Prof: Prabhakar Bhimalapuram

# Science - 1

## Group - B

Marks :

Endsem : 35 %

Midsem : 20 % $\pm 5\%$

Quiz : 10 + 10 %

In Class / Tut Quiz : 25 %

Tutorial Quizzes

# Science - 1

- Science explains everything on the time-scales, the length scales and the energy scales.

Time Scales : Climate Change (Decade - scale)
Bullet travel (Second - scale)
Molecular movement (Picosecond - scale)

Length Scales : High Energy Physics (Quark/lepton Scales)
Expansion of Universe ($10^{35}$ m scale)

Energy Scale : Energy to lift a laptop
" " build a building

- There are a variety of things on these scales that we use in our daily life.

→ **What is Science?** :

- Systematic investigation of natural phenomenon.
- Reproducibility is a key feature of Science. Any experiment must be reproducable. Even a single datapoint is sufficient to disprove a theory.

- We can use the data gathered from the experiment, to create a mathematical model, than can be used to predict things. Mathematical modelling has enabled the development of sciences.

- Science will try to explain the behavior / reason behind the parameters present in the model, and also the limitations / inaccur-acies in the model.

  Newton's first law is not directly provable, since it requires highly ideal conditions. It is only true at asymptotical conditions.

  ⟶ Limitation of Newton's laws.

- Reductionism: Separation of the most important things and neglecting every other factors. ex: The radius of planets while observing their orbital behavior around the sun, through Kepler's planetary laws.

"A System is a sum of its parts" (Superposition-like)

By decomposing a system into its pieces, we can figure out the interactions between the parts and also the most important parts of the system.

- **Microscopics**: The study behind the various interactions between the parts of a system.

- A reductionism model can be used to accurately predict the behavior of the model.

- Reductionism goes from microscopic behavior to macroscopic features.

- Even microscopic interactions follow laws, like conservation of energy.

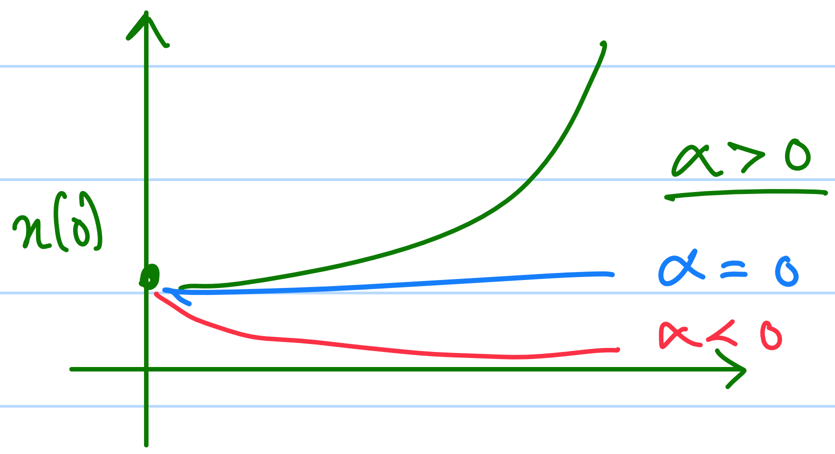
→ **Mathematics in Science:**

- Geometry and Linear Algebra

- Change and Calculus

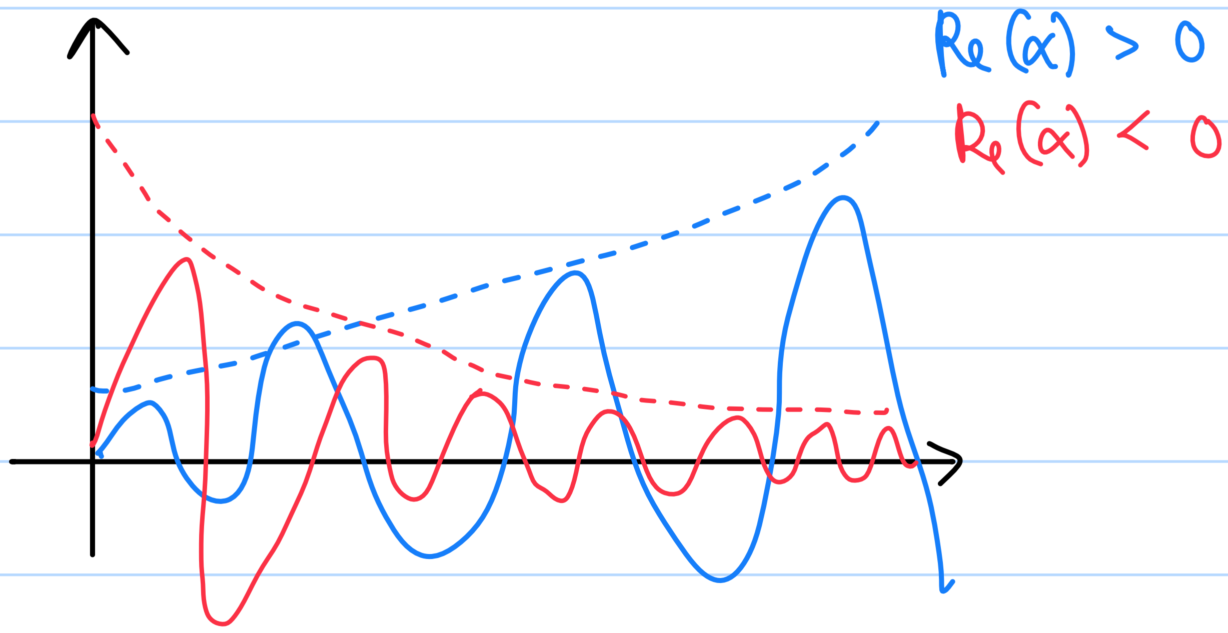
- Chance, probability and statistics. → Uncertainty Principle

ex: The Behavior of a system with $x(t) = x(0)\, e^{\alpha t}$



In the above system, the value of the parameter can fundamentally change the behavior of the system.

If $\alpha \in \mathbb{C}$




→ **The Prey-Predator Model** :-

- $x$ - No. of Rabbits
  $y$ - No. of Tigers

- **Parameters** :
  $\alpha$ - Growth rate of Rabbits
  $\beta$ - Effect of foxes on the rabbits (Rabbits are killed)
  $\gamma$ - Death rate of Foxes
  $\delta$ - Effect of rabbits on foxes (Foxes can reproduce with nutrition)

$$\frac{dx}{dt} = \alpha x - \beta xy \qquad \frac{dy}{dt} = -\gamma y + \delta xy$$

$$\Rightarrow \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha x - \beta xy \\ -\gamma y + \delta xy \end{pmatrix} \begin{matrix} \longrightarrow f \\ \longrightarrow g \end{matrix}$$

$$\hookrightarrow \gamma \begin{pmatrix} x \\ y \end{pmatrix} = \nearrow$$

- The equilibrium points of the system,

$$\frac{dx}{dt} = 0 \qquad\qquad \frac{dy}{dt} = 0$$

$$\Rightarrow \alpha x - \beta xy = 0 \qquad\qquad \Rightarrow -\gamma y + \delta xy = 0$$

$$\Rightarrow x(\alpha - \beta y) = 0 \qquad\qquad \Rightarrow y(-\gamma + \delta x) = 0$$

$$\Rightarrow x = 0 \, , \; y = \frac{\alpha}{\beta} \qquad\qquad \Rightarrow y = 0 \, , \; x = \frac{\gamma}{\delta}$$

$\Rightarrow (0,0), \left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$ are the equilibrium points.

$f(x,y)$ $\quad \left[ \; \delta f_x \;\; \delta f_y \right.$

Jacobian of the system,

$$\begin{pmatrix} \alpha x - \beta xy \\ -\gamma y + \delta xy \end{pmatrix} \xrightarrow{J} \begin{pmatrix} \alpha - \beta y & -\beta x \\ \delta y & -\gamma + \delta x \end{pmatrix} = J$$

$$J\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right) = \begin{pmatrix} 0 & -\beta\gamma/\delta \\ \delta\alpha/\beta & 0 \end{pmatrix} \longrightarrow \gamma'$$

Let $(x_0, y_0) = \left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$ and define $\varepsilon = x - x_0 \, , \; \eta = y - y_0$

Around the equilibrium point, the function can be approximated as,

$$\frac{d}{dt}\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} = J\Big|_{x_0, y_0} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + \cancel{\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}}^{0}\Bigg|_{x_0, y_0}$$

$$\Rightarrow \boxed{\frac{d}{dt}\begin{pmatrix} \varepsilon \\ \eta \end{pmatrix} = \begin{pmatrix} 0 & -\beta\gamma/\delta \\ \delta\alpha/\beta & 0 \end{pmatrix} \begin{pmatrix} \varepsilon \\ \eta \end{pmatrix}}$$

$$\approx \boxed{\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -\beta\gamma/\delta \\ \delta\alpha/\beta & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}}$$ $\rightarrow$ near the equilibrium of $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

Note: The approximation of the function around $(x_0, y_0)$ is done using the result,

Linearization

$$h(x,y) = h(x_0, y_0) + \left.\frac{\delta h}{\delta x}\right|_{x_0}(x - x_0) + \left.\frac{\delta h}{\delta y}\right|_{y_0}(y - y_0)$$

$$= h(x_0, y_0) + \left[\frac{\delta h}{\delta x} \quad \frac{\delta h}{\delta y}\right]\Bigg|_{x_0, y_0}\begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$$

$\uparrow$ is from the Jacobian

III analysis can be done near $(0,0)$.

Jacobian near $(0,0)$, $J\Big|_{0,0} = \begin{pmatrix} \alpha & 0 \\ 0 & -\delta \end{pmatrix}$

$$\Rightarrow \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & -\delta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$ $dt = \frac{dx}{\alpha x}$

$$\Rightarrow \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha x \\ -\delta y \end{pmatrix} \rightarrow \frac{dx}{dt} = \alpha x, \quad \frac{dy}{dt} = -\delta y$$

$$\Rightarrow x = c_1 e^{\alpha t}, \quad y = c_2 e^{-\delta t}$$

$$\Rightarrow \boxed{x = x(0)\, e^{\alpha t} \qquad y = y(0)\, e^{\alpha t}}$$

For the other Eq. point, $\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$,

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -\beta\gamma/\delta \\ \delta\alpha/\beta & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \quad \frac{d\bar{\gamma}}{dt} = \bar{J}\,\bar{\gamma}$$

The general solutions of such a system is of the form,

$$\bar{\gamma}(t) = c_1 e^{\lambda_1 t}\,\bar{v}_1 + c_2 e^{\lambda_2 t}\,\bar{v}_2$$

Where, $\bar{v}_1, \bar{v}_2$ are the Eigenvectors of $J$, $\lambda_1, \lambda_2$ are their respective Eigenvalues.

The Eigenvalues of $J$ are,

$$\det\begin{pmatrix} -\lambda & -\beta\gamma/\delta \\ \delta\alpha/\beta & -\lambda \end{pmatrix} = 0$$

$$= \lambda^2 - \left(\frac{-\cancel{\beta}\gamma}{\cancel{\delta}}\right)\left(\frac{\cancel{\delta}\alpha}{\cancel{\beta}}\right) = 0$$

$$= \lambda^2 - (-\alpha\gamma) = 0$$

$$= \lambda^2 = -\alpha\gamma$$

$$= \lambda = \pm i\sqrt{\alpha\gamma}$$

For $\lambda_1 = i\sqrt{\alpha\gamma}$,

$$\begin{pmatrix} -i\sqrt{\alpha\gamma} & -\frac{\beta\gamma}{\delta} \\ \frac{\delta\alpha}{\beta} & -i\sqrt{\alpha\gamma} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad -x_1 i\sqrt{\alpha\gamma} - \frac{\beta\gamma}{\delta} x_2 = 0$$

$$\Rightarrow \quad x_1 i\sqrt{\alpha\gamma} = \frac{-\beta\gamma}{\delta} x_2$$

$$x_1 = \frac{-\beta\gamma}{i\delta\sqrt{\alpha\gamma}} x_2 = -\frac{\beta}{i\delta}\sqrt{\frac{\gamma}{\alpha}}\, x_2$$

$$\frac{\delta\alpha}{\beta} x_1 - i\sqrt{\alpha\gamma}\, x_2 = 0$$

$$= \quad x_1 = \frac{i\sqrt{\alpha\gamma}\,\beta}{\delta\alpha} x_2 = \frac{i\beta}{\delta}\sqrt{\frac{\gamma}{\alpha}}\, x_2$$

$$\Rightarrow \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{i\beta}{\delta}\sqrt{\frac{\gamma}{\alpha}} \\ 1 \end{pmatrix} = \bar{v}_1$$

For $\lambda_2 = -i\sqrt{\alpha\gamma}$

$$\begin{pmatrix} i\sqrt{\alpha\gamma} & -\beta\gamma/\delta \\ \delta\alpha/\beta & i\sqrt{\alpha\gamma} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$= \quad i\sqrt{\alpha\gamma}\, x_1 - \frac{\beta\gamma}{\delta} x_2 \Rightarrow x_1 = \frac{\beta}{i\delta}\sqrt{\frac{\gamma}{\alpha}} = \frac{-i\beta}{\delta}\sqrt{\frac{\gamma}{\alpha}}$$

$$\Rightarrow \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -i\beta/\delta\sqrt{\gamma/\alpha} \\ 1 \end{pmatrix} = \bar{v}_2$$

$$\Rightarrow \quad \bar{\gamma}(t) = c_1 e^{i\sqrt{\alpha\gamma}t} \begin{pmatrix} i\beta/\delta\sqrt{\gamma/\alpha} \\ 1 \end{pmatrix} + c_2 e^{-i\sqrt{\alpha\gamma}t} \begin{pmatrix} -i\beta/\delta\sqrt{\gamma/\alpha} \\ 1 \end{pmatrix}$$

$$= \bar{r}(t) = c_1\left(\cos\left(\sqrt{\alpha\gamma}\,t\right) + i\sin\left(\sqrt{\alpha\gamma}\,t\right)\right)\begin{pmatrix}\frac{i\beta\sqrt{\gamma}}{\delta\sqrt{\alpha}}\\ 1\end{pmatrix}$$

$$+ c_2\left(\cos\left(\sqrt{\alpha\gamma}\,t\right) - i\sin\left(\sqrt{\alpha\gamma}\,t\right)\right)\begin{pmatrix}-\frac{i\beta\sqrt{\gamma}}{\delta\sqrt{\alpha}}\\ 1\end{pmatrix}$$

$$\Rightarrow \bar{r}(t) = \left[c_1\begin{pmatrix}\frac{i\beta\sqrt{\gamma}}{\delta\sqrt{\alpha}}\\ 1\end{pmatrix} + c_2\begin{pmatrix}\frac{-i\beta\sqrt{\gamma}}{\delta\sqrt{\alpha}}\\ 1\end{pmatrix}\right]\cos\left(\sqrt{\alpha\gamma}\,t\right)$$

$$+ \left[c_1\begin{pmatrix}\frac{i\beta\sqrt{\gamma}}{\delta\sqrt{\alpha}}\\ 1\end{pmatrix} + c_2\begin{pmatrix}\frac{-i\beta\sqrt{\gamma}}{\delta\sqrt{\alpha}}\\ 1\end{pmatrix}\right] i\sin\left(\sqrt{\alpha\gamma}\,t\right)$$

## → Common Differential Equations :-

- $$\frac{dx}{dt} = \alpha x \Rightarrow x = c_1 e^{\alpha t}$$

- $$\frac{\delta^2 f}{\delta t^2} = c^2 \frac{\delta^2 f}{\delta x^2} \longrightarrow \text{Diffusion Equation}$$

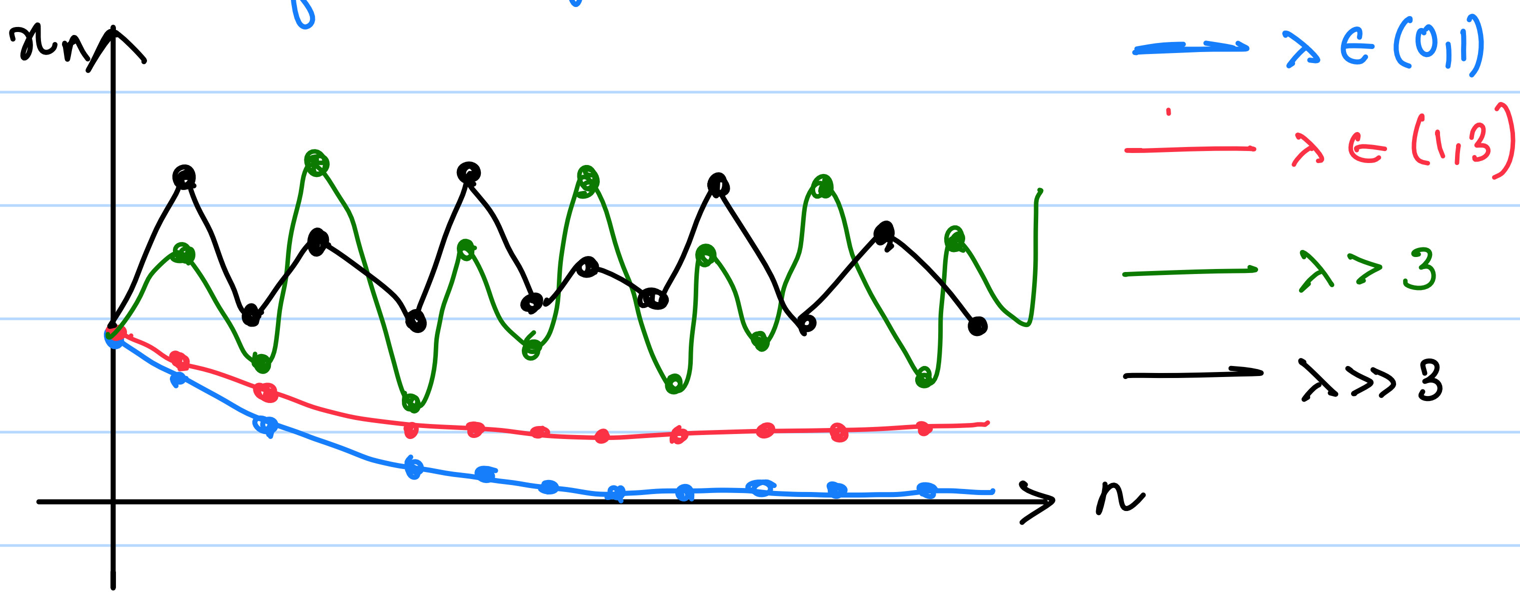
  $$\frac{\delta f}{\delta t} = c^2 \frac{\delta^2 f}{\delta x^2} \longrightarrow \text{Quantum Mechanics}$$

## → Logistics Mapping :-

- A sequence such that,

$$x_{n+1} = \lambda x_n (1 - x_n), \quad \forall n \in \mathbb{N}$$

- Analyzing the terms of the sequence,




Therefore, depending on the value of $\lambda$, the same initial value can result in very different sequences, and for some $\lambda$, small difference in initial value can result in large difference in the final value.

