Assignment 2 solutions

Question 1

Q1

subsection*(a) $X(\omega) = \omega^2$

The pre-image sets of X are

$$\{X \le x\} = \begin{cases} \varnothing & x < 0, \\ \{0\} & 0 \le x < 1, \\ \{-1, 0, 1\} & 1 \le x < 4, \\ \Omega = \{-2, -1, 0, 1, 2\} & x \ge 4. \end{cases}$$

From these sets we obtain the following mutually exclusive and exhaustive subsets of Ω:

$$\{0\}, \quad \{-1, 1\}, \quad \{-2, 2\}.$$

The smallest σ-field is the collection of all possible unions of these three sets. Explicitly, we have

$$\sigma(X) = \big\{\varnothing, \{0\}, \{-1, 1\}, \{-2, 2\}, \{0, -1, 1\}, \{0, -2, 2\}, \{-1, 1, -2, 2\}, \Omega\big\}.$$

Since there are 3 atoms, the total number of sets is

$$|\sigma(X)| = 2^3 = 8.$$

(b) $X(\omega) = \omega + 1$

The pre-image sets of X are

$$\{X \le x\} = \begin{cases} \varnothing & x < -1, \\ \{-2\} & -1 \le x < 0, \\ \{-2, -1\} & 0 \le x < 1, \\ \{-2, -1, 0\} & 1 \le x < 2, \\ \{-2, -1, 0, 1\} & 2 \le x < 3, \\ \Omega = \{-2, -1, 0, 1, 2\} & x \ge 3. \end{cases}$$

From these sets we obtain the following mutually exclusive and exhaustive subsets of Ω:

$$\{-2\}, \quad \{-1\}, \quad \{0\}, \quad \{1\}, \quad \{2\}.$$

The smallest σ-field is the collection of all possible unions of these five sets. Since there are 5 such sets, the total number of sets is

$$|\sigma(X)| = 2^5 = 32.$$

Question 2

Solution

Formulating the Random Variable X

Let a point in the unit square be denoted by $\omega = (x, y)$, where $x, y \in [0, 1]$. The four edges of the square are the lines $x = 0$, $x = 1$, $y = 0$, and $y = 1$. The distances from the point (x, y) to these edges are:

- Distance to $x = 0$ (left edge): x
- Distance to $x = 1$ (right edge): $1 - x$
- Distance to $y = 0$ (bottom edge): y
- Distance to $y = 1$ (top edge): $1 - y$

The random variable $X(\omega)$ is the distance to the *nearest* edge. Therefore, it is the minimum of these four values:

$$X(x, y) = \min(x, 1 - x, y, 1 - y)$$

The minimum value of X is 0, which occurs for any point on the boundary of the square. The maximum value of X occurs at the point furthest from all edges, which is the center of the square, $(0.5, 0.5)$. At this point:

$$X(0.5, 0.5) = \min(0.5, 1 - 0.5, 0.5, 1 - 0.5) = 0.5$$

Thus, the range of the random variable X is the interval $[0, 0.5]$.

The CDF is defined as $F_X(t) = P(X \le t)$. Let's consider the different possible values of t.

- **Case 1:** $t < 0$
  Since the distance X cannot be negative, the event $X \le t$ is impossible.
  $$F_X(t) = P(X \le t) = 0$$
- **Case 2:** $t > 0.5$
  Since the maximum value of X is 0.5, the event $X \le t$ is a certain event.
  $$F_X(t) = P(X \le t) = 1$$
- **Case 3:** $0 \le t \le 0.5$
  We want to find $P(X \le t)$. It is simpler to first compute the probability of the complement event, $P(X > t)$, and then use the relation $P(X \le t) = 1 - P(X > t)$.

