

Prof: Prabhakar Bhimlapuram

Science - 1

Group - B

Marks :

Tutorial Quizzes

Endsem : 35 %

Midsem : 20 % $\pm 5\%$

Quiz : 10 + 10 %

In Class / Tut Quiz : 25 %

Science - 1

- Science explains everything on the time-scales, the length scales and the energy scales.

Time Scales : Climate Change (Decade - scale)
Bullet travel (Second - scale)
Molecular movement (Picosecond - scale)

Length Scales : High Energy Physics (Quark / lepton Scales)
Expansion of Universe (10^{29} m scale)

Energy Scale : Energy to lift a laptop
" " build a building

- There are a variety of things on these scales that we use in our daily life .

→ What is Science ? :

- Systematic investigation of natural phenomenon .

- Reproducibility is a key feature of Science . Any experiment must be reproducible . Even a single datapoint is sufficient to disprove a theory .

- We can use the data gathered from the experiment, to create a mathematical model, than can be used to predict things.

Mathematical modelling has enabled the development of sciences.

- Science will try to explain the behavior / reason behind the parameters present in the model, and also the limitations / inaccuracies in the model.



→ Newton's first law is not directly provable, since it requires highly ideal conditions. It is only true at asymptotical conditions.

→ Limitation of Newton's laws.

- Reductionism: Separation of the most important things and neglecting every other factor. ex: The radius of planets while observing their orbital behavior around the sun, through Kepler's planetary laws.

" A System is a sum of its parts " (Superposition-like)

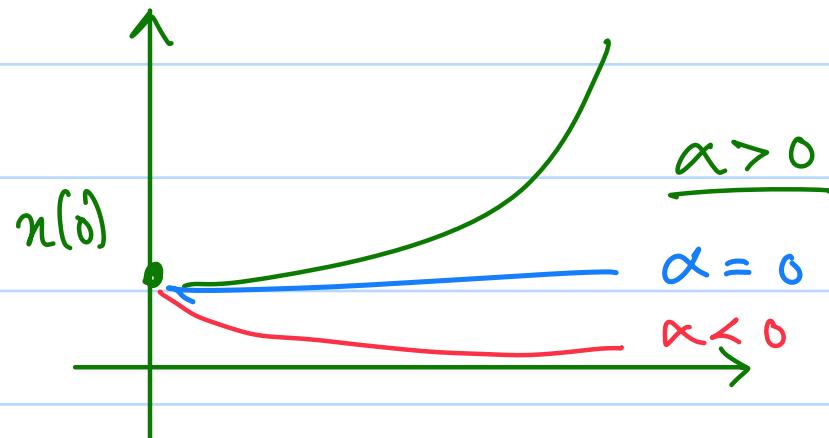
By decomposing a system into its pieces, we can figure out the interactions between the parts and also the most important parts of the system.

- Microscopics: The study behind the various interactions between the parts of a system.
- A reductionism model can be used to accurately predict the behavior of the model.
- Reductionism goes from microscopic behavior to macroscopic features.
- Even microscopic interactions follow laws, like conservation of energy.

→ Mathematics in Science:

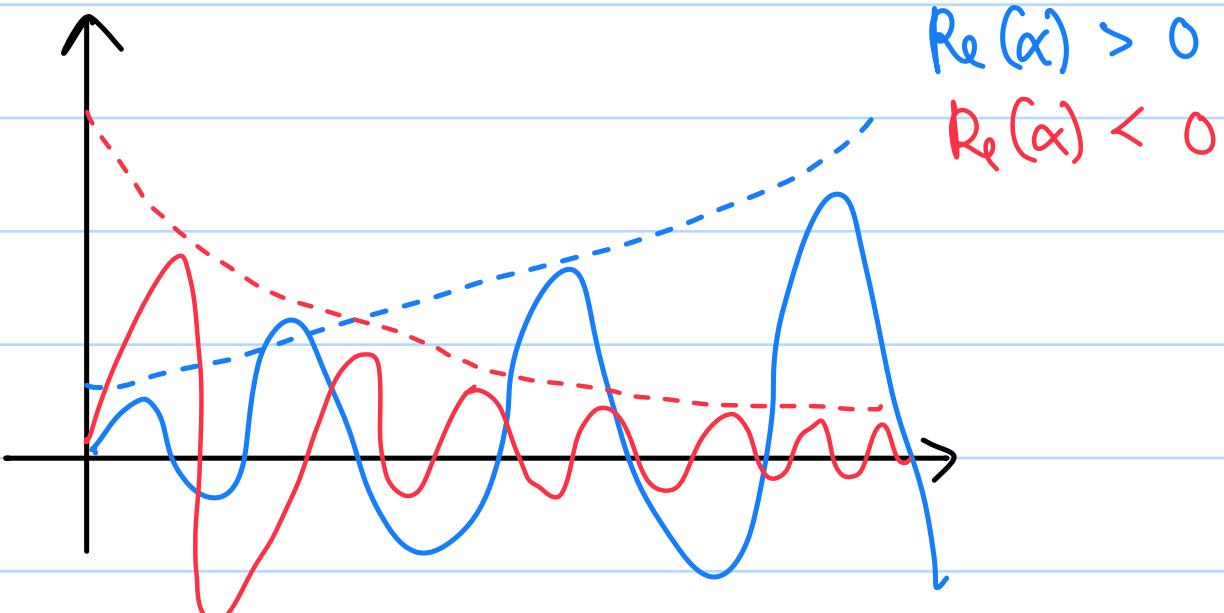
- Geometry and Linear Algebra
- Change and Calculus
- Chance, probability and statistics. → Uncertainty Principle

ex: The Behavior of a system with $n(t) = n(0) e^{\alpha t}$



In the above system, the value of the parameter can fundamentally change the behavior of the system.

If $\alpha \in \mathbb{C}$



→ The Prey-Predator Model :-

- x - No. of Rabbits

- y - No. of Tigers

- Parameters :

- α - Growth rate of Rabbits

- β - Effect of tigers on the rabbits (Rabbits are killed)

- γ - Death rate of Foxes

- δ - Effect of rabbit on foxes (Foxes can reproduce with nutrition)

$$\frac{dx}{dt} = \alpha x - \beta xy \quad \frac{dy}{dt} = -\gamma y + \delta xy$$

$$\Rightarrow \frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha x - \beta xy \\ -\gamma y + \delta xy \end{pmatrix} \rightarrow \begin{matrix} f \\ g \end{matrix}$$

$$\hookrightarrow \gamma \begin{pmatrix} x \\ y \end{pmatrix} = \nearrow$$

- The equilibrium points of the system,

$$\frac{dx}{dt} = 0$$

$$\frac{dy}{dt} = 0$$

$$\Rightarrow \alpha x - \beta ny = 0$$

$$\Rightarrow -\gamma y + \delta xy = 0$$

$$\Rightarrow n(\alpha - \beta y) = 0$$

$$\Rightarrow y(-\gamma + \delta x) = 0$$

$$\Rightarrow x = 0, y = \frac{\alpha}{\beta}$$

$$\Rightarrow y = 0, x = \frac{\gamma}{\delta}$$

$\Rightarrow (0,0), \left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$ are the equilibrium points.

$$f(x,y) \quad \begin{cases} SF_x & SF_y \end{cases}$$

Jacobian of the system,

$$\begin{pmatrix} \alpha x - \beta ny \\ -\gamma y + \delta xy \end{pmatrix} \xrightarrow{J} \begin{pmatrix} \alpha - \beta y & -\beta n \\ \delta y & -\gamma + \delta x \end{pmatrix} = J$$

$$J\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right) = \begin{pmatrix} 0 & -\beta\gamma/\delta \\ \delta\alpha/\beta & 0 \end{pmatrix} \rightarrow \gamma'$$

Let $(x_0, y_0) = \left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$ and define $\varepsilon = x - x_0, n = y - y_0$.

Around the equilibrium point, the function can be approximated as,

$$\frac{d}{dt} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} = J \Big|_{(x_0, y_0)} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + \begin{pmatrix} \overset{0}{(x-x_0)} \\ \overset{0}{(y-y_0)} \end{pmatrix}$$

$$\Rightarrow \frac{d}{dt} (\varepsilon) = \begin{pmatrix} 0 & -\beta\gamma/\delta \\ \delta\alpha/\beta & 0 \end{pmatrix} (\varepsilon)$$

$$\approx \frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -\beta y/\delta \\ \delta x/\beta & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \text{near the equilibrium of } \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

Note: The approximation of the function around (x_0, y_0) is done using the result,

$$\begin{aligned} h(x, y) &= h(x_0, y_0) + \left[\frac{\partial h}{\partial x} \Big|_{(x_0, y_0)} (x - x_0) + \frac{\partial h}{\partial y} \Big|_{(x_0, y_0)} (y - y_0) \right] \\ &= h(x_0, y_0) + \left[\begin{pmatrix} \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{pmatrix} \Big|_{(x_0, y_0)} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} \right] \\ &\quad \text{is from the Jacobian} \end{aligned}$$

Linearization

All analysis can be done near $(0, 0)$.

$$\text{Jacobian near } (0, 0), \quad J \Big|_{(0, 0)} = \begin{pmatrix} \alpha & 0 \\ 0 & -\delta \end{pmatrix}$$

$$\Rightarrow \frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & -\delta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad dt = \frac{dn}{\alpha n}$$

$$\Rightarrow \frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha n \\ -\delta y \end{pmatrix} \rightarrow \begin{aligned} \frac{dn}{dt} &= \alpha n \\ \frac{dy}{dt} &= -\delta y \end{aligned}$$

$$\Rightarrow x = c_1 e^{\alpha t}, \quad y = c_2 e^{-\delta t}$$

$$\Rightarrow x = x(0) e^{\alpha t}, \quad y = y(0) e^{-\delta t}$$

For the other Eq. point, $\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$,

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -\beta\gamma/\delta \\ \delta\alpha/\beta & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \frac{d\bar{x}}{dt} = \bar{J}\bar{x}$$

The general solution of such a system is of the form,

$$\bar{x}(t) = c_1 e^{\lambda_1 t} \bar{v}_1 + c_2 e^{\lambda_2 t} \bar{v}_2$$

where, \bar{v}_1, \bar{v}_2 are the Eigenvectors of J , λ_1, λ_2 are their respective Eigenvalues.

The Eigenvalues of J are,

$$\det \begin{pmatrix} -\lambda & -\beta\gamma/\delta \\ \delta\alpha/\beta & -\lambda \end{pmatrix} = 0$$

$$= \lambda^2 - \left(\frac{-\beta\gamma}{\delta}\right) \left(\frac{\delta\alpha}{\beta}\right) = 0$$

$$= \lambda^2 - (-\alpha\gamma) = 0$$

$$= \lambda^2 = -\alpha\gamma$$

$$= \lambda = \pm i\sqrt{\alpha\gamma}$$

for $\lambda_1 = i\sqrt{\alpha\gamma}$,

$$\begin{pmatrix} -i\sqrt{\alpha\gamma} & -\frac{\beta\gamma}{s} \\ \frac{s\alpha}{\beta} & -i\sqrt{\alpha\gamma} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow -x_1 i\sqrt{\alpha\gamma} - \frac{\beta\gamma}{s} x_2 = 0$$

$$\Rightarrow x_1 i\sqrt{\alpha\gamma} = -\frac{\beta\gamma}{s} x_2$$

$$x_1 = -\frac{\beta\gamma}{i s \sqrt{\alpha\gamma}} x_2 = -\frac{\beta}{i s \sqrt{\alpha}} x_2$$

$$\frac{s\alpha}{\beta} x_1 - i\sqrt{\alpha\gamma} x_2 = 0$$

$$= x_1 = \frac{i\sqrt{\alpha\gamma}\beta}{s\alpha} x_2 = \frac{i\beta}{s} \sqrt{\frac{\gamma}{\alpha}} x_2$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{i\beta}{s} \sqrt{\frac{\gamma}{\alpha}} \\ 1 \end{pmatrix} = \vec{v}_1$$

For $\lambda_2 = -i\sqrt{\alpha\gamma}$

$$\begin{pmatrix} i\sqrt{\alpha\gamma} & -\frac{\beta\gamma}{s} \\ \frac{s\alpha}{\beta} & i\sqrt{\alpha\gamma} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$= i\sqrt{\alpha\gamma} x_1 - \frac{\beta\gamma}{s} x_2 \Rightarrow x_1 = \frac{\beta}{is} \sqrt{\frac{\gamma}{\alpha}} = \frac{-i\beta}{s} \sqrt{\frac{\gamma}{\alpha}}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -i\beta/s \sqrt{\gamma/\alpha} \\ 1 \end{pmatrix} = \vec{v}_2$$

$$\Rightarrow \vec{r}(t) = c_1 e^{i\sqrt{\alpha\gamma}t} \begin{pmatrix} i\beta/s \sqrt{\gamma/\alpha} \\ 1 \end{pmatrix} + c_2 e^{-i\sqrt{\alpha\gamma}t} \begin{pmatrix} -i\beta/s \sqrt{\gamma/\alpha} \\ 1 \end{pmatrix}$$

$$= \bar{r}(t) = c_1 \left(\cos(\sqrt{\alpha\gamma}t) + i \sin(\sqrt{\alpha\gamma}t) \right) \begin{pmatrix} \frac{i\beta\sqrt{r}}{s\sqrt{\alpha}} \\ 1 \end{pmatrix} \\ + c_2 \left(\cos(\sqrt{\alpha\gamma}t) - i \sin(\sqrt{\alpha\gamma}t) \right) \begin{pmatrix} -\frac{i\beta\sqrt{r}}{s\sqrt{\alpha}} \\ 1 \end{pmatrix}$$

$$\Rightarrow \bar{r}(t) = \left[c_1 \left(\frac{i\beta\sqrt{r}}{s\sqrt{\alpha}} \right) + c_2 \left(\frac{-i\beta\sqrt{r}}{s\sqrt{\alpha}} \right) \right] \cos(\sqrt{\alpha\gamma}t) \\ + \left[c_1 \left(\frac{i\beta\sqrt{r}}{s\sqrt{\alpha}} \right) + c_2 \left(\frac{-i\beta\sqrt{r}}{s\sqrt{\alpha}} \right) \right] i \sin(\sqrt{\alpha\gamma}t)$$

→ Common Differential Equations :-

- $\frac{dx}{dt} = \alpha x \Rightarrow x = c_1 e^{\alpha t}$

- $\frac{\delta^2 f}{\delta t^2} = c^2 \frac{\delta^2 f}{\delta x^2} \rightarrow \text{Diffusion Equation}$

