EC5.102 Information and Communication **Spring 2020**

Lecture 4: Sampling Theorem

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Theorem 4.1 (Sampling Theorem)**.** *Let $x(t)$ be a band-limited signal with $X(j\omega) = 0$ for $|\omega| > \omega_M$. Then, $x(t)$ is uniquely determined by its samples $x(nT), n \in \mathbb{Z}$ is*

$$\omega_s > 2\omega_M, \quad \textit{where} \quad \omega_s = \frac{2\pi}{T}. \tag{4.1}$$

Signal	Fourier transform	Property
$x(t)$	$X(j\omega)$	
$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$	Fourier Transform of Periodic Signals
$x_p(t) = x(t)p(t)$ $= \sum_{n=-\infty}^{\infty} x(nT)\delta(n - tT)$	$X_p(j\omega) = \frac{1}{2\pi} X(j\omega) * P(j\omega)$ $= \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$	Fourier Transform of Convolution is Multiplication and Convolution with impulse shifts a signal

Proof.

- $X_p(j\omega)$ is a periodic function of ω consisting of superposition of shifted replicas of $X(j\omega)$ scaled by $\frac{1}{T}$.
- If $\omega_M < (\omega_s - \omega_M)$, there is no overlap between the shifted replicas of $X(j\omega)$ (see Fig. 4.1)
- If $\omega_s < 2\omega_M$, there is overlap and signal is said to be aliased. (See Fig. 4.2).
- If $\omega_M < (\omega_s - \omega_M)$, the impulse train $= \sum_{n=-\infty}^{\infty} x(nT)\delta(n - tT)$ is passed through an ideal lowpass filter with gain T and cutoff frequency ω_c between ω_M and $\omega_s - \omega_M$. The resulting output signal is exactly $x(t)$.
- $x(t)$ is expressed in terms of the $x(nT), n \in \mathbb{Z}$ as follows:

$$x(t) = x_p(t) * h(t),$$

Where $h(t)$ is the impulse response of the ideal low pass filter with gain T and cutoff frequency ω_c.

$$h(t) = T\frac{sin(\omega_c t)}{\pi t}$$

Hence, the interpolation equation is derived as follows:

$$\begin{aligned} x(t) &= x_p(t) * h(t) \\ &= \left(\sum_{n=-\infty}^{\infty} x(nT)\delta(n - tT) \right) * h(t) \\ &= \left(\sum_{n=-\infty}^{\infty} x(nT)\delta(n - tT) \right) * T\frac{sin(\omega_c t)}{\pi t} \\ &= \sum_{n=-\infty}^{\infty} x(nT)T\frac{sin(\omega_c(t - nT))}{\pi(t - nT)} \end{aligned}$$

□

If a signal is sampled at $\omega_s > 2\omega_M$, the signal is said to be oversampled. If a signal is sampled at $\omega_s = 2\omega_M$, the signal is said to be critically sampled. If a signal is sampled at $\omega_s < 2\omega_M$, the signal is said to be under sampled.

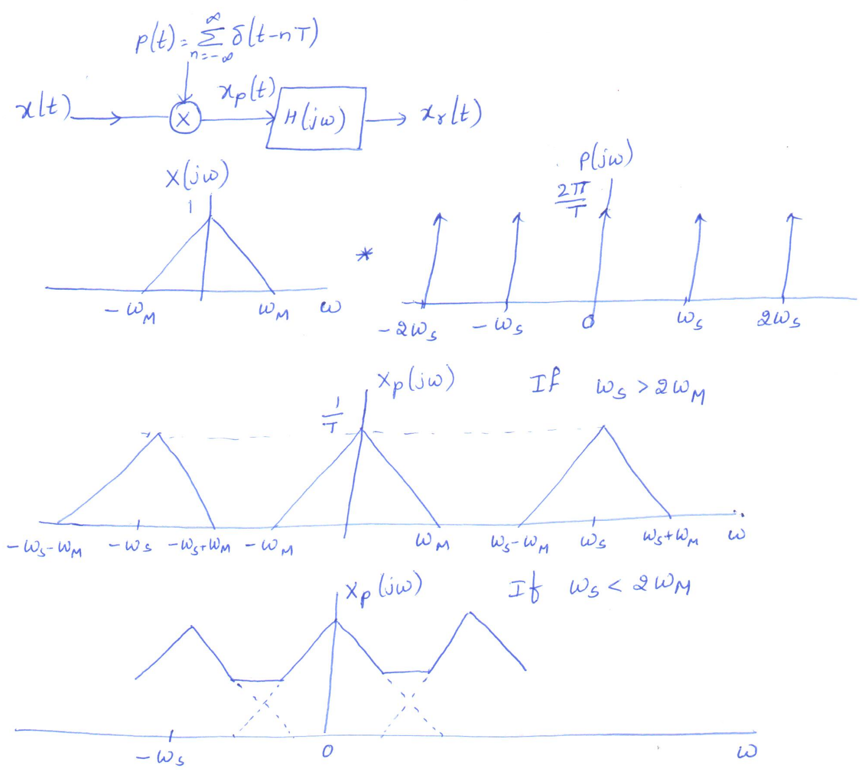

Figure 4.1: Spectrum of sampled signal in case of $\omega_s > 2\omega_M$ and $\omega_s < 2\omega_M$.

