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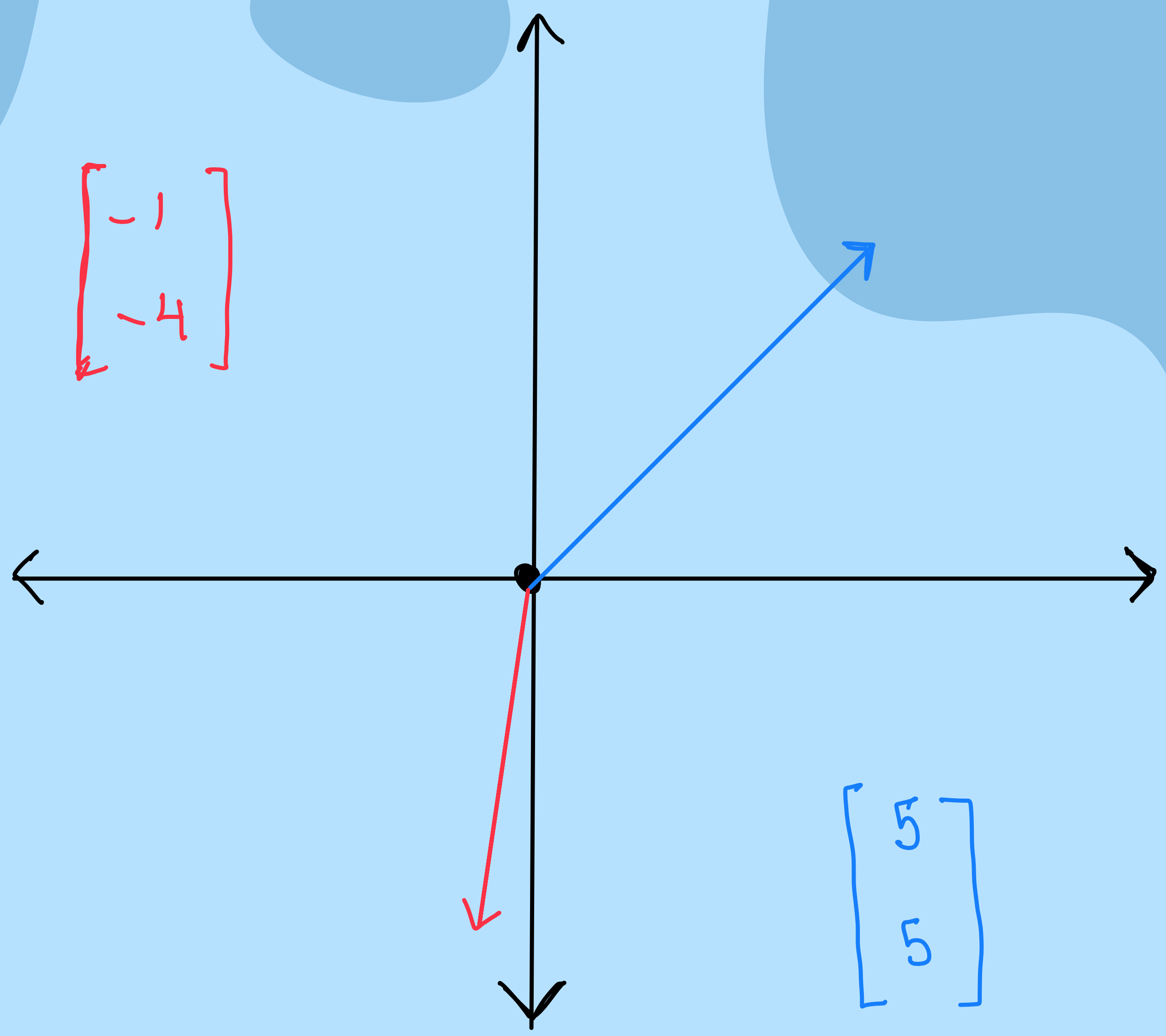
Book: Linear Algebra by Hoffman & Kunze.



Problems will be of the <u>Proving</u> type.

<u>Grading</u>

Tutorial Quiz	=	15 %	
Assignment	=	15 %	20 - 25 %
Quiz 1	=	10 %	10 %
Quiz 2	=	10 %	
Midsem	=	20 %	15 - 20 %
Endsem	=	30 %	

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Linear Algebra

$\rightarrow \infty =$ functional algebra

- It is the study of linear maps on finite dimensional vector space.

↳ function / relation

→ Problems solved by Linear Algebra :-

1) System of linear equations :-

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \dots + a_{1n}x_n = b_1$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 \dots + a_{mn}x_n = b_m$$

- a_{11}, a_{12}, a_{m1}, etc are called the coefficients
- This is a system of lin. eqs as all the individual equation are of max degree 1
  - ⇒ line in 2D
  - ⇒ plane in 3D
- To solve this problem, we need to find a list of numbers $(s_1, s_2, s_3 \dots s_n)$ that satisfy the above system of equation.

list is ordered (also tuple)
set is unordered.

- Such a list of numbers is termed as a solution to this system. Set of solutions = Solution set

- A solution may not exist for a system, or a system may have ∞ solutions.

- When the no. of independent variables (n) is large, it is difficult to systematically visualize the system.

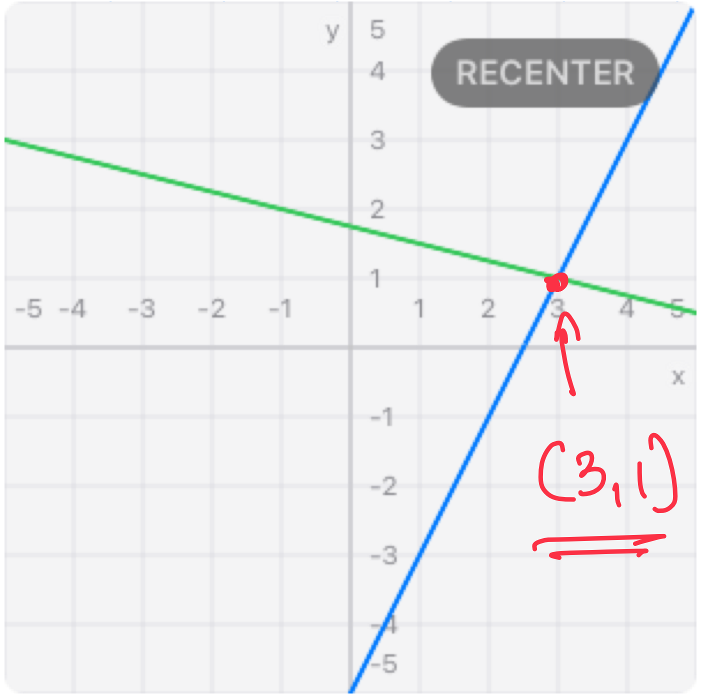
$$① \text{ — } 2x_1 - x_2 = 5 \qquad (n=2)$$
$$② \text{ — } x_1 + 4x_2 = 7$$

This system is easy to solve by elimination and also easy to visualize as well

$$① \times 4 = \quad 8x_1 - 4x_2 = 20$$
$$x_1 + 4x_2 = 7$$
$$\overline{9x_1 = 27}$$
$$\Rightarrow x_1 = 3$$
$$3 + 4x_2 = 7$$
$$\Rightarrow x_2 = 1$$

$\therefore$ (3, 1) is a solution to the given system.

The system can also be represented geometrically and solved.



- **Echelon Form :**

Assume a system of equations,

$$
\begin{aligned}
c_{11}x_1 + c_{12}x_2 + c_{13}x_3 + c_{14}x_4 &= b_1 \\
c_{22}x_2 + c_{23}x_3 + c_{24}x_4 &= b_2 \\
c_{33}x_3 + c_{34}x_4 &= b_3 \\
c_{44}x_4 &= b_4
\end{aligned}
$$

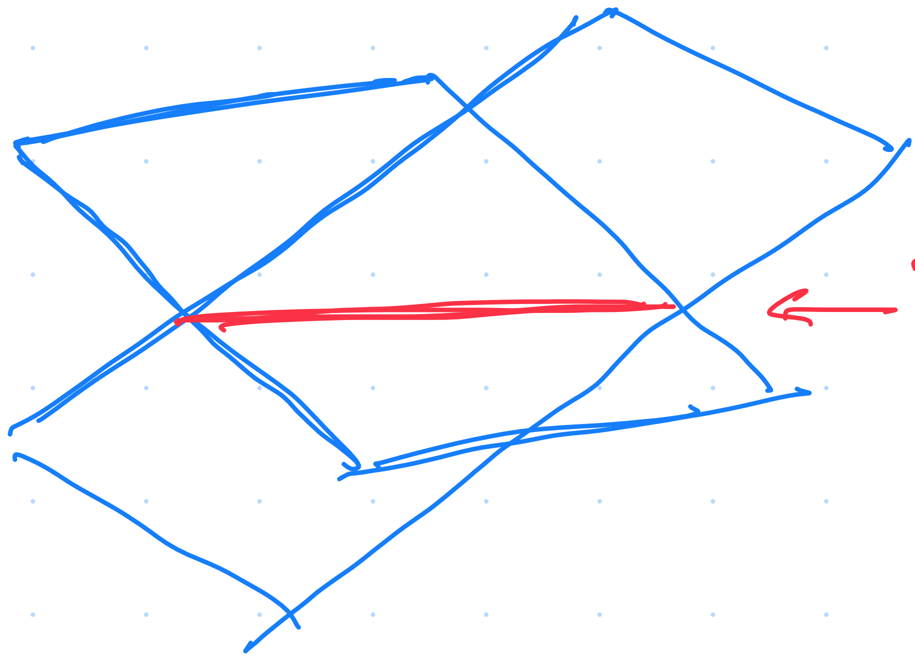
This system is said to be in Echelon form

- A system in Echelon form can be solved easily using Gaussian Elimination.

- **Condition for No Solutions :-**

- In 2D, if the lines (equations) are in ||lel, then $\nexists$ no solution for that system.

- For 2 equations in 3D, $\exists$ infinite solutions for the system.



The solution set is a line

- **Representation of A System of Linear Eqns in Matrices** :-

- Take the system of equations first mentioned. (n no. of eqns).
- That system can be represented in matrix form as,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$\Rightarrow \quad AX = B$$

Coeff matrix   Variable matrix   Output matrix

This is like a function on X.

X is transformed

- Also, take,

$$\bar{V}_1 = \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} \quad \bar{V}_2 = \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix} \cdots \bar{V}_n = \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

The system can be rep.d as,

$$x_1 \cdot \bar{V}_1 + x_2 \cdot \bar{V}_2 + x_3 \cdot \bar{V}_3 \cdots + x_n \cdot \bar{V}_n = B$$

ie, B is a linear combination of vectors $V_1, V_2 \ldots V_n$

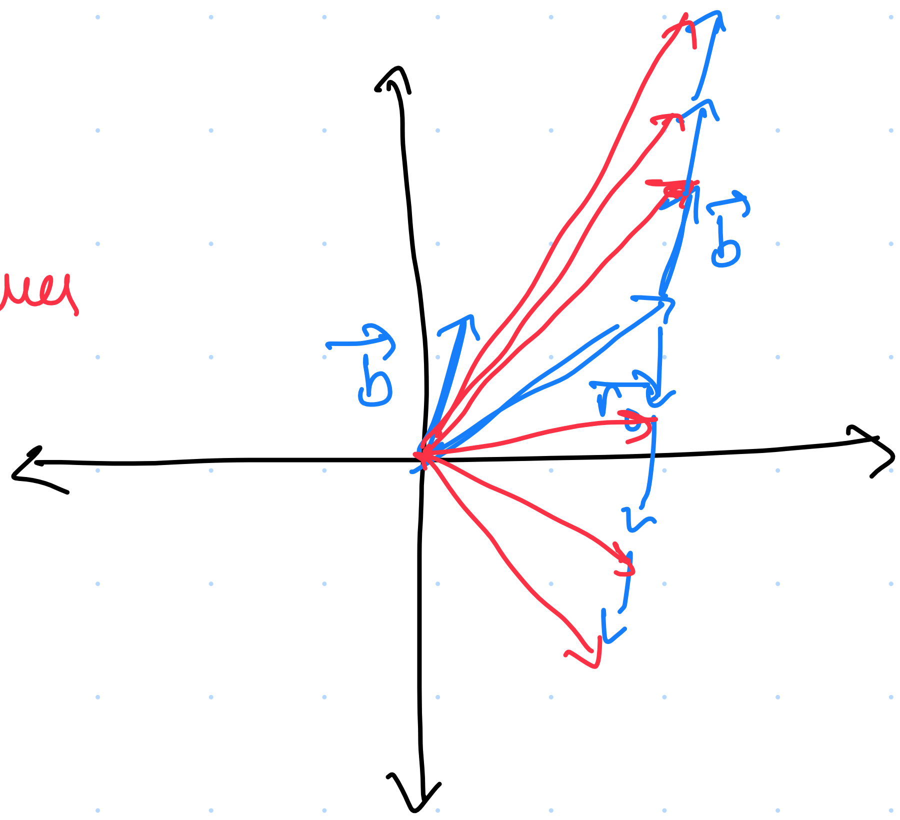
2) Finding the Parameters of a Linear Combination :-

- Given the vectors $\bar{V}_1, \bar{V}_2 \ldots \bar{V}_n$, is $\bar{B}$ a linear combination of these vectors?
- This combination of vectors is dealt with in a vector space.
- A vector need not always be a arrow with a length (Physics definition). Your vector space can be defined by any arbitrary axioms.

- Linear Combination of vectors in a vector space cannot reach higher dimensions.

Q. Take 2 vectors $\vec{r_0}$ and $\vec{b}$ and let $\vec{r} = \vec{r_0} + \lambda\vec{b}$, where λ is a parameter. How does the tip of $\vec{r}$ behave?

The tip of $\vec{r}$ behaves like a line.



→ Linear Map :-

- A map is a relation from one set to another.

$$f : A \longrightarrow B$$

f is a map from A to B.

- If :

$$f(c_1 x_1 + c_2 x_2) = c_1 f(x_1) + c_2 f(x_2)$$

holds true

linearity

Then, the mapping is said to be a linear map.

- Linear maps keep the ratios intact, ie, any ratio in the input space is retained in the output space.

ie,

$$a : b = f(a) : f(b)$$

- Matrix multiplication is a linear operation.

$$A(\lambda_1 X_1 + \lambda_2 X_2) = \lambda_1 A X_1 + \lambda_2 A X_2$$

where λ_1, λ_2 are scalars. (ie, plain numbers)

→ Vector Space :-

A set of objects where the following axioms hold.

1) Addition of 2 objects is valid, commutative, closed

$\bar{v}_1 + \bar{v}_2 = \bar{v}_2 + \bar{v}_1$

2) Zero vector exists

closed = vector + vector = vector

3) Inverse vector exists

4) Scalar multiplication is valid, distributive, closed.

c = scalar

$c(v_1 + v_2) = cv_1 + cv_2$

- Set of Real Numbers ($\mathbb{R}$) is a vector space.

- <u>Dimension</u> :-

The minimum no. of vectors, whose set of all possible linear combinations fill the entire space.

- The vectors taken need not be orthogonal always
- 2 vectors that are not ||lel or anti ||lel, can fill entire 2D space.
- 3 vectors which are such that they all do not lie on the same plane, can cover the entire 3D space.

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→ <u>Linear Algebra</u> :-

- <u>Syllabus</u>: Linear Equations, Vector Spaces, Linear Transformati-ons,
- <u>Textbook</u>: Hoffman & Kunze (Linear Algebra)
Algebra by Artin, LA by Kumaresan, LA done right.

→ **Field:-** $(F, +, \cdot)$

A set of elements (scalars) that follow 2 binary operations, addition $(+)$ and multiplication $(\cdot)$.

- The binary operations $(+)$ and $(\cdot)$ can be defined in any way, but they must follow the below properties:

1) Addition is commutative:

$$x + y = y + x \quad \forall\ x, y \in F$$

2) Addition is associative:

$$(x+y) + z = x + (y + z) \quad \forall\ x, y, z \in F$$

3) $\exists$ an element 0 (zero) such that,

$$x + 0 = x \quad \forall\ x \in F$$

0 is called the Additive identity of F.

4) $\forall\ x \in F\ \exists\ (-x) \in F$ such that,

$$x + (-x) = 0$$

$(-x)$ is the Additive Inverse of x

5) Multiplication is commutative,

$$x \cdot y = y \cdot x \quad \forall\ x, y \in F$$

6) Multiplication is associative,

$$(x \cdot y) \cdot z = x \cdot (y \cdot z) \quad \forall\ x, y, z \in F$$

7) $\exists$ a non-zero element 1 (one) $\in F$, such that,

$$x \cdot 1 = x \quad \forall\ x \in F$$

1 is the multiplicative idetity of F

8) $\forall x \in F$ st $x \neq 0$, $\exists\, x^{-1} \in F$ such that,

$$x \cdot x^{-1} = 1$$

x^{-1} is the multiplicative inverse of x.

