



# Probability And Random Processes

→ Modules :

M1) Basics Of Probability

M2) Discrete Random Variables

M3) Continuous Random Variables

M4) Tail Bounds & Limit Theorems

M5) Random Processes

→ Module 1:

1) Approach to define probability

6) Counting

2) Probability Space

3) Continuity of Probability

4) Conditional Probability, Independence

5) Bayes Theorem & Total Probability Theorem

## → Classical Approach of Probability :-

- For an event E,

$$P(E) = \frac{\text{no. of outcomes favourable to } E}{\text{total no. of outcomes}}$$

- Issues :

- Fails when the outcomes are not equally likely.
- Fails when the no. of possible outcome is infinite.

## → Frequency Approach of Probability :-

- For an event E, where the experiment has been performed n times,

$$P(E) = \frac{n_E}{n}$$

$n_E$  - no. of times E has occurred

$$\Rightarrow P(E) = \lim_{n \rightarrow \infty} \frac{n_E}{n}$$

- Issues :

- We cannot perform an experiment infinite times.
- The ratio may not converge.

→ Axiomatic Approach of Probability :-

- The approach depends on a set of axioms.
- Probability space is a set represented by 3 entities.

$$P.S = (\Omega, \mathcal{F}, P)$$



Sample Space   Event Space   Probability law

- Sample Space : Set of all outcomes of the random experiment.
- Event Space : Set of all favorable outcomes
- Probability Law : Function that defines probability.

• Set Theory :-

$$A \setminus B = \{x \in A \text{ st } x \notin B\}$$

$$A \cup B = \{x \in A \text{ or } x \in B\}$$

$$\text{Power} : \left( \bigcup_{i=1}^{\infty} A_i \right)^c = \bigcap_{i=1}^{\infty} A_i^c$$

$$\bigcup_{i=1}^{\infty} A_i = \{x \in A_i \text{ for some } i \in \mathbb{N}\} \rightarrow \text{By defn } \not\subseteq \cup$$

$$\Rightarrow \left( \bigcup_{i=1}^{\infty} A_i \right)^c = \{x \notin A_i \text{ } \forall i \in \mathbb{N}\} \quad \begin{matrix} \text{inverse of the} \\ \text{prev. statement} \end{matrix}$$

$$A_i^c = \{x \notin A_i\}$$

$$\bigcap_{i=1}^{\infty} A_i^c = \{x \in A_i \text{ } \forall i \in \mathbb{N}\} \rightarrow \text{By defn. of } n$$

$$\Rightarrow \bigcap_{i=1}^{\infty} A_i^c = \underline{\underline{\left( \bigcup_{i=1}^{\infty} A_i \right)^c}}$$

### • Sample Space ( $\Omega$ ) :

- The elements of  $\Omega$ , are all the possible outcomes of the random experiment.
- The elements of  $\Omega$  must be mutually exclusive, (disjoint from each other) and collectively exhaustive (cover all possibility)
- Countably infinite sample space : Tossing a coin until we see tail.

Uncountably infinite sample space : Throwing a dart on a square,  $\Omega = \{(x,y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$

### • Event Space ( $\mathcal{F}$ ) :

- An event is a subset of a sample space
- An event space is a set of subsets of  $\Omega$  that form a

$\sigma$ -field.

- Axioms of  $\sigma$ -field:

i)  $\Omega \in \mathcal{F}$

ii) If  $A \in \mathcal{F}$ , then  $A^c \in \mathcal{F}$  (Closure under complements)

iii)  $A_1, A_2, \dots \in \mathcal{F}$ , then  $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$  (Closure under countable unions)

↳ if we take  $A_i = \emptyset$  &  $i \geq k$ , then we can use

(this statement for finite union as well)

↳ Same can be said for  $\cap$ , using (ii) and De-Morgan's

Note:  $A \Delta B = A \setminus B \cup B \setminus A$

iv)  $A, B \in \mathcal{F} \Rightarrow A \Delta B \in \mathcal{F}$  → Not an axiom. Implied by (ii) - (iii)

$$A \Delta B = (A \cap B^c) \cup (B \cap A^c)$$

$$A \in \mathcal{F}, B \in \mathcal{F} \Rightarrow A \cap B^c \in \mathcal{F}, \text{ (by } B \cap A^c \in \mathcal{F}) \\ \therefore A \Delta B \in \mathcal{F}$$

• Smallest  $\sigma$ -field with  $E$ :  $\{\Omega, \emptyset, E, E^c\}$

" " " "  $A, B : \{\Omega, \emptyset, A, B, A^c, B^c, A \cup B, A \cup B^c, A \cap B, A \cap B^c, A^c \cup B, A^c \cap B, A^c \cap B^c, A^c \cup B^c, A \Delta B, (A \Delta B)^c\}$

Example: Complete  $\{\Omega, \emptyset, \{1\}, \{2, 3\}\}$  for  $\Omega = \{1, 2, 3, 4\}$

$$\mathcal{F} = \{\Omega, \emptyset, \{1\}, \{2, 3\}, \{2, 3, 4\}, \{1, 4\}, \{1, 2, 3\}, \{4\}\}$$

- Define the smallest  $\sigma$ -set of  $X, Y, Z$  as  $\sigma(X, Y, Z)$ , then,

$$\sigma(\Omega, \emptyset, A, B) = \sigma(A \cap B, A \cap B^c, A^c \cap B, A^c \cap B^c)$$

Disjoint sets that collectively exhaust  $\Omega$ .

Any union of the 4 sets, have their complements as a union of the other sets.

- Probability Law (P) :-  $P : \mathcal{F} \rightarrow \mathbb{R}$  is a probability law if it follows the below axioms.

i)  $P(\Omega) = 1$  (Normalization)

ii)  $P(E) \geq 0 \quad \forall E \in \mathcal{F}$  (Non-negativity)

iii) If  $A_1, A_2, \dots$  are mutually exclusive (disjoint), then,

$$P\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n P(A_i)$$

Properties:

i)  $P(\emptyset) = 0$ .

Proof: Let  $A_i = \emptyset \quad \forall i$

$$\begin{aligned} P\left(\bigcup_i A_i\right) &= \sum_i P(A_i) \\ &= P(\emptyset) = \sum P(\emptyset) \Rightarrow P(\emptyset) = 0 \end{aligned}$$

Example: Construct a probability law for rolling a die.

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

Defn

$$\left\{ \begin{array}{l} P(\{\text{i}\}) = p_i \quad \forall i \in \{1, 2, 3, 4, 5, 6\} \quad \text{st. } \sum_{i=1}^6 p_i = 1 \quad * p_i \geq 0 \\ P(A) = \sum_{i \in A} P(\{\text{i}\}), \quad A \subseteq \Omega \end{array} \right.$$

$$P(\Omega) = \sum_{i \in \Omega} P(\{\text{i}\}) = \sum_{i \in \Omega} p_i = 1 \quad (\text{Normalization})$$

$$p_i \geq 0 \Rightarrow \sum_{i \in A} p_i \geq 0 \quad \forall A \subseteq \Omega \Rightarrow P(A) \geq 0 \quad \forall A \subseteq \Omega$$

(Non-negativity)

By defn. additivity is satisfied.

ii)  $P(A) \leq 1 \quad \forall A \subseteq \Omega$

iii) If  $A \subseteq B$ , then  $P(A) \leq P(B)$

iv)  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

v)  $P(A) + P(A^c) = 1$

Proof:

iv)  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cup B) = P(A \cup (B|A)) = P(A) + P(B|A)$

$$P(B) = P((A \cap B) \cup (B|A))$$

$$= P(A \cap B) + P(B|A)$$

$$\Rightarrow P(B|A) = P(B) - P(A \cap B)$$

$$\Rightarrow P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

iii) If  $A \subseteq B$ ,  $P(A \cup B) = P(B)$

$$\begin{aligned} \Rightarrow P(B) &= P(A) + P(B) - P(A \cap B) \\ &= P(A) + P(B/A) + P(\cancel{A \cap B}) - \cancel{P(A \cap B)} \end{aligned}$$

$$\Rightarrow P(B) = P(A) + P(B/A)$$

$$\Rightarrow \underline{\underline{P(B) \geq P(A)}}$$

• Continuity Of Probability :-

- A continuous function has a definite limit for all points in its domain.