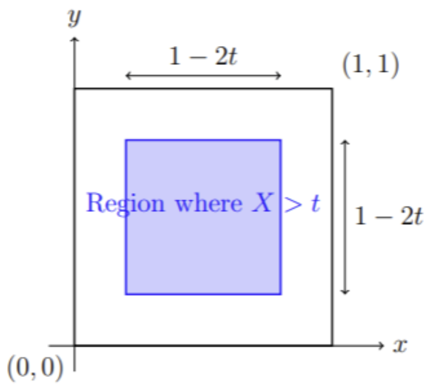
The condition $X > t$ means:

$$\min(x, 1-x, y, 1-y) > t$$

This inequality holds if and only if all four values are greater than t:

$$\begin{aligned} x > t \quad \text{and} \quad 1-x > t \implies x < 1-t \\ y > t \quad \text{and} \quad 1-y > t \implies y < 1-t \end{aligned}$$

Combining these, the event $\{X > t\}$ corresponds to the set of points (x, y) such that $t < x < 1-t$ and $t < y < 1-t$. This region is a smaller square located in the center of the unit square Ω, as depicted below.



The side length of this inner square is $(1-t) - t = 1-2t$.

Since the probability is defined as the area, we have:

$$P(X > t) = \text{Area(inner square)} = (1-2t)^2$$

Now we can find the CDF for this range:

$$F_X(t) = P(X \le t) = 1 - P(X > t) = 1 - (1-2t)^2$$

Expanding this expression, we get:

$$F_X(t) = 1 - (1 - 4t + 4t^2) = 4t - 4t^2$$

Combining all the cases, we can write the complete CDF for the random variable X as a piecewise function:

$$F_X(t) = \begin{cases} 0 & \text{if } t < 0 \\ 4t - 4t^2 & \text{if } 0 \le t \le 0.5 \\ 1 & \text{if } t > 0.5 \end{cases}$$

Question 3

Q.3 Given $F_X(x)$ is a valid CDF.
Properties to show:

1. Limits

$$\lim_{x \to -\infty} G(x) = 0, \quad \lim_{x \to +\infty} G(x) = 1$$

2. Non-decreasing

$$x < y \implies G(x) \leq G(y)$$

3. Right continuity

$$\lim_{\epsilon \to 0^+} G(x + \epsilon) = G(x)$$

(a) $G(x) = 1 - (1 - F_X(x))^r$, $r \in \mathbb{N}$
1. Limits:
As $x \to -\infty$, $F_X(x) \to 0$, so $G(x) \to 1 - (1-0)^r = 0$

$$\lim_{x \to -\infty} G(x) = 0$$

As $x \to +\infty$, $F_X(x) \to 1$, so $G(x) \to 1 - (1-1)^r = 1$

$$\lim_{x \to +\infty} G(x) = 1$$

Thus, limits are satisfied.
2. Non-decreasing property:
Since $F_X(x)$ is non-decreasing, consider

$$g(t) = 1 - (1-t)^r, \quad t \in [0, 1],\ t = F_X(x)$$

Differentiate:

$$g'(t) = r(1-t)^{r-1} \geq 0, \quad \text{for } r \in \mathbb{N},\ 1 - t \leq 1$$

Thus, $g(t)$ is non-decreasing. The composite of non-decreasing functions is non-decreasing, so $G(x)$ is non-decreasing.
3. Right continuity:
$F_X(x)$ is right-continuous and $g(t) = 1 - (1-t)^r$ is continuous (polynomial in t).
The composite of a continuous function with a right-continuous function is right-continuous.
To show this directly:

$$\lim_{\varepsilon \to 0^+} G(x + \varepsilon) = \lim_{\varepsilon \to 0^+} [1 - (1 - F_X(x + \varepsilon))^r] \tag{1}$$

$$= 1 - \left(1 - \lim_{\varepsilon \to 0^+} F_X(x + \varepsilon)\right)^r \quad \text{(since } F_X \text{ is right-continuous)} \tag{2}$$

