Probability and Random Processes - Assignment - 3

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Q1. $\Omega = \{1, 2, 3 \ldots\} = \mathbb{N}$

$\mathcal{F} = \sigma$ Algebra of Ω

$P(\{\omega\}) = 2^{-\omega} \quad \forall \; \omega \in \Omega$

RVs $X(\omega) = \omega$, $Y(\omega) = (-1)^{\omega}$, $E[X|Y] = ?$

$$E[X|Y=y] = \sum_{x \in \mathcal{X}} x P_{X|Y}(x|y) = \sum_{x \in \mathcal{X}} x \frac{P_{XY}(x,y)}{P_Y(y)}$$

$$= \sum_{x \in \mathbb{N}} x \frac{P(X = x \cap Y = y)}{P(Y=y)}$$

$$= \sum_{x \in \mathbb{N}} x \frac{P(\{\omega | X(\omega) = x\} \cap \{\omega | Y(\omega) = y\})}{P(\{\omega | Y(\omega) = y\})}$$

$$= \sum_{x \in \mathbb{N}} x \frac{P(\{\omega | \omega = x\} \cap \{\omega | (-1)^{\omega} = y\})}{P(\{\omega | (-1)^{\omega} = y\})}$$

$$\{\omega | (-1)^{\omega} = y\} = \begin{cases} \{2, 4, 6, 8 \ldots\}, & y = 1 \\ \{1, 3, 5, 7 \ldots\}, & y = -1 \\ \phi, & \text{otherwise} \end{cases}$$

$$\Rightarrow \quad P_Y(y) = \begin{cases} 2^{-2} + 2^{-4} + 2^{-6} \dots, & y = 1 \\ 2^{-1} + 2^{-3} + 2^{-5} \dots, & y = -1 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1/4 \,/\, 1 - 1/4, & y = 1 \\ 1/2 \,/\, 1 - 1/4, & y = -1 \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow \quad P_Y(y) = \begin{cases} 1/3, & y = 1 \\ 2/3, & y = -1 \\ 0, & \text{otherwise} \end{cases} \longrightarrow E[X|Y=y] \text{ is defined only for } y \in \{-1,1\}$$

$$P(\omega \mid \omega = x) = P(\{x\}) = 2^{-x}$$

$$\therefore P_X(x) = 2^{-x} \quad \forall \; x \in \mathbb{N}$$

$$P(\{\omega \mid \omega = x\} \cap \{\omega \mid (-1)^{\omega} = y\}) = P(\{x\} \cap \{\omega \mid (-1)^{\omega} = y\})$$

If $y = 1$,

$$P_{XY}(x,y) = P(\{x\} \cap \{2,4,6,8 \dots\})$$

$$= \begin{cases} P(\{x\}), & x \text{ is even} \\ 0, & x \text{ is odd} \end{cases}$$

Illly if $y = -1$, $P_{XY}(x,y) = \begin{cases} P(\{x\}), & x \text{ is odd} \\ 0, & x \text{ is even} \end{cases}$

If $y \in \{-1,1\}$, $P_{XY}(x,y) = P(\{x\} \cap \phi) = 0$

$$\Rightarrow P_{XY}(x,y) = \begin{cases} 2^{-x}, & (y=1, x \text{ is even}) \text{ or } (y=-1, x \text{ is odd}) \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow E[X \mid Y=y] = \begin{cases} \sum_{x \in \text{even } \mathbb{N}} \frac{x\,2^{-x}}{1/3}, & y = 1 \\ \sum_{x \in \text{odd } \mathbb{N}} \frac{x\,2^{-x}}{2/3}, & y = -1 \\ \text{undefined}, & \text{otherwise} \end{cases}$$

$$= \begin{cases} \sum_{z \in \mathbb{N}} 2z\,2^{-2z} \cdot 3, & y = 1 \\ \sum_{z \in \mathbb{N}} (2z+1)\,2^{-(2z+1)} \frac{3}{2}, & y = -1 \\ \text{undefined}, & \text{otherwise} \end{cases}$$

$$\sum_{z \in \mathbb{N}} 2z\,2^{-2z} \cdot 3 = 6 \sum_{z \in \mathbb{N}} z\,2^{-2z} \longrightarrow \text{Infinite AGP}$$

$$= 6\left(\frac{2^{-2}}{1-2^{-2}} + \frac{1 \cdot 2^{-2}}{(1-2^{-2})^2}\right)$$

