EC5.102 Introduction to Information Systems **Spring 2020**

Lecture 6: Random Variables

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Given an experiment and the corresponding set of possible outcomes (the sample space), a random variable associates a particular number with each outcome (See Fig. 6.1). Mathematically, a random variable is a real-valued function of the experimental outcome. Note that random variable is not a variable but a function.

$$X : \Omega \to \mathbb{R}.$$

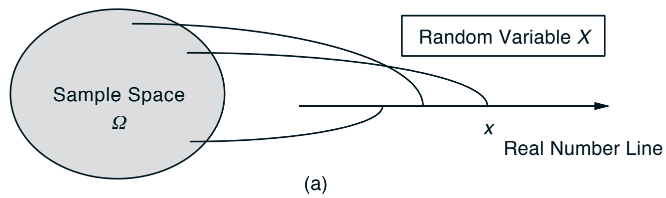

Figure 6.1: Illustrating a sample space and a random variable.

X is a function such that preimage of every half interval $(-\infty, x]$ under X is an event in the event space $\mathcal{F}$.

Consider the following function which maps the outcomes of 3 three coin tosses to the real line:

$$HHH- > 0, HHT- > 1, HTH- > 2, HTT- > 3, THH- > 4, THT- > 5, TTH- > 6, TTT- > 7$$

The above function is a random variable if the event space $\mathcal{F}$ is the power set of the sample space. It is not a random variable if the event space is the following:

$$\mathcal{F} = \{\phi, \{HHH, HTT, THT, TTH\}, \{HHT, HTH, THH, TTT\}, \Omega\}.$$

Suppose the sample space is finite and the event space if the power set of the sample space. Then every function X is a random variable.

6.1 Cumulative Distribution Function

The probability law defined on the events in the sample space translates to a probability law on corresponding events on the real line. Of particular interest are the events of the form $\{X \leq x, x \in \mathbb{R}\}$ and the probability law corresponding to these events is summarized in the form of a function known as cumulative distribution function. The cumulative distribution function (cdf) $F_X(.)$ of the random variable X is defined as

$$F_X(x) = P(X \leq x) = P(\{\omega | X(\omega) \leq x\}) \tag{6.1}$$

The cumulative distribution function (cdf) $F_X(.)$ satisfies the following properties:

1. $F_X(.)$ is monotonically nondecreasing. This is because for $a < b$, the event $X \leq a$ is contained int he event $X \leq b$ and hence must have a smaller probability.

2. $F_X(-\infty) = 0$ and $F_X(-\infty) = 1$. This follows since X must take some finite value (from the real line).
3. $F_X(.)$ is right continuous, i.e., the value of $F_X(.)$ at any point is equal to right hand limit of the function at that point.

6.2 Discrete Random Variables

A random variable is called discrete if its range (the set of values that it can take) is finite or at most countably infinite. A random variable that can take an uncountably infinite number of values is not discrete. For an example, consider the experiment of choosing a point a from the interval $[-1, 1]$. The random variable that associates the numerical value $X(a) = a^2$ to the outcome a is not discrete since the range is $[0, 1]$. On the other hand, the random variable that associates with a the numerical value

$$X(a) = \begin{cases} 1, & a > 0 \\ 0, & a = 0 \\ -1, & a < 0 \end{cases} \tag{6.2}$$

is discrete. For a discrete random variable X, we define the probability mass function (pmf) of X by

$$p_X(a) = P(X = a) = P(\{\omega | X(\omega) = a\}).$$

Note that $p_X(.)$ is a valid pmf if and only if the following condition is satisfied.

$$\sum_{I=1}^{\infty} p_X(x_i) = 1,$$

where $\{x_1, x_2, \ldots, \}$ is the range of the random variable X.