- A function  $f: \mathbb{R} \rightarrow \mathbb{R}$  if a sequence  $x_n \rightarrow x^* \Rightarrow f(x_n) \rightarrow f(x^*)$  as  $n \rightarrow \infty$  (Formal defn), ie,

$$\lim_{n \rightarrow \infty} f(x_n) = f\left(\lim_{n \rightarrow \infty} x_n\right)$$

- But, probability law is a function on sets. For such a function, continuity is defined as,

Let  $A_1, A_2, A_3, \dots, n \in \mathbb{N}$  be a sequence of events, then

$$\lim_{n \rightarrow \infty} P\left(\bigcup_{i=1}^n A_i\right) = P\left(\lim_{n \rightarrow \infty} \bigcup_{i=1}^n A_i\right)$$

Proof:

Claim 1: Consider 3 sets  $A_1, A_2, A_3$ . Then

$$A_1 \cup A_2 \cup A_3 = A_1 \cup (A_2 \setminus A_1) \cup (A_3 \setminus (A_1 \cup A_2))$$

As a union of disjoint sets  $B_1, B_2, B_3$ .

$$\Rightarrow B_i = A_i \setminus \bigcup_{k=1}^{i-1} A_k \longrightarrow \bigcap_{i=1}^{\infty} B_i = \emptyset$$

Proof of Claim 1:

To prove,  $B_i \cap B_j = \emptyset \quad \forall i \neq j$

WLOG let  $i < j$ . Let  $x \in B_i$

$$\Rightarrow x \in A_i \setminus \bigcup_{k=1}^{i-1} A_k \Rightarrow x \notin \bigcup_{k=1}^{i-1} A_k \quad \& \quad x \in A_i$$

$$\begin{aligned} \text{Assume } x \in B_j &\Rightarrow x \in A_j \setminus \bigcup_{k=1}^{j-1} A_k \\ &\Rightarrow x \notin \bigcup_{k=1}^{j-1} A_k \end{aligned}$$

But we proved  $x \in A_i \longrightarrow \text{Contradiction}$

$\therefore x \in B_i \Rightarrow x \notin B_j, \text{ if } i < j$

Similarly can be stated for  $i > j$ .

$\therefore x \in B_i \Rightarrow x \notin B_j \text{ if } i > j$

$$\Rightarrow \underline{B_i \cap B_j = \emptyset \text{ } \forall i \neq j}$$

$$\underline{\text{Claim 2: } \bigcup_{i=1}^n A_i = \bigcup_{i=1}^n B_i \text{ } \forall n \in \mathbb{N}} \quad \textcircled{1}$$

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i \quad \longrightarrow \textcircled{2}$$

Proof of Claim 2:

For  $\textcircled{1}$ , if  $n = 1$ ,  $B_1 = A_1 \rightarrow$  trivially proved

Assume true for  $n = k$ , ie,  $\bigcup_{i=1}^k A_i = \bigcup_{i=1}^k B_k := C_k$

To prove for  $n = k+1$ ,

$$\bigcup_{i=1}^{k+1} A_i = A_{k+1} \cup C_k$$

$$\begin{aligned} \bigcup_{i=1}^{k+1} B_i &= B_{k+1} \cup C_k = (A_{k+1} \bigcup_{i=1}^k A_i) \cup C_k \\ &= (A_{k+1} \setminus C_k) \cup C_k \end{aligned}$$

$$\Rightarrow \boxed{\bigcup_{i=1}^{k+1} B_i = \bigcup_{i=1}^{k+1} A_i} \Rightarrow \textcircled{1} \text{ is valid.}$$

To extend the equality till  $\infty$ , we can prove that any element in LHS will belong to RHS. (Induction is not valid for  $n \rightarrow \infty$ )

Final Proof: To prove:

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n \rightarrow \infty} P\left(\bigcup_{i=1}^n A_i\right)$$

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = P\left(\bigcup_{i=1}^{\infty} B_i\right) \quad \text{As defined and proved above}$$

$$= \sum_{i=1}^{\infty} P(B_i) \quad [\text{Additivity Axiom}]$$

Applies only because summation is defined  
of  $\mathbb{R}$  and limit is well  
- defined for  $\mathbb{R}$ .

$$\begin{aligned} &= \lim_{n \rightarrow \infty} \sum_{i=1}^n P(B_i) \\ &= \lim_{n \rightarrow \infty} P\left(\bigcup_{i=1}^n B_i\right) \quad [\text{Additivity Axiom}] \\ &= \lim_{n \rightarrow \infty} P\left(\bigcup_{i=1}^n A_i\right) \quad [\text{Claim 2}] \end{aligned}$$

$$\Rightarrow P\left(\bigcup_{i=1}^{\infty} A_i\right) = \underbrace{\lim_{n \rightarrow \infty} P\left(\bigcup_{i=1}^n A_i\right)}$$

Note: A sequence  $\{x_n\}$  is said to converge to  $x^*$  iff  
 $\forall \varepsilon > 0$ ,  $\exists n_0 \in \mathbb{N}$  et  $|x_n - x^*| < \varepsilon \forall n > n_0$ .

Corollary 1:

i)  $A_i \subseteq A_{i+1} \forall i \Rightarrow P\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n \rightarrow \infty} P(A_n)$

ii)  $A_i \supseteq A_{i+1} \forall i \Rightarrow P\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{n \rightarrow \infty} P(A_n)$

Corollary 2 : Union Bound for infinite events,

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} P(A_i)$$

→ For finite union, induction can be used for proof  
for infinite, continuity.

→ Conditional Probability :-

•  $P(A|B)$  is the probability of A given that B has occurred.

•  $P(A|B) \propto P(A \cap B)$  If  $P(A \cap B) = 0$ ,  $P(A|B) = 0$

$$= P(A|B) = k P(A \cap B), \text{ if } A=B, P(B|B) = k P(B \cap B)$$
$$\Rightarrow l = k P(B)$$
$$\Rightarrow k = l/P(B)$$

$$\Rightarrow P(A|B) = P(A \cap B) / P(B)$$

→ Independence :

• 2 events A and B are independent if,

$$P(A \cap B) = P(A) \cdot P(B)$$

$$\Rightarrow P(A|B) = P(A)$$

- 3 events  $A, B, C$  are independent if,

$$P(A \cap B) = P(A)P(B)$$

$$P(B \cap C) = P(B)P(C)$$

$$P(A \cap C) = P(A)P(C)$$

$$P(A \cap B \cap C) = P(A)P(B)P(C)$$

- For  $n$  events,

$$P\left(\bigcap_{i \in I} A_i\right) = \prod_{i \in I} P(A_i) \quad \# I \subseteq \{1, 2, 3, \dots, n\}$$

- A collection of sets is a partition of  $\Omega$  if they are mutually exclusive and exhaustive.

$$(A_i \cap A_j = \emptyset \quad \forall i \neq j), \quad (\bigcup_{i=1}^n A_i = \Omega)$$

→ Total Probability Theorem :-

Let  $\{A_1, A_2, A_3, \dots, A_n\}$  be a partition of  $\Omega$ , and  $B$  be any event. Then,

$$P(B) = \sum_i P(B \cap A_i)$$

$$\Rightarrow P(B) = \sum_i P(B|A_i)P(A_i), \quad \text{if } P(A_i) \neq 0 \quad \forall i$$

Proof:

$$B = \bigcup_{i=1}^n (B \cap A_i)$$

$$\Rightarrow P(B) = P\left(\bigcup_{i=1}^n (B \cap A_i)\right)$$

$$\Rightarrow P(B) = \sum_{i=1}^n P(B \cap A_i)$$

$\rightarrow$  Bayes' Theorem :-

$\{A_1, A_2, A_3, \dots, A_n\}$  be a partition of  $\Omega$ .  $B$  be any event.

$$P(A_i | B) = \frac{P(A_i \cap B)}{P(B)}$$

$$\Rightarrow P(A_i | B) = \frac{P(B|A_i) P(A_i)}{\sum_i P(B|A_i) P(A_i)}$$

$\rightarrow$  Multiplication Rule :

$$P(A_1 \cap A_2 \cap A_3 \dots \cap A_n) = P(A_1) P(A_2 | A_1) P(A_3 | A_2 \cap A_1) \dots P(A_n | A_1 \cap A_2 \dots \cap A_{n-1})$$

$\rightarrow$  Conditional Independence :

