

EC5.102: Information and Communication

(Lec-4)

Source coding-4

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Summary of the last class

Recap

- Introduction to source coding:
 - ▶ Definition
 - ▶ Expected length of code $L(C)$
- Types of source codes:
 - ▶ Singular/Non-singular codes
 - ▶ Uniquely decodable codes
 - ▶ Prefix or instantaneous

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- Algorithms for constructing source codes:
 - ▶ Huffman codes
 - ▶ Lempel-Ziv algorithm

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 - ▶ Encoding

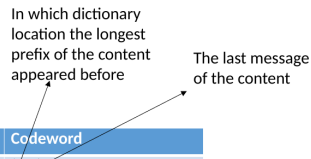
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 - ▶ Encoding
 - ▶ Decoding

Lempel Ziv coding

Lempel-Ziv (L-Z) Coding: Example 1

- Consider the sequence: 1 0 1 0 1 1 0 1 1 0 1 0 1 0 1 0
- Parsing: Identify **phrases of the smallest length** that haven't appeared before.
- After parsing the phrases are: 1, 0, 1 0, 1 1, 0 1, 1 0 1, 0 1 0, 1 0 1 0
- Notice: **New phrase is concatenation of a previous phrase and a new source message.**
- Encoding: Lexicographic (dictionary) ordering of the previous phrase and the new source message are concatenated.



Dictionary location	Contents	Codeword
1	1	(0,1)
2	0	(0,0)
3	10	(1,0)
4	11	(1,1)
5	01	(2,1)
6	101	(3,1)
7	010	(5,0)
8	1010	(6,0)

Lempel-Ziv (L-Z) Coding

The steps of L-Z algorithm are as follows:

- Any sequence of the source output is uniquely parsed into phrases of varying length and these phrases are encoded using codewords of equal length. (This is a **variable-to-fixed length** coding scheme.)
- Parsing is done by identifying **phrases of the smallest length** that have not appeared before.
- The new phrase is the concatenation of a previous phrase and a new source message.
- Encoding: Lexicographic (dictionary) ordering of the previous phrase and the new source message are concatenated.

Today's agenda

- Kraft inequality
- Show that: For Prefix codes, $L(C) \geq H(X)$
- Statement of source coding theorem

Kraft inequality

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- Let $\ell_1, \ell_2, \dots, \ell_m$ be codeword lengths of an instantaneous code.
- **Kraft inequality: Any binary instantaneous code with lengths $\ell_1, \ell_2, \dots, \ell_m$ should satisfy:**

$$\sum_{i=1}^m 2^{-\ell_i} \leq 1.$$

- Proof: In Class

Kraft inequality: Proof outline

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- A codeword at level ℓ_i has $2^{\ell_m - \ell_i}$ descendants at level ℓ_m .

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$$\sum_{i=1}^m 2^{\ell_m - \ell_i} \leq 2^{\ell_m}$$

For Prefix codes, $L(C) \geq H(X)$

Theorem

- Show that: The average length of any prefix-free source code is lower bounded by entropy of the source, i.e.,

$$L(C) \geq H(X).$$

- Proof: In Class

Source Coding Theorem (SCT) (Formal statement)

Huffman codes with
 $L(C)$ approaching $H(X)$

Huffman code: Example

- A zero-memory source emits messages m_1 and m_2 with probabilities 0.8 and 0.2 respectively.
 - ▶ Find binary Huffman code.
 - ▶ Find Huffman code for its second and third order extensions.
 - ▶ Determine code efficiency in each case.
- **Solution:** In class

Huffman code: Solution of Example

- Huffman code for $n = 1$ will be 0 and 1.
 - $L(C) = 1$ and $H(X) = -(0.8 \log_2(0.8) + 0.2 \log_2(0.2)) = 0.72$ bits
 - $\eta = H(X)/L(C) = 0.72$
- Huffman code for $n = 2$:

m_1	m_1	0.64	0
m_1	m_2	0.16	11
m_2	m_1	0.16	100
m_2	m_2	0.04	101

- $L_1(C) = 1.56$. But this word length for two messages of the original source.
- Word length per message will be $L(C) = 1.56/2 = 0.78$
- $\eta = 0.72/0.78 = 0.923$

Huffman code: Solution of Example

- Huffman code for $n = 3$:

m_1	m_1	m_1	0.512	0
m_1	m_1	m_2	0.128	100
m_1	m_2	m_1	0.128	101
m_2	m_1	m_1	0.128	110
m_1	m_2	m_2	0.032	11100
m_2	m_1	m_2	0.032	11101
m_2	m_2	m_1	0.032	11110
m_2	m_2	m_2	0.008	11111

- $L_3(C) = 2.184$. But this word length for three messages of the original source.
- Word length per message will be $L(C) = 2.184/3 = 0.728$
- $\eta = 0.72/0.728 = 0.989$

Observations from Example

- Summary:
 - ▶ For $n = 1$, $\eta = 0.72$
 - ▶ For $n = 2$, $\eta = 0.923$
 - ▶ For $n = 3$, $\eta = 0.989$
- Thus, as the block length increases, the coding efficiency improves and approaches to 1.
- As we use the Huffman coding algorithm over longer and longer blocks of symbols, the average number of bits required to encode each symbol approaches the entropy of the source.
- We will next see why is it so.

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- ▶ **Converse:**

If $L(C) < H(X)$, then $P_e^{(n)} > 0$ for any n .

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- It states that, a source with entropy rate H can be encoded with arbitrarily small error probability at any rate R (bits/source output) provided, $R > H$.
- If $R < H$, the error probability will be bounded away from zero, independent of the complexity of the encoder and decoder employed.