Textbooks:

1) Papoulis
2) Bertsekas
3) Grimmett

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Probability And Random Processes

Grading:

- Assignments 15 %
- Quiz 1 & 2 15 % + 15 %
- In-Class Quiz 5 %
- Midsem 20 %
- Endsem 30 %

Probability And Random Processes

→ **Modules** :

M1) Basics Of Probability

M2) Discrete Random Variables

M3) Continuous Random Variables

M4) Tail Bounds & Limit Theorems

M5) Random Processes

→ **Module 1**:

1) Approach to define probability
2) Probability Space
3) Continuity of Probability
4) Conditional Probability , Independence
5) Bayes Theorem & Total Probability Theorem
6) Counting

→ **Classical Approach of Probability :-**

- For an event E,

$$P(E) = \frac{\text{no. of outcomes favourable to } E}{\text{total no. of outcomes}}$$

- **Issues :**

- Fails when the outcomes are not equally likely.
- Fails when the no. of possible outcomes is infinite.

→ **Frequency Approach of Probability :-**

- For an event E, where the experiment has been performed n times,

$$P(E) = \frac{n_E}{n}$$

n_E – no. of times E has occurred

$$\Rightarrow \quad P(E) = \lim_{n \to \infty} \frac{n_E}{n}$$

- **Issues :**

- We cannot perform an experiment infinite times.
- The ratio may not converge.

→ **Axiomatic Approach of Probability :-**

- The approach depends on a set of axioms.
- Probability space is a set represented by 3 entities.

$$P.S = (\Omega, \mathcal{F}, P)$$

Sample Space   Event Space   Probability Law

- Sample Space : Set of all outcomes of the random experiment.
- Event Space : Set of all favourable outcomes
- Probability Law : Function that defines probability.

- **Set Theory :-**

- $A \setminus B = \{ x \in A \text{ st } x \in B \}$
- $A \cup B = \{ x \in A \text{ or } x \in B \}$

Prove : $\left(\bigcup_{i=1}^{\infty} A_i \right)^c = \bigcap_{i=1}^{\infty} A_i^{\,c}$

$\bigcup_{i=1}^{\infty} A_i = \{ x \in A_i \text{ for some } i \in \mathbb{N} \}$ → By defn of $\cup$

$$\Rightarrow \left(\bigcup_{i=1}^{\infty} A_i\right)^c = \{x \notin A_i \ \forall\ i \in \mathbb{N}\}$$ reverse of the prev. statement

$$A_i^c = \{x \notin A_i\}$$

$$\bigcap_{i=1}^{\infty} A_i^c = \{x \in A_i^c \ \forall\ i \in \mathbb{N}\} \longrightarrow \text{By defn. of } \cap$$

$$\Rightarrow \bigcap_{i=1}^{\infty} A_i^c = \left(\bigcup_{i=1}^{\infty} A_i\right)^c$$

- **Sample Space (Ω):**

  - The elements of Ω, are all the possible outcomes of the random experiment.

  - The elements of Ω must be mutually exclusive (disjoint from each other) and collectively exhaustive (cover all possibilities)

  - Countably infinite sample space : Tossing a coin until we see tails.
    Uncountably infinite sample space : Throwing a dart on a square, $\Omega = \{(x,y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$

- **Event Space ($\mathcal{F}$):**

  - An event is a subset of a sample space

  - An event space is a set of subsets of Ω that form a

σ - field.

