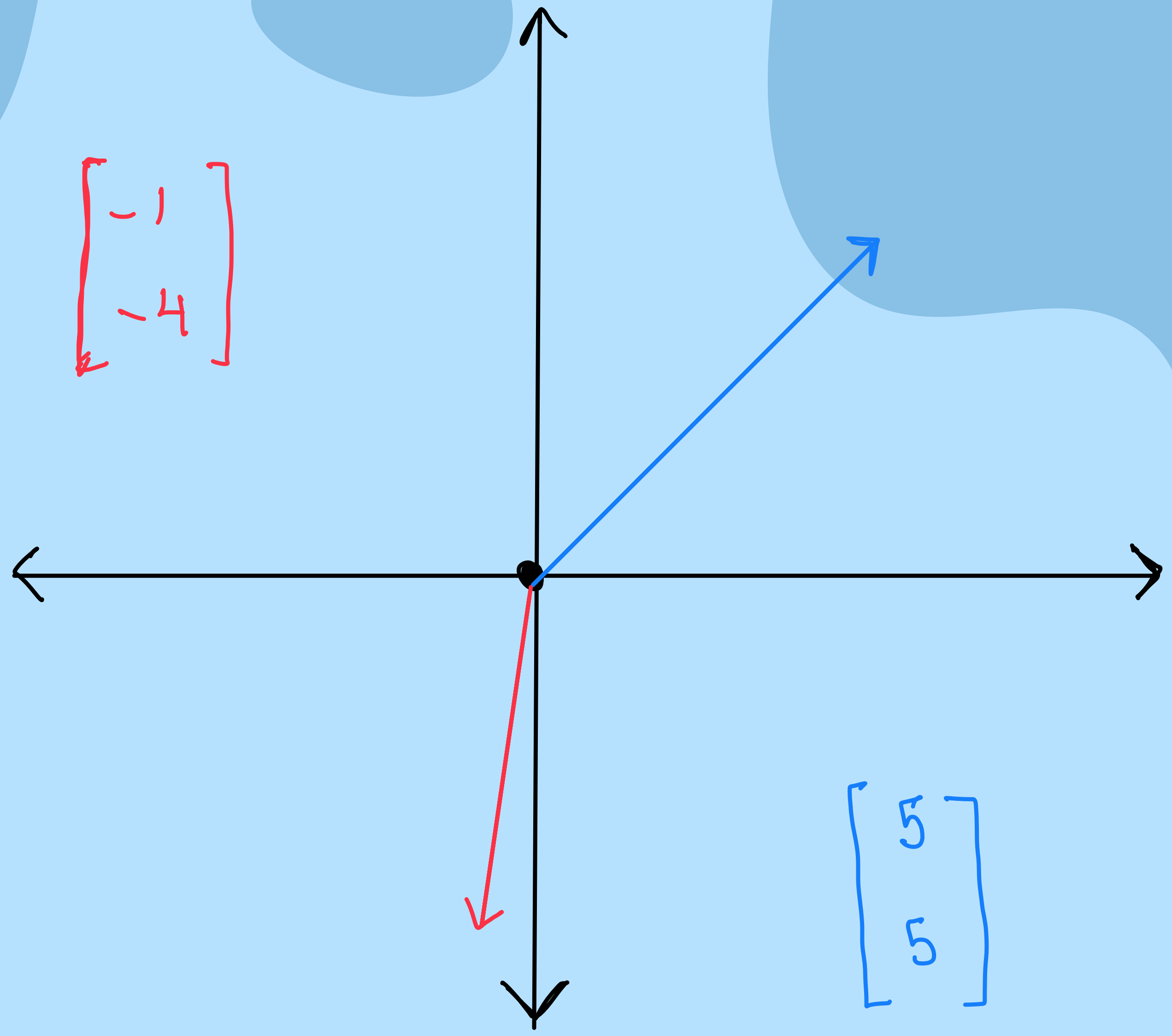
Prof. Siddhartha Das

Book: Linear Algebra by Hoffman & Kunze.



Problems will be of the <u>Proving</u> type.

<u>Grading</u>

		H1
Tutorial Quiz	= 15%	n
Assignment	= 15%	20-25%
Quiz 1	= 10%	10%
Quiz 2	= 10%	
Midsem	= 20%	15-20%
Endsem	= 30%	

3/1/25

Linear Algebra

∞ = functional algebra

- It is the study of linear maps on finite dimensional vector space.

