Quiz 1 - 10
Quiz 2 - 10
Assn - 20
Mid - 25
Final - 35

Theory = 100

Submission - 15
Mid Exam - 10
Project / End Exam - 15

Lab = 40

Textbooks:

Signals & Systems by Oppenheim
DSP, JG Proakis

Addn: Principles of SP, BP Lathi
DSP : Sanjit K Mitra

Signal Processing

Prof. Santosh Nannuru

A > 130
F < 60

Labs:

MATLAB Based
Github Submission

Tutorial: 10:30 - 11:30 Saturday

Prof: Santosh Nannuru

Signal Processing

→ Overview of NeSS :-

- Time variant - function of time
- Signals can be a function of any variable, need not be time always.
- Multi-Dimensional signals - images and videos.
- Anything that carries information is a signal.
- The course focuses more on discrete time signals, since most digital devices work in a discrete manner.
- System process signals, $x(t) \longrightarrow \text{System}(H) \longrightarrow y(t)$.
- LTI system - Linear and Time-Invariant Systems.
- Any circuit with only R, L, C will always form an LTI system, since R, L, C are all linear components.

Diodes and Transistors form non-linear systems.

- In any LTI system, the output $y(t)$ of a signal $x(t)$ is,

$$\boxed{y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau} \rightarrow \underline{\text{Convolution}}$$

$h(t)$ = impulse response of the system.

- <u>Laplace Transform</u> :- Representation of the signal in the s / complex frequency domain.

- Poles of the system function determine the behaviour / properties of the system.

- RoC - Region in the complex s-plane where the Laplace transform converges. Is always a vertical strip, defined by the location of poles.

- <u>Fourier Series</u> :-

- Any periodic signal can be represented as a sum of sines and cosines.

→ Course Topics:-

- Fourier Analysis, Z transform
- Sampling and Quantization - DSP
- Computational Aspect - fast Fourier Transform
- Manipulating Signals - filter design.

→ Fourier Analysis:-

- Periodic and Continuous-time - Fourier Series Rep.
- Aperiodic and Continuous-time - Fourier Transform
- Aperiodic and Discrete-time - Discrete Time Fourier Transform
- Finite Length and Discrete-time - DFT

  Faster → FFT

- Fourier Series Representation:

  - Representation of any periodic signal as a sum of sinusoidal harmonics.

$$x(t) = b_0 + \sum_{k \in \mathbb{N}} \left(a_k \sin(k\omega_0 t) + b_k \cos(k\omega_0 t) \right)$$

ω_0 – Angular frequency of the signal. $\left(2\pi/T\right)$

$$x(t) = \sum_{k \in \mathbb{Z}} c_k e^{jk\omega_0 t}$$

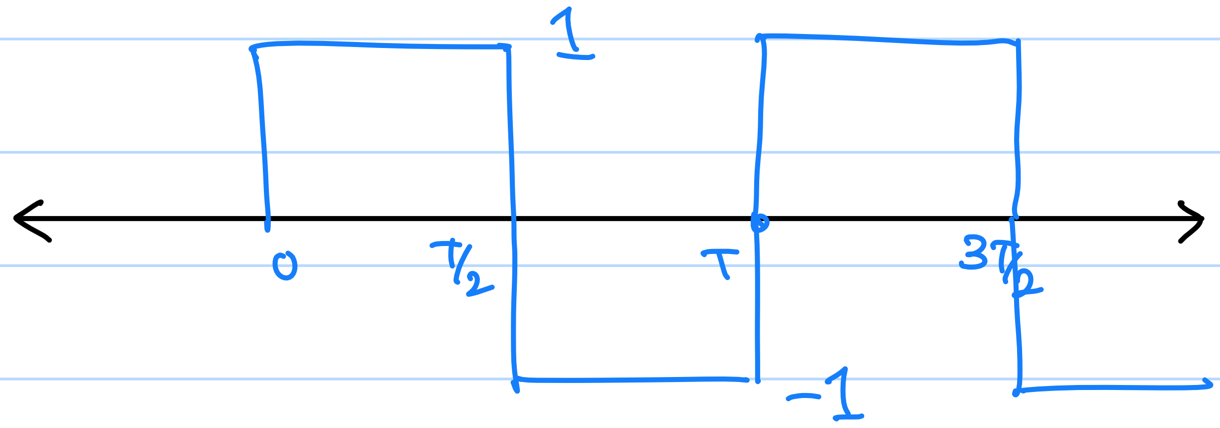
- Analysis : Computing the coefficients of the given signal
  Synthesis : Given the signal, computing the coefficients.

$$a_k = \frac{2}{T} \int_T x(t) \sin(k\omega_0 t)\, dt \quad , \quad k \neq 0$$

$$b_k = \frac{2}{T} \int_T x(t) \cos(k\omega_0 t)\, dt$$

$$c_k = \frac{1}{T} \int_T x(t)\, e^{-jk\omega_0 t}\, dt$$

Compute the FS Coeffs for a Sq. wave.



$$x(t) = \begin{cases} 1 \ , & t \in (0, T/2) \\ -1 \ , & t \in (T/2, T) \end{cases}$$

$$c_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\left[\int_0^{T/2} e^{-jk\omega_0 t}\, dt - \int_{T/2}^{T} e^{-jk\omega_0 t}\, dt\right]$$

$$= \frac{1}{T}\left(\frac{-1}{k\omega_0}\right)\left[\left[e^{-jk\omega_0 t}\right]_0^{T/2} - \left[e^{-jk\omega_0 t}\right]_{T/2}^{T}\right]$$

$$= \frac{-1}{k\cancel{T}\frac{2\pi}{\cancel{T}}}\left(\left[e^{-jk\left(\frac{2\pi}{T}\right)\left(\frac{T}{2}\right)} - e^{-jk\omega_0(0)}\right] - \left[e^{-jk\left(\frac{2\pi}{T}\right)T} - e^{-jk\left(\frac{2\pi}{T}\right)\left(\frac{T}{2}\right)}\right]\right)$$

$$= \frac{-1}{k2\pi}\left[\left(e^{-jk\pi} - 1\right) - \left(e^{-jk2\pi} - e^{-jk\pi}\right)\right]$$

$$= \frac{-1}{k2\pi}\left(2e^{-jk\pi} - 1 - e^{-jk2\pi}\right)$$

$$= \frac{-1}{k2\pi}\left(2e^{-jk\pi} - 2\right)$$

$$= \frac{-1}{k\pi}\left(e^{-jk\pi} - 1\right) \longrightarrow \underline{0 \text{ for even } k}$$

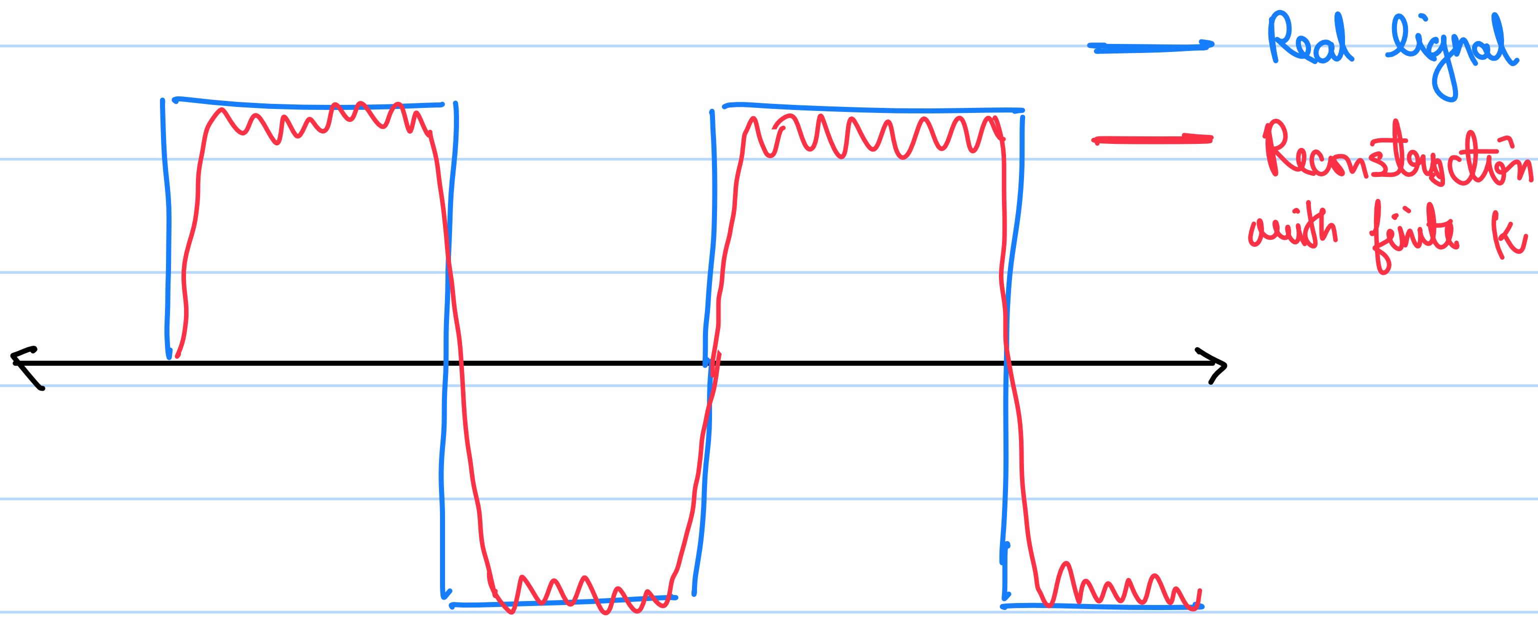
$$= \frac{1-\cos k\pi}{k\pi} \quad \forall \text{ odd } k$$

- **Partial Reconstruction** :-

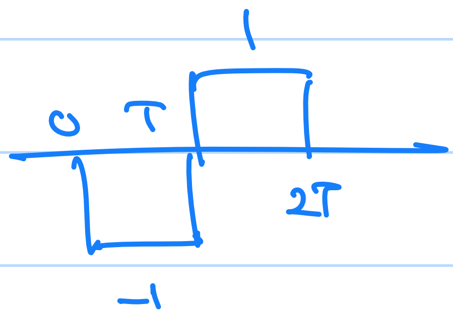
- Not all the coefficients are known to us, so only an approximation of the signal is constructed.

- Reconstruction Error $e(t) = x(t) - \hat{x}(t)$

- Discontinuous signals will have an infinite number of non-zero coefficients.



FS Coeffs of Sq. Wave Continuation,



$$a_k = \frac{2}{2T}\int_{2T} x(t)\sin(k\omega_0 t)\,dt$$

$$= \frac{1}{T}\left[\int_0^T -\sin(k\omega_0 t)\,dt + \int_T^{2T} \sin(k\omega_0 t)\,dt\right]$$

$$= \frac{1}{T}\left(\frac{1}{k\omega_0}\right)\left[\left[\cos(k\omega_0 t)\right]_0^T + \left[-\cos(k\omega_0 t)\right]_T^{2T}\right]$$

$$= \frac{1}{k\pi}\left[\left(\cos\left(k\frac{2\pi}{\cancel{2T}}(\cancel{T})\right) - 1\right) + \left(\cos\left(k\left(\frac{2\pi}{\cancel{2T}}\right)\cancel{T}\right) - \cos\left(k\left(\frac{2\pi}{\cancel{2T}}\right)\cancel{2T}\right)\right)\right]$$

$$= \frac{1}{k\pi}\left[(\cos k\pi - 1) + (\cos k\pi - 1)\right]$$

$$a_k = \frac{2(\cos k\pi - 1)}{k\pi}$$

$$b_k = \frac{2}{2T}\int_{2T} x(t)\cos(k\omega_0 t)\,dt$$

$$= \frac{1}{T}\left[\int_0^T -\cos(k\omega_0 t)\,dt + \int_T^{2T}\cos(k\omega_0 t)\,dt\right]\left(\frac{1}{k\omega_0}\right)$$

$$= \frac{1}{T}\left[\left(-\sin(k\omega_0 t)\right)_0^T + \left(\sin(k\omega_0 t)\right)_T^{2T}\right]\left(\frac{1}{k\omega_0}\right)$$

$$= \frac{1}{T}\left[\left(0 - \cancelto{0}{\sin(k\pi)}\right) + \left(\cancelto{0}{\sin k\frac{2\pi}{2T}2T} - \cancelto{0}{\sin k\frac{2\pi}{2T}T}\right)\right]$$

$$= \underline{\underline{0}} \quad \forall\, k$$

- Note: If $a_k \in \mathbb{R}\ \forall\, k$, then the signal is Even.

→ **Discrete-Time Signals :-** 4/8/25

- Represented by $x[n]$, where $n \in \mathbb{Z}$, n is an index where time duration is the sampling interval.