$$= 6\left(\frac{1/4}{1-1/4} + \frac{1/4}{(1-1/4)^2}\right)$$

$$= 6\left(1/3 + \frac{1/4}{9/16}\right)$$

$$= 6\left(\frac{1}{3} + \frac{4}{9}\right) = 6\left(7/9\right)$$

$$= \underline{\underline{14/3}}$$

$$\sum_{z \in \mathbb{N}} (2z+1)\, 2^{-2z-1} \cdot \frac{2}{3} = \frac{1}{3} \sum_{z \in \mathbb{N}} (2z+1)\, 2^{-2z}$$

$$= \frac{1}{3}\left(2 \sum_{z \in \mathbb{N}} z\, 2^{-2z} + \sum_{z \in \mathbb{N}} 2^{-2z} \right)$$

$$= \frac{2}{3}\left(\frac{1/4}{1-1/4} + \frac{1 \cdot 1/4}{(1-1/4)^2} + \frac{1/4}{1-1/4} \right)$$

$$= \frac{2}{3}\left(\frac{1}{3} + \frac{1/4}{9/16} + \frac{1}{3} \right)$$

$$= \frac{2}{3}\left(\frac{1}{3} + \frac{4}{9} + \frac{1}{3} \right)$$

$$= \frac{2}{3}\left(\frac{3+4+3}{9} \right)$$

$$= \frac{2}{3}\left(\frac{10}{9} \right) = \frac{20}{27}$$

$$\Rightarrow \quad E[X \mid Y = y] = \begin{cases} 14/3 & , \; y = 1 \\ 20/27 & , \; y = -1 \\ \text{undefined} & , \; \text{otherwise} \end{cases}$$

Q2. a) $E[X \mid X] = X \triangleq E[X \mid X = x] = x \;\; \forall\, x \in \mathcal{X}$

Take any $x_i \in \mathcal{X}$,

$$E[X \mid X = x_i] = \sum_{x \in \mathcal{X}} x\, P_{X|X}(x \mid x_i)$$

$$= \sum_{x \in \mathcal{X}} x \frac{P_{XX}(x, x_i)}{P_X(x_i)}$$

$$= \sum_{x \in \mathcal{X}} x \frac{P(X = x \cap X = x_i)}{P_X(x_i)}$$

Clearly, $P(X = x \cap X = x_i) = \begin{cases} P(X = x_i \cap X = x_i), & x = x_i \\ 0, & x \neq x_i \end{cases}$

$$\Rightarrow \sum_{x \in \mathcal{X}} x \frac{P(X = x \cap X = x_i)}{P_X(x_i)} = \frac{x_i P(X = x_i \cap X = x_i)}{P_X(x_i)}$$

$$= \frac{x_i P(X = x_i)}{P_X(x_i)} = \frac{x_i P_X(x_i)}{P_X(x_i)}$$

$$= \underline{\underline{x_i}}$$

$$\Rightarrow E[X | X = x_i] = x_i$$

Since x_i is any general element of $\mathcal{X}$,

$$\Rightarrow E[X | X = x] = x \quad \forall x \in \mathcal{X}$$

$$\Rightarrow \underline{\underline{E[X|X] = X}}$$ Hence Proved.

b) <u>To Prove:</u> $E[X \cdot g(Y) | Y] = g(Y) E[X|Y]$

$$\triangleq E[X \cdot g(Y) | Y = y] = g(y) E[X | Y = y] \quad \forall y \in \mathcal{Y}$$

Since $Xg(Y)$ is conditioned on $Y=y$, the RV becomes $X \cdot g(y)$

$$\Rightarrow E[Xg(Y) \mid Y=y] = \sum_{x \in \mathcal{X}} x g(y) P_{X|Y}(x|y)$$

$$= g(y) \sum_{x \in \mathcal{X}} x P_{X|Y}(x|y)$$

$$\Rightarrow E[Xg(Y) \mid Y=y] = g(y) E[X \mid Y=y] \quad \forall y \in \mathcal{Y}$$

$$\Rightarrow \underline{E[Xg(Y) \mid Y] = g(Y) E[X \mid Y]}$$ Hence Proved.