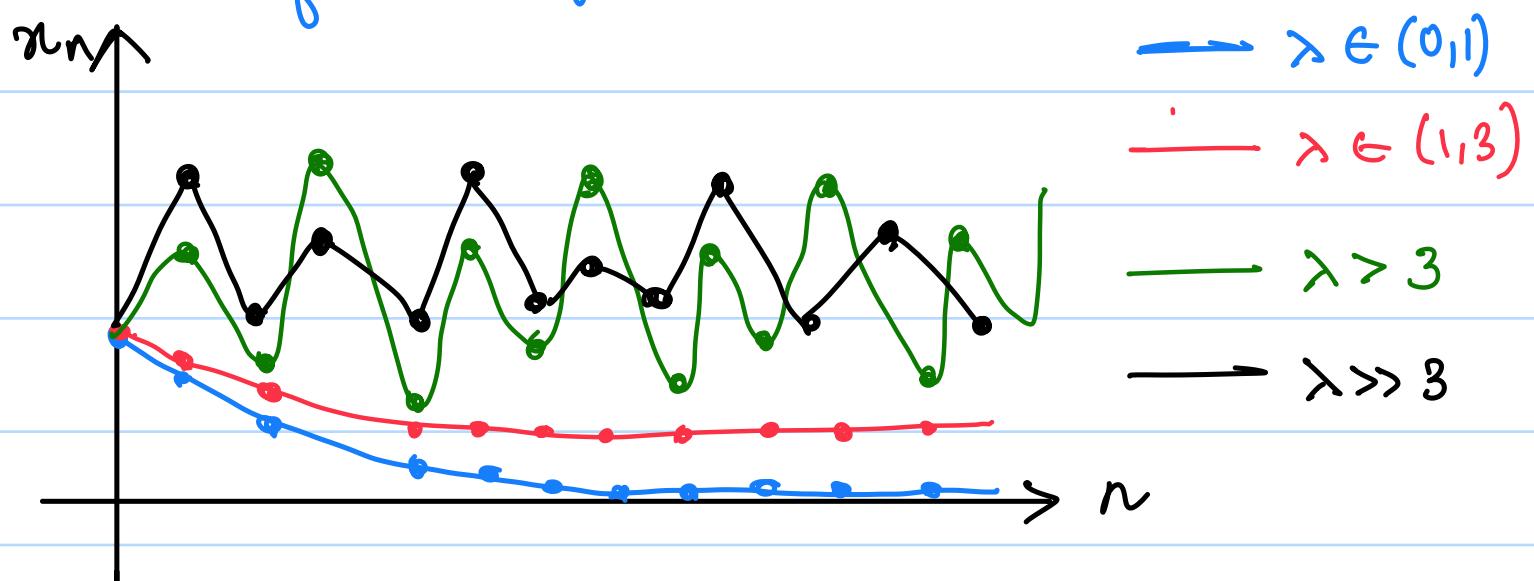
- $\frac{\delta f}{\delta t} = c^2 \frac{\delta^2 f}{\delta x^2} \longleftrightarrow \text{Quantum Mechanics}$

→ Logistics Mapping :-

- A sequence such that,

$$x_{n+1} = \lambda x_n (1 - x_n), \quad \forall n \in \mathbb{N}$$

- Analyzing the terms of the sequence,



Therefore, depending on the value of λ , the same initial value can result in very different sequences, and for some λ , small differences in initial value can result in large differences in the final value.

- Therefore, this sequence is said to exhibit chaotic behavior.

→ Classical Mechanics :-

- Scalar Field: Each point in space is associated with a scalar quantity. ex: Temperature
- Vector Field: Each point in space is associated with a vector of some direction and magnitude. ex: Air Speeds.
- Gradient: The change of a quantity within its field.

$$\nabla f(x_0, y_0, z_0) = \left[\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y} \quad \frac{\partial f}{\partial z} \right]$$

- Divergence: Scalar that measures the outflowness of a vector field at a point.

Laws Of Motion :

- 1) In an inertial frame, an object undergoes uniform rectilinear motion in the absence of external force.
- 2) Rate of change of momentum is equal to the net force on the object.
- 3) Reaction force is equal and opposite to the action force.

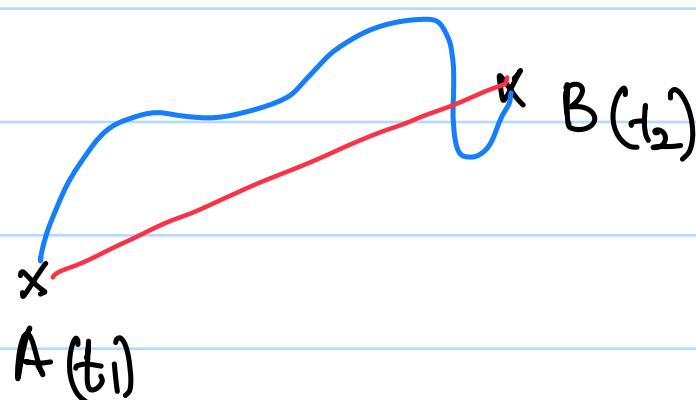
- Work: The work done by the application of an external force is given by,

$$W = \int_{t_1}^{t_2} \bar{F}(t) d\bar{r}(t) \quad \text{infinitesimal } F_x$$

Using the 2nd Law,

$$W = \int_{t_1}^{t_2} \frac{d\bar{p}}{dt} v(t) dt = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

$\Rightarrow W = KE_2 - KE_1 \rightarrow$ Work is just change in kinetic energy.



Different paths may have different amounts of work done.
(No. difference in conservative fields)

In vectorial form, $KE = \frac{1}{2}m(\bar{v} \cdot \bar{v})$

- Conservative Forces: Forces that can be modelled as a gradient of a scalar field, i.e.,

$$\bar{F}(\bar{r}) = -\nabla U(\bar{r})$$

$$\Rightarrow W = \int \bar{F}(\bar{r}) d\bar{r} = - (U_2 - U_1)$$

- The scalar quantity (field $U(\bar{r})$) is the potential energy due to the force.

- Since Work now only depends on PE at 1 and 2, it does not depend on the exact path travelled between 1 and 2.

$$W = K_2 - K_1 = - (U_2 - U_1)$$

$$\Rightarrow K_2 + U_2 = K_1 + U_1 \rightarrow \text{Law of Conservation of Mechanical Energy}$$

- Conservation of Linear Momentum :-

- For an object, if $\bar{F} \cdot \bar{s} = 0$ (\bar{F} is zero or \perp dir to \bar{s})

$$\bar{F} \cdot \bar{s} = 0 \Rightarrow \frac{d\bar{p}}{dt} \cdot \bar{s} = 0 \Rightarrow \frac{d(\bar{p} \cdot \bar{s})}{dt} = 0 \quad (\text{Assuming } \bar{s} \text{ is const})$$

$\Rightarrow \bar{p} \cdot \bar{s}$ is constant \Rightarrow Momentum in the direction of s is conserved

∴ If force in a direction is zero, momentum is conserved in that direction.

- Torque and Conservation of Angular Momentum :-

$$\bullet \text{Angular momentum } \bar{L} = \bar{r} \times \bar{p}$$

$$\bullet \text{Torque } \bar{\tau} = \frac{d}{dt} \bar{L} = \frac{d}{dt} \bar{r} \times \bar{p} = \frac{d\bar{r}}{dt} \times \bar{p} + \bar{r} \times \frac{d\bar{p}}{dt}$$

$$\bar{\tau} = \cancel{\bar{r} \times \bar{p}}^0 + \bar{r} \times \bar{F}$$

$$\Rightarrow \bar{\tau} = \bar{r} \times \bar{F}$$

- ## • Conservation of Angular Momentum,

If $\bar{\tau} \cdot \bar{s} = 0$, then $\bar{E} \cdot \bar{s}$ is constant.

\therefore Angular momentum is conserved along the axis of zero torque.

$$E = T(\bar{v}(t)) + U(\bar{r}(t), t)$$

$$\frac{dT}{dt} = m\bar{v} \frac{d\bar{v}}{dt} = m\bar{v} \cdot \bar{a} = \bar{V} \cdot \bar{f}$$

$$\frac{dU(\vec{r}, t)}{dt} = \frac{\partial U}{\partial \vec{r}} \frac{d\vec{r}}{dt} + \frac{\partial U}{\partial t} = \vec{V} \cdot \Delta \vec{v} + \frac{\partial U}{\partial t}$$

$$\Rightarrow \frac{dE}{dt} = \bar{v} (\vec{F} + \cancel{\Delta U}) + \frac{\partial U}{\partial t}$$

$$\Rightarrow \frac{dE}{dt} = \frac{\partial U}{\partial t} \Rightarrow \text{If PE is constant, E is conserved}$$

- ## o Galilean Relativity :-

Let there be 2 inertial frames 'K' and 'K'
 and K_r is at relative velocity \bar{v} wrt K. Let an object be
 moving with velocity \bar{J} and \bar{v}_r , ie

$$\overline{c_1} = \overline{c_1} - \overline{v}$$

$$\Rightarrow \frac{d\bar{V}_I}{dt} = \frac{d(\bar{U} - \bar{V})}{dt} = \frac{d\bar{U}}{dt} \quad \bar{V} \text{ is const}$$

$$\Rightarrow \bar{F}_I = \bar{F}$$

\therefore All mechanical laws are the same in any inertial frame.

\rightarrow Simple Harmonic Motion :-

- Hooke's Law : $F(x) = -kx \Rightarrow m \frac{d^2x}{dt^2} = -kx$

To solve the DE,

$$m \frac{d^2x}{dt^2} = -kx$$

$$\text{Take } p = m \frac{dx}{dt} \Rightarrow m \frac{d^2x}{dt^2} = \frac{dp}{dt}$$

$$\Rightarrow \frac{dp}{dt} = -kx \Rightarrow dp = -kx dt$$

$$\therefore p = [-kxt]_0^t = -kxl$$

$$\frac{dx}{dt} = \frac{p}{m}$$

\Rightarrow



Is too tuff :(

→ Limitations of Newtonian Mechanics :-

- Galilean relativity violated by EM waves.
- Stability of atoms cannot be explained by Newtonian mechanics.

→ Special Theory of Relativity :-

The postulates of STR are as follows,

- i) The laws of physical phenomena are the same in all inertial reference frames.
- ii) The velocity of light (in free space) is a universal constant, independent of any relative motion.

◦ Galilean Invariance :-

Consider a frame K and another frame K' moving at a speed of v_r along the x-axis.

- Let (x, y, z) be coordinates in K and (x', y', z') in K'. The transformation of coordinates is given by,

$$x' = x - v_r t$$

$$y' = y$$

$$z' = z$$

$$t' = t$$

- The above transformation is termed as a Galilean Transformation.
- Length is conserved in a Galilean transformation,

$$ds^2 = \sum dx_i^2 = \sum dx_i'^2 = ds'^2$$

Newton's laws are also conserved,

$$F = ma = ma' = F'$$

- Velocity in K' is given by,

$$v' = v - v_r$$

That means the velocity of light in K' is $c - v_r \neq c$, in violation of STR.

- Lorentz Transformation :-

- If a light pulse is emitted from the common origin of the systems K and K₁ when they are coincident, then,

$$x = ct$$

$$x_1 = c t_1$$

$$\text{Let } x_1 = \gamma(x - v_r t) \Rightarrow x = \gamma'(x_1 + v_r t_1)$$

Since the laws of physics must be the same in K and K',
 $\gamma = \gamma'$.

$$\Rightarrow x = \gamma(v(x - v_r t) + v_r t_1)$$

$$\Rightarrow x = \gamma^2(x - v_r t) + \gamma v_r t_1$$

$$\Rightarrow x = x\gamma^2 - \gamma^2 v_r t + \gamma v_r t_1$$

$$\Rightarrow \gamma v_r t_1 = x - x\gamma^2 + \gamma^2 v_r t \\ = x(1 - \gamma^2) + \gamma^2 v_r t$$

$$\Rightarrow t_1 = \frac{x}{v_r} (1 - \gamma^2) + rt$$

$$= t_1 = \frac{x}{v_r} \left(\frac{1 - \gamma^2}{\gamma} \right) + rt \quad \text{--- (1)}$$

$$x = ct \Rightarrow x' = \gamma(ct - v_r t) - \gamma(c - v_r)t$$

$$\Rightarrow t_1 = \frac{ct}{v_r} \left(\frac{1 - \gamma^2}{\gamma} \right) + rt$$

$$x_1 = ct_1 \Rightarrow c = \frac{x_1}{t_1} = \frac{\gamma(c - v_r)t}{\frac{ct}{v_r} \left(\frac{1 - \gamma^2}{\gamma} \right) + rt}$$

$$\Rightarrow c = \frac{\gamma(c - v_r)}{\frac{c}{v_r} \left(\frac{1 - \gamma^2}{\gamma} \right) + \gamma}$$

$$\Rightarrow \gamma(c - v_r) = \frac{c^2}{v_r} \left(\frac{1 - \gamma^2}{\gamma} \right) + cr$$

$$\Rightarrow \gamma c - \gamma v_r = \frac{c^2}{v_r} \left(\frac{1 - r^2}{r} \right) + \gamma$$

$$= \gamma v_r = \frac{c^2}{v_r} \left(\frac{r^2 - 1}{r} \right)$$

$$= \frac{\gamma v_r}{c^2} = \frac{1}{v_r} \left(\frac{r^2 - 1}{r} \right) \quad \textcircled{2}$$

Substitute \textcircled{2} in \textcircled{1},

$$\Rightarrow t_1 = \gamma \left(-\frac{\gamma v_r}{c^2} \right) + rt$$

$$= t_1 = \gamma \left(t - \frac{v_r}{c^2} \gamma \right)$$

$$\Rightarrow \gamma = \frac{1}{\sqrt{1 - v_r^2/c^2}}$$

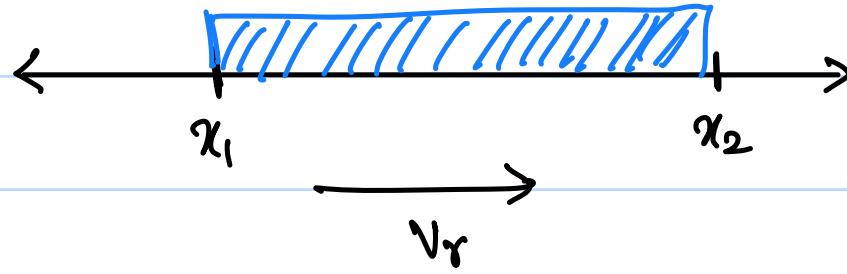
With this, the transformation can be described as,

$$x' = \frac{x - vt}{\sqrt{1 - v_r^2/c^2}}$$

$$t' = \frac{t - v_r x/c^2}{\sqrt{1 - v_r^2/c^2}} = \gamma \left(t - \frac{v_r x}{c^2} \right)$$

The above transformation is described as the Lorentz transformation.

- Length Contraction:



- An observer is moving velocity v_r towards x_2 , from x_1 . He measures the length of the above rod ($x_2 - x_1$) at a time instant t .

