Probability and Random Processes — Assignment - 4

Sricharan Vinoth Kumar, RNo : 2024112022

Q1. Given: Rv X st.

$$f_X(x) = \begin{cases} p\lambda e^{-\lambda x}, & x \geq 0 \\ (1-p)\lambda e^{\lambda x}, & x < 0 \end{cases}$$

λ, p are scalars, $p \in [0,1]$.

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_0^{\infty} x p \lambda e^{-\lambda x}\, dx + \int_{-\infty}^{0} x(1-p)\lambda e^{\lambda x}\, dx$$

$$= p\lambda \underbrace{\int_0^{\infty} x e^{-\lambda x}\, dx}_{I_1} + (1-p)\lambda \underbrace{\int_{-\infty}^{0} x e^{\lambda x}\, dx}_{I_2}$$

I_1,
$$\int_0^{\infty} x e^{-\lambda x}\, dx = \left[x \int e^{-\lambda x}\, dx - \iint e^{-\lambda x}\, dx\, dx \right]_0^{\infty}$$

$$= \left[\frac{-x}{\lambda} e^{-\lambda x} + \frac{1}{\lambda} \int e^{-\lambda x}\, dx \right]_0^{\infty}$$

$$= \left[-\frac{x e^{-\lambda x}}{\lambda} - \frac{e^{-\lambda x}}{\lambda^2} \right]_0^{\infty}$$

$$= \left[\frac{x e^{-\lambda x}}{\lambda} + \frac{e^{-\lambda x}}{\lambda^2} \right]_{\infty}^{0}$$

$$= \left[\left(0 + \frac{1}{\lambda^2}\right) - (0+0) \right] = \frac{1}{\lambda^2}$$

I_2,

$$\int_{-\infty}^{0} x e^{\lambda x} dx = \left[x \int e^{\lambda x} dx - \int\int e^{\lambda x} dx\, dx \right]_{-\infty}^{0}$$

$$= \left[\frac{x e^{\lambda x}}{\lambda} - \frac{1}{\lambda} \int e^{\lambda x} dx \right]_{-\infty}^{0}$$

$$= \left[\frac{x e^{\lambda x}}{\lambda} - \frac{e^{\lambda x}}{\lambda^2} \right]_{-\infty}^{0}$$

$$= \left[\left(0 - \frac{1}{\lambda^2}\right) - (0 - 0) \right]$$

$$= \frac{-1}{\lambda^2}$$

$$\Rightarrow E[X] = p\lambda\left(\frac{1}{\lambda^2}\right) + (1-p)\lambda\left(\frac{-1}{\lambda^2}\right)$$

$$= p/\lambda - (1-p)/\lambda$$

$$E[X] = 2p - 1/\lambda$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx = \int_0^{\infty} x^2 p \lambda e^{-\lambda x} dx + \int_{-\infty}^{0} x^2 (1-p) \lambda e^{\lambda x} dx$$

$$= p\lambda \underbrace{\int_0^{\infty} x^2 e^{-\lambda x} dx}_{I_1} + (1-p)\lambda \underbrace{\int_{-\infty}^{0} x^2 e^{\lambda x} dx}_{I_2}$$

I_1,

$$\int_0^{\infty} x^2 e^{-\lambda x} dx = \left[x^2 \int e^{-\lambda x} dx - 2 \int x \int e^{-\lambda x} dx\, dx \right]_0^{\infty}$$

$$= \left[\frac{-x^2 e^{-\lambda x}}{\lambda} + \frac{2}{\lambda}\left(-\frac{x e^{-\lambda x}}{\lambda} - \frac{e^{-\lambda x}}{\lambda^2} \right) \right]_0^{\infty}$$

$$= \left[\frac{-x^2 e^{-\lambda x}}{\lambda} - \frac{2 x e^{-\lambda x}}{\lambda^2} - \frac{2 e^{-\lambda x}}{\lambda^3} \right]_0^{\infty}$$

$$= \left[(0+0+0) - \left(0+0-\frac{2}{\lambda^3}\right) \right]$$

$$= \frac{2}{\lambda^3}$$

$I_2)$

$$\int_{-\infty}^{0} x^2 e^{\lambda x}\,dx = \left[x^2 \int e^{\lambda x}\,dx - 2\int x \int e^{\lambda x}\,dx\,dx \right]_{-\infty}^{0}$$