9) Multiplication is distributive over addition,

$$x(y+z) = x \cdot y + x \cdot z \quad \forall\, x, y, z \in F$$

- Due to the above properties, the smallest field that can exist is $(\{0,1\}, +, \cdot)$ where $+, \cdot$ are defined as,

+	0	1
0	0	1
1	1	0

XOR binary op.

·	0	1
0	0	0
1	0	1

AND binary op.

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→ System of Linear Equations :-

m eqns
$$\begin{cases} A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n = y_1 \\ A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n = y_2 \\ \vdots \\ A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n = y_m \end{cases} \quad (1.1)$$

- The above equations have max. degree of 1.
- $A_j, x \in (F, +, \cdot)$, to make the equations consistent.
- In a system of linear eqns, the unknown scalar must satisfy all the linear eqns.
- If $y_k = 0 \;\forall\; k \in \mathbb{N}$ in the system, then the system is said to be homogeneous.

Q. $2x_1 - x_2 + x_3 = 0$ $\Rightarrow$ $2x_1 - x_2 + x_3 = 0$

$x_1 + 3x_2 + 4x_3 = 0$ $\xrightarrow{\times 2}$ $\overset{(-)}{2x_1} \overset{(-)}{+ 6x_2} \overset{(-)}{+ 8x_3} = 0$

$-7x_2 - 7x_3 = 0$

$\Rightarrow \underline{x_2 = -x_3}$

$x_1 - 3x_3 + 4x_3 = 0$

$x_1 + x_3 = 0$

$x_1 = -x_3$

Solution is of the form : $\underline{(-x_3, -x_3, x_3)}$

Clearly there are ∞ solns.

- Assume we multiply the equations each by a constant $c_1, c_2 \cdots c_m$, and add all the eq.ns together, we get,

$$(c_1 A_{11} + c_2 A_{21} + \cdots c_m A_{m1}) x_1 + (c_1 A_{12} + c_2 A_{22} + \cdots c_m A_{m2}) x_2 + \cdots\cdots + (c_1 A_{1n} + c_2 A_{2n} \cdots + c_m A_{mn}) x_n \quad (1.2)$$

$$= c_1 y_1 + c_2 y_2 + c_3 y_3 \cdots + c_m y_m$$

→ Linear Combination of eqns.

- Any solution $(x_1, x_2, \ldots x_n)$ that satisfies the system 1.1 will also satisfy the equation 1.2.
- But the inverse is not true, since the combination is also a linear equation with ∞ solutions.
- Let,

$$c_1 A_{11} + c_2 A_{12} \cdots\cdots = B_1$$
$$c_1 A_{21} + c_2 A_{22} \cdots\cdots = B_2$$

by varying $c_1, c_2 \cdots c_n$, we can get new B_{11}, B_{12}, B_{21}, etc.

By making that into another system,

$$B_{11} x_1 + B_{12} x_2 \cdots + B_{1n} x_n = z_1$$
$$B_{21} x_1 + B_{22} x_2 \cdots\cdots + B_{2n} x_n = z_2 \quad (1.3)$$
$$\vdots$$
$$B_{k1} x_1 + B_{k2} x_2 \cdots\cdots + B_{kn} x_n = z_k$$

This is a system of linear combinations of system 1.1

- The solution to 1.1 will also be a solution to system 1.3. But the solutions of 1.3 need not be a solution to 1.1.

- If 1.1 can be written as a system of linear combinations of system 1.3, then all the solutions of 1.3 are also solutions of 1.1, ie, the systems are equivalent.

- **Matrix Representation of Linear Systems :-**

$$\underbrace{\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ \vdots & \vdots & & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix}_{m \times n}}_{\text{coefficient matrix}} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}_{m \times 1} \qquad 1.4$$

- The matrix equation 1.4 is a representation of the system 1.1.

→ **Matrices :-**

- A matrix is defined over a field.
- It is a function or map from a pair of countable numbers to the scalars in a field.

$$M : (i, j) \longrightarrow (F, +, \cdot), \quad i, j \in \mathbb{N}, \quad 1 \leq i \leq m, \; 1 \leq j \leq n$$

- **Row Operations :-**

Operations made on the rows of matrix A, such that the resulting matrix B is equivalent to A.

- There are 3 such elementary operations :

1) Multiplying a row by a non-zero scalar
2) Multiplying a row with a scalar and adding it to another rows.
3) Swapping 2 rows.

- If we assume our matrix is $A_{m \times n} = [A_{ij}]$, then the rows can be defined in function notation as,

1) $e(A_{ij}) = \begin{cases} cA_{ij}, & i = r \\ A_{ij}, & \text{otherwise} \end{cases}$

r, s = rows upon which the operation are performed.

2) $e(A_{ij}) = \begin{cases} A_{ij} + cA_{sj}, & i = r \\ A_{ij}, & \text{otherwise} \end{cases}$

3) $e(A_{ij}) = \begin{cases} A_{rj}, & i = s \\ A_{sj}, & i = r \\ A_{ij}, & \text{otherwise} \end{cases}$

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- **Theorem:** To each row operation e, $\exists$ an elementary row operation e' such that $e'(e(A)) = e(e'(A)) = A$

e' is the 'inverse' of e

- Let A, B be 2 $m \times n$ matrices, A is said to be row equivalent to B, if by doing a finite number of row operation on A, B can be obtained, i.e.,

$$B = e_k(\ldots\ldots e_2(e_1(A))\ldots)$$

$$\Rightarrow A = e_1'(e_2'(e_3' \cdots e_k'(B)))\cdots)$$

Thus, if A is row-equivalent to B, B is also row-equivalent to A.

Q. Prove that Row-Equivalence is an Equivalence relation.

- Theorem:

If A and B are row-equivalent matrices, the homogenous system of linear eqns $AX = 0$ and $BX = 0$ have exactly the same solution.

- Elementary row operations will not change the solution of the system

$$\left.\begin{aligned} A \cdot X &= 0 \\ \Rightarrow e(A) \cdot X &= 0 \\ \Rightarrow e_2(e(A)) \cdot X &= 0 \end{aligned}\right\} \text{Same solution of } X$$

Q. $A = \begin{bmatrix} 2 & -1 & 3 & 2 \\ 1 & 4 & 0 & -1 \\ 2 & 6 & -1 & 5 \end{bmatrix}$

① $-$ 2②

$$A = \begin{bmatrix} 0 & -9 & 3 & 4 \\ 1 & 4 & 0 & -1 \\ 2 & 6 & -1 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 0 & 1 & -11/3 \\ 1 & 0 & 0 & 17/3 \\ 0 & 1 & 0 & -5/3 \end{bmatrix}$$

$\underline{\underline{B}}$

B is a row-reduced matrix

- A $m \times n$ matrix is said to be row-reduced if

a) The first non-zero entry in each non-zero row is 1.

b) Each column which contain the leading non zero element (1) all other entries as zero.

Identity matrix is row reduced.

- Theorem: Every $m \times n$ matrix over the field is row-equivalent to a row reduced matrix. (IQ. Proof)

- **Row Reduced Echelon Matrices :-** A matrix $R_{m \times n}$ is said to be a row-reduced Echelon matrix if.

a) R is row-reduced

b) All the zero rows are below/after non-zero rows.

c) If the rows $1, 2, \dots r$ are non zero rows of R and if the leading non-zero entry of row i occurs in column k_i, $i = 1, 2, 3$ then $k_1 < k_2 < k_3 \dots < k_r$. (Upper Δ matrix)

Either every entry in $R_{m \times n}$ is 0, or $\exists$ $r \in \mathbb{N}$, $1 \leq r \leq m$, & $k_1, k_2, \dots k_r \in \mathbb{N}$ with $1 \leq k_i \leq n$ (Same definition in mathe-matical language.)

a) $R_{ij} = 0 \quad \forall \ i > r$ 	c) $R_{ik_j} = \delta_{ij} \quad \begin{matrix} 1 \le i \le r \\ 1 \le j \le r \end{matrix}$

b) $R_{ij} = 0 \quad \forall \ j < k_i$ 	d) $k_1 < k_2 < k_3 \dots < k_r$

δ_{ij} is called conical delta matrix and is defined as,

$$\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \quad \text{(Identity)}$$

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- **Theorem**:-

Every $m \times n$ matrix A is row equivalent to a row reduced Echelon matrix.

- Every homogenous system $AX = 0$ will have a trivial solution 0. To see if there are any other soln. s,

- If the no. of non zero rows is r and the RREF of A is R, then each unknown variable x_{k_i} (where k_i is the column in which the i^{th} leading one occurs) will only appear in the i^{th} row.

- Since there are r k_i's, there are r unknown dependent variables. $(n-r)$ variables will never have the leading 1 coefficient.

↳ free variables

independent of each other

$$\Rightarrow \quad x_{k_1} + \sum_{j=1}^{n-r} C_{1j} v_j = 0$$

$$x_{k_2} + \sum_{j=1}^{n-r} C_{2j} v_j = 0$$

$$\vdots$$

$$x_{k_r} + \sum_{j=1}^{n-r} C_{rj} v_j = 0$$

where v_j are the $(n-r)$ variables and C is any scalar.

independent

ex: $R = \begin{bmatrix} 0 & 1 & -3 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$ $RX = 0$

There are 2 possible k_i's, 2 & 4

$\Rightarrow$ x_2 and x_4 are dependent variables

x_1, x_3, x_5 are independent variables $\rightarrow v$

$$x_2 + 0x_1 + (-3)x_3 + (1/2)x_5 = 0 \Rightarrow x_2 + \sum c_{1j} v_j = 0$$

$$x_4 + 0x_1 + 0x_3 + 2x_5 = 0 \Rightarrow x_2 + \sum c_{2j} v_j = 0$$

x_2, x_4 are dependent on the other variables but are independent of each other.

- x_1, x_3, x_5 are free scalars. If the field has ∞ scalars, then there are ∞ solns to the equation.

- **Theorem:-**

  If $A_{m \times n}$ & $m < n$ then $AX = 0$ always has a non-trivial solution (Assn: Prove)

- **Theorem:-**

  A square matrix $n \times n$ is row equivalent to I iff $AX = 0$ has only trivial solns. (Assn: Prove)

Non Homogenous Systems :-

$$A_{m\times n} X_{n\times 1} = Y_{m\times 1} \neq 0$$

- Since we cannot apply row transformations on only A, as Y will be affected, we apply the transformations together.

$$[A|Y]_{m\times(n+1)} = A'_{m\times(n+1)}$$

$$= [R|Z]_{m\times(n+1)} \qquad Z = \begin{bmatrix} z_i \\ z_2 \\ \vdots \\ z_m \end{bmatrix}$$

$\Rightarrow$ $RX = Z$ where R is in RREF

- If there are r non-zero rows, $(m-r$ zero rows$)$, and if

$$z_i \neq 0 \text{ for any } i > r$$

then there will be no solutions. (zero = non-zero condition).

ex: $A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 1 \\ 0 & 5 & -1 \end{bmatrix}$ $\quad Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

RTs.

$$R' = \left[\begin{array}{ccc|c} 1 & 0 & 3/5 & \frac{1}{5}(y_1 + 2y_2) \\ 0 & 1 & -1/5 & \frac{1}{5}(y_2 - \\ 0 & 0 & 0 & y_3 - y_2 + 2y_1 \end{array}\right]$$

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→ Matrix Multiplication :-

$$C = A \times B$$

$$C_{ij} = \sum_{r=1}^{n} A_{ir} B_{rj}$$

- For 2 matrices A, B to be multiplicable,

  no. of rows in A = no. of colums in B

- **Theorem :-**

  A, B, C are matrices over field F st BC and $A(BC)$ are well defined. Then, AB, $(AB)C$ are also defined and

$$(AB)C = A(BC)$$

Proof :

$$[A(BC)]_{ij} = \sum_{r=1}^{n} A_{ir} [BC]_{rj}$$

$$[BC]_{rj} = \sum_{s=1}^{n} B_{rs} C_{sj}$$

$$\Rightarrow \quad [A(BC)]_{ij} = \sum_{r=1}^{n} \left[A_{ir} \left(\sum_{s=1}^{n} B_{rs} C_{sj} \right) \right] \quad \text{——} \; (1)$$

$$[(AB)C]_{ij} = \sum_{r'=1}^{n} [AB]_{ir'} C_{r'j}$$

$$[AB]_{ir'} = \sum_{s'=1}^{n} A_{is'} B_{s'r'}$$

$$\Rightarrow [(AB)C]_{ij} = \sum_{r'=1}^{n} \left(\sum_{s'=1}^{n} A_{is'} B_{s'r'} \right) C_{r'j} \quad \text{——} \quad (2)$$

- **Square Matrices :-**

- Square matrices can be multiplied by itself.

$$A \times A = A^2$$

- $$A^p A^q A^r = A^{p+q+r}$$

- A square matrix is said to be an elementary matrix if it is obtained by performing a single elementary row operation on the identity matrix.

For a 2×2 matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$\begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}, \ c \neq 0, \quad \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}, \ c \neq 0 \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

are elementary matrices

- **Theorem :-**

Let e be an elementary row operation & let E be an $m \times m$ elementary matrix, $E = e(I)$, then $\forall\ A_{m \times n}$,

$$e(A) = E \times A$$

(Proof is Assn)

- **Theorem :**

If B is row equivalent to A, then $\exists$ a matrix P s.t $B = PA$, since,

$$B = e_1(e_2(\dots(e_n(A))\dots))$$

$$\Rightarrow B = E_1 E_2 \dots E_n \cdot A$$

- **Invertible Matrices :-**

Consider $n \times n$ matrix A.

- P is said to be left inverse of A if $PA = I$
- Q is said to be right inverse of A if $AQ = I$

- A matrix is said to be invertible if both left and right inverses of it exists.
- For a square matrix, if it is invertible, then its left and right matrices are the same.

$$PA = AQ = I \Rightarrow P = Q$$

$$PA = \mathbb{1}$$

$$\Rightarrow P(AQ) = \mathbb{1} \cdot Q$$

$$\Rightarrow P(\mathbb{1}) = Q$$

$$\Rightarrow P = Q$$

- **Theorem:-**

Let A, B be square matrices over a field F,

1) If A is invertible, so is A^{-1}, and $(A^{-1})^{-1} = A$.

2) If both A and B are invertible, then $(AB)^{-1} = B^{-1}A^{-1}$

Proof:

1) $$A \cdot A^{-1} = \mathbb{1}$$

$$A^{-1} \cdot B = \mathbb{1} \Rightarrow B = (A^{-1})^{-1}$$

$$A \cdot A^{-1} \cdot B = A$$

$$\therefore B = A$$

$$\Rightarrow (A^{-1})^{-1} = A$$

2) $$X \cdot AB = \mathbb{1}$$

$$A \cdot A^{-1} = \mathbb{1} \quad \& \quad B \cdot B^{-1} = \mathbb{1}$$

$$\Rightarrow X \cdot A \cdot B \cdot B^{-1} \cdot A^{-1} = \mathbb{1} \cdot B^{-1} A^{-1}$$

$$\Rightarrow \quad (AB)^{-1} = B^{-1}A^{-1}$$

- **Theorem:**

  All elementary matrices are invertible.

  Proof:

  $$E = e(\mathbb{I})$$

  $$\Rightarrow \mathbb{I} = e^{-1}(E)$$

  $$\Rightarrow \mathbb{I} = E^{-1} \times E$$

- **Theorem:** Consider a square matrix $A_{m \times m}$, the following statements are equivalent (Assn: Proof)

  1) A is invertible
  2) $AX = 0$ has only trivial solution
  3) $AX = Y$ has a solution for each $Y_{m \times 1}$.

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- Since w.k.t A having trivial solns $\Rightarrow$ A is row equivalent to $\mathbb{I}$, using above theorem, we can state that,

  Invertible Matrices are row equivalent to $\mathbb{I}$.

→ <u>Vector Space :-</u> (V)

A vector space (linear space) consists of

a) A field F of scalars

b) A set of objects (vectors)

c) A rule called Addition which associates a vector w/ a pair of vectors such that,

A vector space V is defined over a field F. The elements of a vector space are <u>vectors</u>.