2 events  $A$  and  $B$  are said to be independent given  $C$  if,

$$P(A \cap B | C) = P(A | C) \cdot P(B | C)$$

- If  $P(C) = 1$ , then  $A$  and  $B$  are truly independent, ie,

$$P(A \cap B) = P(A) \cdot P(B)$$

→ Counting Techniques :-

Suppose there are  $n$  objects

i) No. of  $k$ -length sequences :  ${}^n P_k = \frac{n!}{(n-k)!}$

ii) No. of  $k$ -length sets :  ${}^n C_k = \frac{n!}{k!(n-k)!}$

iii) No. of ways of choosing  $r$  sets of sizes  $n_1, n_2, n_3, \dots, n_r$ ,

$$= {}^n C_{n_1} \cdot {}^{n-n_1} C_{n_2} \cdot {}^{n-n_1-n_2} C_{n_3} \cdots {}^{n-n_1-n_2-\cdots-n_{r-1}} C_{n_r}$$

$$= \frac{n!}{(n-n_1)! n_1!} \frac{(n-n_1)!}{(n-n_1-n_2)! n_2!} \cdots \cdots$$

$$= \frac{n!}{n_1! n_2! n_3! \cdots n_r!}$$

→ Random Variables :-

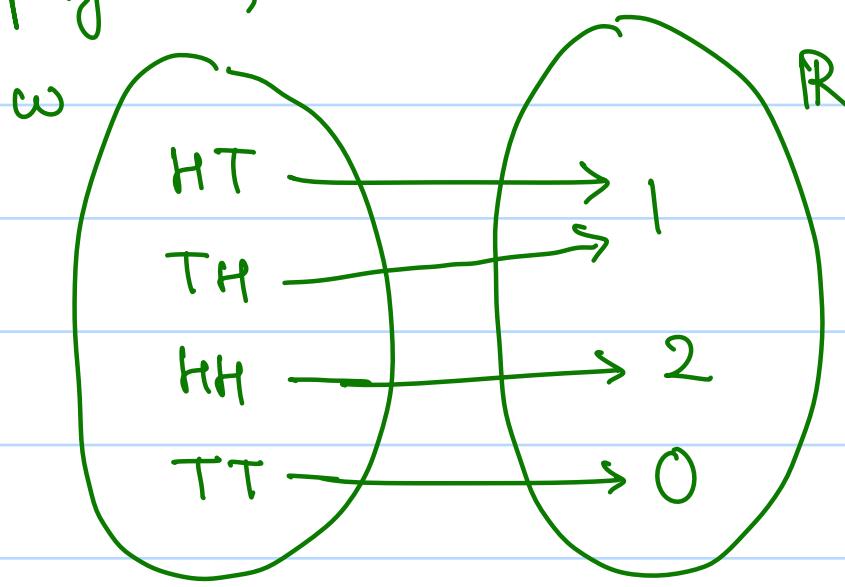
° A random variable is a function  $X: \Omega \mapsto \mathbb{R}$  such that,

$$\{X \leq x\} \triangleq \{\omega \mid X(\omega) \leq x\} \in \mathcal{F}, \forall x \in \mathbb{R}$$

Example:  $\Omega = \{\text{HT, TH, HH, TT}\}$

$X: \Omega \mapsto \mathbb{R}$  st  $X(\omega) = \text{No. of heads}$

The mapping is,



$$\{x \leq x_0\} = \begin{cases} \emptyset, & x_0 < 0 \\ \{\text{TT}\}, & x_0 \in [0, 1) \\ \{\text{HT, TH, TT}\}, & x_0 \in [1, 2) \\ \Omega, & x_0 \geq 2 \end{cases}$$

Theorem:  $X: \Omega \rightarrow \mathbb{R}$  on  $\mathcal{F}$ , then

i)  $X^{-1}((-\infty, x]) \in \mathcal{F}$

ii)  $X^{-1}([x_1, x_2]) \in \mathcal{F} \quad \forall x_1, x_2 \in \mathbb{R}$

iii)  $X^{-1}(\{x\}) \in \mathcal{F} \quad \forall x \in \mathbb{R}$

Proof of (i):

wkt  $X^{-1}(-\infty, x] \in \mathcal{F} \quad \forall x \in \mathbb{R}$

$$(-\infty, x) = \bigcup_{n=1}^{\infty} \left( -\infty, x - \frac{1}{n} \right]$$

$$X^{-1}\left(\left(-\infty, x - \frac{1}{n}\right]\right) \in \mathcal{F} \quad \forall n \Rightarrow X^{-1}\left(\bigcup_{n=1}^{\infty} \left( -\infty, x - \frac{1}{n} \right]\right) \in \mathcal{F}$$

Proof of (ii) :

$$\text{wkt } x^{-1}(-\infty, x_1] \in \mathcal{F}$$

$$x^{-1}(-\infty, x_2) \in \mathcal{F}$$

Since subtraction is closed in  $\mathcal{F}$ ,  $x^{-1}(-\infty, x_2) \in \mathcal{F}$

$$\Rightarrow x^{-1}([x_2, \infty)) \in \mathcal{F}$$

$$\Rightarrow x^{-1}(-\infty, x_1] \cap x^{-1}([x_2, \infty)) \in \mathcal{F}$$

$$\Rightarrow x^{-1}([x_2, x_1]) \in \mathcal{F}$$

iv)  $x^{-1}((x_1, x_2)) \in \mathcal{F}$

$$(x_1, x_2) = (x_1, \infty) \cap (-\infty, x_2)$$

$$x^{-1}(-\infty, x_1] \in \mathcal{F} \Rightarrow x^{-1}(x_1, \infty) \in \mathcal{F}$$

$$x^{-1}([x_2, \infty)) \in \mathcal{F} \Rightarrow x^{-1}(-\infty, x_2) \in \mathcal{F}$$

$$\Rightarrow x^{-1}((x_1, \infty) \cap (-\infty, x_2)) \in \mathcal{F}$$

$$\Rightarrow x^{-1}((x_1, x_2)) \in \mathcal{F}$$

• Borel  $\sigma$ -Algebra :

Smallest  $\sigma$ -field on  $\mathbb{R}$  containing sets of the form  $(-\infty, x]$   $\forall x \in \mathbb{R}$ , ie,

$$B(\mathbb{R}) = \sigma((-\infty, x] \forall x \in \mathbb{R})$$

- Contains all possible subsets of  $\mathbb{R}$ .

• Cumulative Distribution Function (CDF) :-

- A CDF is a function  $F_X : \mathbb{R} \mapsto [0,1]$  such that,

$$F_X(x) = P(\{X < x\})$$

Example: For,

$$\{X \leq x\} = \begin{cases} \emptyset, & x < 0 \\ \{\text{TT}\}, & x \in [0,1) \\ \{\text{TT, HT, TH}\}, & x \in [1,2) \\ \Omega, & x \geq 2 \end{cases}$$

$$F_X(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{4}, & x \in [0,1) \\ \frac{3}{4}, & x \in [1,2) \\ 1, & x \geq 2 \end{cases}$$

Example: Let  $X(\omega) = c$ ,  $c \in \mathbb{R}$ . Then,

$$\{X \leq x\} = \begin{cases} \emptyset, & x < c \\ \Omega, & x \geq c \end{cases}$$

$$\Rightarrow F_X(x) = \begin{cases} 0, & x < c \\ 1, & x \geq c \end{cases}$$

## • Indicator R.V :-

Consider an  $A \in \mathcal{F}$ . Then  $I_A: \Omega \rightarrow \mathbb{R}$  such that,

$$I_A(\omega) = \begin{cases} 1, & \omega \in A \\ 0, & \omega \notin A \end{cases}$$

$$\Rightarrow \{I_A < x\} = \begin{cases} \emptyset, & x < 0 \\ A^c, & x \in [0,1) \\ \Omega, & x \geq 1 \end{cases}$$

$\{I_A < x\} \in \mathcal{F} \quad \forall x \in \mathbb{R}$ . So  $I_A$  is a valid r.v.

$I_A$  is an indicator r.v for the event  $A$ .

•  $B_1 \cap B_2 = \emptyset$ , then,

$$I_{B_1 \cup B_2}(\omega) = I_{B_1}(\omega) + I_{B_2}(\omega) \quad \forall \omega \in \Omega$$

$B_1, B_2, B_3, \dots$  form a partition of  $\Omega$ , then,

$$I_{\bigcup_{i=1}^n B_i} = \sum_{i=1}^n I_{B_i} = 1$$

## • Theorems of CDF :-

i)  $\lim_{x \rightarrow \infty} F_X(x) = 1, \quad \lim_{x \rightarrow -\infty} F_X(x) = 0$

$$\lim_{x \rightarrow \infty} F_X(x) = \lim_{x \rightarrow \infty} P_X(X \leq x) = P(\Omega) = 1$$

$$\lim_{x \rightarrow -\infty} F_X(x) = \lim_{x \rightarrow -\infty} P_X(X \leq x) = P(\emptyset) = 0$$

$P_X$  - Probability law of  $X$

$$2) x < y \Rightarrow F_X(x) \leq F_X(y)$$

3)  $F_X(x)$  is always right continuous, i.e.,

$$\lim_{\epsilon \rightarrow 0^+} F_X(x + \epsilon) = F_X(x) \quad \forall x \in \mathbb{R}$$

Proof:

$$\lim_{n \rightarrow \infty} F_X\left(x + \frac{1}{n}\right) = \lim_{n \rightarrow \infty} P\left(X \leq x + \frac{1}{n}\right)$$

