Probability and Random Processes - Assignment - 2

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P1. By definition of an RV $X : \Omega \longrightarrow \mathbb{R}$,

$$\{ \omega \mid X(\omega) \leq x \} \in \mathcal{F} \quad \forall \quad x \in \mathbb{R}$$

a) $X(\omega) = \omega^2$. The mapping is,

$$X(\omega) = \begin{cases} 4 , & \omega \in \{-2,2\} \\ 1 , & \omega \in \{-1,1\} \\ 0 , & \omega = 0 \end{cases}$$

$$\Rightarrow \quad \{ \omega \mid X(\omega) \leq x \} = \begin{cases} \phi & , x < 0 \\ \{0\} & , 0 \leq x < 1 \\ \{-1,0,1\} & , 1 \leq x < 4 \\ \Omega & , x \geq 4 \end{cases}$$

$\therefore$ $\mathcal{F}$ st X is a valid RV is $\sigma(\Omega, \phi, \{0\}, \{-1,0,1\})$

$$\sigma(\Omega, \phi, \{0\}, \{-1,0,1\}) = \sigma(\{0\}, \{-1,1\}, \{-2,2\})$$
$$= \{ \{0\}, \{-1,1\}, \{-2,2\}, \{-1,0,1\}, \{-2,0,2\}, \{-2,-1,1,2\}, \Omega, \phi \}$$

$$\Rightarrow \mathcal{F} = \{ \phi, \Omega, \{0\}, \{-1,1\}, \{-2,2\}, \{-1,0,1\}, \{-2,0,2\}, \{-2,-1,1,2\} \}$$

b) $X(\omega) = \omega + 1$. The mapping is

$$X(\omega) = \begin{cases} -1, & \omega = -2 \\ 0, & \omega = -1 \\ 1, & \omega = 0 \\ 2, & \omega = 1 \\ 3, & \omega = 2 \end{cases}$$

$$\therefore \{\omega \mid X(\omega) \leq x\} = \begin{cases} \phi, & x < -1 \\ \{-2\}, & -1 \leq x < 0 \\ \{-2,-1\}, & 0 \leq x < 1 \\ \{-2,-1,0\}, & 1 \leq x < 2 \\ \{-2,-1,0,1\}, & 2 \leq x < 3 \\ \Omega, & 3 \leq x \end{cases}$$

$\Rightarrow \mathcal{F}$ st X is a valid RV is $\sigma(\Omega, \phi, \{-2\}, \{-2,-1\}, \{-2,1,0\}, \{-2,-1,0,1\})$

$$\sigma(\Omega, \phi, \{-2\}, \{-2,-1\}, \{-2,-1,0\}, \{-2,-1,0,1\})$$

$$= \sigma(\{-2\}, \{-1\}, \{0\}, \{1\}, \{2\})$$

$$\therefore \mathcal{F} = \{ I \mid I \subseteq \{-2,-1,0,1,2\} \}$$

P2. $\Omega = [0,1] \times [0,1] \longrightarrow$ The Unit Square in $\mathbb{R}^2$.