- <u>Axioms of σ - field</u>:

i) $\Omega \in \mathcal{F}$

ii) If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$ (Closure under complements)

iii) $A_1, A_2, \ldots\ldots \in \mathcal{F}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$ (Closure under countable unions)

↪ if we take $A_i = \phi \;\forall\, i \geq$ some k, then we can use this statement for finite unions as well

(↪ Same can be said for $\cap$, using (ii) and De-Morgans)

<u>Note</u>: $A \Delta B = A \setminus B \cup B \setminus A$

iv) $A, B \in \mathcal{F} \Rightarrow A \Delta B \in \mathcal{F}$ → Not an axiom. Implied by (i) - (iii)

$$A \Delta B = (A \cap B^c) \cup (B \cap A^c)$$

$A \in \mathcal{F}$, $B \in \mathcal{F} \Rightarrow A \cap B^c \in \mathcal{F}$, III^{ly} $B \cap A^c \in \mathcal{F}$

$\therefore A \Delta B \in \mathcal{F}$

- Smallest σ - field with E : $\{\Omega, \phi, E, E^c\}$
  " " " A, B : $\{\Omega, \phi, A, B, A^c, B^c, A \cup B, A \cup B^c, A \cap B, A \cap B^c, A^c \cup B, A^c \cap B, A^c \cap B^c, A^c \cup B^c, A \Delta B, (A \Delta B)^c\}$

<u>Example</u>: Complete $\{\Omega, \phi, \{1\}, \{2,3\}\}$ for $\Omega = \{1,2,3,4\}$

$$\mathcal{F} = \{\Omega, \phi, \{1\}, \{2,3\}, \{2,3,4\}, \{1,4\}, \{1,2,3\}, \{4\}\}$$

- Define the smallest σ-set of X, Y, Z as $\sigma(X, Y, Z)$, then,

$$\sigma(\Omega, \phi, A, B) = \sigma(\underbrace{A \cap B, A \cap B^c, A^c \cap B, A^c \cap B^c}_{\text{Disjoint sets that collectively exhaust } \Omega.})$$

Any union of the 4 sets, have their complements as a union of the other sets.

- <u>Probability Law (P)</u> :- $P : \mathcal{F} \longmapsto \mathbb{R}$ is a probability law if it follows the below axioms.

i) $P(\Omega) = 1$ (Normalization)
ii) $P(E) \geq 0 \quad \forall\, E \in \mathcal{F}$ (Non-negativity)
iii) If $A_1, A_2 \ldots$ are mutually exclusive / disjoint, then,

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

<u>Properties</u> :

i) $P(\phi) = 0$.

<u>Proof</u>: Let $A_i = \phi \ \forall\, i$

$$P\left(\bigcup_i A_i\right) = \sum_i P(A_i)$$

$$= P(\phi) = \sum P(\phi) \Rightarrow P(\phi) = 0$$

<u>Example</u>: Construct a probability law for rolling a die.

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

Defn $\begin{cases} P(\{i\}) = p_i \quad \forall\ i \in \{1,2,3,4,5,6\} \text{ st } \sum_{i=1}^{6} p_i = 1 \ \& \ p_i \geq 0 \\ P(A) = \sum_{i \in A} P(\{i\}), \quad A \subseteq \Omega \end{cases}$

$$P(\Omega) = \sum_{i \in \Omega} P(\{i\}) = \sum_{i \in \Omega} p_i = 1 \quad \text{(Normalization)}$$

$$p_i \geq 0 \Rightarrow \sum_{i \in A} p_i \geq 0 \ \forall\ A \subseteq \Omega \Rightarrow P(A) \geq 0 \ \forall\ A \subseteq \Omega$$
(Non-negativity)

By defn. additivity is satisfied.

ii) $P(A) \leq 1 \quad \forall\ A \subseteq \Omega$

iii) If $A \subseteq B$, then $P(A) \leq P(B)$

iv) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

v) $P(A) + P(A^c) = 1$

Proof:

iv) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$P(A \cup B) = P(A \cup (B \setminus A)) = P(A) + P(B \setminus A)$$

$$P(B) = P((A \cap B) \cup (B \setminus A))$$
$$= P(A \cap B) + P(B \setminus A)$$

$$\Rightarrow P(B \setminus A) = P(B) - P(A \cap B)$$

$$\Rightarrow P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

iii) If $A \subseteq B$, $P(A \cup B) = P(B)$

$$\Rightarrow P(B) = P(A) + P(B) - P(A \cap B)$$
$$= P(A) + P(B/A) + \cancel{P(A \cap B)} - \cancel{P(A \cap B)}$$
$$\Rightarrow P(B) = P(A) + P(B/A)$$
$$\Rightarrow \underline{P(B) \geq P(A)}$$

- <u>Continuity Of Probability</u> :-

- A continuous function has a definite limit for all points in its domain.