- Therefore, this sequence is said to exhibit chaotic behavior.

→ **Classical Mechanics :-**

- **Scalar Field** : Each point in space is associated with a scalar quantity. ex: Temperature

- **Vector Field** : Each point in space is associated with a vector of some direction and magnitude. ex: Air Speeds.

- **Gradient** : The change of a quantity within its field.

$$\nabla f(x_0, y_0, z_0) = \left[ \frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y} \quad \frac{\partial f}{\partial z} \right]$$

- **Divergence** : Scalar that measures the outflowness of a vector field at a point.

- **Laws Of Motion** :

1) In an inertial frame, an object undergoes uniform rectilinear motion in the absence of external forces.

2) Rate of change of momentum is equal to the net force on the object.

3) Reactive force is equal and opposite to the action force.

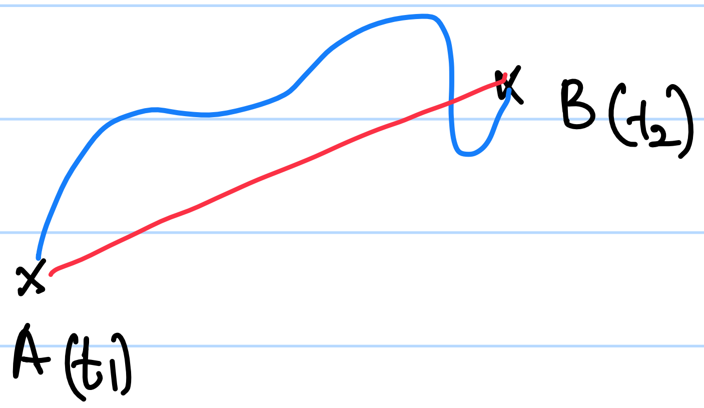
- **Work:** The work done by the application of an external process is given by,

$$W = \int_{t_1}^{t_2} \bar{F}(t)\, d\bar{r}(t)$$ infinitesimal $F_r$

Using the 2nd Law,

$$W = \int_{t_1}^{t_2} \frac{d\bar{p}}{dt} v(t)\, dt = \frac{1}{2} m v_2^2 - \frac{1}{2} m v_2^1$$

$$\Rightarrow \quad W = KE_2 - KE_1 \longrightarrow$$ Work is just change in kinetic energy.




Different paths may have different amounts of work done.
( No difference in conservative fields )

In vectorial form, $\boxed{KE = \frac{1}{2} m (\bar{v} \cdot \bar{v})}$

- **Conservative Forces:** Forces that can be modelled as a gradient of a scalar field, ie,

$$\bar{F}(\bar{r}) = -\nabla U(\bar{r})$$

$$\Rightarrow \quad W = \int_1^2 (-\nabla U(\bar{r}))\, d\bar{r} = \boxed{-(U_2 - U_1)}$$

- The scalar quantity / field $U(\bar{r})$ is the potential energy due to the force.

- Since Work now only depends on PE at 1 and 2, it does not depend on the exact path travelled between 1 and 2.

$$W = K_2 - K_1 = -(U_2 - U_1)$$

$$\Rightarrow \boxed{K_2 + U_2 = K_1 + U_1} \rightarrow \text{Law of Conservation of Mechanical Energy}$$

- <u>Conservation of Linear Momentum</u> :-

- For an object, if $\bar{F}\cdot\bar{s} = 0$ ($\bar{F}$ is zero or $\perp$lar to $\bar{s}$)

$$\bar{F}\cdot\bar{s} = 0 \Rightarrow \frac{d\bar{p}}{dt}\cdot\bar{s} = 0 \Rightarrow \frac{d(\bar{p}\cdot\bar{s})}{dt} = 0 \quad \text{(Assuming } \bar{s} \text{ is const)}$$

$\Rightarrow$ $\bar{p}\cdot\bar{s}$ is constant $\Rightarrow$ Momentum in the direction of $s$ is conserved

$\therefore$ If force in a direction is zero, momentum is conserved in that direction.

- <u>Torque and Conservation of Angular Momentum</u> :-

- Angular momentum $\bar{L} = \bar{r} \times \bar{p}$

- Torque $\bar{\tau} = \frac{d}{dt}\bar{L} = \frac{d}{dt}\bar{r}\times\bar{p} = \frac{d\bar{r}}{dt}\times\bar{p} + \bar{r}\times\frac{d\bar{p}}{dt}$

$$\bar{\tau} = \cancelto{0}{\bar{v}\times\bar{p}} + \bar{r}\times\bar{F}$$

$$\Rightarrow \boxed{\bar{\tau} = \bar{r}\times\vec{F}}$$

- Conservation of Angular Momentum,

If $\bar{\tau} \cdot \bar{s} = 0$, then $\bar{L} \cdot \bar{s}$ is constant.

$\therefore$ Angular momentum is conserved along the axis of zero torque.

- $$E = T(\bar{v}(t)) + U(\bar{r}(t), t)$$

$$\frac{dT}{dt} = m\bar{v}\, d\bar{v}/dt = m\bar{v} \cdot \bar{a} = \bar{v} \cdot \bar{F}$$

$$\frac{dU(\bar{r},t)}{dt} = \frac{\partial U}{\partial \bar{r}} \frac{d\bar{r}}{dt} + \frac{\partial U}{\partial t} = \bar{v} \cdot \nabla U + \frac{\partial U}{\partial t}$$

$\bar{F} = -\nabla U$

$$\Rightarrow \frac{dE}{dt} = \bar{v}\,(\cancelto{0}{\bar{F} + \nabla U}) + \frac{\partial U}{\partial t}$$

$$\Rightarrow \frac{dE}{dt} = \frac{\partial U}{\partial t} \Rightarrow$$ If PE is constant, E is conserved

- <u>Galilean Relativity</u> :-

Let there be 2 inertial frames K and K' and K' is at relative velocity $\bar{V}$ wrt K. Let an object be moving with velocity $\bar{v}$ and $\bar{v}_1$, ie

$$\bar{v}_1 = \bar{v} - \bar{V}$$

$$\Rightarrow \frac{d\bar{v}_1}{dt} = \frac{d(\bar{v} - \bar{V})}{dt} = \frac{d\bar{v}}{dt}$$ $\bar{V}$ is const

$$\Rightarrow \quad \boxed{\bar{F}_1 = \bar{F}}$$

$\therefore$ All mechanical laws are the same in any inertial frame.

→ Simple Harmonic Motion :-

- Hooke's law : $F(x) = -kx \Rightarrow m\dfrac{d^2x}{dt^2} = -kx$

To solve the DE,

$$m\frac{d^2x}{dt^2} = -kx$$

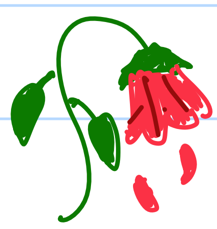
Take $\quad p = m\dfrac{dx}{dt} \Rightarrow m\dfrac{d^2x}{dt^2} = \dfrac{dp}{dt}$

$$\Rightarrow \quad \frac{dp}{dt} = -kx \quad \Rightarrow \quad dp = -kx\,dt$$

$$= \quad p = [-kxt]_0^t = -kxt$$

$$\frac{dx}{dt} = \frac{p}{m}$$

$\Rightarrow$



ts too tuff :(

## → Limitations of Newtonian Mechanics :-

- Galilean relativity violated by EM waves.
- Stability of atoms cannot be explained by Newtonian mechanics.

## → Special Theory of Relativity :-

The postulates of STR are as follows,

i) The laws of physical phenomena are the same in all inertial reference frames.

ii) The velocity of light (in free space) is a universal constant, independent of any relative motion.

- **Galilean Invariance :-**

Consider a frame K and another frame K' moving at a speed of $v_r$ along the x-axis.

- Let $(x, y, z)$ be coordinates in K and $(x', y', z')$ in K'. The transformation of coordinates is given by,

$$x' = x - v_r t$$
$$y' = y$$
$$z' = z$$
$$t' = t$$

- The above transformation is termed as a Galilean transformation.
- Length is conserved in a Galilean transformation,

$$ds^2 = \Sigma dx_1^2 = \Sigma dx_1'^2 = ds'^2$$

Newton's laws are also conserved,

$$F = ma = ma' = F'$$

- Velocity in K' is given by,

$$v' = v - v_r$$

That means the velocity of light in K' is $c - v_r \neq c$, in violation of STR.

- <u>Lorentz Transformation</u> :-

- If a light pulse is emitted from the common origin of the systems K and $K_1$ when they are coincident, then,

$$x = ct$$

$$x_1 = ct_1$$

Let $x_1 = \gamma(x - v_r t) \Rightarrow x = \gamma'(x_1 + v_r t_1)$

Since the laws of physics must be the same in K and K',
$\gamma = \gamma'$.

$$\Rightarrow \quad x = \gamma\left(\gamma\left(x - v_\gamma t\right) + v_\gamma t_1\right)$$

$$\Rightarrow \quad x = \gamma^2\left(x - v_\gamma t\right) + \gamma v_\gamma t_1$$

$$\Rightarrow \quad x = x\gamma^2 - \gamma^2 v_\gamma t + \gamma v_\gamma t_1$$

$$\Rightarrow \quad \gamma v_\gamma t_1 = x - x\gamma^2 + \gamma^2 v_\gamma t$$

$$= x\left(1-\gamma^2\right) + \gamma^2 v_\gamma t$$

$$\Rightarrow \quad t_1 = \frac{x}{\gamma v_\gamma}\left(1-\gamma^2\right) + \gamma t$$

$$= \quad t_1 = \frac{x}{v_\gamma}\left(\frac{1-\gamma^2}{\gamma}\right) + \gamma t \quad \text{——— } (1)$$

$$x = ct \quad \Rightarrow \quad x' = \gamma\left(ct - v_\gamma t\right) = \gamma\left(c - v_\gamma\right)t$$

$$\Rightarrow \quad t_1 = \frac{ct}{v_\gamma}\left(\frac{1-\gamma^2}{\gamma}\right) + \gamma t$$

$$x_1 = ct_1 \quad \Rightarrow \quad c = \frac{x_1}{t_1} = \frac{\gamma\left(c - v_\gamma\right)t}{\frac{ct}{v_\gamma}\left(\frac{1-\gamma^2}{\gamma}\right) + \gamma t}$$

$$\Rightarrow \quad c = \frac{\gamma\left(c - v_\gamma\right)}{\frac{c}{v_\gamma}\left(\frac{1-\gamma^2}{\gamma}\right) + \gamma}$$

$$\Rightarrow \quad \gamma\left(c - v_\gamma\right) = \frac{c^2}{v_\gamma}\left(\frac{1-\gamma^2}{\gamma}\right) + c\gamma$$

$$\Rightarrow \cancel{\gamma c} - \gamma v_r = \frac{c^2}{v_r}\left(\frac{1-\gamma^2}{\gamma}\right) + \cancel{c\gamma}$$

$$= \gamma v_r = \frac{c^2}{v_r}\left(\frac{\gamma^2 - 1}{\gamma}\right)$$

$$= \frac{\gamma v_r}{c^2} = \frac{1}{v_r}\left(\frac{\gamma^2 - 1}{\gamma}\right) \quad \text{——} \quad (2)$$

Substitute (2) in (1),

$$\Rightarrow t_1 = x\left(\frac{-\gamma v_r}{c^2}\right) + \gamma t$$

$$= t_1 = \gamma\left(t - \frac{v_r}{c^2} x\right)$$

$$\Rightarrow \gamma = \frac{1}{\sqrt{1 - v_r^2/c^2}}$$

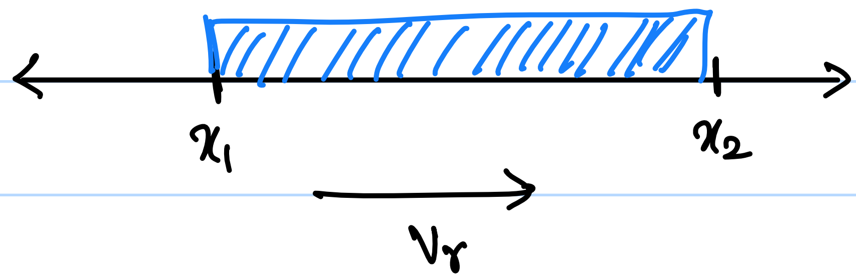
With this, the transformation can be described as,

$$x' = \frac{x - v_r t}{\sqrt{1 - v_r^2/c^2}}$$

$$t' = \frac{t - v_r x/c^2}{\sqrt{1 - v_r^2/c^2}} = \gamma\left(t - \frac{v_r x}{c^2}\right)$$

The above transformation is described as the Lorentz transformation.

- **Length Contraction:**




- An observer is moving velocity $v_r$ towards $x_2$, from $x_1$. He measures the length of the above rod $(x_2 - x_1)$ at a time instant $t$.

The length of the rod in the observer's frame is,

$$l' = x_2' - x_1'$$

$$= \gamma(x_2 - v_r t_2) - \gamma(x_1 - v_r t_1)$$

$$= \gamma(x_2 - x_1 - v_r(t_2 - t_1))$$

Since $t_2' = t_1'$

$$\Rightarrow \gamma\left(t_2 - \frac{v_r x_2}{c^2}\right) = \gamma\left(t_1 - \frac{v_r x_1}{c^2}\right)$$

$$= \quad t_2 - t_1 = \frac{v_r}{c^2}(x_2 - x_1)$$

$$\Rightarrow l' = \gamma\left(x_2 - x_1 - \frac{v_r^2}{c^2}(x_2 - x_1)\right)$$

$$= \frac{1}{\sqrt{1 - \frac{v_r^2}{c^2}}}(x_2 - x_1)\left(1 - \frac{v_r^2}{c^2}\right)$$

$$= (x_2 - x_1)\sqrt{1 - \frac{v_r^2}{c^2}}$$

$$\Rightarrow \quad \underline{l' = l/\gamma} \quad \Rightarrow \quad l' < l$$

$$\sqrt{1-\frac{v_r^2}{c^2}} < 1 \quad \hookrightarrow \gamma > 1$$

This is termed as length contraction.

- <u>Time Dilation</u> :-

– A clock, at a fixed position $x$, ticks at 2 time instants $t_1$ and $t_2$. An observer moving in frame $K'$, with relative velocity $v_r$ measures the time interval.

$$\Delta t' = t_2' - t_1'$$

$$\Delta t' = \gamma\left(t_2 - \frac{v_r}{c^2}x_2\right) - \gamma\left(t_1 - \frac{v_r}{c^2}x_1\right)$$

$$= \gamma\left(t_2 - t_1 - \frac{v_r}{c^2}\cancelto{0}{(x_2 - x_1)}\right)$$

$$= \gamma(t_2 - t_1)$$

$$\Rightarrow \quad \underline{\Delta t' = \gamma \Delta t} \quad \Rightarrow \quad \Delta t' > \Delta t$$

This phenomenon is termed as time dilation.