$$= 1 - (1 - F_X(x))^r \tag{3}$$

$$= G(x) \tag{4}$$

Hence, $G(x)$ is right-continuous.
Therefore, $G(x)$ is a valid CDF.
(b) $G(x) = F_X(x) + (1 - F_X(x)) \log(1 - F_X(x))$
1. Limits:
As $x \to -\infty$, $F_X(x) \to 0$, so

$$G(x) \to 0 + (1 - 0) \log(1 - 0) = 0 + 1 \cdot \log(1) = 0$$

Thus,

$$\lim_{x \to -\infty} G(x) = 0$$

As $x \to +\infty$, $F_X(x) \to 1$. Let $F_X(x) = t \to 1$:

$$\lim_{t \to 1^-} [t + (1 - t) \log(1 - t)]$$

Set $u = 1 - t$, so $u \to 0^+$:

$$\lim_{u \to 0^+} [(1 - u) + u \log u] = 1 + \lim_{u \to 0^+} u \log u \tag{5}$$

Now, using L'Hôpital's Rule:

$$\begin{aligned}
\lim_{u \to 0^+} u \log u &= \lim_{u \to 0^+} \frac{\log u}{1/u} &(6)\\
&= \lim_{u \to 0^+} \frac{1/u}{-1/u^2} &(7)\\
&= \lim_{u \to 0^+} (-u) = 0 &(8)
\end{aligned}$$

Therefore,

$$\lim_{x \to +\infty} G(x) = 1$$

Limits satisfied.
2. Non-decreasing property:
Let $g(u) = u + (1 - u) \log(1 - u)$ for $u \in [0, 1)$.

$$\begin{aligned}
g'(u) &= 1 + \frac{d}{du}[(1 - u) \log(1 - u)] &(9)\\
&= 1 + (-1) \log(1 - u) + (1 - u) \cdot \frac{-1}{1 - u} &(10)\\
&= 1 - \log(1 - u) - 1 &(11)\\
&= -\log(1 - u) &(12)
\end{aligned}$$

For $u \in [0, 1)$, we have $1 - u \in (0, 1]$, so $\log(1 - u) \leq 0$.
Therefore, $g'(u) = -\log(1 - u) \geq 0$.
This shows $g(u)$ is non-decreasing, and since $F_X(x)$ is non-decreasing, $G(x) = g(F_X(x))$ is non-decreasing.
3. Right continuity:

Let $u = F_X(x)$ and $G(u) = u + (1-u)\log(1-u)$ for $u \in [0,1)$.

The function $\log(1-u)$ is continuous for $u \in [0,1)$. At $u = 1$, we define $G(1) = 1$ by continuity (as shown in the limit calculation above).

Case (a): At $x = a$ such that $F_X(a) < 1$

For $u \in [0,1)$, $G(u)$ is continuous. Since $F_X(x)$ is right-continuous and G is continuous at $u = F_X(a) < 1$, the composition $G(F_X(x))$ is right-continuous at $x = a$.

Case (b): At $x = a$ such that $F_X(a) = 1$

$$\lim_{x \to a^+} G(F_X(x)) = \lim_{u \to 1^-} [u + (1-u)\log(1-u)] \tag{13}$$

$$= 1 \quad \text{(as calculated above)} \tag{14}$$

$$= G(1) = G(F_X(a)) \tag{15}$$

Therefore, $G(F_X(x))$ is right-continuous at $x = a$.