Let  $\{X \leq x + \frac{1}{n}\} = B_n \Rightarrow B_n$  is a decreasing sequence

By continuity of probability  $P\left(\bigcap_{n=1}^{\infty} B_n\right) = \lim_{n \rightarrow \infty} P(B_n)$

$$\Rightarrow \lim_{n \rightarrow \infty} F_X\left(x + \frac{1}{n}\right) = P\left(\bigcap_{n=1}^{\infty} \{X \leq x + \frac{1}{n}\}\right)$$

$$= P(\{X \leq x\}) = F_X(x)$$

$$\therefore \lim_{n \rightarrow \infty} F_X\left(x + \frac{1}{n}\right) = F_X(x)$$

$$\Rightarrow \underset{\longrightarrow}{\text{RHL}} = F_X(x)$$

$$4) F_X(x) - \lim_{\epsilon \rightarrow 0^+} F_X(x - \epsilon) = P(\{X = x\})$$

$$5) P_X(x_1 \leq X \leq x_2) = P(\omega : x_1 \leq X(\omega) \leq x_2) \\ = P(X$$

## $\rightarrow$ Discrete Random Variable :-

A random variable is discrete if Range( $X$ )  $\subseteq \mathbb{R}$  is finite / countably infinite.

### • Probability Mass Function :

$P_X : X \mapsto [0,1]$ , given by

$$P_X(x) = P(X = x) = P(\{\omega : X(\omega) = x\})$$

- Lemma: For an rv st  $x \in \{x_1, x_2, x_3, \dots\}$

$$i). \sum_{i=1}^{\infty} P_X(x_i) = \sum_{i=1}^{\infty} P(X = x_i)$$

$$= P\left(\bigcup_{i=1}^{\infty} \{X = x_i\}\right) \quad [\text{Additivity}]$$

$$= P(\{X \in \{x_1, x_2, x_3, \dots\}\}) = P(\Omega)$$

$$\therefore \sum_{i=1}^{\infty} P_X(x_i) = \underline{\underline{1}}$$

### • CDF of a Discrete R.V :-

$$F_X(x) = P(\{X \leq x\}) = P\left(\bigcup_{i \in \mathbb{N}: x_i \leq x} \{X = x_i\}\right)$$

$$\Rightarrow F_X(x) = \sum_{i \in \mathbb{N}: x_i \leq x} P_X(x_i)$$

• Functions on R.V :-

Let  $X: \Omega \rightarrow \mathbb{R}$  and  $Y: \mathbb{R} \rightarrow \mathbb{R}$ , ie  
a function on  $X$ .  $Y$  is also a r.v.

$$\text{Proof: } Y^{-1}(B) = \{y : Y(y) \in B\} \quad \forall B \in \mathcal{B}(\mathbb{R})$$

$$= \{X(\omega) : Y(X(\omega)) \in B\}$$

$$X^{-1}(Y^{-1}(B)) = \{x : Y(X(x)) \in B\}$$

$$= \{\omega : X(\omega) \in Y^{-1}(B)\}$$

Since  $X^{-1}(s) \in \mathcal{F}$   $\forall s \in \mathcal{B}(\mathbb{R})$

$$Y^{-1}(B) \in \mathcal{X}(\mathcal{F}) = \mathcal{F}' \quad \forall B \in \mathcal{B}(\mathbb{R})$$

Better proof  
needed.

$\Rightarrow Y$  is a valid r.v

• PMF of  $y$  :  $P_y(Y = y) = \sum_{Y(\omega)=y} P_X(X = \omega)$

• Expectation :

$$E[X] = \sum_{x \in \mathbb{R}} x P_X(x)$$

• If  $y = Y(x)$ ,

Law of the Unconscious Statistician :  $E[Y] = \sum_{x \in \mathbb{R}} y(x) P_X(x)$

Proof:  $E[Y] = \sum_{y \in Y} y P_Y(y) = \sum_{y \in Y} y \sum_{Y(\omega)=y} P_X(\omega)$

$$= \sum_{y \in Y} \sum_{Y(\omega)=y} y(\omega) P_X(\omega) = \sum_{x \in \mathbb{R}} y(x) P_X(x)$$

All possible  $x$  ←

◦ Variance:

$$\text{Var}[X] = E[(X - E[X])^2]$$

$$\Rightarrow \text{Var}[X] = E[X^2] - E[X]^2$$

◦  $n^{\text{th}}$  Moment of an RV =  $E[X^n]$ . (defn)

◦ Examples of RV :-

1) Bernoulli RV: Binary opp. RV. (ex: Coin Toss)

$$P(\{H\}) = p, P(\{T\}) = 1-p$$

$$X(H) = 1, X(T) = 0 \longrightarrow X \text{ is a Bernoulli R.V}$$

$$E[X] = p, \text{Var}[X] = p - p^2$$

2) Binomial RV: (ex: Coin Tossed n times)

Any event  $\omega$ : Sequence of H's and T's of length  $n$ .

$$P(\{H\}) = p, P(\{T\}) = 1-p. X(\omega) = \text{No. of heads in } \omega$$

$$\Rightarrow P_X(k) = {}^n C_k p^k (1-p)^{n-k} \quad \forall k \in \mathbb{N}$$

$$E[X] = np \quad \text{Var}[X] = np(1-p)$$

3) Geometric RV: (ex: Toss a coin till heads)

$X(\omega)$  = No. of coin tosses in  $\omega$ , to get a head ;

$$P_X(k) = p(1-p)^{k-1} \quad E[X] = \frac{1}{p} \quad \text{Var}[X] = \frac{1-p}{p^2}$$

4) Poisson RV:  $X \in \{0, 1, 2, \dots\}$

$$P_X(k) = \frac{e^{-\lambda} \lambda^k}{k!}, \quad k = 0, 1, 2, \dots \quad \lambda \in \mathbb{R}$$

$$E[X] = \text{Var}[X] = \lambda.$$

• Let  $Y \sim \text{Binomial}(n, p)$ , As  $n \rightarrow \infty$  st  $np = \lambda$ , a constant, we have,

$$\lim_{n \rightarrow \infty} P_Y(k) = \frac{e^{-\lambda} \lambda^k}{k!}, \quad k = 0, 1, 2, 3, \dots$$

Proof:

$$\lim_{n \rightarrow \infty} P_Y(k) = \lim_{n \rightarrow \infty} {}^n C_k p^k (1-p)^{n-k}$$

$$= \lim_{n \rightarrow \infty} \frac{n!}{(n-k)! k!} p^k (1-p)^{n-k}$$

$$= \lim_{n \rightarrow \infty} \frac{n!}{(n-k)! k!} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

$$= \frac{\lambda^k}{k!} \lim_{n \rightarrow \infty} \frac{n(n-1)(n-2) \cdots (n-k+1)}{n^k} \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

$$= \frac{\lambda^k}{k!} \lim_{n \rightarrow \infty} \left( \frac{n}{n} \right) \left( \frac{n-1}{n} \right) \left( \frac{n-2}{n} \right) \dots \left( \frac{n-k+1}{n} \right)^1$$

$$\left( 1 - \frac{\lambda}{n} \right)^{n-k}$$

$$= \frac{\lambda^k}{k!} \lim_{n \rightarrow \infty} \left( 1 - \frac{\lambda}{n} \right)^{n-k}$$

$$= \frac{\lambda^k}{k!} e^{\lim_{n \rightarrow \infty} \left( \frac{-\lambda}{n} \right) (n-k)} \quad (1 + f(n))^{g(n)}$$

$$= \frac{\lambda^k}{k!} e^{\lim_{n \rightarrow \infty} -\lambda + \frac{\lambda k}{n}} \quad f(n) \rightarrow 0$$

$$g(n) \rightarrow \infty$$

as  $n \rightarrow \infty$ ,

then

$$\Rightarrow \lim_{n \rightarrow \infty} P_Y(k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

$$e^{\lim_{n \rightarrow \infty} g(n) \ln(1 + f(n))}$$

$$= e^{\lim_{n \rightarrow \infty} g(n) f(n)}$$

## → Multiple Random Variables :-

- $(X, Y)$  are said to be jointly discrete if  $(x, y)$  takes values in some countable subset of  $\mathbb{R}^2$

$$\text{Joint PMF } P_{X,Y}(x,y) = P(X=x \cap Y=y)$$

$$P_X(x) = \sum_{y \in Y} P_{X,Y}(x,y)$$

$$\begin{aligned} \text{Proof: } P_X(x) &= P(X=x) \\ &= P(X=x \cap \bigcup_{y \in Y} (Y=y)) \\ &= P\left(\bigcup_y (X=x \cap Y=y)\right) \\ &= \sum_{y \in Y} P(X=x \cap Y=y) = \sum_{y \in Y} P_{X,Y}(x,y) \end{aligned}$$

- $P((X,Y) \in B) = \sum_{(x,y) \in B} P_{XY}(x,y) \quad \forall B \subseteq \mathbb{R}^2$

- Functions on 2 Rv's :-

If  $X: \Omega \rightarrow \mathbb{R}$ ,  $Y: \Omega \rightarrow \mathbb{R}$  Rv's,  $Z = g(X, Y)$ ,  
 $g: \mathbb{R}^2 \rightarrow \mathbb{R}$  is also a Rv.

$$P_Z(z) = \sum_{x,y : g(x,y) = z} P_{XY}(x,y)$$

- Independence :-

2 Rv's  $X$  and  $Y$  are said to be independent if

$$P_{XY}(x,y) = P_X(x) P_Y(y) \quad \forall x, y \in X, Y$$

- If  $x, y \in \{0,1\}$ , then  $X, Y$  are independent.