↳ function / relation

→ **Problems solved by Linear Algebra:-**

1) **System of linear equations :-**

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \cdots + a_{1n}x_n = b_1$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 \cdots + a_{mn}x_n = b_m$$

- a_{11}, a_{12}, a_{m1}, etc are called the coefficients
- This is a system of lin. eqs as all the individual equation are of max degree 1
  - ⇒ line in 2D
  - ⇒ plane in 3D
- To solve this problem, we need to find a list of numbers $(s_1, s_2, s_3 \ldots s_n)$ that satisfy the above system of equation.

list is ordered (also tuple)
set is unordered.

- Such a list of numbers is termed as a solution to this system. set of solutions = Solution set

- A solution may not exist for a system, or a system may have ∞ solutions.

- When the no. of independent variables (n) is large, it is difficult to systematically visualise the system.

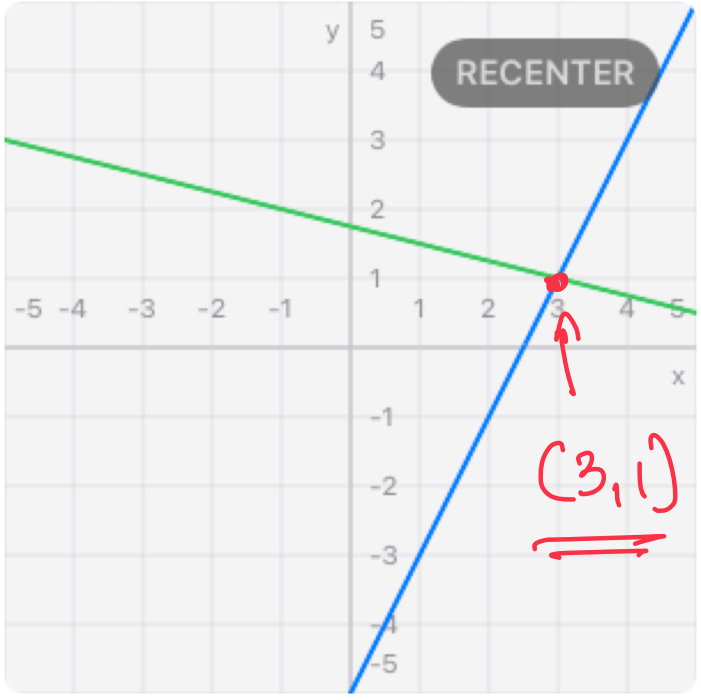
$$① \text{ — } 2x_1 - x_2 = 5 \qquad (n=2)$$
$$② \text{ — } x_1 + 4x_2 = 7$$

This system is easy to solve by elimination and also easy to visualise as well

$$① \times 4 = 8x_1 - 4x_2 = 20$$
$$x_1 + 4x_2 = 7$$
$$9x_1 = 27$$
$$\Rightarrow x_1 = 3$$
$$3 + 4x_2 = 7$$
$$\Rightarrow x_2 = 1$$

$\therefore$ $(3, 1)$ is a solution to the given system.

The system can also be represented geometrically and solved.



- **Echelon Form:**

Assume a system of equations,

$$c_{11}x_1 + c_{12}x_2 + c_{13}x_3 + c_{14}x_4 = b_1$$
$$c_{22}x_2 + c_{23}x_3 + c_{24}x_4 = b_2$$
$$c_{33}x_3 + c_{34}x_4 = b_3$$
$$c_{44}x_4 = b_4$$

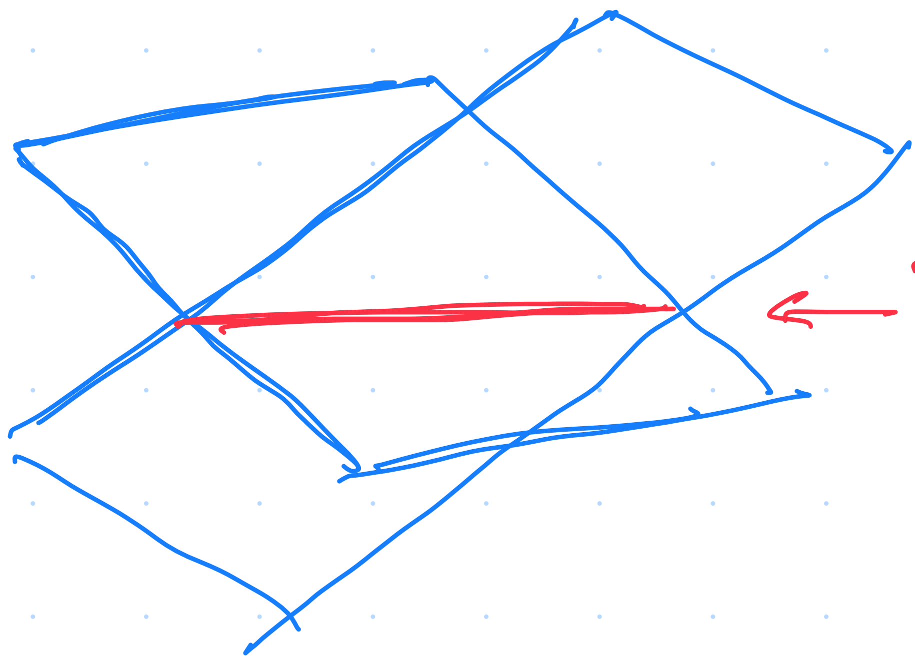
This system is said to be in Echelon form