- **Unit Impulse :**

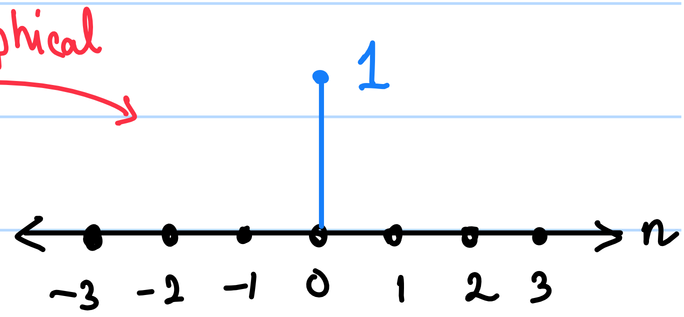
- In continuous time,

$$\delta(t) = 0\ \forall\, t \neq 0,\ \text{s.t.} \int_{-\infty}^{\infty}\delta(t)\,.\,dt = 1$$

- In discrete time,

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

Graphical

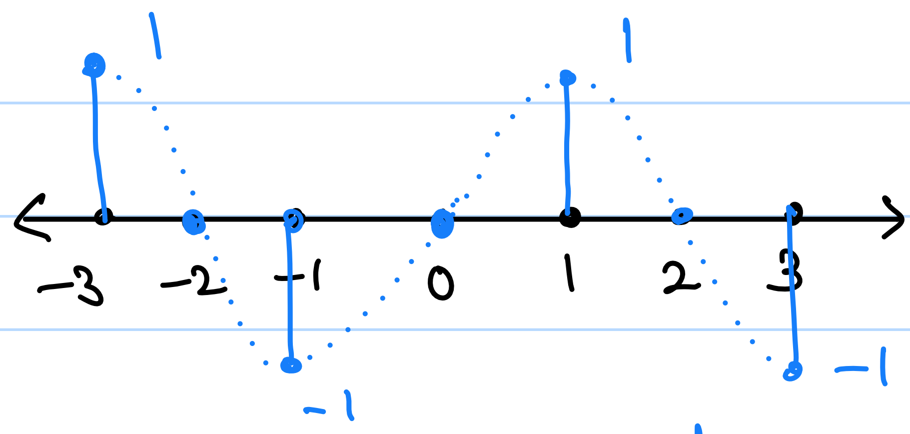


- **Unit Step:** $u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$

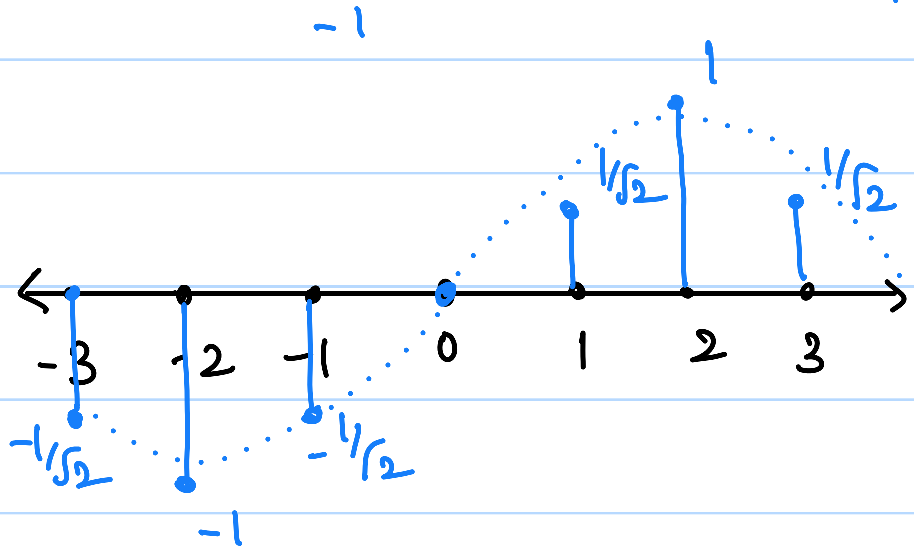
- **Exponential Signal:** $x[n] = a^n, \; a \in \mathbb{R}$

Example: Plot $x[n] = \sin(\omega_0 n)$ for $\omega_0 = \frac{\pi}{2}, \frac{\pi}{4}$

1)



2)



- A signal that is periodic in continuous time may not be always be periodic in discrete time, due to the sampling frequency. ($x[n] = \sin(n)$)

- **Discrete Time Periodic Signals:-**

$x[n]$ is periodic if,

$$x[n+P] = x[n] \;\; \forall n, \text{ for some } P \in \mathbb{Z}$$

- **Complex Exponentials :-**

$$x[n] = e^{sn}, \quad s \in \mathbb{C}$$, i.e. $s = \sigma + j\omega$, $\sigma, \omega \in \mathbb{R}$

$$\Rightarrow \quad x[n] = e^{\sigma n}\left(\cos(\omega n) + j\sin(\omega n)\right)$$

Note: Sinusoids are periodic if $\omega_0 = \frac{2\pi}{P}$ for some $P \in \mathbb{Z}$

- **Even and Odd:**

Even : $x[n] = x[-n] \quad \forall\, n \in \mathbb{Z}$

Odd : $x[n] = -x[-n] \quad \forall\, n \in \mathbb{Z}$

→ **Energy of a Discrete Time Signal :-**

$$E = \sum_{n \in \mathbb{Z}} |x[n]|^2$$

. It may not relate to any physical quantity / property of the signal.

$$\text{Power} = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

Energy Signals : Signals that carry finite energy, and hence, zero power

Power Signals: Signals that carry finite power.

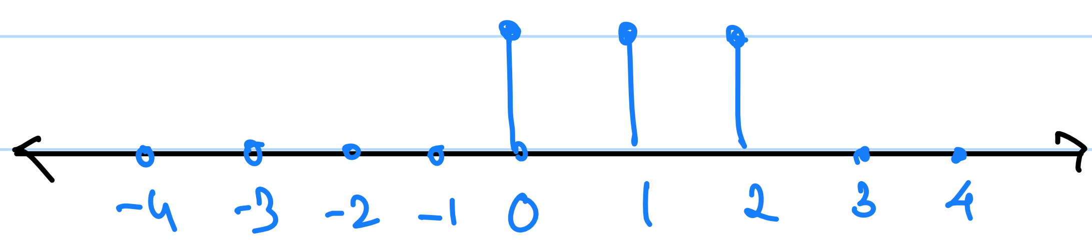
→ Time axis Manipulation for Discrete time Signals :-

- $x[n-n_0]$ – Shifts right by n_0
  $x[n+n_0]$ – Shifts left by n_0

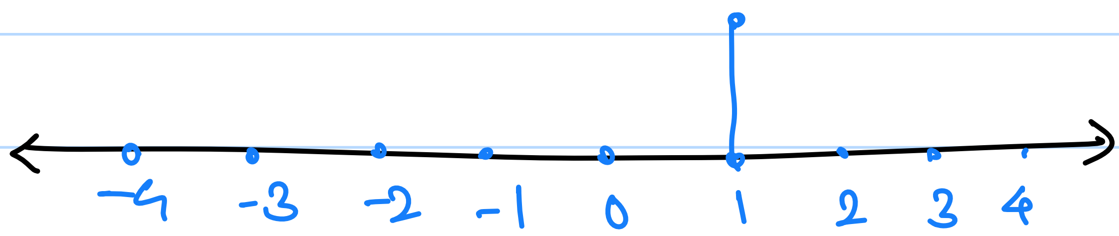
- $x[kn]$ – Only every k^{th} sample is taken

Example: $x[n] = u[n] - u[n-3]$
$y[n] = x[2n-1] = ?$

$x[n]$



$x[2n-1]$



- Also, due to the nature of discrete signals, $\omega_0 = \pi$ is the highest possible angular frequency for a sinusoid.

Beyond $\omega_0 = \pi$, the signal will be the same as those of a lower frequency, due to the properties of sinusoids.

$\cos(\pi n) = (-1)^n \longrightarrow$ Oscillations b/w 1 and -1 at the freq of sampling

$$\sin\left((\pi+\phi)n\right) = \sin(\pi n + \phi n)$$
$$= \sin \pi n \cos \phi n + \cos \pi n \sin \phi n$$
$$= (-1)^n \sin \phi n \longrightarrow \text{freq} < \pi$$

$$\cos\left((\pi+\phi)n\right) = \cos(\pi n)\cos(\phi n) - \sin(\pi n)\sin(\phi n)$$
$$= (-1)^n \cos \phi n \longrightarrow \text{freq} < \pi$$

- **Signal Representation By Unit Impulses :-**

$$x[n] = \sum_{m \in \mathbb{Z}} x[m]\,\delta[n-m] \quad \forall\, n \in \mathbb{Z}$$

Since $\delta[n-m] = 0 \;\forall\; n \neq m$, all the other values of the signal are nullified at each n.

→ **System :**

$$x[n] \longrightarrow \boxed{\text{System}} \longrightarrow y[n]$$

- **Delay :**

$$y[n] = x[n-\Delta] \;.\; \Delta \in \mathbb{Z}$$

- **Amplitude Scaling:**

$$y[n] = \alpha x[n], \quad \alpha \in \mathbb{C}$$

- **Integrator / Summation :**

$$y[n] = \sum_{m \in -\infty}^{n} x[m]$$

- **Difference:**

$$y[n] = x[n] - x[n-1]$$

Q. $x(t) = \cos(4\pi t)$ is sampled with sampling frequency below. Identify the frequency of the obtained DT signal.

f_s: 1 Hz, 2 Hz, 4 Hz, 8 Hz

$$x(t) = \cos(4\pi t)$$

In Sampling, $t \longrightarrow nT$, $T = \frac{1}{f_s}$

$$\Rightarrow \quad x[n] = \cos\left(\frac{4\pi n}{f_s}\right), \quad 2\pi f = \frac{4\pi}{f_s}$$

$$\Rightarrow \quad f = \frac{2}{f_s}$$

→ **Properties of Systems:-**

- **Linearity:** If x_1 gives o/p y_1, x_2 gives o/p y_2 then,

$$\alpha x_1 + \beta x_2 \longrightarrow \alpha y_1 + \beta y_2 \; . \; \alpha, \beta \in \mathbb{C} \text{ constants}$$

- **Time Invariant:** If $x[n]$ gives o/p $y[n]$, then,

$$x[n-n_0] \longrightarrow y[n-n_0] \quad \forall \; n_0 \in \mathbb{Z}$$

Also termed as shift invariant.

Example: Analyze time invariance of

1) $y[n] = x[n-\Delta]$, $\Delta \in \mathbb{Z}$
2) $y[n] = x^2[n]$
3) $y[n] = x[n] + x[n-1]$
4) $y[n] = n\,x[n]$

1) $y[n] = x[n-\Delta] \longrightarrow y[n-n_0] = x[n-n_0-\Delta]$

eq.

$x[n-n_0] \xrightarrow{Sys} x[n-n_0-\Delta]$ $\therefore$ Time Invariant

2) $y[n] = x^2[n] \longrightarrow y[n-n_0] = (x[n-n_0])^2$

eq.

$y_2[n] = (x[n-n_0])^2$ $\therefore$ Time Invariant

3) $y[n] = x[n] + x[n-1] \longrightarrow y[n-n_0] = x[n-n_0] + x[n-n_0-1]$

eq

$y_2[n] = x[n-n_0] + x[n-n_0-1]$ $\therefore$ Time Invariant

4) $y[n] = n\,x[n] \longrightarrow y[n-n_0] = (n-n_0)\,x[n-n_0]$

n. eq

$y_2[n] = n\,x[n-n_0]$ $\therefore$ Time Variant

- **Causality**: A system is said to be causal if the output $y[n]$ $\forall\, n \in \mathbb{Z}$ should only depend on $x[m]$ for $m \leq n$.

- **Stability:** If $x[n] \leq B \;\forall\, n \in \mathbb{Z}$ for some $B \in \mathbb{C}$, then $y[n] \leq B_0 \;\forall\, n \in \mathbb{Z}$, for some $B_0 \in \mathbb{C}$. Bounded Input, Bounded Output.

→ **Linear-Time Systems:-**

- LTI systems are linear and time invariant.

- **Impulse Response:** Response of the system to the impulse signal $\delta[n]$. Denoted by $h[n]$.

- **Step Response:** Response of the system to the step signal $u[n]$. Denoted by $s[n]$

$$\alpha\,\delta[n-n_0] \xrightarrow{H} \alpha\, h[n-n_0]$$

- Since we can denote any signal in the form,

$$x[n] = \sum_m \underbrace{x[m]\,\delta[n-m]}_{}$$

scaled and delayed impulses.

$$\Rightarrow \boxed{y[n] = \sum_m x[m]\, h[n-m]}$$

Applying the above 2 results.

$x[m]$ is a constant wrt n.

$$\Rightarrow \boxed{y[n] = x[n] * h[n]}$$

- **Properties of Convolution :** ||| to multiplication

1) Commutativity : $x[n] * y[n] = y[n] * x[n]$ Proved using change of variable

2) Distributive of Addition : $x[n] * (y[n] + z[n]) = x[n] * y[n] + x[n] * z[n]$

3) Associative : $(x[n] * y[n]) * z[n] = x[n] * (y[n] * z[n])$

- **Properties of $\delta[n]$ in Convolution :**

1) $x[n] * \delta[n] = x[n]$

2) $x[n] * \delta[n-\Delta] = x[n-\Delta] \quad \forall \Delta \in \mathbb{Z}$

3) $x[n] \times \delta[n-\Delta] = \underbrace{x[\Delta]\,\delta[n-\Delta]}_{\text{Scaled impulse}} \quad \forall \Delta \in \mathbb{Z}$

→ **Description of a System :-**

1) Working description : Describe the working of the system.

2) Mathematical description : Describe the mathematical function relating input and output.

3) Impulse Response (LTI) : Describe the impulse response of the LTI system. Output can be calculated through convolution.

- Properties of a System through Impulse Response :-

- Causal Systems : $h[n] = 0 \ \forall \ n \in \mathbb{Z}^-$ for a Causal system.

$$y[n] = x[n] * h[n]$$

$$= \sum_m x[m]\,h[n-m]$$

should be zero

$$= \sum_{m=-\infty}^{n} x[m]\,h[n-m] + \sum_{m=n+1}^{\infty} x[m]\,h[n-m]$$

$$\Rightarrow h[n-m] = 0 \ \forall \ m \geq n+1$$

$$m - n \geq 1$$

$$n - m \leq -1 \Rightarrow n - m < 0$$

$$\Rightarrow h[r] = 0 \ \forall \ r < 0$$

- Stable Systems : $\sum_n |h[n]|$ should be bounded for stability

$$|y[n]| = \left| \sum_m x[m]\,h[n-m] \right|$$

$$\Rightarrow |y[n]| \leq \sum_m |x[m]|\,|h[n-m]|$$ Δle inequality

$$\Rightarrow |y[n]| \leq \sum_m B\,|h[n-m]| = B \sum_k |h[k]|$$

$\therefore$ If $\sum |h[k]|$ is bounded, $|y[n]|$ becomes bounded. (Sufficient Condition)

Note:

If $A \Rightarrow B$, then A is sufficient for B

If $A \Leftarrow B$, then A is Necessary for B

$A \Leftrightarrow B$, A is Necessary and sufficient (iff)

Example: Derive the impulse response of $y[n] = x[n] + \alpha y[n-1]$

$$y[n] = x[n] + \alpha y[n-1]$$

$$= x[n] + \alpha\left(x[n-1] + \alpha y[n-2]\right)$$

$$= x[n] + \alpha\left(x[n-1] + \alpha\left(x[n-2] \ldots\ldots\right.\right.$$

$$= x[n] + \alpha x[n-1] + \alpha^2 x[n-2] + \alpha^3 x[n-3] + \ldots$$

$$= \sum_{i=0}^{n} \alpha^i x[n-i]$$

[Assuming initial rest, ie $y[n] = 0 \ \forall \ n < 0$]

$$h[n] = \sum_{i=0}^{n} \alpha^i \delta[n-i]$$

$$h[n] = \alpha^n$$

→ Linear Constant Coefficient Difference Equations :- (LCCDE)

- First order difference equation : $y[n] = x[n] - x[n-1]$
- Generic Difference Eqn : $y[n] = \sum_{p=0}^{n_1} a_p x[n-p] + \sum_{q=1}^{n_2} b_q y[n-q]$

- If $x[n] = e^{j\omega n}$ is the i/p signal for an LTI system H,

$$y[n] = h[n] * e^{j\omega n}$$

$$= \sum_{m=-\infty}^{\infty} h[m]\, e^{j\omega(n-m)}$$

$$= e^{j\omega n} \underbrace{\sum_{m} h[m]\, e^{-j\omega m}}_{C}$$

$$y[n] = C e^{j\omega n} = \underline{\underline{C x[n]}}$$

For any LTI system H, $e^{j\omega n}$ is an Eigensignal of H. The output waveform will be a scaled and phase-shifted input waveform.