- If  $X, Y$  are independent, then  $E[XY] = E[X]E[Y]$

- Independent  $\Leftrightarrow$  Uncorrelated

- $n$  Rv's  $X_1, X_2, X_3 \dots X_n$  are said to be independent if,

$$P_{X[1:n]}(x_{[1:n]}) = \prod_{i \in [1:n]} P_{X_i}(x_i)$$

- If  $X$  and  $Y$  are independent,  $h(X)$  and  $g(Y)$  are independent

• Let  $Z = g(X, Y)$  be a rv, then,

$$E[Z] = \sum_{x,y} g(x,y) P_{XY}(x,y)$$

→ Properties of  $E[X]$  and  $\text{Var}[X]$  :-

$$1) Y = aX + b \Rightarrow E[Y] = aE[X] + b$$

$$\text{Var}[Y] = a^2 \text{Var}[X]$$

$$2) Y = X_1 + X_2 \Rightarrow E[Y] = E[X_1] + E[X_2] \rightarrow \text{Prove by LOTUS}$$

$$\Rightarrow E[\sum X_i] = \sum E[X_i]$$

$$\Rightarrow \text{Var}[Y] = \text{Var}[X_1] + \text{Var}[X_2] + 2 \text{Cov}[X_1, X_2]$$

Where,

$$\text{Cov}[X_1, X_2] = E[X_1 \cdot X_2] - E[X_1]E[X_2]$$

Definition: Correlation Coefficient =  $\rho(X_1, X_2)$

$$= \frac{\text{Cov}[X_1, X_2]}{\sqrt{\text{Var}[X_1]} \sqrt{\text{Var}[X_2]}}$$

Theorem:  $|\rho(X_1, X_2)| \leq 1$ . Equality iff  $X_1 = aX_2 + b$ ,  $a, b \in \mathbb{R}$   
ie  $X_1$  and  $X_2$  are linearly dependent.

Proof: By Cauchy-Schwarz Inequality,

$$E[XY] \leq \sqrt{E[X^2]E[Y^2]}$$

Proof of Cauchy-Schwarz,

$$E[(x - \alpha y)^2] \geq 0 \quad (\text{Equality if } x = \alpha y \text{ (lin. dep)})$$

$$\rightarrow E[x^2] - 2\alpha E[xy] + \alpha^2 E[y^2] \geq 0 \rightarrow \text{To a Quadratic in } \alpha$$

$$\rightarrow \sqrt{(2E[xy])^2 - 4(E[y^2])(E[x^2])} \leq 0 \rightarrow \text{Discriminant} < 0$$

$$\Rightarrow 4E[xy]^2 - 4E[y^2]E[x^2] \leq 0$$

$$\Rightarrow E[xy] \leq \sqrt{E[y^2]E[x^2]}$$

To Prove the theorem,

$$x \longrightarrow (x - E[x]) \quad y \longrightarrow (y - E[y])$$

$$\rightarrow E[(x - E[x])(y - E[y])] \leq \sqrt{E[(x - E[x])^2]E[(y - E[y])^2]}$$

$$\Rightarrow |\text{Cov}[x, y]| \leq \sqrt{\text{Var}[x]\text{Var}[y]}$$

$$\Rightarrow \rho[x, y]^2 \leq 1 \Rightarrow |\rho[x, y]| \leq 1$$

$$3) \quad X, Y, \text{ st } Z = X + Y$$

$$P_Z(z) = \sum_{\substack{x, y \\ \text{st } x+y \in Z}} P_{XY}(x, y) = \sum_x P_{XY}(x, z-x)$$

$$= \sum_y P_{XY}(z-y, y)$$

$$\Rightarrow \underline{P_X * P_Y} \quad \text{If } X, Y \text{ are independent}$$

Example:  $X, Y$  are Geometric RVs. Find  $P_Z(z)$  where  $Z = X + Y$   
if  $X, Y$  are independent.

$$\begin{aligned}
 P_Z(z) &= P_X(x) * P_Y(y) \\
 &= \sum_x P_X(x) P_Y(z-x) \\
 &= \sum_x p(1-p)^{x-1} q(1-q)^{z-x-1} \\
 &= pq \frac{(1-q)^{z-1}}{(1-p)} \sum_x \left(\frac{1-p}{1-q}\right)^x
 \end{aligned}$$

If  $p = q$ ,

$$P_Z(z) = (z-1) p^2 (1-p)^{z-2}$$

→ Conditioning :-

Conditioning on RV  $X: \Omega \rightarrow \mathbb{R}$  on an event  $A \subseteq \Omega$

$$P_{X|A}(x) \triangleq P(X=x|A)$$

$$P_{X|A}(x) = \frac{P(X=x, A)}{P(A)}$$

° Theorem: If  $A_1, A_2, \dots, A_n$  forms a partition in  $\Omega$ , then

$$P_X(x) = \sum_i P_{X|A_i}(x) P(A_i)$$

- Conditioning on Rv  $X$  on another Rv  $Y$

$$P_{X|Y}(x) = P(X=x | Y=y) = \frac{P(X=x, Y=y)}{P(Y=y)}$$

$$\Rightarrow P_{X|Y}(x) = \frac{P_{XY}(x,y)}{P_Y(y)}$$

- Conditional Expectance :-

$$E[X|Y=y] = \sum_{x \in X} x P_{X|Y}(x|y)$$

$$E[X|Y] = \sum_{y \in Y} \sum_{x \in X} x P_{X|Y}(x|y)$$

$$E[X|A] = \sum_{x \in X} x P_{X|A}(x) \quad \# A \subseteq \Omega$$

$$E[g(x)|A] = \sum_{x \in X} g(x) P_{X|A}(x) \quad \# A \subseteq \Omega$$

- Total Expectation Theorem :-

If  $A_1, A_2, \dots, A_n$  form a partition of  $\Omega$ , with  $P(A_i) > 0 \ \forall i$ , then

$$E[X] = \sum_{i=1}^n E[X|A_i] P(A_i)$$

Proof:  $\sum_{i=1}^n E[X|A_i] P(A_i)$

$$= \sum_{i=1}^n \sum_{x \in X} x P_{X|A_i}(x) P(A_i)$$

$$= \sum_{x \in X} x \sum_{i=1}^n P_{X|A_i}(x) P(A_i)$$

$$= \sum_{x \in X} x \sum_{i=1}^n \frac{P(X=x \cap A_i)}{P(A_i)}$$

$$= \sum_{x \in X} x \sum_{i=1}^n P(X=x \cap A_i)$$

$$= \sum_{x \in X} x P(X=x)$$

$$= \sum_{x \in X} x P_X(x)$$

$$= \underline{\underline{E[X]}}$$

III by  $E[X] = \sum_y E[X|y=y] P_Y(y)$

- Conditional Expectance as a RV :-

$$\phi(y) \triangleq E[X|y=y] = \sum_{x \in X} x P_{X|Y}(x|y)$$

$$\phi : \text{Range}(y) \mapsto \mathbb{R}$$

$\phi(y)$  is a function of an RV. Therefore it is also an RV.