- A system in Echelon form can be solved easily using Gaussian Elimination.

- **Condition for No Solutions:-**

- In 2D, if the lines (equations) are in ||lel, then ∄ no solution for that system.

- For 2 equations in 3D, ∃ infinite solutions for the system.



- Representation of A System of Linear Eqns in Matrices :-

- Take the system of equations first mentioned. (n no. of eqns).
- That system can be represented in matrix form as,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$\Rightarrow \quad AX = B$$

Coeff matrix

Variable matrix

Output matrix

This is like a function on X.

X is transformed

- Also, take,

$$\bar{V}_1 = \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} \quad \bar{V}_2 = \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix} \cdots \bar{V}_n = \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

The system can be rep.d as,

$$x_1 \cdot \bar{V}_1 + x_2 \cdot \bar{V}_2 + x_3 \cdot \bar{V}_3 \cdots + x_n \cdot \bar{V}_n = B$$

ie, B is a linear combination of vectors $V_1, V_2 \ldots V_n$

2) Finding the Parameters of a Linear Combination :-

- Given the vectors $\bar{V}_1, \bar{V}_2 \ldots \bar{V}_n$, is $\bar{B}$ a linear combination of these vectors?

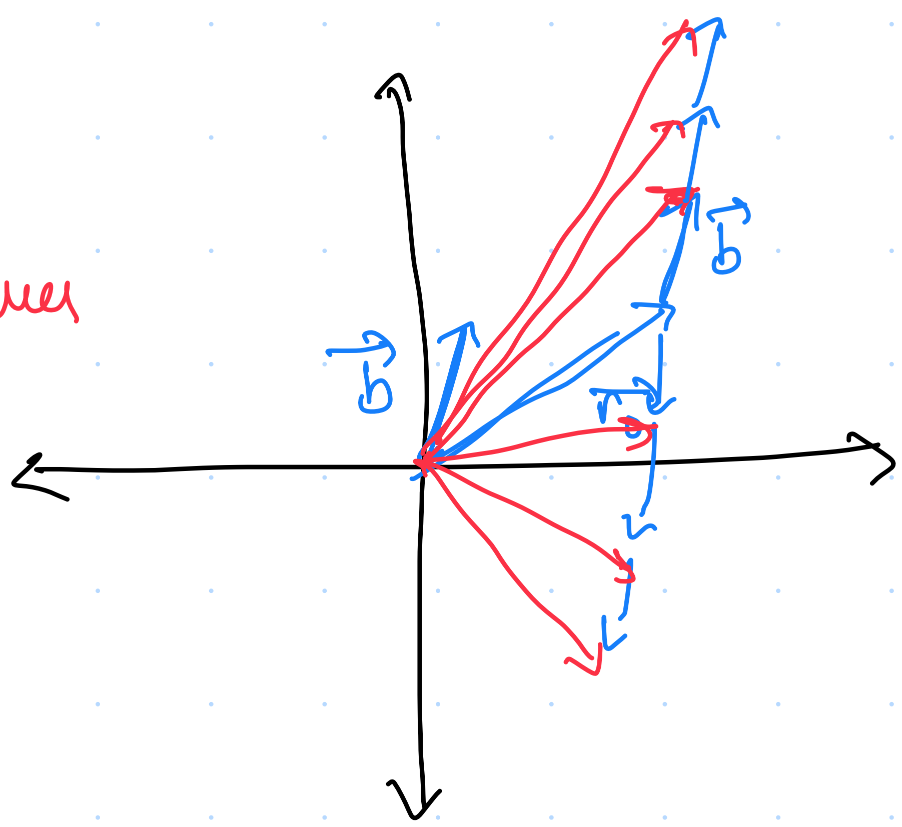
- This combination of vectors is dealt with in a vector space.