1) Addition is commutative

$$\bar{\alpha} + \bar{\beta} = \bar{\beta} + \bar{\alpha} \quad \forall \ \bar{\alpha}, \bar{\beta} \in V$$

2) Addition is closed,

$$\bar{\alpha} + \bar{\beta} \in V \quad \forall \ \bar{\alpha}, \bar{\beta} \in V$$

3) Addition is associative,

$$(\bar{\alpha} + \bar{\beta}) + \bar{\gamma} = \bar{\alpha} + (\bar{\beta} + \bar{\gamma})$$

4) $\exists$ a unique vector called zero $\in V$, st

$$\bar{\alpha} + 0 = \bar{\alpha} \quad \forall \ \alpha \in V$$

5) For each $\bar{\alpha} \in V$, $\exists$ a unique vector $-\bar{\alpha} \in V$ st

$$\bar{\alpha} + (-\bar{\alpha}) = 0$$

d) A rule called scalar multiplication that associates a vector to a pair of a scalar and vector, such that

$$\forall \ c \in F, \ \bar{\alpha} \in \bar{V}, \quad c \cdot \bar{\alpha} \in \bar{V} \ \text{st}$$

1) $1 \cdot \bar{\alpha} = \bar{\alpha}$

2) $c_1(c_2(\bar{\alpha})) = c_2(c_1(\bar{\alpha}))$

3) $c(\bar{\alpha}+\bar{\beta}) = c\bar{\alpha} + c\bar{\beta}$ linearity

4) $(c_1 + c_2)\bar{\alpha} = c_1\bar{\alpha} + c_2\bar{\alpha}$

- $c \cdot \bar{0} = c(\bar{0}+\bar{0})$

  $\Rightarrow c\cdot\bar{0} = c\cdot\bar{0} + c\cdot\bar{0}$

  $\Rightarrow c\cdot\bar{0} = \bar{0}$

- $c\cdot\bar{\alpha} = \bar{0} \Rightarrow c = 0$ or $\bar{\alpha} = \bar{0}$

- A vector $\bar{\alpha} \in V$ is said to be a linear combination of $\bar{\beta_1}, \bar{\beta_2}, \ldots \bar{\beta_n} \in V$ if $\bar{\alpha} = c_1\bar{\beta_1} + c_2\bar{\beta_2} \ldots + c_n\bar{\beta_n}$, for some, $c_1, c_2 \ldots c_n \in F$.

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- Vector Subspace :-

  A vector space S is said to be the subspace of vector space V if $S \subseteq V$ and S is a valid vector space over the same field F as V and with the same vector addn. and multiplication defined for V.

  $S(F', +, \cdot) \subseteq V(F, +, \cdot)$ st $F' \subseteq F$

- $\forall$ vector spaces V, it is a subspace of itself.

- $\forall$ vector spaces V, $\{\bar{0}\} \subseteq V$

- S must contain the same operations as V.

- **Theorem :-**

  A non empty subset $W \subset V$ is a subspace of V iff for each pair of vectors $\bar{\alpha}, \bar{\beta} \in W$ & each c in Field of V, the vector $c\bar{\alpha} + \bar{\beta} \in W$. (Full Proof is Assn.) (Can be took as alternate def.n of subspc).

Proof :

Since W is non-empty, $\exists\ \bar{\alpha} \in W$.

Assume $\forall\ c \in F$, $c\bar{\alpha} + \bar{\beta} \in W$.

Let $c = -1$, $\bar{\beta} = \bar{\alpha}$

$\Rightarrow \quad -\bar{\alpha} + \bar{\alpha} \in W$

$\bar{0} \in W$ (W contains zero vector)

$\Rightarrow \quad c\bar{\alpha} + \bar{0} \in W$ $(\bar{\beta} = \bar{0})$

$\Rightarrow \quad c\bar{\alpha} \in W$ $\longrightarrow$ Using this we can prove commutativity, associativity, etc.

Scalar multiplication is closed within the subset.

- An $n \times m$ matrix A over $\mathbb{C}$ is Hermitian if,

$$A_{jk} = \overline{A_{kj}}$$

- A 2×2 matrix is Hermitian iff it has the form,

$$\begin{bmatrix} z & x+iy \\ x-iy & w \end{bmatrix} \quad \text{where } w, x, y, z \in \mathbb{R}.$$

The set of all Hermitian matrices is not a subspaces of all $n\times n$ matrices over $\mathbb{C}$. (Proof is Assn + What if field is $\mathbb{R}$ instead of $\mathbb{C}$).

- <u>Theorem</u>:

Let V be a vector space over F. The intersection of any collection of subspaces of V is a subspace of V.

Any 2 subspaces of the same vec. space will always have $\bar{0}$.

Let $\bar{\alpha}, \bar{\beta} \in \bigcap_i W_i$

$\Leftarrow \bar{\alpha}, \bar{\beta} \in W_i \ \forall i$

$\Rightarrow c\bar{\alpha} + \bar{\beta} \in W_i \ \forall i, \forall c \in F$

$\Rightarrow c\bar{\alpha} + \bar{\beta} \in \bigcap_i W_i \ \forall c \in F$

$\Rightarrow \bigcap_i W_i$ is a subspace of V (alt. defn of subspace).

- <u>Theorem</u>:-

$\forall$ subsets S of V, $\exists$ a subspace V' of V s.t. $S \subseteq V'$.

- Let $W_1, W_2, W_3 \dots W_n$ be the collection of subspaces of V that contains a subset S of V. The smallest subspace among them is defined as W_i st $W_i = \bigcap_{k=1}^{n} W_k$.

$$\bigcap_{k=1}^{n} W_k \in \{W_1, W_2, W_3 \dots W_n\}.$$

The subspace W_i is said to be the subspace of V spanned by S.

$\therefore$ Let S be a subset of V. The subspace spanned by S is the intersection of all subspaces of V that contains S.

- Theorem:

The subspace spanned by a non-empty subset S of a vector space V is the set of all linear combinations of the vectors in S. (Proof is Assn).

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- If $S_1, S_2, S_3 \dots S_k$ are subsets of a vector space V, the set of all the sums $\bar{\alpha}_1 + \bar{\alpha}_2 + \dots + \bar{\alpha}_k$ of $\bar{\alpha}_i \in S_i$ is called the sum of the subsets $S_1, S_2, S_3 \dots S_k$ and is denoted as $\sum_{i=1}^{k} S_i$.

$$S_1 = \{\bar{\alpha}, \bar{\beta}\} \qquad S_2 = \{\bar{x}, \bar{y}\}$$

$$S_1 + S_2 = \{\bar{x} + \bar{\alpha}, \bar{x} + \bar{\beta}, \bar{y} + \bar{\alpha}, \bar{y} + \bar{\beta}\}$$

- **Theorem :-**

The sum of subspaces $W_1 + W_2 + \dots + W_k$, is also a subspace.

Proof :

Let $\bar{\alpha}, \bar{\beta} \in \sum_{i=1}^{k} W_i$ and $c \in F$, then

$$\bar{\alpha} = \sum_{i=1}^{k} \bar{\alpha_i}, \quad \bar{\alpha}_i \in W_i$$

$$\bar{\beta} = \sum_{i=1}^{k} \bar{\beta_i}, \quad \bar{\beta_i} \in W_i$$

$$c\bar{\alpha} + \bar{\beta} = \sum_{i=1}^{k} c\bar{\alpha}_i + \sum_{i=1}^{k} \bar{\beta_i}$$

$$= \sum_{i=1}^{k} (c\bar{\alpha_i} + \bar{\beta_i})$$

$$c\bar{\alpha}_i + \bar{\beta}_i \in W_i$$

$$\Rightarrow \sum_{i=1}^{k} (c\bar{\alpha_i} + \bar{\beta_i}) \in \sum_{i=1}^{k} W_i$$

$\therefore \sum_{i=1}^{k} W_i$ is a valid subspace. (as per the alternate defn).

- **Theorem :-**

The union $\bigcup_{i=1}^{k} W_i$ is not a valid subspace always. $\sum_{i=1}^{k} W_i$ is spanned by $\bigcup_{i=1}^{k} W_i$. (Proof is Assn).

- **Theorem :-**

The solution to the system $AX=0$ form a subspace

Proof :-

Let Y and Z be any 2 soln.s of the system.

$\Rightarrow \quad AY = 0 \;,\quad AZ = 0$

$A(cY+Z) = c \cdot AY + AZ = \underline{0}$

$\Rightarrow \quad cY + Z$ is also a soln to the system.

$\Rightarrow$ The solution set forms a valid vector space.

→ Bases and Dimension :-

- For a vector field V defined over F, A subset $S \subset V$ is linearly dependent if $\exists$ distinct vectors $\bar{\alpha}_1, \bar{\alpha}_2 \ldots, \bar{\alpha}_n \in S$ & scalars $c_1, c_2, c_3 \ldots, c_n \in F$, not all of which are zero st,

$$c_1\bar{\alpha}_1 + c_2\bar{\alpha}_2 + \cdots + c_n\bar{\alpha}_n = 0$$

comple-ments

even if $\bar{\alpha}_i = 0$ $\forall i$, the statement holds.

If this is not followed, S is linearly independent, ie

$$c_1\bar{\alpha}_1 + c_2\bar{\alpha}_2 + \cdots + c_n\bar{\alpha}_n = 0$$

$$\Rightarrow c_1, c_2, c_3 \cdots c_n = 0$$

- **Theorem :**

Any set which has a linearly dependent subset is linearly dependent.

- **Theorem:**

  Any set that contains $\bar{0}$ is linearly dependent.

- **Theorem:**

  Any subset of linearly independent set is linearly independent.

Consequences of def.n

- **Theorem:**

  A set S is linearly independent iff each finite subset of S is linearly independent.

- A set of independent vectors that span the entire vector space is called a basis.

- The space V is finite-dimensional if $\exists$ a finite basis.

Q. $S = \{(3,0,-3), (-1,1,2), (4,2,-2), (2,1,1)\}$ Check if this is linearly independent and dependent.

A: Since $S \subset \mathbb{R}_3$ and $n(S) = 4 > 3$, S must be linearly dependent.

Also, $2\bar{\alpha}_1 + 2\bar{\alpha}_2 - \alpha_3 + 0\bar{\alpha}_4 = 0$

- $\{(0,0,1), (0,1,0), (1,0,0)\}$ is a standard basis. There is only one non-zero value in each vector, which is 1.

Q. Consider a $m \times n$ matrix A and $n \times 1$ X st (Assn)

$$AX = 0$$

How many linearly independent vectors are there in solution space.

subspace of the solution for X.

- **Theorem** :-

If V is a vector space spanned by $\bar{\beta}_1, \bar{\beta}_2, \ldots \ldots \bar{\beta}_n$, then any independent set of vectors in V is finite and contains no more than n elements. (Proof is Assn).

i.e. length of longest subset of X space st it is lin. indp.

size of basis will be n always.

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- **Corollary** :

For a vector space, all its bases will have the same cardinality. (All the vectors in the basis are linearly independent)

- **Corollary** :

V is finite dimensional vector space. The dimension of V is the cardinality of its basis.

Let $\dim(V) = n$

a) Any subset of V which contains more than n elements is linearly dependent.

b) No subset of V which contains strictly less than n elements can span V.

- **Lemma** :

S is a linearly independent subset of V. Suppose $\bar{\beta} \in V$ is not in the subspace spanned by S. Then the set formed by adding $\bar{\beta}$ to S $(S \cup \{\bar{\beta}\})$ is linearly independent.

Proof :

Let $S = \{\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_n\} \subset V$

If $S \cup \{\bar{\beta}\}$ is lin. dep,

$$\sum_{i=1}^{n} c_i \bar{\alpha}_i + b\bar{\beta} = \bar{0}$$

$$\Rightarrow \quad \bar{\beta} = -\frac{1}{b} \sum_{i=1}^{n} c_i \bar{\alpha}_i$$

If $c_i = 0 \ \forall i$, $\bar{\beta} = \bar{0}$, which is not possible since it is a part of any subspace.

Also the above eqn tells us that for $b \neq 0$, $\bar{\beta}$ is a linear combination of $\bar{\alpha}$ vectors, which is not possible since $\bar{\beta} \notin$ subspace spanned by S.

$$\therefore \quad \sum_{i=1}^{n} c_i \bar{\alpha}_i + b\bar{\beta} \neq 0$$

$\Rightarrow \quad S \cup \{\bar{\beta}\}$ is lin. indep.

- Theorem :-

If W is a subspace of a finite dimensional V, every independent of W is finite & is part of a finite basis of V.

- Corollary :-

Let $A_{n\times n}$ over F. Suppose rows of A form linearly independent vectors in F. Then $A_{n\times n}$ is invertible. (Proof is Assn)

Proof:

Let E_i be a vector in F^n such that its i^{th} element is 1 and all the other entries are 0. $(0,0,0\dots,1,0,0,0\dots)$

Clearly the set of E_i vectors is linearly independent and also form the basis for F^n.

Now, let,

$$E_i = \sum B_{ij} \bar{\alpha}_j ,$$

If we form a matrix using E_i, we get an identity matrix, so,

$$\mathbb{1} = B \cdot A$$

- **Theorem**:-

If W_1 & W_2 are finite dimensional subspaces of V, then $W_1 + W_2$ is finite dimensional & $\dim W_1 + \dim W_2 = \dim(W_1 \cap W_2) + \dim(W_1 \cup W_2)$. (Proof is Assn)

→ **Coordinates**:-

V is a vector space. Consider a basis $\beta = \{\bar{\alpha}_1, \bar{\alpha}_2 \dots \bar{\alpha}_n\}$. Any $\bar{\alpha}$ in V, $\bar{\alpha} = \sum_{i=1}^{n} x_i \bar{\alpha}_i$ for some scalars x_i, then the coordinates of $\bar{\alpha}$ in V wrt β is $(x_1, x_2 \dots, x_n)$.

- The basis must be ordered. ⟶ since order matters in coordinates.

- 2 coordinates to $\bar{\alpha}$ w.r.t to the same ordered basis B, will be the same.

$$\bar{\alpha} = \sum_{i=1}^{n} x_i \bar{\alpha}_i$$

$$\bar{\alpha} = \sum_{i=1}^{n} y_i \bar{\alpha}_i$$

$$\Rightarrow \bar{\alpha} - \bar{\alpha} = \sum_{i=1}^{n} x_i \bar{\alpha}_i - \sum_{i=1}^{n} y_i \bar{\alpha}_i$$

$$= \sum_{i=1}^{n} (x_i \bar{\alpha}_i - y_i \bar{\alpha}_i) = \bar{0}$$

$$= \sum_{i=1}^{n} (x_i - y_i) \bar{\alpha}_i = \bar{0}$$

Since the set of $\bar{\alpha}_i$ is linearly independent,

$$\sum_{i=1}^{n} (x_i - y_i) \bar{\alpha}_i = \bar{0} \Rightarrow x_i - y_i = 0 \ \forall i$$

$$\Rightarrow x_i = y_i \ \forall i$$

- An ordered basis is a sequence of linearly independent vectors that span the entire vector space. order matters in sequence.

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- If $(x_1, x_2 \ldots x_n)$ are the coordinates of $\bar{\alpha}$ and $(y_1, y_2, \ldots y_n)$ are the coordinates of $\bar{\beta}$ w.r.t to the same basis, then the coordinates of $\bar{\alpha} + \bar{\beta}$ is $(x_1 + y_1, x_2 + y_2, x_3 + y_3, \ldots\ldots, x_n + y_n)$, and the coordinates of $c \cdot \bar{\alpha}$, where c is a scalar, is $(cx_1, cx_2 \ldots, cx_n)$.

- Let $\bar{\alpha} \in V$, and B and B' be 2 ordered bases in V, st;

$$B = \{\bar{\alpha}_1, \bar{\alpha}_2, \ldots\ldots, \bar{\alpha}_n\}$$

$$B' = \{\bar{\alpha}_1', \bar{\alpha}_2', \ldots\ldots, \bar{\alpha}_n'\}$$

$$\bar{\alpha} = \sum_{i=1}^{n} x_i \bar{\alpha}_i = \sum_{i=1}^{n} x_i' \bar{\alpha}_i'$$

$\exists$ unique scalars st,

$$\bar{\alpha}_j' = \sum_{i=1}^{n} P_{ij} \bar{\alpha}_i \quad \forall j$$

$$\Rightarrow \boxed{X' = PX}$$

the rows of P are the coordinates of B' vectors in B.

- **Theorem**:

V is a n dimensional vector space over a field F. B and B' are ordered bases. Then $\exists$ a unique necessarily invertible matrix P over F such that,

1) $[\bar{\alpha}]_B = P[\bar{\alpha}]_{B'} \quad \forall \bar{\alpha} \in V$
2) $[\bar{\alpha}_{B'}] = P^{-1}[\bar{\alpha}_B]$

- In $\mathbb{R}^2$,

$$P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$\{(\cos\theta, \sin\theta), (-\sin\theta, \cos\theta)\}$ is a basis in $\mathbb{R}^2$.

