

Probability and Random Processes - Assignment 1

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Q1. Let Ω and \mathcal{F} be the sample space and event space of some probability space.

Let A_1, A_2 be any 2 atoms in \mathcal{F} . Then, we can say that,

$$\textcircled{1} \quad A_1, A_2 \neq \emptyset, \quad \textcircled{2} \quad \begin{aligned} & \nexists B \subseteq A_1, \quad B = A_1 \text{ or } B = \emptyset \\ & \nexists C \subseteq A_2, \quad C = A_2 \text{ or } C = \emptyset \end{aligned}$$

Assume, that A_1, A_2 are not disjoint; ie, $\exists x \in \Omega$ st $x \in A_1$ and $x \in A_2$, or $A_1 \cap A_2 \neq \emptyset$.

Let $B = \{x\}$. Clearly $B \subseteq A_1$ and $B \subseteq A_2$, and $B \neq \emptyset$.

Since $B \neq \emptyset$, by $\textcircled{2}$, we can say that $B = A_1$ and $B = A_2$

$$\Rightarrow A_1 = A_2$$

$$\Rightarrow \text{If } A_1 \cap A_2 \neq \emptyset, \text{ then } A_1 = A_2$$

$$\Rightarrow \text{If } A_1 \neq A_2, \text{ then } A_1 \cap A_2 = \emptyset \quad [\text{Negation of the prev. statement}]$$

∴ Two distinct atoms are necessarily disjoint from each other.

Q2. $E_1, E_2, E_3, \dots, E_n$ are mutually exclusive and exhaustive events in sample space Ω .

$$E_i \cap E_j = \emptyset \quad \forall i \neq j \quad [\text{Mutually Exclusive}]$$

∴ Smallest σ -field: $\mathcal{F} = \left\{ \bigcup_{i \in I} E_i : I \subseteq \{1, 2, 3, \dots, n\} \right\}$

To determine the validity of \mathcal{F} as a σ -field.

1) $\Omega \in \mathcal{F}$.

If $I = \{1, 2, 3, \dots, n\}$ [A set is a subset of itself]

$$\bigcup_{i \in I} E_i = E_1 \cup E_2 \dots \cup E_n = \Omega$$

∴ $\Omega \in \mathcal{F}$

2) Closure under complement

Since each E_i is disjoint from another and they are collectively exhaustive, the complement of the union of some E_i 's, is the union of all other E_i 's, ie,

$$\left(\bigcup_{i \in I} E_i \right)^c = \bigcup_{i \notin I} E_i \text{ or } \bigcup_{i \in I^c} E_i$$

Since $I^c \subseteq \{1, 2, 3, \dots, n\}$, $\bigcup_{i \in I^c} E_i \in \mathcal{F}$

\therefore Closure under complement is satisfied.

3) Closure under Union.

Let A_1, A_2, A_3, \dots be a sequence of events, such that

$$A_k = \bigcup_{i \in I_k} E_i, \quad I_k \subseteq \{1, 2, 3, \dots, n\} \quad \forall k \in \mathbb{N}$$

$$\bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} \left(\bigcup_{i \in I_k} E_i \right) = \bigcup_{i \in \bigcup_{k=1}^{\infty} I_k} E_i$$

Since $I_k \subseteq \{1, 2, 3, \dots, n\} \quad \forall k$, $\bigcup_{k=1}^{\infty} I_k \subseteq \{1, 2, 3, \dots, n\}$

$\therefore \bigcup_{i \in \bigcup_{k=1}^{\infty} I_k} E_i \in \mathcal{F} \Rightarrow$ Closure under Union is satisfied.

Therefore, the above mentioned σ -field is valid.