- **Relativistic Doppler Effect :-**

Consider a source of light and a reciever, with approach velocity $v$.

Let the reciever be in a frame $K$ and the source in $K'$.

In time $\Delta t$, the source emits $n$ waves. The distance between the front and back ie,

$$l = c\Delta t - v\Delta t$$

$$\Rightarrow \quad \lambda = \frac{c\Delta t - v\Delta t}{n}$$

$$\Rightarrow \quad f = \frac{nc}{c\Delta t - v\Delta t}$$

Since the source emits with frequency $f_0$,

$$n = f_0 \Delta t' = f_0 \frac{\Delta t}{\gamma} \quad [\text{time dilation}]$$

$$\Rightarrow \quad f = \frac{f_0 c\Delta t}{\gamma(c\Delta t - v\Delta t)}$$

$$= \frac{1}{\gamma\left(1 - \frac{v}{c}\right)} f_0$$

$$\underline{f = \frac{\sqrt{1 - v^2/c^2}}{1 - v/c} f_0}$$

If the source and receiver are separating with velocity v,

$$l = c\Delta t + v\Delta t$$

$$\Rightarrow \lambda = \frac{c\Delta t + v\Delta t}{n}$$

$$\Rightarrow f = \frac{nc}{c\Delta t + v\Delta t}$$

$$n = f_0 \Delta t' = f_0 \frac{\Delta t}{\gamma}$$

$$\Rightarrow f = \frac{c\Delta t}{\gamma(c\Delta t + v\Delta t)} f_0$$

$$\Rightarrow f = \frac{\sqrt{1 - v^2/c^2}}{1 + v/c} f_0$$

**Note:**

Lorentz transformation for velocities,

Let K and K' be 2 inertial frames with relative velocity $v_r$, with an object moving at velocities v and v' in the frames respectively.

$$v' = \frac{dx'}{dt'} = \frac{dx'}{dt} \Big/ \frac{dt'}{dt}$$

WKT,

$$x' = \gamma(x - v_r t) \qquad t' = \gamma\left(t - \frac{v_r x}{c^2}\right)$$

$\gamma$ is a constant wrt $t$.

$$\Rightarrow \quad \frac{dx'}{dt} = \gamma\left(\frac{dx}{dt} - v_r\right) \qquad \frac{dt'}{dt} = \gamma\left(1 - \frac{v_r}{c^2}\frac{dx}{dt}\right)$$

$$= \gamma(v - v_r) \qquad\qquad = \gamma\left(1 - \frac{v_r v}{c^2}\right)$$

$$\Rightarrow \quad v' = \frac{\cancel{\gamma}(v - v_r)}{\cancel{\gamma}\left(1 - \frac{v_r v}{c^2}\right)} = \frac{v - v_r}{1 - \frac{v_r v}{c^2}}$$

Verification by taking $v = c$

$$v' = \frac{c - v_r}{1 - \frac{v_r \cancel{c}}{c^{\cancel{2}}}} = \frac{c - v_r}{\frac{(c - v_r)}{c}} = \underline{\underline{c}}$$

$\therefore$ $v = v'$ if $v = c$. STR is not violated.

- <u>Proper Time</u>:-

Time as measured in the observer's frame of reference, with respect to events that occur at the same time.

- <u>Proper Length</u> :-

Length as measured in the observer's frame of reference, with respect to an object that is at rest in his frame.

- Events that happen at different times will undergo length contraction between them.

<u>Note</u>: An event is defined with both space and time.

- <u>The Twin Paradox</u> :-

- If there are 2 twins, one on Earth and one on an interstellar voyage travelling near the speed of light, the space travelling twin will age much slower than the twin on Earth.

- The paradox is that, since with respect to the traveller, the twin on Earth is the one moving, the Earth twin must be the younger one, due to symmetry.

- The solution, is that since the traveller is going away, and then coming back to Earth, he is actually in 2 different reference frames. (One while separation, one while approach)

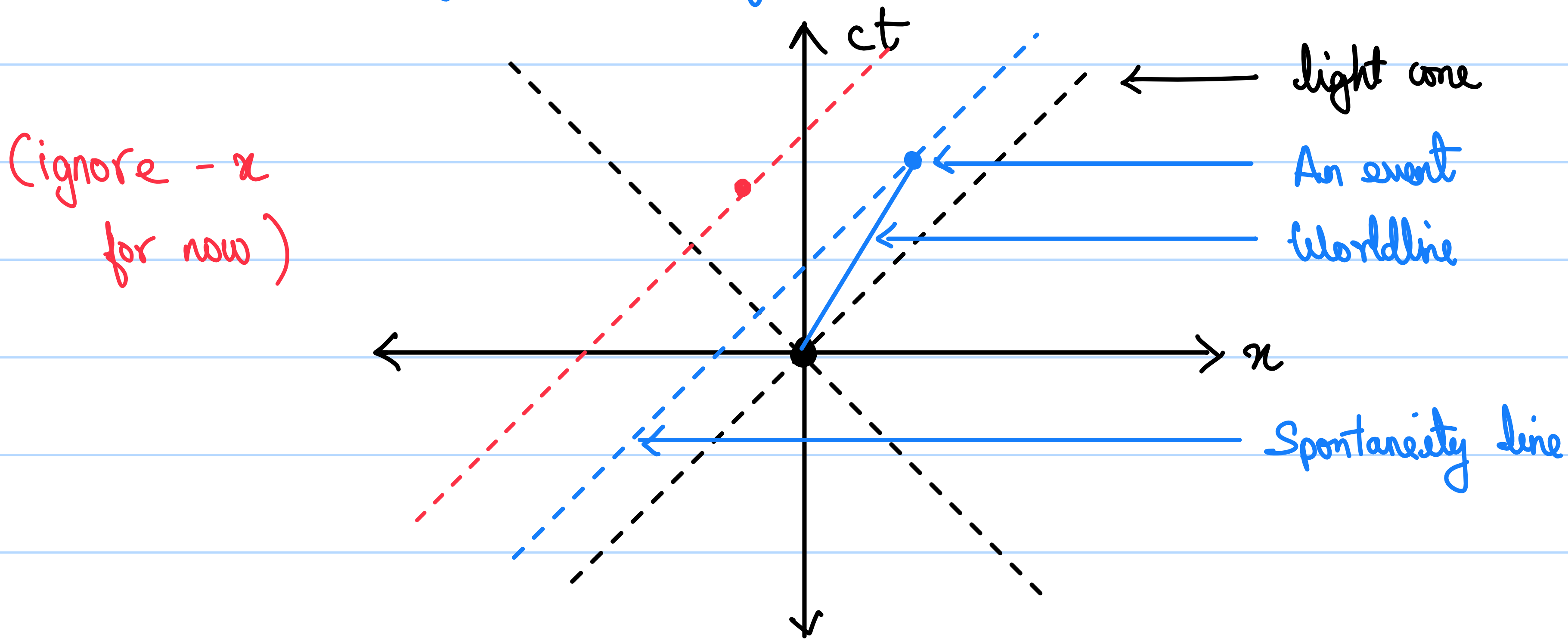
- To explain this further, we can use space time diagrams.

## ∘ Spacetime Diagrams (Minkowski Geometry) :-

- In Minkowski Geometry, the spacetime interval between 2 events is defined as,

$$ds^2 = dx^2 + dy^2 + dz^2 - (cdt)^2$$

- A spacetime diagram is as follows, (on one spatial dimension $x$)




<u>Event</u> : A point on the spacetime diagram

<u>Worldline</u>: The path travelled by an event, ie the history of an object through space and time.

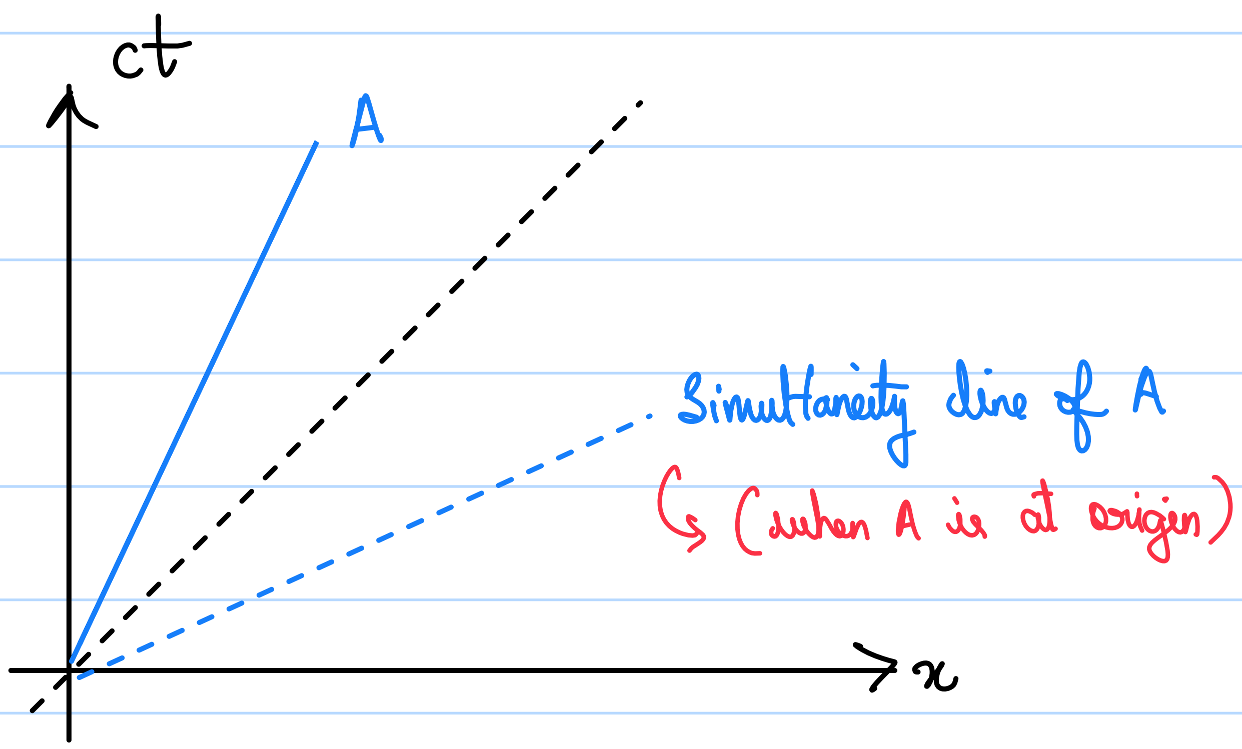
<u>Simultaneity line</u>: Set of all events that are spontaneous with the event, in the observer's frame (if observer is at rest ∥ to light cone)

<u>Light cone</u>: Speed of light representation (Worldlines cannot cross light cone, as nothing can cross the speed of light). Also, the set of events that are simultaneous to origin.

- The slope of the simultaneity line depends on the velocity of the object being analysed in the observer frame. Higher the speed, higher the slope, but cannot cross $m = 1$ or $-1$ (speed of light)
  $-x$ direction

- Length contraction and time dilation distort the object's spacetime diagram, messing up the spontaneity line.

Let A represent a moving observer wrt ground, the ground's space time diagram is as follows.




- To determine the simultaneity line of A, let velocity of A be v.

Applying Lorentz Transformation,

$$x' = \gamma(x - vt)$$
$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

Simultaneity means $t' =$ const (let it be k).

The slope of a world line is given by $\frac{d(ct)}{dx}$.

$$K = \gamma\left(t - \frac{vx}{c^2}\right)$$

$$\Rightarrow \quad t - \frac{v}{c^2}x = \frac{K}{\gamma}$$

$$\Rightarrow \quad ct = \frac{v}{c}x + \frac{cK}{\gamma}$$

$\therefore$ Slope of $ct$ vs $x$ = $\boxed{\frac{v}{c}}$.

$\therefore$ The simultaneity line of an object at $(ct_0, x_0)$ moving at velocity $v$ with respect to the observer is

$$\boxed{(t - t_0) = \frac{v}{c}(x - x_0)}$$

Verification: If $v = 0$, slope $= 0$, ie, set of all points $(x, t)$ such that $t = t_0$. This is clearly correct.

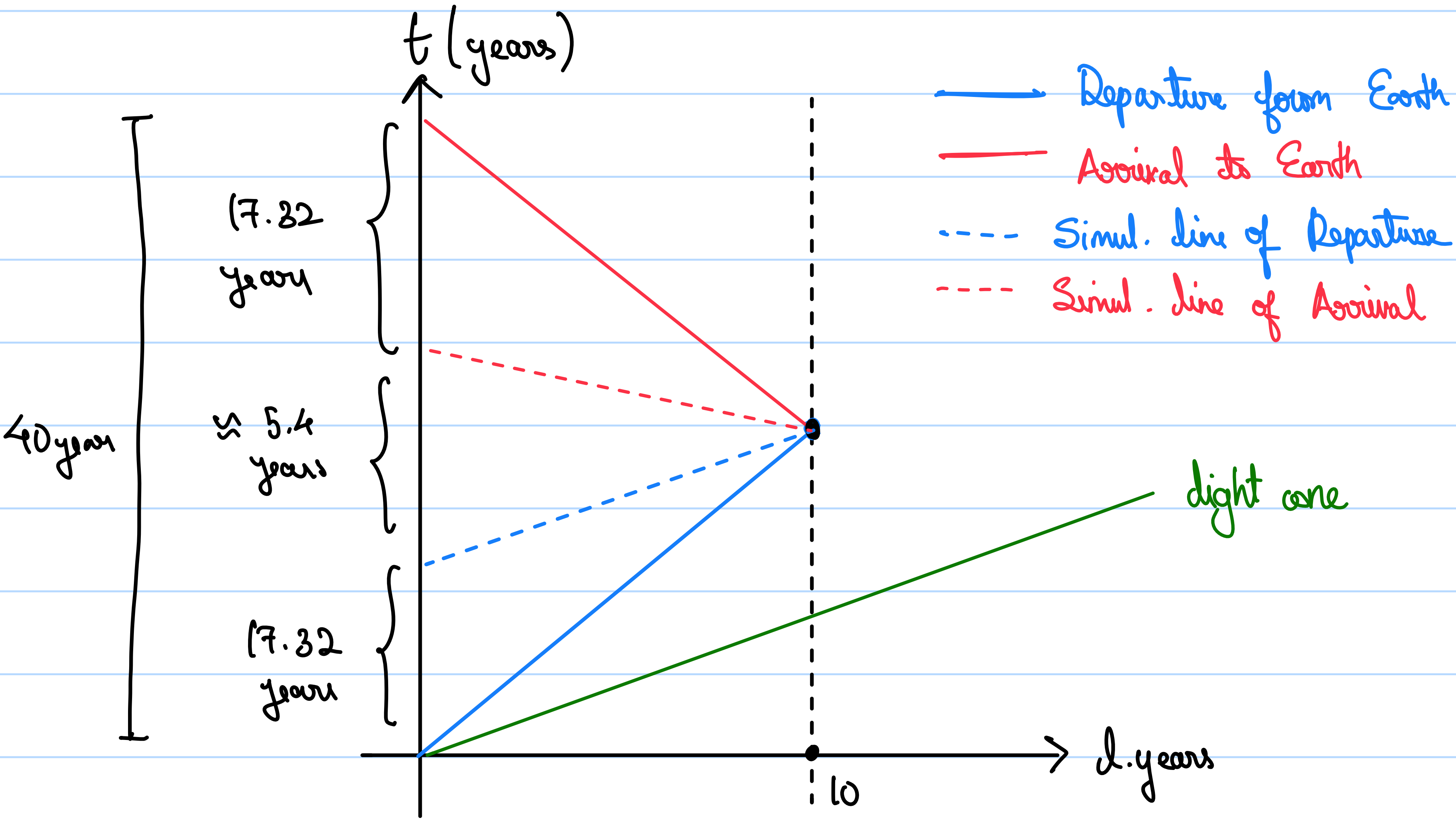
- <u>Solving the Twin Paradox using Spacetime Paradox</u> :-

- Let the speed of the traveller be $0.5c$, ie, $50\%$ of the speed of light, and let his destination be 10 light years away.

- Time taken to travel $= \frac{10}{0.05} = 20$ years
  $\Rightarrow$ Total time $= 40$ years

} In Earth frame

Drawing the spacetime diagram on the Earth frame,




- Here, we see why the switching of reference frames causes the difference in aging.

- The simultaneity line of the travelling twin changes when he changes direction. Therefore, there is a "time jump" with respect to Earth for the traveller. The aging of the Earth twin happens in that jump.

- The Earth twin ages 40 years. To calculate the age of the traveller twin, we can use length contraction to determine the duration of the trip as per him,

$$l' = l/\gamma = 10\sqrt{1-\frac{(0.5c)^2}{c^2}}$$

$$= 10\sqrt{0.75} \approx 8.66 \text{ light years}$$

$$t = \frac{8.66}{0.5} = 17.32 \text{ years.}$$

$$\text{Total } t = 34.64$$

$\therefore$ The traveller twin will age 34.64 years.