→ <u>Discrete Time Fourier Transform</u> :-

- The Discrete Time Fourier Transform of a discrete signal $x[n]$ is defined as,

$$X(\omega) = \sum_{-\infty}^{\infty} x[n]\, e^{-j\omega n}$$ → Analysis Equation

Applying this in the Eigensignal concept,

$$y[n] = e^{j\omega_0 n} \sum_{m=-\infty}^{\infty} h[m]\, e^{-j\omega_0 m}$$

$$\Rightarrow y[n] = e^{j\omega_0 n} H(\omega_0)$$

$H(\omega)$ → FT of $h[n]$, ie, system function

- For a discrete signal,

$$X(\omega + 2\pi) = X(\omega)$$

$X(\omega)$ is continuous in ω.

- <u>Inverse DTFT</u> :-

$$x[n] \xleftrightarrow{\text{DTFT}} X(\omega)$$

- We only need the values of $X(\omega)$ for $\omega \in [-\pi, \pi]$ or $[0, 2\pi]$, due to its periodicity.

- Inverse DTFT is done as follows,

$$X(\omega) e^{j\omega m} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} e^{j\omega m}$$

$$= \int_0^{2\pi} X(\omega) e^{j\omega m} d\omega = \int_0^{2\pi} \sum_{n=-\infty}^{\infty} x[n] e^{j\omega(m-n)} d\omega$$

$$= \sum_{n=-\infty}^{\infty} x[n] \int_0^{2\pi} e^{j\omega(m-n)} d\omega$$

$$\Rightarrow x[m] = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X(\omega) e^{j\omega m} d\omega$$

$\rightarrow$ Synthesis Equation

<u>Examples</u>: Compute DTFT of

1) $x[n] = a^n u[n] \qquad |a| < 1$

$$X(\omega) = \sum x[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \frac{1}{1 - ae^{-j\omega}}$$ [Sum of ∞ GP]

2) $x[n] = \delta[n]$

$$X(\omega) = \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\omega n}$$

$$= e^{-j\omega(0)} = 1 \quad \forall \omega$$

3) $x[n] = \begin{cases} 1, & -M \leq n \leq M \\ 0, & \text{otherwise} \end{cases}$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = \sum_{n=-M}^{M} e^{-j\omega n}$$

$$= \frac{e^{j\omega M}(1 - e^{-j\omega 2M})}{1 - e^{-j\omega}}$$

$$X(\omega) = \frac{e^{j\omega M} - e^{-j\omega M}}{1 - e^{-j\omega}}$$

- If $X(\omega)$ is not periodic with period 2π, it cannot be a valid DTFT of a discrete signal.

<u>Note:</u> $\int_{-\infty}^{\infty} \delta(t)\, x(t)\, dt = x(0)$

<u>Note 2:</u> If one waveform in time domain gives another waveform in Fourier domain, the same waveform in Fourier domain gives the latter

waveform in time domain, i.e., the shape of waveforms can be switched from time to Fourier and viceversa.

→ **Properties of DTFT:-**

1) **Linearity**:

$$x_1[n] \xleftrightarrow{DTFT} X_1(\omega)$$
$$x_2[n] \xleftrightarrow{DTFT} X_2(\omega)$$

$$\Rightarrow \alpha x_1[n] + \beta x_2[n] \xleftrightarrow{DTFT} \alpha X_1(\omega) + \beta X_2(\omega)$$

Can be proved by the linear nature of Integration.

2) **Time Shift**:-

$$x[n] \xleftrightarrow{DTFT} X(\omega)$$
$$\Rightarrow x[n-n_0] \xleftrightarrow{DTFT} X(\omega)e^{-j\omega n_0}$$

Proof:

$$X'(\omega) = \sum_n x[n-n_0]\, e^{-j\omega n}$$

Take $n - n_0 = m$

$$\Rightarrow X'(\omega) = \sum_m x[m]\, e^{-j\omega(m+n_0)}$$
$$= \sum x[m]\, e^{-j\omega m} \cdot e^{-j\omega n_0}$$
$$\Rightarrow \underline{X'(\omega) = X(\omega)\, e^{-j\omega n_0}}$$

Since $|e^{-j\omega n_0}| = 1$, the magnitude spectrum is unaffected. Only the phase spectrum changes.

3) Frequency Shift :-

$$X(\omega) \xleftrightarrow{DTFT} x[n]$$

$$\Rightarrow \boxed{X(\omega-\omega_0) \xleftrightarrow{DTFT} x[n]\,e^{j\omega_0 n}}$$

4) Symmetry:

$$x[n] \xleftrightarrow{DTFT} X(\omega)$$

real	$X(\omega) = X^*(-\omega)$
real and even	real and even
real and odd	imag. and odd.

Example: Is frequency shifting an LTI system?

$$Y(\omega) = X(\omega-\omega_0)$$

$$\Rightarrow \quad y[n] = x[n]\,e^{j\omega_0 n}$$

o/p delay:

$$y[n-n_0] = x[n-n_0]\,e^{j\omega_0(n-n_0)}$$

i/p delay:

$$y_2[n] = x[n-n_0]\,e^{j\omega_0 n}$$

$$\Rightarrow \quad y[n-n_0] \neq y_2[n]$$

System is not time-invariant.

5) Differentiation in Frequency :-

$$x[n] \xleftrightarrow{DTFT} X(\omega)$$

$$-jnx[n] \xleftrightarrow{DTFT} \frac{d}{d\omega} X(\omega)$$

$$X(\omega) = \sum_n x[n]\, e^{-j\omega n}$$

$$\frac{d}{d\omega} X(\omega) = \sum_n \frac{d}{d\omega} x[n] e^{-j\omega n} = \sum_n x[n] \frac{d}{d\omega}\, e^{-j\omega n}$$

$$= \sum_n x[n]\, (-jn)\, e^{-j\omega n}$$

$$= \sum_n \left(-jnx[n]\right) e^{-j\omega n}$$

6) Convolution :-

$$x[n] \xleftrightarrow{DTFT} X(\omega) \qquad y[n] \xleftrightarrow{DTFT} Y(\omega)$$

$$x[n] * y[n] \xleftrightarrow{DTFT} X(\omega) \cdot Y(\omega)$$

$$F\left(x[n] * y[n]\right) = \sum_n \left(\sum_m x[m]\, y[n-m]\right) e^{-j\omega n}$$

$$= \sum_n \left(\sum_m x[m]\, y[n-m]\; e^{-j\omega(m+n-m)}\right)$$

$$= \sum_n \left(\sum_m x[m]\, y[n-m]\, e^{-j\omega m}\; e^{-j\omega(n-m)}\right)$$

$$= \sum_n \left(\sum_m x[m]\, e^{-j\omega m}\; y[n-m]\, e^{-j\omega(n-m)}\right)$$

$$= \sum_m x[m] e^{-j\omega m} \left(\sum_n y(n-m)\, e^{-j\omega(n-m)} \right)$$

$$= \left(\sum_m x[m]\, e^{-j\omega m} \right) Y(\omega)$$

$$= \underline{X(\omega) \cdot Y(\omega)}$$

7) Parseval's Relation :-

Energy is conserved across time and frequency domains, ie,

$$\sum_n |x[n]|^2 = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} |X(\omega)|^2 d\omega$$

8) Multiplication in Time Domain :-

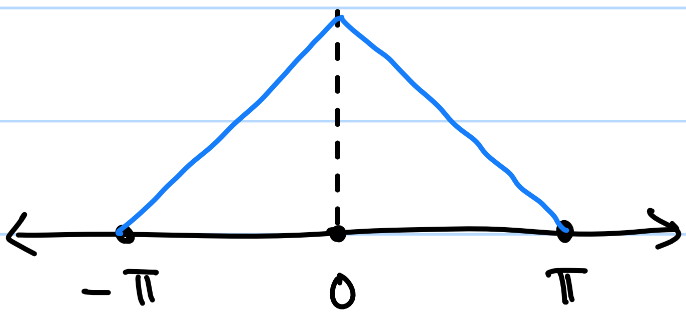
$$x[n] \xleftrightarrow{DTFT} X(\omega) \quad , \quad y[n] \xleftrightarrow{DTFT} Y(\omega)$$

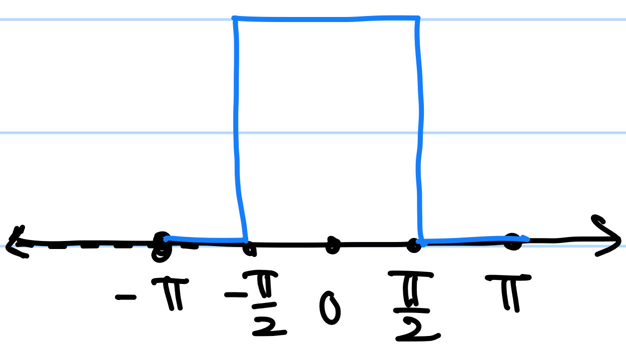
$$x[n] \cdot y[n] \xleftrightarrow{DTFT} \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X(\alpha) Y(\omega - \alpha)\, d\alpha$$

Periodic Convolution

Example:

Perform periodic convolution of





→ Convolution and Frequency Selectivity :-

- We know that,

$$e^{j\omega_0 n} \longrightarrow \boxed{H} \longrightarrow H(\omega_0)\, e^{j\omega_0 n}$$

$$x[n] \longrightarrow \boxed{H} \longrightarrow y[n] = x[n] * h[n]$$

$$x[n] \updownarrow X(\omega) \qquad\qquad y[n] \updownarrow Y(\omega) = X(\omega)\, H(\omega)$$

$\therefore$ For all LTI system $Y(\omega) = X(\omega)\, H(\omega)$

Because of the above relation, $Y(\omega)$ cannot contain frequencies that $X(\omega)$ does not have, ie, $H(\omega)$ is like a gate / filter.

If $X(\omega_0) = 0$, then $Y(\omega_0) = X(\omega_0)\, H(\omega_0) = \underline{\underline{0}}$.

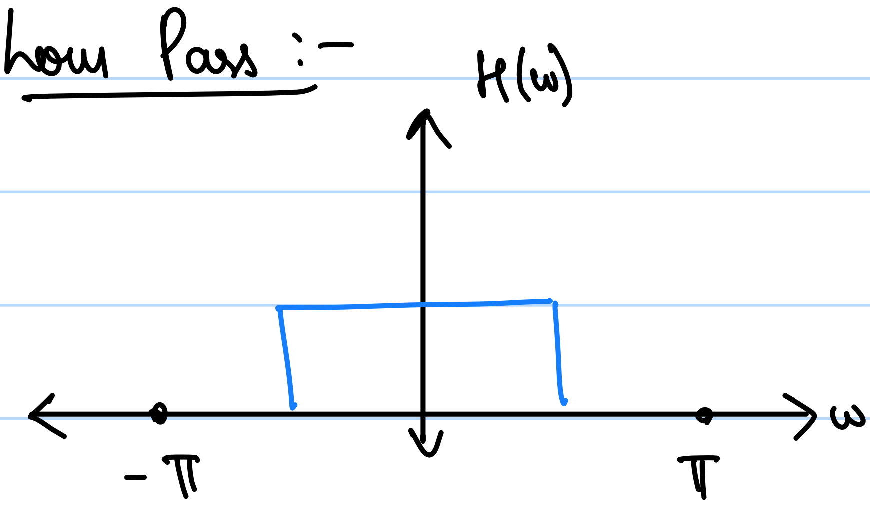
$H(\omega)$ – Frequency Response
$|H(\omega)|$ – Magnitude Response
$\measuredangle H(\omega)$ – Phase Response

The above mentioned multiplication consequence is termed as frequency selectivity.

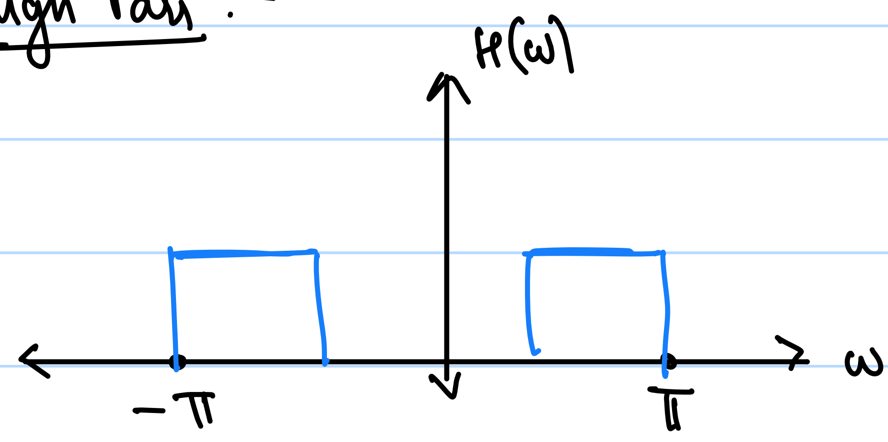
- The behaviour of $H(\omega)$ can be modified to make filters.
  1) Low Pass & High Pass Filter
  2) Band stop & Band pass Filter

3) Notch & All pass filter.