- A function $f: \mathbb{R} \mapsto \mathbb{R}$ if a sequence $x_n \rightarrow x^* \Rightarrow f(x_n) \longrightarrow f(x^*)$ as $n \longrightarrow \infty$ (Formal defn), ie,

$$\lim_{n \to \infty} f(x_n) = f\left(\lim_{n \to \infty} x_n\right)$$

- But, probability law is a function on sets. For such a function, continuity is defined as,

Let $A_1, A_2, A_3 \ldots \forall\, n \in \mathbb{N}$ be a sequence of events, then

$$\lim_{n \to \infty} P\left(\bigcup_{i=1}^{n} A_i\right) = P\left(\lim_{n \to \infty} \bigcup_{i=1}^{n} A_i\right)$$

Proof:

Claim 1: Consider 3 sets A_1, A_2, A_3. Then

$$A_1 \cup A_2 \cup A_3 = A_1 \cup (A_2 \setminus A_1) \cup (A_3 \setminus (A_1 \cup A_2))$$

As a union of disjoint sets B_1, B_2, B_3.

$$\Rightarrow B_i = A_i \setminus \bigcup_{k=1}^{i-1} A_k \longrightarrow \boxed{\bigcap_{i=1}^{\infty} B_i = \phi}$$

Proof of Claim 1:

To prove, $B_i \cap B_j = \phi \quad \forall\, i \neq j$

WLOG let $i < j$. Let $x \in B_i$

$$\Rightarrow x \in A_i \setminus \bigcup_{k=1}^{i-1} A_k \Rightarrow x \notin \bigcup_{k=1}^{i-1} A_k \ \& \ x \in A_i$$

Assume $x \in B_j \Rightarrow x \in A_j \setminus \bigcup_{k=1}^{j-1} A_k$

$$\Rightarrow x \notin \bigcup_{k=1}^{j-1} A_k$$

But we proved $x \in A_i \longrightarrow$ Contradiction

$\therefore x \in B_i \Rightarrow x \notin B_j$, if $i < j$

Ⅲly can be stated for $i > j$.

$\therefore x \in B_i \Rightarrow x \notin B_j$ if $i > j$

$$\Rightarrow \quad B_i \cap B_j = \phi \quad \forall \; i \neq j$$

Claim 2: $\displaystyle\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} B_i \quad \forall \; n \in \mathbb{N}$ —— ①

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i \quad \text{———} \quad ②$$

Proof of Claim 2:

For ①, if $n = 1$, $B_1 = A_1 \longrightarrow$ trivially proved

Assume true for $n = k$, ie, $\displaystyle\bigcup_{i=1}^{k} A_i = \bigcup_{i=1}^{k} B_k := C_k$

To prove for $n = k+1$,

$$\bigcup_{i=1}^{k+1} A_i = A_{k+1} \cup C_k$$

$$\bigcup_{i=1}^{k+1} B_i = B_{k+1} \cup C_k = \left(A_{k+1} \setminus \bigcup_{i=1}^{k} A_i\right) \cup C_k$$

$$= (A_{k+1} \setminus C_k) \cup C_k$$

$$= A_{k+1} \cup C_k = \bigcup_{i=1}^{k+1} A_i$$

$$\Rightarrow \boxed{\bigcup_{i=1}^{k+1} B_i = \bigcup_{i=1}^{k+1} A_i} \Rightarrow ① \text{ is valid.}$$

To extend the equality till ∞, we can prove that any element in LHS will belong to RHS. (Induction is not valid for $n \to \infty$)