$\Rightarrow$ The "time jump" is $\approx$ 5.4 years.

- The travelling twin does not "feel" the time jump, but the Earth twin does, hence causing the difference in aging.

- Since the time jump is +ve for all speeds $< c$, the Earth twin will always be older than the travelling twin.

Converse: If speed is $> c$, Earth twin will be younger than traveller twin ( Yay Time Travel !!! )

- Spacetime Invariance :

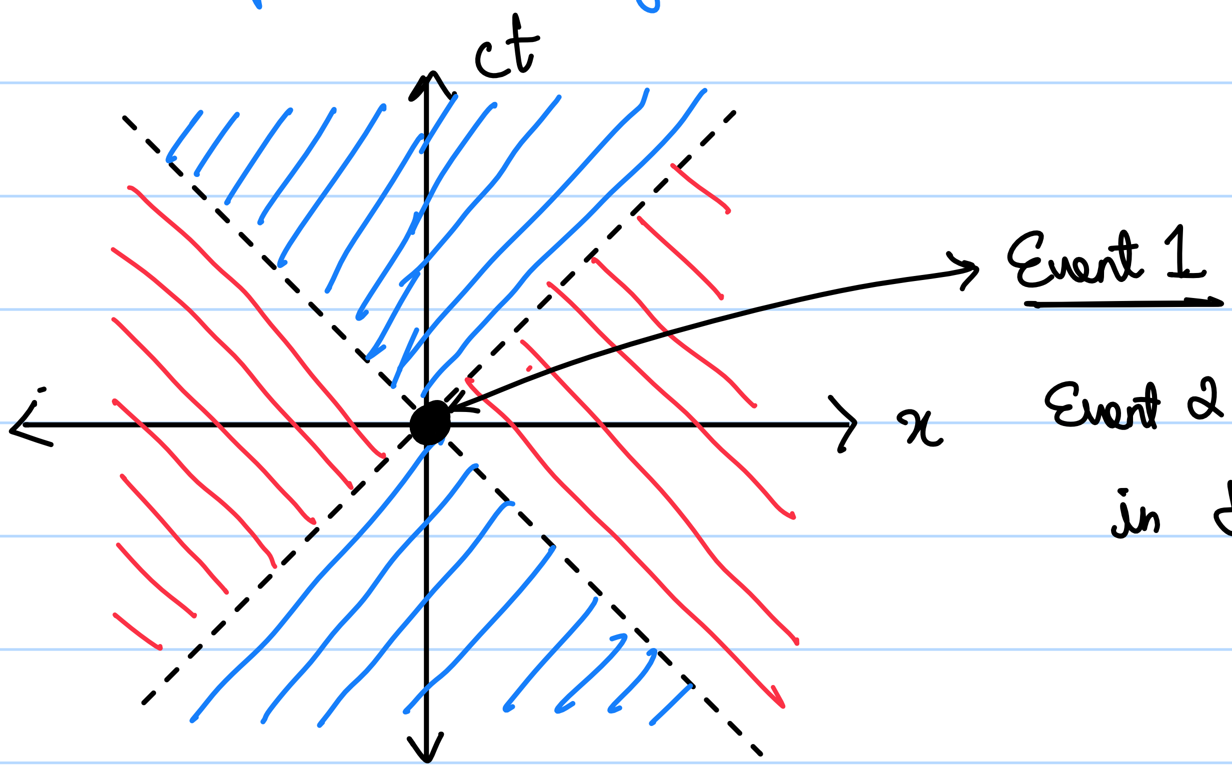
- The spacetime interval between 2 events is,

$$ds^2 = dx^2 + dy^2 + dz^2 - (c\,dt)^2$$

- Since our examples only have $x$ coordinate and time,

$$ds^2 = dx^2 - (cdt)^2$$

- In the below spacetime diagram,




Event 2 lies anywhere in the plane

We know that the slope of any line is,

$$m = \frac{\Delta(ct)}{\Delta(x)} = \frac{ct}{x} \text{ at Event 2, but lite since need to prove for gen. case}$$

$$\Rightarrow \quad \Delta(ct) = m\,\Delta(x)$$

If $m = 1 \longrightarrow$ The light cone,

$$\Delta(ct) = \Delta x \Rightarrow \Delta(ct)^2 = \Delta x^2$$

$$\Rightarrow \quad \underline{\Delta s^2 = 0}$$

Any event with $\Delta s^2$ is simultaneous to the origin event. They are termed as lightlike events

If $m > 1 \longrightarrow$ The Blue region

$$\Delta(ct) < \Delta(x)$$
$$\Rightarrow \Delta(ct)^2 < \Delta(x)^2$$
$$\Rightarrow \underline{\Delta s^2 > 0}$$

$\exists$ an inertial frame where the 2 events happen at the same position but diff. times

Any event which is causal to the origin event has $\Delta s^2 > 0$, but all such events with $\Delta s^2 > 0$ are not causal to origin. Such events with $\Delta s^2 > 0$ are termed as timelike events.

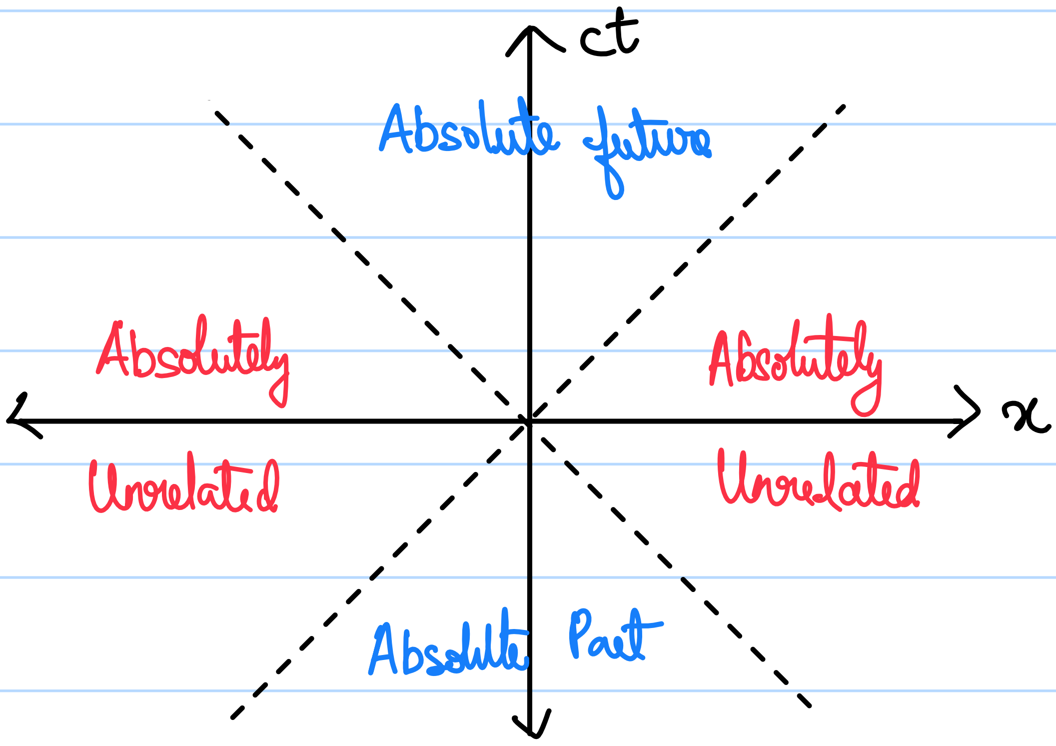
If $m < 1 \longrightarrow$ The Red region

$$\Delta(ct)^2 > \Delta x^2$$
$$\Rightarrow \underline{\Delta s^2 < 0}$$

$\exists$ an inertial frame where the events occur at the same time but diff. positions.

Events with $\Delta s^2 < 0$ with the origin, cannot be related to the origin. Such events are called space like events.

- These results apply to any pair of 2 events, not only origin.

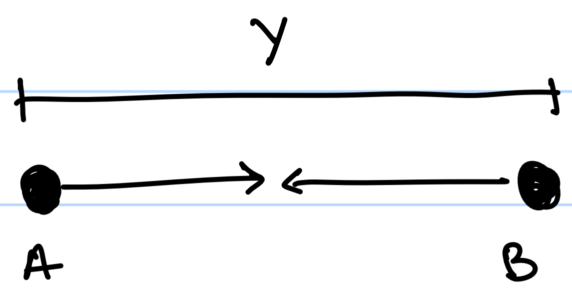


- The significance of these results lies in the fact that the spacetime interval is unaffected by the Lorentz transformation, ie, $\Delta s^2$ is the same in all frames for any 2 events.

$\therefore$ If 2 events are absolutely unrelated in a frame, they are absolutely unrelated in any frame

→ <u>STR in the Real World</u> :-

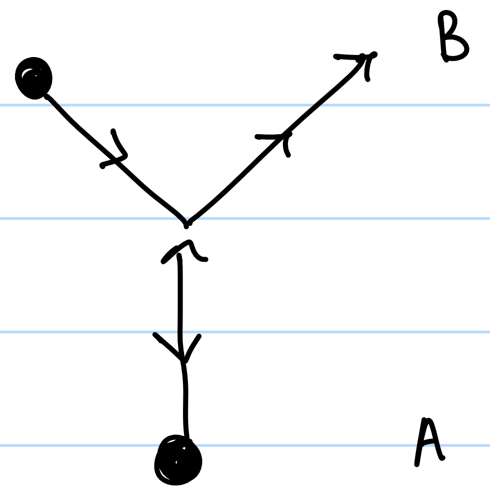
1) <u>Relativistic Momentum</u> :-



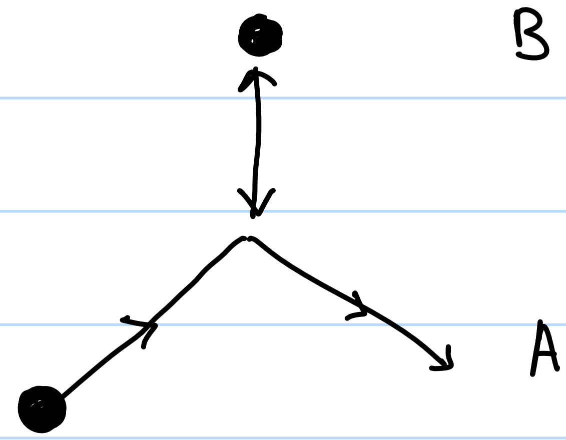

Let there be 2 balls A and B with velocities $V_A, V_A'$ and $V_B, V_B'$, in the reference frames S and S' respectively, along the y axis. Let S' move at V velocity along +x wrt S.

If $V_A = V_B'$ and the balls are colliding, the collision will be,

S :




S' :




Time taken for A's round trip in S $= \dfrac{Y}{V_A}$

" " B's round trip in S' $= \dfrac{Y}{V_B'} = \dfrac{Y}{V_A}$

} Let it be $T_0$

In S, $v_B = \frac{y}{T}$, $T$ = Time for B's round trip in S

$$\Rightarrow \quad T = \gamma T_0 \quad (T_0 \text{ is B's time in } S')$$

$$= \quad T = \frac{T_0}{\sqrt{1 - v^2/c^2}}$$

$$\Rightarrow \quad v_B = \frac{y \sqrt{1 - v^2/c^2}}{T_0}$$

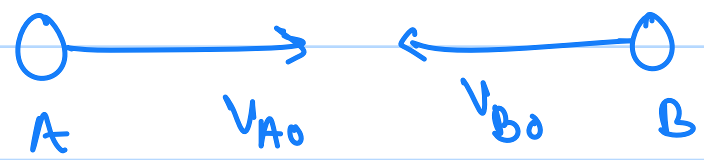
wkt $\quad v_A = \frac{y}{T_0}$

Using the classical definition of momentum, we get,

$$p_A = m_A v_A = m_A \left(\frac{y}{T_0}\right)$$

$$p_B = m_B v_B = m_B \left(\sqrt{1 - v^2/c^2}\right)\left(\frac{y}{T_0}\right)$$

We know that, in the lab frame, the situation is as below.




Here, the total momentum of the system is zero if $m_A = m_B$

$$\Rightarrow \quad p_A - p_B = 0$$

[Physical laws must be the same in all frames]

$$= m_A\left(\frac{Y}{T_0}\right) - m_B\left(\sqrt{1-v^2/c^2}\right)\left(\frac{Y}{T_0}\right) = 0$$

$$= m_A = m_B\sqrt{1-v^2/c^2}$$

$$\Rightarrow \boxed{m_B = \frac{m_A}{\sqrt{1-v^2/c^2}}}$$

Now, redefining $p$, we get

$$\boxed{P_A = \frac{m_A v_A}{\sqrt{1-v_A^2/c^2}}}$$

- frame velocity

$$\Rightarrow \boxed{p_A = \gamma(v_A)\, m_A v_A}$$

just accept the formula, I can't understand the proof anymore :(

In the above equation, since $\gamma \to \infty$ as $v \to c$, as velocity approaches $c$, the momentum of the object nears infinity, making it harder for further increase in momentum.

$\Rightarrow$ $v = c$ is practically impossible

- Relativistic Second Law :

- In Newton's second law,

$$\bar{F} = \frac{d\bar{p}}{dt} = \frac{d}{dt}\gamma(\bar{v})m\bar{v}$$

$$= m\frac{d}{dt}\gamma(\bar{v})\,\bar{v}$$

$$= m \frac{d}{dt} \frac{v^2}{\sqrt{1-v^2/c^2}}$$

(On solving) $$= \frac{m}{(1-v^2/c^2)^{3/2}} \frac{dv}{dt}$$

$$\Rightarrow \quad F = \frac{ma}{(1-v^2/c^2)^{3/2}}$$

- Energy in STR :-

$$KE = \int F.ds$$

$$= \int \frac{d}{dt} \gamma m v \, ds$$

$$= \int v \, d(\gamma m v)$$

$$= \int v \, d\left(\frac{mv}{\sqrt{1-v^2/c^2}}\right)$$

Integrating by Parts,

$$\Rightarrow KE = \left[ v\left(\frac{mv}{\sqrt{1-v^2/c^2}}\right) - m \int \frac{v\,dv}{\sqrt{1-v^2/c^2}} \right]$$

$$= \frac{mv^2}{\sqrt{1-v^2/c^2}} + mc^2 \left[\sqrt{1-v^2/c^2}\right]_0^v$$

$$= \frac{mv^2}{\sqrt{1-v^2/c^2}} + mc^2\sqrt{1-v^2/c^2} - mc^2$$

$$= \frac{mv^2 + mc^2\left(1 - \frac{v^2}{c^2}\right) - mc^2\sqrt{1 - v^2/c^2}}{\sqrt{1 - v^2/c^2}}$$

$$= \frac{mc^2 - mc^2\sqrt{1 - v^2/c^2}}{\sqrt{1 - v^2/c^2}}$$

$$\Rightarrow \quad KE = (\gamma - 1)\, mc^2$$

When $v = 0$, ie $\gamma = 1$, $KE = 0$. If we interpret $\gamma mc^2$ as the total energy of the object, we get

$$E = KE + E_0$$

$$E_0 = E - KE = \gamma mc^2 - (\gamma - 1)mc^2$$

$$\Rightarrow \quad E_0 = mc^2$$

$E_0$ is defined as the rest mass energy of the system, ie, energy of the object while stationary.

