EC5.102: Information and Communication

(Lec-12)

Modulation, Channel capacity

(21-April-2025)

Arti D. Yardi

Email address: arti.yardi@iiit.ac.in
Office: A2-204, SPCRC, Vindhya A2, 1st floor

Recap

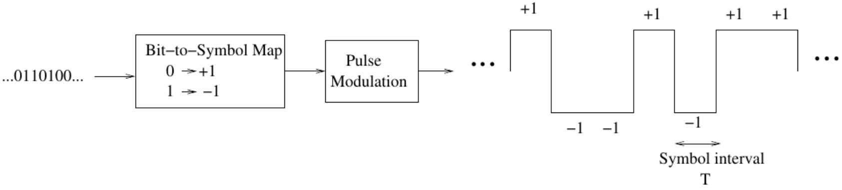
- Analog modulation
  - $m(t) \rightarrow \begin{bmatrix} u_c(t) & u_s(t) \end{bmatrix}$, $m(t), u_c(t), u_s(t)$ are baseband signals
  - Up-conversion: $u_p(t) = u_c(t)\cos(2\pi f_c t) - u_s(t)\sin(2\pi f_c t)$
  - Complex envelop of passband signal $u_p(t)$ is $u_c(t) + ju_s(t)$
  - DSB-SC AM: $m(t) \rightarrow \begin{bmatrix} Am(t) & 0 \end{bmatrix}$
- Digital modulation
  - Message $\mathbf{m}[n] = \begin{bmatrix} a_1 & a_2 & \ldots & a_m \end{bmatrix}$, each $a_i \in \{0, 1\}$
  - $\mathbf{m}[n] \rightarrow b_c[n] + jb_s[n]$, where $b_c[n] + jb_s[n]$ is complex number.
  - BPSK ($m = 1$), QPSK ($m = 2$), 8-PSK ($m = 3$), 2^m-ary PSK
  - For up-conversion: $u_c(t) = \sum_n b_c[n]p(t - nT)$, $u_s(t) = \sum_n b_s[n]p(t - nT)$
  - Note: $b_c[n] + jb_s[n]$ and $u_c(t) + ju_s(t)$ are equivalent.

Digital modulation

Our focus (BPSK, QPSK)

Pulse modulation

- Recall: Pulse modulation



- Mathematical representation of pulse modulation:

$$u(t) = \sum_n b[n]p(t - nT), \tag{1}$$

  where each $b[n] \in \{+1, -1\}$ and $p(t)$ is the “modulating” pulse.

- Note: Waveform $u(t)$ in Eq. (1) is a baseband signal!
- Question: How to convert it to a passband signal?

BPSK

- Aim: Convert $u(t) = \sum_n b[n]p(n - nT)$ to a passband signal?
- One easy approach would be to send the passband signal

$$u_p(t) = u(t)\cos(2\pi f_c t)$$

- Picture for this modulation.
- Observe:

$$u_p(t) = \begin{cases} \cos(2\pi f_c t) \text{ if } b[n] = +1 \\ -\cos(2\pi f_c t) \text{ if } b[n] = -1 \end{cases}$$

- Since the phase of the carrier switches between two values 0 and π, this modulation scheme is termed Binary Phase Shift Keying (BPSK).

Complex envelop of a passband signal

- Recall: Passband signal is given by

$$u_p(t) = u_c(t)\cos(2\pi f_c t) - u_s(t)\sin(2\pi f_c t).$$

- The complex envelope of $u_p(t)$ is given by $u(t) = u_c(t) + ju_s(t)$.
- One can consider an equivalent complex envelope: $b[n] = b_c[n] + jb_s[n]$
  - $b_c[n]$ will modulate the I-component
  - $b_s[n]$ will modulate the Q-component
- For BPSK, we have $b[n] \in \{+1, -1\}$. What will be the I and Q components for BPSK?
- We will not always ingore Q-component: Example QPSK