- A vector need not always be a arrow with a length (Physics definition). Your vector space can be defined by any arbitrary axioms.

- Linear Combination of vectors in a vector space cannot reach higher dimensions.

Q. Take 2 vectors $\vec{r_0}$ and $\vec{b}$ and let $\vec{r} = \vec{r_0} + \lambda\vec{b}$, where λ is a parameter. How does the tip of $\vec{r}$ behave?

The tip of $\vec{r}$ behaves like a line.



→ Linear Map :-

- A map is a relation from one set to another.

$$f : A \longrightarrow B$$

f is a map from A to B.

- If :

$$f(c_1x_1 + c_2x_2) = c_1 f(x_1) + c_2 f(x_2)$$

holds true

linearity

Then, the mapping is said to be a linear map.

- Linear maps keep the ratios intact, ie, any ratio in the input space is retained in the output space.

ie,

$$a : b = f(a) : f(b)$$

- Matrix multiplication is a linear operation.

$$A(\lambda_1 X_1 + \lambda_2 X_2) = \lambda_1 A X_1 + \lambda_2 A X_2$$

where λ_1, λ_2 are scalars. (ie, plain numbers)

→ Vector Space :-

A set of objects where the following axioms hold.

1) Addition of 2 objects is valid, commutative, closed

$\bar{v}_1 + \bar{v}_2 = \bar{v}_2 + \bar{v}_1$

2) Zero vector exists

closed = vector + vector = vector

3) Inverse vector exists

4) Scalar multiplication is valid, distributive, closed.

c = scalar

$c(v_1 + v_2) = cv_1 + cv_2$

- Set of Real Numbers ($\mathbb{R}$) is a vector space.

- <u>Dimension</u> :-

The minimum no. of vectors, whose set of all possible linear combinations fill the entire space.

- The vectors taken need not be orthogonal always
- 2 vectors that are not ||lel or anti ||lel, can fill entire 2D space.
- 3 vectors which are such that they all do not lie on the same plane, can cover the entire 3D space.

<u>7/1/25</u>

-> <u>Linear Algebra</u> :-

- <u>Syllabus</u>: Linear Equations, Vector Spaces, Linear Transformati-ons,
- <u>Textbook</u>: Hoffman & Kunze (Linear Algebra)
  Algebra by Artin, LA by Kumaresan, LA done right.

→ **Field:-** $(F, +, \cdot)$

A set of elements (scalars) that follow 2 binary operations, addition (+) and multiplication (·).

- The binary operations (+) and (·) can be defined in any way, but they must follow the below properties:

1) Addition is commutative:

$$x + y = y + x \quad \forall \; x, y \in F$$

2) Addition is associative:

$$(x+y) + z = x + (y+z) \quad \forall \; x, y, z \in F$$

3) $\exists$ an element 0 (zero) such that,

$$x + 0 = x \quad \forall \; x \in F$$

0 is called the Additive identity of F.

4) $\forall \; x \in F \; \exists \; (-x) \in F$ such that,

$$x + (-x) = 0$$

$(-x)$ is the Additive Inverse of x

5) Multiplication is commutative,

$$x \cdot y = y \cdot x \quad \forall \; x, y \in F$$

6) Multiplication is associative,

$$(x \cdot y) \cdot z = x \cdot (y \cdot z) \quad \forall \; x, y, z \in F$$

7) $\exists$ a non-zero element 1 (one) $\in F$, such that,

$$x \cdot 1 = x \quad \forall \; x \in F$$

1 is the multiplicative identity of F

8) $\forall x \in F$ st $x \neq 0$, $\exists\, x^{-1} \in F$ such that,

$$x \cdot x^{-1} = 1$$ x^{-1} is the multiplicative inverse of x.

9) Multiplication is distributive over addition,

$$x(y+z) = x \cdot y + x \cdot z \quad \forall\, x, y, z \in F$$

- Due to the above properties, the smallest field that can exist is $(\{0,1\}, +, \cdot)$ where $+, \cdot$ are defined as,

+	0	1
0	0	1
1	1	0

XOR binary op.

·	0	1
0	0	0
1	0	1

AND binary op.

10/1/25

→ System of Linear Equations :-

m eqns
$$\begin{cases} A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n = y_1 \\ A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n = y_2 \\ \vdots \\ A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n = y_m \end{cases} \quad (1.1)$$

- The above equations have max. degree of 1.
- $A, x \in (F, +, \cdot)$, to make the equations consistent.
- In a system of linear eqns, the unknown scalar must satisfy all the linear eqns.
- If $y_k = 0 \ \forall \ k \in \mathbb{N}$ in the system, then the system is said to be homogeneous.