Final Proof: To prove:

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} P\left(\bigcup_{i=1}^{n} A_i\right)$$

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = P\left(\bigcup_{i=1}^{\infty} B_i\right)$$ As defined and proved above

$$= \sum_{i=1}^{\infty} P(B_i) \quad [\text{Additivity Axiom}]$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} P(B_i)$$

Applies only because summation is defined of $\mathbb{R}$ and limit is well-defined for $\mathbb{R}$.

$$= \lim_{n\to\infty} P\left(\bigcup_{i=1}^{n} B_i\right) \quad [\text{Additivity Axiom}]$$

$$= \lim_{n\to\infty} P\left(\bigcup_{i=1}^{n} A_i\right) \quad [\text{Claim 2}]$$

$$\Rightarrow \underline{P\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} P\left(\bigcup_{i=1}^{n} A_i\right)}$$

Note: A sequence $\{x_n\}$ is said to converge to x^* iff $\forall\ \varepsilon > 0$, $\exists\ n_0 \in \mathbb{N}$ st $|x_n - x^*| < \varepsilon\ \forall\ n > n_0$.

Corollary 1:

i) $A_i \subseteq A_{i+1}\ \forall i \Rightarrow$ $\boxed{P\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} P(A_n)}$

ii) $A_i \supseteq A_{i+1}\ \forall i \Rightarrow$ $\boxed{P\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} P(A_n)}$

Corollary 2 : Union Bound for infinite events,

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} P(A_i)$$

→ For finite union, induction can be used for proof
For infinite, continuity.

→ Conditional Probability :-

- $P(A|B)$ is the probability of A given that B has occurred.
- $P(A|B) \propto P(A \cap B)$ If $P(A\cap B) = 0$, $P(A|B) = 0$

$= \quad P(A|B) = kP(A\cap B)$, if $A = B$, $P(B|B) = kP(B\cap B)$

$\Rightarrow \quad 1 = kP(B)$

$\Rightarrow \quad k = 1/P(B)$

$$\Rightarrow \quad P(A|B) = P(A\cap B)/P(B)$$

→ Independence :

- 2 events A and B are independent if,

$$P(A\cap B) = P(A)\cdot P(B)$$

$$\Rightarrow \quad P(A|B) = P(A)$$

- 3 events A, B, C are independent if,

$$P(A \cap B) = P(A)\,P(B)$$
$$P(B \cap C) = P(B)\,P(C)$$
$$P(A \cap C) = P(A)\,P(C)$$
$$P(A \cap B \cap C) = P(A)\,P(B)\,P(C)$$

- For n events,

$$P\left(\bigcap_{i \in I} A_i\right) = \prod_{i \in I} P(A_i) \quad \forall\, I \subseteq \{1, 2, 3 \ldots n\}$$

- A collection of sets is a partition of Ω if they are mutually exclusive and exhaustive.

$$\left(A_i \cap A_j = \phi \;\; \forall\, i \neq j\right), \; \left(\bigcup_{i=1}^{n} A_i = \Omega\right)$$

→ <u>Total Probability Theorem</u> :- Let $\{A_1, A_2, A_3 \ldots A_n\}$ be a partition of Ω, and B be any event. Then,

$$P(B) = \sum_i P(B \cap A_i)$$

$$\Rightarrow \; P(B) = \sum_i P(B \mid A_i)\,P(A_i), \quad \text{if } P(A_i) \neq 0 \;\; \forall\, i$$

<u>Proof</u>:

$$B = \bigcup_{i=1}^{n} (B \cap A_i)$$

$$\Rightarrow P(B) = P\left(\bigcup_{i=1}^{n} (B \cap A_i)\right)$$

$$\Rightarrow P(B) = \sum_{i=1}^{n} P(B \cap A_i)$$

→ Bayes' Theorem :-

$\{A_1, A_2, A_3 \ldots A_n\}$ be a partition of Ω. B be any event.