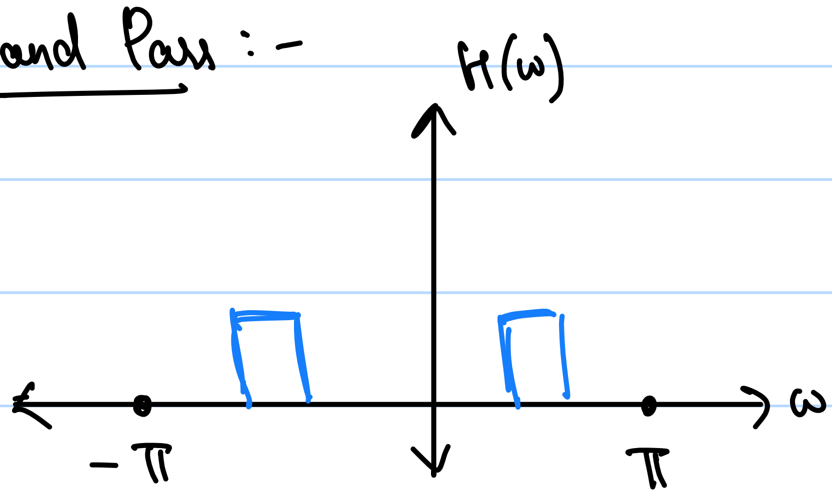
Low Pass :-



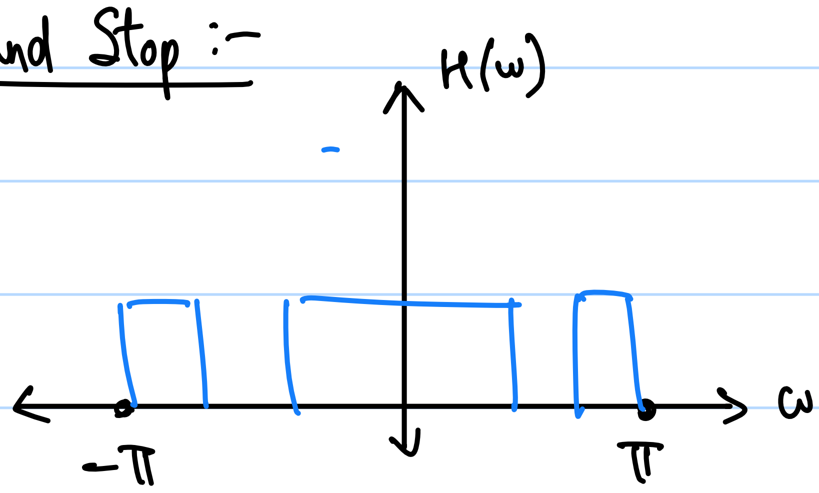
High Pass :-



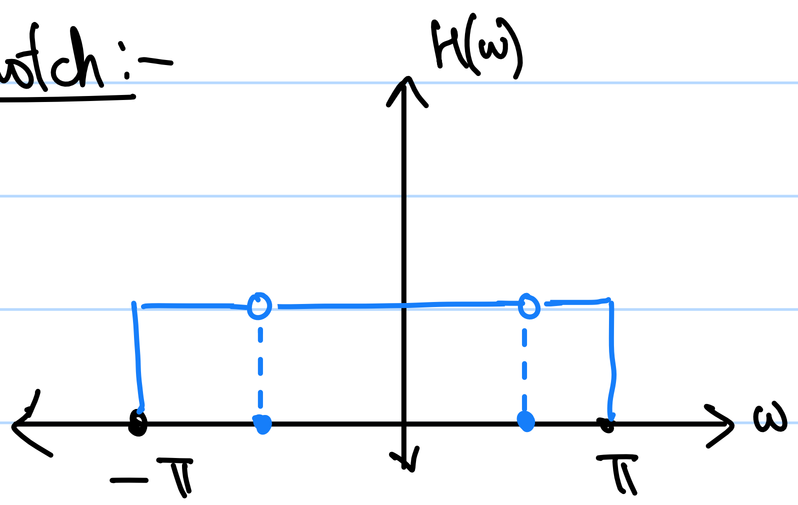
Band Pass :-



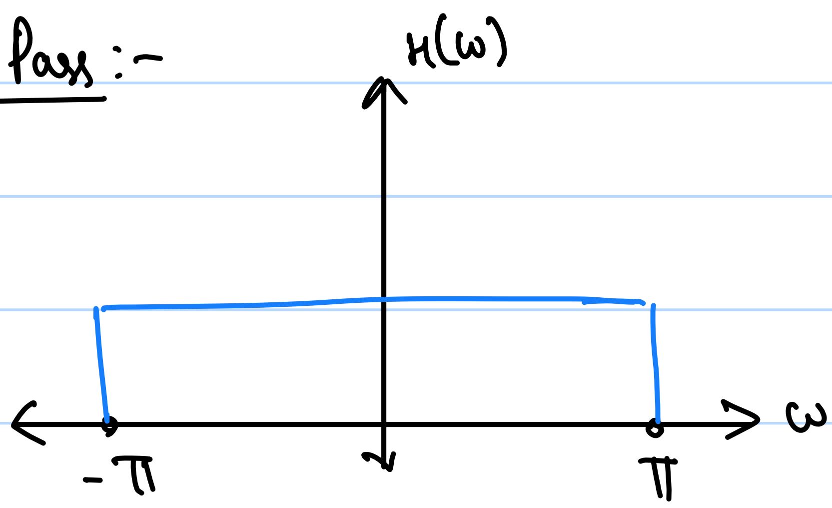
Band Stop :-



Notch :-



All Pass :-



H(ω) is symmetric in all the filters since h[n] is a real valued impulse response.

All-pass filters are used in delay systems, since phase can be non-zero.

- **Constraints for implementing Ideal filters :-**

1) They are not BIBO stable. (Bounded input $\neq$ Bounded output)

2) They have infinite impulse responses.

3) They are not causal.

Example: Identify the filter type of the following LTI systems.

a) $y[n] = \frac{1}{2}(x[n] + x[n-1])$

$$\Rightarrow Y(s) = \frac{1}{2}\left(X(s) + X(s)e^{-j\omega}\right) \Rightarrow$$

$$H(s) = \frac{1}{2}(1 + e^{-j\omega})$$

$$= \frac{1}{2}(1 + \cos(\omega) - j\sin(\omega))$$

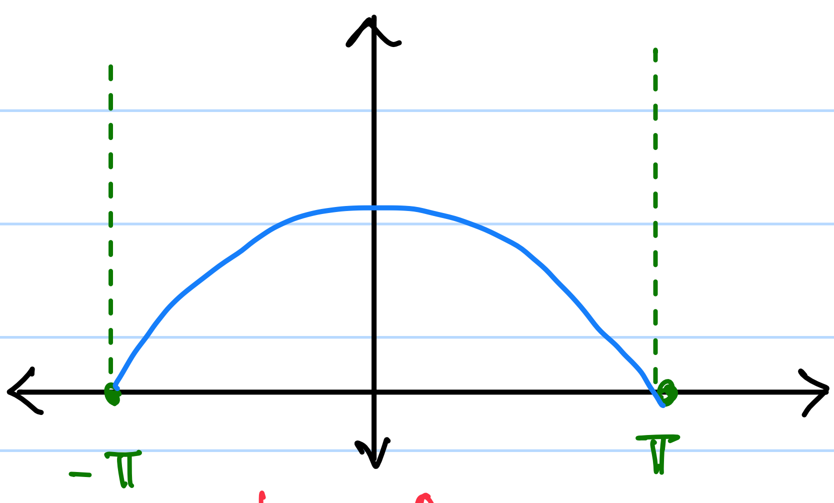
$$= \frac{1}{2} + \frac{1}{2}\cos(\omega) - \frac{j}{2}\sin(\omega)$$

$$|H(s)| = \sqrt{\frac{1}{4}(1+\cos\omega)^2 + \frac{1}{4}\sin^2\omega}$$

$$= \sqrt{\frac{1}{4} + \frac{1}{2}\cos\omega + \frac{1}{4}\cos^2\omega + \frac{1}{4}\sin^2\omega}$$

$$= \sqrt{\frac{1}{2} + \frac{1}{2}\cos\omega}$$

$$= \frac{1}{\sqrt{2}}\sqrt{1 + \cos(\omega)}$$



Low Pass

b) $y[n] = \frac{1}{2}(x[n] - x[n-1])$

$$\Rightarrow Y(s) = \frac{X(s)}{2}(1 - e^{-j\omega})$$

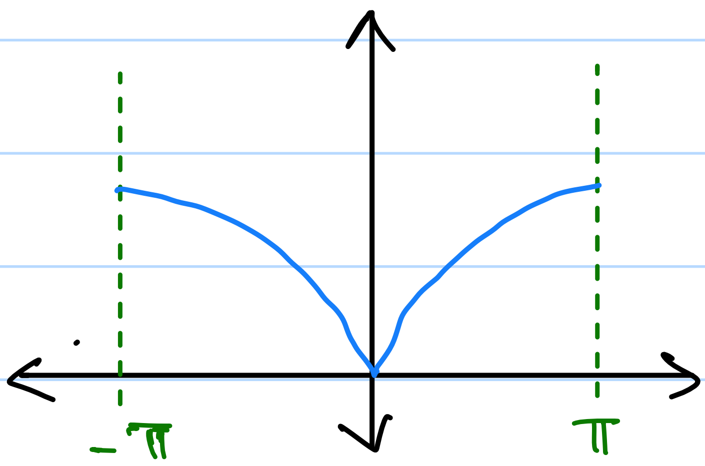
$$\Rightarrow H(s) = \frac{1 - e^{-j\omega}}{2}$$

$$|H(s)| = \frac{1}{2}\sqrt{(1 - e^{-j\omega})(1 - e^{j\omega})}$$

$$= \frac{1}{2}\sqrt{1 - e^{-j\omega} - e^{j\omega} + \cancel{e^{-j\omega + j\omega}}\,1}$$

$$= \frac{1}{2}\sqrt{2 - (2\cos(\omega))}$$

$$= \frac{1}{\sqrt{2}}\sqrt{1 - \cos(\omega)}$$



High Pass

c) $h[n] = a^n u[n]$

$$H(\omega) = \sum_n a^n u[n] e^{-j\omega n} = \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

$$= \frac{1}{1 - ae^{-j\omega}} \times \frac{(1 - ae^{j\omega})}{(1 - ae^{j\omega})}$$

$$= \frac{1 - ae^{j\omega}}{1 - ae^{j\omega} - ae^{-j\omega} + a} = \frac{1 - ae^{j\omega}}{1 + a - a(2\cos\omega)}$$

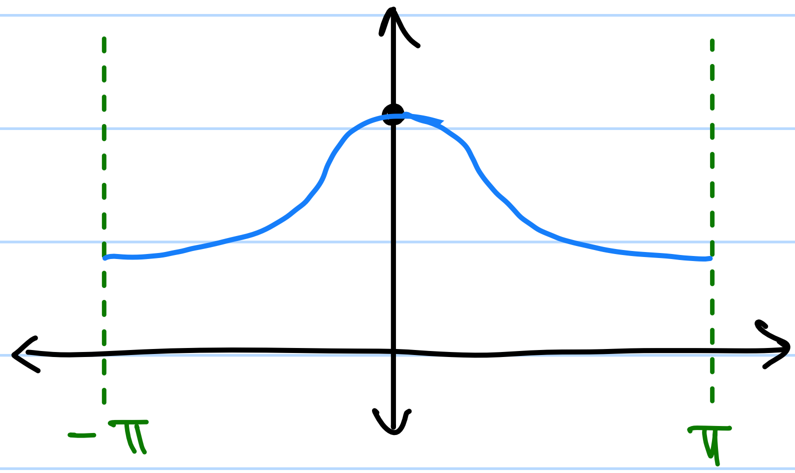
$$= \frac{1 - ae^{j\omega}}{1 + a - 2a\cos\omega}$$

$$|H(\omega)| = \frac{1}{1+a-2a\cos(\omega)} \sqrt{(1-ae^{j\omega})(1-ae^{-j\omega})}$$

$$= \frac{1}{1+a-2a\cos\omega} \sqrt{1+a-ae^{j\omega}-ae^{-j\omega}}$$

$$= \frac{1}{1+a-2a\cos\omega} \sqrt{1+a-2a\cos\omega}$$

$$= \frac{1}{\sqrt{1+a-2a\cos\omega}}$$



Low Pass

→ Continuous Time Fourier Transform :-

$$x(t) \xleftrightarrow{CTFT} X(\omega)$$

- An extension of FSR to non-periodic signals.
- $$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$$

$$e^{j\omega_0 t} \longrightarrow \boxed{LTI} \longrightarrow H(\omega_0)e^{j\omega_0 t}$$

ie, $e^{j\omega t}$ is an Eigensignal $\forall$ continuous LTI systems.

Important DTFT pairs :-

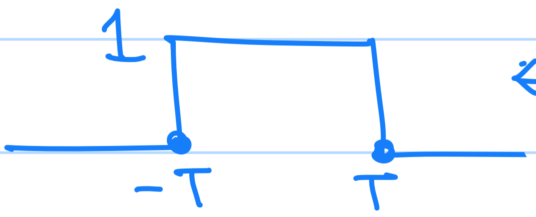
1) $a^n u[n] \xleftrightarrow{DTFT} \dfrac{1}{1-ae^{-j\omega}}$ $\quad \underline{\forall\, |a|<1}$

2) $-a^n u[-n-1] \xleftrightarrow{DTFT} \dfrac{1}{1-ae^{-j\omega}}$ $\quad \underline{\forall\, |a|>1}$

Important CTFT pairs :-

1) $\delta(t) \xleftrightarrow{CTFT} 1$

2) $\delta(t-b) \xleftrightarrow{CTFT} e^{-j\omega b}$

3) 

$\xleftrightarrow{CTFT} \text{sinc}\left(\frac{2\pi}{T}\omega\right)$

4) If $x(t)$ is periodic,

wkt $\quad x(t) = \sum_{k\in\mathbb{Z}} a_k e^{jk\omega_0 t}$ $\quad$ [ω_0 is the angular freq. of $x(t)$]

$$\Rightarrow X(\omega) = \int_{-\infty}^{\infty} \sum_{k\in\mathbb{Z}} a_k e^{jk\omega_0 t} e^{-j\omega t}\, dt$$

$$= \sum_{k\in\mathbb{Z}} a_k \int_{-\infty}^{\infty} e^{jk\omega_0 t} e^{-j\omega t}\, dt$$

This is why FT exists.