Q3. $P(A) = \frac{3}{4}, P(B) = \frac{1}{3}$

a) $P(A \cap B)$ is maximum when $B \subseteq A$. [$A \not\subseteq B$ as $P(A) > P(B)$]

$$\Rightarrow P(A \cap B) = P(B) = \frac{1}{3}$$

$P(A \cap B)$ is minimum when $A \cup B = \Omega$ [ie, exhaustive]

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\Rightarrow P(A \cap B) = \left(\frac{3}{4} + \frac{1}{3}\right) - 1 = \frac{13}{12} - 1$$

$$\Rightarrow P(A \cap B) = \frac{1}{12}$$

$$\Rightarrow \underline{\frac{1}{12} \leq P(A \cap B) \leq \frac{1}{3}}$$

b) i) Maxima condition.

$$\text{Let } \Omega = \{x, y, z\}, P(\{x\}) = \frac{4}{12} = \frac{1}{3}, P(\{y\}) = \frac{5}{12},$$

$$P(\{z\}) = \frac{3}{12} = \frac{1}{4}. P(A) = \sum_{k \in A} P(\{k\})$$

Define $A = \{x, y\}, B = \{x\}. \Rightarrow B \subseteq A$

$$P(A) = \frac{4}{12} + \frac{5}{12} = \frac{9}{12} = \frac{3}{4}$$

$$P(B) = \frac{4}{12} = \frac{1}{3}$$

$$P(A \cap B) = \underline{\underline{P(B)}} = \frac{1}{3}$$

ii) Minima Condition

$$\text{Let } \Omega = \{x, y, z\}, P(\{x\}) = \frac{1}{12}, P(\{y\}) = \frac{8}{12}, P(\{z\}) = \frac{3}{12}$$

$$P(A) = \sum_{k \in A} P(\{k\}).$$

Define $A = \{x, y\}, B = \{x, z\}$.

$$P(A) = \frac{1}{12} + \frac{8}{12} = \frac{9}{12} = \frac{3}{4}$$

$$P(B) = \frac{1}{12} + \frac{3}{12} = \frac{4}{12} = \frac{1}{3}$$

$$P(A \cap B) = P(\{x\}) = \underline{\frac{1}{2}}$$

Q4. C1 - Heads on both faces

C2 - Tail on both faces

C3 - Head and Tail

To find: If the outcome is head, find the probability of the other side being tail, ie, $P(C_3|H) = ?$

$$P(C_3|H) = \frac{P(H|C_3) P(C_3)}{P(H)}$$

[Bayes' Theorem]

$$= \frac{P(H|C_3) P(C_3)}{P(H|C_1) P(C_1) + P(H|C_2) P(C_2) + P(H|C_3) P(C_3)}$$

$$= \frac{\frac{1}{2} \times \frac{1}{3}}{1 \times \frac{1}{3} + 0 \times \frac{1}{3} + \frac{1}{2} \times \frac{1}{3}}$$

$$= \frac{\frac{1}{6}}{\frac{1}{3} + \frac{1}{6}}$$

$$= \frac{\frac{1}{6}}{\frac{3}{6}} = \underline{\frac{1}{3}}$$

∴ Probability of choosing C_3 given that the outcome was H is $\frac{1}{3}$

Q5. $C = \{A_1, A_2, A_3, \dots, A_n\}$ are mutually independent events, ie,

$$P(A_i \cap A_j) = P(A_i) \cdot P(A_j) \quad \forall i \neq j$$

To prove: $(C \setminus \{A_i\}) \cup \{A_i^c\}$ is a collection of mutually independent events for any i .

$(C \setminus \{A_i\}) \cup \{A_i^c\} \rightarrow$ Essentially, we replace A_i with A_i^c in the collection C .