The length of the rod in the observer's frame is,

$$l' = x'_2 - x'_1$$

$$\begin{aligned} &= \gamma(x_2 - v_r t_2) - \gamma(x_1 - v_r t_1) \\ &= \gamma(x_2 - x_1 - v_r(t_2 - t_1)) \end{aligned}$$

$$\text{Since } t'_2 = t'_1$$

$$\Rightarrow \gamma\left(t_2 - \frac{v_r x_2}{c^2}\right) = \gamma\left(t_1 - \frac{v_r x_1}{c^2}\right)$$

$$t_2 - t_1 = \frac{v_r}{c^2}(x_2 - x_1)$$

$$\Rightarrow l' = \gamma\left(x_2 - x_1 - \frac{v_r^2}{c^2}(x_2 - x_1)\right)$$

$$= \frac{1}{\sqrt{1 - \frac{v_r^2}{c^2}}} (x_2 - x_1) \left(1 - \frac{v_r^2}{c^2}\right)$$

$$= (x_2 - x_1) \sqrt{1 - \frac{v_r^2}{c^2}}$$

$$\Rightarrow \underline{l'} = l/\gamma \Rightarrow l' < l \quad \sqrt{1 - \frac{v_r^2}{c^2}} < 1$$

$\hookrightarrow \gamma > 1$

This is termed as length contraction.

- Time Dilation :-

- A clock, at a fixed position x , ticks at 2 time instants t_1 and t_2 . An observer moving in frame K' , with relative velocity v_r measures the time interval.

$$\Delta t' = t_2' - t_1'$$

$$\Delta t' = \gamma \left(t_2 - \frac{v_r}{c^2} x_2 \right) - \gamma \left(t_1 - \frac{v_r}{c^2} x_1 \right)$$

$$= \gamma \left(t_2 - t_1 - \frac{v_r}{c^2} (x_2 - x_1) \right)$$

$$= \gamma (\Delta t)$$

$$\Rightarrow \underline{\Delta t'} = \gamma \Delta t \Rightarrow \Delta t' > \Delta t$$

This phenomenon is termed as time dilation.

- Relativistic Doppler Effect :-

Consider a source of light and a receiver, with approach velocity v .

Let the receiver be in a frame K and the source in K' .

In time Δt , the source emits n waves. The distance between the front and back is,

$$l = c\Delta t - v\Delta t$$

$$\Rightarrow \lambda = \frac{c\Delta t - v\Delta t}{n}$$

$$\Rightarrow f = \frac{nc}{c\Delta t - v\Delta t}$$

Since the source emits with frequency f_0 ,

$$n = f_0 \Delta t' = f_0 \frac{\Delta t}{\gamma} \quad [\text{time dilation}]$$

$$\Rightarrow f = \frac{f_0 c \Delta t}{\gamma (c \Delta t - v \Delta t)}$$

$$= \frac{1}{\gamma \left(1 - \frac{v}{c}\right)} f_0$$

$$f = \frac{\sqrt{1 - v^2/c^2}}{1 - v/c} f_0$$

If the source and receiver are separating with velocity v ,

$$l = c\Delta t + v\Delta t$$

$$\Rightarrow \lambda = \frac{c\Delta t + v\Delta t}{n}$$

$$\Rightarrow f = \frac{nc}{c\Delta t + v\Delta t}$$

$$n = f_0 \Delta t' = f_0 \frac{\Delta t}{\gamma}$$

$$\Rightarrow f = \frac{c\Delta t}{\gamma(c\Delta t + v\Delta t)} f_0$$

$$\Rightarrow f = \frac{\sqrt{1 - v^2/c^2}}{1 + v/c} f_0$$

Note:

Lorentz transformation for velocities,

Let K and K' be 2 inertial frames with relative velocity v_r , with an object moving at velocities v and v' in the frames respectively.

$$v' = \frac{dx'}{dt'} = \frac{\frac{dx'}{dt}}{\frac{dt'}{dt}}$$

Wkt,

$$x' = \gamma(x - v_r t) \quad t' = \gamma(t - \frac{v_r x}{c^2})$$

γ is a constant w.r.t t .

$$\Rightarrow \frac{dx'}{dt} = \gamma \left(\frac{dx}{dt} - v_r \right) \quad \frac{dt'}{dt} = \gamma \left(1 - \frac{v_r}{c^2} \frac{dx}{dt} \right)$$
$$= \gamma (v - v_r) \quad = \gamma \left(1 - \frac{v_r v}{c^2} \right)$$

$$\Rightarrow v' = \frac{\gamma (v - v_r)}{\gamma \left(1 - \frac{v_r v}{c^2} \right)} = \frac{v - v_r}{1 - \frac{v_r v}{c^2}}$$

Verification by taking $v = c$

$$v' = \frac{c - v_r}{1 - \frac{v_r c}{c^2}} = \frac{c - v_r}{\frac{(c - v_r)}{c}} = \underline{\underline{c}}$$

$\therefore v = v'$ if $v = c$. STR is not violated.

o Proper Time:-

Time as measured in the observer's frame of reference, with respect to events that occur at the same time.

- Proper length :-

Length as measured in the observer's frame of reference, with respect to an object that is at rest in his frame.

- Events that happen at different times will undergo length contraction between them.

Note: An event is defined with both space and time.

- The Twin Paradox :-

- If there are 2 twins, one on Earth and one on an interstellar voyage travelling near the speed of light, the space travelling twin will age much slower than the twin on Earth.
- The paradox is that, since with respect to the traveller, the twin on Earth is the one moving, the Earth twin must be the younger one, due to symmetry.
- The solution, is that since the traveller is going away, and then coming back to Earth, he is actually in 2 different reference frames. (One while separation, one while approach)
- To explain this further, we can use space time diagrams.

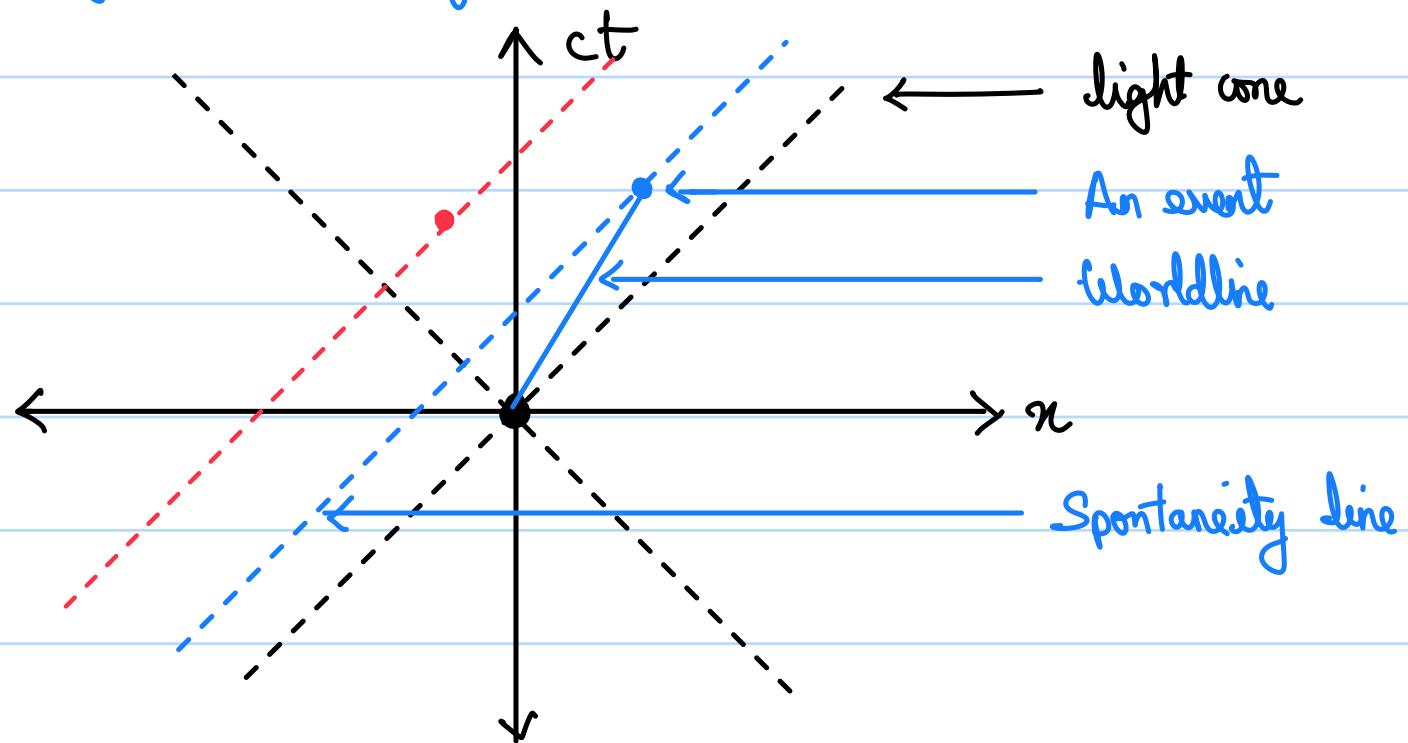
Spacetime Diagrams (Minkowski Geometry) :-

- In Minkowski Geometry, the spacetime interval between 2 events is defined as,

$$ds^2 = dx^2 + dy^2 + dz^2 - (cdt)^2$$

- A spacetime diagram is as follows, (on one spatial dimension x)

(ignore $-x$ for now)



Event : A point on the spacetime diagram

Worldline : The path travelled by an event, i.e. the history of an object through space and time.

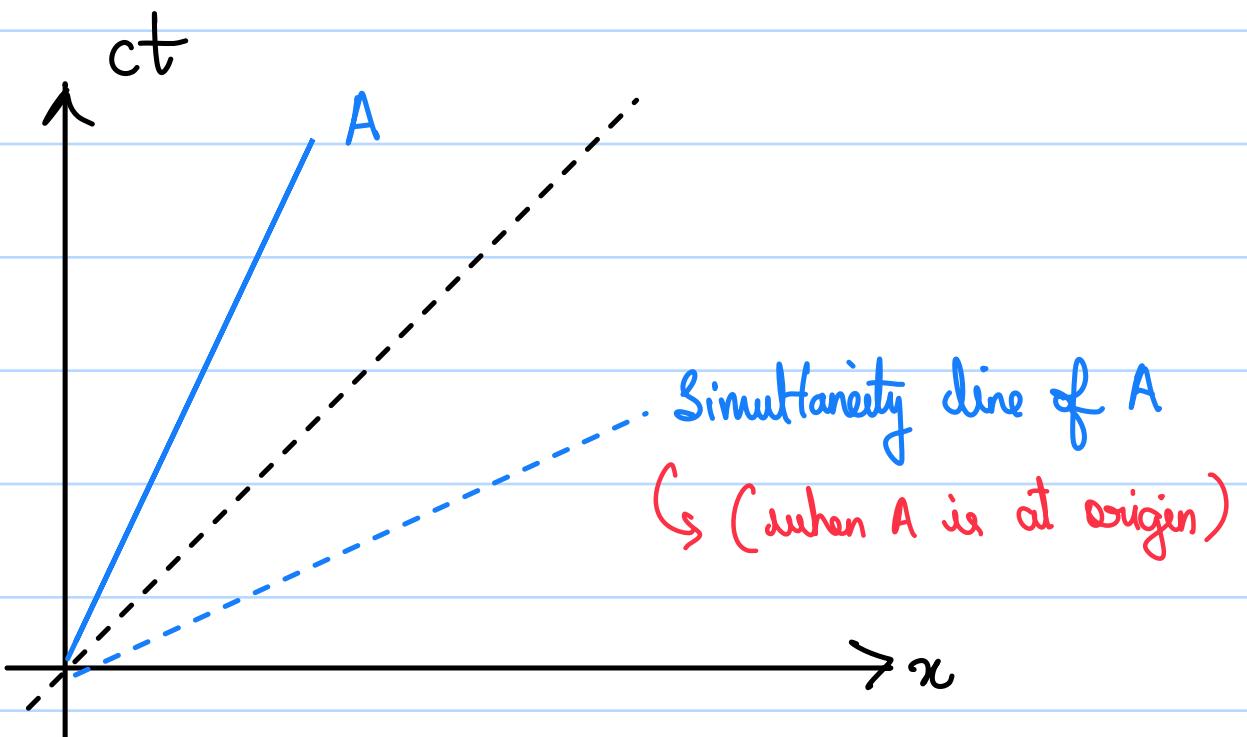
Simultaneity line : Set of all events that are simultaneous with the event 1 in the observer's frame (if observer is at rest w.r.t. light cone)

Light cone : Speed of light representation (worldlines cannot cross light cone, as nothing can cross the speed of light). Also, the set of events that are simultaneous to origin.

- The slope of the simultaneity line depends on the velocity of the object being analysed in the observer frame. Higher the speed, higher the slope, but cannot cross $m = 1$ or -1 (speed of light)

$\uparrow \rightarrow$ direction
- Length contraction and time dilation distort the object's spacetime diagram, smearing up the spontaneity line.

Let A represent a moving observer w.r.t ground. The ground's space-time diagram is as follows.



- To determine the simultaneity line of A, let velocity of A be v .

Applying Lorentz Transformation,

$$x' = \gamma(x - vt)$$

$$t' = \gamma(t - \frac{vx}{c^2})$$

Simultaneity means $t' = \text{const}$ (let it be k).

The slope of a worldline is given by $\frac{d(ct)}{dx}$.

$$\kappa = \gamma \left(t - \frac{vx}{c^2} \right)$$

$$\Rightarrow t - \frac{v}{c^2}x = \frac{\kappa}{\gamma}$$

$$\Rightarrow ct = \frac{v}{c}x + \frac{ck}{\gamma}$$

\therefore Slope of ct vs $x = \underline{\frac{v}{c}}$.