- Theorem:  $E[\phi(y)] = E[X]$

$$E[\phi(y)] = \sum_{y \in Y} \phi(y) P_Y(y)$$

$$= \sum_{y \in Y} E[X|Y=y] P_Y(y)$$

$$= E[X] \quad (\text{Total Expectation Theorem})$$

- Conditional Independence of Rvs :-

$X$  and  $Y$  are conditionally independent given  $A$  with  $P(A) > 0$

if

$$P_{X,Y|A}(x,y) = P_{X|A}(x) P_{Y|A}(y)$$

- Conditional Variance :

$$\text{Var}[X|Y=y] = E[(X - E[X|Y=y])^2 | Y=y]$$

Let  $\Psi(y) = \text{Var}[X|Y=y]$ ,  $\Psi(y)$  is an Rv.

$$\text{Var}[X|Y=y] = E[X^2|Y=y] - (E[X|Y=y])^2 \rightarrow \begin{matrix} \text{Same Proof as} \\ \text{in Non-Condition} \\ - al Case \end{matrix}$$

- Law Of Total Variance :-

$$\text{Var}[X] = E[\Psi(Y)] \neq \text{Var}[\Phi(Y)]$$

$$E[\Psi(Y)] = \sum_{y \in Y} \Psi(y) P_Y(y) \quad (\text{LOTUS})$$

$$= \sum_{y \in Y} (E[X^2|Y=y] - E[X|Y=y]^2) P_Y(y)$$

$$= \sum_{y \in Y} E[x^2 | y=y] P_Y(y) - \sum_{y \in Y} E[x | y=y]^2 P_Y(y)$$

$$\sum_{y \in Y} E[x^2 | y=y] P_Y(y) = E[x^2] \quad (\text{LOTUS})$$

$$\sum_{y \in Y} E[x | y=y]^2 P_Y(y) =$$

$\rightarrow$  Memoless Property of a Geometric RV :-

$$P_X(k) = (1-p)^{k-1} p, \quad k \in \mathbb{N}$$

$$\begin{aligned} P(X > n) &= \sum_{k=n+1}^{\infty} (1-p)^{k-1} p \\ &= p \frac{(1-p)^{n+1}}{1-(1-p)} = \underline{(1-p)^n} \end{aligned}$$

Property:  $P(X > m+n | X > m) = P(X > n)$

$$P(X > m+n | X > m) = \frac{P(X > m+n \cap X > m)}{P(X > m)}$$

$$= \frac{P(X > m+n)}{P(X > m)}$$

$$= \frac{(1-p)^{m+n}}{(1-p)^m}$$

$$= \underline{(1-p)^n}$$

- CDF of an RV is given by,

$$F_X(x) = P(X < x) = \sum_{x_i \leq x} P_X(x_i)$$

$$\Rightarrow P_X(x) = F_X(x) - F_X(x-1)$$

Example:  $X = \max\{X_1, X_2, X_3\}$ ,  $X_1, X_2, X_3$  are independent.

$$P_{X_i}(k) = \frac{1}{100} \quad \forall i \in \{1, 2, 3\}, \quad \forall k \in [1:10]$$

$$P_X(k) = P(X = k) = P(\max\{X_1, X_2, X_3\} = k)$$

$$F_X(k) = P(X \leq k) = P(\max\{X_1, X_2, X_3\} \leq k)$$

→ Application of Discrete RVs :-

- Minimum Mean Square Error Estimation :-

$$X \sim P_X \longrightarrow P_{Y|X} \longrightarrow Y \xrightarrow{f} \hat{X} = f(Y)$$

- Theorem: The function  $\phi(y) = E[X|Y=y]$ , gives the min. value of  $E[(X-f(y))^2]$ .

- Proof:

$$E[(X-f(y))^2] = E[E[(X-f(y))^2|Y]]$$

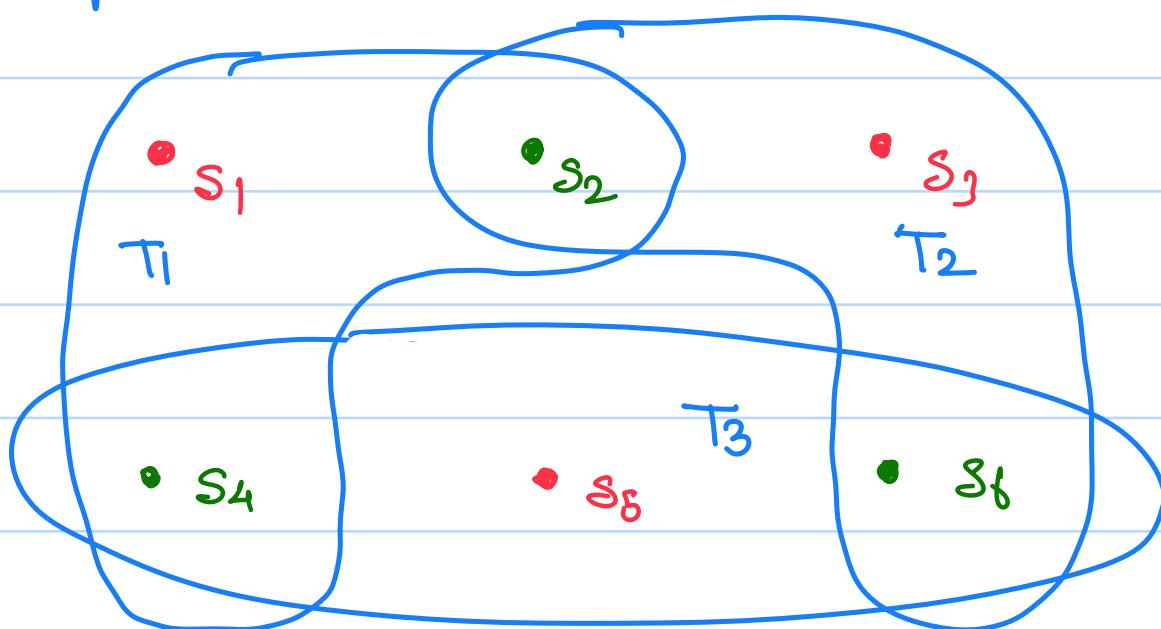
$$\begin{aligned}
 &= \sum_y P_y(y) E[(X - f(y))^2 | Y = y] \\
 &= \sum_y P_y(y) E[x^2 + f(y)^2 - 2xf(y) | Y = y] \\
 &= \sum_y P_y(y) (E[x^2 | Y = y] + f(y)^2 - 2E[X | Y = y]f(y))
 \end{aligned}$$

Each of the terms in the summation is minimized when,

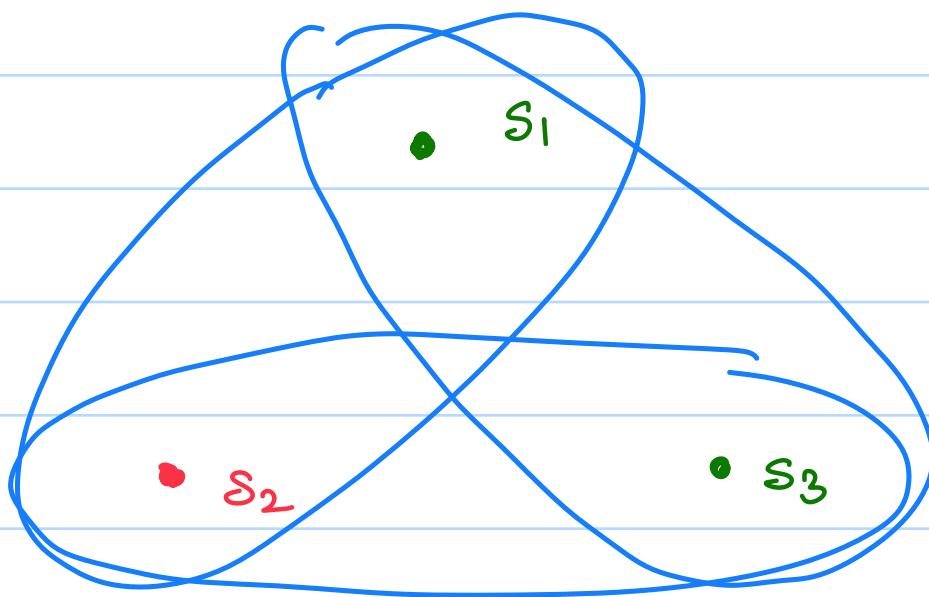
$$f(y) = \phi(y) = E[X | Y = y]$$

### o Combinatorics and Graph Theory :-

Let  $S$  be a set of some elements and  $T_1, T_2, T_3 \dots, T_m \subseteq S$  st.  $n(T_i) = l + i \in [1:m]$ , where  $l \in \mathbb{N}$ , then 2-coloring of  $S$  is coloring each element of  $S$  such that  $T_i \neq$  Monochromatic  $\forall i \in [1:m]$ , i.e.,  $T_i$  has elements of both colors  $\forall i \in [1:m]$ .



In the above  $S$ , 2-coloring is possible.



In the above  $S$ , 2-coloring is not possible.

- Theorem:

If  $m < 2^{d-1}$ , then  $\exists$  a valid 2 coloring of the set  $S$ .