$$= \left[\frac{x^2 e^{\lambda x}}{\lambda} - \frac{2}{\lambda} \int x e^{\lambda x}\,dx \right]_{-\infty}^{0}$$

$$= \left[\frac{x^2 e^{\lambda x}}{\lambda} - \frac{2}{\lambda}\left(\frac{x e^{\lambda x}}{\lambda} - \frac{e^{\lambda x}}{\lambda^2} \right) \right]_{-\infty}^{0}$$

$$= \left[\frac{x^2 e^{\lambda x}}{\lambda} - \frac{2x e^{\lambda x}}{\lambda^2} + \frac{2 e^{\lambda x}}{\lambda^3} \right]_{-\infty}^{0}$$

$$= \left[\left(0 - 0 + \frac{2}{\lambda^3}\right) - (0+0+0) \right]$$

$$= \frac{2}{\lambda^3}$$

$$\Rightarrow E[X^2] = p \times \left(\frac{2}{\lambda^2}\right) + (1-p) \times \left(\frac{2}{\lambda^2}\right)$$

$$= \frac{2p}{\lambda^2} + \frac{2(1-p)}{\lambda^2} = \underline{\underline{\frac{2}{\lambda^2}}}$$

$$\mathrm{Var}[X] = E[X^2] - (E[X])^2 = \frac{2}{\lambda^2} - \left(\frac{2p-1}{\lambda}\right)^2$$

$$= \frac{2}{\lambda^2} - \frac{4p^2+1-4p}{\lambda^2}$$

$$\Rightarrow \mathrm{Var}[X] = \frac{4p - 4p^2 + 1}{\lambda^2}$$

Q2. To Prove: 2 Rvs X and Y are independent iff $F_{XY}(x,y) = F_X(x)F_Y(y)$ $\forall$ x,y.

Case 1: X, Y are Discrete Rv.s,

Let X, Y be independent, then

$$F_{XY}(x,y) = P(X<x, Y<y) = \sum_{u<x}\sum_{v<y} P_{XY}(u,v)$$

$$= \sum_{u<x}\sum_{v<y} P_X(u)P_Y(v) \quad \text{[By independence]}$$

$$= \sum_{u<x} P_X(u) \sum_{v<y} P_Y(v) = \sum_{u<x} P_X(u) F_Y(y)$$

$$= F_Y(y) \sum_{u<x} P_X(u)$$

$$\Rightarrow \quad F_{XY}(x,y) = F_Y(y)F_X(x)$$

$\therefore$ X, Y are independent $\Rightarrow$ $F_{XY}(x,y) = F_X(x)F_Y(y)$

Let X, Y be 2 Rv.s st $F_{XY}(x,y) = F_X(x)F_Y(y)$

$$P_{XY}(x,y) = F_{XY}(x,y) - F_{XY}(x-1,y) - F_{XY}(x,y-1) + F_{XY}(x-1,y-1)$$

$$= F_X(x)F_Y(y) - F_X(x-1)F_Y(y) - F_X(x)F_Y(y-1) + F_X(x-1)F_Y(y-1)$$

$$= F_X(x)\left(F_Y(y) - F_Y(y-1)\right) - F_X(x-1)\left(F_Y(y) - F_Y(y-1)\right)$$

$$= \left(F_X(x) - F_X(x-1)\right)\left(F_Y(y) - F_Y(y-1)\right)$$

$$= P_X(x)\, P_Y(y)$$

$\therefore F_{XY}(x,y) = F_X(x)F_Y(y) \Rightarrow P_{XY}(x,y) = P_X(x)P_Y(y) \Rightarrow X, Y$ are independent.