\therefore The simultaneity line of an object at (ct_0, x_0) moving at velocity v with respect to the observer is

$$(t - t_0) = \frac{v}{c} (x - x_0)$$

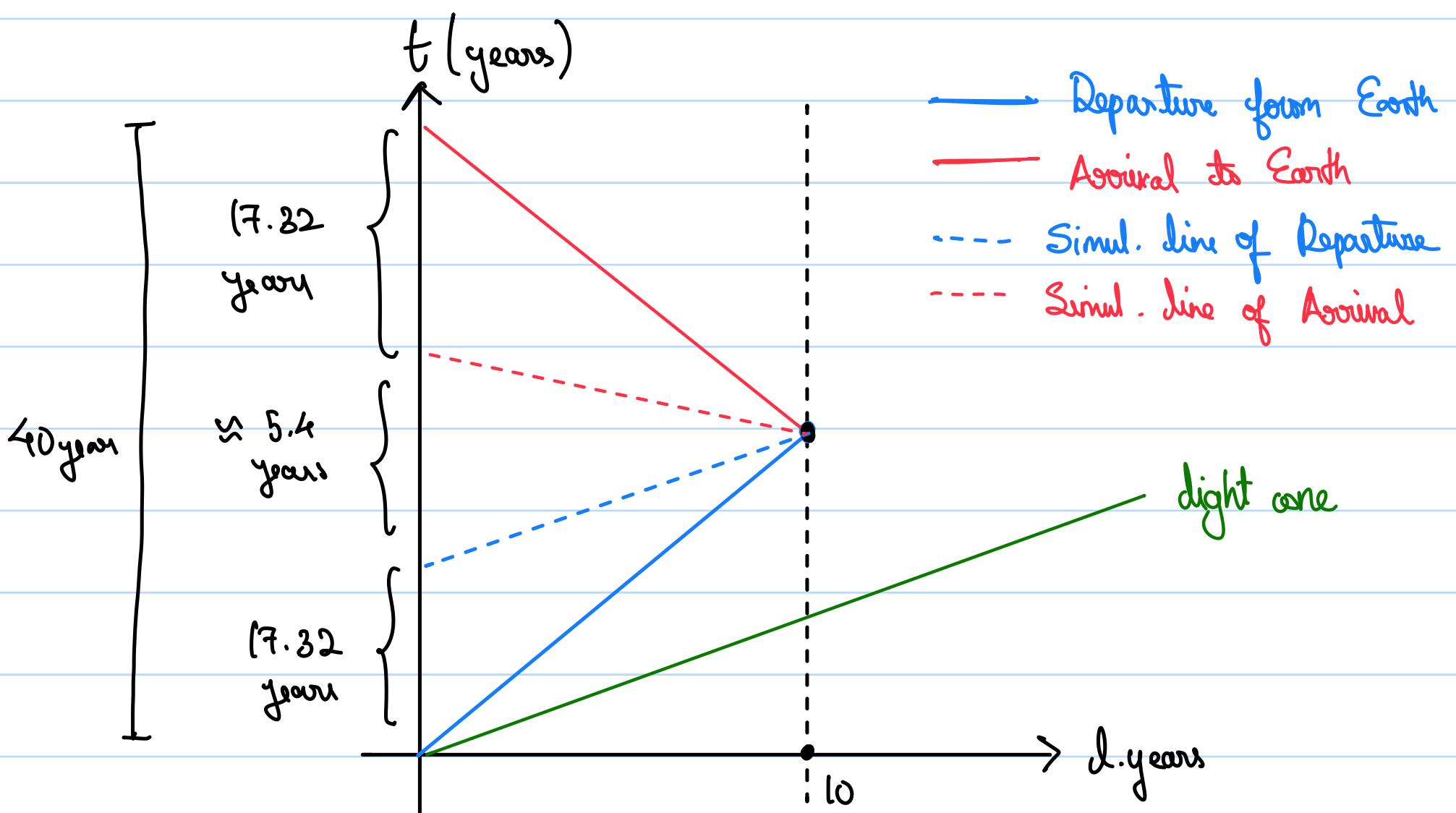
Verification: If $v = 0$, slope = 0, ie, set of all points (x, t) such that $t = t_0$. This is clearly correct.

- Solving the Twin Paradox using Spacetime Paradox :-

- Let the speed of the traveller be $0.5c$, ie, 50% of the speed of light, and let his destination be 10 light years away.

- Time taken to travel = $\frac{10}{0.5} = 20$ years } In Earth frame
 \Rightarrow Total time = 40 years

Drawing the spacetime diagram on the Earth frame,



- Here, we see why the switching of reference frames causes the difference in aging.
- The simultaneity line of the travelling twin changes when he changes direction. Therefore, there is a "line jump" with respect to Earth for the traveller. The aging of the Earth twin happens in that jump.
- The Earth twin ages 17.32 years. To calculate the age of the traveller twin, we can use length contraction to determine the duration of the trip as per him,

$$l' = l/\gamma = 10 \sqrt{1 - \frac{(0.5c)^2}{c^2}}$$

$$= 10 \sqrt{0.75} \approx 8.66 \text{ light years}$$

$$t = \frac{8.66}{0.5} = 17.32 \text{ years.}$$

$$\text{Total } t = 34.64$$

\therefore The traveller twin will age 34.64 years.

\Rightarrow The "time jump" is ≈ 5.4 years.

- The travelling twin does not "feel" the time jump, but the Earth twin does, hence causing the difference in aging.
- Since the time jump is +ve for all speeds $< c$, the Earth twin will always be older than the travelling twin.

 Converse: If speed is $> c$, Earth twin will be younger than traveller twin (Yay Time Travel !!!)

• Spacetime Invariance :

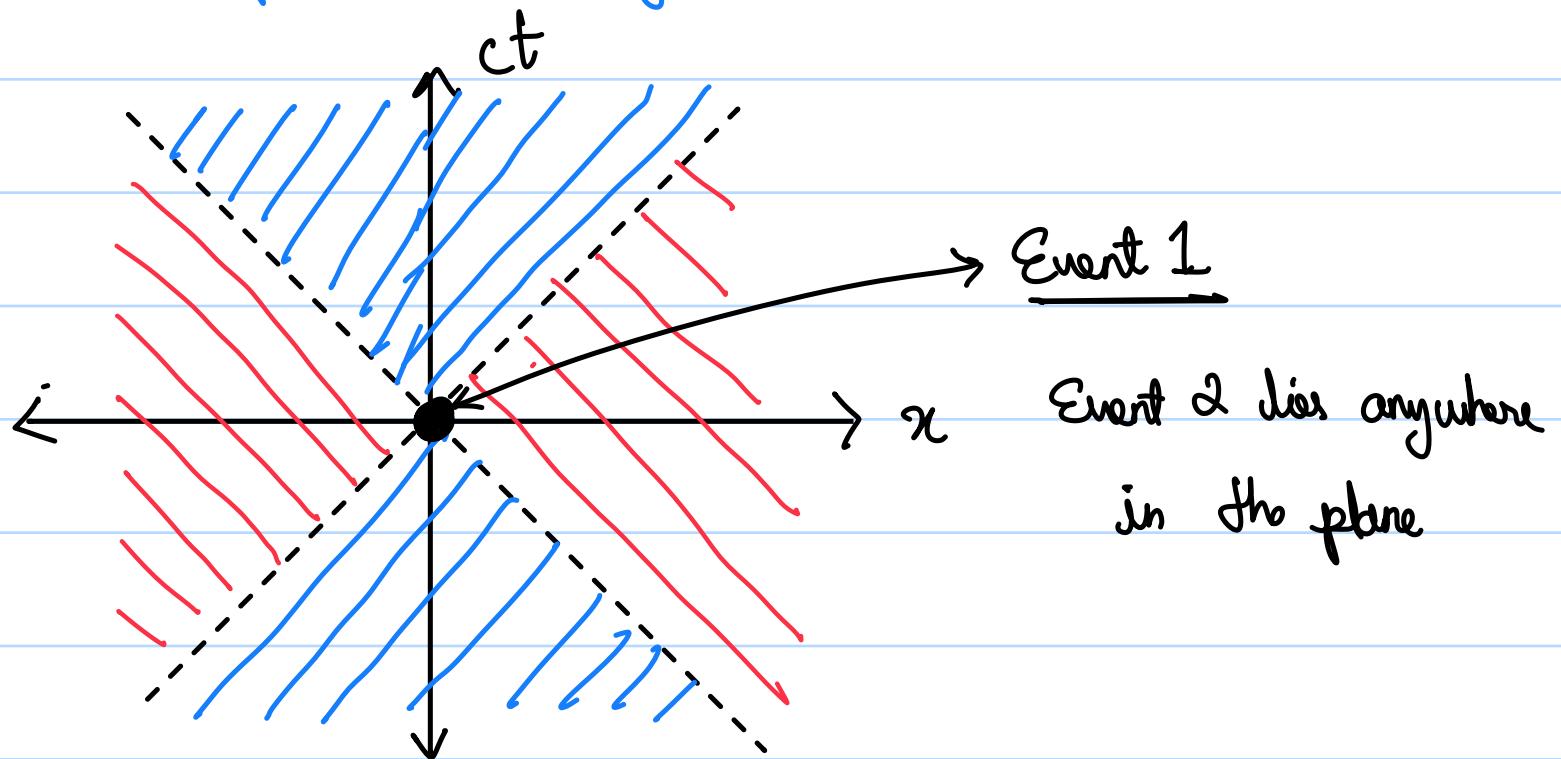
- The spacetime interval between 2 events is,

$$ds^2 = dx^2 + dy^2 + dz^2 - (cdt)^2$$

- Since our examples only have x coordinate and time,

$$ds^2 = dx^2 - (c dt)^2$$

- In the below spacetime diagram,



We know that the slope of any line is,

$$m = \frac{\Delta(ct)}{\Delta(x)} = \frac{ct}{x} \text{ at Event 2, but since need to prove for gen. case}$$

$$\Rightarrow \Delta(ct) = m \Delta(x)$$

If $m = 1 \rightarrow$ The light cone,

$$\Delta(ct) = \Delta x \Rightarrow \Delta(ct)^2 = \Delta x^2$$

$$\Rightarrow \underline{\Delta s^2 = 0}$$

Any event with $\Delta s^2 = 0$ is simultaneous to the origin event. They are termed as lightlike events.

If $m > 1 \rightarrow$ The Blue region

$$\begin{aligned}\Delta(ct) &< \Delta(x) \\ \Rightarrow \Delta(ct)^2 &< \Delta(x)^2 \\ \Rightarrow \underline{\Delta s^2} &> 0\end{aligned}$$

\exists an inertial frame where the 2 events happen at the same position but diff. times

Any event which is causal to the origin event has $\Delta s^2 > 0$, but all such events with $\Delta s^2 > 0$ are not causal to origin. Such events with $\Delta s^2 > 0$ are termed as timelike events.

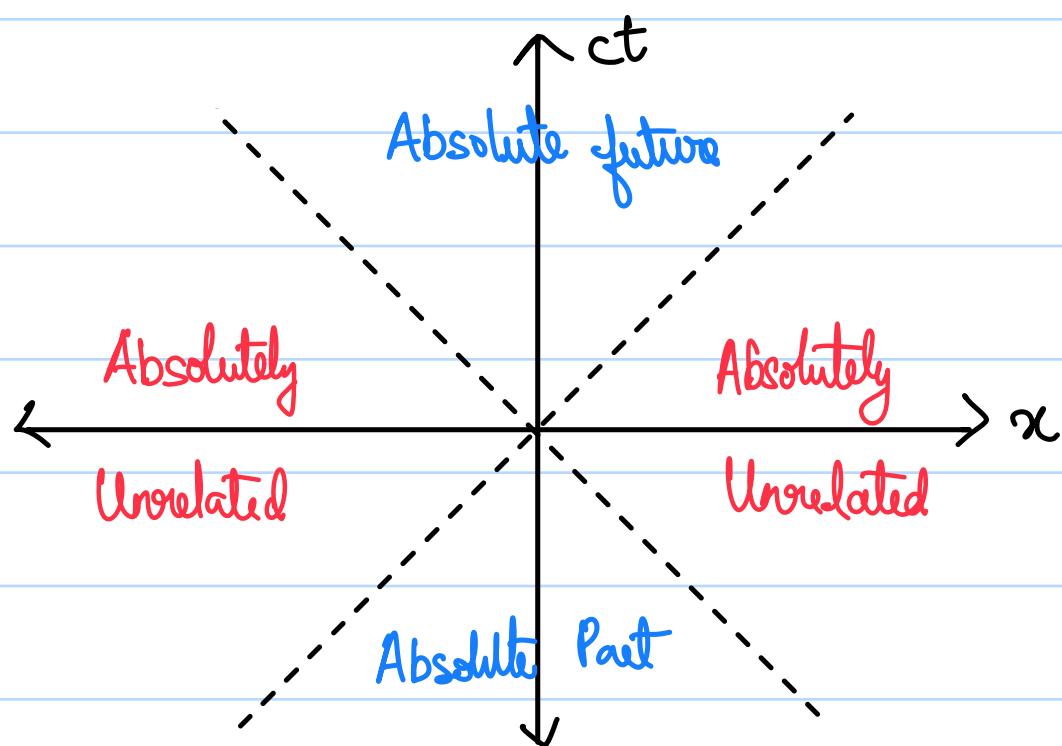
If $m < 1 \rightarrow$ The Red region

$$\begin{aligned}\Delta(ct)^2 &> \Delta x^2 \\ \Rightarrow \underline{\Delta s^2} &< 0\end{aligned}$$

\exists an inertial frame where the events occur at the same time but diff. position.

Events with $\Delta s^2 < 0$ with the origin, cannot be related to the origin. Such events are called space-like events.

- These results apply to any pair of 2 events, not only origin.

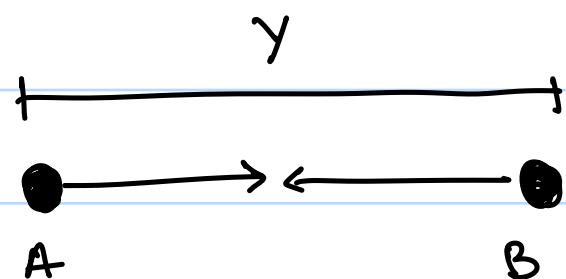


- The significance of these results lies in the fact that the spacetime interval is unaffected by the Lorentz transformation, i.e., Δs^2 is the same in all frames for any 2 events.

\therefore If 2 events are absolutely unrelated in a frame, they are absolutely unrelated in any frame

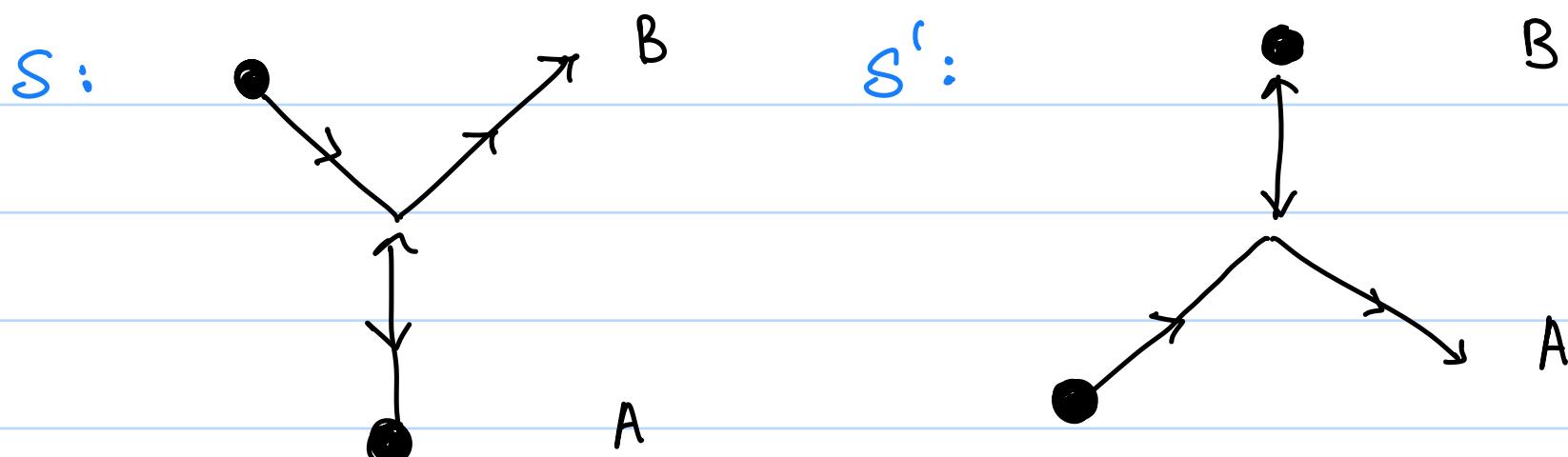
→ STR in the Real World :-

1) Relativistic Momentum :-



Let there be 2 balls A and B with velocities v_A, v_A' and v_B, v_B' , in the reference frames S and S' respectively, along the y-axis. Let S' move at v velocity along $+x$ w.r.t S.