→ Linear Transformations :-

- Given 2 vector spaces V and W over field F. A linear transformation $T : V \rightarrow W$ is a map / function that maps each $\bar{\alpha}$ in V to an element in W st,

$$\boxed{T(c\bar{\alpha} + \bar{\beta}) = c \cdot T(\bar{\alpha}) + T(\bar{\beta})} \qquad \forall\, \bar{\alpha}, \bar{\beta} \in V \text{ and } c \in F$$

- A matrix is a linear transformation function, since it can satisfy the above property

$$\boxed{R(cA + B) = c \cdot RA + RB}$$

- $$T(\bar{0} + \bar{0}) = T(\bar{0}) + T(\bar{0})$$

$$= \quad T(\bar{0}) = 2 \cdot T(\bar{0}) \qquad \bar{0}_V = \text{zero vector in } V$$

$$= \quad \boxed{T(\bar{0}_V) = \bar{0}_W} \qquad \bar{0}_W = \text{zero vector in } W$$

- $$\boxed{T\left(\sum x_i \bar{\alpha}_i\right) = \sum x_i T(\bar{\alpha}_i)}$$

- **Theorem :**

  V is a finite dimensional vector space over F and B an ordered basis $\{\bar{\alpha}_1, \bar{\alpha}_2, \dots, \bar{\alpha}_n\}$. Let W be a vector space also over F and $\{\bar{\beta}_1, \bar{\beta}_2, \dots \bar{\beta}_n\}$, then there is precisely one linear transformation $T : V \rightarrow W$ st $T\bar{\alpha}_i = \bar{\beta}_i \;\; \forall\, i$

- **Proof:**

Any $\bar{\alpha} \in V$

$$\bar{\alpha} = \sum_{i=1}^{n} x_i \bar{\alpha_i} \quad , \quad x_i \in F \;\forall\, i \in \{1, 2, \dots n\}$$

Define,

$$T(x_1\bar{\alpha_1} + x_2\bar{\alpha_2} + \dots + x_n\bar{\alpha_n}) = x_1\bar{\beta_1} + x_2\bar{\beta_2} + \dots x_n\bar{\beta_n}$$

We can see that there is no contradiction due to the existence of such a function.

$$\bar{\gamma} = \sum_{i=1}^{n} y_i \bar{\alpha_i}$$

$$T(c\bar{\alpha} + \bar{y}) = \sum_{i=1}^{n} (cx_i + y_i)\, T(\bar{\alpha_i})$$

$$= \sum_{i=1}^{n} c x_i T(\bar{\alpha_i}) + \sum_{i=1}^{n} y_i T(\bar{\alpha_i})$$

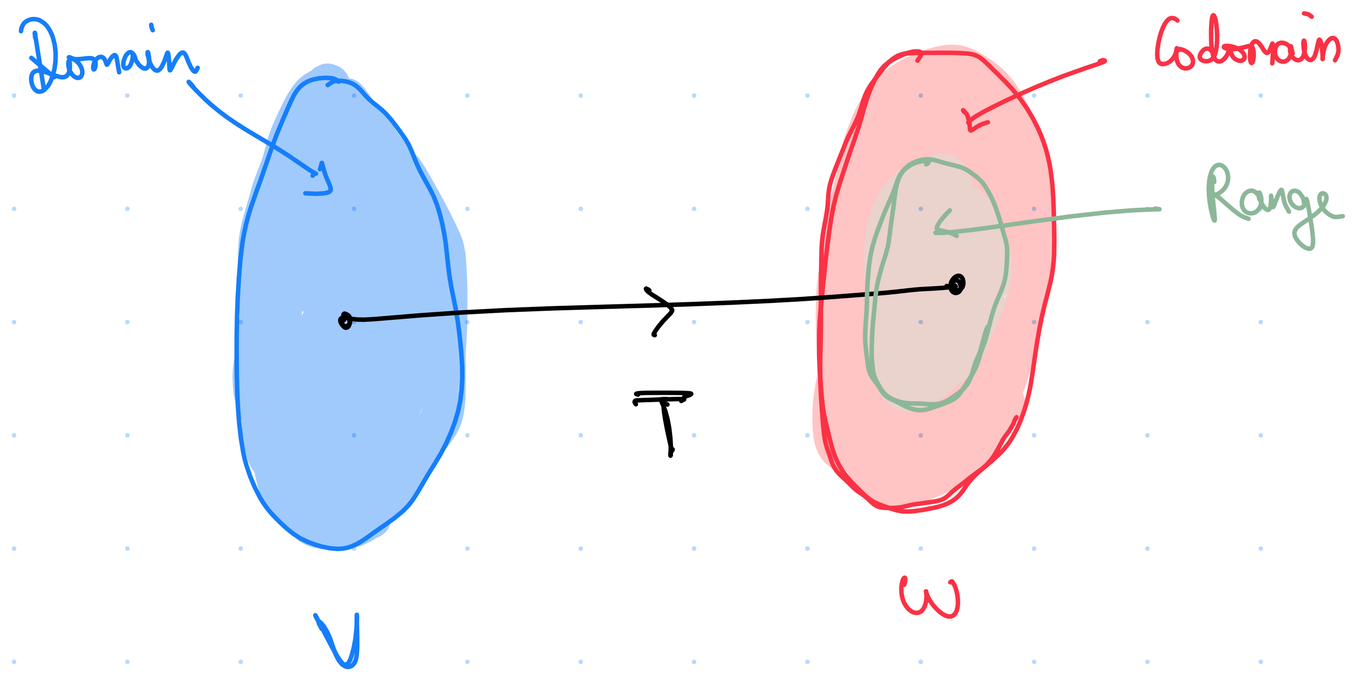
$$= c \cdot T(\bar{\alpha}) + T(\bar{\gamma})$$

Consider a linear transformation $U : V \mapsto W$ st $U(\bar{\alpha_1}) = \bar{\beta_1}$

$$U(\bar{\alpha}) = \sum x_i U(\bar{\alpha_i})$$

$\bar{\beta}$ does <u>not</u> have to be linearly independent.

Clearly T and U are the same function.



- The range of $T: V \mapsto W$ in W will be a subspace of W.

- The set $\{\bar{\alpha} : T(\bar{\alpha}) = \bar{0}_W\}$ is defined as the null space of T.

- <u>Theorem</u>:

If V and W are vector spaces and $T: V \mapsto W$ is a linear transformation and V is finite dimensional, then,

$$\text{rank}(T) + \text{nullity}(T) = \dim(V)$$

<u>(Proof in Assn)</u>

Rank(T) = Dimension of Image of T

Nullity (T) = Dimension of Null space of T

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- $\text{Range}(T) = \{T(\bar{\alpha}) \; \forall \; \bar{\alpha} \in V\}$ $(T: V \mapsto W)$

- $A_{m \times n} X_{n \times 1} = Y_{m \times 1}$ Linear Transformation of n-dimensional vector into m-dimensional vector.

- Basis change is also a linear transformation.

• Algebra Of Linear Transformations :-

- The set of all linear transformations from V to W, is also a vector space with the same field as V, W.

- If $T: V \mapsto W$ and $U: V \mapsto W$ are 2 linear transformations, then the linear transformation $(T+U)$ defined as $(T+U)\bar{\alpha} = T\bar{\alpha} + U\bar{\alpha}$ $\forall \bar{\alpha} \in V$, is a linear transformation $(T+U): V \mapsto W$.

- If $c \in F$, the function cT defined by $(cT)(\bar{\alpha}) = c(T\bar{\alpha})$ $\forall \bar{\alpha} \in V$. is a linear transformation $cT: V \mapsto W$.

- The set of all linear transformations b/w V and W $\rightarrow L(V,W)$, along with addition and scalar multiplication as defined above, forms a vector space over F.

(Prove (Detailed))

- Null transformation : $N: V \mapsto \{\bar{0}_W\}$

- Theorem :

Let V and W be 2 vector spaces over a field F. $\dim(V) = n$, $\dim(W) = m$. Then $\dim(L(V,W)) = n \cdot m$.

(Proof is Assn)

Example:

$$A_{m \times n} V_{n \times 1} = W_{m \times 1}$$

$$L(V,W) = \{A_{m \times n}\}$$

- If we are only given the basis of V and the result of the linear transformation of that basis, we can exactly know what linear transformation has been applied.

- Theorem:

V, W, Z be vector spaces over F, st $\exists$,

$$T : V \to W$$
$$U : W \to Z$$

where T, U are linear transformations,

Then $U \cdot T : V \longrightarrow Z$ defined by $(U \cdot T)\bar{\alpha} = U(T\bar{\alpha})$ is a linear transformation.

$$(UT)(c\bar{\alpha} + \bar{\beta}) = U(cT\bar{\alpha} + T\bar{\beta}) = cUT\bar{\alpha} + UT\bar{\beta}$$

- $T : V \mapsto V$ is termed as a linear operator. $T \in L(V, V)$

- Composition of linear transformations = Multiplication of the 2 matrices.

× scalar multiplication

- If $T \in L(V, V)$, then T^n is well defined $\forall\, n \in \mathbb{N}$.

If $T, U \in L(V, V)$, then UT and TU are well defined, and $UT \neq TU$

- Lemma:

V is a vector space over F. Let $U, T_1, T_2 \in L(V, V)$, $c \in F$.

1) $\mathbb{I} \cdot U = U \cdot \mathbb{I} = U$

2) $U \cdot (T_1 + T_2) = U \cdot T_1 + U \cdot T_2$

3) $(T_1 + T_2) \cdot U = T_1 \cdot U + T_2 \cdot U$

4) $c(UT_1) = U(cT_1) = (cU)T_1$

- Linear operator may or may not be invertible (Null transformation)
- If the transformation matrix is invertible, the corresponding linear transforma-tion is invertible

- Theorem:

$T : V \mapsto W$ is invertible if $\exists$ a $U : W \mapsto V$ st UT is identity on V and TU is identity on W.

If T is invertible, the function U is unique and denoted by T^{-1}.

$T : V \mapsto W$ is invertible iff.

1) $T\bar{\alpha} = T\bar{\beta} \Rightarrow \bar{\alpha} = \bar{\beta}$
2) $\forall \bar{\beta} \in W \; \exists \; \bar{\alpha} \in V$ st $T(\bar{\alpha}) = \bar{\beta}$, ie, Range $(T) = W$.

- Theorem:

V & W be vector spaces over F. $T : V \mapsto W$ is a linear transformation. If T is invertible then inverse function T^{-1} is a linear function from W to V. Inverse of a LT is a LT.

- If $T\bar{\alpha} = \bar{0}$ iff $\bar{\alpha} = \bar{0}$ then $T: V \to W$ is said to be non-singular.

$$\Updownarrow$$

$$T \text{ is } 1:1$$

$$\Updownarrow$$

$$\text{nullspace}(T) = \{\bar{0}\}$$

- If T is non singular, T maps each linearly independent subset of V into a linearly independent subset of W.

- **Proof**:

$T: V \to W$ is non singular,

$\{\bar{\alpha}_1, \bar{\alpha}_2 \ldots, \bar{\alpha}_n\}$ is a linearly independent subset of V.

$$T\{\bar{\alpha}_1, \bar{\alpha}_2 \ldots, \bar{\alpha}_n\} = \{T\bar{\alpha}_1, T\bar{\alpha}_2, \ldots T\bar{\alpha}_n\}$$

$$c_1 T\bar{\alpha}_1 + c_2 T\bar{\alpha}_2 + \cdots + c_n T\bar{\alpha}_n = \bar{0}_W$$

$$= \quad T\sum_{i=1}^{n} c_i \bar{\alpha}_i = \bar{0}_W$$

Since $\{\bar{\alpha}_1, \bar{\alpha}_2 \ldots \bar{\alpha}_n\}$ is lin. indp,

$$T\sum_{i=1}^{n} c_i \bar{\alpha}_i = \bar{0}_W \Rightarrow c_i = 0 \ \forall i$$

$$= \sum_{i=1}^{n} c_i (T\bar{\alpha}_i) = \bar{0}_W \Rightarrow c_i = 0 \ \forall i$$

$$= \{T\bar{\alpha}_1, T\bar{\alpha}_2, \ldots T\bar{\alpha}_n\} \text{ is lin. indp.}$$

For other direction,

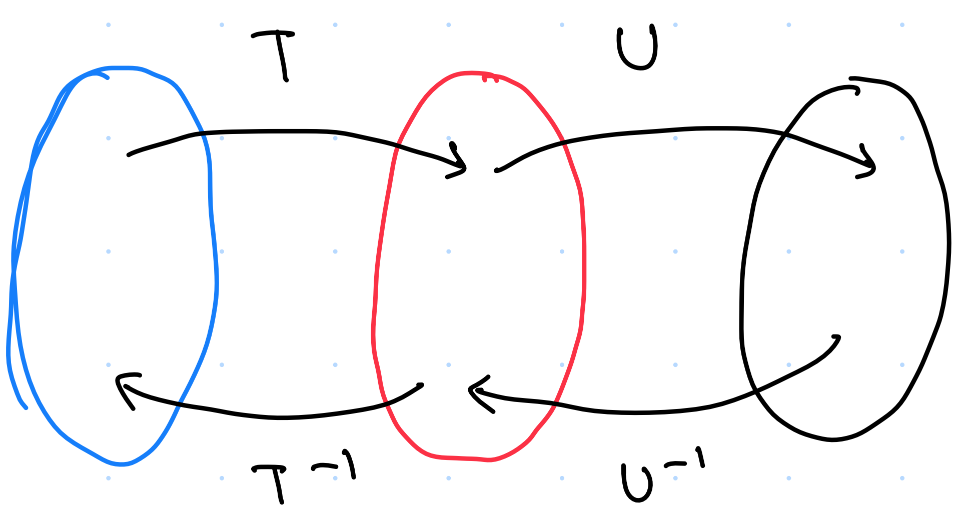
If T maps linearly independent sets to other linearly independent sets, then only $\bar{0}_V$ can be mapped to $\bar{0}_W$.

- If T is invertible, then T is both non singular and onto.

- Theorem:

V and W are finite dimensional vector spaces over F st. $\dim(V) = \dim(W)$, $T: V \longrightarrow W$ is a linear transformation, then the following statements are equivalent.

1) T is invertible,
2) T is non singular
3) T is onto, $\text{Range}(T) = W$
4) If $\{\bar{\alpha}_1, \bar{\alpha}_2, \dots, \bar{\alpha}_n\}$ is a basis in V, $\{T\bar{\alpha}_1, T\bar{\alpha}_2 \dots, T\bar{\alpha}_n\}$ is a basis in W.
5) There is some basis $\{\bar{\alpha}_1, \bar{\alpha}_2 \dots \bar{\alpha}_n\}$ st $\{T\bar{\alpha}_1, T\bar{\alpha}_2, \dots, T\bar{\alpha}_n\}$ is a basis in W.

- 

$(TU)^{-1}$

$= U^{-1}T^{-1}$

- A set G is a group w/ a binary operation '$\cdot$' if $\forall\, x, y \in G$,

1) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ Associativity

2) $\exists\, e \in G$ st $x \cdot e = e \cdot x = x \;\forall\, x \in G$ Identity

3) $x \cdot x^{-1} = e$ Inverse

- The set of invertible linear transformations in $L(V,V)$ forms a group with respect to composition

matrix multiplication.

- If $U, T \in L(V,V)$, then $UT \in L(V,V)$

- A linear transformation $T: V \to W$ is an isomorphism if it is invertible.

- If $\exists$ an invertible $T: V \to W$, then V is said to be isomorphic to W.

- Theorem:

If $\dim(V) = n$, then V over F is isomorphic to F^n.

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→ **Matrices :**

- A rectangular array of numbers
- **Order:** No. of rows × No. of columns.
- Entries of the matrix given by $\{a_{ij}\}$ i^{th} row j^{th} column.
- A matrix with only 1 column is called a column matrix / vector
  " " 1 row is called a row matrix / vector.
- 2 matrices are equal if their orders & their corresponding elements are equal, ie,

$$a_{ij} = b_{ij} \quad \forall\, i,j \, , \qquad \{a_{ij}\}_{m\times n} \quad \{b_{ij}\}_{m\times n}$$

→ **Matrices To Represent a System of Equations :-**

$$\begin{aligned} 2x + 3y &= 5 \\ 4x + 7y &= 8 \end{aligned} \longrightarrow \underset{2\times 2}{\begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} x \\ y \end{bmatrix}} = \underset{2\times 1}{\begin{bmatrix} 5 \\ 8 \end{bmatrix}}$$

→ **Special Matrices :-**

- **Square matrix:** $m = n$
- **Zero matrix:** $a_{ij} = 0 \;\forall\, i,j$

- **Diagonal Matrix:** $a_{ij} = 0 \ \forall \ i \neq j$

- **Elementary Matrices:** Matrix that is due to one elementary op.n on a identity matrix.

- **Upper Δ Matrix:** $a_{ij} = 0 \ \forall \ i > j$

- **Lower Δ Matrix:** $a_{ij} = 0 \ \forall \ i < j$

→ **Matrix Operations :-**

- **Addition of Matrices:**

Two matrices of the same order can be added if they are of the same order. The sum is given by,

$$A + B = C$$

$$\Rightarrow \quad c_{ij} = a_{ij} + b_{ij} \quad \forall \ i,j$$

- **Scalar Multiplication:**

$$kA = k[a_{ij}] = [ka_{ij}]$$

– **Properties:**

1. $A + B = B + A$
2. $A + (B+C) = (A+B) + C$
3. $k(CA) = (kC)A$

Note: Abelian Group: Normal group def.n with commutativity.

Q. Are the following valid Abelian groups?

1) $(M_n(\mathbb{R}), +)$

2) $(M_{n\times n}(\mathbb{R}), \cdot)$

3) $(M_{n\times n}(\mathbb{C}), +$

→ Matrix Multiplication:

- 2 matrices can be multiplied if the no. of columns of the first matrix is equal to the no. of rows of the second.

$$A_{m\times n} \times B_{n\times k} = C_{m\times k}$$

- Matrix multiplication is not commutative in general.