Q. $2x_1 - x_2 + x_3 = 0$ $\Rightarrow$ $2x_1 - x_2 + x_3 = 0$

$x_1 + 3x_2 + 4x_3 = 0$ $\xrightarrow{\times 2}$ $2x_1 \overset{(-)}{+} 6x_2 \overset{(-)}{+} 8x_3 = 0$

$-7x_2 - 7x_3 = 0$

$\Rightarrow \underline{x_2 = -x_3}$

$x_1 - 3x_3 + 4x_3 = 0$

$x_1 + x_3 = 0$

$x_1 = -x_3$

Solution is of the form : $\underline{(-x_3, -x_3, x_3)}$

Clearly there are ∞ solns.

- Assume we multiply the equations each by a constant $c_1, c_2 \cdots c_m$ and add all the eq.ns together, we get,

$$(c_1 A_{11} + c_2 A_{21} + \cdots c_m A_{m1}) x_1 + (c_1 A_{12} + c_2 A_{22} + \cdots c_m A_{m2}) x_2$$
$$+ \cdots\cdots + (c_1 A_{1n} + c_2 A_{2n} \cdots + c_m A_{mn}) x_n \qquad (1.2)$$
$$= c_1 y_1 + c_2 y_2 + c_3 y_3 \cdots + c_m y_m \longrightarrow \text{Linear Combination of eqns.}$$

- Any solution $(x_1, x_2, \cdots x_n)$ that satisfies the system 1.1 will also satisfy the equation 1.2.
- But the inverse is not true, since the combination is also a linear equation with ∞ solutions.
- Let,

$$c_1 A_{11} + c_2 A_{12} \cdots = B_1$$
$$c_1 A_{21} + c_2 A_{22} \cdots = B_2$$

by varying $c_1, c_2 \cdots c_n$, we can get new B_{11}, B_{12}, B_{21}, etc.

By making that into another system,

$$B_{11} x_1 + B_{12} x_2 \cdots + B_{1n} x_n = z_1$$
$$B_{21} x_1 + B_{22} x_2 \cdots + B_{2n} x_n = z_2 \qquad (1.3)$$
$$\vdots$$
$$B_{k1} x_1 + B_{k2} x_2 \cdots + B_{kn} x_n = z_k$$

This is a system of linear combinations of system 1.1

- The solution to 1.1 will also be a solution to system 1.3. But the solutions of 1.3 need not be a solution to 1.1.

- If 1.1 can be written as a system of linear combinations of system 1.3, then all the solutions of 1.3 are also solutions of 1.1, ie, the systems are equivalent.

- <u>Matrix Representation of Linear Systems :-</u>

$$\underbrace{\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ \vdots & \vdots & & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix}_{m\times n}}_{\text{coefficient matrix}} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}_{m\times 1} \qquad (1.4)$$

- The matrix equation 1.4 is a representation of the system 1.1.

→ <u>Matrices :-</u>

- A matrix is defined over a field.
- It is a function or map from a pair of countable numbers to the scalars in a field.

$$M : (i, j) \longrightarrow (F, +, \cdot), \quad i, j \in \mathbb{N}, \; 1 \le i \le m, \; 1 \le j \le n$$

- <u>Row Operations :-</u>

Operations made on the rows of matrix A, such that the resulting matrix B is equivalent to A.

- There are 3 such elementary operations :

1) Multiplying a row by a non-zero scalar
2) Multiplying a row with a scalar and adding it to another rows.
3) Swapping 2 rows.

- If we assume our matrix is $A_{m \times n} = [A_{ij}]$, then the rows can be defined in function notation as,

1) $e(A_{ij}) = \begin{cases} cA_{ij}, & i = r \\ A_{ij}, & \text{otherwise} \end{cases}$

r, s = rows upon which the operation are performed.

2) $e(A_{ij}) = \begin{cases} A_{ij} + cA_{sj}, & i = r \\ A_{ij}, & \text{otherwise} \end{cases}$

3) $e(A_{ij}) = \begin{cases} A_{rj}, & i = s \\ A_{sj}, & i = r \\ A_{ij}, & \text{otherwise} \end{cases}$

17/1/25

- **Theorem:** To each row operation e, $\exists$ an elementary row operation e' such that $e'(e(A)) = e(e'(A)) = A$

e' is the 'inverse' of e

- Let A, B be 2 $m \times n$ matrices, A is said to be row equivalent to B, if by doing a finite number of row operation on A, B can be obtained, i.e.,

$$B = e_k(\ldots\ldots e_2(e_1(A))\ldots)$$

$$\Rightarrow A = e_1'(e_2'(e_3' \cdots e_k'(B)))\cdots)$$

Thus, if A is row-equivalent to B, B is also row-equivalent to A.