$$P(A_i | B) = \frac{P(A_i \cap B)}{P(B)}$$

$$\Rightarrow P(A_i | B) = \frac{P(B|A_i) P(A_i)}{\sum_i P(B|A_i) P(A_i)}$$

→ Multiplication Rule :

$$P(A_1 \cap A_2 \cap A_3 \ldots \cap A_n) = P(A_1) P(A_2 | A_1) P(A_3 | A_2 \cap A_1) \cdots P(A_n | A_1 \cap \ldots \cap A_{n-1})$$

→ Conditional Independence :

2 events A and B are said to be independent given C if,

$$P(A \cap B | C) = P(A|C) \cdot P(B|C)$$

- If $P(C) = 1$, then A and B are truly independent, ie,

$$P(A \cap B) = P(A) \cdot P(B)$$

→ Counting Techniques :-

Suppose there are n objects

i) No. of k-length sequences : $^nP_k = \dfrac{n!}{(n-k)!}$

ii) No. of k-length sets : $^nC_k = \dfrac{n!}{k!\,(n-k)!}$

iii) No. of ways of choosing r sets of sizes $n_1, n_2, n_3 \dots n_r$,

$$= {}^{n}C_{n_1} \cdot {}^{n-n_1}C_{n_2} \cdot {}^{n-n_1-n_2}C_{n_3} \cdots {}^{n-n_1-n_2 \cdots -n_{r-1}}C_{n_r}$$

$$= \frac{n!}{(n-n_1)!\,n_1!} \; \frac{(n-n_1)!}{(n-n_1-n_2)!\,n_2!} \; \cdots \cdots$$

$$= \frac{n!}{n_1!\,n_2!\,n_3! \cdots n_r!}$$

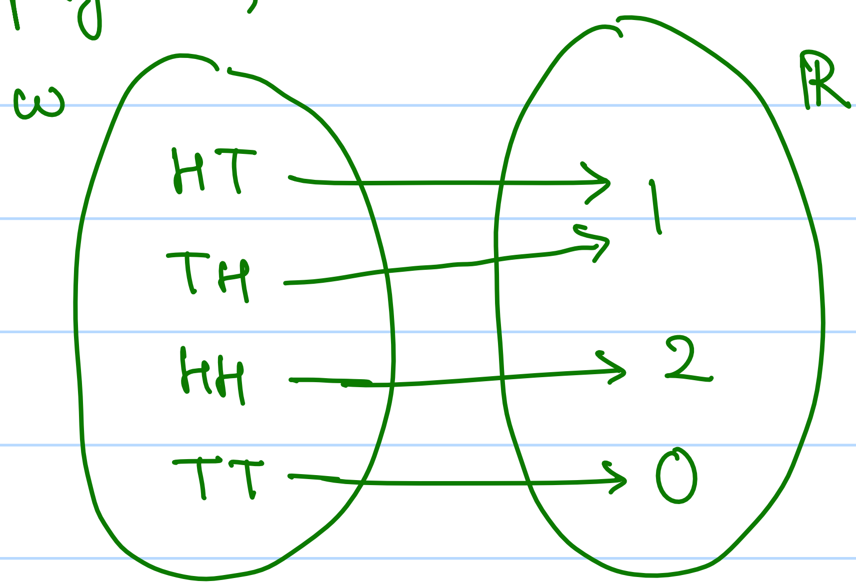
→ Random Variables :-

- A random variable is a function $X : \Omega \mapsto \mathbb{R}$ such that, $\{X \leq x\} \triangleq \{\omega \mid X(\omega) \leq x\} \in \mathcal{F}, \ \forall x \in \mathbb{R}$

Example: $\Omega = \{HT, TH, HH, TT\}$

$X : \Omega \mapsto \mathbb{R}$ st $X(\omega) =$ No. of heads

The mapping is,



$$\{x \le x_0\} = \begin{cases} \phi, & x_0 < 0 \\ \{TT\}, & x_0 \in [0,1) \\ \{HT, TH, TT\}, & x_0 \in [1,2) \\ \Omega, & x_0 \ge 2 \end{cases}$$