Q. A, B, C are 3 matrices, pt $(AB)C = A(BC)$.

→ Inverse of a Matrix:

If A is a $n\times n$ matrix, an inverse of A is a $n\times n$ matrix A' such that,

$$AA' = A'A = I$$

Q. $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ $A' = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$ Check if A' is an inverse of A.

- Zero matrix does not have an inverse.

- <u>Theorem</u>:

  If $A_{n\times n}$ is an invertible matrix, then the inverse is unique.

  <u>Proof</u>:

  Suppose A has 2 inverses A' and A'', then

$$AA' = I \text{ --- } (1) \quad , \quad AA'' = I \text{ --- } (2)$$

$$A' \underset{\text{Identity}}{=} A'(I) \underset{(2)}{=} A'(AA'') \underset{\text{Associativity}}{=} (A'A)A'' \underset{(1)}{=} IA'' \underset{\text{Identity}}{=} A''$$

$$\Rightarrow \underline{A' = A''}$$

- $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible iff $ad - bc \neq 0$.

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} a & -b \\ -c & d \end{bmatrix}$$

- <u>Properties</u> :-

  - If A is an invertible matrix, then A^{-1} is invertible, and

$$(A^{-1})^{-1} = A$$

  - If A is an invertible matrix and c is a non zero scalar, then,

$$(cA)^{-1} = \frac{1}{c}A^{-1}$$

  - If A and B are invertible and of same size, then AB is also invertible,

$$(AB)^{-1} = B^{-1}A^{-1}$$

- If A is invertible, then A^n is invertible for all non negative integers n,

$$(A^n)^{-1} = (A^{-1})^n$$

<u>Proof:</u>

If $n=1$,

$$(A^1) = A \text{, and clearly } A^{-1} = A^{-1}$$

For $n=2$,

$$(A^2)^{-1} = (AA)^{-1}$$
$$= (A^{-1})(A^{-1})$$
$$= (A^{-1})^2$$

Assume it is true for $n=k$, ie

$$(A^k)^{-1} = (A^{-1})^k$$

Then for $n=k+1$,

$$(A^{k+1})^{-1} = (A^k A^1)^{-1}$$
$$= A^{-1}(A^k)^{-1}$$
$$= A^{-1}(A^{-1})^k$$
$$= (A^{-1})^{k+1}$$

$$\Rightarrow (A^{k+1})^{-1} = (A^{-1})^{k+1}$$

$\therefore$ By induction, the statement has been verified.

- <u>Proof of $(A^{-1})^{-1} = A$:</u>

Given that A^{-1} is the inverse of A. Let A'' be the inv. of A^{-1}
$\hookrightarrow (A^{-1})^{-1}$

$$\Rightarrow AA^{-1} = I \quad \text{—— ①}$$

$$A^{-1}A = I \quad \text{also} \quad A^{-1}A'' = I$$

$$= A^{-1}A = A^{-1}(A^{-1})^{-1}$$

$$= (AA^{-1})A = (AA^{-1})(A^{-1})^{-1}$$

$$= A = (A^{-1})^{-1}$$

→ Elementary Matrices :-

A matrix that is obtained by doing a single elementary row operation on the identity matrix.

$$E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \quad E_1A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 3a_{21} & 3a_{22} & 3a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}$$

$$E_2A = \begin{bmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \quad E_3A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ -2a_{21}+a_{41} & -2a_{22}+a_{42} & -2a_{23}+a_{43} \end{bmatrix}$$

- **Theorem:**

Let E be the elementary matrix created by performing elementary row operations on I. If the same set of row operations is performed on A, then the result is EA.

- <u>Inverse of an Elementary Matrix</u> :-

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} E_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = E_1$$

- <u>Fundamental Theory Of Invertible Matrices</u> :-

Let A be an invertible matrix of order $n \times n$, then the following statements are equivalent.

1) A is invertible.
2) $AX = B$ has a unique solution
3) $AX = 0$ has only trivial solution
4) The reduced row Echelon form of A is I
5) A is a product of Elementary matrices.

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→ <u>Transpose</u> :

The transpose of a matrix A_{ij} is defined as

$$[A^T]_{ij} = A_{ji}$$

- Properties:

  - $(A^T)^T = A$
  - $(AB)^T = B^T A^T$
  - $(A+B)^T = A^T + B^T$
  - $(kA)^T = kA^T$, k is a scalar.

→ **Determinant:**

- For a 2×2 matrix, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant is defined as,

$$\det(A) = ad - bc$$

- For a 3×3 matrix,

$$\det(A) = \sum_{j=1}^{3} (-1)^{i+j} a_{ij} \det(A_{ij})$$

where A_{ij} is the minor matrix of a_{ij} in A

- Laplace Expansion:

The determinant of an $n\times n$ matrix $A = [A_{ij}]_{n\times n}$ where $n \geq 2$ is computed as,

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

$$= \sum_{j=1}^{n} a_{ij} C_{ij}$$

i is fixed

we can fix either row or column.

where $C_{ij} = (-1)^{i+j} \det(A_{ij})$

- If A is upper or lower triangular, then,

$$\det(A) = \prod_{i=1}^{n} a_{ii}$$

- **Properties:**

  - If A has a zero row/column, then $\det(A) = 0$.
  - If B is obtained by interchanging rows of A, $\det(B) = \det(A)$.
  - If A has 2 identical rows, then $\det(A) = 0$
  - If B is obtained by multiplying a row/column of A by k, then $\det(B) = k\det(A)$.
  - If A, B, C are such that i^{th} row of C is the sum of i^{th} row of A and B, and all the other rows are same, then,

$$\det(C) = \det(A) + \det(B)$$

Proof:

cofactor matrix

$$\det(C) = \sum_{j=1}^{n} c_{ij} C_{oij}$$

Given that $c_{ij} = a_{ij} + b_{ij} \ \forall j$

$$\Rightarrow \det(C) = \sum_{j=1}^{n} (a_{ij} + b_{ij}) C_{oij}$$

$$= \det(C) = \sum_{j=1}^{n} a_{ij} C_{oij} + \sum_{j=1}^{n} b_{ij} C_{oij}$$

Since all the other rows of A, B, C are the same, their cofactor matrices wrt to the i^{th} row are the same.

$$\therefore \quad \det(C) = \det(A) + \det(B)$$

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- Let B be a matrix and E be an elementary matrix, then

$$\det(EB) = \det(E) \times \det(B)$$

→ Cramer's Rule :-

Let A be an invertible matrix $(n \times n) \in \mathbb{R}^n$. Then the unique soln. of the system of eq.n $AX = B$ is given by

$$\boxed{x_i = \frac{\det(A_i(B))}{\det(A)}}$$

$A_i(B) \longrightarrow$ Replace the i^{th} column of A with B, which is a column matrix.

- $\det(A) \neq 0$ to get unique solns.

Example:

$$x_1 + 2x_2 = 2$$
$$-x_1 + 4x_2 = 1$$

$$\longrightarrow \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$x_1 = \frac{\det\left(\begin{bmatrix} 2 & 2 \\ 1 & 4 \end{bmatrix}\right)}{\det(A)}$$

$$\det(A) = 4 - (-2) = 6$$

$$x_1 = \frac{8-2}{6} = \frac{6}{6} = \underline{\underline{1}}$$

$$x_2 = \frac{\det\left(\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}\right)}{\det(A)} = \frac{1-(-2)}{6} = \underline{\underline{\frac{1}{2}}}$$

- Let A be an invertible matrix, then

$$\boxed{A^{-1} = \frac{1}{\det(A)}\,\mathrm{adj}(A).}$$

where $\mathrm{adj}(A)$ represents the Adjoint of A, which is the transpose of the cofactor matrix of A.

Example:

Find the inverse of $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{bmatrix}$

$$\mathrm{adj}(A) = \begin{bmatrix} -18 & 10 & 4 \\ 3 & -2 & -1 \\ 10 & -6 & -2 \end{bmatrix}^T$$

$$\begin{aligned} \det(A) &= 1(-18) + 2(10) + (-1)(4) \\ &= -18 + 20 - 4 \\ &= -2 \end{aligned}$$

$$A^{-1} = \frac{-1}{2}\begin{bmatrix} -18 & 10 & 4 \\ 3 & -2 & -1 \\ 10 & -6 & -2 \end{bmatrix}^T = \begin{bmatrix} 9 & -5 & -2 \\ -3/2 & 1 & 1/2 \\ -5 & 3 & 1 \end{bmatrix}^T$$

$$= \begin{bmatrix} 9 & -3/2 & -5 \\ -5 & 1 & 3 \\ -2 & 1/2 & 1 \end{bmatrix}$$

→ <u>Inner Product :-</u>

- It is an operation on a vector space V, that assigns to any pair of vectors $\alpha, \beta \in V$, a scalar $\langle \alpha, \beta \rangle$.

- An inner product space is a vector space V over F together with a inner product that follows the map,

$$\langle , \rangle : V \times V \longrightarrow F$$

that satisfies the following properties

i) $\langle \alpha, \beta \rangle = \overline{\langle \beta, \alpha \rangle}$ (Conjugate Property)

ii) $\langle c\alpha + \gamma, \beta \rangle = c \langle \alpha, \beta \rangle + \langle \gamma, \beta \rangle$ (Linearity in first argument)

iii) $\langle \alpha, \alpha \rangle \geq 0$, $\langle \alpha, \alpha \rangle = 0 \iff \alpha = 0$

- $u, v \in V$ are said to be orthogonal if $\langle u, v \rangle = 0$

- <u>Standard Inner Product :-</u>

$$\bar{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \quad \bar{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

- The standard inner product is defined as,

$$\langle \bar{u}, \bar{v} \rangle = \sum_{i=1}^{n} (u_i v_i)$$

→ **Theorem:**

If $\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n\} \subseteq \mathbb{R}^n$ is an orthogonal set of vector in $\mathbb{R}^n$, then they form a linearly independent set.

∘ **Proof:**

If $c_1, c_2, \ldots, c_n$ are scalars such that,

$$c_1\bar{v}_1 + c_2\bar{v}_2 + \cdots + c_n\bar{v}_n = \bar{0}$$

$$\Rightarrow \left\langle \sum_{j=1}^{n} c_j \bar{v}_j, \bar{v}_i \right\rangle = \bar{0}, \quad \forall\, i \in \mathbb{N}$$

$$\Rightarrow \sum_{j=1}^{n} c_j \langle \bar{v}_j, \bar{v}_i \rangle = \bar{0} \quad \forall\, i \in \mathbb{N}$$

$$\Rightarrow c_i \langle \bar{v}_i, \bar{v}_i \rangle = \bar{0} \quad \forall\, i \in \mathbb{N}$$

Since $\bar{v}_i \neq 0 \ \forall\, i$

$$\Rightarrow \underline{c_i = 0 \quad \forall\, i \in \mathbb{N}}$$

$\Rightarrow$ The set is linearly independent.

Note: Length of $\bar{v} = \|\bar{v}\| = \sqrt{\langle v, v \rangle}$ (Def.n)

Distance b/w $\bar{u}$ and $\bar{v}$ = $\|\bar{u} - \bar{v}\|$

- For a standard product over $\mathbb{R}^2 \times \mathbb{R}^2$,

$$\left\langle \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\rangle = u_1 v_1 + u_2 v_2$$

Since $u_1, v_1, u_2, v_2 \in \mathbb{R}$, $u_1 v_1 + u_2 v_2 \in \mathbb{R}$ (Closure of multiplication and addition over real no.s)

$\Rightarrow$ $\langle , \rangle$ is defined from $\mathbb{R}^2 \times \mathbb{R}^2 \mapsto \mathbb{R}$

- To show that linearity over the first operation holds in standard product,

$$\left\langle \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\rangle = u_1 v_1 + u_2 v_2$$

$$\left\langle c \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\rangle = c \left\langle \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\rangle + \left\langle \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\rangle$$

$$= \left\langle \begin{bmatrix} cu_1 + w_1 \\ cu_2 + w_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\rangle = c(u_1 v_1 + u_2 v_2) + w_1 v_1 + w_2 v_2$$

$$= (cu_1 + w_1)(v_1) + (cu_2 + w_2)(v_2) = cu_1 v_1 + cu_2 v_2 + w_1 v_1 + w_2 v_2$$

$$= cu_1v_1 + w_1v_1 + cu_2v_2 + w_2v_2 = cu_1v_1 + cu_2v_2 + w_1v_1 + w_2v_2$$

$$\underline{LHS = RHS}$$

- If $\bar{u}, \bar{v} \in \mathbb{C}^n$, then the complex dot product is defined as,

$$\bar{u} \cdot \bar{v} = \sum_{i=1}^{n} u_i^c v_i$$

Example:

$$\bar{u} = \begin{bmatrix} i \\ 1 \end{bmatrix} \quad \bar{v} = \begin{bmatrix} 2-3i \\ 1+5i \end{bmatrix}$$

$$\langle \bar{u}, \bar{v} \rangle = (-i)(2-3i) + 1(1+5i)$$

$$= -2i - 3 + 1 + 5i$$

$$= \underline{-2+3i}$$

$$\|\bar{u}\| = \sqrt{\bar{u} \cdot \bar{u}} = \sqrt{(-i)(i) + (1)(1)} = \sqrt{1 - i^2}$$

$$= \underline{\underline{\sqrt{2}}}$$

$$\|\bar{v}\| = \sqrt{(2+3i)(2-3i) + (1-5i)(1+5i)}$$

$$= \sqrt{4 + 9 + 1 + 25}$$

$$= \sqrt{5+9+25} = \underline{\underline{\sqrt{39}}}$$

$$d(\bar{u},\bar{v}) = \|\bar{u}-\bar{v}\|$$

$$= \left\| \begin{bmatrix} -2+4i \\ -5i \end{bmatrix} \right\|$$

- Let A be a complex matrix, the conjugate transpose of A is

$$A^* = \bar{A}^T$$

where $\bar{A}$ is the matrix in which each entry is the conjugate of the entry in the same position in A.

- Properties of $\bar{A}$:-

1) $\overline{(\bar{A})} = A$
2) $\overline{(cA)} = \bar{c}\bar{A}$
3) $\overline{(A^T)} = (\bar{A})^T$
4) $\overline{(A+B)} = \bar{A}+\bar{B}$
5) $\overline{(AB)} = (\bar{A})(\bar{B})$

- Properties of A^* :-

1) $((A^*)^*) = A$
2) $(cA)^* = \bar{c}A^*$
3) $(A+B)^* = A^*+B^*$
4) $(AB)^* = B^*A^*$

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Example:

Find an orthogonal basis for the subspace W of $\mathbb{R}^3$ given by,

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \; x-y+2z=0 \right\}$$

$$x - y + 2z = 0$$

Since W is 2D, it will have only 2 indp. variables.

$$x = y - 2z$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y - 2z \\ y \\ z \end{bmatrix} = \begin{bmatrix} y - 2z \\ y + 0z \\ 0y + z \end{bmatrix} = y \overset{\bar{y}}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} + z \overset{\bar{z}}{\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}}$$

Let $\bar{\omega} = \begin{bmatrix} i \\ j \\ k \end{bmatrix}$ be a vector orthogonal to $\bar{y}$ and in W

$$\Rightarrow \bar{\omega} \cdot \bar{y} = i + j = 0 \qquad \Rightarrow \begin{matrix} i - j + 2k = 0 \\ i + j \quad\;\; = 0 \end{matrix}$$

$$= \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} i \\ j \\ k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -2 \end{bmatrix} \begin{bmatrix} i \\ j \\ k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} i \\ j \\ k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$j - k = 0$$

$$\Rightarrow j = k$$

$$i - j + 2k = 0$$

$$i + k = 0$$

$$i = -k$$

$$\Rightarrow \bar{\omega} = \begin{bmatrix} k \\ -k \\ k \end{bmatrix} = k \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$\bar{\omega}$ and $\bar{y}$ are lin. indp and orthogonal.

$\therefore \{\bar{\omega}, \bar{y}\}$ is the required basis.

→ Gram Schmidt Process :-

- A process to find an orthogonal basis for a subspace W.
- The algorithm begins with an arbitrary basis and orthogonalize it.

Example :

An arbitrary basis for the subspace W in prev. scenario is,

$$\bar{y} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \bar{z} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

- We need to create a vector that is orthogonal to $\bar{y}$ and $\bar{z}$ while also being lin. indp.
- $\text{proj}_{\bar{x}_1}(\bar{x}_2) = \dfrac{\langle \bar{x}_1 \cdot \bar{x}_2 \rangle}{\langle \bar{x}_1 \cdot \bar{x}_1 \rangle} \bar{x}_1$

In terms of coord. geo, you are removing the component of $\bar{y}$ that is $\parallel$ed to $\bar{z}$.

- The new vector is,

$$\bar{\omega} = \bar{y} - \text{proj}_{\bar{z}}(\bar{y})$$

→ For a 2D subspace.