If $v_A = v_B'$ and the balls are colliding, the collision will be,



Time taken for A's round trip in S = $\frac{y}{v_A}$
 " " B's round trip in $S' = \frac{y}{v_B'} = \frac{y}{v_A}$

} Let it be T_0

In S, $v_B = \gamma/T$, T = Time for B's round trip in S

$$\Rightarrow T = \gamma T_0 \quad (T_0 \text{ is B's time in } S')$$

$$= T = \frac{T_0}{\sqrt{1-v^2/c^2}}$$

$$\Rightarrow v_B = \frac{\gamma \sqrt{1-v^2/c^2}}{T_0}$$

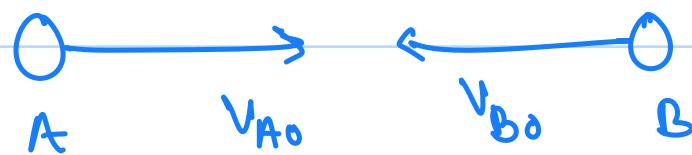
likewise $v_A = \frac{\gamma}{T_0}$

Using the classical definition of momentum, we get,

$$p_A = m_A v_A = m_A \left(\frac{\gamma}{T_0} \right)$$

$$p_B = m_B v_B = m_B \left(\sqrt{1-v^2/c^2} \right) \left(\frac{\gamma}{T_0} \right)$$

We know that, in the lab frame, the situation is as below.



Here, the total momentum of the system is zero

$$\text{if } m_A = m_B$$

$$\Rightarrow p_A - p_B = 0 \quad [\text{Physical laws must be the same in all frames}]$$

$$= m_A \left(\frac{v}{T_0} \right) - m_B \left(\sqrt{1 - v^2/c^2} \right) \left(\frac{v}{T_0} \right) = 0$$

$$= m_A = m_B \sqrt{1 - v^2/c^2}$$

$$\Rightarrow m_B = \frac{m_A}{\sqrt{1 - v^2/c^2}}$$

Now, redefining P , we get

$$P_A = \frac{m_A v_A}{\sqrt{1 - v_A^2/c^2}}$$

$$\Rightarrow P_A = \gamma(v_A) m_A v_A$$

- frame velocity

← just accept the formula,
I can't understand the
proof anymore :)

In the above equation, since $\gamma \rightarrow \infty$ as $v \rightarrow c$, as velocity approaches c , the momentum of the object nears infinity, making it harder for further increase in momentum.

$\Rightarrow v = c$ is practically impossible

• Relativistic Second Law:

- In Newton's second law,

$$\bar{F} = \frac{d\bar{p}}{dt} = \frac{d}{dt} \gamma(\bar{v}) m \bar{v}$$

$$= m \frac{d}{dt} \gamma(\bar{v}) \bar{v}$$

$$= m \frac{d}{dt} \frac{v^2}{\sqrt{1-v^2/c^2}}$$

$$(\text{On solving}) = \frac{m}{(1-v^2/c^2)^{3/2}} \frac{dv}{dt}$$

$$\Rightarrow F = \frac{ma}{(1-v^2/c^2)^{3/2}}$$

- Energy in STR :-

$$KE = \int F \cdot ds$$

$$= \int \frac{d}{dt} \gamma mv \, ds$$

$$= \int v \, d(\gamma mv)$$

$$= \int v \, d\left(\frac{mv}{\sqrt{1-v^2/c^2}}\right)$$

Integrating by parts,

$$\Rightarrow KE = \left[v \left(\frac{mv}{\sqrt{1-v^2/c^2}} \right) - m \int \frac{v dv}{\sqrt{1-v^2/c^2}} \right]$$

$$= \frac{mv^2}{\sqrt{1-v^2/c^2}} + mc^2 \left[\sqrt{1-v^2/c^2} \right]_0^v$$

$$= \frac{mv^2}{\sqrt{1-v^2/c^2}} + mc^2 \sqrt{1-v^2/c^2} - mc^2$$

$$\begin{aligned}
 &= \frac{mv^2 + mc^2(1 - \frac{v^2}{c^2}) - mc^2\sqrt{1 - v^2/c^2}}{\sqrt{1 - v^2/c^2}} \\
 &= \frac{mc^2 - mc^2\sqrt{1 - v^2/c^2}}{\sqrt{1 - v^2/c^2}}
 \end{aligned}$$

$$\Rightarrow KE = (\gamma - 1) mc^2$$

When $v = 0$, ie $\gamma = 1$, $KE = 0$. If we interpret γmc^2 as the total energy of the object, we get

$$E = KE + E_0$$

$$E_0 = E - KE = \gamma mc^2 - (\gamma - 1)mc^2$$

$$\Rightarrow E_0 = mc^2$$

E_0 is defined as the rest mass energy of the system, ie, energy of the object while stationary.

If $v \ll c$,

$$\begin{aligned}
 KE &= mc^2(\gamma - 1) \\
 &= mc^2 \left((1 - \frac{v^2}{c^2})^{-1/2} - 1 \right) \\
 &= mc^2 \left(1 + \frac{1}{2} \frac{v^2}{c^2} - 1 \right) \quad [\text{Binomial Approximation}]
 \end{aligned}$$

$$\underline{KE = \frac{1}{2}mv^2} \longrightarrow \text{Classical formula.}$$

$$\bullet E = \frac{mc^2}{\sqrt{1-v^2/c^2}}$$

$$P = \frac{mv}{\sqrt{1-v^2/c^2}}$$

$$E^2 = \frac{m^2 c^4}{1-v^2/c^2}$$

$$P^2 = \frac{m^2 v^2}{1-v^2/c^2}$$

$$P^2 c^2 = \frac{m^2 v^2 c^2}{1-v^2/c^2}$$

$$E^2 - P^2 c^2 = \frac{m^2 c^2}{1-v^2/c^2} (c^2 - v^2)$$

$$= \frac{m^2 c^4 (1 - v^2/c^2)}{1-v^2/c^2}$$

$$\Rightarrow E^2 - P^2 c^2 = (mc^2)^2$$

By the above equation, if \exists a massless particle, its energy will be given by $E = pc$. (ex: photon)

→ General Theory of Relativity :-

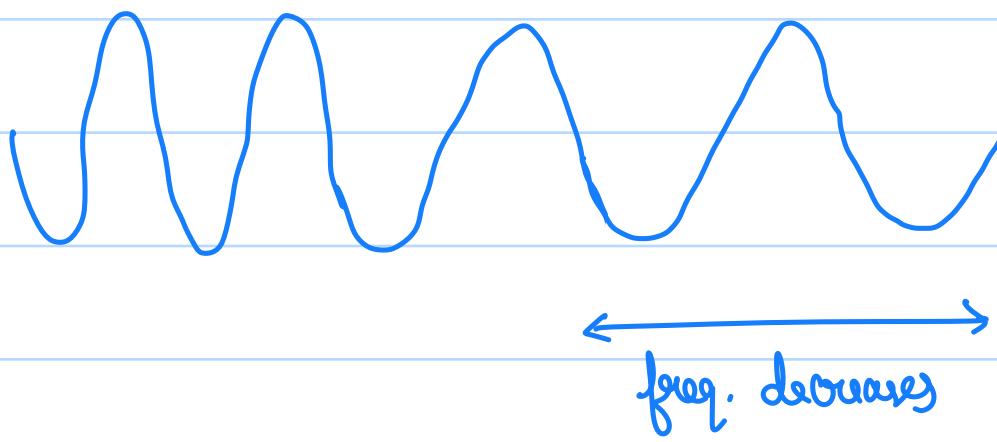
° General Theory of Relativity is based on the principle of equivalence, that states,

"An observer in a closed laboratory cannot distinguish between the effects produced by a gravitational field and those produced by the acceleration of the laboratory"

- GTR extends STR into non-inertial frames, ie, the effect of acceleration on simultaneity.
- In a non-inertial frame, light bends against the direction of acceleration. This is termed as "lensing".



If direction of acceleration is in the direction of light, the light is "redshifted" (the detector moves away from the light source)



- GTR states that, since by the principle of equivalence, gravitational acceleration is the same as any acceleration; gravitation will also cause lensing and redshifting.

∴ Light bends due to gravity.

→ Calculus of Variations :-

- The basic problem in Calculus of Variation is to find the value such that the integral,

$$J = \int_{x_1}^{x_2} f\{y(x), y'(x); x\} dx$$

is an extremum (maximum or minimum).

- We are to vary $y(x)$ until J is maximum or minimum.
- If J is minimum, that means if $y(x)$ is changed into a neighbouring function, then J increases.

- Let the neighbouring function of $y(x)$ be $y(\alpha, x)$, $[y(0, x) = y(x)]$ which turns the functional J into a parameterized functional,

$$J(\alpha) = \int_{x_1}^{x_2} f(y(\alpha, x), y'(\alpha, x); x) dx$$

Such that,

$$y(\alpha, x) = y(x) + \alpha \eta(x)$$

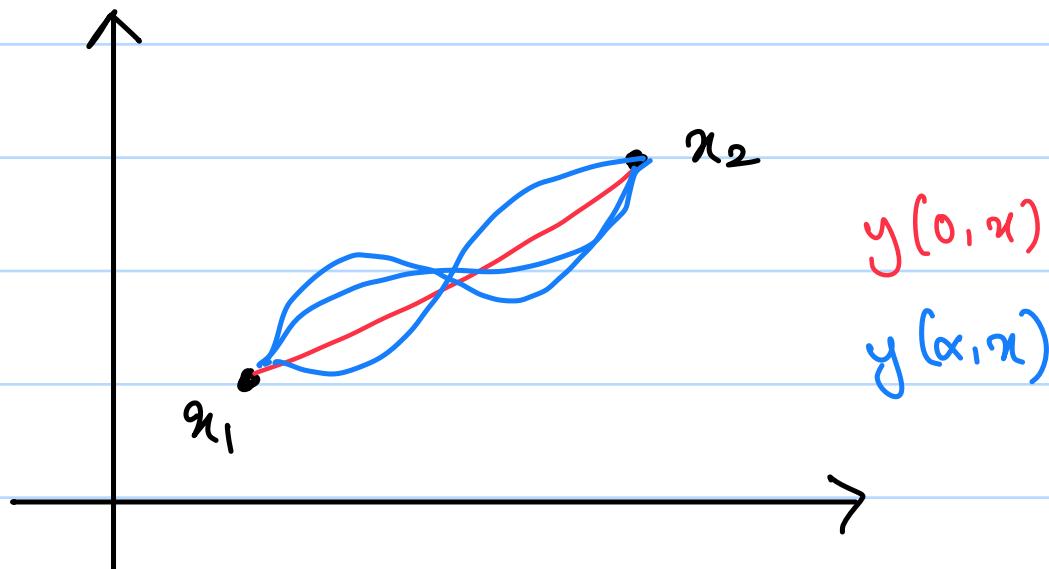
Where $\eta(x)$ is some function with a continuous first derivative and vanishes at x_1 and x_2 , since the function $y(\alpha, x)$ should be the same at the endpoints x_1 and x_2 + α , i.e.

$$\eta(x_1) = \eta(x_2) = 0$$

- If $y(0, x)$ is the location of the extremum of J , then,

$$\frac{dJ}{dx} \Big|_{x=0} = 0$$

But the above condition is only necessary, not sufficient.



Interpretation of $y(\alpha, n)$

In a nutshell, we are finding a path between x_1 and x_2 such that a parameter (represented by the functional J) is minimized or maximized.

- Euler's Equation :-

• We know an extremum is achieved when $\frac{\delta J}{\delta x} = 0$

$$\frac{\delta J}{\delta x} - \frac{\delta}{\delta x} \int_{x_1}^{x_2} f(y, y'; x) dx$$

$$= \int \left(\frac{\delta f}{\delta y} \frac{\delta y}{\delta x} + \frac{\delta f}{\delta y'} \frac{\delta y'}{\delta x} \right) dx$$

→ We represent this as $\int_{x_1}^{x_2} f(y, y'; x) dx$

$$y(x,n) = y(0,n) + \alpha \eta(x)$$

$$\Rightarrow \frac{\delta y}{\delta \alpha} = \eta(x)$$

$$y'(x,n) = y'(0,n) + \alpha \eta'(n)$$

$$\frac{\delta y'}{\delta \alpha} = \eta'(n)$$

$$\Rightarrow \frac{\delta J}{\delta \alpha} = \int_{n_1}^{n_2} \frac{\delta F}{\delta y} h(x) + \frac{\delta F}{\delta y'} \eta'(x) dx$$

$$\int_{n_1}^{n_2} \frac{\delta F}{\delta y'} \eta'(x) dx = \left. \frac{\delta F}{\delta y'} \eta(x) \right|_{n_1}^{n_2} - \int_{n_1}^{n_2} \frac{d}{dx} \frac{\delta F}{\delta y'} \eta(x) dx \rightarrow \eta(n_1) = \eta(n_2)$$

$$\frac{\delta J}{\delta \alpha} = \int_{n_1}^{n_2} \left(\frac{\delta F}{\delta y} \eta(n) - \frac{d}{dn} \frac{\delta F}{\delta y'} \eta(n) \right) dn$$

$$\frac{\delta J}{\delta \alpha} = \int_{n_1}^{n_2} \left(\frac{\delta F}{\delta y} - \frac{d}{dn} \frac{\delta F}{\delta y'} \right) \eta(n) dn$$

If $\frac{\delta J}{\delta \alpha} = 0$, then $\frac{\delta F}{\delta y} - \frac{d}{dn} \frac{\delta F}{\delta y'} = 0$

- The equation $\frac{\delta F}{\delta y} - \frac{d}{dn} \frac{\delta F}{\delta y'}$ is termed as Euler's equation, and the solution to this equation gives the extremum of J .

