LA Assignment - 4 - 13/2/25

1. If $A_{n\times n}$ over F, and the rows of A form linearly independent vectors in F, then prove that $A_{n\times n}$ is invertible.

Let E_i in F^n be a vector of n elements in it such that the i^{th} scalar is 1 and the others are zero.

Clearly the set of $E_i \ \forall\, i \leq n$ is a linearly independent vector, and also forms the basis of F^n, since any vector in F^n can be represented as a linear combination of the vectors $E_i \ \forall\, i$.

Let $\bar{\alpha}_1, \bar{\alpha}_2 \ldots \bar{\alpha}_n$ be the row vectors of A. Since $\bar{\alpha}_1, \bar{\alpha}_2 \ldots \bar{\alpha}_n$ are n linearly independent vectors, they span the full vector space F_n. Therefore $\exists$ B_{ij} in F such that

$$E_i = \sum_{j=1}^{n} B_{ij}\, \bar{\alpha}_j$$

ie E_i is a linear combination of the vectors $\bar{\alpha}_i \ \forall\, i$,

If we form a matrix B using B_{ij}'s, we get

$$I = BA$$

E_i is the i^{th} row of the identity matrix I.

$\therefore \exists$ a matrix B for A st $BA = I \iff A$ is invertible

$\therefore$ If the row vectors of A are linearly independent, A is invertible.

2. If W_1 and W_2 are finite dimensional subspaces of V, then prove that $W_1 + W_2$ is finite dimensional and $\dim(W_1) + \dim(W_2)$
$= \dim(W_1 \cap W_2) + \dim(W_1 + W_2)$.

Let $\{\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3 \dots, \bar{\alpha}_n\}$ be a basis vector in $W_1 \cap W_2$,
$\Rightarrow \dim(W_1 \cap W_2) = n$

Since $W_1 \cap W_2$ is a subspace of W_1 and W_2, every independent set of vectors in $W_1 \cap W_2$ is finite and every basis of $W_1 \cap W_2$ is part of a basis in W_1 and W_2

ie, $\exists$ a basis in W_1 that is
$$\{\bar{\alpha}_1, \bar{\alpha}_2 \dots, \bar{\alpha}_n, \bar{\beta}_1 \dots \bar{\beta}_m\} \Rightarrow \dim(W_1) = n+m$$
and $\exists$ a basis in W_2 that is
$$\{\bar{\alpha}_1, \bar{\alpha}_2 \dots \bar{\alpha}_n, \bar{\gamma}_1, \bar{\gamma}_2 \dots \bar{\gamma}_k\} \Rightarrow \dim(W_2) = n+k$$

Also,
$$\{\bar{\alpha}_1, \bar{\alpha}_2 \dots \bar{\alpha}_n, \bar{\beta}_1, \bar{\beta}_2, \dots \bar{\beta}_m, \bar{\gamma}_1, \bar{\gamma}_2 \dots \bar{\gamma}_k\} \in \underline{\underline{W_1 + W_2}}$$

Suppose,
$$\sum_{i=1}^{n} x_i \bar{\alpha}_i + \sum_{j=1}^{m} y_j \bar{\beta}_j + \sum_{l=1}^{k} z_l \bar{\gamma}_l = 0 \quad \text{for some } x_i, y_j, z_l \in F \ \forall\, i, j, l$$
$$\Rightarrow \sum_{i=1}^{n} x_i \bar{\alpha}_i + \sum_{j=1}^{m} y_j \bar{\beta}_j = -\sum_{l=1}^{k} z_l \bar{\gamma}_l$$
$$\Rightarrow \sum_{l=1}^{k} z_l \bar{\gamma}_l \text{ belongs to } W_1$$

$$\Rightarrow \sum_{l=1}^{k} z_l \bar{\gamma}_l = \sum_{i=1}^{n} c_i \bar{\alpha}_i \quad \text{for some } c_i \in F \ \forall i$$

$$\Rightarrow \sum_{l=1}^{k} z_l \bar{\gamma}_l + \sum_{i=1}^{n} (-c_i) \bar{\alpha}_i = 0$$

But since $\{\bar{\alpha}_1, \bar{\alpha}_2 \ldots \bar{\alpha}_n, \bar{\gamma}_1, \bar{\gamma}_2 \ldots \bar{\gamma}_k\}$ is independent,

$$\Rightarrow z_l = 0 \ \forall \ l$$

$$\Rightarrow \sum_{i=1}^{n} x_i \bar{\alpha}_i + \sum_{j=1}^{m} y_j \bar{\beta}_j = 0$$

Since $\{\bar{\alpha}_1, \bar{\alpha}_2 \ldots \bar{\alpha}_n, \bar{\beta}_1, \bar{\beta}_2 \ldots \bar{\beta}_m\}$ is also independent

$$\Rightarrow x_i = 0 \ \forall \ i \text{ and } y_j = 0 \ \forall \ j$$

$$\therefore \sum_{i=1}^{n} x_i \bar{\alpha}_i + \sum_{j=1}^{m} y_j \bar{\beta}_j + \sum_{l=1}^{k} z_l \bar{\gamma}_l = 0 \Rightarrow x_i = 0 \ \forall i, \ y_j = 0 \ \forall j, \ z_l = 0 \ \forall \ l$$

$\Rightarrow \{\bar{\alpha}_i, \bar{\beta}_j, \bar{\gamma}_l \mid i \leq n, j \leq m, l \leq k \text{ and } i, j, k \in \mathbb{N}\}$ is an independent set in $W_1 + W_2$

The above set is also a basis in $W_1 + W_2$ since it is formed by the bases of W_1 and W_2

$\Rightarrow \dim(W_1 + W_2) = n + m + k$, ie <u>$\dim(W_1 + W_2)$ is finite</u>

$\dim(W_1) = n + m$, $\dim(W_2) = n + k$, $\dim(W_1 \cap W_2) = n$

$$\Rightarrow \dim(W_1) + \dim(W_2) = n + m + n + k$$
$$= n + (n + m + k)$$
$$= \underline{\underline{\dim(W_1 \cap W_2) + \dim(W_1 + W_2)}}$$

Hence Proved.