If $v \ll c$,

$$KE = mc^2(\gamma - 1)$$
$$= mc^2\left(\left(1 - v^2/c^2\right)^{-1/2} - 1\right)$$
$$= mc^2\left(\cancel{1} + \frac{1}{2}\frac{v^2}{c^2} - \cancel{1}\right) \quad \text{[Binomial Approximation]}$$

$$\underline{KE = \frac{1}{2}mv^2} \longrightarrow \text{Classical formula.}$$

- $$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}} \qquad p = \frac{mv}{\sqrt{1 - v^2/c^2}}$$

$$E^2 = \frac{m^2 c^4}{1 - v^2/c^2} \qquad p^2 = \frac{m^2 v^2}{1 - v^2/c^2}$$

$$p^2 c^2 = \frac{m^2 v^2 c^2}{1 - v^2/c^2}$$

$$E^2 - p^2 c^2 = \frac{m^2 c^2}{1 - v^2/c^2}\left(c^2 - v^2\right)$$

$$= \frac{m^2 c^4 \left(1 - \frac{v^2}{c^2}\right)}{1 - v^2/c^2}$$

$$\Rightarrow \boxed{E^2 - p^2 c^2 = (mc^2)^2}$$

By the above equation, if $\exists$ a massless particle, its energy will be given by $E = pc$. (ex: photon)

→ <u>General Theory of Relativity</u> :-

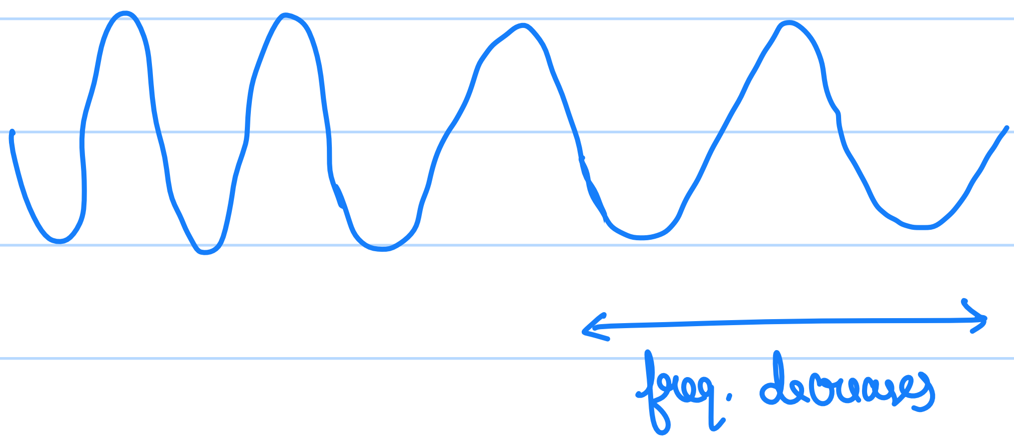
- General Theory of Relativity is based on the principle of equivalence, that states,

"An observer in a closed laboratory cannot distinguish between the effects produced by a gravitational field and those produced by the acceleration of the laboratory"

- GTR extends STR into non-inertial frames, ie, the effect of acceleration on simultaneity.

- In a non-inertial frame, light bends against the direction of acceleration. This is termed as "lensing".


If direction of acceleration is in the direction of light, the light is "redshifted" (the detector moves away from the light source)




- GTR states that, since by the principle of equivalence, gravitational acceleration is the same as any acceleration, gravitation will also cause lensing and redshifting.

$\therefore$ Light bends due to gravity.

→ <u>Calculus of Variations</u> :-

- The basic problem in Calculus of Variation is to find the value such that the integral,

$$J = \int_{x_1}^{x_2} f\{y(x), y'(x); x\}\, dx$$

is an extremum (maximum or minimum).

- We are to vary $y(x)$ until $J$ is maximum or minimum.

- If $J$ is minimum, that means if $y(x)$ is changed into a neighbouring function, then $J$ increases.

- Let the neighbouring function of $y(x)$ be $y(\alpha, x)$, $[y(0,x) = y(x)]$ which turns the functional $J$ into a parameterized functional,

$$J(\alpha) = \int_{x_1}^{x_2} f\left(y(\alpha,x), \dot{y}(\alpha,x); x\right) dx$$

Such that,

$$y(\alpha, x) = y(x) + \alpha\eta(x)$$

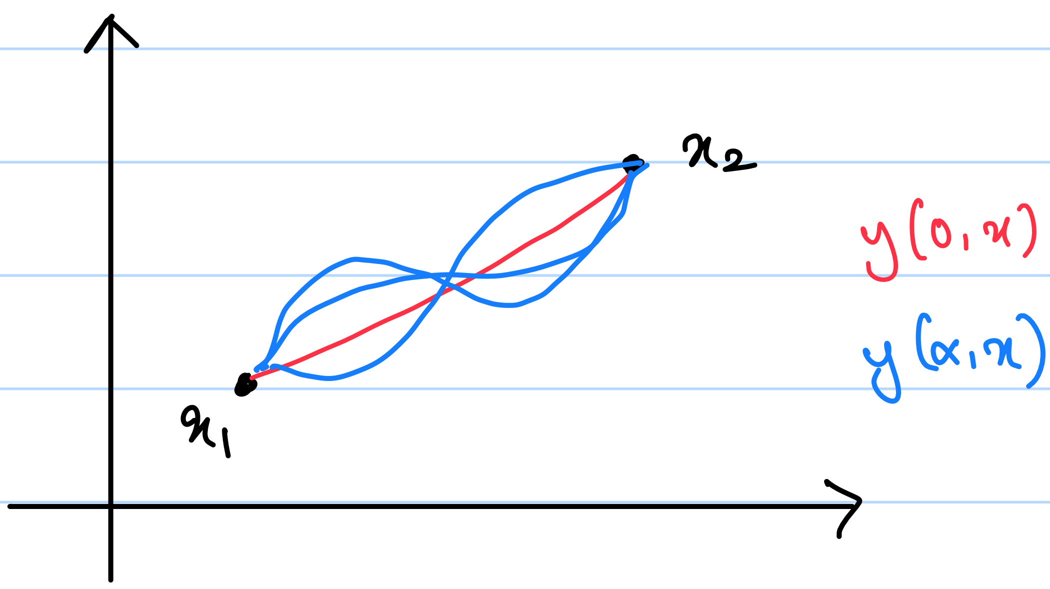
Where $\eta(x)$ is some function with a continuous first derivative and vanishes $x_1$ and $x_2$, since the function $y(\alpha,x)$ should be the same at the endpoints $x_1$ and $x_2$ $\forall\ \alpha$, i.e.

$$\eta(x_1) = \eta(x_2) = 0$$

- If $y(0,x)$ is the location of the extremum of $J$, then,

$$\left.\frac{dJ}{d\alpha}\right|_{\alpha=0} = 0$$

But the above condition is only necessary, not sufficient.




Interpretation of $y(\alpha,x)$

In a nutshell, we are finding a path between $x_1$ and $x_2$ such that a parameter (represented by the functional $J$) is minimized or maximized.

- Euler's Equation :-

- We know an extremum is achieved when $\frac{\delta J}{\delta \alpha} = 0$

$$\frac{\delta J}{\delta \alpha} = \frac{\delta}{\delta \alpha} \int_{x_1}^{x_2} f(y, y'; x)\, dx$$

$$= \int \left( \frac{\delta f}{\delta y} \frac{\delta y}{\delta \alpha} + \frac{\delta f}{\delta y'} \frac{\delta y'}{\delta \alpha} \right) dx$$

We represent this as $\delta \int_{x_1}^{x_2} f(y, y'; x)\, dx$

$$y(\alpha,x) = y(0,x) + \alpha\eta(x)$$

$$\Rightarrow \frac{\delta y}{\delta \alpha} = \eta(x)$$

$$y'(\alpha,x) = y'(0,x) + \alpha\eta'(x)$$

$$\frac{\delta y'}{\delta \alpha} = \eta'(x)$$

$$\Rightarrow \frac{\delta J}{\delta \alpha} = \int_{x_1}^{x_2} \frac{\delta f}{\delta y}\eta(x) + \frac{\delta f}{\delta y'}\eta'(x)\, dx$$

$$\int_{x_1}^{x_2} \frac{\delta f}{\delta y'}\eta'(x)\, dx = \cancelto{0}{\frac{\delta f}{\delta y'}\eta(x)\Big|_{x_1}^{x_2}} - \int_{x_1}^{x_2} \frac{d}{dx}\frac{\delta f}{\delta y'}\eta(x)\, dx$$

$\rightarrow \eta(x_1) = \eta(x_2)$

$$\frac{\delta J}{\delta \alpha} = \int_{x_1}^{x_2}\left(\frac{\delta f}{\partial y}\eta(x) - \frac{d}{dx}\frac{\delta f}{\delta y'}\eta(x)\right)dx$$

$$\frac{\delta J}{\delta \alpha} = \int_{x_1}^{x_2}\left(\frac{\delta f}{\delta y} - \frac{d}{dx}\frac{\delta f}{\delta y'}\right)\eta(x)\, dx$$

If $\frac{\delta J}{\delta \alpha} = 0$, then $\boxed{\frac{\delta f}{\delta y} - \frac{d}{dx}\frac{\delta f}{\delta y'} = 0}$

- The equation $\frac{\delta f}{\delta y} - \frac{d}{dx}\frac{\delta f}{\delta y'}$ is termed as Euler's equation, and the solution to this equation gives the extremum of $J$.

- For any functions that do not directly depend on $x$, we can use another form of Euler's equation.

We know that,

$$\frac{df}{dx} = \frac{d}{dx} f(y, y'; x) = \frac{\delta f}{\delta y}\frac{dy}{dx} + \frac{\delta f}{\delta y'}\frac{dy'}{dx} + \frac{\delta f}{\delta x}$$

$$= y'\frac{\delta f}{\delta y} + y''\frac{\delta f}{\delta y'} + \frac{\delta f}{\delta x}$$

Also, $$\frac{d}{dx}\left(y'\frac{\delta f}{\delta y'}\right) = y''\frac{\delta f}{\delta y'} + y'\frac{d}{dx}\frac{\delta f}{\delta y'}$$

$$\Rightarrow \frac{d}{dx}\left(y'\frac{\delta f}{\delta y'}\right) = \frac{df}{dx} - \frac{\delta f}{\delta x} - y'\frac{\delta f}{\delta y} + y'\frac{d}{dx}\frac{\delta f}{\delta y'}$$

$$\Rightarrow \frac{d}{dx}\left(y'\frac{\delta f}{\delta y'}\right) = \frac{df}{dx} - \frac{\delta f}{\delta x} - y'\left(\cancelto{0}{\frac{\delta f}{\delta y} - \frac{d}{dx}\frac{\delta f}{\delta y'}}\right)$$

first form of Euler's Equation

$$\Rightarrow \frac{d}{dx}\left(y'\frac{\delta f}{\delta y'}\right) + \frac{\delta f}{\delta x} - \frac{df}{dx} = 0$$

$$\Rightarrow \underline{\frac{\delta f}{\delta x} - \frac{d}{dx}\left(f - y'\frac{\delta f}{\delta y'}\right) = 0}$$

When $f$ does not explicitly depend on $x$ $\left(\frac{\delta f}{\delta x} = 0\right)$

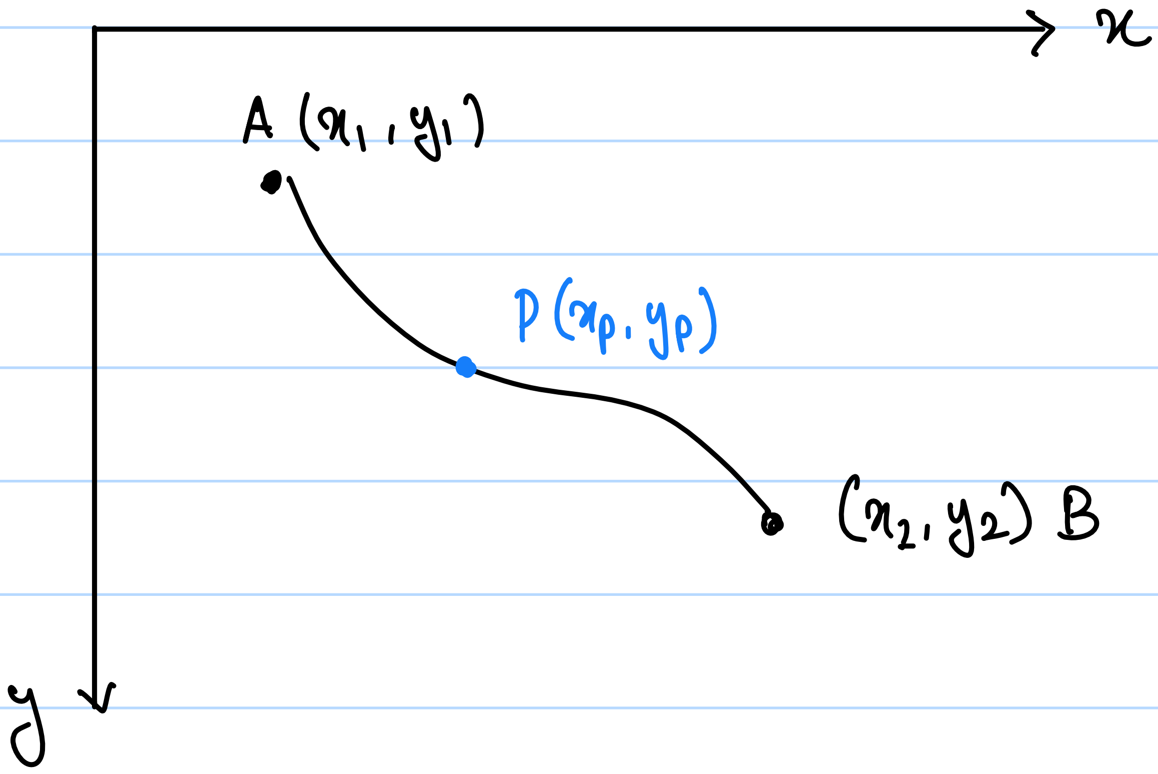
$$\Rightarrow \frac{d}{dx}\left(f - y'\frac{\delta f}{\delta y'}\right) = 0$$

$$\Rightarrow f - y'\frac{\delta f}{\delta y'} = \text{constant}$$

## • Solving Problems Using Euler's Equation :-

### i) The Brachistochrone Problem :-

Consider a particle moving in a constant force field, starting at rest from $(x_1, y_1)$ to some point $(x_2, y_2)$. Find the path that would take the least amount of time




KE at P − KE at A = Work done b/w PA

$$\frac{1}{2}\cancel{m}v^2 - 0 = \cancel{m}g(y_1 - y_p) = mg\Delta y$$

$$\Rightarrow \left(\frac{ds}{dt}\right)^2 = 2g\Delta y$$

$$\frac{ds}{dt} = \sqrt{2gy} \qquad (\text{let } x_1, y_1 = 0, 0)$$

$$dt = \frac{ds}{\sqrt{2gy}}$$

$$t = \int_A^B \frac{ds}{\sqrt{2gy}}$$

$$= \int_A^B \left( \frac{dx^2 + dy^2}{2gy} \right)^{1/2}$$

$$= \int_A^B \left( \frac{1 + \dot{y}^2}{2gy} \right)^{1/2} dx = \frac{1}{\sqrt{2g}} \int_A^B \frac{\sqrt{1+\dot{y}^2}}{\sqrt{y}} dx$$

$$\Rightarrow f(y, \dot{y}; x) = \frac{\sqrt{1+\dot{y}^2}}{\sqrt{y}}$$

Since $f$ does not directly depend on $x$ $\left(\frac{\delta f}{\delta x} = 0\right)$, using the second form of Euler's equation,

$$f - y' \frac{\delta f}{\delta y'} = c$$

$$\Rightarrow \frac{\sqrt{1+y'^2}}{\sqrt{y}} - y' \frac{\delta}{\delta y'} \left( \frac{\sqrt{1+y'^2}}{\sqrt{y}} \right) = c$$

$$\Rightarrow \frac{\sqrt{1+y'^2}}{\sqrt{y}} - y' \frac{1}{\sqrt{y}} \frac{1}{2\sqrt{1+y'^2}} 2y' = c$$

$$= \frac{\sqrt{1+y'^2}}{\sqrt{y}} - \frac{y'^2}{\sqrt{y}\sqrt{1+y'^2}} = c$$

$$= \frac{1+y'^2 - y'^2}{\sqrt{y}\sqrt{1+y'^2}} = c$$

$$= \frac{1}{\sqrt{y}\sqrt{1+y'^2}} = c$$

$$= \sqrt{y}\sqrt{1+y'^2} = 1/c$$

$$= y(1+y'^2) = 1/c^2$$

$$y\left(1+\left(\frac{dy}{dx}\right)^2\right) = 1/c^2 = k$$

$$\frac{dy}{dx} = \sqrt{\frac{k}{y}-1}$$

$$dx = \int \frac{\sqrt{y}}{\sqrt{k-y}} \cdot dy$$

$$\int_0^x dx = \int_0^y \frac{\sqrt{y}}{\sqrt{k-y}} dy$$

$$y = k\sin^2\theta \qquad dy = k\, 2\sin\theta\cos\theta\, d\theta$$

$$= \int_0^\theta \frac{\cancel{\sqrt{k}}\sin\theta \quad k\, 2\sin\theta \cancel{\cos\theta}\, d\theta}{\cancel{\sqrt{k}\cos\theta}}$$

$$= 2k\int_0^\theta \sin^2\theta\, d\theta$$

$$= 2k\int_0^\theta (1-\cos 2\theta)\, d\theta$$

$$= 2k\left(\theta - \frac{\sin 2\theta}{2}\right)$$

$$= 4k(2\theta - \sin 2\theta)$$

$$\Rightarrow x = 4k(\phi - \sin\phi) \qquad \phi = 2\theta$$

$$\Rightarrow y = k\sin^2\phi/2 = k(1-\cos\phi)$$

$\therefore$ The path is formed by the polar coordinates,

$$x = 4k(\phi - \sin\phi), \quad y = k(1-\cos\phi)$$

- **Euler's Equation of Several Dependent Variables :-**

$$J = \int_{x_1}^{x_2} f(y_1, y_2, y_3 \dots y_n, y_1', y_2', y_3' \dots y_n' ; x)\, dx$$

The extremum of $J$ satisfies

$$\frac{\delta f}{\delta y_i} - \frac{d}{dx}\frac{\delta f}{\delta y_i'} = 0 \quad \forall\ i \in [1:n]$$

→ **Hamilton's Principle :-**

- " Of all the possible paths along which a dynamical system may move from one point to another within a specified time interval, the actual path followed is that which minimizes the time integral of the difference between the kinetic and potential energies ".

$$\Rightarrow \quad \delta \int_{t_1}^{t_2} (T - U)\, dt = 0$$

$\delta \rightarrow$ Notation to describe the variation in paths.

