EC5.102: Information and Communication

Introduction to probability and random variables

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Reference books

- "Probabiilty and measure" by P. Billingsley
- "Probabiilty, Random variables, and Stochastic processes", by A. Papoulis

About my teaching style

- I will be using a combination of slides and board.
- Be super interactive.. ask questions..
- Discuss learnings of the class with your friends.
- Refer to the suggested reference books.
- There will be breakout sessions to solve problems (very important).
- There will be self-quizes/surprize-quizes in the class.. :P
- NO LAPTOPS in the class!

Recap of the previous class

Recap

- We now live a digital world!

  Analog signal $\rightarrow$ Sampling $\rightarrow$ Quantization $\rightarrow$ Bit-sequence

- **How to do further processing of this bit-sequence?**
  - Source encoding:

    Bit-sequence $\rightarrow$ Source encoding $\rightarrow$ Bit-sequence'
  - Channel encoding:

    Bit-sequence' $\rightarrow$ Channel encoding $\rightarrow$ Bit-sequence''

- We shall study source and channel coding soon.
- Digression: Basics of probability

Recap: Introduction to probability

- Need for probability: Our inability to say “something” about an outcome of an event in a “deterministic” manner!
- Applications: Communication systems, Forecasting, Finance, Machine learning, Computer science...
- Next agenda: Basics of probability and random variables
  - Probability space
  - Random variables (RVs)
  - Types of RVs: Discrete and continuous RVs
  - Joint and conditional RVs

Review: Set theory

Review: Set theory

- Universal set: $\mathcal{S}$
- Subset of set $\mathcal{S}$: $A \subseteq \mathcal{S}$
- Complement of A: A^c
- Intersection: $A \cap B$
- Union: $A \cup B$
- Empty (or null) set: Φ
- Element and simpleton sets

- $A \backslash B = A \cap B^c$
- Disjoint sets: $A \cap B = \Phi$
- $\Phi^c = \mathcal{S}$
- $(A \cup B)^c = A^c \cap B^c$
- $(A \cap B)^c =?$
- $A \cap A^c =?$
- $A \cup A^c =?$

Review: Set theory

- Countably finite set
- Countably infinite set
- Uncountably infinite set
- Power set of a set with finite elements
- Can we talk about power set of a set with infinite elements?

Set theory and probability space

Set theory	Probability space (to be defined precisely soon)
Universe	Sample space
Subset	Event
Element	Outcome
Simpleton set	Simple event
Null set	Impossible event
Disjoint sets	Mutually exclusive events

- Probability:
  - Assign a value called as "probability" to an event.
  - Can you relate this to a function/map?
- **Probability space**: Formalize the notions of sample space, events, and probability.

Probability space

Motivation for defining a probability space

- Let us try to “formalize” a random experiment.
- Example: Rolling a dice
  - How to “describe” this experiment mathematically?
  - Make a list of things that one needs do towards this.
  - Suppose our focus is on only two events A and B. Can you simplify this description?
- Can you think something similar for a coin toss experiment?
- Can you think something similar for an experiment of randomly choosing a real number between an interval $[0, 1]$?
- Probability space provides a formal model of a random experiment.

Probability space $(\Omega, \mathcal{F}, P)$

- Probability space provides a formal model of a random experiment.
- A probability space consists of three elements $(\Omega, \mathcal{F}, P)$:
  1. Sample space Ω: Set of all possible outcomes
  2. Event space $\mathcal{F}$: Collection of sets outcomes in Ω such that
     - ⋆ Contains both empty set ϕ and Ω
     - ⋆ Closed under complement:

       $$\text{If} \;\; A \in \mathcal{F} \;\; \text{then} \;\; A^c \in \mathcal{F}$$

     - ⋆ Closed under countable union

       $$\text{If} \;\; A_1, A_2, \ldots \in \mathcal{F} \;\; \text{then} \;\; \cup_{i=1}^{\infty} A_i \in \mathcal{F}$$

  3. Probability measure $P(.)$: Function from event space $\mathcal{F}$ to $[0, 1]$
- **Think: Probability space does capture "essential" model of a random experiment!**

Event space: Example

- Event space $\mathcal{F}$: 1. Contains ϕ and Ω
  2. If $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$
  3. If $A_1, A_2 \in \mathcal{F}$ then $A_1 \cup A_2 \in \mathcal{F}$

- Example: Rolling a six-sided dice ($\Omega = \{1, 2, 3, 4, 5, 6\}$)
  - Consider two events: $A = \{1 \cup 2 \cup 3\}$ and $B = \{1\}$
  - Event space $\mathcal{F}_A$ generated by A will be

$$\mathcal{F}_A = \{\Omega, \phi, A, A^c\} = \Big\{\Omega, \phi, \{1 \cup 2 \cup 3\}, \{4 \cup 5 \cup 6\}\Big\}$$

  - Event space $\mathcal{F}_B$ generated by B will be

$$\mathcal{F}_B = \{\Omega, \phi, B, B^c\} = \Big\{\Omega, \phi, \{1\}, \{2 \cup 3 \cup 4 \cup 5 \cup 6\}\Big\}$$