Let X be a constant RV 1, ie, $X(\omega) = 1 \ \forall \omega \in \Omega$.

$$\Rightarrow X(\omega) g(Y(\omega)) = g(Y(\omega)) \ \forall \omega \in \Omega$$

$$\Rightarrow \underline{X g(Y) = g(Y)}$$

$$E[Xg(Y) \mid Y=y] = g(y) E[X \mid Y=y] \ \forall y \in \mathcal{Y}$$

$$\Rightarrow E[g(Y) \mid Y=y] = g(y) \sum_{x \in \mathcal{X}} x P_{X|Y}(x|y)$$

$$= g(y) \left((1) \cdot P_{X|Y}(1|y) \right)$$

$$P_{X|Y}(1, y) = \frac{P_{XY}(1, y)}{P_Y(y)} = \frac{P_Y(y)}{P_Y(y)} = 1$$

$$\Rightarrow E[g(Y) \mid Y=y] = g(y) \ \forall y \in \mathcal{Y}$$

$$\Rightarrow \quad \underline{E[g(Y)|Y] = g(Y)}$$ Hence Proved.

c) To Prove: $E[E[X|Y,Z]|Y] = E[X|Y]$

$$\triangleq \quad E[E[X|Y=y_1, Z=z]|Y=y_1] = E[X|Y=y_1] \quad \forall \; y, y_1 \in \mathcal{Y}, \; z \in \mathcal{Z}$$

$$E[E[X|Y,Z]|Y=y] = \sum_{z \in Z} E[X|Y=y, Z=z] \, P_{Z|Y}(z|y)$$

$$E[X|Y=y, Z=z] = \sum_{x \in X} x \, P_{X|YZ}(x|y,z)$$

$$\Rightarrow E[E[X|Y,Z]|Y=y] = \sum_{z \in Z} \left(\sum_{x \in X} x \, P_{X|YZ}(x|y,z) \right) P_{Z|Y}(z|y)$$

$$= \sum_{x \in X, z \in Z} x \, P_{X|YZ}(x|y,z) \, P_{Z|Y}(z|y)$$

W.k.t,

$$P_{X|Y}(x|y) = \sum_{z \in Z} P_{X|YZ}(x|y,z) \, P_{Z|Y}(z|y)$$ [Total Probability Theorem]

$$\Rightarrow E[E[X|Y,Z]|Y=y] = \sum_{x \in X} x \, P_{X|Y}(x|y)$$

$$\Rightarrow E[E[X|Y,Z]|Y=y] = E[X|Y=y] \quad \forall \; y \in \mathcal{Y}$$

$$\Rightarrow \underline{E[E[X|Y,Z]|Y] = E[X|Y]}$$ Hence Proved.

Q3. Given: $F_Y(y) = 1 - \dfrac{2}{(y+1)(y+2)}$, for integer $y \geq 0$

$$P_{Z|Y}(z|y) = \frac{1}{y^2}, \text{ for } 1 \leq z \leq y^2$$

To find: $E[Z] = ?$

$$P_Y(y) = F_Y(y) - F_Y(y-1)$$

$$= 1 - \frac{2}{(y+1)(y+2)} - \left(1 - \frac{2}{(y-1+1)(y-1+2)}\right)$$

$$= \cancel{1} - \frac{2}{(y+1)(y+2)} - \cancel{1} + \frac{2}{y(y+1)}$$

$$= \frac{2}{y(y+1)} - \frac{2}{(y+1)(y+2)}$$

$$= \frac{2y+4-2y}{y(y+1)(y+2)} = \frac{4}{y(y+1)(y+2)}$$

$$E[Z] = \sum_{z \in Z} z P_Z(z)$$

$$= \sum_{z \in Z} z \sum_{y \in Y} P_{Z|Y}(z|y) P_Y(y) \quad \text{[Total Probability Theorem]}$$

$$= \sum_{z \in Z} z \sum_{y \in Y} \left(\frac{1}{y^2}\right)\left(\frac{4}{y(y+1)(y+2)}\right)$$

$$E[Z] = \sum_{z \in Z} z \sum_{y \in Y} \frac{4}{y^3(y+1)(y+2)}$$

Take the inner sum $S = \sum_{y \in Y} \frac{4}{y^3(y+1)(y+2)}$. Let each term of the sum be s_y, i.e.,

$$s_y = \frac{4}{y^3(y+1)(y+2)}$$

Clearly, we can see that

$s_y \longrightarrow \infty$ as $y \longrightarrow 0, -1$ or -2

Therefore the sum S is divergent and cannot be simplified.

Q4. Given: Discrete RVs X and Y with mean = 0, variance = 1, covariance = ρ.