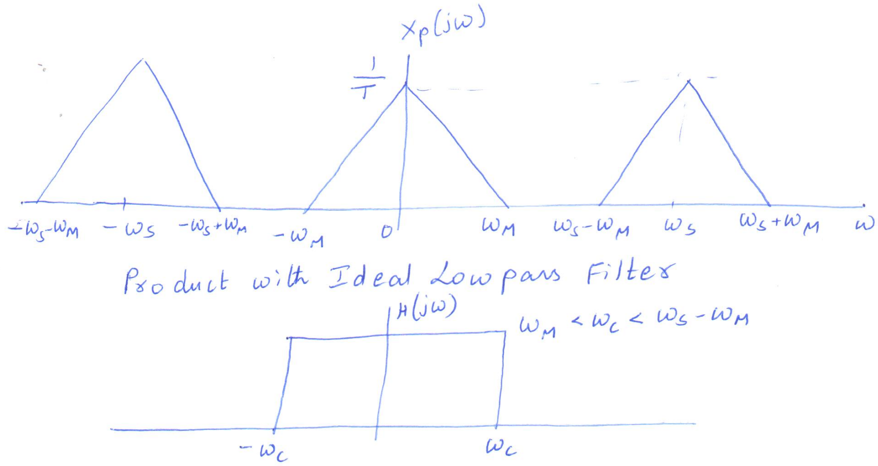

Figure 4.2: Signal Reconstruction by passing the sampled signal through an ideal low pass filter with gain T and cutoff frequency ω_c.

4.1 Examples of Aliasing

Aliasing of audio and images causes perceivable distortion of the signal. We give two more examples of aliasing (due to sampling sinusoidal signals) below:

- **Strobe Effect:** Consider an example in which we have a disc rotating at a constant rate with a single radial line marked on the disc. The flashing strobe acts as a sampling system, since it illuminates the discussion for extremely brief time intervals at a periodic rate. When the strobe frequency is much higher than the rotational speed of the disc, the speed of rotation of the disc is perceived correctly. When the strobe frequency becomes less than twice the rotational frequency of the disc, the rotation appears to be at a lower frequency than is actually the case. Furthermore, because of phase reversal, the disc will appear to be rotating in the wrong direction.

- **Oscilloscope:** Consider a 2.2kHZ signal sampled at 20 kHz, which is almost ten times the frequency of the input waveform. The digitized approximation in Fig. 4.3 has the same shape as the input waveform. Fig. 4.4 shows the same 2200 Hz signal, but the sampling frequency has been reduced to 2 kHz. Note that the input waveform is sampled only about once each cycle, and the digitized waveform appears to be, at least approximately, a sinusoidal signal of much lower frequency than the input waveform

4.2 Examples of Signals and Sampling Rates

The range of human hearing is quite large, both in terms of sound intensity and sound frequency. Humans can hear sounds between about 20 Hz and 20000 Hz; music and speech typically covers the range from 100 Hz to 3000 Hz. The ear is most sensitive to sound of about 3000 Hz, as shown in Fig. 4.5.

4.2.1 Anti-alias Filter

According to the Nyquist sampling theorem, the sampling rate should be at least twice the maximum frequency component of the signal of interest. In other words, the maximum frequency of the input signal should be less than or equal to half of the sampling rate.

How do you ensure that this is definitely the case in practice? Even if you are sure that the signal being measured has an upper limit on its frequency, pickup from stray signals (such as the powerline frequency or from local radio stations) could contain frequencies higher than the Nyquist frequency. These frequencies may then alias into the appropriate frequency range and thus give you erroneous results.

To be sure that the frequency content of the input signal is limited, a low pass filter (a filter that passes low frequencies but attenuates the high frequencies) is added before the sampler and the ADC. This filter is an anti-alias filter because by attenuating the higher frequencies (greater than the Nyquist frequency), it prevents the aliasing components from being sampled. Because at this stage (before the sampler and the ADC) you are still in the analog world, the anti-aliasing filter is an analog filter.

Signal	Sampling Rate
Telephony Standards	8 kHz
G.722 compression standards	16 kHz
Digital Radio, NICAM (Nearly Instantaneous Compandable Audio matrix)	32 kHz
Audio CDs	44.1 kHz
Android Devices	8, 16, 24, 44.1 or 48 kHz (support all the above)
Common pixel sampling Rate	13.5 MHz
High Definition	96 kHz

Table 4.1: Signals and their corresponding Sampling Frequencies.