Direct approach:

$$\lim_{\epsilon \to 0^+} G(x+\epsilon) = \lim_{\epsilon \to 0^+} [F_X(x+\epsilon) + (1 - F_X(x+\epsilon))\log(1 - F_X(x+\epsilon))] \tag{16}$$

Since $\lim_{\epsilon \to 0^+} F_X(x+\epsilon) = F_X(x)$ and the function $t + (1-t)\log(1-t)$ is continuous at $t = F_X(x)$ (for $F_X(x) < 1$) or has the correct limiting value (for $F_X(x) = 1$), we obtain:

$$\lim_{\epsilon \to 0^+} G(x+\epsilon) = F_X(x) + (1 - F_X(x))\log(1 - F_X(x)) = G(x)$$

$$\boxed{G(x) \text{ is right-continuous, hence it is a valid CDF.}}$$

Question 4

Solution to Problem 4

We want to show that for a non-negative integer-valued random variable N,

$$\mathbb{E}[N] = \sum_{i=1}^{\infty} \Pr(N \geq i).$$

Step 1: Expand the right-hand side

We write the first few terms explicitly:

$$\begin{aligned}
\Pr(N \geq 1) &= \Pr(N = 1) + \Pr(N = 2) + \Pr(N = 3) + \cdots, \\
\Pr(N \geq 2) &= \Pr(N = 2) + \Pr(N = 3) + \cdots, \\
\Pr(N \geq 3) &= \Pr(N = 3) + \Pr(N = 4) + \cdots, \\
&\vdots
\end{aligned}$$

Step 2: Add up the rows

Thus,

$$\sum_{i=1}^{\infty} \Pr(N \geq i) = \big(\Pr(N=1)+\Pr(N=2)+\Pr(N=3)+\cdots\big)+\big(\Pr(N=2)+\Pr(N=3)+\cdots\big)+\cdot$$

Step 3: Count the repetitions

- $\Pr(N = 1)$ appears only once.
- $\Pr(N = 2)$ appears twice: in $\Pr(N \geq 1)$ and $\Pr(N \geq 2)$.
- $\Pr(N = 3)$ appears three times: in $\Pr(N \geq 1), \Pr(N \geq 2), \Pr(N \geq 3)$.
- In general, $\Pr(N = k)$ appears exactly k times.

Hence,

$$\sum_{i=1}^{\infty} \Pr(N \geq i) = \sum_{k=1}^{\infty} k \Pr(N = k).$$

Step 4: Recognize the expectation

By definition of expectation for an integer-valued random variable,

$$\mathbb{E}[N] = \sum_{k=0}^{\infty} k \Pr(N = k).$$

Therefore,

$$\mathbb{E}[N] = \sum_{i=1}^{\infty} \Pr(N \geq i),$$

as required.

Problem 5

Give an example of a non-constant random variable X such that

$$\mathbb{E}\left[\tfrac{1}{X}\right] \;=\; \frac{1}{\mathbb{E}[X]}.$$

Solution

We need a non-constant random variable X with the property above.

Idea: Try to make $\mathbb{E}[X] = 1$, and choose values of X in reciprocal pairs so that $\mathbb{E}[X]$ and $\mathbb{E}[1/X]$ line up naturally. Numbers like 2 and $\frac{1}{2}$ are reciprocals, and -1 is its own reciprocal (up to sign), which allows us to fine-tune the mean.

Example construction:

$$X = \begin{cases} -1 & \text{with probability } \frac{1}{9}, \\ \frac{1}{2} & \text{with probability } \frac{4}{9}, \\ 2 & \text{with probability } \frac{4}{9}. \end{cases}$$

Verification:

$$\begin{aligned} \mathbb{E}[X] &= (-1)\cdot\tfrac{1}{9} + \tfrac{1}{2}\cdot\tfrac{4}{9} + 2\cdot\tfrac{4}{9} \\ &= -\tfrac{1}{9} + \tfrac{2}{9} + \tfrac{8}{9} \\ &= 1. \end{aligned}$$

$$\begin{aligned} \mathbb{E}\left[\tfrac{1}{X}\right] &= \tfrac{1}{-1}\cdot\tfrac{1}{9} + \tfrac{1}{1/2}\cdot\tfrac{4}{9} + \tfrac{1}{2}\cdot\tfrac{4}{9} \\ &= -\tfrac{1}{9} + \tfrac{8}{9} + \tfrac{2}{9} \\ &= 1. \end{aligned}$$

