= \lim_{n\to\infty}\left(\frac{1}{4}\right)\left(\frac{n-1}{n}\right)\\
&= \lim_{n\to\infty}\left(\frac{1}{4}\right)\left(1-\frac{1}{n}\right)\\
&= \frac{1}{4}
\end{aligned}$$

$$\therefore \lim_{n\to\infty}\sum_{k=1}^{n}\left(\sqrt{1+\frac{k}{n^2}}-1\right)=\frac{1}{4}$$

Question 3

It is given that $a_n \geq 1$ and,

$$\left(a_n+\frac{1}{a_n}\right) \text{ is convergent to (let's say) L} \tag{1}$$

By AM-GM inequality, we get,

$$\begin{gathered}
\left(a_n+\frac{1}{a_n}\right)\geq 2\\
\implies L\geq 2
\end{gathered} \tag{2}$$

Assume $a_n \to A$ such that $A \notin \mathbb{R}$, that is, a_n is divergent. Then,

$$\begin{aligned}
&\lim_{n\to\infty}\left(a_n+\frac{1}{a_n}\right)=\lim_{n\to\infty}a_n+\frac{1}{\lim_{n\to\infty}a_n}\\
&\implies L=A+\frac{1}{A}\\
&\implies A^2+1-LA=0
\end{aligned} \tag{3}$$

In equation (3), we get a quadratic equation in A. The discriminant of this equation, let's say, D must be such that $D < 0$. for our assumption to hold.

$$D=L^2-4(1)(1) \tag{4}$$

W.k.t $L \geq 2 \implies L^2 \geq 4$

$$\implies D\geq 0 \tag{5}$$

But this is a contradiction with what was stated previously. Therefore our assumption is incorrect

$$\therefore A \in \mathbb{R}$$
$$\therefore a_n \text{ is convergent}$$

Question 4

We know that

$$\left(1 + \frac{1}{n}\right)^n \to e \text{ as } n \to \infty \tag{1}$$

Let $\{x_n\}$ be any sequence such that, $\{x_n\} \to \infty$ as $n \to \infty$, ie, $\{x_n\}$ is a divergent sequence.
Also, let us define a fucntion f: $\mathbb{R} \to \mathbb{R}$ as

$$f(x) = \left(1 + \frac{1}{x}\right)^x \tag{2}$$

Using equation (1), we can see that,

$$\{f(x_n)\} \to e \text{ as } n \to \infty \tag{3}$$

Since $\{x_n\}$ is any divergent sequence, using the sequential criterion theorem of limits, we can state that,

$$\lim_{x\to\infty} f(x) = e \tag{4}$$

$$\therefore \lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = e \tag{5}$$

Question 5

$$\lim_{x\to 0} \frac{(1+x)^a - 1}{x} = a$$

Using Binomial Theorem, we get,

$$(1+x)^a = 1^a + ax + \frac{a(a-1)}{2}x^2 + \text{..........} + x^a$$
$$\implies (1+x)^a = 1 + ax + \frac{a(a-1)}{2}x^2 + \text{..........} + x^a$$
$$\implies (1+x)^a - 1 = ax + \frac{a(a-1)}{2}x^2 + \text{..........} + x^a$$
$$\implies \frac{(1+x)^a - 1}{x} = a + \frac{a(a-1)}{2}x + \text{..........} + x^{a-1}$$
$$\therefore \lim_{x\to 0} \frac{(1+x)^a - 1}{x} = a$$

Question 6

The summation $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is known as the Basel Problem. Using Euler's approach to this problem.

$$\sin(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \text{. . . . .} \tag{1}$$

$$\implies \frac{\sin(x)}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots\dots\dots \tag{2}$$

In equation (2), we can see that the coefficient of $x^2 = \frac{-1}{3!}$. Using the Weiestrauss Factorization Theorem to write $\frac{\sin(x)}{x}$ as a linear product of its roots, we get,

$$\frac{\sin(x)}{x} = \left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 + \frac{x}{2\pi}\right)\dots\dots\dots \tag{3}$$