QPSK

- Recall: The equivalent complex envelope: $b[n] = b_c[n] + jb_s[n]$
- Let us see what happens to the passband signal when $b_c[n], b_s[n]$ each take values in $\{+1, -1\}$.

$$u_p(t) = u_c(t)\cos(2\pi f_c t) - u_s(t)\sin(2\pi f_c t).$$

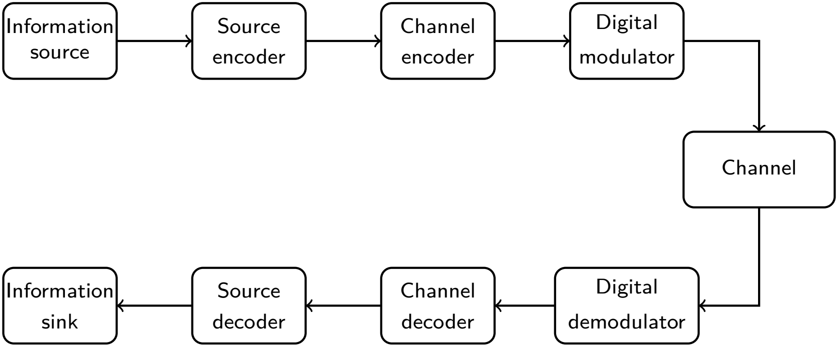
with

$$u_c(t) = \sum_n b_c[n]p(t - nT) \quad \text{and} \quad u_s(t) = \sum_n b_s[n]p(t - nT)$$

- What will be $u_p(t)$ if $b_c[n] = +1$ and $b_s[n] = +1$? Similarly for other values.
- Since the phase of the carrier switches between four values $\pi/4, \pi/4, 3\pi/4$ and $-3\pi/4$, this is termed Quadrature Phase Shift Keying (QPSK).
- Constellations for BPSK, QPSK, 8-PSK, 2^m-PSK

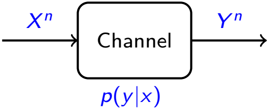
Discrete memoreless channel (DMC)

Communication channel



We will ignore up-conversion and consider complex-input channel

Discrete memoryless channel (DMC)



- Discrete channel: It is defined as a system consisting of an input alphabet $\mathcal{X}$ and output alphabet $\mathcal{Y}$ and a probability transition matrix $p(y|x)$ that expresses the probability of observing the output symbol y given that we send the symbol x.
- Memoryless channel: The channel is said to be memoryless if the probability distribution of the output depends only on the input at that time and is conditionally independent of previous channel inputs or outputs.
- We focus on discrete memoreless channel (DMC).
- For DMC we have,

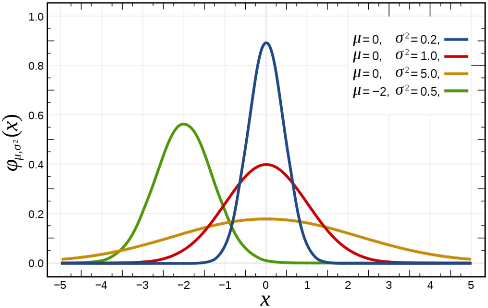
$$p(y^n|x^n) = \prod_{i=1}^{n} p(y_i|x_i)$$

- Examples of DMC: BSC, BEC, AWGN

Additive white Gaussian noise (AWGN) channel

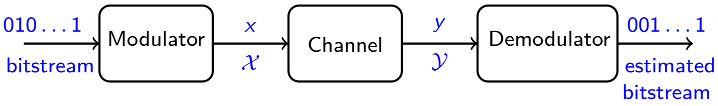
- For AWGN channel: $Y = X + Z$ where $Z \sim \mathcal{N}(0, \sigma^2)$
- Support set of X depends on the underlying modulation scheme.
- Recall: PDF of a Gaussian r.v. $\mathcal{N}(\mu, \sigma^2)$ with parameters μ and σ is given by

$$f_Z(z) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(z-\mu)^2}{2\sigma^2}}$$