- **Generalization :-**

Let $\{\bar{x}_1, \bar{x}_2, \dots, \bar{x}_k\}$ be a basis for the subspace W of V, then,

$$\bar{v}_1 = \bar{x}_1$$

$$\bar{v}_2 = \bar{x}_2 - \text{proj}_{\bar{v}_1} \bar{x}_2$$

$$\bar{v}_3 = \bar{x}_3 - \text{proj}_{\bar{v}_1} \bar{x}_3 - \text{proj}_{\bar{v}_2} \bar{x}_3$$

$$\Rightarrow \quad \boxed{\bar{v}_k = \bar{x}_k - \sum_{i=1}^{k-1} \text{proj}_{\bar{v}_i}(\bar{x}_k)}$$

$$\Rightarrow \quad \boxed{\bar{v}_k = \bar{x}_k - \sum_{i=1}^{k-1} \frac{\langle \bar{v}_i, \bar{x}_k \rangle}{\langle \bar{v}_i, \bar{v}_i \rangle} \bar{v}_i}$$

- The set $\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k\}$ is orthogonal to $\{\bar{x}_1, \bar{x}_2 \ldots \bar{x}_k\}$ by construction.
- To prove $\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k\}$ is a basis,

  If $k = 1$, the set will trivially be a basis

  Assume that for $k = n$, the vectors are all lin. indp, then for $k = n+1$,

$$\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n, \bar{v}_{n+1}\}$$

$$\bar{v}_{n+1} = \bar{x}_{n+1} - \sum_{i=1}^{n} \text{proj}_{\bar{v}_i}(\bar{x}_{n+1})$$

Example:

Find an orthogonal basis for the subspace given by the basis,

$$\bar{x}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \quad \bar{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \quad \bar{x}_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \\ 2 \end{bmatrix}$$

$$\bar{v}_1 = \bar{x}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

$$\bar{v}_2 = \bar{x}_2 - \frac{\langle \bar{v}_1, \bar{x}_2 \rangle}{\langle \bar{v}_1, \bar{v}_1 \rangle} \bar{v}_1$$

$$= \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{(2-1+0+1)}{(1+1+1+1)} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 3/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

$$\bar{v}_3 = \bar{x}_3 - \left(\operatorname{proj}_{\bar{v}_1} \bar{x}_3 + \operatorname{proj}_{\bar{v}_2} \bar{x}_3 \right)$$

$$= \begin{bmatrix} 2 \\ -2 \\ 1 \\ 2 \end{bmatrix} - \left(\frac{(2+2-1+2)}{(1+1+1+1)} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} + \frac{(3-3+1/2+1)}{(3/4+3/4+1/4+1/4)} \begin{bmatrix} 3/2 \\ 3/2 \\ 1/2 \\ 1/2 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 2 \\ -2 \\ 1 \\ 2 \end{bmatrix} - \left(\begin{bmatrix} 5/4 \\ -5/4 \\ -5/4 \\ 5/4 \end{bmatrix} + \begin{bmatrix} 9/8 \\ 9/8 \\ 3/8 \\ 3/8 \end{bmatrix} \right)$$

→ **Cauchy - Schwartz Inequality :-**

Let u and v be vectors in the inner product space V. Then,

$$\boxed{|\langle u, v\rangle| \leq \|u\| \cdot \|v\|}$$

- Equality holds if u, v are scalar multiples of each other.

→ **Triangle Inequality :-**

Let u and v be vectors in an inner product space V, then,

$$\boxed{\|u + v\| \leq \|u\| + \|v\|}$$

Example: Construct an orthogonal basis for $\mathbb{P}_2$ wrt to the inner product, $\langle f, g\rangle = \int_{-1}^{1} f(x) \cdot g(x)\, dx$, by applying Gram Schmidt process on the vectors $\{1, x, x^2\}$

$$\bar{u}_1 = 1$$

$$\bar{u}_2 = x - \text{proj}_{u_1}(x)$$

$$= x - \frac{\langle u_1, x\rangle}{\langle u_1, u_1\rangle} u_1$$

$$= x - (1) \frac{\int_{-1}^{1} x \cdot dx}{\int_{-1}^{1} dx} = x - \frac{\left(\frac{1}{2} - \frac{1}{2}\right)}{(1 - (-1))} = x - 0 = \underline{\underline{x}}$$

$$\bar{v}_3 = \bar{x}_3 - \frac{\langle \bar{v}_2, \bar{x}_3 \rangle}{\langle \bar{v}_2, \bar{v}_2 \rangle} \bar{v}_2 - \frac{\langle \bar{v}_1, \bar{x}_3 \rangle}{\langle \bar{v}_1, \bar{v}_1 \rangle} \bar{v}_1$$

$$= x^2 - x \frac{\int_{-1}^{1} x^3 \, dx}{\int_{-1}^{1} x^2 \, dx} - (1) \frac{\int_{-1}^{1} x^2 \, dx}{\int_{-1}^{1} 1 \, dx}$$

$$= x^2 - x \frac{\left(\frac{1}{4} - \frac{1}{4}\right)}{\frac{1}{3} - \left(\frac{-1}{3}\right)} - \frac{\left(\frac{1}{3} - \left(\frac{-1}{3}\right)\right)}{1 - (-1)}$$

$$= x^2 - \frac{1}{3}$$

- **Theorem:**

  The columns of an $m \times n$ matrix Q form an orthogonal matrix iff $QQ^T = I$.

- An $n \times n$ matrix whose columns form an orthonormal set is an orthogonal matrix
- A square matrix Q is orthogonal if $Q^{-1} = Q^T$.

→ <u>Orthogonal Decomposition Theorem</u> :-

- Let W be a subspace of $\mathbb{R}^n$ and $\bar{v} \in \mathbb{R}^n$. Then there are unique vectors $\bar{w} \in W$ and $\bar{w}' \in W'$ such that,

$$\bar{v} = \bar{w} + \bar{w}'$$
$$= proj_{\bar{w}}(\bar{v}) + proj_{\bar{w}'}(\bar{v})$$

- Where W' is the <u>orthogonal complement</u> of W.

- <u>Orthogonal Complement</u> :-
  Let W be a subspace of $\mathbb{R}^n$.
  We say that a vector $\bar{v}$ in $\mathbb{R}$ is orthogonal to W if it is orthogonal to $\forall \bar{w} \in W$.

- The set of all $\bar{v} \in \mathbb{R}^n$ st $\bar{v}$ is orthogonal to W, is termed as the orthogonal complement of W.

- <u>Theorem</u>:
  If W is a valid subspace of $\mathbb{R}^n$, the W' is also a subspace of $\mathbb{R}^n$.

  <u>Proof</u>:

  Since $\langle \bar{0}, \bar{w} \rangle = 0 \ \forall \ \bar{w} \in W$, $\bar{0} \in W'$

  Let $\bar{x}, \bar{y} \in W'$

$\Rightarrow \langle \bar{x}, \bar{\alpha} \rangle = 0$ & $\langle \bar{y}, \bar{\alpha} \rangle = 0 \quad \forall \; \bar{\alpha} \in W$

$\Rightarrow \langle c\bar{x} + \bar{y}, \bar{\alpha} \rangle = 0 \quad \forall \; c \in F, \bar{\alpha} \in W$ (By linearity in the first operand)

$\Rightarrow \underline{c\bar{x} + \bar{y} \in W'}$

$\therefore \; \underline{\bar{x}, \bar{y} \in W' \Rightarrow c\bar{x} + \bar{y} \in W'}$, $\therefore W'$ is a vector space.

- <u>Theorem</u>:

$$(W')' = W$$

<u>Proof</u>:

W' is the set of all vectors orthogonal to W.

$\Rightarrow \forall \; \bar{\alpha} \in W'$, $\bar{\alpha}$ is orthogonal to W.

$\Rightarrow \forall \; \bar{\alpha} \in W'$, $\bar{\alpha}$ is orthogonal to $\bar{\beta}$, $\forall \bar{\beta} \in W$.

$\Rightarrow \forall \; \bar{\beta} \in W$, $\bar{\beta}$ is orthogonal to $\bar{\alpha}$, $\forall \; \bar{\alpha} \in W'$

(Since orthogonality is bidirectional)

$\Rightarrow \forall \; \bar{\beta} \in W$, $\bar{\beta}$ is orthogonal to W'

$\Rightarrow W$ is orthogonal to W'

$\Rightarrow \underline{(W')' = W}$

- **Theorem :-**

The columns of an $m \times n$ matrix Q forms an orthonormal set iff $Q^T Q = I_{n \times n}$

↓ ie, Q is orthogonal

- **Proof :**

Since $Q^T Q = I$

$$[Q^T Q]_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

$$[Q^T Q]_{ij} = \sum_{r=0}^{m} Q^T_{ir} Q_{rj}$$

$$= (i^{th} \text{ row of } Q^T) \cdot (j^{th} \text{ column of } Q)$$

$$= (i^{th} \text{ column of } Q) \cdot (j^{th} \text{ column of } Q)$$

If columns of Q are orthonormal,

$$(i^{th} \text{ column of } Q)(j^{th} \text{ colum of } Q) = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

$$\Rightarrow [Q^T Q]_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

|||ly backward direction can be proved.

$\therefore$ Columns of Q are orthonormal $\Leftrightarrow Q^T Q = I$

- **Definition:**

An $n \times n$ matrix Q whose colums form an orthonormal set is called an orthogonal matrix.

- A square matrix is orthogonal iff $Q^{-1} = Q^T$.

→ **QR Factorization:-**

Let A be an $m \times n$ matrix with linearly independent columns as $A = QR$, where,

Q - $m \times n$ matrix with orthogonal columns

R - Invertible upper triangular matrix.

- To find Q, we use Gram Schmidt Process on A.
- $\underline{R = Q^T A}$.

Example:

Find the QR factorization of $A = \begin{bmatrix} 1 & 2 & 2 \\ -1 & 1 & 2 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

$$\bar{q_1} = \bar{a_1} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

$$\bar{q_2} = \bar{a_2} - \text{proj}_{\bar{a_1}} \bar{a_2} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{(2-1+0+1)}{4+1+1} \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/3 \\ 1/6 \\ 0 \\ 1/6 \end{bmatrix} = \begin{bmatrix} 5/3 \\ 5/6 \\ 0 \\ 5/6 \end{bmatrix}$$

$$\overline{q_3} = \overline{a_3} - \text{proj}_{\overline{q_2}}\, a_3 - \text{proj}_{\overline{q_1}}\, \overline{a_3}$$

$$= \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix} - \frac{\left(\frac{10}{3} + \frac{5}{3} + \frac{5}{3}\right)}{\left(\frac{25}{9} + \frac{25}{36} + \frac{25}{36}\right)} \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix} - \frac{(2-2-1+2)}{(1+1+1+1)} \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix} - \frac{\frac{20}{3}}{\frac{150}{36}} \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix}$$

$\cancel{24}\ 8$

$\frac{\cancel{72}}{\cancel{45}}$ $\cancel{15}$

5

$$= \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix} - \frac{72\cancel{0}}{45\cancel{0}} \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 1/2 \\ 1/4 \\ 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 - \frac{16}{5} - \frac{1}{2} \\ 2 - \frac{16}{5} - \frac{1}{2} \\ 1 - \frac{8}{5} - \frac{1}{4} \\ 2 - \frac{16}{5} - \frac{1}{2} \end{bmatrix} =$$

→ Eigenvalues And Eigenvectors :-

Consider an $n \times n$ matrix A. A scalar λ is called the Eigenvalue of A if there is a non zero vector $\bar{x}$ such that,

$$\boxed{A\bar{x} = \lambda \bar{x}}$$

Notes of 1/4/25 – In Rough Note

4/4/25

→ **Theorem :-**

Let A be a $n \times n$ matrix and let $\lambda_1, \lambda_2, \dots \lambda_m$ be the set of distinct Eigenvalues of A, and $\overline{x}_1, \overline{x}_2 \dots \overline{x}_m$ be their corresponding Eigenvectors. Then $\overline{x}_1, \overline{x}_2, \dots \overline{x}_m$ is linearly independent.

∘ **Proof:**

Let $\overline{x}_1, \overline{x}_2, \overline{x}_3, \dots, \overline{x}_m, \overline{x}_{m+1}$ be linearly independent.

Then wkt $\exists$ scalars $c_1, c_2, \dots c_m$ st,

$$\overline{x}_m = c_1 \overline{x}_1 + c_2 \overline{x}_2 + \dots c_{m-1} \overline{x}_{m-1} \quad \text{—— (1)}$$

$$\Rightarrow A\overline{x}_m = c_1 A\overline{x}_1 + c_2 A\overline{x}_2 + \dots c_{m-1} A\overline{x}_{m-1}$$

$$\Rightarrow \lambda_m \overline{x}_m = c_1 \lambda_1 \overline{x}_1 + c_2 \lambda_2 \overline{x}_2 + \dots + c_{m-1} \lambda_{m-1} \overline{x}_{m-1} \quad \text{(2)}$$

Multiplying eq (1) with λ_{m+1},

$$\Rightarrow \lambda_{m+1} \overline{x}_m = \lambda_{m+1} \left(\sum_{i=1}^{m-1} c_i \overline{x}_i \right) \quad \text{—— (3)}$$

(3) − (1),

$$(\lambda_{m+1} - \lambda_m) \overline{x}_m = \sum_{i=1}^{m-1} c_i (\lambda_{m+1} - \lambda_i) \overline{x}_i$$

Since the Eigenvalues are distinct, $\lambda_{m+1} - \lambda_i \neq 0 \ \forall i$

$$\Rightarrow \sum_{i=1}^{m} c_i(\lambda_{m+1}-\lambda_i)\bar{x}_i = 0$$

But since $(\lambda_{m+1}-\lambda_i) \neq 0$ and $\bar{x}_i \neq \bar{0} \ \forall i$

$$\Rightarrow c_i = 0 \ \forall i$$

$$\Rightarrow \underline{\{\bar{x}_i \mid i \in \{1,2,\dots m\}\} \text{ is lin. indp}}$$

→ **Similar Matrices** :-

Let A and B be $n \times n$ matrices. We say that A is similar to B if $\exists$ an invertible matrix such that

$$\boxed{P^{-1}AP = B} \Rightarrow A \sim B$$

- If A is similar to B, then

$$P^{-1}AP = B$$

$$\Rightarrow PP^{-1}APP^{-1} = PBP^{-1}$$

$$\Rightarrow A = PBP^{-1}$$

$$\Rightarrow \underline{B \sim A} \longrightarrow \underline{\text{Equivalence Relation}}$$

- **Theorem** :- Let A, B, C be 3 matrices, then

i) $A \sim A$

ii) If $A \sim B$, then $B \sim A$

iii) If $A \sim B$, $B \sim C$, then $A \sim C$

Similarity is an equivalence relation.

Proof:

i) Let $P = I$

$\Rightarrow \quad I A I^{-1} = A \longrightarrow I = I^{-1}$

$\therefore A \sim A$

iii) If $A \sim B$ and $B \sim C$,

$A \sim B \Rightarrow B = P^{-1} A P$

$B \sim C \Rightarrow C = Q^{-1} B Q$

$\Rightarrow C = Q^{-1} (P^{-1} A P) Q$

$= (Q^{-1} P^{-1}) A (PQ)$

wkt $(PQ)^{-1} = Q^{-1} P^{-1}$

$\Rightarrow C = (PQ)^{-1} A (PQ)$

$\therefore A \sim C$.

- Theorem:

If $A \sim B$, then

- $\det(A) = \det(B)$
- A, B have the same characteristic polynomial.

- Proof:

i) If $A \sim B$, then

$$B = PAP^{-1}$$

$$\Rightarrow \det(B) = \det(P) \times \det(A) \times \det(P^{-1})$$ ($\det(AB) = \det(A)\det(B)$)

$$= \frac{\det(P)}{\det(P)} \times \det(A)$$ ($\det(P^{-1}) = \frac{1}{\det(P)}$)

$$= \det(A)$$

$$\therefore \underline{\det(B) = \det(A)}$$

ii) The characteristic eqn of a matrix A is given by,

$$\det(A - \lambda I) = 0$$

If $A \sim B$, $B = PAP^{-1}$

Characteristic Eqn of B,

$$\det(B - \lambda I) = 0$$

$$\Rightarrow \det(PAP^{-1} - \lambda I) = 0$$

$$\Rightarrow \det(PAP^{-1} - P\lambda I P^{-1}) = 0$$

$$\Rightarrow \det(P(A - \lambda I)P^{-1}) = 0$$

$$\Rightarrow \det(P)\det(A - \lambda I)\det(P^{-1}) = 0$$

$$\Rightarrow \det(A - \lambda I) = 0$$ ($\det(P^{-1}) \cdot \det(P)$)

→ Diagonalization :-

An $n \times n$ matrix A is diagonalizable if $\exists$ a diagonal matrix D such that $A \sim D$, ie, $\exists$ P such that

$D = P^{-1}AP$

- **Theorem :**

Let A be a $n \times n$ matrix. Then A is diagonalizable iff A has n linearly independent Eigenvectors.

Proof :

Let A be diagonalizable

$\Rightarrow$ $\exists$ a diagonal matrix D st $A \sim D$.