Q. Prove that Row-Equivalence is an Equivalence relation.

- Theorem:

If A and B are row-equivalent matrices, the homogenous system of linear eqns $AX = 0$ and $BX = 0$ have exactly the same solution.

- Elementary row operations will not change the solution of the system

$$\left.\begin{aligned} &A \cdot X = 0 \\ \Rightarrow\ &e(A) \cdot X = 0 \\ \Rightarrow\ &e_2(e(A)) \cdot X = 0 \end{aligned}\right\} \text{Same solution of } X$$

Q. $A = \begin{bmatrix} 2 & -1 & 3 & 2 \\ 1 & 4 & 0 & -1 \\ 2 & 6 & -1 & 5 \end{bmatrix}$

① -2 ②

$A = \begin{bmatrix} 0 & -9 & 3 & 4 \\ 1 & 4 & 0 & -1 \\ 2 & 6 & -1 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 0 & 1 & -11/3 \\ 1 & 0 & 0 & 17/3 \\ 0 & 1 & 0 & -5/3 \end{bmatrix}$

$\underline{\underline{B}}$

B is a row-reduced matrix

- A $m \times n$ matrix is said to be row-reduced if

a) The first non-zero entry in each non-zero row is 1.

b) Each column which contains the leading non zero element (1) all other entries as zero.

Identity matrix is row reduced.

- <u>Theorem</u>: Every $m \times n$ matrix over the field is row-equivalent to a row reduced matrix. (<u>Q. Proof</u>)

- <u>Row Reduced Echelon Matrices</u>:- A matrix $R_{m \times n}$ is said to be a row-reduced Echelon matrix if.

a) R is row-reduced

b) All the zero rows are below/after non-zero rows.

c) If the rows $1, 2, \dots r$ are non zero rows of R and if the leading non-zero entry of row i occurs in column k_i, $i = 1, 2, 3$ then $k_1 < k_2 < k_3 \dots < k_r$. (Upper Δ matrix)

Either every entry in $R_{m \times n}$ is 0, or $\exists$ $r \in \mathbb{N}$, $1 \leq r \leq m$, & $k_1, k_2, \dots k_r \in \mathbb{N}$ with $1 \leq k_i \leq n$ (Same definition in mathematical language.)

a) $R_{ij} = 0 \quad \forall\ i > r$ c) $R_{ik_j} = \delta_{ij} \quad \begin{matrix} 1 \le i \le r \\ 1 \le j \le r \end{matrix}$

b) $R_{ij} = 0 \quad \forall\ j < k_i$ d) $k_1 < k_2 < k_3 \dots < k_r$

δ_{ij} is called conical delta matrix and is defined as,

$$\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \quad \text{(Identity)}$$

21 | 1 / 25

- Theorem :-

Every $m \times n$ matrix A is row equivalent to a row reduced Echelon matrix.

- Every homogenous system $AX = 0$ will have a trivial solution 0. To see if there are any other soln. s,

- If the no. of non zero rows is r and the RREF of A is R, then each unknown variable x_{k_i} (where k_i is the column in which the ith leading one occurs) will only appear in the ith row.

- Since there are r k_i's, there are r unknown dependent variables. $(n - r)$ variables will never have the leading 1 coefficient.

↳ free variables

independent of each other

$$\Rightarrow \quad x_{k_1} + \sum_{j=1}^{n-r} C_{1j} v_j = 0$$

$$x_{k_2} + \sum_{j=1}^{n-r} C_{2j} v_j = 0$$

$$\vdots$$

$$x_{k_r} + \sum_{j=1}^{n-r} C_{rj} v_j = 0$$

where v_j are the $(n-r)$ variables and C is any scalar.

independent

Eg: $R = \begin{bmatrix} 0 & 1 & -3 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$ $RX = 0$

There are 2 possible k_i's, 2 & 4

$\Rightarrow$ x_2 and x_4 are dependent variables

x_1, x_3, x_5 are independent variables $\rightarrow$ v

$x_2 + 0x_1 + (-3)x_3 + (1/2)x_5 = 0 \Rightarrow \boxed{x_2 + \sum C_{1j} v_j = 0}$

$x_4 + 0x_1 + 0x_3 + 2x_5 = 0 \Rightarrow \boxed{x_2 + \sum C_{2j} v_j = 0}$

x_2, x_4 are dependent on the other variables but are independent of each other.