Demodulator for BPSK

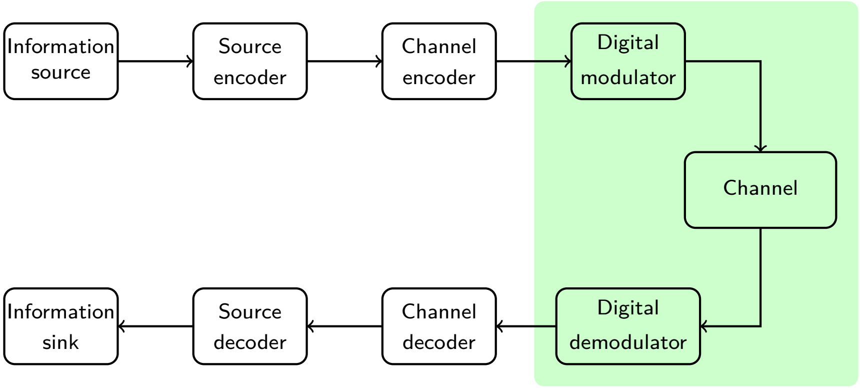
BPSK Demodulator

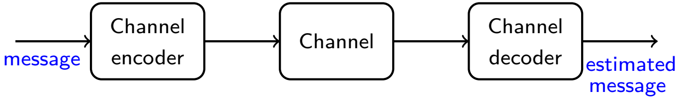


- BPSK modulator: Binary phase shift keying modulator
  - 0 is mapped to $+1$ Volts
  - 1 is mapped to -1 Volts
- For BPSK we have: $\mathcal{X} = \{+1, -1\}$
- Optimal detection and Bit error rate for BPSK
- What to do when 0 and 1 are mapped to $+\sqrt{E}$ and $-\sqrt{E}$ respectively?
- How to extend these ideas to QPSK?
- Relation to BSC, BEC

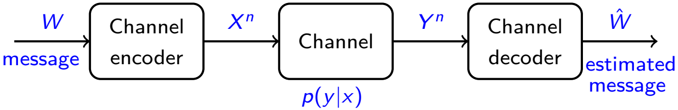
Introduction to Channel Capacity

Discrete channel





Discrete memoryless channel (DMC)



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- Memoryless channel: The channel is said to be memoryless if the probability distribution of the output depends only on the input at that time and is conditionally independent of previous channel inputs or outputs.
- We focus on discrete memoreless channel (DMC).
- "Information" channel capacity (C) of a discrete memoryless channel defined as

$$C := \max_{p(x)} I(X;Y)$$

where the maximum is taken over all possible input distributions $p(x)$.

Properties of channel capacity

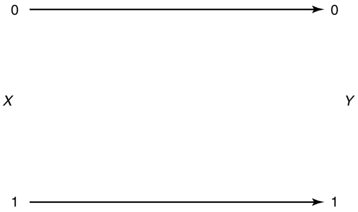
- Capacity: $C := \max_{p(x)} I(X;Y)$
- $I(X;Y) = H(X) - H(X|Y) = H(Y) - H(Y|X)$.
- $C \geq 0$
- $C \leq \log|\mathcal{X}|$
- $C \leq \log|\mathcal{Y}|$

Examples of Channel Capacity

Capacity of some simple DMCs

- Noiseless binary channel
- Noisy channel with non-overlapping inputs
- Binary symmetric channel (BSC)
- Binary erasure channel channel (BEC)

Noiseless binary channel

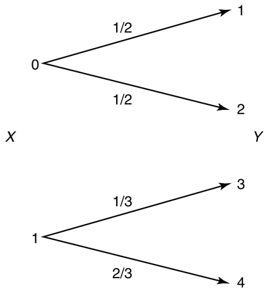


- Any transmitted bit is received without error.
- What is its capacity? Answer: 1 bit
- Verify using the formula for channel capacity.