Thus,

$$\mathbb{E}\left[\tfrac{1}{X}\right] = 1 = \frac{1}{\mathbb{E}[X]}.$$

General recipe: Pick any $a > 0,\ a \neq 1$. Place equal probability mass p on a and $1/a$, and the remaining mass on -1. Because of the symmetry,

$$\mathbb{E}[X] = \mathbb{E}\left[\tfrac{1}{X}\right] = p\left(a + \tfrac{1}{a}\right) - (1 - 2p).$$

To force this equal to 1, choose

$$p = \frac{2}{a + \frac{1}{a} + 2}.$$

For $a = 2$, this gives $p = \frac{4}{9}$, reproducing the example above.

Question 6

Problem: Show that, if X is a binomial or Poisson random variable, then the probability mass function (PMF) P_X has the property that $P_X(k-1)P_X(k+1) \leq P_X(k)^2$. Also, give an example of a PMF P_X such that $P_X(k)^2 = P_X(k-1)P_X(k+1)$.

Solution

Part (a): Binomial Distribution

Let $X \sim \text{Binomial}(n, p)$ where $n \geq 0$ is an integer and $0 \leq p \leq 1$. The PMF is:

$$P_X(k) = \binom{n}{k} p^k (1-p)^{n-k} \quad \text{for } k = 0, 1, 2, \ldots, n$$

Firstly, note that $k \geq 1$ since we have the term $P_X(k-1)$ which has the term $\binom{n}{k-1}$

Next, for $p = 1$, or $p = 0$, we get a trivial $0 = 0$ equation. So we now show for $p \in (0, 1)$ that $P_X(k-1)P_X(k+1) \leq P_X(k)^2$.

Computing the ratio, $\frac{P_X(k+1)}{P_X(k)}$:

$$\begin{aligned}
\frac{P_X(k+1)}{P_X(k)} &= \frac{\binom{n}{k+1} p^{k+1} (1-p)^{n-k-1}}{\binom{n}{k} p^k (1-p)^{n-k}} \\
&= \frac{n-k}{k+1} \cdot \frac{p}{1-p}
\end{aligned}$$

Similarly, $\frac{P_X(k)}{P_X(k-1)}$:

$$\begin{aligned}
\frac{P_X(k)}{P_X(k-1)} &= \frac{\binom{n}{k} p^k (1-p)^{n-k}}{\binom{n}{k-1} p^{k-1} (1-p)^{n-k+1}} \\
&= \frac{n-k+1}{k} \cdot \frac{p}{1-p}
\end{aligned}$$

Now, the inequality $P_X(k-1)P_X(k+1) \leq P_X(k)^2$ is equivalent to:

$$\frac{P_X(k-1)P_X(k+1)}{P_X(k)^2} \leq 1$$

This can be rewritten as:

$$\frac{P_X(k+1)}{P_X(k)} \cdot \frac{P_X(k-1)}{P_X(k)} \leq 1$$

Substituting our ratios (since $p \neq 0, 1$)

$$\frac{n-k}{k+1} \cdot \frac{\cancel{p}}{\cancel{1-p}} \cdot \frac{k}{n-k+1} \cdot \frac{\cancel{1-p}}{\cancel{p}} \leq 1$$

Simplifying:

$$\frac{k(n-k)}{(k+1)(n-k+1)} \leq 1$$

This is equivalent to:

$$k(n-k) \leq (k+1)(n-k+1)$$

Expanding both sides:

$$\cancel{kn} - \cancel{k^2} \leq \cancel{kn} - \cancel{k^2} + \cancel{k} + n - \cancel{k} + 1$$
$$n + 1 \geq 0$$
$$n \geq -1$$

Which is true anyways, hence we prove the given inequality is true for a binomial PMF.