Combining both results,

$$F_{XY}(x,y) = F_X(x)\,F_Y(y) \iff X, Y \text{ are independent RVs.}$$

Case 2: X, Y are continuous.

$$F_{XY}(x,y) = \int_{u<x}\int_{v<y} f_{XY}(u,v)\,dv\,du$$

Let X, Y be independent, then,

$$f_{XY}(x,y) = f_X(x)\,f_Y(y)$$

$$\Rightarrow F_{XY}(x,y) = \int_{u<x}\int_{v<y} f_X(u)\,f_Y(v)\,dv\,du$$

$$= \int_{u<x} f_X(u) \int_{v<y} f_Y(v)\,dv\,du$$

$$= F_Y(y) \int_{u<x} f_X(u)\,du$$

$$= F_Y(y)\,F_X(u)$$

$\therefore$ X, Y are independent $\Rightarrow F_{XY}(x,y) = F_X(x) F_Y(y)$

Let X, Y be 2 RVs such that $F_{XY}(x,y) = F_X(x) F_Y(y)$

$$f_{XY}(x,y) = \frac{\delta^2 F_{XY}(x,y)}{\delta xy}$$

$$= \frac{\delta^2}{\delta xy}\left(F_X(x) F_Y(y)\right)$$

$$= \frac{\delta}{\delta x} F_X(x) \left(\frac{\delta}{\delta y} F_Y(y)\right)$$

$$= f_Y(y) \frac{\delta}{\delta x} F_X(x)$$

$$= f_X(x) f_Y(y)$$

$\therefore F_{XY}(x,y) = F_X(x) F_Y(y) \Rightarrow f_{XY}(x,y) = f_X(x) f_Y(y) \Rightarrow$ X, Y are independent

Combining both results,

$F_{XY}(x,y) = F_X(x) F_Y(y) \iff$ X, Y are independent RVs.

Q3. Let X be the RV corresponding to the event of choosing a random point in the interval $[a,b]$.

$\forall A \subseteq [a,b]$, let $P(A) = \dfrac{\text{length of } A}{b-a}$.

$$\Rightarrow F_X(x) = \frac{x-a}{b-a} \quad \forall x \in [a,b]$$

$$\Rightarrow f_X(x) = \frac{\delta F_X(x)}{\delta x} = \frac{1}{b-a}$$

Let L represent the RV of getting a segment of length l on choosing the points.

$$f_L(l) = \int_a^{b-l} f_X(x) f_X(x+l)\, dx + \int_{a+l}^{b} f_X(x) f_X(x-l)\, dx$$

$[\forall\, l \in [0, b-a]]$

$$= \int_a^{b-l} \frac{1}{(b-a)^2}\, dx + \int_{a+l}^{b} \frac{1}{(b-a)^2}\, dx$$

$$= \frac{1}{(b-a)^2}\left[(b-l-a) + (b-a-l)\right]$$

$$= \frac{2(b-l-a)}{(b-a)^2}$$

$$E[L] = \int_{-\infty}^{\infty} l\, f_L(l)\, dl$$

Clearly $f_L(l) = 0 \;\forall\; l < 0$ and $l > b-a$ [length cannot be -ve and cannot exceed the interval]

$$\Rightarrow E[L] = \int_0^{b-a} \frac{2l(b-l-a)}{(b-a)^2}\, dl$$

$$= \frac{2}{(b-a)^2} \int_0^{b-a} (b-a)l - l^2\, dl$$

$$= \frac{2}{(b-a)^2} \left[\frac{(b-a)l^2}{2} - \frac{l^3}{3}\right]_0^{b-a}$$

$$= \frac{2}{(b-a)^2}\left(\frac{(b-a)^3}{2} - \frac{(b-a)^3}{3}\right) = \frac{\cancel{2}}{\cancel{(b-a)^2}}\,\frac{(b-a)^{\cancel{3}}}{\cancel{6}\cdot 3}$$

$$\therefore\ E[L] = \frac{b-a}{3}$$

Q4 a) <u>Given:</u> Discrete RV X, event A st $P(A) > 0$, $\mathbb{I}_A$ – indicator RV of A.

<u>To Prove:</u> $E[X|A] = \dfrac{E[\mathbb{I}_A X]}{P(A)}$

$$E[X|A] = \sum_{x \in \mathcal{X}} x\, P(X = x \mid A)$$

$$= \sum_{x \in \mathcal{X}} x\, \frac{P(X = x \cap A)}{P(A)}$$

$$= \frac{1}{P(A)} \sum_{x \in \mathcal{X}} x\, P(X = x \cap A) \quad \text{——— (1)}$$

Let $Y = \mathbb{I}_A X$

$$E[Y] = \sum_{y \in \mathcal{Y}} y\, P(Y = y) = \sum_{y \in \mathcal{Y}} y\, P(\mathbb{I}_A X = y)$$