- Proof: Let  $S = \{x_1, x_2, x_3 \dots, x_n\}$ . Randomly color each element of  $S$  black or white, independently and identically distributed, each with probability  $\frac{1}{2}$ .

Let  $E_i$  be the event that  $T_i$  is monochromatic.

$\Omega = \{W, B\}^n$ , i.e., each element of  $\Omega, w$ , is an  $n$ -length sequence of W and B.

$$P(\{w\}^n) = P(\{B\}^n) = \frac{1}{2^n}$$

$$\Rightarrow P(E_i) = \frac{1}{2^d} \times 2 = \frac{1}{2^{d-1}}$$

$$P(\exists \text{ monochromatic } T_i) = P\left(\bigcup_{i=1}^m E_i\right) = \sum_{i=1}^m P(E_i)$$

$$= \sum_{i=1}^m \frac{1}{2^{d-1}} = \frac{m}{2^{d-1}}$$

$$\frac{m}{2^{d-1}} < 1 \Rightarrow m < 2^{d-1}$$

$$P(\text{no monochromatic } Ti) = 1 - \frac{m}{2^{d-1}} > 0 \text{ if } m < 2^{d-1}$$

$P(\text{no monochromatic } Ti) > 0 \Rightarrow \exists \omega \in \Omega \text{ st the associated coloring is a valid 2-coloring.}$

[ Let  $A = \{\omega \in \Omega : \omega \text{ satisfies } E\}$ . If  $P(A) > 0$ , then  $\exists \omega \in \Omega \text{ st } \omega \text{ satisfies } E$  ]  $\rightarrow$  Probabilistic Method

- The converse of the above theorem is false.

- Entropy (Uncertainty) :-

- Consider an RV with,

$$P_X(0) = P_X(1) = \frac{1}{2} (X_1)$$

and another RV with,

$$P_X(0) = 0.9, P_X(1) = 0.1 (X_2)$$

Intuitively, we see that the uncertainty of realization in  $X_1$  is higher than  $X_2$ . This uncertainty is measured by the Entropy of  $X$ .

$$H(X) = \sum_{x \in X} P_X(x) \log \frac{1}{P_X(x)}$$

- If  $X \sim \text{Binomial RV}$ ,

$$H(X) = p \log \frac{1}{p} + (1-p) \log \frac{1}{1-p} := \underline{h(p)}$$

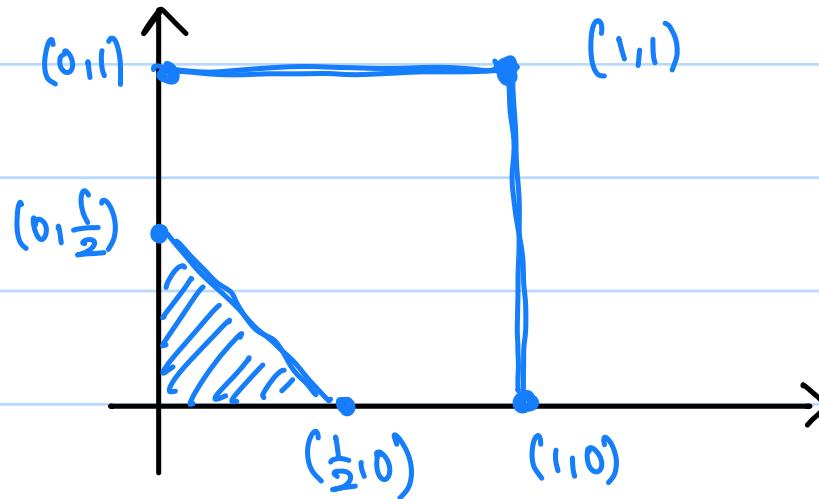
→ Continuous Random Variable :-

- We know that an RV is a map from  $(\Omega, \mathcal{F}, P)$  to  $(\mathbb{R}, \mathcal{B}(\mathbb{R}), P_X)$  such that,

$$\{ \omega : X(\omega) \leq x \} \in \mathcal{F} \quad \forall x \in \mathbb{R}.$$

- Now, if  $\Omega$  is an uncountably infinite set,

Take  $\Omega = [0,1]^2$ , ie,  $\Omega$  is a unit square.



Let  $P(A) = \text{area}(A) \quad \forall A \subseteq [0,1]^2$

$$P((x,y) : x+y \leq \frac{1}{2}) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$$

$$P((0.4, 0.5)) = 0, \quad P(\Omega) = 1$$

$$\text{But } \Omega = \bigcup_{x,y \in [0,1]} \{(x,y)\}$$

$$\Rightarrow P(\Omega) = P\left(\bigcup_{x,y \in [0,1]} \{(x,y)\}\right)$$

$$= \sum_{x,y \in [0,1]} P(\{(x,y)\}) = \underline{0} \neq 1$$

$\therefore$  Additivity only holds for countable number of disjoint sets.

- Definition:

A random variable  $X$  is said to be continuous if its CDF can be expressed as,

$$F_X(x) = \int_{-\infty}^x f_X(u) du \quad \forall x \in \mathbb{R}$$

for some integrable function  $f_X(x) : \mathbb{R} \mapsto [0, \infty)$  called the probability density function (PDF) of  $X$ .

$$f_X(x) = F_X'(x)$$

$$P(x \in [a,b]) = \int_a^b f_X(u) du$$

} [Fundamental Theorem of Calculus]

$f_X(x)$  directly does not give us probability. We can think of  $f_X(x) \Delta x$  as the probability of  $(x, x + \Delta x)$ , which becomes probability of  $x$  for small  $\Delta x$ .

- Theorem: If a continuous Rv  $X$  has a PDF  $f_x$ , then

$$a) \int_{-\infty}^{\infty} f_x(x) dx = 1$$

$$b) P(X=x) = 0 \quad \forall x \in \mathbb{R}$$

- Proof:

$$a) \int_{-\infty}^{\infty} f_x(x) dx = P(x \in (-\infty, \infty)) = P(\Omega) = \underline{\underline{1}}$$

$$b) P(X=x) = \int_x^x f_x(x) dx = \underline{\underline{0}}$$

- Expectation:

The expectation of a continuous Rv  $X$  with PDF  $f_x$   
is given by,

$$E[X] = \int_{-\infty}^{\infty} x f_x(x) dx$$

- If  $X$  is a continuous Rv, then  $g(X)$  can be either a discrete or a continuous Rv, depending on the definition of  $g$ .

- Theorem:

If  $X$  and  $g(X)$  are continuous Rv's, then,

$$E[g(x)] = \int_{-\infty}^{\infty} g(x) f_x(x) dx \quad [\text{LOTUS}]$$

• Proof:

Lemma: For a non-negative continuous RV  $Y$ , (ie  $f_Y(y) = 0 \forall y < 0$ )

$$E[Y] = \int_0^\infty P(Y > y) dy$$

$$\begin{aligned} \text{Proof: } \int_0^\infty P(Y > y) dy &= \int_0^\infty \int_y^\infty f_Y(t) dt dy \\ &= \int_{t=0}^\infty \int_{y=0}^t f_Y(t) dy dt \\ &= \int_{t=0}^\infty f_Y(t) \left( \int_{y=0}^t dy \right) dt \\ &= \int_{t=0}^\infty t f_Y(t) dt \\ &= \underline{\underline{E[Y]}} \end{aligned}$$

Proving the theorem assuming  $g(x)$  is non-negative,

$$\begin{aligned} E[g(x)] &= \int_0^\infty P(g(x) > y) dy \\ &= \int_0^\infty P(X \in \{x : g(x) > y\}) dy \\ &= \int_{y=0}^\infty \int_{x: g(x) > y} f_X(x) dx dy \\ &= \int_{x: g(x) > 0} f_X(x) \int_{y=0}^{g(x)} dy dx \end{aligned}$$

$$= \int_{x: g(x) > y} g(x) f_x(x) dx$$

$$= \int_{-\infty}^{\infty} g(x) f_x(x) dx$$

For any general  $g(x)$ ,

$$\text{Prove by showing that } E[Y] = \int_0^\infty P(Y > y) dy - \int_0^\infty P(Y < -y) dy$$

$$\int_0^\infty P(Y > y) dy - \int_0^\infty P(Y < -y) dy$$

$$= \int_0^\infty \int_y^\infty f_y(t) dt dy - \int_0^\infty \int_{-\infty}^{-y} f_y(t) dt dy$$

$$= \int_0^\infty f_y(+t) \int_0^t dy dt - \int_{-\infty}^0 f_y(+t) \int_0^{-t} dy dt$$

$$= \int_0^\infty f_y(t) t dt - \int_{-\infty}^0 f_y(t) (-t) dt$$

$$= \int_0^\infty f_y(t) \cdot t dt + \int_{-\infty}^0 f_y(t) \cdot t dt = \int_{-\infty}^\infty t f_y(t) dt$$