Theorem: $X : \Omega \mapsto \mathbb{R}$ on $\mathcal{F}$, then

i) $X^{-1}((-\infty, x)) \in \mathcal{F}$

ii) $X^{-1}([x_1, x_2]) \in \mathcal{F} \quad \forall\, x_1, x_2 \in \mathbb{R}$

iii) $X^{-1}(\{x_1\}) \in \mathcal{F} \quad \forall\, x_1 \in \mathbb{R}$

Proof of (i):

wkt $X^{-1}(-\infty, x] \in \mathcal{F} \quad \forall\, x \in \mathbb{R}$

$$(-\infty, x) = \bigcup_{n=1}^{\infty} \left(-\infty, x - \frac{1}{n}\right]$$

$$X^{-1}\left(\left(-\infty, x - \frac{1}{n}\right]\right) \in \mathcal{F} \quad \forall\, n \Rightarrow X^{-1}\left(\bigcup_{n=1}^{\infty} \left(-\infty, x - \frac{1}{n}\right]\right) \in \mathcal{F}$$

Proof of (ii) :

wkt $X^{-1}((-\infty, x_1]) \in \mathcal{F}$

$X^{-1}((-\infty, x_2)) \in \mathcal{F}$

Since subtraction is closed in $\mathcal{F}$, $\Omega / X^{-1}(-\infty, x_2) \in \mathcal{F}$

$\Rightarrow X^{-1}([x_2, \infty)) \in \mathcal{F}$

$\Rightarrow X^{-1}((-\infty, x_1]) \cap X^{-1}([x_2, \infty)) \in \mathcal{F}$

$\Rightarrow X^{-1}([x_2, x_1]) \in \mathcal{F}$

iv) $X^{-1}((x_1, x_2)) \in \mathcal{F}$

$$(x_1, x_2) = (x_1, \infty) \cap (-\infty, x_2)$$

$X^{-1}((-\infty, x_1]) \in \mathcal{F} \Rightarrow X^{-1}((x_1, \infty)) \in \mathcal{F}$

$X^{-1}([x_2, \infty)) \in \mathcal{F} \Rightarrow X^{-1}((-\infty, x_2)) \in \mathcal{F}$

$\Rightarrow X^{-1}((x_1, \infty) \cap (-\infty, x_2)) \in \mathcal{F}$

$\Rightarrow X^{-1}((x_1, x_2)) \in \mathcal{F}$

- **Borel σ-Algebra :**

Smallest σ field on $\mathbb{R}$ containing sets of the form $(-\infty, x] \ \forall x \in \mathbb{R}$, ie,

$$B(\mathbb{R}) = \sigma((-\infty, x] \ \forall x \in \mathbb{R})$$

- Contains all possible subsets of $\mathbb{R}$.

- Cumulative Distribution Function (CDF) :-

- A CDF is a function $F_X : \mathbb{R} \mapsto [0,1]$ such that,

$$F_X(x) = P(\{X < x\})$$

Example: For,

$$\{X \leq x\} = \begin{cases} \phi , & x < 0 \\ \{TT\}, & x \in [0,1) \\ \{TT, HT, TH\} . & x \in [1,2) \\ \Omega, & x \geqslant 2 \end{cases}$$

$$F_X(x) = \begin{cases} 0 , & x < 0 \\ 1/4 , & x \in [0,1) \\ 3/4 , & x \in [1,2) \\ 1, & x \geqslant 2 \end{cases}$$

Example: Let $X(\omega) = c$, $c \in \mathbb{R}$. Then,

$$\{X \leq x\} = \begin{cases} \phi , & x < c \\ \Omega , & x \geqslant c \end{cases}$$

$$\Rightarrow \qquad F_X(x) = \begin{cases} 0 , & x < c \\ 1 , & x \geqslant c \end{cases}$$