- x_1, x_3, x_5 are free scalars. If the field has ∞ scalars, then there are ∞ solns to the equation.

- <u>Theorem</u> :-
  If $A_{m\times n}$ & $m<n$ then $AX=0$ always has a non-trivial solution (<u>Assn</u>: Prove)

- <u>Theorem</u> :-
  A square matrix $n\times n$ is row equivalent to I iff $AX=0$ has only trivial solns. (<u>Assn</u>: Prove)

Non Homogenous Systems :-

$$A_{m\times n} X_{n\times 1} = Y_{m\times 1} \neq 0$$

- Since we cannot apply row transformations on only A, as Y will be affected, we apply the transformations together.

$$[A|Y]_{m\times(n+1)} = A'_{m\times(n+1)}$$

$$= [R|Z]_{m\times(n+1)} \qquad Z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_m \end{bmatrix}$$

$$\Rightarrow \quad RX = Z \quad \text{where } R \text{ is in RREF}$$

- If there are r non-zero rows, $(m-r$ zero rows$)$, and if

$$z_i \neq 0 \text{ for any } i > r$$

then there will be <u>no solutions</u>. (zero = non-zero condition).

ex: $A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 1 \\ 0 & 5 & -1 \end{bmatrix} \qquad Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

RTs.

$$R' = \left[\begin{array}{ccc|c} 1 & 0 & 3/5 & \frac{1}{5}(y_1 + 2y_2) \\ 0 & 1 & -1/5 & \frac{1}{5}(y_2 - \\ 0 & 0 & 0 & y_3 - y_2 + 2y_1 \end{array}\right]$$

24|1|25

→ Matrix Multiplication :-

$$C = A \times B$$

$$C_{ij} = \sum_{r=1}^{n} A_{ir} B_{rj}$$

- For 2 matrices A, B to be multiplicable,

  no. of rows in A = no. of colums in B

- Theorem :-

  A, B, C are matrices over field F st BC and A(BC) are well defined. Then, AB, (AB)C are also defined and

$$(AB)C = A(BC)$$

Proof:

$$[A(BC)]_{ij} = \sum_{r=1}^{n} A_{ir} [BC]_{rj}$$

$$[BC]_{rj} = \sum_{s=1}^{n} B_{rs} C_{sj}$$

$$\Rightarrow \quad [A(BC)]_{ij} = \sum_{r=1}^{n} \left[A_{ir} \left(\sum_{s=1}^{n} B_{rs} C_{sj} \right) \right] \longrightarrow (1)$$

$$[(AB)C]_{ij} = \sum_{r'=1}^{n} [AB]_{ir'} C_{r'j}$$

$$[AB]_{ir'} = \sum_{s'=1}^{n} A_{is'} B_{s'r'}$$

$$\Rightarrow \quad [(AB)C]_{ij} = \sum_{r'=1}^{n} \left(\sum_{s'=1}^{n} A_{is'} B_{s'r'} \right) C_{r'j} \quad \text{——} \quad (2)$$

- <u>Square Matrices :-</u>

- Square matrices can be multiplied by itself.

$$A \times A = A^2$$

- $$A^p A^q A^r = A^{p+q+r}$$

- A square matrix is said to be an elementary matrix if it is obtained by performing a single elementary row operation on the identity matrix.

For a 2×2 matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$\begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}, \; c \neq 0, \quad \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}, \; c \neq 0 \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

are elementary matrices

- Theorem :-

Let e be an elementary row operation & let E be an $m \times m$ elementary matrix, $E = e(I)$, then $\forall A_{m \times n}$,

$$e(A) = E \times A$$

(Proof is Assn)

- Theorem:

If B is row equivalent to A, then $\exists$ a matrix P s.t $B = PA$, since,

$$B = e_1(e_2(\dots(e_n(A))\dots))$$

$$\Rightarrow B = E_1 E_2 \dots E_n \cdot A$$

- Invertible Matrices :-

Consider $n \times n$ matrix A.

- P is said to be left inverse of A if $PA = I$
- Q is said to be right inverse of A if $AQ = I$

- A matrix is said to be invertible if both left and right inverses of it exists.
- For a square matrix, if it is invertible, then its left and right matrices are the same.

$$PA = AQ = I \Rightarrow P = Q$$

$$PA = \mathbb{1}$$

$$\Rightarrow P(AQ) = \mathbb{1} \cdot Q$$

$$\Rightarrow P(\mathbb{1}) = Q$$

$$\Rightarrow P = Q$$

- **Theorem:-**

Let A, B be square matrices over a field F,

1) If A is invertible, so is A^{-1}, and $(A^{-1})^{-1} = A$.