Since $T$ is a function of $\dot{x}$ and $U$ is a function of $x$,

$$L = T - U = L(x, \dot{x}; t)$$

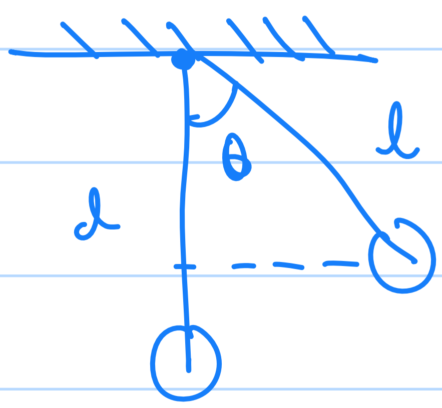
$$\Rightarrow \delta \int_{t_1}^{t_2} L(x, \dot{x}; t)\, dt = 0$$

- Since L is similar to the function used in Euler's equation, we can define the Euler-Lagrangian equations of any system as,

$$\frac{\delta L}{\delta x_i} - \frac{d}{dt}\left(\frac{\delta L}{\delta x'_i}\right) = 0 \qquad i = \underset{x}{1}, \underset{y}{2}, \underset{z}{3}$$

The function is defined as the Lagrangian of the particle being analysed.

<u>Example 1</u>: Lagrangian of a Simple Pendulum.




$$L = \frac{1}{2} m v^2 - mg(l - l\cos\theta)$$

$$L = \frac{1}{2} m l^2 \omega^2 - mg(l - l\cos\theta) \qquad [\omega = \dot{\theta}]$$

$$\frac{\delta L}{\delta \theta} - \frac{d}{dt}\left(\frac{\delta L}{\delta \omega}\right) = 0$$

$$\frac{\delta L}{\delta \theta} = -mgl\sin\theta \qquad \frac{\delta L}{\delta \omega} = \frac{1}{\cancel{2}} m l^2 \cancel{2}\omega$$

$$\frac{d}{dt}\left(\frac{\delta L}{\delta \omega}\right) = m l^2 \frac{d\omega}{dt} = m l^2 \alpha$$

$$\Rightarrow \quad -mgl\sin\theta - ml^2\alpha = 0$$

$$\Rightarrow \quad -g\sin\theta = l\alpha$$

$$\Rightarrow \quad \theta'' = -\frac{g}{l}\sin\theta$$

For small $\theta$, $\sin\theta \approx \theta$

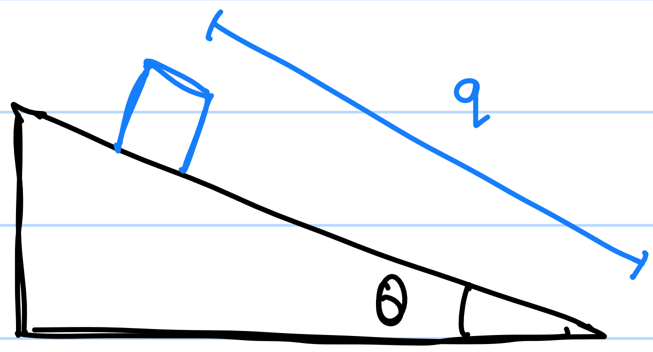
$$\Rightarrow \quad \underline{\theta'' = -\frac{g}{l}\theta} \longrightarrow \text{SHM solution}$$

$$x'' = -\omega^2$$

**Example 2:** EOM of a block sliding down a rising incline.

The incline $\theta$ increases linearly with time as $\theta = at$

Let the distance of the block from the axis of rotation be $q$.




$$T = \frac{1}{2}m\dot{q}^2 + \frac{1}{2}m(q\dot{\theta})^2 \qquad U = -mgq\sin\theta$$

Due to linear motion — Due to rising of incline

$$\mathcal{L} = T - U$$
$$= \frac{1}{2}m\dot{q}^2 + \frac{1}{2}mq^2\dot{\theta}^2 - mgq\sin\theta$$
$$= \frac{1}{2}m\dot{q}^2 + \frac{1}{2}mq^2\alpha^2 + mgq\sin\alpha t = \mathcal{L}(q, \dot{q}; t)$$

$$\frac{\delta L}{\delta q} - \frac{d}{dt}\left(\frac{\delta L}{\delta \dot{q}}\right) = 0$$

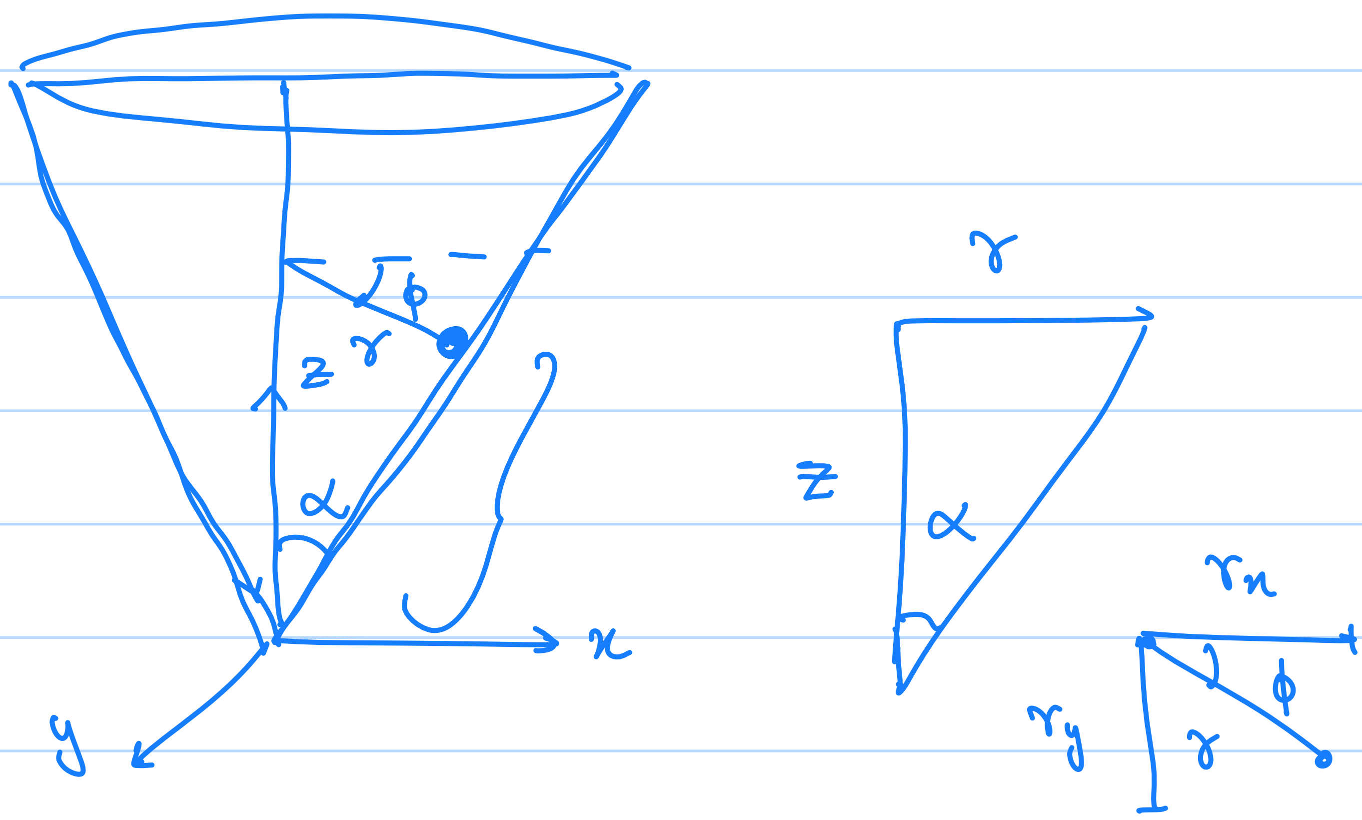
$$\frac{\delta L}{\delta q} = mq\alpha^2 - mg\sin\alpha t$$

$$\frac{\delta L}{\delta \dot{q}} = m\dot{q}, \quad \frac{d}{dt}\left(\frac{\delta L}{\delta \dot{q}}\right) = m\frac{d\dot{q}}{dt} = m\ddot{q}$$

$$\Rightarrow mq\alpha^2 - mg\sin\alpha t - m\ddot{q} = 0$$

$$\Rightarrow \underline{\ddot{q} = q\alpha^2 - g\sin\alpha t}$$

3) Example 3: Wall of Death



Here we see one more advantage of Lagrangian mechanics, ie, the use of arbitrary coordinates.

$$x = r\cos\phi \qquad y = r\sin\phi \quad , \quad z = r\cot\alpha$$

$$\dot{x} = \dot{r}\cos\phi - r\sin\phi\cdot\dot{\phi}$$
$$\dot{y} = \dot{r}\sin\phi + r\cos\phi\cdot\dot{\phi}$$
$$\dot{z} = \dot{r}\cot\alpha$$

$$\dot{v}^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$$
$$= \dot{r}^2\cos^2\phi + r^2\dot{\phi}^2\sin^2\phi - 2\dot{\phi}\dot{r}r\cos\phi\sin\phi$$
$$+ \dot{r}^2\sin^2\phi + r^2\dot{\phi}^2\cos^2\phi - 2\dot{\phi}\dot{r}r\cos\phi\sin\phi$$
$$+ \dot{r}^2\cot^2\alpha$$

$$\Rightarrow \dot{v}^2 = \dot{r}^2 + r^2\dot{\phi}^2 + \dot{r}^2\cot^2\alpha$$
$$= \dot{r}^2(1+\cot^2\alpha) + r^2\dot{\phi}^2$$
$$= \dot{r}^2\operatorname{cosec}^2\alpha + r^2\dot{\phi}^2$$

$$KE = \frac{1}{2}m\dot{r}^2\operatorname{cosec}^2\alpha + \frac{1}{2}mr^2\dot{\phi}^2$$

$$U = mgz = mgr\cot\alpha$$

$$\Rightarrow L = \frac{1}{2}m\dot{r}^2\operatorname{cosec}^2\alpha + \frac{1}{2}mr^2\dot{\phi}^2 - mgr\cot\alpha$$

$\hookrightarrow L(r, \phi, \dot{r}, \dot{\phi}; t)$

$$\Rightarrow \quad \frac{\delta L}{\delta r} - \frac{d}{dt}\left(\frac{\delta L}{\delta \dot r}\right) = 0 \quad \& \quad \frac{\delta L}{\delta \phi} - \frac{d}{dt}\left(\frac{\delta L}{\delta \dot\phi}\right) = 0$$

$$\frac{\delta L}{\delta r} = m r \dot\phi^2 - mg\cot\alpha$$

$$\frac{\delta L}{\delta \dot r} = m \dot r \operatorname{cosec}^2\alpha$$

$$\frac{d}{dt}\left(\frac{\delta L}{\delta \dot r}\right) = m\operatorname{cosec}^2\alpha \frac{d\dot r}{dt} = m\ddot r \operatorname{cosec}^2\alpha$$

$$\Rightarrow \cancel{m} r\dot\phi^2 - \cancel{m}g\cot\alpha = \cancel{m}\ddot r \operatorname{cosec}^2\alpha$$

$$\Rightarrow \quad \ddot r = r\dot\phi^2 \sin^2\alpha - g\cos\alpha\sin\alpha = r\dot\phi^2 \sin^2\alpha - \frac{g}{2}\sin 2\alpha$$

$$\frac{\delta L}{\delta \phi} = 0 \quad \Rightarrow \quad \frac{d}{dx}\left(\frac{\delta L}{\delta \dot\phi}\right) = 0$$

$$\Rightarrow \quad \frac{\delta L}{\delta \dot\phi} = \text{const}$$

$$\frac{\delta L}{\delta \dot\phi} = m r^2 \dot\phi \quad \Rightarrow \quad \underline{\underline{\dot\phi = \frac{c}{m r^2}}}$$

$\Rightarrow$ <u>EOM:</u>

$$\ddot r = r\dot\phi^2 \sin^2\alpha - \frac{g}{2}\sin 2\alpha$$

$$\dot\phi = c/mr^2$$

## Lagrangian Equation in the Presence of Constraints :-

- For a Lagrangian defined as $\delta L(y, \dot{y}; x)$, the solution is given by Euler's equation as,

$$\frac{\delta L}{\delta y} - \frac{d}{dx}\frac{\delta L}{\delta \dot{y}} = 0$$

- But if we have some constraints on $y$ as $g(y; x)$, the Euler's equation becomes,

Lagrange Undetermined Multiplier

$$\frac{\delta L}{\delta y} - \frac{d}{dx}\left(\frac{\delta L}{\delta \dot{y}}\right) + \lambda(x)\frac{\delta g}{\delta y} = 0$$

- For the general case of several dependent variables and constraints

$$\frac{\delta f}{\delta y_i} - \frac{d}{dx}\left(\frac{\delta f}{\delta \dot{y}_i}\right) + \sum_j \lambda_j(x)\frac{\delta g_j}{\delta y_i} = 0 \quad \forall$$

dependent vars $i$ and constraints $j$ on $i$.

## Requirements for Constructing the Lagrangian :-

1) The forces acting on the system must be derivable from a potential / several potentials. (ie, conservative forces)

2) The equations of motion must connect the coordinates of the system and can be functions of time.

- Essence of Lagrangian Dynamics :-

- Lagrangian dynamics is not separate from Newtonian dynamics. They will yield the same results for the same physical system.

- Newtonian dynamics deals with quantities acting on an object (the forces), Lagrangian dynamics deals with quantities of the object (the energies).

- Lagrangian dynamics deals with scalar quantities and is invariant to coordinate transformations, allowing the use of generalized coordinates.

- A Theorem on Kinetic Energy :-

- The Kinetic energy of a system is given by,

$$T = \frac{1}{2} \sum_{\alpha=1}^{n} \sum_{i=1}^{k} m_\alpha \dot{x}_{\alpha i}^2$$

Let $x_{\alpha i} = x_{\alpha i}(q_i, t)$, where $q_i$ is the generalized coordinate of $x$.

$$\Rightarrow \quad \dot{x}_{\alpha 1} = \sum_j \frac{\delta x_{\alpha i}}{\delta q_j} \dot{q}_j + \frac{\delta x_{\alpha i}}{\delta t}$$

$$\Rightarrow \quad \dot{x}_{\alpha 1}^2 = \sum_{j,k} \left( \frac{\delta x_{\alpha i}}{\delta q_j} \frac{\delta x_{\alpha i}}{\delta q_k} \dot{q}_j \dot{q}_k \right) + 2 \sum_j \frac{\delta x_{\alpha i}}{\delta q_j} \dot{q}_j \frac{\delta x_{\alpha i}}{\delta t}$$

$$+ \left( \frac{\delta x_{\alpha i}}{\delta t} \right)^2$$

$$\Rightarrow T = \frac{1}{2}\sum_\alpha m_\alpha \sum_{i,j,k}\left(\frac{\delta x_{\alpha i}}{\delta q_j}\frac{\delta x_{\alpha i}}{\delta q_k}\dot{q}_j \dot{q}_k\right) + \sum_\alpha m_\alpha \sum_{i,j}\frac{\delta x_{\alpha i}}{\delta q_j}\dot{q}_j \frac{\delta x_{\alpha i}}{\delta t}$$

$$+ \frac{1}{2}\sum_\alpha m_\alpha \sum_i \left(\frac{\delta x_{\alpha i}}{\delta t}\right)^2$$

If time does not explicitly apply when $x$ is represented in the generalized coordinate transformation, i.e. $\frac{\delta x_{\alpha i}}{\delta t} = 0$

$$\Rightarrow T = \frac{1}{2}\sum_\alpha m_\alpha \sum_{i,j,k}\left(\frac{\delta x_{\alpha i}}{\delta q_j}\frac{\delta x_{\alpha i}}{\delta q_k}\dot{q}_j \dot{q}_k\right)$$

$$T = \sum_{j,k} a_{jk}\dot{q}_j\dot{q}_k$$

[Summing over $i$ to create $a_{jk}$] — (1)

$$\frac{\delta T}{\delta \dot{q}_i} = \sum_k a_{ik}\dot{q}_k + \sum_j a_{ji}\dot{q}_i$$

Multiply on both sides by $\dot{q}_i$ and sum over $i$,

$$\Rightarrow \sum_i \dot{q}_i \frac{\delta T}{\delta \dot{q}_i} = \underbrace{\sum_{i,k} a_{ik}\dot{q}_i\dot{q}_k + \sum_{i,j} a_{ij}\dot{q}_i\dot{q}_k}_{\text{Each of these is } \parallel \text{ to } (1)}$$

$$\Rightarrow \boxed{\sum_i \dot{q}_i \frac{\delta T}{\delta \dot{q}_i} = 2T}$$

Therefore, for gen. coord transformation such that time is not a direct variable of the coordinate, the above result can be used.