$\Rightarrow$ $D = P^{-1}AP$

$\Rightarrow$ $PD = AP$

Let the columns of P be $\bar{p_1}, \bar{p_2}, \bar{p_3} \dots \bar{p_n}$, and the diagonal values of D be $\lambda_1, \lambda_2, \lambda_3 \dots$

$$A[\bar{p_1}\ \bar{p_2}\ \dots\ \bar{p_n}] = [\bar{p_1}\ \bar{p_2}\ \dots\ \bar{p_n}] \begin{bmatrix} \lambda_1 & 0 & 0 & \dots \\ 0 & \lambda_2 & 0 & \dots \\ 0 & \vdots & & \\ \vdots & \vdots & & \end{bmatrix}$$

$$\Rightarrow A[\bar{p_1}\ \bar{p_2} \dots \bar{p_n}] = [\lambda_1\bar{p_1}\ \ \lambda_2\bar{p_2}\ \dots\ \lambda_n\bar{p_n}]$$

Equating the columns, we get,

$$A\bar{p}_1 = \lambda_1 \bar{p}_1, \quad A\bar{p}_2 = \lambda_2 \bar{p}_2 \quad \dots \quad A\bar{p}_n = \lambda_n \bar{p}_n$$

$\Rightarrow$ $\bar{p}_1, \bar{p}_2 \dots \bar{p}_n$ are Eigenvectors of A wrt to Eigen-values $\lambda_1, \lambda_2 \dots \lambda_n$.

→ **Fundamental Theorem Of Invertible Matrices :-**

Let A be a $n \times n$ matrix. The following statements are equivalent.

1. A is invertible
2. $AX = Y$ has unique soln $\forall\ Y \in \mathbb{R}^n$
3. $AX = 0$ has only trivial soln.
4. RREF of A is I_n
5. A is the product of Elementary Matrices.
6. Rank of A is n
7. Nullity of A is 0
8. The column vectors of A are linearly independent.

Example: Diagonalize A into D and find P st $P^{-1}AP = D$

a) $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{pmatrix}$

- **Algorithm For Diagonalization :-**

1. Find the Eigenvalues and Eigenvectors of A
2. Verify if the Eigenvectors are linearly independent.
3. If linearly independent, then $D = \text{dia}(\lambda_1, \lambda_2, \lambda_3, \dots)$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{pmatrix}$$

Characteristic Eqn : $\det(A - \lambda I) = 0$

$$\det\begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 2 & -5 & 4-\lambda \end{pmatrix} = 0$$

$$-\lambda(-\lambda(4-\lambda)+5) - 1(0-2) = 0$$

$$\lambda^2(4-\lambda) - 5\lambda + 2 = 0$$

$$4\lambda^2 - \lambda^3 - 5\lambda + 2 = 0$$

$$(\lambda-1)(-\lambda^2+3\lambda-2) = 0$$

$$(\lambda-1)(-\lambda^2+\lambda+2\lambda-2) = 0$$

$$(\lambda-1)(\lambda(-\lambda+1)-2(-\lambda+1)) = 0$$

$$(\lambda-1)(\lambda-2)(1-\lambda) = 0$$

$$-(\lambda-1)^2(\lambda-2) = 0$$

$$-\lambda^2 + 3\lambda - 2$$

$$\lambda-1 \,\big)\, -\lambda^3 + 4\lambda^2 - 5\lambda + 2$$

$$\overset{(+)}{-}\lambda^3 \overset{(-)}{+} \lambda^2$$

$$3\lambda^2 - 5\lambda$$

$$\overset{(-)}{}3\lambda^2 \overset{(+)}{-} 3\lambda$$

$$-2\lambda + 2$$

$\Rightarrow \quad \lambda = 1_{AM=2}, \; 2_{AM=1}$

There are only 2 Eigenvalues, but we need 3 lin. indp Eigenvec-tors. $\therefore$ Not Diagonalizable.

b) $A = \begin{pmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{pmatrix}$

$$\det(A - \lambda I) = \det \begin{pmatrix} -1-\lambda & 0 & 1 \\ 3 & -\lambda & -3 \\ 1 & 0 & -1-\lambda \end{pmatrix}$$

$$= (-1-\lambda)(-\lambda(-1-\lambda)) + 1(0-(-\lambda))$$

$$= -(1+\lambda)(\lambda+\lambda^2) + \lambda$$

$$= -\lambda(1+\lambda)^2 + \lambda$$

$$= \lambda(-(1+\lambda)^2 + 1)$$

$$= \lambda(-(\lambda^2+2\lambda+1)+1)$$

$$= \lambda(-\lambda^2 - 2\lambda \cancel{-1} \cancel{+1})$$

$$= \lambda(-\lambda^2 - 2\lambda)$$

$$= -\lambda^2(\lambda+2)$$

$\therefore \lambda = 0_{AM=2}, \; -2_{AM=1}$

Eigenspace of $\lambda_1 = 0$,

$$(A - 0(\mathbb{I}))X = 0$$

$$\Rightarrow AX = 0$$

$$= \begin{pmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$-x + z = 0$$

$$x = z$$

$$E_{\lambda=0} = \begin{pmatrix} x \\ y \\ x \end{pmatrix}, \quad x, y \in \mathbb{R}$$

$$= x\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad x, y \in \mathbb{R}$$

Eigenvalue of $\lambda = -2$

$$= (A + 2\mathbb{I})X = 0$$

On solving we will get an Eigenvector lin indp to the prev 2 eigenvectors. Solve it later

Example: Compute A^{100} if $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$. Eigenvalues are $2, -1$.

Corresponding Eigenvectors = $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

- We know that if A is diagonalizable, $A = P^{-1}DP$

Then $A^n = P^{-1}D\cancel{P}\cancel{P^{-1}}DPP^{-1}DP \ldots$ n times

$$\boxed{A^n = P^{-1}D^nP}$$

$$D = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow D^n = \begin{pmatrix} 2^n & 0 \\ 0 & (-1)^n \end{pmatrix}$$

$$\Rightarrow D^{100} = \begin{pmatrix} 2^{100} & 0 \\ 0 & 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$

$$P^{-1} = \frac{1}{\det(P)} \begin{pmatrix} -1 & -2 \\ -1 & 1 \end{pmatrix}^T$$

$$= \frac{1}{-1-2} \begin{pmatrix} -1 & -1 \\ -2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1/3 & 1/3 \\ 2/3 & -1/3 \end{pmatrix}$$

$$\Rightarrow A^{100} = P^{-1}D^{100}P = \begin{pmatrix} 1/3 & 1/3 \\ 2/3 & -1/3 \end{pmatrix} \begin{pmatrix} 2^{100} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$

→ Orthogonal Diagonalization Of Symmetric Matrix :-

A square matrix A is termed as Orthogonally Diagonalizable if $\exists$ an orthogonal matrix Q and a diagonal matrix D such that,

$$\boxed{Q^T A Q = D}$$

Example: $A = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}$

$$\det(A - \lambda I) = \begin{pmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{pmatrix}$$

$$= (1-\lambda)(-2-\lambda) - 4$$
$$= (\lambda-1)(2+\lambda) - 4$$
$$= 2\lambda + \lambda^2 - 2 - \lambda - 4$$
$$= \lambda^2 + \lambda - 6 = 0$$
$$= \lambda^2 + 3\lambda - 2\lambda - 6 = 0$$
$$= \lambda(\lambda+3) - 2(\lambda+3) = 0$$
$$= \underline{\underline{\lambda = 2, -3}}$$

$$(A - 2I)X = 0$$

$$\Rightarrow \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-x + 2y = 0, \quad 2x - 4y = 0$$

$$x = 2y \qquad 2x = 4y$$

$$\Rightarrow x = 2y$$

$$\Rightarrow E_{\lambda=2} = \begin{bmatrix} 2y \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} y, \quad y \in \mathbb{R}$$

$$(A + 3I)x = 0$$

$$= \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$= \quad 4x + 2y = 0 \qquad 2x + y = 0$

$\qquad 4x = -2y \qquad 2x = -y$

$\qquad 2x = -y \qquad \underline{y = -2x}$

$$\Rightarrow \quad E_{\lambda=-3} = \begin{bmatrix} x \\ -2x \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} x, \quad x \in \mathbb{R}$$

$\therefore$ Eigenvectors : $\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$$P = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \qquad D = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$$

Already orthogonal.

Normalized $\Rightarrow$ $P = \begin{pmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{pmatrix} \rightarrow \underline{\underline{Q}}$

- <u>Theorem</u> :

If A is symmetric, then A is orthogonally diagonalizable.

Proof:

If A is orthogonally diagonalizable,

$$A = Q^T D Q$$

$$A^T = ((Q^T D) Q)^T$$

$$= Q^T (Q^T D)^T$$

$$= Q^T D^T Q$$

$$= Q^T D Q = A$$

$\Rightarrow A^T = A \Rightarrow A$ is symmetric.

- Theorem: If A is real symmetric matrix, then the Eigenvalues of A are real.

- Theorem: If A is symmetric then any 2 Eigenvectors of A are orthogonal.

Proof:

$$Av = \lambda v$$

Let $\overline{A}$ be the conjugate of A.

$$\Rightarrow \overline{Av} = \overline{\lambda v}$$

$$= \overline{A}\,\overline{v} = \overline{\lambda}\,\overline{v}$$

$$= A\bar{v} = \bar{\lambda}\bar{v}$$

$$\bar{v}^T A = \bar{v}^T A^T$$

$$= (A\bar{v})^T$$

$$= (\bar{\lambda}\bar{v})^T$$

$$= \bar{\lambda}\bar{v}^T$$

$$\lambda \bar{v}^T v = \bar{v}^T(\lambda v) = \bar{v}^T A v = (\bar{v}^T A) v = (\bar{\lambda}\bar{v}^T) v$$

$$= \bar{\lambda}\bar{v}^T v$$

$$\Rightarrow \lambda \bar{v}^T v = \bar{\lambda}\bar{v}^T v$$

$$\Rightarrow \lambda = \bar{\lambda} \Rightarrow \lambda \in \mathbb{R}$$

Proof (2):

Let v_1 and v_2 be 2 Eigenvectors of 2 distinct Eigenvalues λ_1, λ_2 of A.

$$\lambda \langle v_1, v_2 \rangle = \langle \lambda v_1, v_2 \rangle$$

$$= \langle A v_1, v_2 \rangle$$

$$= (A v_1)^T v_2$$

$$= v_1^T A^T v_2$$

$$= v_1^T A v_2 = v_1^T \lambda_2 v_2$$

$$= \lambda_2 v_1^T v_2 = \lambda_2 \langle v_1, v_2 \rangle$$

$$\Rightarrow (\lambda_1 - \lambda_2)\langle v_1, v_2\rangle = 0$$

$$\Rightarrow \lambda_1 = \lambda_2 \quad (\text{or}) \quad \underline{\langle v_1, v_2\rangle = 0}$$

→ **Spectral Theorem :-**

Let A be a $n \times n$ matrix. Then A is symmetric iff it is orthogonally diagonalizable.

- The Spectral decomposition theorem allows us to write a real symmetric matrix in the form,

$$A = QDQ^T \quad (Q - \text{Orthogonal}, D - \text{Diagonal})$$

$$\underset{n\times 1}{\bar{q}_1} \ \underset{1\times n}{\bar{q}_1^T}$$

$$= \underset{n\times n}{\begin{bmatrix} \bar{q}_1 & \bar{q}_2 & \cdots & \bar{q}_n \end{bmatrix}} \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \cdots & & & & \cdots \end{bmatrix} \begin{bmatrix} \bar{q}_1^T \\ \bar{q}_2^T \\ \vdots \end{bmatrix}$$

$$= \lambda_1 \bar{q}_1 \bar{q}_1^T + \lambda_2 \bar{q}_2 \bar{q}_2^T \ldots$$

$$\Rightarrow \boxed{A = \sum_{i=1}^{n} \lambda_i \bar{q}_i \bar{q}_i^T} \qquad \bar{q}_i = i\text{th column of } Q.$$

Example: Find the spectral decomposition form of $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

On calculation of Eigenvalues and vectors, we get,

$$\lambda_1 = 4, \ \lambda_2 = 1, \ \lambda_3 = 1$$

$$\mathcal{E}_{\lambda_1} = t\begin{pmatrix}1\\1\\1\end{pmatrix}, \ \forall t \in \mathbb{R}$$

$$\mathcal{E}_{\lambda_2} = t\begin{pmatrix}-1\\0\\1\end{pmatrix} + p\begin{pmatrix}-1\\1\\0\end{pmatrix}, \ \forall t, p \in \mathbb{R}$$

$$\Rightarrow \ \overline{v_1} = \begin{pmatrix}1\\1\\1\end{pmatrix} \quad \overline{v_2} = \begin{pmatrix}-1\\0\\1\end{pmatrix} \quad \overline{v_3} = \begin{pmatrix}-1\\1\\0\end{pmatrix}$$

Gram Schmidt

$$\Rightarrow \ \overline{q_1} = \begin{pmatrix}1\\1\\1\end{pmatrix} \quad \overline{q_2} = \begin{pmatrix}-1\\0\\1\end{pmatrix} \quad \overline{q_3} = \begin{pmatrix}-1/2\\1\\-1/2\end{pmatrix}$$

Normalize

$$\Rightarrow \ \overline{q_1} = \begin{pmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix} \quad \overline{q_2} = \begin{pmatrix}-1/\sqrt{2}\\0\\1/\sqrt{2}\end{pmatrix} \quad \overline{q_3} = \begin{pmatrix}-1/\sqrt{6}\\2/\sqrt{6}\\-1/\sqrt{6}\end{pmatrix}$$

$$\Rightarrow \ Q = \begin{bmatrix}1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6}\\ 1/\sqrt{3} & 0 & 2/\sqrt{6}\\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6}\end{bmatrix}$$

$$\overline{q_1}\,\overline{q_1}^T = \begin{bmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix}\begin{bmatrix}1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3}\end{bmatrix} = \begin{bmatrix}1/3 & 1/3 & 1/3\\ 1/3 & 1/3 & 1/3\\ 1/3 & 1/3 & 1/3\end{bmatrix}$$

$$\bar{q}_2 \bar{q}_2^T = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & -1/2 \\ 0 & 0 & 0 \\ -1/2 & 0 & 1/2 \end{bmatrix}$$

$$\bar{q}_3 \bar{q}_3^T = \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix} \begin{bmatrix} -1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} \end{bmatrix} = \begin{bmatrix} 1/6 & -2/6 & 1/6 \\ -2/6 & 4/6 & -2/6 \\ 1/6 & -2/6 & 1/6 \end{bmatrix}$$

$$\Rightarrow \quad A = 4 \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} + 1 \begin{bmatrix} 1/2 & 0 & -1/2 \\ 0 & 0 & 0 \\ -1/2 & 0 & 1/2 \end{bmatrix}$$

$$+ 1 \begin{bmatrix} 1/6 & -1/3 & 1/6 \\ -1/3 & 2/3 & -1/3 \\ 1/6 & -1/3 & 1/6 \end{bmatrix}$$

<u>Example</u>: Find a 2×2 matrix whose eigenvalues are $\lambda_1 = 3$, $\lambda_2 = -2$, and the corresponding eigenvectors are $\bar{v}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ $\bar{v}_2 = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$

$$\bar{v}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \quad \bar{v}_2 = \begin{bmatrix} -4 \\ 3 \end{bmatrix} \Rightarrow \quad \bar{q}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$\bar{q_2} = \bar{v_2} + \frac{\langle \bar{v_2} \cdot \bar{q_1} \rangle}{\langle \bar{q_1} \cdot \bar{q_1} \rangle} \bar{q_1}$$

$$= \begin{bmatrix} -4 \\ 3 \end{bmatrix} + \frac{\cancel{(-12) + 12}}{\langle \bar{q_1} \cdot \bar{q_1} \rangle} \bar{q_1}$$

$$= \underline{\begin{bmatrix} -4 \\ 3 \end{bmatrix}}$$

$$\bar{q_1} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad \bar{q_2} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

Normalization

$$\Rightarrow \bar{q_1} = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix} \qquad \bar{q_2} = \begin{bmatrix} -4/5 \\ 3/5 \end{bmatrix}$$

$$\bar{q_1}\bar{q_1}^T = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix} \begin{bmatrix} 3/5 & 4/5 \end{bmatrix} = \begin{bmatrix} 9/25 & 12/25 \\ 12/25 & 16/25 \end{bmatrix}$$

$$\bar{q_2}\bar{q_2}^T = \begin{bmatrix} -4/5 \\ 3/5 \end{bmatrix} \begin{bmatrix} -4/5 & 3/5 \end{bmatrix} = \begin{bmatrix} 16/25 & -12/25 \\ -12/25 & 9/25 \end{bmatrix}$$

$$\Rightarrow A = 3 \begin{bmatrix} 9/25 & 12/25 \\ 12/25 & 16/25 \end{bmatrix} - 2 \begin{bmatrix} 16/25 & -12/25 \\ -12/25 & 9/25 \end{bmatrix}$$

→ <u>Quadratic form</u> :-

An expression of the form $ax^2 + bxy + cy^2$ is said to be in quadratic form in x, y.