- For any function that do not directly depend on x , we can use another form of Euler's equation.
We know that,

$$\frac{df}{dx} = \frac{d}{dx} f(y, y'; x) = \frac{\delta f}{\delta y} \frac{dy}{dx} + \frac{\delta f}{\delta y'} \frac{dy'}{dx} + \frac{\delta f}{\delta x}$$

$$= y' \frac{\delta f}{\delta y} + y'' \frac{\delta f}{\delta y'} + \frac{\delta f}{\delta x}$$

Also,

$$\frac{d}{dx} \left(y' \frac{\delta f}{\delta y'} \right) = y'' \frac{\delta f}{\delta y'} + y' \frac{d}{dx} \frac{\delta f}{\delta y'}$$

$$\Rightarrow \frac{d}{dx} \left(y' \frac{\delta f}{\delta y'} \right) = \frac{df}{dx} - \frac{\delta f}{\delta x} - y' \frac{\delta f}{\delta y} + y' \frac{d}{dx} \frac{\delta f}{\delta y'}$$

$$\Rightarrow \frac{d}{dx} \left(y' \frac{\delta f}{\delta y'} \right) = \frac{df}{dx} - \frac{\delta f}{\delta x} - y' \left(\frac{\delta f}{\delta y} - \cancel{\frac{d}{dx} \frac{\delta f}{\delta y'}} \right)$$

$$\Rightarrow \frac{d}{dx} \left(y' \frac{\delta f}{\delta y'} \right) + \frac{\delta f}{\delta x} - \frac{df}{dx} = 0$$

first form of
Euler's Equation

$$\Rightarrow \underbrace{\frac{\delta f}{\delta x} - \frac{d}{dx} \left(f - y' \frac{\delta f}{\delta y'} \right)}_{} = 0$$

When f does not explicitly depend on x ($\frac{\delta f}{\delta x} = 0$)

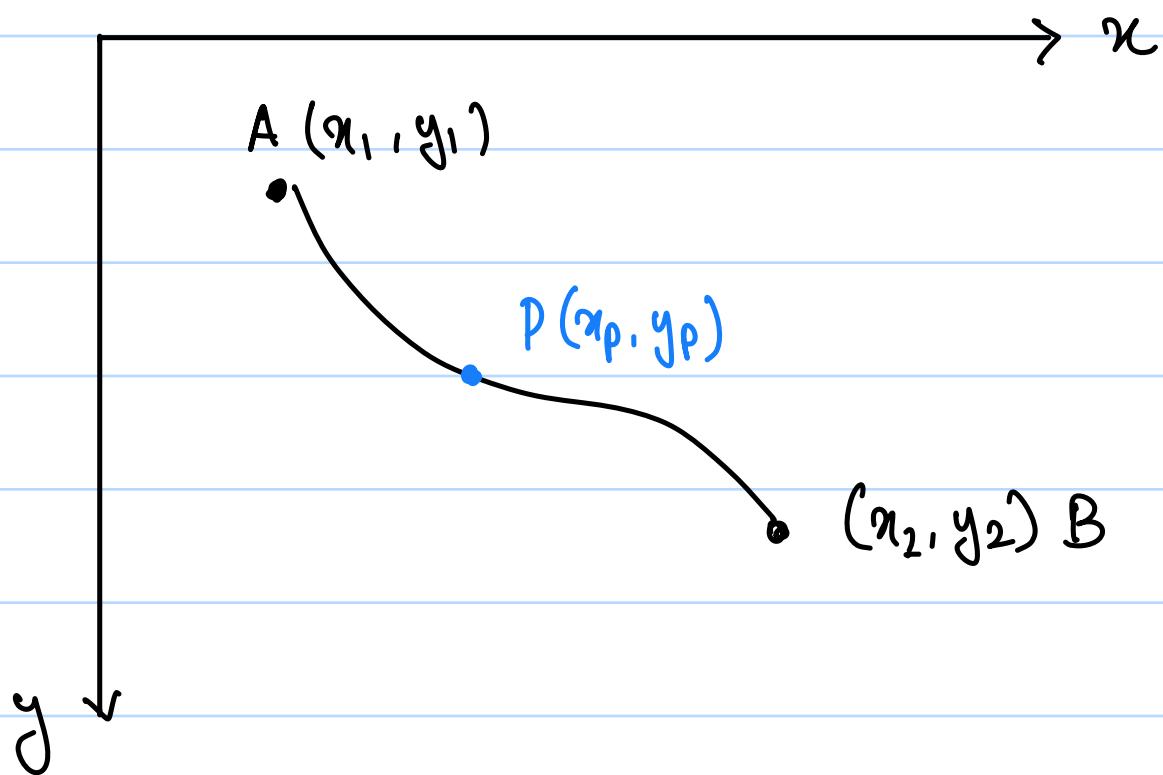
$$\Rightarrow \frac{d}{dx} \left(f - y' \frac{\delta f}{\delta y'} \right) = 0$$

$$\Rightarrow f - y' \frac{\delta f}{\delta y'} = \text{constant}$$

- Solving Problems Using Euler's Equation :-

- 1) The Brachistochrone Problem :-

Consider a particle moving in a constant force field, starting at rest from (x_1, y_1) to some point (x_2, y_2) . Find the path that would take the least amount of time



$$\text{KE at } P - \text{KE at } A = \text{Work done b/w PA}$$

$$\frac{1}{2}mv^2 - 0 = mgh(y_1 - y_p) = mg\Delta y$$

$$\Rightarrow \left(\frac{ds}{dt}\right)^2 = 2g\Delta y$$

$$\frac{ds}{dt} = \sqrt{2gy} \quad (\text{let } x_1, y_1 = 0, 0)$$

$$dt = \frac{ds}{\sqrt{2gy}}$$

$$t = \int_A^B \frac{ds}{\sqrt{2gy}}$$

$$= \int_A^B \left(\frac{dx^2 + dy^2}{2gy} \right)^{1/2}$$

$$= \int_A^B \left(\frac{1 + \dot{y}^2}{2gy} \right)^{1/2} dx = \frac{1}{\sqrt{2g}} \int_A^B \frac{\sqrt{1 + \dot{y}^2}}{\sqrt{y}} dx$$

$$\Rightarrow f(y, \dot{y}; x) = \frac{\sqrt{1 + \dot{y}^2}}{\sqrt{y}}$$

Since f does not directly depend on x ($\frac{\partial f}{\partial x} = 0$), using the second form of Euler's equation,

$$f - y' \frac{\delta f}{\delta y'} = c$$

$$\Rightarrow \frac{\sqrt{1 + \dot{y}'^2}}{\sqrt{y}} - y' \frac{\delta}{\delta y'} \left(\frac{\sqrt{1 + \dot{y}'^2}}{\sqrt{y}} \right) = c$$

$$\Rightarrow \frac{\sqrt{1 + \dot{y}'^2}}{\sqrt{y}} - y' \frac{1}{\sqrt{y}} \frac{1}{2\sqrt{1 + \dot{y}'^2}} \cdot 2\dot{y}' \frac{dy'}{dx} = c$$

$$= \frac{\sqrt{1 + \dot{y}'^2}}{\sqrt{y}} - \frac{y'^2}{\sqrt{y} \sqrt{1 + \dot{y}'^2}} = c$$

$$= \frac{1 + \dot{y}'^2 - y'^2}{\sqrt{y} \sqrt{1 + \dot{y}'^2}} = c$$

$$= \frac{1}{\sqrt{y} \sqrt{1 + \dot{y}'^2}} = c$$

$$= \sqrt{y} \sqrt{1+y'^2} = \frac{1}{c}$$

$$= y(1+y'^2) = \frac{1}{c^2}$$

$$y\left(1 + \left(\frac{dy}{dx}\right)^2\right) = \frac{1}{c^2} = k$$

$$\frac{dy}{dx} = \sqrt{\frac{k}{y} - 1}$$

$$dx = \int \frac{\sqrt{y}}{\sqrt{k-y}} \cdot dy$$

$$\int_0^x dx = \int_0^y \frac{\sqrt{y}}{\sqrt{k-y}} dy$$

$$y = k \sin^2 \theta \quad dy = k 2 \sin \theta \cos \theta d\theta$$

$$= \int_0^\theta \frac{\sqrt{k} \sin \theta}{\sqrt{k \cos \theta}} k 2 \sin \theta \cos \theta d\theta$$

$$= 2k \int_0^\theta \sin^2 \theta d\theta$$

$$= 2k \int_0^\theta (1 - \cos 2\theta) d\theta$$

$$= 2k \left(\theta - \frac{\sin 2\theta}{2} \right)$$

$$= 4k (\theta - \sin \theta)$$

$$\Rightarrow x = 4k (\phi - \sin \phi) \quad \phi = \theta$$

$$\Rightarrow y = k \sin^2 \theta / 2 = k (1 - \cos \phi)$$

∴ The path is formed by the polar coordinates,

$$x = 4k(\phi - \sin\phi), y = k(1 - \cos\phi)$$

- Euler's Equation of Several Dependent Variables :-

$$J = \int_{y_1}^{y_n} f(y_1, y_2, y_3, \dots, y_n, y'_1, y'_2, y'_3, \dots, y'_n; x) dx$$

The extremum of J satisfies

$$\frac{\delta F}{\delta y_i} - \frac{d}{dx} \frac{\delta F}{\delta y'_i} = 0 \quad + i \in [1:n]$$

→ Hamilton's Principle :-

- " Of all the possible paths along which a dynamical system may move from one point to another within a specified time interval, the actual path followed is that which minimizes the time integral of the difference between the kinetic and potential energies".

$$\Rightarrow \delta \int_{t_1}^{t_2} (T - U) dt = 0 \quad \text{S} \rightarrow \text{Notation to describe the variation in paths.}$$

Since T is a function of \dot{x} and U is a function of x ,

$$L = T - U = L(x, \dot{x}; t)$$

$$\Rightarrow \int_{t_1}^{t_2} L(x, \dot{x}; t) dt = 0$$

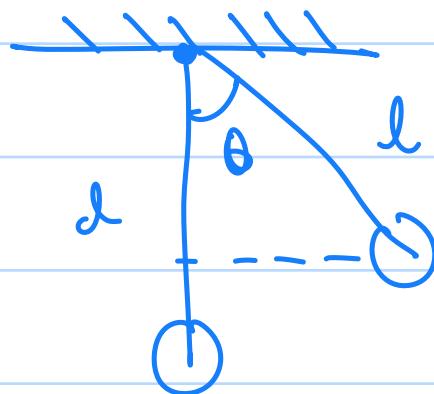
- Since L is similar to the function used in Euler's equation, we can define the Euler-Lagrangian equations of any system as,

$$\frac{\delta L}{\delta x_i} - \frac{d}{dt} \left(\frac{\delta L}{\delta \dot{x}_i} \right) = 0 \quad i = 1, 2, 3$$

x, y, z

The function is defined as the Lagrangian of the particle being analysed.

Example 1: Lagrangian of a Simple Pendulum.



$$L = \frac{1}{2}mv^2 - mg(l - l\cos\theta)$$

$$L = \frac{1}{2}ml^2\omega^2 - mg(l - l\cos\theta) \quad [\omega = \dot{\theta}]$$

$$\frac{\delta L}{\delta \theta} - \frac{d}{dt} \left(\frac{\delta L}{\delta \dot{\theta}} \right) = 0$$

$$\frac{\delta L}{\delta \theta} = -mg l \sin\theta$$

$$\frac{\delta L}{\delta \dot{\theta}} = \frac{1}{2}ml^2\ddot{\theta}$$

$$\frac{d}{dt} \left(\frac{\delta L}{\delta \dot{\theta}} \right) = ml^2 \frac{d\dot{\theta}}{dt} = ml^2\ddot{\theta}$$

$$\Rightarrow -mgd\sin\theta - md^2\alpha = 0$$

$$\Rightarrow -g\sin\theta = d\alpha$$

$$\Rightarrow \theta'' = -\frac{g}{d}\sin\theta$$

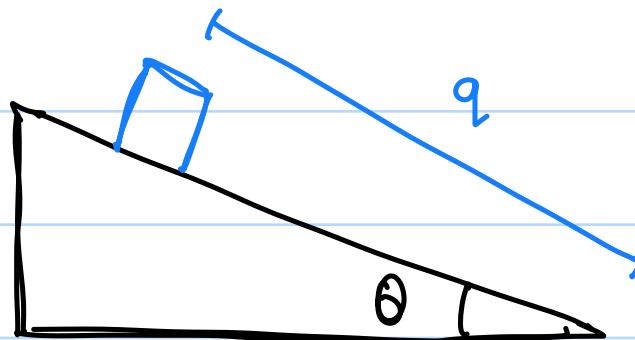
For small θ , $\sin\theta \approx \theta$

$$\Rightarrow \theta'' = -\frac{g}{d}\theta \xrightarrow{\text{SHM oscillation}} \underline{\theta'' = -\omega^2}$$

Example 2: EOM of a block sliding down a swinging incline.

The incline θ increases linearly with time as $\theta = at$

Let the distance of the block from the axis of rotation be q .



$$T = \frac{1}{2}m\dot{q}^2 + \frac{1}{2}m(q\dot{\theta})^2 \quad U = -mgq\sin\theta$$

↑
Due to linear motion

↑
Due to swing of incline

$$\begin{aligned}
 L &= T - U \\
 &= \frac{1}{2}m\dot{q}^2 + \frac{1}{2}m\dot{\theta}^2 - mgq\sin\theta \\
 &= \frac{1}{2}m\dot{q}^2 + \frac{1}{2}m\dot{\theta}^2\alpha^2 + mgq\sin\alpha t = L(q, \dot{q}; t)
 \end{aligned}$$