2) If both A and B are invertible, then $(AB)^{-1} = B^{-1}A^{-1}$

Proof:

1) $A \cdot A^{-1} = \mathbb{1}$

$A^{-1} \cdot B = \mathbb{1} \Rightarrow B = (A^{-1})^{-1}$

$A \cdot A^{-1} \cdot B = A$

$\therefore B = A$

$\Rightarrow (A^{-1})^{-1} = A$

2) $X \cdot AB = \mathbb{1}$

$A \cdot A^{-1} = \mathbb{1}$ & $B \cdot B^{-1} = \mathbb{1}$

$\Rightarrow X \cdot A \cdot B \cdot B^{-1} \cdot A^{-1} = \mathbb{1} \cdot B^{-1} A^{-1}$

$$\Rightarrow \quad (AB)^{-1} = B^{-1}A^{-1}$$

- Theorem:

All elementary matrices are invertible.

Proof:

$$E = e(\mathbb{I})$$

$$\Rightarrow \mathbb{I} = e^{-1}(E)$$

$$\Rightarrow \mathbb{I} = E^{-1} \times E$$

- Theorem: Consider a square matrix $A_{m \times m}$, the following statements are equivalent (Assn : Proof)

1) A is invertible
2) $AX = 0$ has only trivial solutions
3) $AX = Y$ has a solution for each $Y_{m \times 1}$.

28/1/25

- Since w.k.t A having trivial solns $\Rightarrow$ A is row equivalent to $\mathbb{I}$, using above theorem, we can state that,

Invertible Matrices are row equivalent to $\mathbb{I}$.

→ Vector Space :- (V)

A vector space (linear space) consists of

a) A field F of scalars

b) A set of objects (vectors)

c) A rule called Addition which associates a vector w/ a pair of vectors such that,

A vector space V is defined over a field F. The elements of a vector space are vectors.

1) Addition is commutative

$$\bar{\alpha} + \bar{\beta} = \bar{\beta} + \bar{\alpha} \quad \forall \; \bar{\alpha}, \bar{\beta} \in V$$

2) Addition is closed,

$$\bar{\alpha} + \bar{\beta} \in V \quad \forall \; \bar{\alpha}, \bar{\beta} \in V$$

3) Addition is associative,

$$(\bar{\alpha} + \bar{\beta}) + \bar{\gamma} = \bar{\alpha} + (\bar{\beta} + \bar{\gamma})$$

4) $\exists$ a unique vector called zero $\in V$, st

$$\bar{\alpha} + 0 = \bar{\alpha} \quad \forall \; \alpha \in V$$

5) For each $\bar{\alpha} \in V$, $\exists$ a unique vector $-\bar{\alpha} \in V$ st

$$\bar{\alpha} + (-\bar{\alpha}) = 0$$

d) A rule called scalar multiplication that associates a vector to a pair of a scalar and vector, such that

$$\forall \; c \in F, \; \bar{\alpha} \in \bar{V}, \quad c \cdot \bar{\alpha} \in \bar{V} \text{ st}$$

1) $1 \cdot \bar{\alpha} = \bar{\alpha}$

2) $c_1(c_2(\bar{\alpha})) = c_2(c_1(\bar{\alpha}))$

3) $c(\bar{\alpha} + \bar{\beta}) = c\bar{\alpha} + c\bar{\beta}$ linearity

4) $(c_1 + c_2)\bar{\alpha} = c_1\bar{\alpha} + c_2\bar{\alpha}$

- $c \cdot \bar{0} = c(\bar{0} + \bar{0})$

  $\Rightarrow c \cdot \bar{0} = c \cdot \bar{0} + c \cdot \bar{0}$

  $\Rightarrow c \cdot \bar{0} = \bar{0}$

- $c \cdot \bar{\alpha} = \bar{0} \Rightarrow c = 0$ or $\bar{\alpha} = \bar{0}$

- A vector $\bar{\alpha} \in V$ is said to be a linear combination of $\bar{\beta_1}, \bar{\beta_2}, \ldots \bar{\beta_n} \in V$ if $\bar{\alpha} = c_1\bar{\beta_1} + c_2\bar{\beta_2} \ldots + c_n\bar{\beta_n}$, for some, $c_1, c_2 \ldots c_n \in F$.