- **Indicator R.V :-**

Consider an $A \in \mathcal{F}$. Then $I_A : \Omega \mapsto \mathbb{R}$ such that,

$$I_A(\omega) = \begin{cases} 1, & \omega \in A \\ 0, & \omega \notin A \end{cases}$$

$$\Rightarrow \{I_A < x\} = \begin{cases} \phi, & x < 0 \\ A^c, & x \in [0,1) \\ \Omega, & x \geqslant 1 \end{cases}$$

$\{I_A < x\} \in \mathcal{F}\ \forall\ x \in \mathbb{R}$. So I_A is a valid r.v.

I_A is an indicator r.v for the event A.

- $B_1 \cap B_2 = \phi$, then,

$$I_{B_1 \cup B_2}(\omega) = I_{B_1}(\omega) + I_{B_2}(\omega) \ \forall\ \omega \in \Omega$$

$B_1, B_2, B_3 \ldots$ form a partition of Ω, then,

$$I_{\bigcup_{i=1}^{n} B_i} = \sum_{i=1}^{n} I_{B_i} = 1$$

- **Theorems of CDF :-**

i) $\lim_{x \to \infty} F_X(x) = 1$, $\lim_{x \to -\infty} F_X(x) = 0$

$$\lim_{x \to \infty} F_X(x) = \lim_{x \to \infty} P_X(X \leqslant x) = P(\mathbb{R}) = \underline{1}$$

$$\lim_{x \to -\infty} F_X(x) = \lim_{x \to -\infty} P_X(X \leqslant x) = P(\phi) = \underline{0}$$

P_X – Probability law of X

2) $x < y \Rightarrow F_X(x) \leq F_X(y)$

3) $F_X(x)$ is always right continuous, ie,

$$\lim_{\varepsilon \to 0^+} F_X(x+\varepsilon) = F_X(x) \quad \forall\, x \in \mathbb{R}$$

<u>Proof:</u>

$$\lim_{n\to\infty} F_X\left(x+\frac{1}{n}\right) = \lim_{n\to\infty} P\left(X \leq x+\frac{1}{n}\right)$$

Let $\left\{X \leq x+\frac{1}{n}\right\} = B_n \Rightarrow B_n$ is a decreasing sequence

We know by continuity of probability $P\left(\bigcap_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} P(B_n)$

$$\Rightarrow \lim_{n\to\infty} F_X\left(x+\frac{1}{n}\right) = P\left(\bigcap_{n=1}^{\infty}\left\{X \leq x+\frac{1}{n}\right\}\right)$$

$$= P\left(\{X \leq x\}\right) = F_X(x)$$

$$\therefore \lim_{n\to\infty} F_X\left(x+\frac{1}{n}\right) = F_X(x)$$

$$\Rightarrow \underline{RHL = F_X(x)}$$

4) $F_X(x) - \lim_{\varepsilon\to 0^+} F_X(x-\varepsilon) = P(\{X = x\})$

5) $P_X(x_1 \leq X \leq x_2) = P(\omega : x_1 \leq X(\omega) \leq x_2)$

$= P(X$

→ **Discrete Random Variable** :-

A random variable is discrete if $Range(X) \subseteq \mathbb{R}$ is finite / countably infinite.

- **Probability Mass Function** :

$P_X : X \longmapsto [0,1]$, given by

$$P_X(x) = P(X = x) = P(\{\omega : X(\omega) = x\})$$

- **Lemma:** For an rv st $x \in \{x_1, x_2, x_3 \dots\}$

i). $\displaystyle\sum_{i=1}^{\infty} P_X(x_i) = \sum_{i=1}^{\infty} P(X = x_i)$

$$= P\left(\bigcup_{i=1}^{\infty} \{X = x_i\}\right) \quad [\text{Additivity}]$$

$$= P(\{X \in \{x_1, x_2, x_3 \dots\}\}) = P(\Omega)$$

$$\therefore \sum_{i=1}^{\infty} P_X(x_i) = \underline{\underline{1}}$$

- **CDF of a Discrete R.V** :-

$$F_X(x) = P(\{X \leq x\}) = P\left(\bigcup_{i \in \mathbb{N} : x_i \leq x} \{X = x_i\}\right)$$