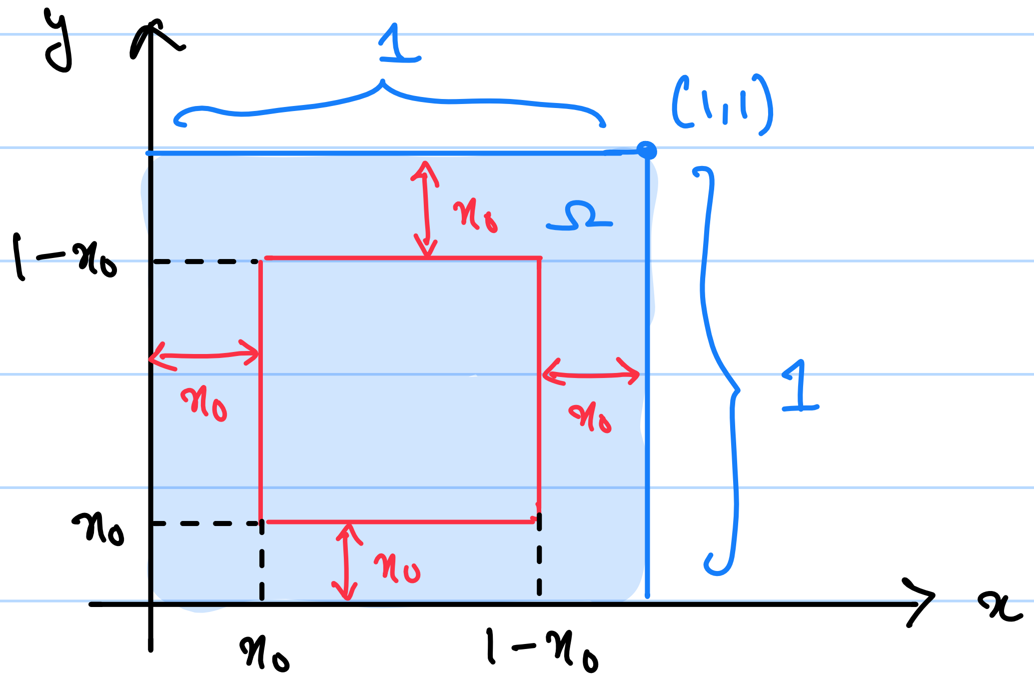
$\forall A \subseteq \Omega$, $P(A) = \text{area}(A)$.
$\forall \omega \in \Omega$, $X(\omega) =$ distance of ω from its nearest edge.

CDF of $X = ?$

CDF of $X = F_X(x) = P(\{\omega \mid X(\omega) \leq x\})$

Since Ω is a unit square, $X(\omega) \in [0, 0.5]$. Let there be some $x_0 \in [0, 0.5]$.

The set of all $\omega \in \Omega$ such that $X(\omega) \geq x_0$, will form a square inside the unit square, as shown below,



The square will be formed by the points (x_0, x_0), $(1-x_0, x_0)$, $(x_0, 1-x_0)$, $(1-x_0, 1-x_0)$. Any $\omega \in \Omega$ inside the square will have $X(\omega) > x_0$. Let the square be S_0.

$$P(\{\omega \mid X(\omega) \leq x_0\}) = 1 - P(\{\omega \mid X(\omega) > x_0)$$

$$\Rightarrow P(\{\omega \mid X(\omega) \leq x_0\}) = 1 - P(S_0)$$

$$\Rightarrow P(\{\omega \mid X(\omega) \leq x_0\}) = 1 - \text{area}(S_0)$$

$$\Rightarrow P(\{\omega \mid X(\omega) \leq x_0\}) = 1 - (1 - 2x_0)^2$$

$$= 1 - (1 + 4x_0^2 - 4x_0)$$

$$\Rightarrow F_X(x_0) = 4x_0 - 4x_0^2$$

Since x_0 is any general point in $[0, \frac{1}{2}]$, which is the range of X, we can define $F_X(x)$ as,

$$F_X(x) = 4x - 4x^2 \quad \forall \; x \in [0, \tfrac{1}{2}]$$

For $x < 0$, since distance cannot be negative,

$$P(\{\omega \mid X(\omega) \leq x\} = 0 \Rightarrow F_X(x) = 0$$

For $x > 1/2$, since $X(\omega) \in [0, 1/2] \; \forall \; \omega \in \Omega$,

$$\{\omega \mid X(\omega) \leq 1/2\} = \Omega \Rightarrow \underline{F_X(x) = 1}$$

$$\Rightarrow \boxed{F_X(x) = \begin{cases} 0, & x < 0 \\ 4x - 4x^2, & 0 \leq x \leq 1/2 \\ 1, & x > 1/2 \end{cases}}$$