To Prove: $E[\max\{X^2, Y^2\}] \leq 1 + \sqrt{1-\rho^2}$

$E[X] = 0$, $E[Y] = 0$

$Var[X] = 1$, $Var[Y] = 1$

$Var[X] = E[X^2] - (E[X])^2$

$\Rightarrow 1 = E[X^2] - (0)^2$

$\Rightarrow E[X^2] = 1$

|||ly $E[Y^2] = 1$

$Cov[X,Y] = E[XY] - \cancel{E[X]E[Y]}^{\,0} \Rightarrow E[XY] = \rho$

$$\max\{X^2, Y^2\} = \frac{X^2+Y^2}{2} + \frac{|X^2-Y^2|}{2}$$

$$\Rightarrow E[\max\{X^2,Y^2\}] = \frac{1}{2}(E[X^2]+E[Y^2]) + \frac{1}{2}E[|X^2-Y^2|]$$

$$= 1 + \frac{1}{2}E[|(X+Y)(X-Y)|]$$

By Cauchy-Schwartz inequality,

$$E[|(X+Y)(X-Y)|] \leq \sqrt{E[(X+Y)^2]E[(X-Y)^2]}$$

$$E[(X+Y)^2] = E[X^2] + E[Y^2] + 2E[XY]$$

$$= 1+1+2\rho = 2+2\rho$$

$$E[(X-Y)^2] = E[X^2] + E[Y^2] - 2E[XY]$$

$$= 1+1-2\rho = 2-2\rho$$

$$\Rightarrow E[|(X+Y)(X-Y)|] \leq \sqrt{(2+2\rho)(2-2\rho)} = \sqrt{4-4\rho^2}$$

$$\Rightarrow E[|(X+Y)(X-Y)|] \leq 2\sqrt{1-\rho^2}$$

$$\Rightarrow E[\max\{X^2,Y^2\}] = 1+\frac{1}{2}E[|(X+Y)(X-Y)|] \leq 1+\frac{1}{2}(2\sqrt{1-\rho^2})$$

$$\Rightarrow \underline{E[\max\{X^2,Y^2\}] \leq 1+\sqrt{1-\rho^2}}$$ Hence Proved.

Q5. $\pi [1:n] \longmapsto [1:n]$ is a function that gives a permutation of $[1:n]$, where $\pi(i)$ is the position of i in the ordering given by the permutation.

Fixed point : $x \in [1:n]$ st $\pi(x) = x$.

X : Rv of no. of fixed points of a permutation.

$E[X] = ?$, $Var[X] = ?$

Define an Indicator Rv I_i such that,

$$I_i(\omega) = \begin{cases} 1, & \omega = i \text{ and } \pi(i) = i \\ 0, & ow \end{cases}$$

where $\omega \in \Omega = [1:n]$

ie, $I_i = 1$ if i is a fixed point and 0 otherwise

Clearly $X = \sum_{i=1}^{n} I_i$

$$\Rightarrow E[X] = \sum_{i=1}^{n} E[I_i]$$

$$E[I_i] = P(\pi(i) = i) = \frac{(n-1)!}{n!} = \underline{\underline{\frac{1}{n}}}$$ [I_i is an indicator Rv]

$$\Rightarrow \quad E[X] = \sum_{i=1}^{n} \left(\frac{1}{n}\right) = \underline{\underline{1}}$$

$$Var[X] = \sum_{i=0}^{n} Var[I_i] + \sum_{i,j \in [1:n]} Cov(I_i, I_j)$$

$$Var[I_i] = E[I_i^2] - E[I_i]^2$$

$$= E[I_i] - E[I_i]^2$$

$$= 1/n - 1/n^2 = (n-1)/n^2 \quad \forall\, i \in [1:n]$$

$$Cov(I_i, I_j) = E[I_i I_j] - E[I_i]E[I_j]$$

$$E[I_i I_j] = P(\pi(i) = i, \pi(j) = j) = \frac{(n-2)!}{n!} = \frac{1}{n(n-1)}$$

$$\Rightarrow \quad Cov(I_i, I_j) = \frac{1}{n(n-1)} - \left(\frac{1}{n}\right)\left(\frac{1}{n}\right)$$

$$= \frac{1}{n(n-1)} - \frac{1}{n^2}$$

$$= \frac{n - (n-1)}{n^2(n-1)} = \frac{1}{n^2(n-1)} \quad \forall\, i,j \in [1:n]$$

$$\sum_{i,j \in [1:n]} Cov(I_i, I_j) = n(n-1) \cdot \frac{1}{n^2(n-1)} = \frac{1}{n}$$

$$\Rightarrow \quad Var[X] = \sum_{i=0}^{n} \frac{n-1}{n^2} + \frac{1}{n} = \frac{n-1}{n} + \frac{1}{n}$$

$$\Rightarrow \quad Var[X] = 1$$

$\therefore$ $E[X] = 1$ and $Var[X] = 1$

Q6. $X_1, X_2, \ldots, X_n$ are independent discrete Rv's, $X = \sum_{i=0}^{n} X_i$.
If each of X_i is a geometric Rv with parameter p_i such that $E[X] = \mu > 0$