By orthogonality of sinusoids, $\rightarrow \left[\forall\, \omega \neq k\omega_0,\ \int_{-\infty}^{\infty} e^{jk\omega_0 t} e^{-j\omega t}\, dt = 0\right]$

$$\Rightarrow X(\omega) = 2\pi \sum_{k\in\mathbb{Z}} a_k\, \delta(\omega - k\omega_0)$$

$\longrightarrow$ Can also be directly proved by point 7

5) $1 \xleftrightarrow{CTFT} 2\pi\,\delta(\omega)$

6) $\sum_{k=-\infty}^{\infty} \delta(t-kT) \xleftrightarrow{CTFT} \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - k\frac{2\pi}{T}\right)$

7) $e^{j\omega_0 t} \xleftrightarrow{CTFT} 2\pi\,\delta(\omega-\omega_0)$

Most of the CTFT pairs are symmetric, with a scaling factor of 2π sometimes.

- For continuous LTI systems, if i/p $x(t)$ is periodic, then o/p $y(t)$ is periodic.

Since $x(t)$ is periodic, $x(t) = \sum_{k\in\mathbb{Z}} a_k e^{jk\omega_0 t}$

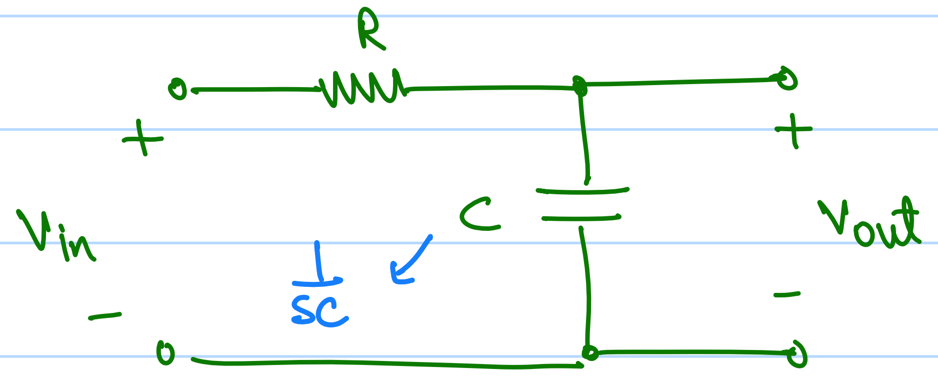
By linearity and Eigensignal nature,

$$y(t) = \sum_{k\in\mathbb{Z}} a_k H(k\omega_0)\, e^{jk\omega_0 t}$$

$$\Rightarrow y(t) = \sum_{k\in\mathbb{Z}} b_k e^{jk\omega_0 t}$$

$\therefore$ $y(t)$ has FSR coefficients $a_k H(k\omega_0)$ $\forall\, k$, making it a periodic signal.

Example: Use FT to find the VTC of the following circuit.



$$Z(\omega) = R + \frac{1}{j\omega C} = \frac{j\omega CR + 1}{j\omega C}$$

$$I(\omega) = \frac{V_{in}(\omega)}{Z(\omega)} = \frac{V_{in}(\omega) j\omega C}{j\omega CR + 1}$$

$$V_{out}(\omega) = I(\omega) \frac{1}{j\omega C} = \frac{V_{in}(\omega)}{j\omega CR + 1}$$

$$\frac{V_{out}(\omega)}{V_{in}(\omega)} = H(\omega) = \frac{1}{j\omega CR + 1}$$

→ Sampling :-

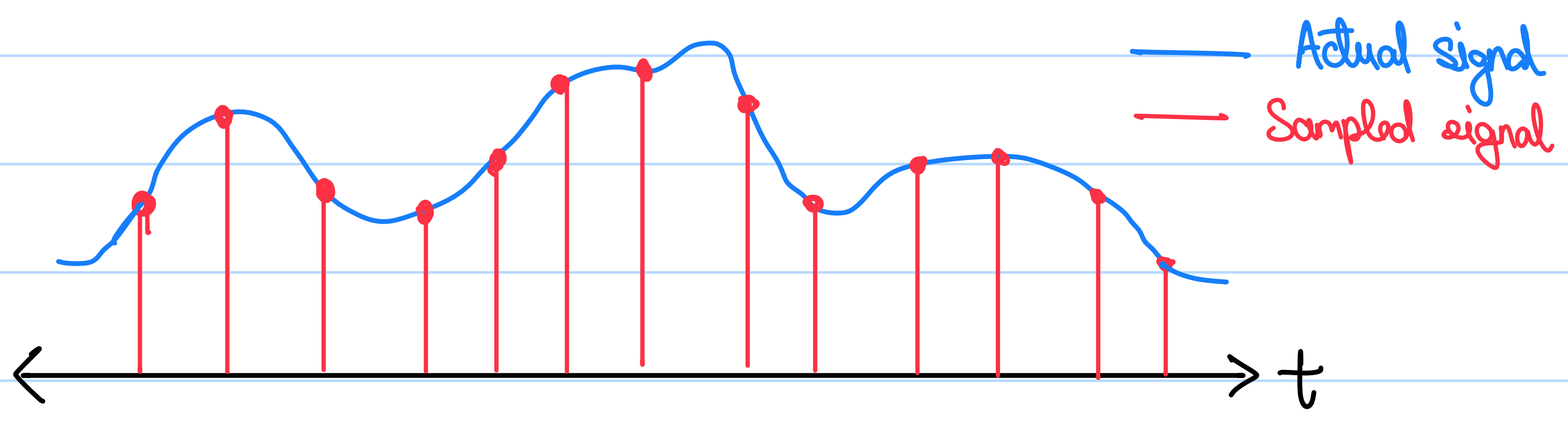
Process of converting continuous time signals into discrete-time signals.

$$x[n] = x_c(nT_s) \quad \forall\, n \in \mathbb{Z} \qquad T_s - \text{Sampling period}$$

- Band-limited Signals :-

- Band : A range of frequencies

- Band-limited signals are those signals for which the spectrum $X(\omega) = 0 \;\forall\; \omega \notin [-B, B]$, where B is a finite constant.



- Clearly, shorter the sampling interval, lesser is the information lost by sampling.

Note:

If $x(t) \xrightarrow{F} X(\omega)$, $X(t) \xrightarrow{F} 2\pi x(-\omega)$

(Duality of FT)

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$$Y(\omega) = \int_{-\infty}^{\infty} X(t)\, e^{-j\omega t}\, dt$$

$$\frac{Y(\omega)}{2\pi} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(t)\, e^{-j\omega t}\, dt$$

Let $\alpha = -t$, $d\alpha = -dt$

$$\Rightarrow \frac{Y(\omega)}{2\pi} = \frac{1}{2\pi}\int_{\infty}^{-\infty} X(-\alpha)\, e^{-j\omega(-\alpha)}\, (-d\alpha)$$

$$\frac{Y(\omega)}{2\pi} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(-\alpha)\, e^{j\omega\alpha}\, d\alpha$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega \xrightarrow{t=\alpha} \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\alpha)\, e^{j\omega\alpha}\, d\alpha = x(\alpha)$$

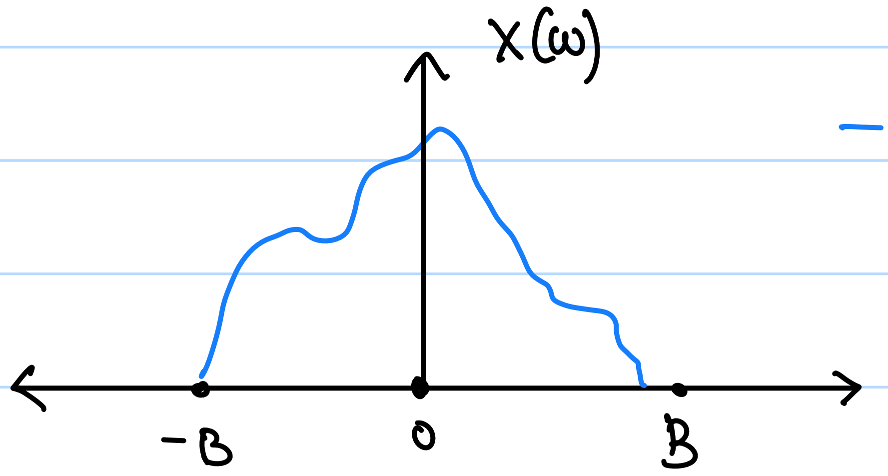
$$x(-\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(-\alpha)\, e^{j\omega\alpha}\, d\alpha$$

$$\Rightarrow \quad \frac{Y(\omega)}{2\pi} = x(-\omega)$$

$$\Rightarrow \quad Y(\omega) = \underline{2\pi x(-\omega)}$$

- If the sampling rate is $f_s = 1/T_s$, the sampled discrete-time signal is of the form,

$$x[n] = x_c(nT_s)$$

- 

$-B, B$ = Frequency Support

<u>Band-limited signals</u>

Examples of Band limited signals : Sinusoids , $\mathrm{sinc}(x)$.

- <u>Sampling Theorem</u> :-

- Sampling generally causes a loss of information. But for band-limited signals, there is a certain sampling frequency above which information is conserved.

- Information loss is due to the high frequency variations in between 2 samples.

- But, band limited signals have a hard limit on how fast these variations will be.

- **Sampling Theorem:** A band-limited signal with max. frequency ω_M can be perfectly recovered from its samples if the sampling frequency ω_s satisfies,

$$\omega_s > 2\omega_M$$

Proof:

Impulse train rep. of sampling : $p(t) = \sum_{n \in \mathbb{Z}} \delta(t - nT_s)$

$$x_p(t) = x_c(t)\, p(t)$$

$$x_p(t) = \sum_{n \in \mathbb{Z}} x_c(nT_s)\, \delta(t - nT_s) \longrightarrow \text{Sampled signal}$$

$$X_p(\omega) = \int_{-\infty}^{\infty} \sum_{n \in \mathbb{Z}} x_c(nT_s)\, \delta(t - nT_s)\, e^{-j\omega t}\, dt$$

$$= \sum_{n \in \mathbb{Z}} \int_{-\infty}^{\infty} x_c(nT_s)\, \delta(t - nT_s)\, e^{-j\omega t}\, dt$$

$$= \sum_{n \in \mathbb{Z}} x_c(nT_s)\, e^{-j\omega n T_s}$$

$$= \sum_{n \in \mathbb{Z}} x_c(nT_s)\, e^{-jk(T_s)\omega} \longrightarrow \underline{\underline{\text{FSR}}}$$

$\Rightarrow$ $X_p(\omega)$ is periodic with period $= \frac{2\pi}{T_s} = \underline{\underline{\omega_s}}$ and FSR coefficients $a_n = x_c(nT_s)$

Also, $X_p(\omega) = X_c(\omega) * P(\omega)$

$$\Rightarrow \quad X_p(\omega) = \int_{-\infty}^{\infty} X_c(\alpha) \, \frac{2\pi}{T_s} \sum_{k \in \mathbb{Z}} \delta\left(\omega - \alpha + k\frac{2\pi}{T_s}\right) d\alpha$$

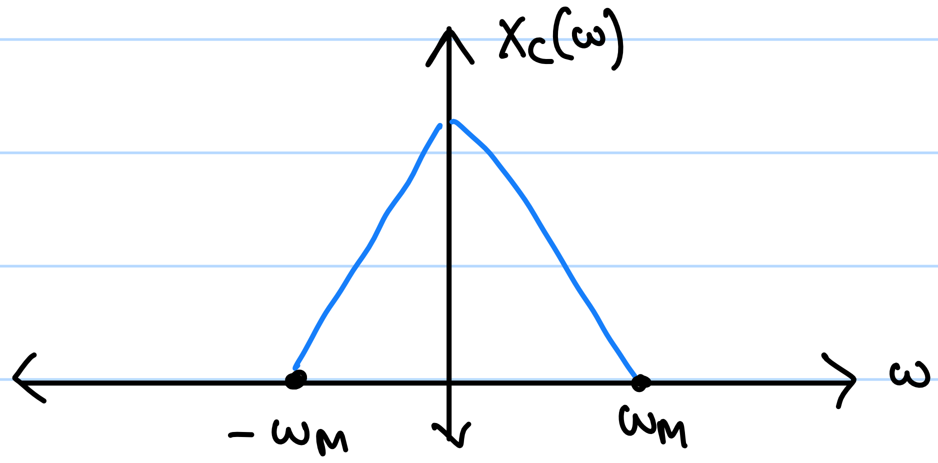
$$= \sum_{k \in \mathbb{Z}} X_c\left(\omega - k\frac{2\pi}{T_s}\right)$$

$$= \sum_{k \in \mathbb{Z}} X_c(\omega - k\omega_s)$$

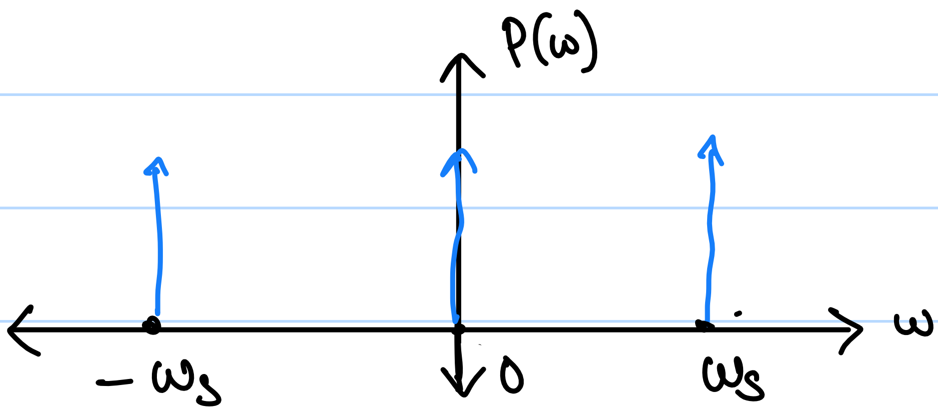
$\Rightarrow$ Within $(-\omega_s, \omega_s)$, $\underline{X_p(\omega) = X_c(\omega)}$

If $\omega_s > \omega_M$, the entire non-zero spectrum of $X_c(\omega)$ will be contained within the interval $(-\omega_s, \omega_s)$ of $X_p(\omega)$, ensuring that all the information is still present in $X_p(\omega)$, in the form of that spectrum.