For any A_j ($\text{st } j \neq i$) in C , we know that

$$P(A_j \cap A_i) = P(A_j) P(A_i)$$

$$\begin{aligned} P(A_j \cap A_i^c) &= P(A_j) - P(A_j \cap A_i) && [A \cap B^c = A \setminus B \\ &&& = A \setminus (A \cap B)] \\ &= P(A_j) - P(A_j)P(A_i) && [A_j \text{ is independent of } A_i \\ &&& \text{ & } j \neq i] \\ &= P(A_j)(1 - P(A_i)) \\ &= P(A_j) P(A_i^c) \end{aligned}$$

$$\Rightarrow P(A_j \cap A_i^c) = P(A_j) P(A_i^c) + \sum_{j \neq i}$$

$\therefore A_i^c$ is independent of all A_j et $j \neq i$.

$\therefore (C \setminus \{A_i\}) \cup \{A_i^c\}$ is a collection of mutually independent sets.

Q6. Let A, B be the events of Alice and Bob saying that E occurred respectively.

To find: $P(E|A \cap B) = ?$

$$P(E|A \cap B) = \frac{P(A \cap B|E) P(E)}{P(A \cap B|E) P(E) + P(A \cap B|E^c) P(E^c)}$$

[Bayes' Theorem]

$$= \frac{\left(\frac{9}{10} \times \frac{9}{10}\right) \left(\frac{1}{1000}\right)}{\frac{9}{10} \times \frac{9}{10} \times \frac{1}{1000} + \frac{1}{10} \times \frac{1}{10} \times \frac{999}{1000}}$$

$$= \frac{81}{81 + 999} = \frac{81}{1080}$$

$$= \underline{\underline{0.075}}$$

\therefore The probability that E actually occurred, given that both Alice and Bob said it occurred is 0.075.

Q7. The man starts with 200 £ and must reach 20,000,000 £. Each round he either wins or loses 1 £, with equal probability.

This represents a classical gambler's ruin problem.

Let the current amount of money be i , and the target be n . Let P_i represent the probability of winning with i as the starting amount.

We can say that,

$$P_i = \frac{1}{2} P_{i-1} + \frac{1}{2} P_{i+1} \quad \# 0 < i < n$$

①

loses current round wins current round

so must win with so must win with
amount $i-1$ amount $i+1$

Also, clearly, $P_0 = 0$ and $P_n = 1$ (Already lost or won the
whole game, respectively)

Solving eq. ①,

$$P_i = \frac{1}{2} P_{i+1} + \frac{1}{2} P_{i-1}$$

$$P_i = \frac{1}{2} \left(\frac{1}{2} P_{i+2} + \frac{1}{2} P_i \right) + \frac{1}{2} \left(\frac{1}{2} P_i + \frac{1}{2} P_{i-2} \right)$$

$$= \frac{1}{4} P_{i+2} + \frac{1}{2} P_i + \frac{1}{4} P_{i-2}$$

$$\Rightarrow P_i = \frac{1}{2} P_{i+2} + \frac{1}{2} P_{i-2}$$

$$\Rightarrow P_i = \frac{1}{2} P_{i+n} + \frac{1}{2} P_{i-n} \quad \# n \leq i$$

$$\Rightarrow P_{200} = \frac{1}{2} P_{200+200} + \frac{1}{2} P_{200-200} = \underline{\frac{1}{2} P_{400}} + \frac{1}{2} \cancel{P_0}^0$$

$$\Rightarrow P_{200} = \frac{1}{2} P_{400}$$

$$\text{Hence } P_{400} = \frac{1}{2} P_{800} \Rightarrow P_{200} = \frac{1}{4} P_{800}$$

$$\therefore P_{200} = \frac{1}{\frac{n}{200}} P_n = \underline{\underline{\frac{200}{n} P_n}}$$

$$n = 2000000$$

$$\Rightarrow P_{200} = \frac{1}{\frac{2000000}{200}} P_{2000000} \quad |$$

$$\Rightarrow P_{200} = \frac{1}{10000} = \underline{\underline{0.0001 \%}}$$

\therefore Probability of winning the game, starting at 200 ₦ and reaching 20,00,000 ₦, is 0.0001 %.

\Rightarrow Probability of Bankruptcy = 0.9999 %