To Show: $Var[X]$ is minimum if $p_i = \frac{n}{\mu}$.

$$P_{X_i}(k) = p_i(1-p_i)^{k-1}$$

$$E[X_i] = \frac{1}{p_i} \qquad Var[X_i] = \frac{1-p_i}{p_i^2}$$

$$E[X] = \mu \Rightarrow \sum_{i=1}^{n} E[X_i] = \mu \Rightarrow \sum_{i=1}^{n} \frac{1}{p_i} = \mu$$

$$Var[X] = \sum_{i=1}^{n} Var[X_i] + \sum_{i,j \in [1:n]} \cancelto{0}{Cov(X_i, X_j)}$$

$$= \sum_{i=1}^{n} Var[X_i] \qquad \left[Cov(X,Y) = 0 \text{ if } X,Y \text{ are independent} \right]$$

$$= \sum_{i=1}^{n} \frac{1-p_i}{p_i^2} = \sum_{i=1}^{n} \frac{1}{p_i^2} - \sum_{i=1}^{n} \frac{1}{p_i}$$

Using the Lagrangian Multiplier, ie,

$L = f(x_1, x_2 .. x_n) - \lambda g(x_1, x_2 .. x_n)$, where f - function to be minimized
g - constraint function

Then $x_1, x_2, x_3 \ldots x_n$, such that

$f'(x_1, x_2, \ldots x_n) = \lambda g'(x_1, x_2 \ldots x_n)$, will give a minima of f in constraint g.

The constraint g is, $\sum_{i=1}^{n} \frac{1}{p_i} = \mu$

$$\Rightarrow \quad g = \sum_{i=1}^{n} \frac{1}{p_i} - \mu$$

$$\Rightarrow \quad L = \sum_{i=1}^{n} \frac{1-p_i}{p_i^2} - \lambda \left(\sum_{i=1}^{n} \frac{1}{p_i} - \mu \right)$$

$$\frac{\delta L}{\delta p_j} = \frac{\delta}{\delta p_j} \left(\frac{1}{p_j^2} - \frac{1}{p_j} \right) - \lambda \frac{\delta}{\delta p_j} \left(\frac{1}{p_j} \right) = 0$$

$$= \frac{-2}{p_j^3} + \frac{1}{p_j^2} + \lambda \frac{1}{p_j^2} = 0$$

$$= \frac{-2}{p_j} + 1 + \lambda = 0$$

$$= p_j = \frac{2}{1+\lambda}$$

Since λ is a constant, $p_i = \frac{2}{1+\lambda} \ \forall \ i \in [1:n]$

Let $p = \frac{2}{1+\lambda} \Rightarrow p_i = p \ \forall \ i \in [1:n]$

$$\sum_{i=1}^{n} \frac{1}{p} = \mu \Rightarrow \frac{n}{p} = \mu \Rightarrow p = \frac{n}{\mu}$$

$\therefore$ Minima of $Var[X] = \sum_{i=1}^{n} \frac{1-p_i}{p_i^2}$, under the constraint $E[X] = \mu$, occurs at $p_1 = p_2 = p_3 \cdots = p_n = \frac{n}{\mu}$.

Hence Proved.

Q7. X_1, X_2, X_3 are independent RV's, taking +ve integer values,

$$P_{X_i}(k) = (1-p_i)\,p_i^{k-1}, \quad \text{for } k \in \mathbb{N},\ i \in \{1,2,3\}$$

Find $P(X_1 < X_2 < X_3)$ and $P(X_1 \leq X_2 \leq X_3)$

$P_{X_i}(k) = (1-p_i)p_i^{k-1} \Rightarrow X_i$ is a geometric RV with param. $1-p_i$.