6.2.1 Bernoulli Random Variable

Consider the toss of a biased coin, which comes up a head with probability p, and a tail with probability $1 - p$. The Bernoulli random variable takes the two values 1 and 0, depending on whether the outcome is a head or a tail:

$$X(T) = 0, \quad X(H) = 1. \tag{6.3}$$

The probability mass function (pmf) of the Bernoulli random variable is given by

$$p_X(0) = 1 - p, \quad p_X(1) = p. \tag{6.4}$$

The cumulative distribution function (cdf) of the Bernoulli random variable is given by

$$F_X(x) = \begin{cases} 0, & x < 0 \\ 1 - p, & 0 \leq x < 1 \\ 1, & x \geq 1 \end{cases}. \tag{6.5}$$

6.2.2 Binomial Random Variable

A biased coin is tossed n times. At each toss, the coin comes up a head with probability p, and a tail with probability $1 - p$, independently of prior tosses. The sample space is given by the set of all 2^n possible tuples of H, T combinations. For the case of $n = 4$, the sample space is as given below:

$$\begin{aligned}\Omega &= \{TTTT, TTTH, TTHT, TTHH, THTT, THTH, THHT, THHH, \\ &\quad HTTT, HTTH, HTHT, HTHH, HHTT, HHTH, HHHT, HHHH\}.\end{aligned}$$

For any $\omega \in \Omega$, $X(\omega)$ is defined as the number of heads in ω. The range of values which the random variable X takes is $\{0, 1, \ldots, n\}$. The probability mass function (pmf) of the random variable X is given by

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}, \ 0 \le k \le n$$

This random variable is known as binomial random variable. Note that the above pmf is a valid pmf as it sums to 1.

$$\sum_{k=0}^{n} p_X(k) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = (p + (1-p))^n = 1.$$

6.2.3 Geometric Random Variable

Suppose that we repeatedly and independently toss a biased coin with probability of a head p, where $0 < p < 1$ till a head comes up for the first time. The sample space corresponding to the experiment is given by

$$\Omega = \{H, TH, TTH, TTTH, \ldots\}.$$

For any $\omega \in \Omega$, $X(\omega)$ is defined as the number of tosses in ω. The range of values which the random variable X takes is $\{1, 2, \ldots, \}$. The probability mass function (pmf) of the random variable X is given by

$$p_X(k) = (1-p)^{k-1} p, \ k \in \{1, 2, \ldots\}.$$

This random variable is known as geometric random variable. Note that the above pmf is a valid pmf as it sums to 1.

$$\sum_{k=1}^{\infty} p_X(k) = \sum_{k=1}^{\infty} (1-p)^{k-1} p == p \sum_{k=0}^{\infty} (1-p)^k = p \frac{1}{1-(1-p)} = 1.$$

6.2.4 Poisson Random Variable

A poisson random variable takes nonnegative integer values. Its pmf is given by

$$p_X(k) = e^{-\lambda} \frac{\lambda^k}{k!}, \ k = 0, 1, 2, \ldots$$

Note that the above pmf is a valid pmf as it sums to 1.

$$\sum_{k=0}^{\infty} p_X(k) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda} e^{\lambda} = 1.$$

The Poisson distribution may be useful to model events such as

- The number of meteorites greater than 1 meter diameter that strike Earth in a year
- The number of patients arriving in an emergency room between 10 and 11 pm

- The number of photons hitting a detector in a particular time interval

An important property of the Poisson random variable is that it may be used to approximate a binomial random variable when the binomial parameter n is large and p is small. To see this, suppose that X is binomial random variable with parameters n, p and let $\lambda = np$. Then

$$\begin{aligned} P(X = k) &= \frac{n!}{(n-k)!k!} p^k (1-p)^{n-k} \\ &= \frac{n!}{(n-k)!k!} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} \\ &= \frac{n(n-1)\dots(n-k+1)}{n^k} \frac{\lambda^k}{k!} \frac{(1-\frac{\lambda}{n})^n}{(1-\frac{\lambda}{n})^k}. \end{aligned}$$

Now, for n large and p small, we have

$$\begin{aligned} \left(1 - \frac{\lambda}{n}\right)^n &\approx e^{-\lambda} \\ \frac{n(n-1)\dots(n-k+1)}{n^k} &\approx 1 \\ \left(1 - \frac{\lambda}{n}\right)^k &\approx 1. \end{aligned}$$

Substituting these approximations in the above equation, we have the pmf of the Poisson random variable.