P3. a) $G_X(x) = 1 - (1 - F_X(x))^{\gamma}$, $\gamma \in \mathbb{N}$

$$\lim_{x\to\infty} G_X(x) = \lim_{x\to\infty} 1 - (1 - F_X(x))^{\gamma}$$
$$= 1 - (1-1)^{\gamma} \quad \left[\lim_{x\to\infty} F_X(x) = 1\right]$$
$$\Rightarrow \underline{\underline{\lim_{x\to\infty} G_X(x) = 1}}$$

$$\lim_{x\to-\infty} G_X(x) = \lim_{x\to-\infty} 1 - (1 - F_X(x))^{\gamma}$$
$$= 1 - (1-0)^{\gamma} \quad \left[\lim_{x\to-\infty} F_X(x) = 0\right]$$
$$\Rightarrow \underline{\underline{\lim_{x\to-\infty} G_X(x) = 0}}$$

$$\lim_{\varepsilon\to0^+} G_X(x+\varepsilon) = \lim_{\varepsilon\to0^+} 1 - (1 - F_X(x+\varepsilon))^{\gamma}$$
$$= 1 - (1 - F_X(x))^{\gamma} \quad \left[F_X(x) \text{ is right cont.}\right]$$
$$\Rightarrow \underline{\underline{\lim_{\varepsilon\to0^+} G_X(x+\varepsilon) = G_X(x) \quad \forall\, x \in \mathbb{R}}}$$

$\Rightarrow$ $\underline{G_X(x) \text{ is right continuous}}$ $\forall\, x \in \mathbb{R}$

If $x_1 > x_2$, then $F_X(x_1) \geq F_X(x_2)$
$$\Rightarrow 1 - F_X(x_1) \leq 1 - F_X(x_2)$$
$$\Rightarrow (1 - F_X(x_1))^{\gamma} \leq (1 - F_X(x_2))^{\gamma}$$
$$\Rightarrow 1 - (1 - F_X(x_1))^{\gamma} \geq 1 - (1 - F_X(x_2))^{\gamma}$$
$$\Rightarrow G_X(x_1) \geq G_X(x_2)$$

$\therefore$ $x_1 > x_2 \Rightarrow G_X(x_1) \geq G_X(x_2)$

$\therefore$ $\underline{G_X(x) \text{ is non decreasing}}$

$G_X(x)$ satisfies all the properties of a valid CDF

$\therefore$ $1 - (1 - F_X(x))^{\gamma}$ is a valid CDF

b) $G_X(x) = F_X(x) + (1 - F_X(x)) \log (1 - F_X(x))$

$$\lim_{x \to \infty} G_X(x) = \lim_{x \to \infty} \left(F_X(x) + (1 - F_X(x)) \log (1 - F_X(x))\right)$$

$$= 1 + \lim_{t \to 0} t \log t \qquad \left[\lim_{x \to \infty} F_X(x) = 1\right]$$

$$= \underline{\underline{1}} \qquad \left[\lim_{t \to 0} t \log t = 0\right]$$

$$\lim_{x \to -\infty} G_X(x) = \lim_{x \to -\infty} \left(F_X(x) + (1 - F_X(x)) \log (1 - F_X(x)) \right)$$

$$= \log 1 \qquad \left[\lim_{x \to -\infty} F_X(x) = 0\right]$$

$$= \underline{\underline{0}}$$

$$\lim_{\varepsilon \to 0^+} G_X(x + \varepsilon) = \lim_{\varepsilon \to 0^+} \left(F_X(x+\varepsilon) + (1 - F_X(x+\varepsilon)) \log (1 - F_X(x+\varepsilon))\right)$$