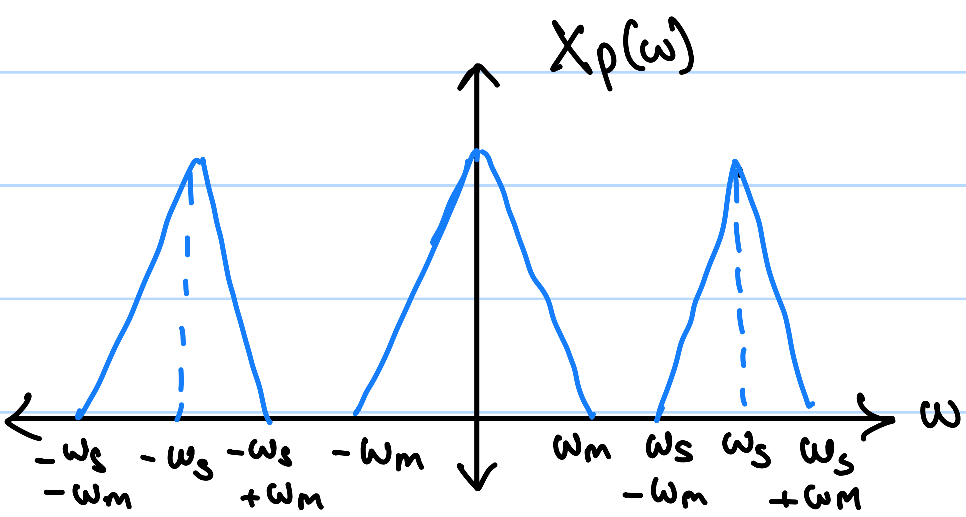
$X_c(\omega)$:



$P(\omega)$:



$X_p(\omega)$:



No overlap (loss of spectrum if $\omega_s > \omega_M$

- To exactly reconstruct $x_c(t)$, we will pass $x_p(t)$ through a low pass filter, with cutoff frequency $\omega_c \in (\omega_M, \omega_s - \omega_M)$

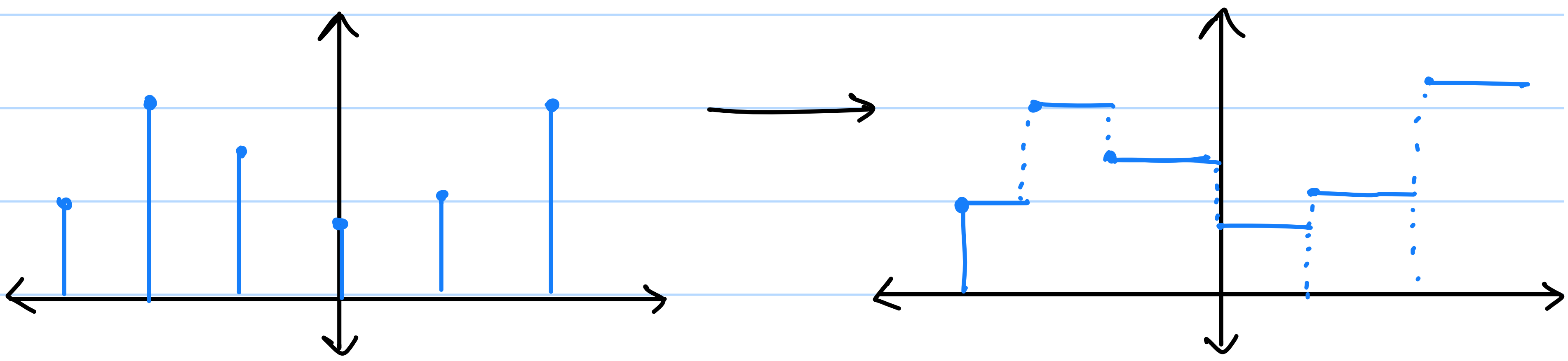
Ideal low pass filter $= \dfrac{\sin(\omega_0 t)}{\pi t}$

$$x_r(t) = \left[\sum_{n=-\infty}^{\infty} x_c(nT_s)\,\delta(t-nT_s)\right] * \frac{\sin(\omega_0 t)}{\pi t} \qquad \omega_0 \in (\omega_M, \omega_s-\omega_M)$$

$$\Rightarrow x_r(t) = \sum_{n=-\infty}^{\infty} x_c(nT_s)\,\frac{\sin(\omega_0(t-nT_s))}{\pi(t-nT_s)} \longrightarrow \text{Weighted sum of sinc functions}$$

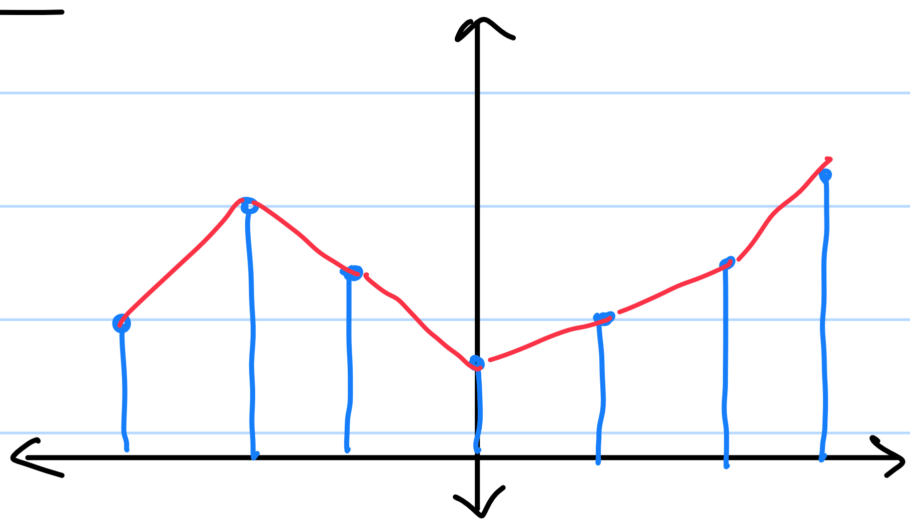
- Reconstruction :-

1) Zero Order Hold / Piecewise Constant :

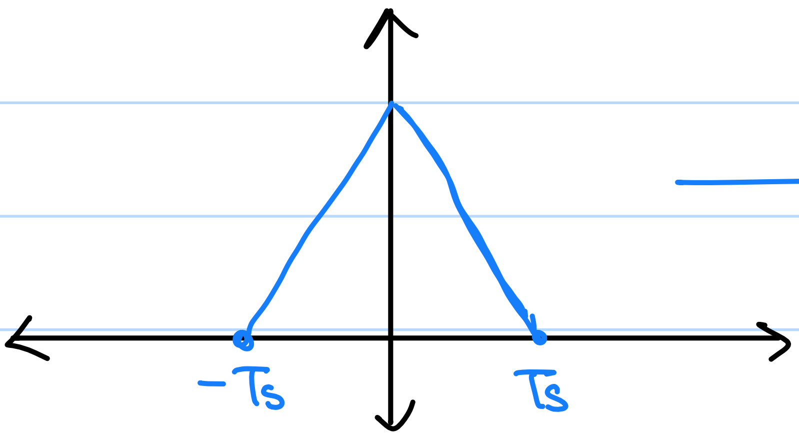


Takes the value of each sample and keeps it constant until the next sample.

2) Linear Interpolation :



$h(t)$:



→ A non ideal low pass filter

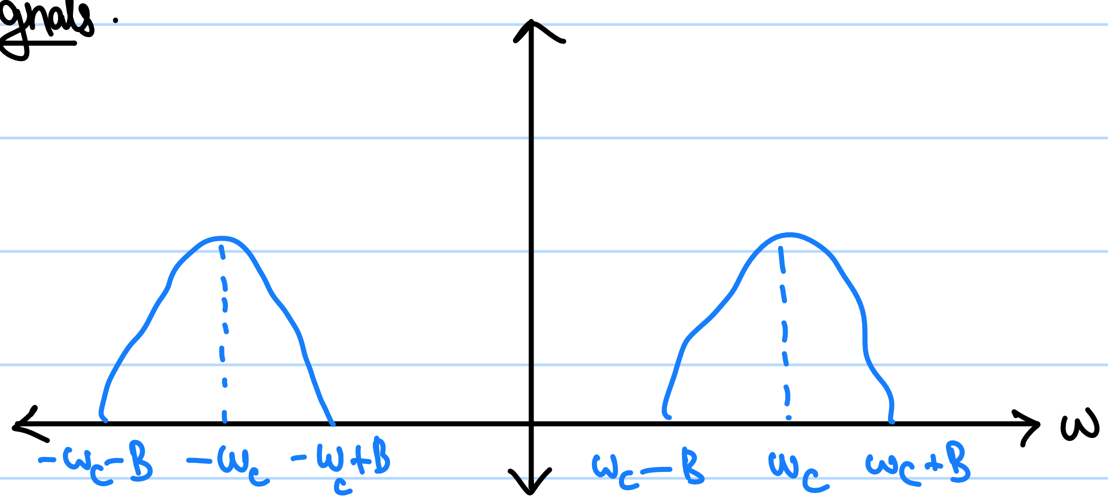
3) Sinc - Based Interpolation :-

$$x_r(t) = x_p(t) * \operatorname{sinc}(\omega_s \cdot t)$$

This works since $\operatorname{sinc}(t)$ acts like a non ideal low pass filter.

- Beyond Sampling Theorem :-

i) Band Pass Signals:



Band pass signals are those signals which are non zero for intervals not centered at origin.

Such signals can be shifted to the origin, reducing the Nyquist frequency of the signal.

2) Frequency Compression :-

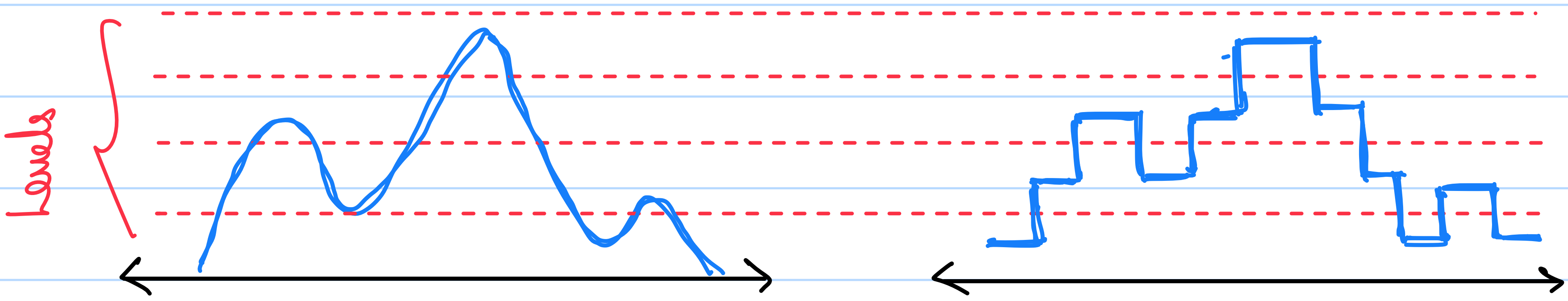
Using certain algorithms, a band limited signal can have a reduced bandwidth, albeit with the possibility of some information loss, but not as much as due to aliasing.

→ Quantization :-

- Sampling of the amplitude.

- Since the amplitude of a signal can be any real number, including irrational no.s, the bits needed to store a signal can become extremely large.

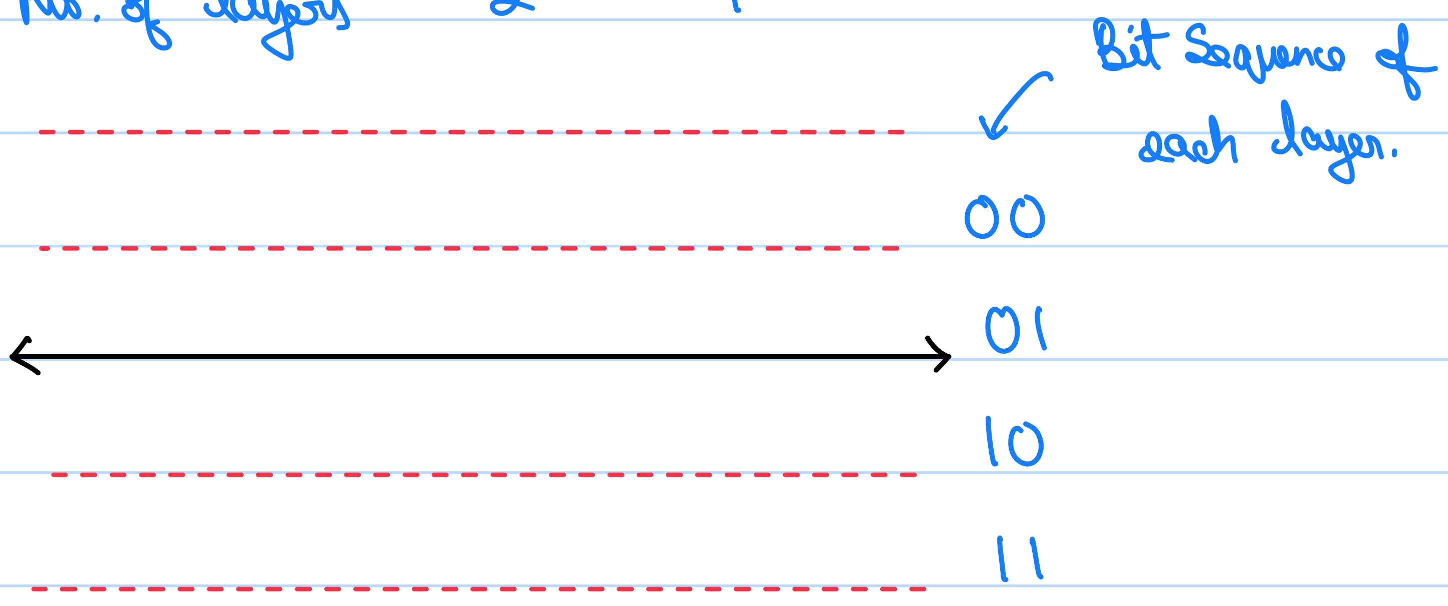
- Quantization divides the Amplitude interval into levels and assigns a binary sequence to each level.