If we take all the x^2 terms of equation (3), we get the x^2 coefficient of the equation to be,

$$-\left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2}\right) = -\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} \tag{4}$$

By comparing equation (4) with the result of equation (2), we get,

$$\begin{aligned} -\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} &= \frac{-1}{3!} \\ \implies \sum_{n=1}^{\infty}\frac{1}{n^2} &= \frac{\pi^2}{6} \end{aligned} \tag{5}$$

$$\begin{aligned} \frac{\pi^2}{6} &= \frac{9.8696.....}{6} = \frac{29.6088.....}{18} \\ \implies \frac{29}{18} &< \frac{\pi^2}{6} < \frac{30}{18} \\ \\ \therefore \quad \frac{29}{18} &< \sum_{n=1}^{\infty}\frac{1}{n^2} < \frac{31}{18} \end{aligned} \tag{6}$$

Question 7

$$\begin{aligned} n^4 + n^2 + 1 &= n^4 + 2n^2 + 1 - n^2 \\ &= (n^2+1)^2 - n^2 \\ &= (n^2+1-n)(n^2+1+n) \\ \implies \sum_{n=1}^{\infty}\frac{n}{n^4+n^2+1} &= \sum_{n=1}^{\infty}\frac{n}{(n^2+1-n)(n^2+1+n)} \end{aligned}$$

Using Partial Fractions Method,

$$\begin{aligned} \sum_{n=1}^{\infty}\frac{n}{(n^2+1-n)(n^2+1+n)} &= \sum_{n=1}^{\infty}\frac{\frac{1}{2}((n^2+1-n)-(n^2+1+n))}{(n^2+1-n)(n^2+1+n)} \\ &= \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n^2+1-n} - \frac{1}{n^2+1+n}\right) \end{aligned}$$

To simplify the expression, take,

$$\begin{aligned} \{a_n\} &= \frac{1}{n^2+1-n} \\ \{b_n\} &= \frac{1}{n^2+1+n} \end{aligned}$$

We can observe that,

$$\begin{aligned}
a_{n+1} &= \frac{1}{(n+1)^2+1-(n+1)} \\
&= \frac{1}{n^2+1+n} \\
&= b_n
\end{aligned}$$

Therefore, the given summation simplifies to,

$$\begin{aligned}
\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n^2+1-n}-\frac{1}{n^2+1+n}\right) &= \frac{1}{2}(a_1-a_2+a_2-a_3.....) \\
&= \frac{1}{2}(a_1) = \frac{1}{2} \\
\implies \sum_{n=1}^{\infty}\frac{n}{n^4+n^2+1} = \frac{1}{2}
\end{aligned}$$

Question 8

1

To test the convergence of $\sum_{n=1}^{\infty}\frac{a^n}{(n!)^{\frac{1}{n}}}$, we can use the Ratio test,

$$\lim_{n\to\infty}\frac{\frac{a^{n+1}}{((n+1)!)^{1/(n+1)}}}{\frac{a^n}{(n!)^{1/n}}} \text{ (Let it be L)} \tag{0}$$

Simplyfying the expression, we get

$$\lim_{n\to\infty}\frac{a(n!)^{\frac{1}{n}}}{(n+1)!^{\frac{1}{n+1}}} \tag{0}$$

As n becomes very large, the ratio between $(n!)^{\frac{1}{n}}$ and $(n+1)!^{\frac{1}{n+1}}$ will tend to 1, as the change in value of these terms with increasing n, reduces. Therefore we get $L = a$ For the series to be convergent,$L < 1$.
Therefore, the given series is convergent for $0 < a < 1$ and divergent otherwise

2

To observe the behavior of the term $\left(1+\frac{1}{n}\right)^n$, for large n, we know from Q4 that

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \tag{0}$$

Therefore, for large terms, the behavior of the series is same as $e\sum_{n=1}^{\infty}a^n$
Since the series is a Geometric Series, we can say that the series is convergent if $a < 1$ If a = 1, the sequence tends to e when n becomes large, which means that the series will not converge. $\therefore$ The given series is convergent for $0 < a < 1$ and divergent otherwise