$$= F_X(x) + (1 - F_X(x)) \log (1 - F_X(x))$$

$$= G_X(x)$$

$\therefore$ $G_X(x)$ is right continuous $\forall\ x \in \mathbb{R}$

$$G_X(x) = F_X(x) + (1 - F_X(x)) \log (1 - F_X(x))$$

$$= F_X(x) + \log (1 - F_X(x)) - F_X(x) \log (1 - F_X(x))$$

$$\frac{dG_X(x)}{dx} = \frac{dF_X(x)}{dx} + \frac{1}{1 - F_X(x)} \left(- \frac{dF_X(x)}{dx} \right) - \frac{dF_X(x)}{dx} \log (1 - F_X(x))$$

$$- F_X(x) \left(\frac{1}{1 - F_X(x)} \left(- \frac{dF_X(x)}{dx} \right) \right)$$

$$= \frac{dF_X(x)}{dx} \left(1 - \frac{1}{1 - F_X(x)} - \log (1 - F_X(x)) + \frac{F_X}{1 - F_X(x)} \right)$$

$$= \frac{dF_X(x)}{dx}\left(\frac{\cancel{1} - \cancel{F_X(x)} - \cancel{1} - (1-F_X(x))\log(1-F_X(x)) + \cancel{F_X(x)}}{1-F_X(x)}\right)$$

$$= \frac{dF_X(x)}{dx}\left(-\frac{(1-F_X(x))\log(1-F_X(x))}{1-F_X(x)}\right)$$

$$\frac{dG_X(x)}{dx} = \frac{dF_X(x)}{dx}\left(-\log(1-F_X(x))\right)$$

$$1 - F_X(x) \leq 1 \qquad [F_X(x) \in [0,1]]$$

$$\Rightarrow \log(1-F_X(x)) \leq 0$$

$$\Rightarrow -\log(1-F_X(x)) \geq 0 \quad \forall x \in \mathbb{R}$$

Since $F_X(x)$ is non decreasing $\frac{dF_X(x)}{dx} \geq 0 \quad \forall x \in \mathbb{R}$

$$\Rightarrow \frac{dF_X(x)}{dx}\left(-\log(1-F_X(x))\right) \geq 0 \quad \forall x \in \mathbb{R}$$

$$\Rightarrow \frac{dG_X(x)}{dx} \geq 0 \quad \forall x \in \mathbb{R}$$

$\Rightarrow$ $G_X(x)$ is non decreasing $\forall x \in \mathbb{R}$

$G_X(x)$ satisfies all the properties of a valid CDF

$\therefore$ $F_X(x) + (1-F_X(x))\log(1-F_X(x))$ is a valid CDF

P4. N = Non negative integer valued RV.

<u>To Prove:</u> $E[N] = \sum_{i=1}^{\infty} P(N \geq i) \rightarrow \sum_{i=1}^{\infty} P(\omega | N(\omega) \geq i)$

$$E[N] = \sum_{x \in \mathbb{N}} x P_N(x)$$

$$P_N(x) = P(\{\omega | N(\omega) = x\}) \triangleq P(N = x)$$

$$\Rightarrow \quad E[N] = \sum_{x \in \mathbb{N}} x P(N = x)$$

$$= P(N=1) + 2P(N=2) + 3P(N=3) + \dots$$

$$= (P(N=1) + P(N=2) + P(N=3) + \dots)$$
$$+ (P(N=2) + P(N=3) + \dots)$$
$$+ (P(N=3) + P(N=4) + \dots) + \dots$$

$$= P(N \geq 1) + P(N \geq 2) + P(N \geq 3) + \dots$$

$[P(N \geq i) = P(N=i) + P(N=i+1) + \dots]$

$$\Rightarrow \quad \boxed{E[N] = \sum_{i \in \mathbb{N}} P(N \geq i)}$$

Hence Proved.