In the general case, if $f(y_k)$ is a homogenous function of $y_k$ of degree $n$, then

$$\boxed{\sum_k \dot{y}_k \frac{\delta f}{\delta \dot{y}_k} = nf}$$

- <u>Conservation of Energy in Lagrangian Mechanics :-</u>

- Time is homogenous within an inertial reference frame. So,

$$\frac{\delta L}{\delta t} = 0$$ [The Lagrangian is invariant with time]

Since L is a function of $q_i$ and $\dot{q}_i$,

$$\frac{dL}{dt} = \sum_i \frac{\delta L}{\delta q_i} \dot{q}_i + \sum_i \frac{\delta L}{\delta \dot{q}_i} \ddot{q}_i$$

By Euler's equation, $\frac{\delta L}{\delta q_i} - \frac{d}{dt}\frac{\delta L}{\delta \dot{q}_i} = 0$

$$\Rightarrow \frac{dL}{dt} = \sum_i \frac{d}{dt}\left(\frac{\delta L}{\delta \dot{q}_i}\right)\dot{q}_i + \sum_i \frac{\delta L}{\delta \dot{q}_i}\ddot{q}_i$$

$$= \sum_i \frac{d}{dt}\left(\dot{q}_i \frac{\delta L}{\delta \dot{q}_i}\right)$$ [Reverse of Product Rule]

$$\Rightarrow \frac{dL}{dt} - \frac{d}{dt}\left(\sum_i \dot{q}_i \frac{\delta L}{\delta \dot{q}_i}\right) = 0$$

$$\Rightarrow \quad \frac{d}{dt}\left(L - \sum \dot{q}_i \frac{\delta L}{\delta \dot{q}_i}\right) = 0$$

$$\Rightarrow \quad L - \sum \dot{q}_i \frac{\delta L}{\delta \dot{q}_i} = \text{Constant} \quad (\text{Let it be } -H)$$

$$\frac{\delta L}{\delta \dot{q}_i} = \frac{\delta}{\delta \dot{q}_i}(T - U) = \frac{\delta T}{\delta \dot{q}_i} - \cancelto{0}{\frac{\delta U}{\delta \dot{q}_i}}$$

[ Potential Energy will not depend explicitly on velocity $\dot{q}_i$ ]

$$\Rightarrow \quad L - \sum \dot{q}_i \frac{\delta T}{\delta \dot{q}_i} = -H$$

$$\Rightarrow \quad T - U - 2T = -H$$

$$\Rightarrow \quad T + U = H$$

$$\Rightarrow \quad \boxed{T + U = \text{Constant}}$$

The function H is termed as the Hamiltonian of the system and is constant only if:

1) The Eqn of transformation to the generalized coordinates must be independent of time, to make T a homogeneous function of velocity $\dot{q}_i$.

2) The potential energy must be independent of velocity of the object (ie must be conservative).

## ∘ Conservation of Linear Momentum in Lagrangian Mechanics :-

– Space is homogenous in an inertial reference frame, ie, the Lagrangian is invariant of any small translation of the whole system,

$$\Rightarrow \quad \delta L = \sum_i \frac{\delta L}{\delta x_i}\, \delta x_i + \sum_i \frac{\delta L}{\delta \dot{x}_i}\, \delta \dot{x}_i = 0$$

where $\delta x_i$ is the small displacement in space, independent of time

$$\delta \dot{x}_i = \delta \frac{d}{dt} x_i = \frac{d}{dt}\, \delta x_i = \underline{\underline{0}}$$

$$\Rightarrow \quad \sum_i \frac{\delta L}{\delta x_i}\, \delta x_i = 0$$

$$\Rightarrow \quad \frac{\delta L}{\delta x_i} = 0$$

[ $x_i$'s are non-zero and linearly independent ]

$$\Rightarrow \quad \frac{d}{dt} \frac{\delta L}{\delta \dot{x}_i} = 0$$

$$\Rightarrow \quad \frac{\delta L}{\delta \dot{x}_i} = \text{constant}$$

$$\Rightarrow \quad \frac{\delta (T-U)}{\delta \dot{x}_i} = \frac{\delta T}{\delta \dot{x}_i} = \frac{\delta}{\delta \dot{x}_i}\left(\frac{1}{2} m \dot{x}_i^2\right) = \text{constant}$$

$$\Rightarrow \quad \underline{m \dot{x}_i = \text{constant}.}$$

- **Conservation of Angular Momentum in Lagrangian Mechanics :-**

- Space is also isotropic within an inertial reference frame, ie, the Lagrangian is invariant of the orientation of the system.

If the system is rotated by a small angle $\delta\theta$, the radius vectors change as,

$$\delta r = \delta\theta \times r$$

$$\Rightarrow \delta\dot{r} = \delta\theta \times \dot{r}$$

The change in L due to small rotation $\delta\theta$ is,

$$\delta L = \sum_i \frac{\delta L}{\delta x_i}\delta x_i + \sum_i \frac{\delta L}{\delta \dot{x}_i}\delta \dot{x}_i$$

From the prev. section, we see that,

$$\frac{\delta L}{\delta \dot{x}_i} = p_i$$

[ The assumptions taken in both sections are the same ]

$$\Rightarrow \frac{\delta L}{\delta x_i} = \dot{p}_i$$

$$\Rightarrow \delta L = \sum_i \dot{p}_i \delta x_i + \sum p_i \delta\dot{x}_i = 0$$

$$\Rightarrow \dot{p}_i \delta x_i + p_i \delta\dot{x}_i = 0$$

$$\Rightarrow \quad \dot{p}_i \, \delta r_i + p_i \, \delta \dot{r}_i = 0$$

$$\Rightarrow \quad \dot{p}_i \cdot (\delta\theta \times r) + p_i \, (\delta\theta \times \dot{r}) = 0$$

$$\Rightarrow \quad \delta\theta \cdot (r \times \dot{p}_i) + \delta\theta \cdot (\dot{r} \times p_i) = 0 \qquad \text{[Cyclic Permutation]}$$

$$\Rightarrow \quad \delta\theta \cdot \left[ (r \times \dot{p}_i) + (\dot{r} \times p_i) \right] = 0$$

$$\Rightarrow \quad \delta\theta \cdot \frac{d}{dt} (r \times p_i) = 0$$

$$\Rightarrow \quad \frac{d}{dt} (r \times p_i) = 0 \qquad [\delta\theta \text{ is arbitrary}]$$

$$\Rightarrow \quad r \times p_i = \text{constant}$$

Since Angular momentum $= r \times p_i$, we see that Angular momentum of the system is constant, given the assumptions hold for the system.

- <u>Hamiltonian Dynamics</u> :-

- We know that, if the potential of an object is independent of its velocity then,

$$p_i = \frac{\delta L}{\delta \dot{x}_i}$$

- If the Lagrangian is expressed in generalized coordinates $q_i$,

$$p_j = \frac{\delta L}{\delta \dot{q}_j} \quad \& \quad \dot{p}_j = \frac{\delta L}{\delta q_j}$$

$p_j$ is defined as the generalized momentum of the system.

- Using generalized momentum, the Hamiltonian of a system is defined as,

$$H = \sum_j p_j \dot{q}_j - L$$

Where the Lagrangian $L$ is a function of the generalized coordinates and generalized velocities, and the time.

- The Hamiltonian of a system is expressed as,

$$H(q_k, p_k ; t) = \sum_j p_j \dot{q}_j - L(q_k, \dot{q}_k ; t)$$

The Hamiltonian is expressed as a function of $(q_k, p_k ; t)$, the Lagrangian is a function of $(q_k, \dot{q}_k ; t)$.

- The differential of $H$ is,

$$dH = \sum_k \left( \frac{\delta H}{\delta q_k} \delta q_k + \frac{\delta H}{\delta p_k} \delta p_k \right) + \frac{\delta H}{\delta t} dt$$

$$= \sum_k \left( \dot{q}_k dp_k + p_k d\dot{q}_k - \frac{\delta L}{\delta q_k} \delta q_k - \frac{\delta L}{\delta \dot{q}_k} \delta \dot{q}_k \right) - \frac{\delta L}{\delta t} dt$$

$$= \sum_{k} \left( \dot{q}_k \, dp_k - \dot{p}_k \, dq_k \right) - \frac{\delta L}{\delta t} dt$$

Using the coefficients of $dH$, we can see that,

$$\underbrace{\frac{\delta H}{\delta p_k} = \dot{q}_k \qquad \frac{\delta H}{\delta q_k} = -\dot{p}_k}_{\text{Hamiltonian / Canonical EOM}} \qquad \frac{\delta L}{\delta t} = -\frac{\delta H}{\delta t}$$

Also, $\frac{dH}{dt} = \frac{\delta H}{\delta t}$, ie, if $H$ does not explicitly contain time, then $H$ is conserved, ie, constant.

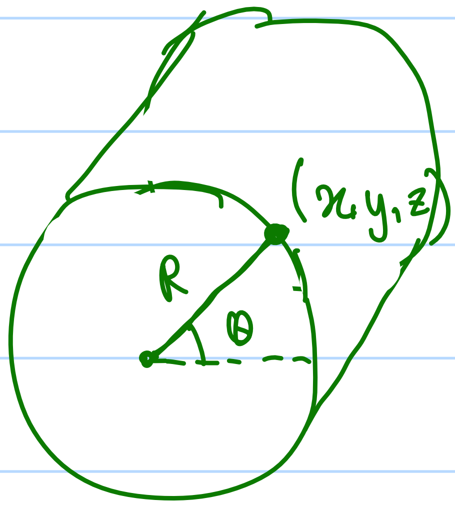
- As prev. seen, $H$ is conserved if PE is velocity dependent and the transformations to the general coordinates do not contain time.
- If there are $s$ Lagrangian Equations, there are $2s$ canonical equations ($s$ - Degrees of Freedom)
- Canonical Equations are first order DE's, while Lagrangians are second order DE's.

Example: Using Hamiltonian Mechanics to find the EOM of a particle of mass $m$ constrained on the surface of a cylinder $x^2 + y^2 = R^2$. The particle has a force towards the origin and proportional to the distance of the particle $F = -kr$.

$$U = -\int F\,dr = -\int(-kr)\,dr = \frac{1}{2}kr^2$$

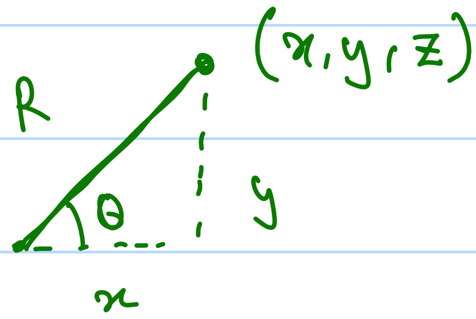
$$\Rightarrow \quad U = \frac{1}{2}k\left(x^2+y^2+z^2\right) = \frac{1}{2}k\left(R^2+z^2\right)$$

$$v^2 = (\dot{x})^2 + (\dot{y})^2 + (\dot{z})^2$$




$$= \left(\dot{\overline{R\cos\theta}}\right)^2 + \left(\dot{\overline{R\sin\theta}}\right) + (\dot{z})^2$$

$$= R^2\left(-\sin\theta\cdot\dot{\theta}\right)^2 + R^2\left(\cos\theta\cdot\dot{\theta}\right) + \dot{z}^2$$




$$= R^2\sin^2\theta\cdot\dot{\theta}^2 + R^2\cos^2\theta\cdot\dot{\theta}^2 + \dot{z}^2$$

$$x = R\cos\theta$$

$$y = R\sin\theta$$

$$v^2 = R^2\dot{\theta}^2 + \dot{z}^2$$

$$T = \frac{1}{2}m\left(R^2\dot{\theta}^2 + \dot{z}^2\right)$$

$$= \frac{1}{2}mR^2\dot{\theta}^2 + \frac{1}{2}m\dot{z}^2$$

$$L = T - U = \frac{1}{2}mR^2\dot{\theta}^2 + \frac{1}{2}m\dot{z}^2 - \frac{1}{2}kR^2 - \frac{1}{2}kz^2$$

$$= L\left(\dot{\theta}, z, \dot{z}; t\right); \quad \frac{\delta L}{\delta t} = 0$$

$$= \quad p_\theta = \frac{\delta L}{\delta\dot{\theta}} = mR^2\dot{\theta} \qquad \Rightarrow T = \frac{p_\theta^2}{2mR^2} + \frac{p_z^2}{2m}$$

$$p_z = \frac{\delta L}{\delta\dot{z}} = m\dot{z}$$

Since the system is conservative and the generalized coordinates $(\theta, z)$ do not involve time, the Hamiltonian is defined as,

$$H = T + U$$

$$= \frac{p_\theta^2}{2mR^2} + \frac{p_z^2}{2m} + \frac{1}{2}k(R^2 + z^2)$$

$$H(z, p_\theta, p_K)$$

$$\Rightarrow \quad \dot{\theta} = \frac{\delta H}{\delta p_\theta} = \frac{p_\theta}{mR^2}$$

$$\dot{z} = \frac{\delta H}{\delta p_z} = \frac{p_z}{m}$$

$$\dot{p_\theta} = -\frac{\delta H}{\delta \theta} = 0 \longrightarrow$$ Angular Momentum is constant

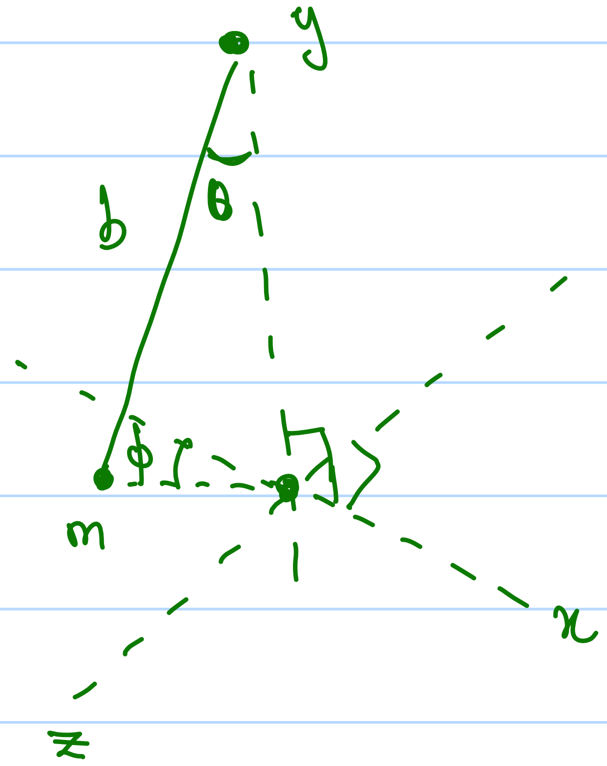
$$\dot{p_z} = -\frac{\delta H}{\delta z} = -kz$$

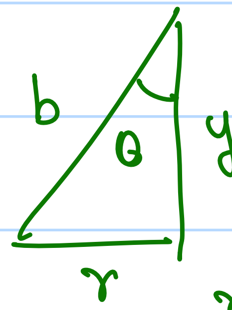
$$\dot{z} = \frac{p_z}{m} \quad , \quad \dot{p_z} = -kz$$

$$\Rightarrow \quad \ddot{z} = \frac{\dot{p_z}}{m} = \frac{-k}{m}z$$

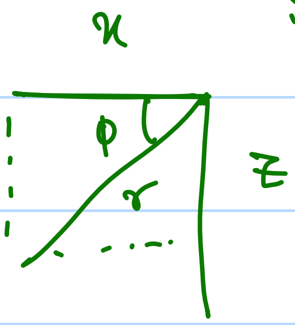
$$\Rightarrow \quad \ddot{z} = \frac{-k}{m}z \longrightarrow$$ $z$ is in SHM.