- Sampling and Quantization are done in the ADC (Analog-Digital Converter)

- The no. of layers used in Quantization is 2^B, where B is the no. of bits assigned to each level.

If $B = 2$, No. of layers $= 2^2 = 4$



- **Channels:** The no. of independent signals in a piece of information.

  An RGB Image has 3 channels – R, G, B.
  A Stereo Audio has 2 channels – Left, Right.

- Reconstruction of a Quantized signal is never guaranteed to be exact, unlike in Sampling, which has the Nyquist theorem.

→ **Overall Digital System :-**

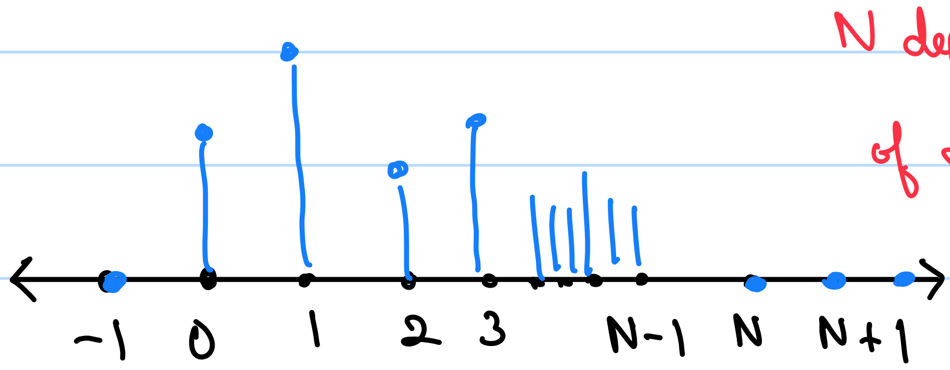
$x(t)$ → Anti-Aliasing filter → ADC → Digital Processor → DAC → $y(t)$

→ Discrete Fourier Transform :- (Different from DTFT)

- For signals $x[n]$ such that,

$$x[n] = 0 \begin{cases} n < 0 \\ n \geq N \end{cases}$$

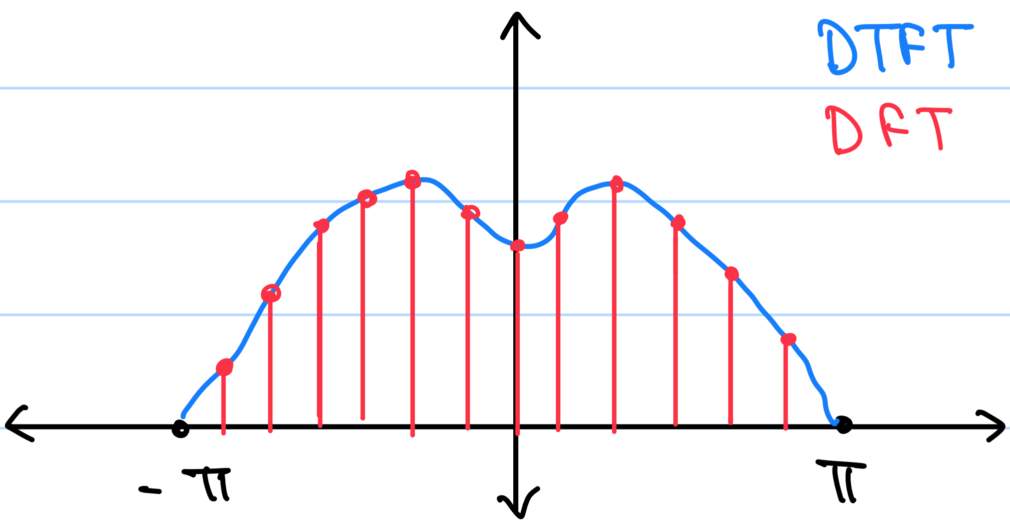
→ ie, a practical time-limited signal of N samples.
N depends on the sampling rate of the signal.



- In DFT, we consider the DTFT of the signal only at discrete frequencies, ie it is a discrete-frequency Fourier Transform.

- Consider $\omega_k = \frac{2\pi}{N} k$, $k \in \{0,1,2,\dots,N-1\}$

$$\boxed{X(\omega_k) = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}}, \quad k \in \{0,1,2,\dots,N-1\}$$



DFT gives $X(\omega_k)$ for $k \in \{0,1,2,\dots N-1\}$, which gives us N linear equations in N unknowns.

$$\left(\begin{array}{lcl} N \text{ linear equations} & = & \text{Each of the } X(\omega_k) \text{ equations} \\ N \text{ unknowns} & = & \text{Each of the samples of } x[n] \end{array}\right)$$

$$x(nT_s) = x[n] \xleftrightarrow{\text{DFT}} X[k] = X\left(\frac{2\pi}{N}k\right)$$

- DFT is a change of basis operation on the $x[n]$ vector (time basis to frequency basis)

- Inverse DFT,

$$s_m = \sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}km}$$

To get $x[n]$,

$$s_m = \sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}km}$$

$$= \sum_{k=0}^{N-1} \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}\, e^{j\frac{2\pi}{N}km}$$

$$= \sum_{n,k=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}\, e^{j\frac{2\pi}{N}km}$$

$$= \sum_{n,k=0}^{N-1} x[n]\, e^{j\frac{2\pi}{N}k(m-n)}$$

$$= \sum_{n=0}^{N-1} x[n] \sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}k(m-n)}$$

$$\sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}k(m-n)} = \text{Sum of } N^{th} \text{ roots of } 1 = \begin{cases} 0, & m \neq n \\ N, & m = n \end{cases}$$

$$\Rightarrow \underline{\underline{s_m = N x[m]}}$$

$$\Rightarrow \quad x[m] = 8m / N$$

$$\Rightarrow \quad \boxed{x[m] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}km}}$$

- Let $b_k[n] = e^{j\frac{2\pi}{N}kn}$. $b_k[n]$ are all periodic signals and all have a common period of N,

$$b_k[n+N] = b_k[n] \quad \forall \; k \in \{0,1,2,\dots,N-1\}$$

- DFT as a Linear Transformation :-

- Denote $W_N = e^{-j\frac{2\pi}{N}}$. All $b_k[n]$ are just powers of W_N. Using W_N, we can represent DFT as,

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ \vdots \\ X[N-1] \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \cdots & & & \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)^2} \end{bmatrix}}_{\text{Change of Basis Matrix}} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{bmatrix}$$

$$\Rightarrow \quad \boxed{\bar{X} = \bar{F}\,\bar{x}}, \quad \bar{F} = \text{DFT Matrix}$$

$$\boxed{\bar{x} = \bar{F}^{-1}\bar{X}} \quad \bar{F}^{-1} = \text{inv. DFT Matrix}$$

$$\boxed{\bar{F} = [W_N^{ij}]}, \quad i,j \in \{0,1,2,\dots,N-1\}$$

↳ zero-indexed matrix.

- We can see that the no. of operations to perform the matrix multiplication is N^2. $\therefore$ The time complexity of DFT is $O(n^2)$.

- However, using certain properties of the DFT matrix, we can reduce the complexity to $O(n\log n)$. This faster algorithm is termed as FFT (Fast Fourier Transform).

Example: Find DFT of $x[n] = \begin{cases} 1, & n=0 \\ 0, & n = 1,2,3 \dots, N-1 \end{cases}$

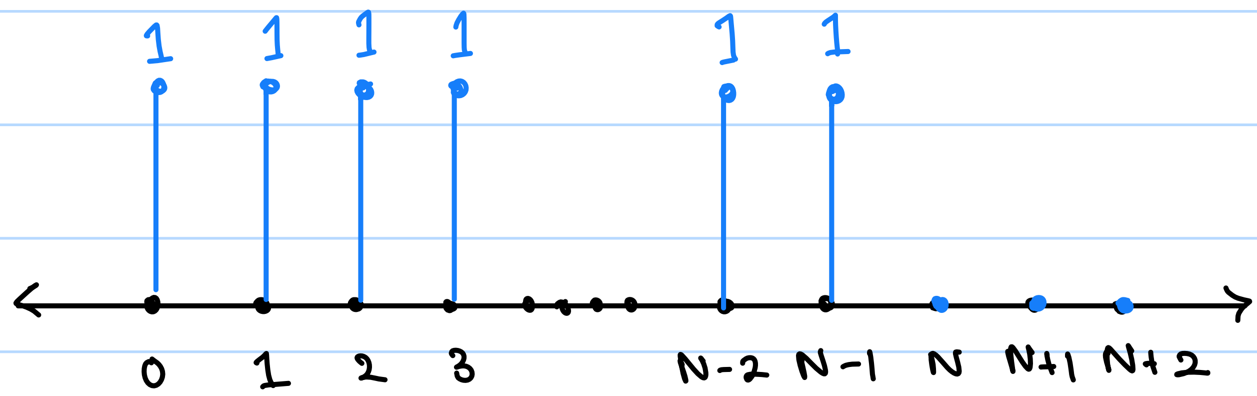
We cannot take $N = 2$, as N is already defined.

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

$$= e^{-j\frac{2\pi}{N}k(0)}$$

$$= 1$$

$$X[k] = \begin{cases} 1, & k = \frac{2\pi}{N}n, \; n \in [0 : N-1] \\ 0, & \text{otherwise} \end{cases}$$



Example: Find DFT of $x[n] = 1, \ n \in \{0, 1, 2, \dots N-1\}$

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}kn}$$

If $k \neq 0$,

$$= 1 \left(\frac{1 - e^{-j\frac{2\pi}{N}kN}}{1 - e^{-j\frac{2\pi}{N}k}} \right)$$

$$= 1 \left(\frac{1 - e^{-j2\pi k}}{1 - e^{-j\frac{2\pi}{N}k}} \right)$$

$$= 1 \left(\frac{1 - 1}{1 - e^{-j\frac{2\pi}{N}k}} \right) = \underline{\underline{0}}$$

If $k = 0$,

$$X[k] = \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}(0)(n)}$$

$$= \sum_{n=0}^{N-1} 1 = N$$

$$\Rightarrow X[k] = \begin{cases} N, & k = 0 \\ 0, & k \neq 0 \end{cases}$$

- 2 point DFT matrix $(N=2)$, $(x[n] : x[0], x[1], \ X[k] : X[0], X[1])$

$$F = \begin{bmatrix} 1 & 1 \\ 1 & \omega_N \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & e^{-j\pi} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

- 4 point DFT matrix,

$$F = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{-j\frac{\pi}{2}} & e^{-j\pi} & e^{-j\frac{3\pi}{2}} \\ 1 & e^{-j\pi} & e^{-j2\pi} & e^{-j3\pi} \\ 1 & e^{-j\frac{3\pi}{2}} & e^{-j3\pi} & e^{-j\frac{9\pi}{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

Note: F matrix is symmetric, ie, $F^T = F$.

- In 2 point DFT,

$$\begin{bmatrix} X[0] \\ X[1] \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \end{bmatrix}$$

$$= \begin{bmatrix} x[0] + x[1] \\ x[0] - x[1] \end{bmatrix}$$

$$\Rightarrow X[0] = x[0] + x[1]$$
$$X[1] = x[0] - x[1]$$

|||ly we can write down expressions for 4 point DFT as well.

- **I-DFT :-**

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}kn}$$

$$x[n+rN] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}kn}\, \overset{1}{\cancel{e^{j\frac{2\pi}{N}krN}}}$$

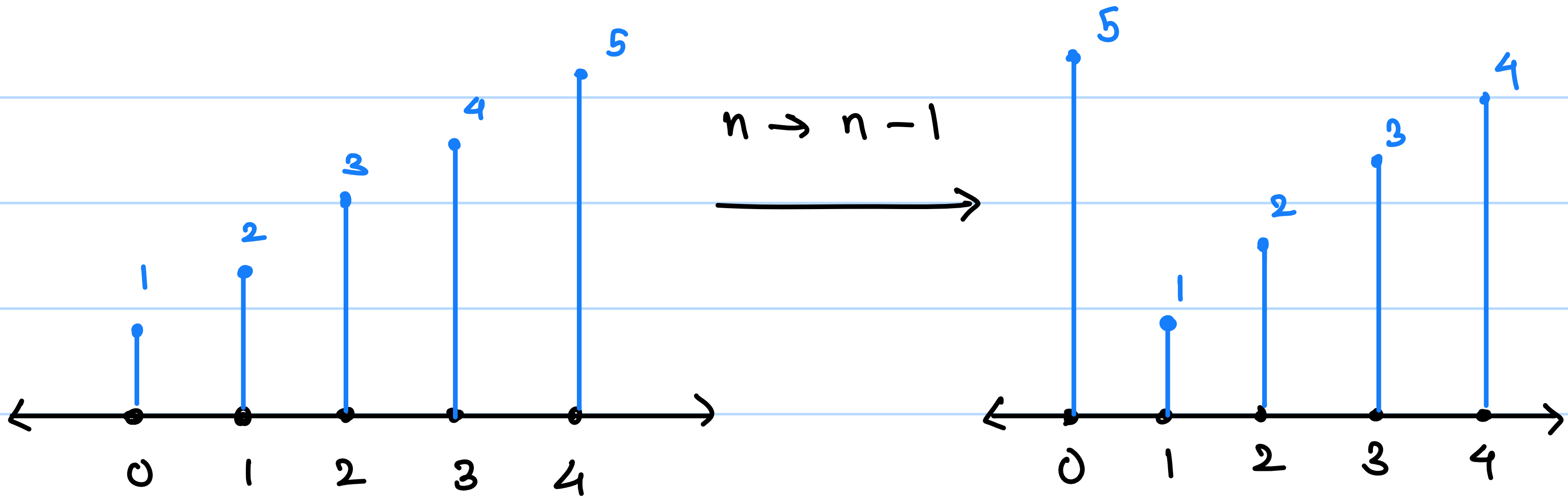
$$\Rightarrow \quad x[n+rN] = x[n]$$

- To prevent that periodic behavior from affecting our analysis, we perform zero padding, ie, add zero scalars to the end of our signal vector, to increase N, giving us a better sampling of $X(\omega)$, to get $X[k]$.