P5. To give an example of an RV such that,

$$E[1/X] = 1/E[X]$$

Let X take values $(1, -2, 3)$ with probabilities p, q, r.
Let $E[X] = 4/3 \Rightarrow E[1/X] = 3/4$, then we get the following system of linear equations,

$$p + q + r = 1$$
$$p - 2q + 3r = 4/3$$
$$p - q/2 + r/3 = 3/4$$

$$\frac{4}{3} - \frac{3}{4} = \frac{16-9}{12} = \frac{7}{12}$$

$$p - 2q + 3r = 4/3$$
$$(-)\; p \overset{(+)}{-} q/2 \overset{(-)}{+} r/3 = 3/4$$
$$-\frac{3q}{2} + \frac{8r}{3} = \frac{7}{12}$$

$$\Rightarrow -18q + 32r = 7 \quad \text{——} \quad (1)$$

$$p + q + r = 1 \Rightarrow p = 1 - q - r$$

$$1 - q - r - q/2 + r/3 = 3/4$$
$$1 - 3q/2 - 2r/3 = 3/4$$
$$3q/2 + 2r/3 = 1/4$$
$$6q + 8r/3 = 1$$
$$18q + 8r = 3 \quad \text{——} \quad (1)$$

$$-18q + 32r = 7$$
$$18q + 8r = 3$$
$$40r = 10$$
$$\Rightarrow r = 1/4$$

$$18q + 8\left(\frac{1}{4}\right) = 3$$
$$\Rightarrow q = 1/18$$

$$p = 1 - \frac{1}{18} - \frac{1}{4}$$
$$p = \frac{25}{36}$$

$$\Rightarrow p = \frac{25}{36}, \quad q = \frac{1}{18}, \quad r = \frac{1}{4}$$

$$E[X] = (1)\left(\frac{25}{36}\right) - 2\left(\frac{1}{18}\right) + 3\left(\frac{1}{4}\right)$$
$$= \frac{25}{36} - \frac{4}{36} + \frac{27}{36} = \frac{48}{36} = 4/3$$
$$E[1/X] = (1)\left(\frac{25}{36}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{18}\right) + \frac{1}{3}\left(\frac{1}{4}\right)$$
$$= \frac{25}{36} - \frac{1}{36} + \frac{3}{36} = \frac{27}{36} = 3/4$$

$$\Rightarrow E[1/X] = 1/E[X]$$

∴ The RV X, that takes 3 non zero values 1, −2, 3 with respective probabilities 25/36, 1/18, 1/4, satisfies the given equality.

P6. <u>To Prove</u>: If X is a Binomial or Poisson RV, then,

$$P_X(k-1)\,P_X(k+1) \leq P_k(k)^2$$

For Binomial RV's, $P_X(k) = {}^nC_k\, p^k (1-p)^{n-k}$

$$P_X(k-1)\,P_X(k+1) = {}^nC_{k-1}\,{}^nC_{k+1}\, p^{k-1}(1-p)^{n-k+1}\, p^{k+1}(1-p)^{n-k-1}$$

$$= \frac{n!}{(n-k+1)!\,(k-1)!}\;\frac{n!}{(n-k-1)!\,(k+1)!}\; p^{2k}(1-p)^{2(n-k)}$$

$$= \frac{\cancel{n!^2}}{(n-k+1)!\,(n-k-1)!\,(k-1)!\,(k+1)!}\;\frac{P_X(k)^2}{\left(\frac{\cancel{n!}}{(n-k)!\,k!}\right)^2}$$

$$= \frac{P_X(k)^2\,(n-k)!\,(n-k)!\,k!\,k!}{(n-k+1)!\,(n-k-1)!\,(k-1)!\,(k+1)!}$$

$$= \frac{P_X(k)^2\,(n-k)\,k}{(n-k+1)(k+1)}$$

$$\frac{k}{k+1} < 1, \quad \frac{n-k}{n-k+1} < 1 \Rightarrow \frac{(n-k)\,k}{(n-k+1)(k+1)} < 1$$