**Example:** Use Hamiltonian Dynamics to find the EOM for a spherical pendulum of mass m and length b,




$r = b\sin\theta$
$\approx b\theta$




$x = r\cos\phi$

$z = r\sin\phi$

$\Rightarrow \quad x = b\sin\theta\cos\phi \approx b\theta\cos\phi$

$z = b\sin\theta\sin\phi \approx b\theta\sin\phi$

$$v^2 = \dot{x}^2 + \dot{z}^2$$

$$= (b\dot{\theta\cos\phi})^2 + (b\dot{\theta\sin\phi})^2$$

$$= b^2\left( (\dot{\theta}\cos\phi - \theta\sin\phi\,\dot{\phi})^2 + (\dot{\theta}\sin\phi + \theta\cos\phi\cdot\dot{\phi})^2 \right)$$

$$= b^2\left( \dot{\theta}^2\cos^2\phi + \theta^2\dot{\phi}^2\sin^2\phi - \cancel{2\theta\dot{\theta}\cos\phi}\sin\phi\,\dot{\phi} \right.$$
$$\left. + \dot{\theta}^2\sin^2\phi + \theta^2\dot{\phi}^2\cos^2\phi + \cancel{2\theta\dot{\theta}\cos\phi}\sin\phi\,\dot{\phi} \right)$$

$$= b^2(\dot{\theta}^2 + \theta^2\dot{\phi}^2)$$

$$T = \frac{1}{2}mb^2(\dot{\theta}^2 + \theta^2\dot{\phi}^2)$$

$$\approx T = \frac{1}{2}mb^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2)$$

[Reversing the small angle approximation]

$$U = -mgb\cos\theta$$

$$\mathcal{L} = T - U$$
$$= \frac{1}{2}mb^2\dot{\theta}^2 + \frac{1}{2}mb^2\sin^2\theta\,\dot{\phi}^2 - mgb\cos\theta$$
$$= L(\theta, \dot{\theta}, \dot{\phi}\,;\, t)$$

$$p_\theta = \frac{\delta L}{\delta \dot{\theta}} = mb^2\dot{\theta}$$

$$p_\phi = \frac{\delta L}{\delta \dot{\phi}} = mb^2\sin^2\theta\,\dot{\phi}$$

$$H = T + U$$
$$= \frac{p_\theta^2}{2mb^2} + \frac{p_\phi^2}{2mb^2\sin^2\theta} - mgb\cos\theta$$

$$\dot{\theta} = \frac{\delta H}{\delta p_\theta} = \frac{p_\theta}{mb^2}$$

$$\dot{\phi} = \frac{\delta H}{\delta p_\phi} = \frac{p_\phi}{mb^2\sin^2\theta}$$

$$\dot{p}_\theta = -\frac{\delta H}{\delta \theta} = \frac{p_\phi^2}{2mb^2}\left(\frac{-3}{\sin^3\theta}\right)\cos\theta - mgb\sin\theta$$
$$= \frac{-3p_\phi^2\cos\theta}{2mb^2\sin^3\theta} - mgb\sin\theta$$

$$\dot{p}_\phi = -\frac{\delta H}{\delta \phi} = 0 \longrightarrow$$ Angular Momentum about the central axis is constant.

## → Central Force Motion :-

- The motion of a two body system affected by a force directed along the line joining the centre.

- ### Reduced Mass :-

- To describe such a system, we require the following quantities,

$r_1$ – Center of first object
$r_2$ – " " second object
$R$ – Center of Mass

- Assuming the force is conservative, the potential energy of the system is a function of $r = |\bar{r}_1 - \bar{r}_2|$.

The Lagrangian for such a system is,

$$L(\dot{r}, r; t) = \frac{1}{2} m_1 \dot{r}_1^2 + \frac{1}{2} m_2 \dot{r}_2^2 - U(r)$$

- Studying the system in the frame of reference of the COM (ie, no translation motion of the system), let the location of the COM be the origin, so,

$$m_1 r_1 + m_2 r_2 = 0$$

$$\Rightarrow \quad r_1 = \frac{m_2}{m_1 + m_2} r$$

$$r_2 = \frac{m_1}{m_1 + m_2} r$$

Substituting this in the Lagrangian,

$$L(r, \dot{r}, t) = \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} \dot{r}^2 + \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} \dot{r}^2 - U(r)$$

$$= \frac{1}{2} \mu \dot{r}^2 - U(r)$$

Where $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$, termed as reduced mass of the system.

- Now, the system can be treated as a single particle of mass $\mu$ at a distance of $r$ from the origin.

- Conservation Theorem :-

- Since the potential energy of the system is a function of $r$, the distance of the particle from the origin, the system possesses spherical symmetry about the origin.

- We know under such conditions, angular momentum $\bar{r} \times \bar{p}$ is conserved.

- $$L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

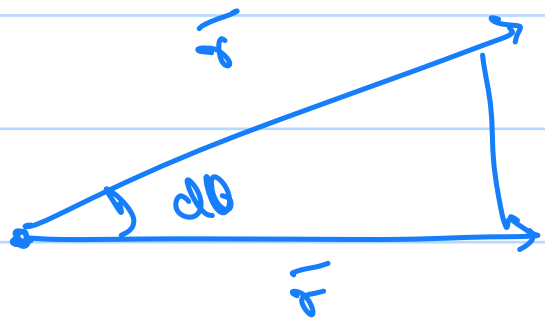
$$\dot{p}_\theta = \frac{\delta L}{\delta \theta} = 0 \qquad p_\theta = \frac{\delta L}{\delta \dot{\theta}} = \mu r^2 \dot{\theta} = \text{constant}$$

$\Rightarrow$ $p_\theta$ = constant.

Let $p_\theta = l$, ie, $l = \mu r^2 \dot{\theta}$ = constant,

The Area sector swept out by the rotating radius vector is,

$$dA = \frac{1}{2} r^2 \, d\theta$$




$$\Rightarrow \quad \dot{A} = \frac{dA}{dt} = \frac{r^2}{2}\frac{d\theta}{dt}$$

$$= \frac{\cancel{r^2}}{2}\frac{l}{\mu \cancel{r^2}}$$

$$\Rightarrow \quad \dot{A} = \frac{l}{2\mu} = \text{constant}$$

$\therefore$ The areal velocity of the simplified single body system is a constant. [||| to Kepler's Second Law of Planetary Motion]

- Due to the assumptions taken for the system, we can directly state that Energy in the system is conserved.

$$\Rightarrow E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\theta}^2 + U(r)$$

$$\Rightarrow E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{l^2}{\mu r^2} + U(r)$$

- Equation of Motion :

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{l^2}{\mu r^2} + U(r)$$

$$\Rightarrow \dot{r} = \pm\sqrt{\frac{2}{\mu}(E-U) - \frac{l^2}{\mu^2 r^2}}$$

$$\Rightarrow r = \pm\int\sqrt{\frac{2}{\mu}(E-U) - \frac{l^2}{\mu^2 r^2}}\, dt$$

Also, $d\theta = \frac{d\theta}{dt}\frac{dt}{dr}dr = \frac{\dot{\theta}}{\dot{r}}dr$, and $\dot{\theta} = \frac{l}{\mu r^2}$

$$\Rightarrow \theta = \int \frac{\pm(l/r^2)\,dr}{\sqrt{2\mu(E-U) - l^2/\mu^2 r^2}}$$

- Using the Lagrangian to get the EOM,

$$\frac{\delta L}{\delta r} - \frac{d}{dt}\frac{\delta L}{\delta \dot{r}} = 0$$

$$\mu r\dot{\theta}^2 - \frac{d}{dt}(\mu\dot{r}) + \frac{\delta U}{\delta r} = 0$$

$$= \mu(r\dot{\theta}^2 - \ddot{r}) = -\frac{\delta U}{\delta r} = F(r)$$

Take $u = \frac{1}{r}$

$$\frac{du}{d\theta} = \frac{du}{dr}\frac{dr}{d\theta} = \frac{-1}{r^2}\frac{dr}{dt}\frac{dt}{d\theta} = \frac{-1}{r^2}\frac{\dot{r}}{\dot{\theta}}$$

Since wkt $\dot{\theta} = \frac{l}{\mu r^2}$

$$\Rightarrow \frac{du}{d\theta} = -\frac{\mu}{l}\dot{r}$$

$$\frac{d^2u}{d\theta^2} = \frac{d}{d\theta}\left(-\frac{\mu}{l}\dot{r}\right)$$

$$= \frac{dt}{d\theta}\frac{d}{dt}\left(-\frac{\mu}{l}\dot{r}\right)$$

$$= \frac{1}{\dot{\theta}}\frac{-\mu}{l}\ddot{r}$$

$$\Rightarrow \frac{d^2u}{d\theta^2} = -\frac{\mu}{l\dot{\theta}}\ddot{r} = -\frac{\mu^2 r^2}{l^2}\ddot{r}$$

$$\Rightarrow \ddot{r} = \frac{-l^2}{\mu^2 r^2}\frac{d^2u}{d\theta^2} =$$

$$\Rightarrow \ddot{r} = -\frac{l^2}{\mu^2}u^2\frac{d^2u}{d\theta^2}$$

$$r\dot{\theta} = \frac{l^2u^3}{\mu^2}$$

$$\Rightarrow \quad \mu\left(\frac{l^2 u^3}{\mu^2} + \frac{l^2 u^2}{\mu^2}\frac{d^2 u}{d\theta^2}\right) = -F\left(\frac{1}{u}\right)$$

$$\Rightarrow \quad \frac{l^2 u^2}{\mu}\left(u + \frac{d^2 u}{d\theta^2}\right) = -F\left(\frac{1}{u}\right)$$

$$\Rightarrow \quad \frac{d^2 u}{d\theta^2} + u = \frac{-\mu}{l^2 u^2} F\left(\frac{1}{u}\right)$$

$$\Rightarrow \quad \boxed{\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{-\mu r^2}{l^2} F(r)}$$

[This equation is useful if we are given the orbit $r(\theta)$ and we need the force law that creates that orbit]

Example: An object is moving in an orbit given by $r = ke^{\alpha\theta}$. Find the force law defining that orbit, and $r(t)$, $\theta(t)$ and energies

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{-\mu r^2}{l^2} F(r)$$

$$\frac{d^2}{d\theta^2}\left(\frac{e^{-\alpha\theta}}{k}\right) + \frac{e^{-\alpha\theta}}{k} = \frac{-\mu k^2 e^{2\alpha\theta}}{l^2} F(r)$$

$$\frac{e^{-\alpha\theta}}{k}(\alpha^2 + 1) = \frac{-\mu k^2 e^{2\alpha\theta}}{l^2} F(r)$$

$$\Rightarrow \quad F(r) = -\frac{l^2 e^{-\alpha\theta}(\alpha^2+1)}{\mu k^3 e^{2\alpha\theta}}$$

$$\Rightarrow \quad F(r) = -\frac{l^2(\alpha^2+1)}{\mu r^3}$$ (An attractive inverse cube law)

To find $r(t)$ and $\theta(t)$

$$\dot{\theta} = \frac{l}{\mu r^2}$$

$$\frac{d\theta}{dt} = \frac{l}{\mu k^2 e^{2\alpha\theta}}$$

$$\Rightarrow \int e^{2\alpha\theta}\, d\theta = \int \frac{l}{\mu k^2}\, dt$$

$$= \quad \frac{1}{2\alpha} e^{2\alpha\theta} = \frac{lt}{\mu k^2} + C$$

$$= \quad e^{2\alpha\theta} = \frac{2\alpha l t}{\mu k^2} + 2\alpha C$$

$$\theta(t) = \frac{1}{2\alpha} \ln\left(\frac{2\alpha l t}{\mu k^2} + 2\alpha C\right)$$

$$r = k e^{\alpha\theta}$$

$$= k e^{\frac{1}{2}\ln\left(\frac{2\alpha l t}{\mu k^2} + 2\alpha C\right)}$$

$$r(t) = \sqrt{\frac{2\alpha l t}{\mu} + 2\alpha k^2 C}$$

To find the energies,

$$U(r) = -\int_{-\infty}^{r} F(r)\,dr$$

$$= -\int_{-\infty}^{r} -\frac{l^2(\alpha^2+1)}{\mu r^3}\,dr$$

$$= \frac{l^2(\alpha^2+1)}{\mu}\int_{-\infty}^{r}\frac{dr}{r^3}$$

$$= \frac{l^2(\alpha^2+1)}{\mu}\left[\frac{r^{-2}}{-2}\right]$$

$$U(r) = -\frac{l^2(\alpha^2+1)}{2\mu r^2}$$

$$\dot{r} = \frac{dr}{d\theta}\dot{\theta} = \frac{d}{d\theta}\left(ke^{\alpha\theta}\right)\frac{l}{\mu r^2}$$

$$= k\alpha e^{\alpha\theta}\frac{l}{\mu r^2}$$

$$= \frac{\alpha r l}{\mu r^2}$$

$$\Rightarrow \quad \dot{r} = \frac{\alpha l}{\mu r} \qquad r\dot{\theta} = \frac{rl}{\mu r^2} = \frac{l}{\mu r}$$

$$T(r) = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu(r\dot{\theta})^2$$

$$= \frac{\mu}{2}\frac{\alpha^2 l^2}{\mu^2 r^2} + \frac{\mu}{2}\frac{l^2}{\mu^2 r^2}$$

$$T(r) = \frac{\alpha^2 l^2}{2\mu r^2} + \frac{l^2}{2\mu r^2}$$

$$= \frac{l^2(\alpha^2+1)}{2\mu r^2} = -U(r)$$

$$\Rightarrow E(r) = T(r) + U(r) = \underline{\underline{0}}$$

- **Orbits in a Central Field :-**

- We know that,

$$\dot{r} = \sqrt{\pm \frac{2}{\mu}(E-U) - \frac{l^2}{\mu^2 r^2}}$$

If $\dot{r} = 0$, then,

$$\frac{2}{\mu}(E-U) - \frac{l^2}{\mu^2 r^2} = 0$$

$$E - U - \frac{l^2}{2\mu r^2} = 0$$

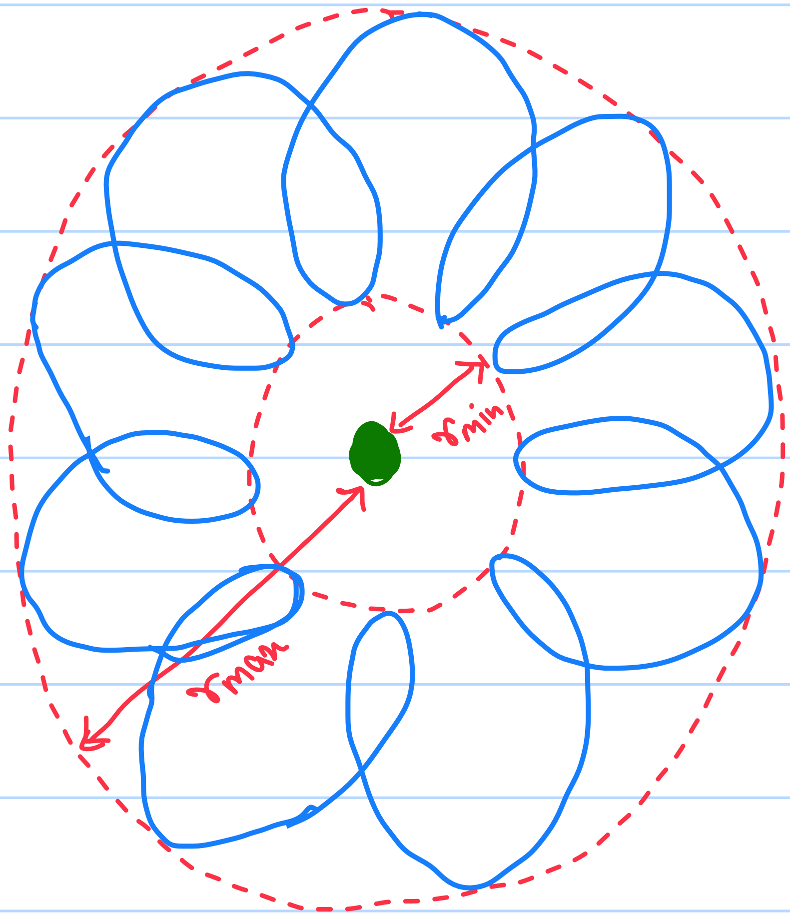
- If $\dot{r} = 0$, those points will act as a turning point for the orbit. The above equation in general will have 2 roots ($r_{max}$ and $r_{min}$). Therefore the motion of the particle is confined to $r \in [r_{max}, r_{min}]$

- Such an orbit can be either periodic or aperiodic. The orbit is periodic if the total angle change after each "oscillation" between $r_{max}$ and $r_{min}$ is a fraction of $2\pi$.

- The angle change is given by,

$$\boxed{\Delta\theta = 2\int_{r_{min}}^{r_{max}} \frac{(l/r^2)\,dr}{\sqrt{2\mu\left(E - U - \frac{l^2}{2\mu r^2}\right)}}}$$

If $\Delta\theta = 2\pi\left(\frac{a}{b}\right)$, $a,b \in \mathbb{N}$, then the orbit is periodic, aka, <u>closed</u>, with $b$ orbits per oscillation, giving a revolution / period.




A closed periodic orbit b/w $r_{min}$ and $r_{max}$

- If $U(r) \propto r^n$, then a closed non-circular orbit can exist only if $n = -1, 1$

$n = 1 \longrightarrow$ Harmonic Oscillations

$n = -1 \longrightarrow$ Inverse Square Law forces (ex: gravity)