$$\frac{SL}{S\dot{q}} - \frac{d}{dt}\left(\frac{SL}{S\dot{q}}\right) = 0$$

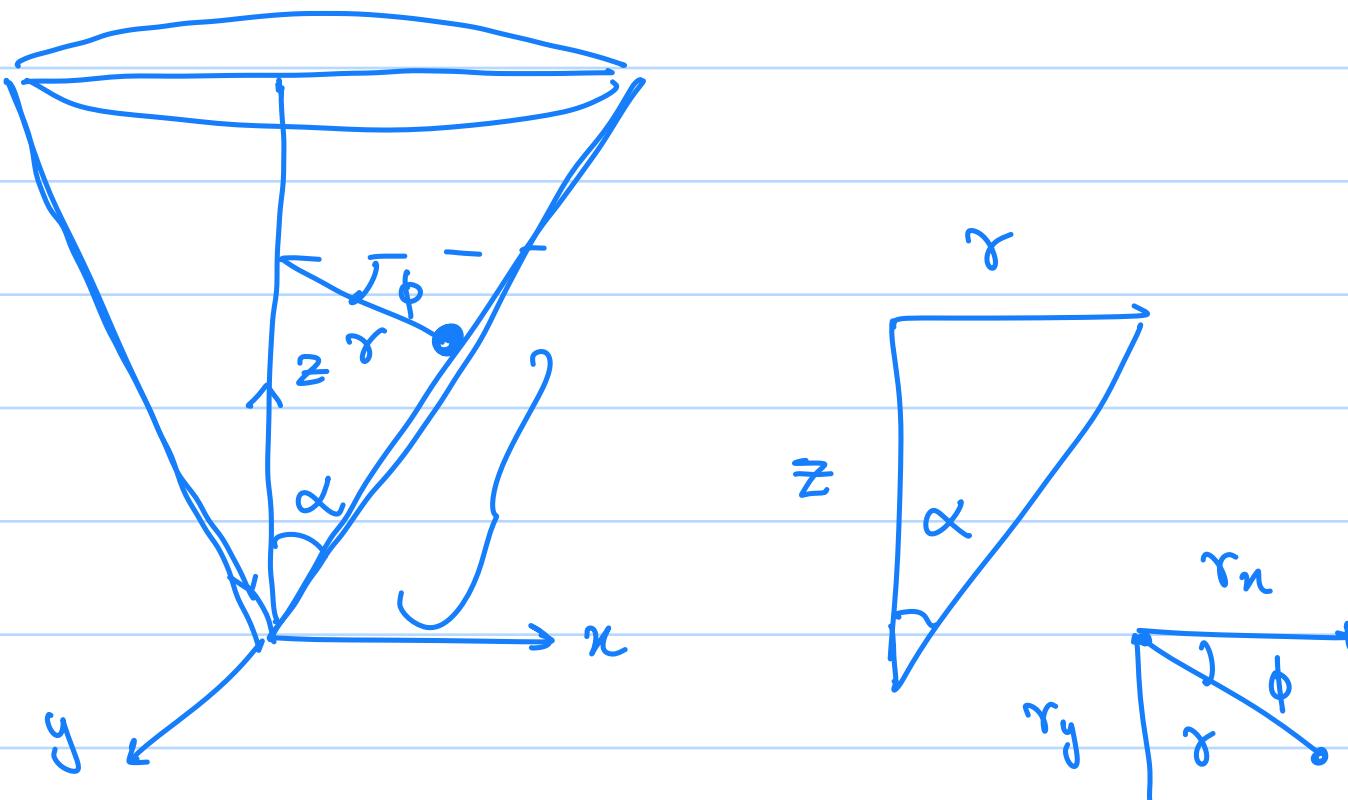
$$\frac{SL}{S\dot{q}} = mg\alpha^2 - mg\sin\alpha t$$

$$\frac{SL}{S\ddot{q}} = m\ddot{q}, \quad \frac{d}{dt}\left(\frac{SL}{S\ddot{q}}\right) = m \frac{d\ddot{q}}{dt} = m\ddot{\ddot{q}}$$

$$\Rightarrow mg\alpha^2 - mg\sin\alpha t - m\ddot{\ddot{q}} = 0$$

$$\Rightarrow \underline{\ddot{\ddot{q}} = g\alpha^2 - g\sin\alpha t}$$

3) Example 3: Wall of Death



Here we see one more advantage of Lagrangian mechanics, i.e.,
the use of arbitrary coordinates.

$$x = r \cos \phi \quad y = r \sin \phi \quad , \quad z = r \cot \alpha$$

$$\dot{x} = \dot{r} \cos \phi - r \sin \phi \cdot \dot{\phi}$$

$$\dot{y} = \dot{r} \sin \phi + r \cos \phi \cdot \dot{\phi}$$

$$\dot{z} = \dot{r} \cot \alpha$$

$$\begin{aligned}\dot{v}^2 &= \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \\ &= \dot{r}^2 \omega^2 \phi + r^2 \dot{\phi}^2 \sin^2 \phi - 2 \dot{\phi} \dot{r} r \omega \phi \sin \phi \\ &\quad + \dot{r}^2 \sin^2 \phi + r^2 \dot{\phi}^2 \cos^2 \phi - 2 \dot{\phi} \dot{r} r \omega \phi \sin \phi \\ &\quad + \dot{r}^2 \cot^2 \alpha\end{aligned}$$

$$\begin{aligned}\Rightarrow \dot{v}^2 &= \dot{r}^2 + r^2 \dot{\phi}^2 + \dot{r}^2 \cot^2 \alpha \\ &= \dot{r}^2 (1 + \cot^2 \alpha) + r^2 \dot{\phi}^2 \\ &= \dot{r}^2 \csc^2 \alpha + r^2 \dot{\phi}^2\end{aligned}$$

$$KE = \frac{1}{2} m \dot{r}^2 \csc^2 \alpha + \frac{1}{2} m r^2 \dot{\phi}^2$$

$$V = mgz = mg r \cot \alpha$$

$$\Rightarrow L = \frac{1}{2} m \dot{r}^2 \csc^2 \alpha + \frac{1}{2} m r^2 \dot{\phi}^2 - mg r \cot \alpha$$

$$\hookrightarrow L(r, \phi, \dot{r}, \dot{\phi}; t)$$

$$\Rightarrow \frac{\delta L}{\delta r} - \frac{d}{dt} \left(\frac{\delta L}{\delta \dot{r}} \right) = 0 \quad \& \quad \frac{\delta L}{\delta \phi} - \frac{d}{dt} \left(\frac{\delta L}{\delta \dot{\phi}} \right) = 0$$

$$\frac{\delta L}{\delta r} = m r \dot{\phi}^2 - mg \cot \alpha$$

$$\frac{\delta L}{\delta \dot{r}} = m \ddot{r} \csc^2 \alpha$$

$$\frac{d}{dt} \left(\frac{\delta L}{\delta \dot{r}} \right) = m \csc^2 \alpha \frac{d \dot{r}}{dt} = m \ddot{r} \csc^2 \alpha$$

$$\Rightarrow m r \dot{\phi}^2 - mg \cot \alpha = m \ddot{r} \csc^2 \alpha$$

$$\Rightarrow \ddot{r} = r \dot{\phi}^2 \sin^2 \alpha - g \cot \alpha \sin \alpha = r \dot{\phi}^2 \sin^2 \alpha - \frac{g}{2} \sin 2\alpha$$

$$\frac{\delta L}{\delta \phi} = 0 \Rightarrow \frac{d}{dt} \left(\frac{\delta L}{\delta \dot{\phi}} \right) = 0$$

$$\Rightarrow \frac{\delta L}{\delta \dot{\phi}} = \text{const}$$

$$\frac{\delta L}{\delta \dot{\phi}} = m r^2 \dot{\phi} \Rightarrow \dot{\phi} = \frac{C}{m r^2}$$

\Rightarrow EOM:

$$\ddot{r} = r \dot{\phi}^2 \sin^2 \alpha - \frac{g}{2} \sin 2\alpha$$

$$\dot{\phi} = C/mr^2$$

• Lagrangian Equation in the Presence of Constraint :-

- For a Lagrangian defined as $\delta L(y, \dot{y}; x)$, the solution is given by Euler's equation as,

$$\frac{\delta L}{\delta y} - \frac{d}{dx} \frac{\delta L}{\delta \dot{y}} = 0$$

- But if we have some constraint on y as $g(y; x)$, the Euler's equation becomes,

$$\frac{\delta L}{\delta y} - \frac{d}{dx} \left(\frac{\delta L}{\delta \dot{y}} \right) + \lambda(x) \frac{\delta g}{\delta y} = 0$$

Lagrange Undetermined Multiplier

- For the general case of several dependent variables and constraints

$$\frac{\delta F}{\delta y_i} - \frac{d}{dx} \left(\frac{\delta F}{\delta \dot{y}_i} \right) + \sum_j \lambda_j(x) \frac{\delta g_j}{\delta y_i} = 0 \quad \begin{matrix} \text{dependent var} \\ i \text{ and constraint} \\ j \text{ on } i. \end{matrix}$$

• Requirements for Constructing the Lagrangian :-

- 1) The forces acting on the system must be derivable from a potential / several potentials. (ie, conservative forces)
- 2) The equations of motion must connect the coordinates of the system and can be functions of time.

◦ Essence of Lagrangian Dynamics :-

- Lagrangian dynamics is not separate from Newtonian dynamics. They will yield the same results for the same physical system.
- Newtonian dynamics deals with quantities acting on an object (the forces), Lagrangian dynamics deals with quantities of the object (the energies).
- Lagrangian dynamics deals with scalar quantities and is invariant to coordinate transformation, allowing the use of generalized coordinates.

◦ A Theorem on Kinetic Energy :-

- The kinetic energy of a system is given by,

$$T = \frac{1}{2} \sum_{\alpha=1}^n \sum_{i=1}^k m_\alpha \dot{x}_{\alpha i}^2$$

Let $x_{\alpha i} = x_{\alpha i}(q_i, t)$, where q_i is the generalized coordinate of x .

$$\Rightarrow \dot{x}_{\alpha i} = \sum_j \frac{\delta x_{\alpha i}}{\delta q_j} \dot{q}_j + \frac{\delta x_{\alpha i}}{\delta t}$$

$$\begin{aligned} \Rightarrow \dot{x}_{\alpha i}^2 &= \sum_{j,k} \left(\frac{\delta x_{\alpha i}}{\delta q_j} \frac{\delta x_{\alpha i}}{\delta q_k} \dot{q}_j \dot{q}_k \right) + 2 \sum_j \frac{\delta x_{\alpha i}}{\delta q_j} \dot{q}_j \frac{\delta x_{\alpha i}}{\delta t} \\ &\quad + \left(\frac{\delta x_{\alpha i}}{\delta t} \right)^2 \end{aligned}$$

$$\Rightarrow T = \frac{1}{2} \sum_{\alpha} m_{\alpha} \sum_{i,j,k} \left(\frac{\delta x_{\alpha i}}{s q_j} \frac{\delta x_{\alpha i}}{s q_k} \dot{q}_j \dot{q}_k \right) + \sum_{\alpha} m_{\alpha} \sum_{i,j} \frac{\delta x_{\alpha i}}{s q_j} \dot{q}_j \frac{\delta x_{\alpha i}}{s t}$$

$$+ \frac{1}{2} \sum_{\alpha} m_{\alpha} \sum_i \left(\frac{\delta x_{\alpha i}}{s t} \right)^2$$

If time does not explicitly apply when x is represented in the generalized coordinate transformation, ie. $\frac{\delta x_{\alpha i}}{s t} = 0$

$$\Rightarrow T = \frac{1}{2} \sum_{\alpha} m_{\alpha} \sum_{i,j,k} \left(\frac{\delta x_{\alpha i}}{s q_j} \frac{\delta x_{\alpha i}}{s q_k} \dot{q}_j \dot{q}_k \right)$$

$$T = \sum_{i,j,k} a_{ijk} \dot{q}_j \dot{q}_k \quad [\text{Summing over } i \text{ to create } a_{jk}] - \textcircled{1}$$

$$\frac{s t}{s \dot{q}_i} = \sum_k a_{ik} \dot{q}_k + \sum_j a_{ij} \dot{q}_i$$

Multiply on both sides by \dot{q}_i and sum over i ,

$$\Rightarrow \sum_i \dot{q}_i \frac{s t}{s \dot{q}_i} = \sum_{i,k} a_{ik} \dot{q}_i \dot{q}_k + \underbrace{\sum_{i,j} a_{ij} \dot{q}_i \dot{q}_k}_{\text{Each of these is III to } \textcircled{1}}$$

$$\Rightarrow \sum_i \dot{q}_i \frac{s t}{s \dot{q}_i} = 2T$$

Therefore, for gen. coord transformation such that time is not a direct variable of the coordinate, the above result can be used.

In the general case, if $f(y_k)$ is a homogenous function of y_k of degree n , then

$$\sum_k \dot{y}_k \frac{\partial f}{\partial y_k} = n f$$

- Conservation of Energy in Lagrangian Mechanics :-

- Time is homogeneous within an inertial reference frame. So,

$$\frac{\delta L}{\delta t} = 0 \quad [\text{The Lagrangian is invariant with time}]$$

Since L is a function of q_i and \dot{q}_i ,

$$\frac{dL}{dt} = \sum_i \frac{\delta L}{\delta q_i} \dot{q}_i + \sum_i \frac{\delta L}{\delta \dot{q}_i} \ddot{q}_i$$

By Euler's equation, $\frac{\delta L}{\delta q_i} - \frac{d}{dt} \frac{\delta L}{\delta \dot{q}_i} = 0$

$$\Rightarrow \frac{dL}{dt} = \sum_i \frac{d}{dt} \left(\frac{\delta L}{\delta \dot{q}_i} \right) \dot{q}_i + \sum_i \frac{\delta L}{\delta \dot{q}_i} \ddot{q}_i$$

$$= \sum_i \frac{d}{dt} \left(\dot{q}_i \frac{\delta L}{\delta \dot{q}_i} \right) \quad [\text{Reverse of Product Rule}]$$

$$\Rightarrow \frac{dL}{dt} - \frac{d}{dt} \left(\sum_i \dot{q}_i \frac{\delta L}{\delta \dot{q}_i} \right) = 0$$

$$\Rightarrow \frac{d}{dt} \left(L - \sum q_i \frac{\dot{q}_i}{\frac{\partial L}{\partial \dot{q}_i}} \right) = 0$$

$$\Rightarrow L - \sum q_i \frac{\dot{q}_i}{\frac{\partial L}{\partial \dot{q}_i}} = \text{constant} \quad (\text{Let it be } -H)$$

$$\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial U}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i} - \cancel{\frac{\partial U}{\partial \dot{q}_i}} \xrightarrow{0}$$

[Potential Energy
will not depend
explicitly on velocity
 \dot{q}_i]

$$\Rightarrow L - \sum q_i \frac{\dot{q}_i}{\frac{\partial L}{\partial \dot{q}_i}} = -H$$

$$\Rightarrow T - U - 2T = -H$$

$$\Rightarrow T + U = H$$

$$\Rightarrow T + U = \text{constant}$$

The function H is termed as the Hamiltonian of the system and is constant only if:

1) The Eqn of transformation to the generalized coordinates must be independent of time, to make T a homogeneous function of velocity \dot{q}_i .

2) The potential energy must be independent of velocity of the object (ie must be conservative).

° Conservation of Linear Momentum in Lagrangian Mechanics :-

- Space is homogeneous in an inertial reference frame, i.e., the Lagrangian is invariant of any small translation of the whole system,

$$\Rightarrow S_L = \sum_i \frac{\delta L}{\delta x_i} \delta x_i + \sum_i \frac{\delta L}{\delta \dot{x}_i} \delta \dot{x}_i = 0$$

where δx_i is the small displacement in space, independent of time

$$\delta x_i = S \frac{d}{dt} x_i = \frac{d}{dt} \delta x_i = 0$$

$$\Rightarrow \sum_i \frac{\delta L}{\delta x_i} \delta x_i = 0$$

$$\Rightarrow \frac{\delta L}{\delta x_i} = 0 \quad [q_i's \text{ are non-zero and linearly independent}]$$

$$\Rightarrow \frac{d}{dt} \frac{\delta L}{\delta \dot{x}_i} = 0$$

$$\Rightarrow \frac{\delta L}{\delta \dot{x}_i} = \text{constant}$$

$$\Rightarrow \frac{S(T-U)}{\delta x_i} = \frac{ST}{\delta x_i} = \frac{S}{\delta x_i} \left(\frac{1}{2} m \dot{x}_i^2 \right) = \text{constant}$$

$$\Rightarrow \underline{m \dot{x}_i = \text{constant.}}$$

◦ Conservation of Angular Momentum in Lagrangian Mechanics :-

- Space is also isotropic within an inertial reference frame, ie, the Lagrangian is invariant of the orientation of the system.