$$\Rightarrow \boxed{F_X(x) = \sum_{i \in \mathbb{N} : x_i \leq x} P_X(x_i)}$$

- **Functions on R.V :-**

Let $X : \omega \longmapsto \mathbb{R}$ and $Y : \mathbb{R} \longmapsto \mathbb{R}$, ie a function on X. Y is also a r.v.

Proof: $Y^{-1}(B) = \{ y : Y(y) \in B \} \ \forall \ B \in \mathcal{B}(\mathbb{R})$

$= \{ X(x) : Y(X(x)) \in B \}$

$X^{-1}(Y^{-1}(B)) = \{ x : Y(X(x)) \in B \}$

$= \{ x : X(x) \in Y^{-1}(B) \}$

Since $X^{-1}(S) \in \mathcal{F} \ \forall \ S \in \mathcal{B}(\mathbb{R})$

$Y^{-1}(B) \in X(\mathcal{F}) = \mathcal{F}' \ \forall \ B \in \mathcal{B}(\mathbb{R})$

$\Rightarrow$ Y is a valid r.v

Better proof needed.

- PMF of y : $P_Y(Y = y) = \sum_{Y(x) = y} P_X(X = x)$

- **Expectation:** $E[X] = \sum_{x \in \mathbb{R}} x P_X(x)$

- If $y = Y(x)$,

Law of the Unconscious Statistician : $E[Y] = \sum_{x \in \mathcal{X}} Y(x) P_X(x)$

Proof: $E[Y] = \sum_{y \in \mathcal{Y}} y P_Y(y) = \sum_{y \in \mathcal{Y}} y \sum_{Y(x) = y} P_X(x)$

$$= \underbrace{\sum_{y \in \mathcal{Y}} \sum_{Y(x) = y}}_{\text{All possible } x} Y(x) P_X(x) = \sum_{x \in \mathcal{X}} Y(x) P_X(x)$$

- **Variance:**

$$\boxed{Var[X] = E[(X - E[X])^2]}$$

$$\Rightarrow \boxed{Var[X] = E[X^2] - E[X]^2}$$

- n^{th} Moment of an RV $= E[X^n]$. (defn)

- **Examples of RV:-**

1) **Bernoulli RV:** Binary o/p Rv. (ex: Coin Toss)

$$P(\{H\}) = p, \quad P(\{T\}) = 1-p$$

$$X(H) = 1, \quad X(T) = 0 \longrightarrow X \text{ is a Bernoulli R.v}$$

$$E[X] = p, \quad Var[X] = p - p^2$$

2) **Binomial RV:** (ex: Coin Tossed n times)

Any event ω : Sequence of H's and T's of length n.

$P(\{H\}) = p$, $P(\{T\}) = 1-p$. $X(\omega) =$ No. of heads in ω

$$\Rightarrow P_X(k) = {}^nC_k \, p^k (1-p)^{n-k} \quad \forall \, k \in \mathbb{N}$$

$$E[X] = np \qquad Var[X] = np(1-p)$$

3) **Geometric RV**: (ex: Toss a coin till heads)

$X(\omega)$ = No. of coin tosses in ω, to get a head ;)

$$P_X(k) = p(1-p)^{k-1} \qquad E[X] = \frac{1}{p} \qquad Var[X] = \frac{1-p}{p^2}$$

4) **Poisson RV**: $X \in \{0, 1, 2, \dots\}$

$$P_X(k) = \frac{e^{-\lambda}\lambda^k}{k!}, \quad k = 0, 1, 2 \dots \qquad \lambda \in \mathbb{R}$$

$$E[X] = Var[X] = \lambda.$$