The range of $\mathbb{I}_A$ is $\{0, 1\}$, since

$$\mathbb{I}_A(\omega) = \begin{cases} 0, & \omega \notin A \\ 1, & \omega \in A \end{cases}$$

$$\Rightarrow \mathbb{I}_A(\omega) X(\omega) = \begin{cases} 0, & \omega \notin A \\ X(\omega), & \omega \in A \end{cases}$$

$\therefore$ Range of $\mathbb{I}_A X$ = Range of $X \cup \{0\}$

$$\Rightarrow E[\mathbb{I}_A X] = \sum_{x \in X} x \, P(\mathbb{I}_A X = x)$$

[Ignoring $y=0$ since value inside integration is 0 if $y=0$, so will not affect final value]

Since for $\mathbb{I}_A(\omega) X(\omega) = x$, we need $\omega \in A$ and $X(\omega) = x$,

$$P(\mathbb{I}_A X = x) = P(X = x \cap A)$$

$$\therefore \sum_{x \in X} x P(\mathbb{I}_A X = x) = \sum_{x \in X} x P(X = x \cap A) = E[\mathbb{I}_A X] \quad \text{— ②}$$

Substituting ② in ①, we get,

$$E[X|A] = \frac{E[\mathbb{I}_A X]}{P(A)}$$

b) Given: X : R.v. of sum of outcomes of rolling a die twice.
A_i : Event that outcome of first die is i, $i \in [1:6]$

To Find: $E[X|A_i] \ \forall \ i \in [1:6]$

$$E[X|A_i] = \sum_{x \in [2:12]} x P(X = x | A_i)$$

$$\Rightarrow E[X|A_i] = \sum_{n=1}^{6} (i+n) P(X = i+n | A_i)$$

$P(X = x \mid A_i) = 1/6 \quad \forall \; i \in [1:6]$ [There are 6 possible sums if outcome of the first roll is given, all of them equally likely]

$$\Rightarrow \boxed{E[X \mid A_i] = \sum_{n=1}^{6} \frac{i+n}{6}}$$

$$E[X \mid A_1] = \frac{2+3+4+5+6+7}{6} = \underline{\underline{\frac{27}{6}}}$$

$$E[X \mid A_2] = \frac{3+4+5+6+7+8}{6} = \underline{\underline{\frac{33}{6}}}$$

$$E[X \mid A_3] = \frac{4+5+6+7+8+9}{6} = \underline{\underline{\frac{39}{6}}}$$

$$E[X \mid A_4] = \frac{5+6+7+8+9+10}{6} = \underline{\underline{\frac{45}{6}}}$$

$$E[X \mid A_5] = \frac{6+7+8+9+10+11}{6} = \underline{\underline{\frac{51}{6}}}$$

$$E[X \mid A_6] = \frac{7+8+9+10+11+12}{6} = \underline{\underline{\frac{57}{6}}}$$

Q5. Given:

$$F_X(x) = \begin{cases} 1 - \frac{a^3}{x^3}, & x \geq a \\ 0, & x < a \end{cases}$$

To Find: $f_X(x)$, $E[X]$, $Var[X]$

$$f_X(x) = \frac{dF_X(x)}{dx} = \begin{cases} \frac{3a^3}{x^4}, & x \geq a \\ 0, & x < a \end{cases}$$

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_{a}^{\infty} x \frac{3a^3}{x^4}\,dx$$

$$= 3a^3 \int_{a}^{\infty} \frac{dx}{x^3} = 3a^3 \left[-\frac{1}{2x^2}\right]_a^{\infty}$$

$$= 3a^3 \left(0 + \frac{1}{2a^2}\right)$$

$$\therefore E[X] = \frac{3}{2}a$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_{a}^{\infty} x^2 \frac{3a^3}{x^4}\,dx = 3a^3 \int_{a}^{\infty} \frac{dx}{x^2}$$

$$= 3a^3 \left[\frac{-1}{x}\right]_a^{\infty} = 3a^3\left(0 + \frac{1}{a}\right)$$

$$= \underline{\underline{3a^2}}$$

$$\mathrm{Var}[X] = E[X^2] - (E[X])^2$$

$$= 3a^2 - \left(\frac{3}{2}a\right)^2 = a^2\left(3 - \frac{9}{4}\right)$$

$$\Rightarrow \mathrm{Var}[X] = \frac{3}{4}a^2$$

Q6. Given:

$$f_{XY}(x,y) = c(y^2 - x^2)e^{-y}, \quad 0 < y < \infty, \; -y \leq x \leq y$$

To find: c, $f_X(x)$, $f_Y(y)$

By the properties of a PDF,

$$\int_{y\in Y}\int_{x\in X} f_{XY}(x,y)\,dx\,dy = 1$$

$$= \int_{y\in Y}\int_{x\in X} c(y^2-x^2)e^{-y}\,dx\,dy = 1$$

$$= c\int_{y=0}^{\infty} e^{-y}\int_{-y}^{y}(y^2-x^2)\,dx\,dy = 1$$

$$= c\int_{0}^{\infty} e^{-y}\left[y^2x-\frac{x^3}{3}\right]_{-y}^{y} dy = 1$$

$$= \int_0^\infty e^{-y}\left[\left(y^3-\frac{y^3}{3}\right)-\left(-y^3+\frac{y^3}{3}\right)\right]dy = \frac{1}{c}$$

$$= \int_0^\infty e^{-y}\left(2y^3-\frac{2y^3}{3}\right)dy = \frac{1}{c}$$

$$= \frac{4}{3}\int_0^\infty y^3e^{-y}\,dy = \frac{1}{c}$$

$\int_0^\infty y^3e^{-y}\,dy$, let $z=-y \Rightarrow \int_0^{-\infty} -z^3e^{z}\,d(-z) = -\int_{-\infty}^{0} z^3e^{z}\,dz$

$$\int_{-\infty}^{0} z^3e^{z}\,dz = \left[z^3e^{z}-3\int z^2e^{z}\,dz\right]_{-\infty}^{0}$$

$$= \left[z^3e^{z}-3\left(z^2e^{z}-2\int ze^{z}\,dz\right)\right]_{-\infty}^{0}$$

$$= \left[z^3e^{z}-3\left(z^2e^{z}-2\left(ze^{z}-e^{z}\right)\right)\right]$$

$$= \left[z^3e^{z}-3z^2e^{z}+6ze^{z}-6e^{z}\right]_{-\infty}^{0}$$

$$= \left[-6-0\right] = \underline{\underline{-6}}$$

$$\therefore \int_0^\infty y^3 e^{-y}\, dy = -(-6) = \underline{\underline{6}}$$

$$\Rightarrow \left(\frac{4}{3}\right) \times 6 = \frac{1}{c} \Rightarrow = c = \frac{3}{24}$$

$$\Rightarrow \boxed{c = \frac{1}{8}}$$

$$f_X(x) = \int_{y \in Y} f_{XY}(x,y)\, dy$$

$$-y \leqslant x \leqslant y \Rightarrow |x| < y \Rightarrow f_X(x) = \int_{|x|}^\infty f_{XY}(x,y)\, dy$$

$$f_X(x) = \int_{|x|}^\infty \frac{(y^2 - x^2)e^{-y}}{8}\, dy = \frac{1}{8}\left(\int_{|x|}^\infty y^2 e^{-y} dy - x^2 \int_{|x|}^\infty e^{-y} dy\right)$$

$$= \frac{1}{8}\left(\left[-y^2 e^{-y} + 2\int y e^{-y} dy\right]_{|x|}^\infty + x^2\left(0 - e^{-|x|}\right)\right)$$

$$= \frac{1}{8}\left(\left[-y^2 e^{-y} + 2\left(-ye^{-y} - e^{-y}\right)\right]_{|x|}^\infty - x^2 e^{-|x|}\right)$$

$$= \frac{1}{8}\left(\left[-e^{-y}\left(y^2 + 2y + 2\right)\right]_{|x|}^\infty - x^2 e^{-|x|}\right)$$

$$= \frac{1}{8}\left(\left[0 + e^{-|x|}\left(x^2 + 2|x| + 2\right)\right] - x^2 e^{-|x|}\right)$$

$$= \frac{1}{8}\left(x^2 e^{-|x|} + 2|x|e^{-|x|} + 2e^{-|x|} - x^2 e^{-|x|}\right)$$