$$\Rightarrow \frac{P_X(k)^2\,(n-k)\,k}{(n-k+1)(k+1)} \leq P_X(k)^2$$

$$\Rightarrow P_X(k-1)\,P_X(k+1) \leq P_X(k)^2$$

For Poisson RVs, $P_X(k) = \dfrac{e^{-\lambda}\lambda^k}{k!}$

$$P_X(k-1)\,P_X(k+1) = \frac{e^{-\lambda}\lambda^{k-1}}{(k-1)!}\;\frac{e^{-\lambda}\lambda^{k+1}}{(k+1)!}$$

$$= \frac{e^{-2\lambda}\lambda^{2k}}{\frac{k!}{k} \times k!\,(k+1)}$$

$$= \frac{e^{-2\lambda}\lambda^{2k}}{k!^2\left(\frac{k+1}{k}\right)}$$

$$P_X(k)^2 = \frac{e^{-2\lambda}\lambda^{2k}}{k!^2}$$

Since $\dfrac{k+1}{k} > 1$, $\dfrac{e^{-2\lambda}\lambda^{2k}}{k!^2\left(\frac{k+1}{k}\right)} \leqslant \dfrac{e^{-2\lambda}\lambda^{2k}}{k!^2}$

$$\Rightarrow P_X(k-1)\,P_X(k+1) \leq P_X(k)^2$$

$\therefore$ For Binomial or Poisson RVs,

$$P_X(k-1)\,P_X(k+1) \leqslant P_X(k)^2$$

The given condition will show equality if X is a constant RV, ie,

$$P_X(k) = c \quad \forall\ k \in \mathbb{N}$$

$$P_X(k-1)\,P_X(k+1) = P_X(k)^2$$

$$\Rightarrow c \cdot c = (c)^2$$

$$\Rightarrow c^2 = c^2 \longrightarrow \text{True}$$

$$\therefore \underline{P_X(k-1)\,P_X(k+1) = P_X(k)^2}$$

P7. Given: X is a Poisson RV. Let $\theta = e^{-3\lambda}$

a) $g(X) = e^{-3X} \Rightarrow E[g(X)] = \sum_{x \in \mathcal{X}} g(x) P_X(x)$ [LOTUS]

$$= \sum_{x \in \mathcal{X}} e^{-3x} \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x \in \mathbb{N}} \frac{e^{-3x} \lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x \in \mathbb{N}} \frac{(\lambda e^{-3})^x}{x!}$$

$$= e^{-\lambda} e^{(\lambda e^{-3})} \qquad \left[\sum_{i \in \mathbb{N}} \frac{c^i}{i!} = e^c \right]$$

$$= \underline{e^{\lambda e^{-3} - \lambda}}$$

Bias $= E[g(X)] - \theta = e^{\lambda e^{-3} - \lambda} - e^{-3\lambda}$

$$e^{\lambda e^{-3} - \lambda} - e^{-3\lambda} = 0$$

$$\Rightarrow \lambda e^{-3} - \lambda = -3\lambda$$

$$\cancel{\lambda} e^{-3} = -2\cancel{\lambda}$$

$$e^{-3} = -2 \longrightarrow \text{Not true}$$

$\therefore$ The given estimator is biased.

b) $g(X) = (-2)^X$

$$E[g(X)] = \sum_{x \in \mathcal{X}} (-2)^x P_X(x) = \sum_{x \in \mathbb{N}} (-2)^x \frac{e^{-\lambda} \lambda^x}{x!} =$$

$$= e^{-\lambda} \sum_{x \in \mathbb{N}} \frac{(-2\lambda)^x}{x!}$$

$$= e^{-\lambda} e^{-2\lambda} = \underline{\underline{e^{-3\lambda}}}$$

$$\text{Bias} = E[g(X)] - \theta = e^{-3\lambda} - e^{-3\lambda} = \underline{\underline{0}}$$

$\therefore$ The given estimator is unbiased.