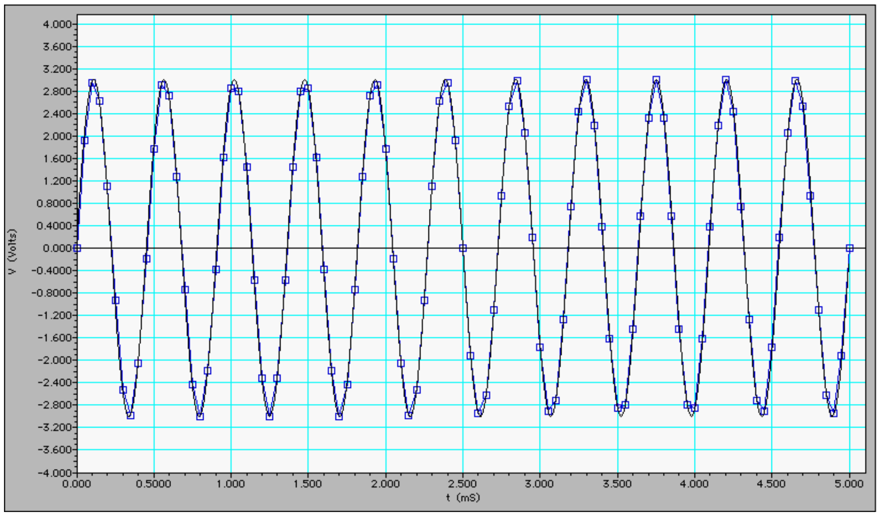

Figure 4.3: 2.2kHz signal sampled at 20kHz.

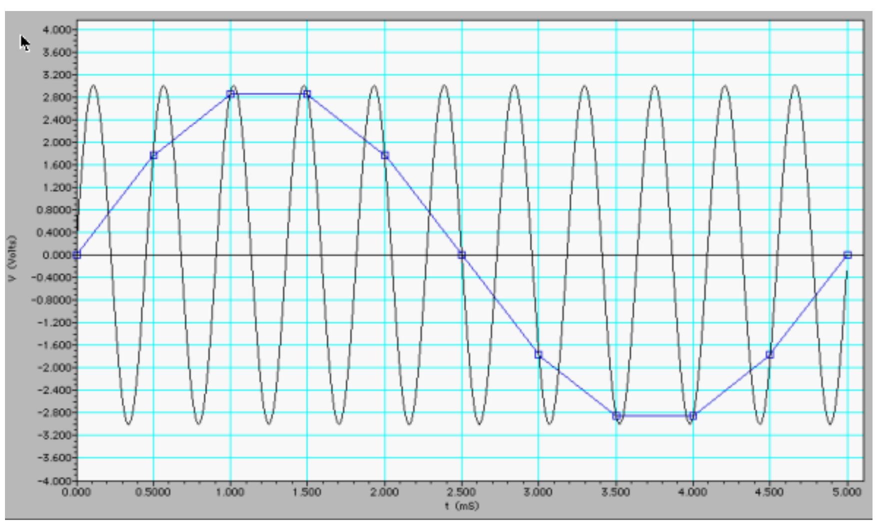

Figure 4.4: 2.2kHz signal sampled at 2kHz.

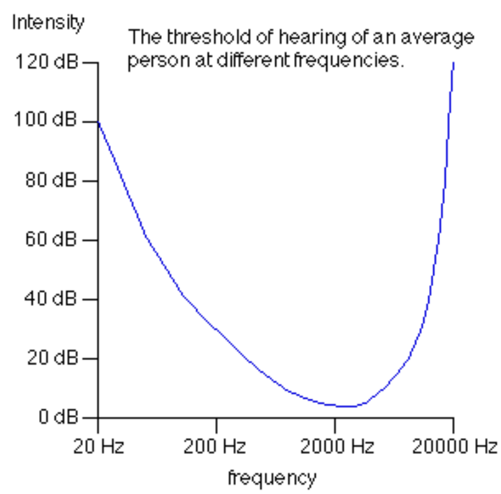

Figure 4.5: Sensitivity of hearing vs frequency of human ear.

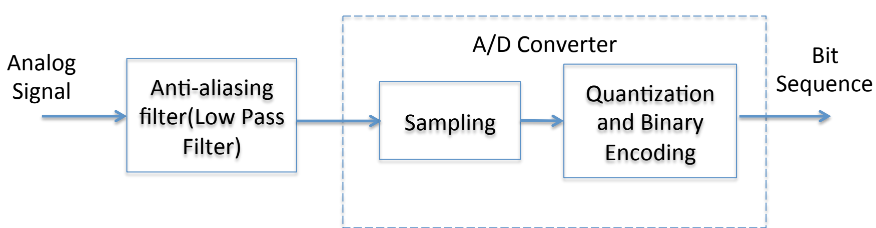

Figure 4.6: Block Diagram for converting analog signal to bit sequence.