Real analysis
Assignment 2 solutions
Due: 9 November 2024 before 11:59 pm

1. (5 points) Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of connected subsets of a space X. Suppose that $A_n \cap A_{n+1} \neq \emptyset$ for each n. Show that the union $\bigcup_n A_n$ is connected.

   Solution:

   This proof will be done by contradiction. Assume that $A = \bigcup_n A_n$ is not connected, i.e, A has a separation consisting two sets C and D. Consider an arbitrary $A_i \in \{A_n\}$. Then A_i lies entirely within either C or D. There are two cases to consider: 1) All the A_i are entirely in either C or they are all entirely in D. or 2) Some are entirely in C and some are entirely in D. In case 1, if they are all in C, then C and D are not a separation of A which is a contradiction. In case 2, if they are split then the $A_i's$ in C are disjoint from the $A_i's$ in D. This contradicts the hypothesis $A_n \cap A_{n+1} \neq \emptyset$ for all n.

2. (5 points) Are closures and interiors of connected sets always connected? Justify with reason.

   Solution:

   The closure of a connected set is connected. For contradiction, assume that $\bar{A}$ is not connected. Then $\bar{A}$ has a separation consisting of two sets G_1 and G_2. Then there exists $r > 0$ such that $U(x, r) \subseteq G_1$.

   Let $x \in \bar{A} \cap G_1$. This implies that $U(x, r) \cap A \neq \emptyset$. Let $y \in U(x, r) \cap A$. Then $y \in G_1 \cap A$. This implies that $A \cap G_1 \neq \emptyset$. Similarly, $A \cap G_2 \neq \emptyset$.

   Consider two tangent closed disks. The union will give us a connected set. But the interior part of it will be two separated open balls.

3. (5 points) Show that the union of a finite number of compact sets in a metric space (X, d) is compact.

   Solution:

   Let G be an open cover of F with $F = \bigcup_{i=i}^{n} F_i$ and $n \geq 1$. Then G is an open cover of each F_i with $i = 1, 2, \cdots, n$.

   Since F_i is compact, we can extract from G, a finite open subcover G_i of F_i. Put now $G_0 = G_1 \cup G_2 \cup \cdots G_n$. G_0 is then a finite open subcover of F .

4. (5 points) Let $f : A \to \mathbb{R}$ be continuous on A. If $K \subseteq A$ is compact, show that $f(K)$ is also compact.

   Solution:

   Let $\{V_\alpha\}_{\alpha\in I}$ be an open cover of $f(K)$. Thus, $f(K) \subseteq \bigcup_{\alpha\in I} V_\alpha$. (2 marks).

   This implies that $K \subseteq f^{-1}(\bigcup_{\alpha\in I} V_\alpha) = \bigcup_{\alpha\in I} f^{-1}(V_\alpha)$. (3 marks)

   Since f is continuous, each $f^{-1}(V_\alpha)$ is an open subset of X (1 mark).

   Since K is compact and $K \subseteq f^{-1}(\bigcup_{\alpha\in I} V_\alpha)$, there exists $n \in \mathbb{N}$, with $K \subseteq f^{-1}(\bigcup_{j=1}^{n} V_{\alpha_j})$ for some $\alpha_1, \alpha_2, \cdots, \alpha_n \in I$. (3 marks)

Hence $f(K) \subseteq \bigcup_{j=1}^{n} V_{\alpha_j}$ and $f(K)$ is compact. (1 mark)

5. (10 points) Give an example of an open cover of $(0,1)$ which has no finite subcover.

   Solution:

   We have $(0,1) = \bigcup_{n=1}^{\infty} (1/n, 1)$. This cover has no finite subcover.

6. (10 points) Find the pointwise limit of the sequence of functions $f_n(x) = x^n (n \in \mathbb{N})$ on the closed segment $[0,1]$. Is this convergence uniform? Justify your answer.

   Solution:

$$f(x) = \left\{ \begin{array}{ll} 0, & \text{for } 0 \leq x < 1 \\ 1, & \text{for } x = 1 \end{array} \right\}$$

   This convergence is not uniform. Assume for contradiction that the convergence is uniform. Then for every $\epsilon > 0$, there exists N_ϵ such that $|x^n - f(x)| < \epsilon$ for all $n \geq N_\epsilon$. Choose $\epsilon = \frac{1}{2}$. Then there exists N_0 such that for every $n \geq N_0$, we have $|x^n - f(x)| < \frac{1}{2}$ for every x. Let $n = N_0$. Let $x = \frac{3}{4}^{\frac{1}{N_0}}$. Note that $f(x) = 0$. Then $|f_{N_0}(x) - f(x)| = x^{N_0} = \frac{3}{4} > \frac{1}{2}$.

7. (10 points) Show that there exist irrational numbers x such that x^n is irrational for all positive integers n.

   Solution:

   Let $X = C[0,1]$ be the space of continuous functions on [0,1] with the uniform norm. This is a complete metric space.

   For each positive integer n and each rational number q in $[0,1]$, define:

$$F(n,q) = \{f \in X : \text{there exists}\, x \in (q - \frac{1}{n}, q + \frac{1}{n}) \cap [0,1]\, \text{such that f is differentiable at x}\} \tag{1}$$

   Each $F(n,q)$ is open in X.

   The union of all $F(n,q)$ for all n and q is the set of all functions that are differentiable at some point.

   We need to show that each $F(n,q)$ is not dense in X. To do this, we can construct a function that is not in $F(n,q)$ but is arbitrarily close to any given function in X.

   Since each $F(n,q)$ is open but not dense, its complement is closed with non-empty interior.

   By the Baire Category Theorem, the intersection of all complements of $F(n,q)$ is dense in X.

   This intersection is precisely the set of nowhere differentiable functions.

   Therefore, we have shown that the set of nowhere differentiable functions is dense in C[0,1].