$$= E[Y]$$

$$\therefore E[Y] = \int_0^\infty P(Y > y) dy - \int_0^\infty P(Y < -y) dy$$

$$E[g(x)] = \int_0^\infty P(g(x) > x) dx - \int_0^\infty P(g(x) < -x) dx$$

$$= \int_0^\infty \int_{t: g(t) > x} f_x(t) dt dx - \int_0^\infty \int_{t: g(t) < -x} f_x(t) dt dx$$

$$= \int_{-\infty}^\infty \int_0^{g^+(t)} f_x(t) dx dt + \int_{-\infty}^\infty \int_0^{g^-(t)} f_x(t) dx dt$$

$$= \int_{-\infty}^\infty f_x(t) \int_0^{g^+(t)} dx dt + \int_{-\infty}^\infty f_x(t) \int_0^{g^-(t)} dx dt$$

$$= \int_{-\infty}^\infty g^+(t) f_x(t) dt + \int_{-\infty}^\infty g^-(t) f_x(t) dt$$

$$g^+(t) = \max\{0, g(t)\}$$

$$g^-(t) = \max\{0, -g(t)\}$$

$$= \int_{-\infty}^\infty (g^+(t) + g^-(t)) f_x(t) dt$$

$$= \int_{-\infty}^\infty g(t) f_x(t) dt$$

$$= \underline{\mathbb{E}[g(x)]}$$

o Variance:

$$\text{Var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2]$$

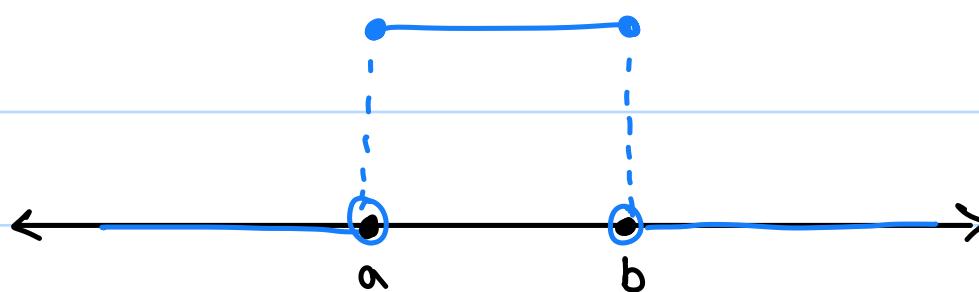
$$= \int_{-\infty}^\infty (x - \mathbb{E}[x]^2) f_x(x) dx$$

$$= \mathbb{E}[x^2] - (\mathbb{E}[x])^2$$

• Examples of Continuous Rvs :-

i) Uniform Continuous Rv :-

$$f_x(x) = \begin{cases} \frac{1}{b-a}, & x \in [a,b] \\ 0, & \text{otherwise} \end{cases}$$



$$E[X] = \frac{a+b}{2}$$

$$Var[X] = \frac{(b-a)^2}{12}$$

2) Exponential Rv:

$$f_x(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases} \quad \lambda \in \mathbb{R}_+$$

Good model for the amount of time taken until a certain event happens.

$$E[X] = \int_{-\infty}^{\infty} x f_x(x) dx = \int_{-\infty}^{\infty} x \lambda e^{-\lambda x} dx$$

$$t = \lambda x, \quad dt = \lambda dx$$

$$\Rightarrow E[X] = \frac{1}{\lambda} \int_0^{\infty} t e^{-t} dt$$

$$= \frac{1}{\lambda} \left[ t \int e^{-t} dt - \int \int e^{-t} dt dt \right]_0^\infty$$

$$= \frac{1}{\lambda} \left[ -te^{-t} + \int e^{-t} dt \right]_0^\infty$$

$$= \frac{1}{\lambda} \left[ -te^{-t} - e^{-t} \right]_0^\infty$$

$$= \frac{1}{\lambda} \left[ -e^{-t}(t+1) \right]_0^\infty$$

$$= \frac{1}{\lambda} (0 - (-1))$$

$$\Rightarrow E[x] = \frac{1}{\lambda}$$

$$E[x^2] = \int_{-\infty}^{\infty} x^2 f_x(x) dx \quad [\text{LOTUS}]$$

$$= \int_0^{\infty} x^2 x e^{-\lambda x} dx$$

$$= \lambda \left[ x^2 \int e^{-\lambda x} dx - \int 2x \int e^{-\lambda x} dx dx \right]_0^\infty$$

$$= \lambda \left[ \left( \frac{-1}{\lambda} \right) x^2 e^{-\lambda x} - 2 \left( \frac{-1}{\lambda} \right) \int x e^{-\lambda x} dx \right]_0^\infty$$

$$= \left[ -x^2 e^{-\lambda x} + 2 \left( x \int e^{-\lambda x} dx - \int \int e^{-\lambda x} dx dx \right) \right]_0^\infty$$

$$= \left[ -x^2 e^{-\lambda x} + 2 \left( -\frac{x e^{-\lambda x}}{\lambda} + \frac{1}{\lambda} \int e^{-\lambda x} dx \right) \right]_0^\infty$$

$$= \left[ -x^2 e^{-\lambda x} - \frac{2x e^{-\lambda x}}{\lambda} - \frac{2 e^{-\lambda x}}{\lambda^2} \right]_0^\infty$$

$$= \left[ -\left( -\frac{2}{\lambda^2} \right) \right]$$

$$\Rightarrow E[X^2] = \frac{2}{\lambda^2}$$

$$\Rightarrow \text{Var}[X] = E[X^2] - E[X]^2 \\ = \frac{2}{\lambda^2} - \frac{1}{\lambda^2}$$

$$\Rightarrow \underline{\text{Var}[X] = \frac{1}{\lambda^2}}$$

3) Gaussian RV:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

where  $\mu \in \mathbb{R}$ ,  $\sigma \in \mathbb{R}^+$

$$\begin{aligned} \int_{-\infty}^{\infty} f_X(x) dx &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) dx \end{aligned}$$

$$\text{Let } \frac{x-\mu}{\sigma} = t, dt = dx\left(\frac{1}{\sigma}\right)$$

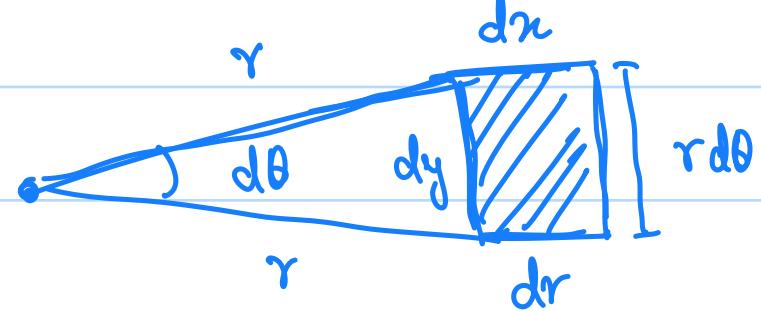
$$\begin{aligned} \Rightarrow \int_{-\infty}^{\infty} f_X(x) dx &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\left(-\frac{t^2}{2}\right) dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{t^2}{2}\right) dt := I \end{aligned}$$

$$I^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx \int_{-\infty}^{\infty} \exp\left(-\frac{y^2}{2}\right) dy$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(-\frac{(x^2+y^2)}{2}\right) dx dy$$

Let  $x = r\cos\theta, y = r\sin\theta$

$$dx dy = \text{Area of a small element} = r dr d\theta$$



$$\Rightarrow I^2 = \frac{1}{2\pi} \int_0^{2\pi} \int_{r=0}^{\infty} \exp\left(-\frac{r^2}{2}\right) r dr d\theta$$

$$= \frac{1}{2\pi} \int_{r=0}^{\infty} r \exp\left(-\frac{r^2}{2}\right) \int_0^{2\pi} d\theta dr$$

$$= \int_{r=0}^{\infty} r \exp\left(-\frac{r^2}{2}\right) dr$$

$$\frac{d}{dr} \exp\left(-\frac{r^2}{2}\right) = -r \exp\left(-\frac{r^2}{2}\right)$$

$$\Rightarrow \int_{r=0}^{\infty} r \exp\left(-\frac{r^2}{2}\right) dr = \left[ -\exp\left(-\frac{r^2}{2}\right) \right]_0^{\infty} = \underline{\underline{1}}$$

$$\Rightarrow I^2 = 1 \Rightarrow I = 1$$

$$\Rightarrow \int_{-\infty}^{\infty} f_x(x) dx = 1 \Rightarrow f_x(x) \text{ is a valid PDF}$$

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