  - Event space $\mathcal{F}_{AB}$ generated by A and B will be

$$\begin{aligned}\mathcal{F}_{AB} = \Big\{&\Omega, \phi, A, A^c, B, B^c, (A \cup B), (A \cup B)^c, (A \cup B^c), (A \cup B^c)^c,\\ &(A^c \cup B), (A^c \cup B)^c, (A^c \cup B^c), (A^c \cup B^c)^c\Big\}\\ = \Big\{&\Omega, \phi, A, A^c, B, B^c, A^c \cup B, (A^c \cup B)^c\Big\}\\ = \Big\{&\Omega, \phi, A, A^c, B, B^c, \{1 \cup 4 \cup 5 \cup 6\}, \{2 \cup 3\}\Big\}\end{aligned}$$

Event space: Example

- Example: Rolling a six-sided dice ($\Omega = \{1, 2, 3, 4, 5, 6\}$)
- Power set of the sample space will be

$$
\begin{aligned}
\mathcal{F} = \Big\{ & \Phi, \{1 \cup 2 \cup 3 \cup 4 \cup 5 \cup 6\}, \\
& \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \\
& \{1 \cup 2\}, \{1 \cup 3\}, \ldots, \{5 \cup 6\}, \\
& \{1 \cup 2 \cup 3\}, \ldots, \{4 \cup 5 \cup 6\}, \\
& \{1 \cup 2 \cup 3 \cup 4\}, \ldots, \{3 \cup 4 \cup 5 \cup 6\}, \\
& \{1 \cup 2 \cup 3 \cup 4 \cup 5\}, \ldots, \{2 \cup 3 \cup 4 \cup 5 \cup 6\} \Big\}
\end{aligned}
$$

- Power set $\mathcal{F}$ is a valid event set: Homework
- How many elements will be there in power set $\mathcal{F}$?

Axioms of probability

Probability measure

- Probability measure $P(.)$: Function from event space $\mathcal{F}$ to $[0, 1]$
- Axioms of probability:
  - For all $A \in \mathcal{F}$, $0 \leq P(A) \leq 1$
  - $P(\phi) = 0$
  - $P(\Omega) = 1$
  - If $A \cap B = \phi$ then $P(A \cup B) = P(A) + P(B)$
- Example: Rolling a six-sided (biased) dice

$$P(\{1\}) = 0.1, \ P(\{2\}) = 0.2, \ P(\{3\}) = 0.1,$$
$$P(\{4\}) = 0.3, \ P(\{5\}) = 0.15, \ P(\{6\}) = 0.15,$$

Various concepts in probability

- Conditional probability
- Independent events
- Mutually exclusive events
- Total probability theorem
- Bayes' theorem

Motivation to random variables

Example of rolling two dice

- Example of rolling two dice where we are interested in the sum of two dice.
- Suppose $X =$ sum of two dice. Then we have

$$\begin{aligned} \Omega = \Big\{ &(1,1), (1,2), \ldots, (1,6) \\ &(2,1), (2,2), \ldots, (2,6) \\ &\qquad\quad \vdots \qquad\qquad\qquad\qquad \xrightarrow{X} \qquad \Omega' = \{2, 3, \ldots, 12\} \\ &(6,1), (6,2), \ldots, (6,6) \Big\} \end{aligned}$$

- Suppose $\mathcal{F}$ and $\mathcal{F}'$ are power sets of Ω and Ω' respectively.
- Can you write down P'?
- We have a map X given by

$$X : (\Omega, \mathcal{F}, P) \to (\Omega', \mathcal{F}', P')$$

- For our application, it is more convenient to work with $(\Omega', \mathcal{F}', P')$ than $(\Omega, \mathcal{F}, P)$.

Example of choosing two real numbers from $[0, 1]$

- Choose any two real numbers from $[0, 1]$
  - What will be Ω?
  - What will be $\mathcal{F}$? (to be discussed soon)
  - What will be P? (e.g. choose any number with uniform probability)
- We are interested only in addition of these two numbers.
  - What will be Ω'?
  - What will be $\mathcal{F}'$?
  - What will be P'? (Will it be uniform? Yes/no?)
- We have a map X given by

$$X : (\Omega, \mathcal{F}, P) \rightarrow (\Omega', \mathcal{F}', P')$$

- It is more convenient to work with $(\Omega', \mathcal{F}', P')$ than $(\Omega, \mathcal{F}, P)$.

Motivation to random variables

- One can write down the event space $\mathcal{F}$, when Ω has finite entries.
- What to do when Ω has infinite (countable/uncountable) entries?
- Given a random experiment with associated $(\Omega, \mathcal{F}, P)$, it is sometimes difficult to deal directly with $\omega \in \Omega$. Example: Rolling a dice twice
- Depending on the experiment, Ω will be different.
- Can we come up with a general platform which is independent of the choice of a particular Ω?
- It is desirable and convenient to study probability space and related advanced concepts using this general platform!

Answer: Random Variables (rv or RV)