$$
\begin{aligned}
C = \max_{p(x)} I(X;Y) &= \max_{p(x)} H(Y) - H(Y|X) \\
&= \max_{p(x)} H(Y) - 0 \\
&= 1
\end{aligned}
$$

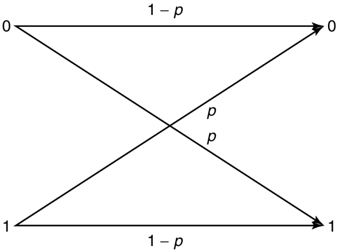
- Which distribution $p(x)$ maximizes $H(Y)$?

Noisy Channel with Non-overlapping Outputs



- What is the capacity of this channel? Class exercise
- **Why?**
- The channel appears to be noisy, but really is not. Even though the output of the channel is a random consequence of the input, the input can be determined from the output, and hence every transmitted bit can be recovered without error.

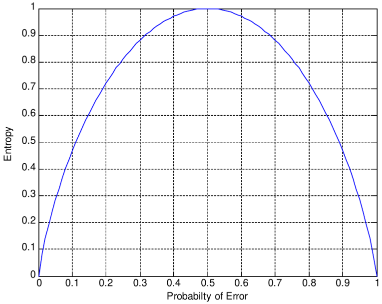
Binary Symmetric Channel



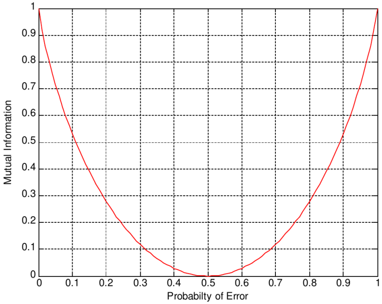
- What is the capacity? Solved in class
- $I(X;Y) = H(Y) - H(Y|X)$
- $C = 1 - H(p)$, where recall that $H(p) = -p\log_2 p - (1-p)\log_2(1-p)$
- Let us see graphical interpretation

Entropy vs capacity of BSC

- Entropy of BSC:
  $H(p) = -p \log_2 p - (1-p) \log_2 (1-p)$



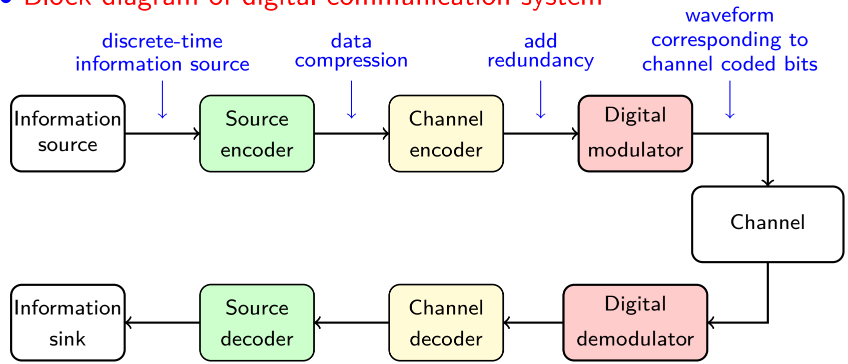
- Capacity of BSC: $1 - H(p)$



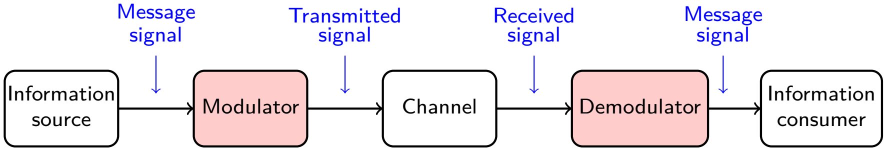
Analog vs Digital communication system

Analog vs Digital communication system

- Block diagram of digital communication system



- Block diagram of analog communication system



Wish you all the best!