- We can write $ax^2 + bxy + cx^2$ as,

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & c/2 \\ c/2 & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- IIIly for 3 variables x, y, z

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz$$

$$\Rightarrow \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a & d/2 & e/2 \\ d/2 & b & f/2 \\ e/2 & f/2 & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

- <u>Definition</u> :

A quadratic expression in n variables is a function $f : \mathbb{R}^n \to \mathbb{R}$, of the form,

$$f(\bar{x}) = \bar{x}^T A \bar{x} \quad , \quad \bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$$

where A is an $n \times n$ symmetric matrix

Example:

1) What is the quadratic form of $\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$

$$[x \;\; y]\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x-3y & -3x+5y \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= 2x^2 - 3xy + y(-3x+5y)$$

$$= 2x^2 - 3xy - 3xy + 5y^2$$

$$= 2x^2 - 6xy + 5y^2$$

→ Principal Axis Theorem :-

Every Quadratic form is diagonalizable.

Let A be an $n \times n$ symmetric matrix with quadratic form $x^T A x$.

$$A = Q^T D Q$$

Take, $x = Qy \Rightarrow y = Q^{-1}x \Rightarrow y = Q^T x$

$$x^T A x = (Qy)^T A (Qy)$$

$$= y^T Q^T A Q y$$

$$= \underline{\underline{y^T D y}}$$

Example: Find the change of variable such that the quadratic form of,

$$f(x_1, x_2) = 5x_1^2 + 4x_1x_2 + 2x_2^2 \Rightarrow \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}$$

such that $f(x_1, x_2) = y^T D y$.

- A Quadratic form is classified as

1) Positive definite : $f(x) > 0 \;\forall\; x \neq 0$ (Eigenvals are +ve)

2) Positive semidefinite : $f(x) \geq 0 \;\forall\; x \neq 0$ (Eigenvals are +ve or 0)

3) Negative Definite : $f(x) < 0 \;\forall\; x \neq 0$ (EV are −ve)

4) Negative Semidefinite : $f(x) \leq 0 \;\forall\; x \neq 0$ (EVs are −ve or 0)

5) Indefinite : $f(x)$ can take both +ve and −ve values. (EVs are both +ve and −ve)

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→ Singular Value Decomposition :-

For any $m \times n$ matrix A, $A^T A$ is a symmetric $n \times n$ matrix, and hence can be orthogonally diagonalized by the spectral theorem.

- The Eigenvalues of A^TA are real and non-negative.

Proof: Let $\{\lambda_i\}$ be the eigenvalues of A^TA with corresponding eigenvectors $\{\bar{v}_i\}$.

$$
\begin{aligned}
0 \leq \|Av\|^2 &= \langle Av \cdot Av \rangle \\
&= (Av)^T(Av) \\
&= v^TA^TAv \\
&= v^T(A^TAv) \\
&= v^T\lambda v \\
&= \lambda \|v\|^2
\end{aligned}
$$

$$\Rightarrow \quad \lambda \|v\|^2 \geq 0$$

Since $\|v\|^2 \geq 0$ (square no.), $\underline{\underline{\lambda > 0}}$

- If A is a $m \times n$ matrix then the singular values of A are the square roots of the eigenvalues of A^TA and are denoted by $s_1, s_2, s_3 \ldots, s_n$.

It is conventional to arrange the singular values in descending order.

Example: Find the singular values of $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$A^T A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

To get eigenvalues,

$$\det(A - \lambda I) = 0$$

$$= \det \begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} = 0$$

$$= (2-\lambda)^2 - 1 = 4 - 4\lambda + \lambda^2 - 1$$

$$= 3 - 4\lambda + \lambda^2$$

$$= 3 - 3\lambda - \lambda + \lambda^2$$

$$= 3(1-\lambda) - \lambda(1-\lambda)$$

$$= (3-\lambda)(1-\lambda)$$

$$\Rightarrow \underline{\lambda = 1, 3}$$

$\therefore$ singular values, $\underline{\sigma_1 = \sqrt{3},\ \sigma_2 = 1}$

We generally cannot compute the singular values for AA^T instead of A^TA.

- <u>Singular Value Decomposition Theorem</u>:

Let A be an $m \times n$ matrix with singular values $\delta_1, \delta_2 \ldots \delta_r$, $(\delta_1 \geq \delta_2 \geq \delta_3 \cdots \geq \delta_n \geq 0)$ and $\delta_{r+1} = \delta_{r+2} = \delta_{r+3} \cdots = \delta_n = 0$. Then there $\exists$ an $m \times m$ orthogonal matrix U, $n \times n$ orthogonal matrix V, $m \times n$ matrix Σ such that

$$\Sigma = \begin{array}{c|cc} & r & n-r \\ \hline r & \multicolumn{2}{c}{\multirow{2}{*}{$\left[\begin{array}{c|c} D & \bar{0} \\ \hline \bar{0} & \bar{0} \end{array}\right]_{m \times n}$}} \\ n-r & & \end{array}, \text{ where } D = \text{dia}(\delta_1, \delta_2, \delta_3 \ldots \delta_r)$$

such that $A = U \Sigma V^T$

- To construct V, we need to find an orthogonal basis $\{v_1, v_2, v_3 \ldots v_n\} \in \mathbb{R}$ consisting of the eigenvectors of the $n \times n$ symmetric matrix A^TA. Then $V = [v_1 \; v_2 \cdots v_n]$

With that, find $\{Av_1, Av_2, Av_3 \ldots, Av_n\}$, is an orthogonal set of vectors in $\mathbb{R}_n$:

Let $\bar{v_i}$ be the Eigenvector of A^TA corresponding to the Eigen-value λ_i. Then for $i \neq j$,

$$(A\bar{v_i}) \cdot (A\bar{v_j}) = (A\bar{v_i}^T) \cdot (A\bar{v_j})$$

$$= \bar{v}_i^T A^T A \bar{v}_j$$

$$A = U \Sigma V^T$$

$$= \bar{v}_i^T (A^T A \bar{v}_j)$$

$$AV = U\Sigma$$

$$= \bar{v}_i^T \lambda_j \bar{v}_j$$

$$= \lambda_j \bar{v}_i^T \cdot \bar{v}_j$$

$$= \lambda_j (\bar{v}_i \cdot \bar{v}_j) = \underline{\underline{0}}$$

Recall that,

$$s_i > 0 \quad \forall \ i \leq r$$ (When placed in descending order)

$$\bar{u}_i = \frac{1}{s_i} A \bar{v}_i \quad \forall \ i \leq r$$ will form the columns of U,

If $r < m$, the remaining columns of U need to be found by applying Gram - Schmidt's process.

(Find the Space of vectors orthogonal to $\bar{u}_1, \bar{u}_2 \ldots \bar{u}_r$ and take their basis and apply GSP)

Example: Find the singular value decomposition of

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{2 \times 3}$$

1) Constructing Σ.

$$A^T A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Eigenvalues of A^TA,

$$\det\begin{pmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{pmatrix} = (1-\lambda)^3 - 1(1-\lambda)$$

$$= (1-\lambda)((1-\lambda)^2-1)$$

$$= (1-\lambda)(1-2\lambda+\lambda^2-1)$$

$$= (1-\lambda)(-2\lambda+\lambda^2)$$

$$= \lambda(1-\lambda)(-2+\lambda)$$

$$= \lambda(1-\lambda)(\lambda-2)$$

$$\underline{\lambda = 0, 1, 2}$$

$\Rightarrow$ δ: $\delta_1 = \sqrt{2}$, $\delta_2 = 1$, $\delta_3 = 0$, $(r = 2)$

$$\Rightarrow \quad \Sigma = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

2) Construct V,

Eigenvectors of A^TA,

$$(A^TA - 2I)X = 0$$

$$= \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$-x+y=0 \Rightarrow x=y$$

$$x-y=0 \Rightarrow x=y$$

$$\underline{\underline{-z=0}} \Rightarrow z=0$$

$$\Rightarrow \quad E_{\lambda_1} = x\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \forall\ x \in \mathbb{R} \longrightarrow \bar{v}_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$

$$(A^TA - \mathbb{I})X = 0$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \quad \underline{y=0}$$

$$\underline{x=0}$$

$$\Rightarrow \quad E_{\lambda_2} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} . z \quad \forall\ z \in \mathbb{R} \longrightarrow \bar{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$A^TAX = 0$$

$$= \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$x+y=0 \Rightarrow x=-y$$

$$z=0$$

$$E_{\lambda_3} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} x \,, \forall x \in \mathbb{R} \longrightarrow \bar{v}_3 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}$$

$$\Rightarrow \bar{V} = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix}$$

$$\bar{u}_1 = \frac{1}{s_1} A \bar{v}_1$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\bar{u}_2 = \frac{1}{1} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix}$$

Example: Find the SVD of

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}_{3\times 2}$$

$\lambda_1 = 3, \ \lambda_2 = 1 \Rightarrow \delta_1 = \sqrt{3}, \ \delta_2 = 1$

$$\Rightarrow \quad \Sigma = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Eigenvectors,

$$A^T A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$(A^T A - 3I)X = 0$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$-x + y = 0 \Rightarrow x = y$

$x - y = 0 \Rightarrow x = y$

$\therefore E_{\lambda_1} = x\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ x \in \mathbb{R}$

$(A^T A - I)X = 0$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$x + y = 0 \Rightarrow x = -y$

$x + y = 0$

$\therefore E_{\lambda_2} = x\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \ x \in \mathbb{R}$

- For a matrix in its SVD form $A = U\Sigma V^T$, the vectors $\{u_1, u_2, u_3 \dots u_r\}$ are left singular vectors and $\{v_1, v_2, v_3 \dots v_r\}$ are right singular vectors.

$$A = U\Sigma V^T$$

$$= [u_1 \; u_2 \; u_3 \cdots u_r] \left[\begin{array}{ccc|c} \sigma_1 & 0 & \cdots & \\ 0 & \sigma_2 & \cdots & 0 \\ \cdots & \cdots & \sigma_r & \\ \hline & 0 & & 0 \end{array}\right] \begin{bmatrix} v_1^T \\ v_2^T \\ v_3^T \\ \vdots \end{bmatrix}$$

$$= \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T \cdots \sigma_r u_r v_r^T$$

$$\Rightarrow \boxed{A = \sum_{i=1}^{r} \sigma_i u_i v_i^T}$$

σ – Singular Values (Previously δ)

Outer Product Form

Example:

Find the outer product form of $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

wkt,

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}_{\Sigma} \underbrace{\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix}}_{V}$$

$$\Rightarrow A = \sqrt{2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

Find the Outer Product Form of $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}_{3\times 2}$

$$A^T A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$\det \begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} = (2-\lambda)^2 - 1$$

$$= 4 - 4\lambda + \lambda^2 - 1$$

$$= \lambda^2 - 4\lambda + 3$$

$$= (\lambda - 1)(\lambda - 3)$$

$\Rightarrow \quad \lambda = 1, \lambda = 3 \Rightarrow \quad \lambda_1 = 3, \lambda_2 = 1, \Rightarrow \quad \sigma_1 = \sqrt{3}, \sigma_2 = 1$

$$\Sigma = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 2-3 & 1 \\ 1 & 2-3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\Rightarrow \quad \underline{\underline{x = y}}$$

$$\Rightarrow \quad E_{\lambda_1} = x \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad x \in \mathbb{R}$$

$$\begin{pmatrix} 2-1 & 1 \\ 1 & 2-1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \quad y = -x$$

$$\Rightarrow \quad E_{\lambda_2} = x \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad x \in \mathbb{R}$$

$$\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix} \longrightarrow \begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}, \begin{pmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{pmatrix}$$

$$\Rightarrow \quad V = \begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2}\\1/\sqrt{2} & -1/\sqrt{2}\end{pmatrix}$$

$$\bar{u}_1 = \frac{1}{\sigma_1} A \bar{v}_1$$

$$= \frac{1}{\sqrt{3}} \begin{pmatrix}1 & 1\\1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}$$

$$= \frac{1}{\sqrt{3}} \begin{pmatrix}2\\1\\1\end{pmatrix} = \begin{pmatrix}2/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix}$$

$$\bar{u}_2 = \frac{1}{\sigma_2} A \bar{v}_2 = \frac{1}{1}\begin{pmatrix}1 & 1\\1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}0\\1\\-1\end{pmatrix}$$

$$\bar{u}_3 : \quad \langle \bar{u}_1, \bar{u}_3 \rangle = 0 \qquad \bar{u}_3 = \begin{pmatrix}a\\b\\c\end{pmatrix}$$

$$\Rightarrow \quad \frac{2}{\sqrt{3}} a + \frac{1}{\sqrt{3}} b + \frac{1}{\sqrt{3}} c = 0$$

$$\Rightarrow \quad b - c = 0 \Rightarrow b = c$$

$$\Rightarrow \quad 2a + 2b = 0$$

$$\Rightarrow a + b = 0 \Rightarrow a = -b$$

$$\Rightarrow \quad \begin{matrix} b = -a \\ c = -a \end{matrix} \quad \Rightarrow \quad a\begin{pmatrix}1\\-1\\-1\end{pmatrix} \quad a \in \mathbb{R}$$

$$\bar{u}_3 = \begin{pmatrix}1\\-1\\-1\end{pmatrix} - \frac{(0-1+1)}{(0+1+1)}\begin{pmatrix}0\\1\\-1\end{pmatrix} - \frac{(2-1-1)}{(4+1+1)}\begin{pmatrix}2\\1\\1\end{pmatrix}$$

$$= \begin{pmatrix}1\\-1\\-1\end{pmatrix} \longrightarrow \begin{pmatrix}1/\sqrt{3}\\-1/\sqrt{3}\\-1/\sqrt{3}\end{pmatrix}$$

$$\Rightarrow \quad U = \begin{pmatrix}2 & 0 & 1/\sqrt{3}\\1 & 1 & -1/\sqrt{3}\\1 & -1 & -1/\sqrt{3}\end{pmatrix} \longrightarrow U = \begin{pmatrix}2/\sqrt{6} & 0 & 1/\sqrt{3}\\1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3}\\1/\sqrt{6} & -1/\sqrt{2} & -1/\sqrt{3}\end{pmatrix}$$

$$\Rightarrow \quad A = \begin{pmatrix}2/\sqrt{6} & 0 & 1/\sqrt{3}\\1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3}\\1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3}\end{pmatrix}\begin{pmatrix}\sqrt{3} & 0\\0 & 1\\0 & 0\end{pmatrix}\begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2}\\1/\sqrt{2} & -1/\sqrt{2}\end{pmatrix}$$

$$\Rightarrow \quad A = \sqrt{3}\begin{pmatrix}2/\sqrt{6}\\1/\sqrt{6}\\1/\sqrt{6}\end{pmatrix}\begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix} + 1\begin{pmatrix}0\\1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}\begin{pmatrix}1/\sqrt{2} & -1/\sqrt{2}\end{pmatrix}$$

- **Pseudoinverse :**

The pseudoinverse of a matrix $A_{m\times n}$ is the $n\times m$ matrix defined by,

$$A^{+}_{n\times m} = V_{n\times n}\,\Sigma^{+}\,U^{T}_{m\times m}, \quad \Sigma^{+}_{n\times m} = \left[\begin{array}{c|c} D^{-1} & 0\\ \hline 0 & 0\end{array}\right]_{n\times m}$$

Example :

Find the pseudoinverse of $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

wkt,

$$A = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}_{\Sigma} \underbrace{\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix}}_{V^T}$$

$$\Rightarrow A^{+} = \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 & 0 \\ 1/2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 & 0 \\ 1/2 & 0 \\ 0 & 1 \end{pmatrix}$$

Example:

Find the pseudoinverse of $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$

wkt,

$$A = \begin{pmatrix} 2/\sqrt{6} & 0 & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

$$\Rightarrow A^+ = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{3} & -1/\sqrt{3} \end{pmatrix}$$

$$A^+ = \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 0 \\ 1/\sqrt{6} & -1/\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} 2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{3} & -1/\sqrt{3} \end{pmatrix}$$

$$= \begin{pmatrix} 2/6 & 4/6 & 4/6 \\ 2/6 & -2/6 & -2/6 \end{pmatrix} = \begin{pmatrix} 1/3 & 2/3 & 2/3 \\ 1/3 & -1/3 & -1/3 \end{pmatrix}$$

→ Complex Matrices :-

- A square complex matrix A will be Hermitian matrix if $\boxed{A^* = A}$.
- A square matrix U is unitary if $\boxed{U^* = U^{-1}}$. Complex Equivalent of Orthogonal matrix.
- In Complex domain, we have Unitary Diagonalization.
- Unitary Diagonalization :-

A complex matrix A is said to be unitarily diagonalizable if ∃ a unitary matrix U and diagonal matrix D such that,

$$\boxed{A = U^T D U}$$

- **Algorithm:**

- Compute The Eigenvalues of A
- Find the basis of each Eigenspace
- Orthogonalize them using Gram Schmidt (wrt Complex Dot Product)
- Form the column vectors of U with the above vectors.
- Profit.