To find $P(X_1 < X_2 < X_3)$, if $X_1 = k_1$, $X_2 = k_2$ and $X_3 = k_3$ then $k_1 < k_2 < k_3$ for any favourable event.

$$\Rightarrow P(X_1 < X_2 < X_3) = \sum_{k_1=1}^{\infty} P_{X_1}(k_1) \sum_{k_2=k_1+1}^{\infty} P_{X_2}(k_2) \sum_{k_3=k_2+1}^{\infty} P_{X_3}(k_3)$$

$$= \sum_{k_1=1}^{\infty} (1-p_1)p_1^{k_1-1} \sum_{k_2=k_1+1}^{\infty} (1-p_2)p_2^{k_2-1} \sum_{k_3=k_2+1}^{\infty} (1-p_3)p_3^{k_3-1}$$

$$= (1-p_1)(1-p_2)(1-p_3) \sum_{k_1=1}^{\infty} p_1^{k_1-1} \sum_{k_2=k_1+1}^{\infty} p_2^{k_2-1} \sum_{k_3=k_2+1}^{\infty} p_3^{k_3-1}$$

$$\sum_{k_3=k_2+1}^{\infty} p_3^{k_3-1} = \frac{p_3^{k_2+1-1}}{1-p_3} = \frac{p_3^{k_2}}{1-p_3}$$

$$\Rightarrow P(X_1 < X_2 < X_3) = (1-p_1)(1-p_2) \sum_{k_1=1}^{\infty} p_1^{k_1-1} \sum_{k_2=k_1+1}^{\infty} p_2^{k_2-1} p_3^{k_2}$$

$$= \left(\frac{1-p_1}{p_1}\right)\left(\frac{1-p_2}{p_2}\right) \sum_{k_1=1}^{\infty} p_1^{k_1} \sum_{k_2=k_1+1}^{\infty} (p_2p_3)^{k_2}$$

$$\sum_{k_2=k_1+1}^{\infty} (p_2p_3)^{k} = \frac{(p_2p_3)^{k_1+1}}{1-p_2p_3}$$

$$\Rightarrow P(X_1 < X_2 < X_3) = \left(\frac{1-p_1}{p_1}\right)\left(\frac{1-p_2}{\cancel{p_2}}\right)\left(\frac{\cancel{p_2}\,p_3}{1-p_2p_3}\right)\sum_{k_1=1}^{\infty} p_1^{k_1}(p_2p_3)^{k_1}$$

$$\sum_{k_1=1}^{\infty}(p_1p_2p_3)^{k_1} = \frac{p_1p_2p_3}{1-p_1p_2p_3}$$

$$\Rightarrow P(X_1 < X_2 < X_3) = \left(\frac{1-p_1}{\cancel{p_1}}\right)\frac{(1-p_2)\,p_3}{1-p_2p_3}\,\frac{\cancel{p_1}\,p_2\,p_3}{1-p_1p_2p_3}$$

$$\Rightarrow P(X_1 < X_2 < X_3) = \frac{(1-p_1)(1-p_2)\,p_2\,p_3^2}{(1-p_2p_3)(1-p_1p_2p_3)}$$

$$P(X_1 \leq X_2 \leq X_3) = \sum_{k_1=1}^{\infty} P_{X_1}(k_1)\sum_{k_2=k_1}^{\infty} P_{X_2}(k_2)\sum_{k_3=k_2}^{\infty} P_{X_3}(k_3)$$

$$= \frac{(1-p_1)}{p_1}\frac{(1-p_2)}{p_2}\frac{(1-p_3)}{p_3}\sum_{k_1=1}^{\infty} p_1^{k_1}\sum_{k_2=k_1}^{\infty} p_2^{k_2}\sum_{k_3=k_2}^{\infty} p_3^{k_3}$$

$$= \frac{(1-p_1)(1-p_2)\cancel{(1-p_3)}}{p_1p_2p_3}\sum_{k_1=1}^{\infty} p_1^{k_1}\sum_{k_2=k_1}^{\infty} p_2^{k_2}\left(\frac{p_3^{k_2}}{\cancel{1-p_3}}\right)$$

$$= \frac{(1-p_1)(1-p_2)}{p_1p_2p_3}\sum_{k_1=1}^{\infty} p_1^{k_1}\,\frac{(p_2p_3)^{k_1}}{1-p_2p_3}$$

$$= \frac{(1-p_1)(1-p_2)}{\cancel{p_1p_2p_3}(1-p_2p_3)}\;\frac{\cancel{p_1p_2p_3}}{1-p_1p_2p_3}$$

$$\Rightarrow P(X_1 \leq X_2 \leq X_3) = \frac{(1-p_1)(1-p_2)}{(1-p_2p_3)(1-p_1p_2p_3)}$$