If the system is rotated by a small angle $\delta\theta$, the radius vector changes as,

$$\begin{aligned} \delta r &= \delta\theta \times r \\ \Rightarrow \delta \dot{r} &= \delta\theta \times \dot{r} \end{aligned}$$

The change in L due to small rotation $\delta\theta$ is,

$$\delta L = \sum_i \frac{\delta L}{\delta x_i} \delta x_i + \sum_i \frac{\delta L}{\delta \dot{x}_i} \delta \dot{x}_i$$

From the prev. section, we see that,

$$\frac{\delta L}{\delta \dot{x}_i} = p_i \quad [\text{The assumptions taken in both sections are the same}]$$

$$\Rightarrow \frac{\delta L}{\delta x_i} = \dot{p}_i$$

$$\Rightarrow \delta L = \sum_i \dot{p}_i \delta x_i + \sum_i p_i \delta \dot{x}_i = 0$$

$$\Rightarrow \dot{p}_i \delta x_i + p_i \delta \dot{x}_i = 0$$

$$\Rightarrow \dot{p}_i S_{ri} + p_i \dot{S}_{ri} = 0$$

$$\Rightarrow \dot{p}_i \cdot (S_\theta \times r) + p_i (S_\theta \times \dot{r}) = 0$$

$$\Rightarrow S_\theta \cdot (r \times \dot{p}_i) + S_\theta \cdot (\dot{r} \times p_i) = 0 \quad [\text{Cyclic Permutation}]$$

$$\Rightarrow S_\theta \cdot [(r \times \dot{p}_i) + (\dot{r} \times p_i)] = 0$$

$$\Rightarrow S_\theta \cdot \frac{d}{dt} (r \times p_i) = 0$$

$$\Rightarrow \frac{d}{dt} (r \times p_i) = 0 \quad [S_\theta \text{ is arbitrary}]$$

$$\Rightarrow r \times p_i = \text{constant}$$

Since Angular momentum = $r \times p_i$, we see that Angular momentum of the system is constant, given the assumptions hold for the system.

• Hamiltonian Dynamics :-

- We know that, if the potential of an object is independent of its velocity then,

$$p_i = \frac{SL}{Sx_i}$$

- If the Lagrangian is expressed in generalized coordinates q_i ,

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \quad \& \quad \dot{p}_j = \frac{\partial L}{\partial q_j}$$

p_j is defined as the generalized momentum of the system.

- Using generalized momentum, the Hamiltonian of a system is defined as,

$$H = \sum_j p_j \dot{q}_j - L$$

Where the Lagrangian L is a function of the generalized coordinates and generalized velocities, and the time.

- The Hamiltonian of a system is expressed as,

$$H(q_k, p_k; t) = \sum_j p_j \dot{q}_j - L(q_k, \dot{q}_k; t)$$

The Hamiltonian is expressed as a function of $(q_k, p_k; t)$, the Lagrangian is a function of $(q_k, \dot{q}_k; t)$.

- The differential of H is,

$$dH = \sum_k \left(\frac{\partial H}{\partial q_k} \dot{q}_k + \frac{\partial H}{\partial p_k} \dot{p}_k \right) + \frac{\partial H}{\partial t} dt$$

$$= \sum_k \left(\dot{q}_k dp_k + p_k dq_k - \frac{\partial L}{\partial q_k} \dot{q}_k - \frac{\partial L}{\partial \dot{q}_k} \dot{p}_k \right) - \frac{\partial L}{\partial t} dt$$

$$= \sum_k (q_k \dot{p}_k - p_k \dot{q}_k) - \frac{\delta L}{\delta t} dt$$

Using the coefficients of dH , we can see that,

$$\frac{\delta H}{\delta p_k} = \dot{q}_k \quad \frac{\delta H}{\delta q_k} = -\dot{p}_k \quad \frac{\delta L}{\delta t} = -\frac{\delta H}{\delta t}$$

Hamiltonian / Canonical EOM

Also, $\frac{dH}{dt} = \frac{\delta H}{\delta t}$, ie., if H does not explicitly contain time, then H is conserved, ie., constant.

- As prev. seen, H is conserved if PE is velocity dependent and the transformation to the general coordinate do not contain time.
- If there are s Lagrangian Equations, there are $2s$ canonical equations (s - Degrees of freedom)
- Canonical Equations are first order DE's, while Lagrangians are second order DE's.

Example: Using Hamiltonian Mechanics to find the EOM of a particle of mass m constrained on the surface of a cylinder $x^2 + y^2 = R^2$. The particle has a force towards the origin and proportional to the distance of the particle $F = -kr$.

$$U = - \int F dr = - \int (-kr) dr = \frac{1}{2} kr^2$$

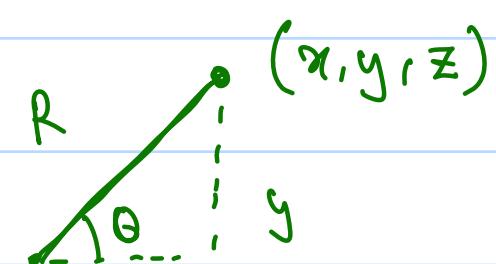
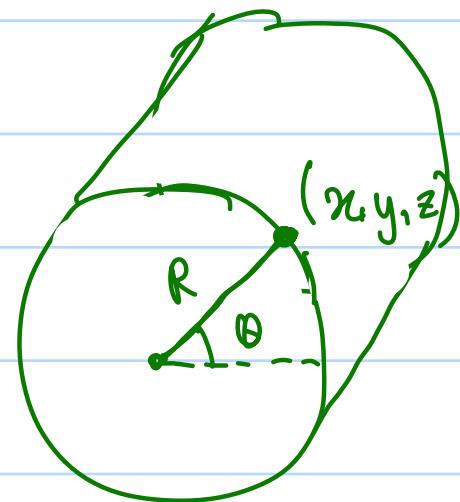
$$\Rightarrow U = \frac{1}{2} k(x^2 + y^2 + z^2) = \frac{1}{2} k(R^2 + z^2)$$

$$V^2 = (\dot{x})^2 + (\dot{y})^2 + (\dot{z})^2$$

$$= (R\dot{\cos\theta})^2 + (R\dot{\sin\theta})^2 + (\dot{z})^2$$

$$= R^2 (-\dot{\sin\theta} \cdot \dot{\theta})^2 + R^2 (\dot{\cos\theta} \cdot \dot{\theta})^2 + \dot{z}^2$$

$$= R^2 \sin^2\theta \cdot \dot{\theta}^2 + R^2 \omega^2 \theta \cdot \dot{\theta}^2 + \dot{z}^2$$



$$V^2 = R^2 \dot{\theta}^2 + \dot{z}^2$$

$$x = R\cos\theta$$

$$y = R\sin\theta$$

$$T = \frac{1}{2}m(R^2\dot{\theta}^2 + \dot{z}^2)$$

$$= \frac{1}{2}mR^2\dot{\theta}^2 + \frac{1}{2}m\dot{z}^2$$

$$L = T - U = \frac{1}{2}mR^2\dot{\theta}^2 + \frac{1}{2}m\dot{z}^2 - \frac{1}{2}kR^2 - \frac{1}{2}kz^2$$

$$= L(\dot{\theta}, z, \dot{z}; t), \frac{dL}{dt} = 0$$

$$= p_\theta = \frac{dL}{d\dot{\theta}} = mR^2\dot{\theta} \quad \Rightarrow \quad T = \frac{p_\theta^2}{2mR^2} + \frac{p_z^2}{2m}$$

$$p_z = \frac{dL}{d\dot{z}} = m\dot{z}$$

Since the system is conservative and the generalized coordinates (θ, z) do not involve time, the Hamiltonian is defined as,

$$H = T + V$$

$$= \frac{p_\theta^2}{2mR^2} + \frac{p_z^2}{2m} + \frac{1}{2}k(R^2+z^2)$$

$$H(z, p_\theta, p_z)$$

$$\Rightarrow \dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mR^2}$$

$$\dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0 \rightarrow \text{Angular Momentum is constant}$$

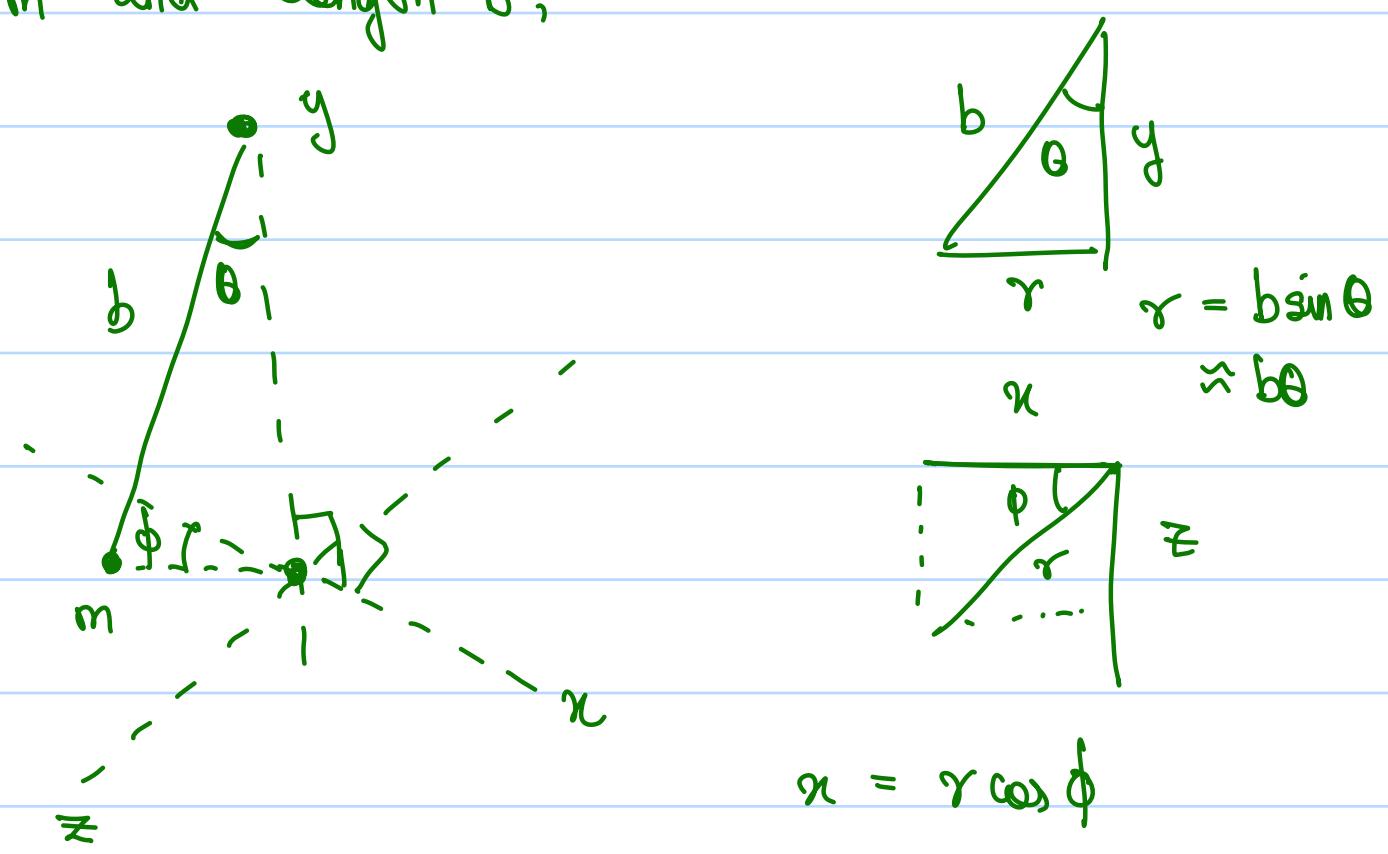
$$\dot{p}_z = -\frac{\partial H}{\partial z} = -kz$$

$$\dot{z} = \frac{p_z}{m}, \quad \dot{p}_z = -kz$$

$$\Rightarrow \ddot{z} = \frac{\dot{p}_z}{m} = -\frac{k}{m}z$$

$$\Rightarrow \ddot{z} = -\frac{k}{m}z \rightarrow \underline{\underline{z \text{ is in SHM}}}$$

Example: Use Hamiltonian Dynamics to find the EOM for a spherical pendulum of mass m and length b ,



$$\begin{aligned} v^2 &= \dot{x}^2 + \dot{z}^2 \\ &= (b\dot{\theta}\cos\phi)^2 + (b\dot{\theta}\sin\phi)^2 \end{aligned} \Rightarrow \begin{aligned} x &= b\sin\theta\cos\phi \approx b\theta\cos\phi \\ z &= b\sin\theta\sin\phi \approx b\theta\sin\phi \end{aligned}$$

$$= b^2 \left((\dot{\theta}\cos\phi - \theta\sin\phi\dot{\phi})^2 + (\dot{\theta}\sin\phi + \theta\cos\phi\dot{\phi})^2 \right)$$

$$= b^2 \left(\dot{\theta}^2\cos^2\phi + \theta^2\dot{\phi}^2\sin^2\phi - \cancel{2\dot{\theta}\theta\cos\phi\sin\phi\dot{\phi}} \right. \\ \left. + \dot{\theta}^2\sin^2\phi + \theta^2\dot{\phi}^2\cos^2\phi + \cancel{2\dot{\theta}\theta\cos\phi\sin\phi\dot{\phi}} \right)$$

$$= b^2 (\dot{\theta}^2 + \theta^2\dot{\phi}^2)$$

$$T = \frac{1}{2}mb^2(\dot{\theta}^2 + \theta^2\dot{\phi}^2)$$

$$\approx T = \frac{1}{2}mb^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) \quad [\text{Reversing the small angle approximation}]$$

$$U = -mgb\cos\theta$$

$$\begin{aligned}
 L &= T - U \\
 &= \frac{1}{2}mb^2\dot{\theta}^2 + \frac{1}{2}mb^2\sin^2\theta\dot{\phi}^2 - mgbc\cos\theta \\
 &= L(\theta, \dot{\theta}, \dot{\phi}; t)
 \end{aligned}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mb^2\dot{\theta}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mb^2\sin^2\theta\dot{\phi}$$

$$\begin{aligned}
 H &= T + U \\
 &= \frac{p_\theta^2}{2mb^2} + \frac{p_\phi^2}{2mb^2\sin^2\theta} - mgbc\cos\theta
 \end{aligned}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mb^2}$$

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mb^2\sin^2\theta}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{p_\phi^2}{2mb^2} \left(\frac{-3}{\sin^3\theta} \right) \cos\theta - mgb\sin\theta$$

$$= -\frac{3p_\phi^2 \cos\theta}{2mb^2\sin^3\theta} - mgb\sin\theta$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0 \rightarrow \text{Angular Momentum about the central axis is constant.}$$