$$\boxed{\therefore f_X(x) = \frac{e^{-|x|}\left(|x| + 1\right)}{4}}$$

$$f_Y(y) = \int_{x \in X} f_{XY}(x,y)\, dx$$

$$= \int_{-y}^{y} \frac{1}{8}(y^2 - x^2) e^{-y}\, dx$$

$$= \frac{1}{8}\left(\int_{-y}^{y} y^2 e^{-y}\, dx - \int_{-y}^{y} x^2 e^{-y}\, dx \right)$$

$$= \frac{1}{8}\left(y^2 e^{-y}(y-(-y)) - e^{-y}\left(\frac{y^3}{3} - \left(\frac{-y^3}{3}\right)\right)\right)$$

$$= \frac{1}{8}\left(2y^3 e^{-y} - \frac{2y^3}{3} e^{-y} \right)$$

$$= \frac{1}{\cancel{8}_{2}}\left(\frac{\cancel{4}}{3} y^3 e^{-y}\right)$$

$$\boxed{\therefore\ f_Y(y) = \frac{y^3 e^{-y}}{6}}$$

Q7. Given: X_1, X_2, X_3 are independent continuous RVs with PDF $f_X(x)$.

To find: $P(X_1 < X_2 < X_3)$

$$P(X_1 < X_2 < X_3) = \int_{u=-\infty}^{\infty} \int_{v=-\infty}^{u} \int_{w=-\infty}^{v} f_{X_1X_2X_3}(u,v,w)\, dw\, dv\, du$$

$$= \int_{u=-\infty}^{\infty} \int_{v=u}^{\infty} \int_{w=v}^{\infty} f_{X_1}(u) f_{X_2}(v) f_{X_3}(w)\, dw\, dv\, du$$

[By independence]

$$= \int_{u=-\infty}^{\infty} f_{X_1}(u) \int_{v=u}^{\infty} f_{X_2}(v) \int_{w=v}^{\infty} f_{X_3}(w)\, dw\, dv\, du$$

$$\therefore P(X_1 < X_2 < X_3) = \int_{-\infty}^{\infty} f_X(u) \int_{u}^{\infty} f_X(v) \int_{v}^{\infty} f_X(w)\, dw\, dv\, du$$

$$= \int_{-\infty}^{\infty} f_X(u) \int_{u}^{\infty} f_X(v) \left(1 - F_X(v)\right) dv\, du$$

$$\int_{u}^{\infty} f_X(v)\left(1 - F_X(v)\right) dv = \int_{u}^{\infty} f_X(v)\, dv - \int_{u}^{\infty} f_X(v) F_X(v)\, dv$$

$$= \left(1 - F_X(u)\right) - \int_{u}^{\infty} f_X(v) F_X(v)\, dv$$

Let $F_X(v) = y$, $dy = f_X(v)\, dv$

$$\int_{F_X(u)}^{1} f_X(v) F_X(v)\, dv = \int y\, dy = \left[\frac{y^2}{2}\right]_{F_X(u)}^{1} = \left[\frac{F_X(v)^2}{2}\right]_{u}^{\infty}$$

$$= \frac{1 - F_X(u)^2}{2}$$

$$\Rightarrow \int_{u}^{\infty} f_X(v)\left(1 - F_X(v)\right) dv = 1 - F_X(u) - \frac{1 - F_X(u)^2}{2}$$

$$= \frac{2\left(1 - F_X(u)\right) - 1 + F_X(u)^2}{2}$$

$$= \frac{1 - 2F_X(u) + F_X(u)^2}{2}$$

$$= \frac{\left(F_X(u) - 1\right)^2}{2}$$

$$\therefore P(X_1 < X_2 < X_3) = \int_{-\infty}^{\infty} f_X(u) \frac{\left(F_X(u) - 1\right)^2}{2}\, du$$

Let $z = F_X(u)$, $dz = f_X(u)\,du$

$$\int_{-\infty}^{\infty} f_X(u)\,\frac{(F_X(u)-1)^2}{2}\,du = \int_0^1 \frac{(z-1)^2}{2}\,dz$$

$$= \frac{1}{2}\left[\frac{(z-1)^3}{3}\right]_0^1$$

$$= \frac{1}{2}\left(\frac{0}{3} - \left(\frac{-1}{3}\right)\right) = \frac{1}{6}$$

$$\therefore\ P(X_1 < X_2 < X_3) = \frac{1}{6}$$