Part (b): Poisson Distribution

Let $X \sim \text{Poisson}(\lambda)$ where $\lambda > 0$. The PMF is:

$$P_X(k) = \frac{e^{-\lambda}\lambda^k}{k!} \quad \text{for } k = 0, 1, 2, \ldots$$

The ratio $\frac{P_X(k+1)}{P_X(k)}$:

$$\frac{P_X(k+1)}{P_X(k)} = \frac{e^{-\lambda}\lambda^{k+1}/(k+1)!}{e^{-\lambda}\lambda^k/k!} = \frac{\lambda}{k+1}$$

The ratio $\frac{P_X(k)}{P_X(k-1)}$:

$$\frac{P_X(k)}{P_X(k-1)} = \frac{\lambda}{k}$$

The inequality becomes, (since the denominators cannot be 0, since $\lambda > 0$)

$$\frac{P_X(k+1)}{P_X(k)} \leq \frac{P_X(k)}{P_X(k-1)}$$
$$\frac{\cancel{\lambda}}{k+1} \leq \frac{\cancel{\lambda}}{k}$$
$$k \leq k+1$$
$$0 \leq 1$$

This inequality also holds for a Poisson PMF!

Part (c): Example with Equality

Consider the geometric distribution with PMF:

$$P_X(k) = p(1-p)^k \quad \text{for } k = 0, 1, 2, \ldots$$

where $0 \leq p \leq 1$.
We have:

$$\begin{aligned} P_X(k-1) &= p(1-p)^{k-1} \\ P_X(k) &= p(1-p)^k \\ P_X(k+1) &= p(1-p)^{k+1} \end{aligned}$$

Now:

$$\begin{aligned} P_X(k-1)P_X(k+1) &= p(1-p)^{k-1} \cdot p(1-p)^{k+1} \\ &= p^2(1-p)^{2k} \\ &= [p(1-p)^k]^2 \\ &= P_X(k)^2 \end{aligned}$$

Therefore, we have equality: $P_X(k)^2 = P_X(k-1)P_X(k+1)$ for all $k \geq 1$.

Question 7

Solution 7

Given a random variable $X \sim \text{Poisson}(\lambda)$, its probability mass function (PMF) is:

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

The parameter to estimate is $\theta = e^{-3\lambda}$. An estimator $g(X)$ is **unbiased** if its expected value, $E[g(X)]$, equals θ. The expected value is calculated by:

$$E[g(X)] = \sum_{k=0}^{\infty} g(k) \cdot P(X = k)$$

(a) Analysis of $g(X) = e^{-3X}$

$$\begin{aligned} E[e^{-3X}] &= \sum_{k=0}^{\infty} e^{-3k} \cdot \frac{e^{-\lambda}\lambda^k}{k!} \\ &= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(e^{-3}\lambda)^k}{k!} \\ &= e^{-\lambda} \cdot e^{\lambda e^{-3}} \qquad \left(\text{using the series expansion } \sum_{k=0}^{\infty} \frac{y^k}{k!} = e^y \right) \\ &= e^{\lambda(e^{-3}-1)} \end{aligned}$$

Since $E[g(X)] = e^{\lambda(e^{-3}-1)} \neq e^{-3\lambda}$, this estimator is **biased**.

(b) Analysis of $g(X) = (-2)^X$

$$\begin{aligned} E[(-2)^X] &= \sum_{k=0}^{\infty} (-2)^k \cdot \frac{e^{-\lambda}\lambda^k}{k!} \\ &= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(-2\lambda)^k}{k!} \\ &= e^{-\lambda} \cdot e^{-2\lambda} \qquad \left(\text{using the series expansion } \sum_{k=0}^{\infty} \frac{y^k}{k!} = e^y \right) \\ &= e^{-\lambda-2\lambda} \\ &= e^{-3\lambda} \end{aligned}$$

Since $E[g(X)] = e^{-3\lambda} = \theta$, this estimator is **unbiased**.