- Zero padding works for finite length signals only.

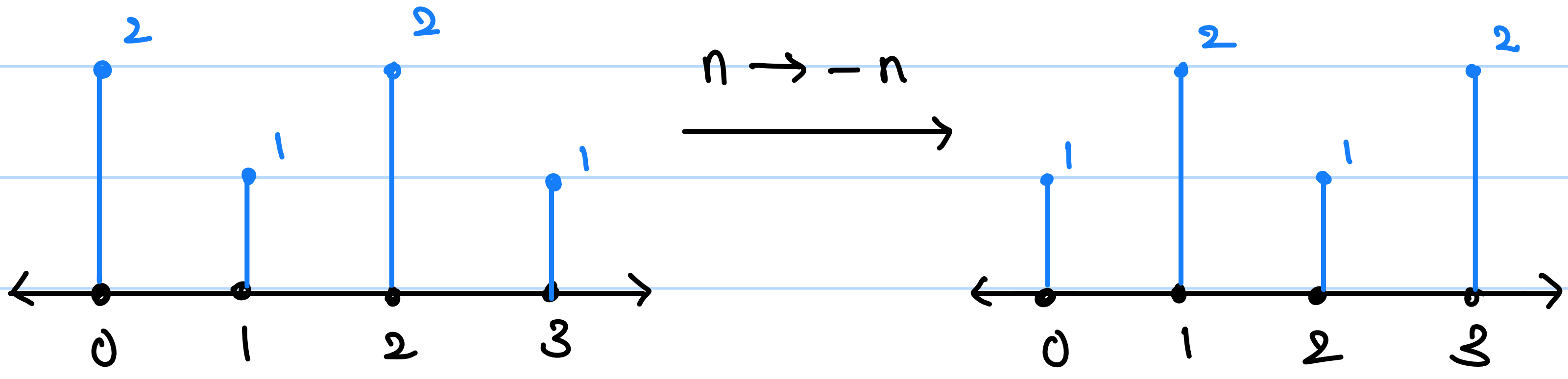
- **Circular Time Shift :-**

$$x[n-1] = x[n-1 \bmod N] \longrightarrow \text{Due to the abv. periodicity}$$



- The signal effectively wraps around the given time domain and loops back when shifted.

- $x[-n] = x[-n \bmod N] = x[N-n]$



- <u>LTI Systems</u>:-

$$x[n] \longrightarrow \boxed{H[k]} \longrightarrow y[n]$$

$$Y[k] = X[k]H[k]$$

$$y[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]H[k]\, e^{j\frac{2\pi}{N}kn}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} x[l]\, e^{-j\frac{2\pi}{N}lk}\, H[k]\, e^{j\frac{2\pi}{N}kn}$$

$$= \frac{1}{N}\sum_{l=0}^{N-1} x[l] \sum_{k=0}^{N-1} H[k]\, e^{-j\frac{2\pi}{N}lk}\, e^{j\frac{2\pi}{N}kn}$$

$$= \frac{1}{N}\sum_{l=0}^{N-1} x[l] \sum_{k=0}^{N-1} H[k]\, e^{j\frac{2\pi}{N}k(n-l)}$$

$$= \frac{1}{N}\sum_{l=0}^{N-1} x[l]\, h[n-l] = \frac{1}{N}\sum_{l=0}^{N-1} x[l]\, h[n-l \bmod N]$$

$$\Rightarrow \quad y[n] = x[n] \circledast h[n]$$

$\circledast \longrightarrow$ Circular convolution

- Let $N = 4$

$$y[0] = x[0]h[0] + x[1]h[3] + x[2]h[2] + x[3]h[1]$$
$$y[1] = x[0]h[1] + x[1]h[0] + x[2]h[3] + x[3]h[2]$$
$$y[2] = x[0]h[2] + x[1]h[1] + x[2]h[0] + x[3]h[3]$$
$$y[3] = x[0]h[3] + x[1]h[2] + x[2]h[1] + x[3]h[0]$$

$$\Rightarrow \begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \end{bmatrix} = \underbrace{\begin{bmatrix} h[0] & h[3] & h[2] & h[1] \\ h[1] & h[0] & h[3] & h[2] \\ h[2] & h[1] & h[0] & h[3] \\ h[3] & h[2] & h[1] & h[0] \end{bmatrix}}_{\text{Circulant Matrix } (H)} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}$$

each row is a circularly shifted variation of another row.

- If we do linear convolution of $x[n]$ and $h[n]$,

$$\hat{y}[n] = \sum_{l=-\infty}^{\infty} x[l]h[n-l] = \sum_{l=0}^{N-1} x[l]h[n-l]$$

$$\hat{y}[0] = x[0]h[0]$$
$$\hat{y}[1] = x[0]h[1] + x[1]h[0]$$
$$\hat{y}[2] = x[0]h[2] + x[1]h[1] + x[2]h[0]$$
$$\hat{y}[3] = x[0]h[3] + x[1]h[2] + x[2]h[1] + x[3]h[0]$$
$$\hat{y}[4] = x[1]h[3] + x[2]h[2] + x[3]h[1]$$
$$\hat{y}[5] = x[2]h[3] + x[3]h[2]$$
$$\hat{y}[6] = x[3]h[3]$$

We can see that,

$$y[3] = \hat{y}[3],$$
$$y[2] = \hat{y}[2] + \hat{y}[6]$$
$$y[1] = \hat{y}[1] + \hat{y}[5]$$
$$y[0] = \hat{y}[0] + \hat{y}[4]$$

$$\begin{bmatrix} \hat{y}[0] \\ \hat{y}[1] \\ \hat{y}[2] \\ \hat{y}[3] \\ \hat{y}[4] \\ \hat{y}[5] \\ \hat{y}[6] \end{bmatrix} = \begin{bmatrix} h[0] & 0 & 0 & 0 \\ h[1] & h[0] & 0 & 0 \\ h[2] & h[1] & h[0] & 0 \\ h[3] & h[2] & h[1] & h[0] \\ 0 & h[3] & h[2] & h[1] \\ 0 & 0 & h[3] & h[2] \\ 0 & 0 & 0 & h[3] \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}$$

To make the matrix circulant,

$$\begin{bmatrix} \hat{y}[0] \\ \hat{y}[1] \\ \hat{y}[2] \\ \hat{y}[3] \\ \hat{y}[4] \\ \hat{y}[5] \\ \hat{y}[6] \end{bmatrix} = \begin{bmatrix} h[0] & 0 & 0 & 0 & h[3] & h[2] & h[1] \\ h[1] & h[0] & 0 & 0 & 0 & h[3] & h[2] \\ h[2] & h[1] & h[0] & 0 & 0 & 0 & h[3] \\ h[3] & h[2] & h[1] & h[0] & 0 & 0 & 0 \\ 0 & h[3] & h[2] & h[1] & h[0] & 0 & 0 \\ 0 & 0 & h[3] & h[2] & h[1] & h[0] & 0 \\ 0 & 0 & 0 & h[3] & h[2] & h[1] & h[0] \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Example: $x[n] = \{1\,1\,0\,0\,1\,1\}$, find a) $x[n]x[-n]$,
b) $x[n-1]$, c) $x[n] \circledast x[n]$

$$x[-n] = x[6-n] = \{1\,1\,1\,0\,0\,1\}$$

$$\Rightarrow \quad x[n]x[-n] = \{1\,1\,0\,0\,0\,1\}$$

$$x[n-1] = x[n-1 \bmod 6] = \{1\,0\,0\,1\,1\,1\}$$

$$x[n] \circledast x[n] = \{3\,4\,3\,2\,2\,2\}$$

$x[n]$:	1	1	0	0	1	1
$-n$:	1	1	1	0	0	1
$-n+1$:	1	1	0	0	1	1
$-n+2$:	1	0	0	1	1	1
$-n+3$:	0	0	1	1	1	1
$-n+4$:	0	1	1	1	1	0
$-n+5$:	1	1	1	1	0	0

- **Properties of DFT:-**

1) Linearity

2) Time Shift:

$$x[n] \xleftrightarrow{DFT} X[k]$$

$$x[n-n_0] \xleftrightarrow{DFT} X[k]\, e^{-j\frac{2\pi}{N} n_0 k}$$

3) Freq. Shift:

$$x[n]\, e^{j\frac{2\pi}{N} n k_0} \xleftrightarrow{DFT} X[k-k_0]$$

$$X[k-k_0] = X(k-k_0 \bmod N)$$ [Circular Shift]

4) Time Reversal :-

$$x[-n] \xleftrightarrow{\text{DFT}} X[-k]$$

5) Conjugate :-

$$x^*[n] \xleftrightarrow{\text{DFT}} X^*[-k]$$

6) Parseval's Relation :-

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X[k]|^2$$

7) Multiplication in Time :-

$$x[n]\, y[n] \xleftrightarrow{\text{DFT}} \frac{1}{N} X[k] \circledast Y[k]$$

8) Circular Convolution in Time :-

$$x[n] \circledast y[n] \xleftrightarrow{\text{DFT}} X[k]\, Y[k]$$

9) Real Signals :-

$$x[n] \in \mathbb{R} \Rightarrow x[n] = x^*[n]$$

$$x[n] \xleftrightarrow{\text{DFT}} X[k]$$

$$x^*[n] \xleftrightarrow{\text{DFT}} X^*[-k]$$

$$\Rightarrow \quad \boxed{X[k] = X^*[-k]}$$

<u>Example:</u> $x[n] = \cos\left(\frac{2\pi}{N} n\right)$, $n = 0, 1, \ldots N-1$

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{n=0}^{N-1} \cos\left(\frac{2\pi}{N} n\right) e^{-j\frac{2\pi}{N}kn}$$

$$\Rightarrow \quad X[k] = \left\{0, \frac{N}{2}, 0, 0 \ldots, 0, \frac{N}{2}\right\}$$

<u>Example:</u> $x[n] = \cos(\omega n)$, $n = 0, 1, 2 \ldots N-1$, $\omega \in (0, 2\pi)$

$$X[k] = \sum_{n=0}^{N-1} \cos(\omega n) e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{n=0}^{N-1} \cos(\omega n)\left(\cos\left(\frac{2\pi}{N}kn\right) - j\sin\left(\frac{2\pi}{N}kn\right)\right)$$

$$= \sum_{n=0}^{N-1} \left(\cos(\omega n)\cos\left(\frac{2\pi}{N}kn\right) - \cancelto{0}{\cos(\omega n)\, j\sin\left(\frac{2\pi}{N}kn\right)}\right)$$

(orthogonality)

$$= \sum_{n=0}^{N-1} \cos(\omega n)\cos\left(\frac{2\pi}{N}kn\right)$$

$$\cos A \cos B = \frac{\cos(A+B) + \cos(A-B)}{2}$$

$$\Rightarrow \quad X[k] = \frac{1}{2} \sum_{n=0}^{N-1} \left(\cos\left(\left(\frac{2\pi}{N}k + \omega\right) n\right) + \cos\left(\left(\frac{2\pi}{N}k - \omega\right) n\right)\right)$$

- **Complexity of N-point DFT :-**

- $x[n] \xleftrightarrow{DFT} X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$

No. of multiplications = N^2 complex multiplications → 4 real multiplications, 2 real addition

" " additions = $N(N-1)$ complex addn → 2 real addn.

=> No of real multiplications = $\underline{\underline{4N^2}}$

No of real additions = $2N^2 + 2N(N-1) = \underline{\underline{2N(2N-1)}}$

- $X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$, $W_N = e^{-j\frac{2\pi}{N}}$ (Twiddle factor)

- **Properties of W_N:**

- $W_N^0 = W_N^N = 1$

- $W_N^{n+N} = W_N^n$

- $W_N^P = W_{N/2}^{P/2}$ ∀ even P

- $\left.\begin{array}{l} W_N^{N/2} = -1 \\ W_N^{k+N/2} = -W_N^k \end{array}\right\}$ ∀ even N

- $X[k] = \sum_{n\ \text{even}} x[n] W_N^{nk} + \sum_{n\ \text{odd}} x[n] W_N^{nk}$

$$\Rightarrow \quad X[k] = \sum_{m=0}^{\frac{N}{2}-1} x[2m]\, W_N^{2mk} + \sum_{r=0}^{\frac{N}{2}-1} x[2r+1]\, W_N^{(2r+1)k}$$

$$= \underbrace{\sum_{m=0}^{\frac{N}{2}-1} x[2m]\, W_{N/2}^{mk}}_{N/2\text{- point DFT of } x[2n]} + \underbrace{\sum_{r=0}^{\frac{N}{2}-1} x[2r+1]\, W_{N/2}^{rk}}_{N/2\text{- point DFT of } x[2r+1]} \cdot W_N^{k}$$

$x[2m]$ – All even samples $x[2r+1]$ – All odd samples.

But a $N/2$ point sample gives a $N/2$ length vector as the output.

$$X\left[k+\frac{N}{2}\right] = \sum_{m=0}^{\frac{N}{2}-1} x[2m]\, W_{N/2}^{m(k+\frac{N}{2})} + \left(\sum_{r=0}^{\frac{N}{2}-1} x[2r+1]\, W_{N/2}^{r(k+\frac{N}{2})}\right) W_N^{k}$$

$$= \sum_{m=0}^{N/2-1} x[2m]\, W_{N/2}^{mk} - \left(\sum_{r=0}^{\frac{N}{2}-1} x[2r+1]\, W_{N/2}^{rk}\right) W_N^{k}$$

$\therefore$ The final DFT is,

$$X[k] = \begin{cases} X_e[k] + W_N^{k} X_o[k], & k = 0, 1, \ldots, \frac{N}{2}-1 \\ X_e[k] - W_N^{k} X_o[k], & k = \frac{N}{2}, \ldots, N-1 \end{cases}$$