EC5.102 Information and Communication **Spring 2020**

Lecture 8: Joint PMF, Conditional PMF, Conditional Expectation

Instructor: Dr. Lalitha Vadlamani

Consider two discrete r.v.s X and Y associated with the same experiment. We can talk about the joint cdf of X and Y as the function $F_{X,Y}(x,y) = P(X \le x, Y \le y) = P(\{\omega | X(\omega) \le x, Y(\omega) \le y\})$. A few properties of the joint cdf are listed below:

- $F_{X,Y}$ should be non-decreasing in both variables.
- $F_{X,Y}(\infty,\infty) = 1$, $F_{X,Y}(-\infty, y) = 0$, $F_{X,Y}(x,-\infty) = 0$, $F_{X,Y}(x,\infty) = F_X(x)$, $F_{X,Y}(\infty, y) = F_Y(y)$.
- $F_{X,Y}$ is right continuous in both the variables.

8.1 Joint PMF of Two Random Variables

Consider two discrete r.v.s X and Y associated with the same experiment. The joint pdf of X and Y is defined by

$$p_{X,Y}(x,y) = P(\{\omega | X(\omega) = x, Y(\omega) = y\}).$$

We will write $P(X = x, Y = y)$ in short for $P(\{\omega | X(\omega) = x, Y(\omega) = y\})$. We can calculate the pmts of X and Y by using the formulas

$$\begin{aligned} p_X(x) &= \sum_{y \in \mathcal{Y}} p_{X,Y}(x,y) \\ p_Y(y) &= \sum_{x \in \mathcal{X}} p_{X,Y}(x,y) \end{aligned}$$

We refer to p_X and p_Y as the marginal pmfs to distinguish them from the joint pmf. An example joint pmf and the corresponding marginal pmfs have been shown in Fig. 8.1.

8.2 Conditioning one random variable on another

Let X and Y be two r.v.s associated with the same experiment. If we assume the experimental value of Y is some particular y with $p_Y(y) > 0$) this provides partial knowledge about the value of X. The conditional probability of X conditioned on $Y = y$ is defined as

$$p_{X|Y}(x|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}.$$

It follows that

$$\sum_{x \in \mathcal{X}} p_{X|Y}(x|y) = 1, \forall y.$$

Two random variables X and Y are said to be independent if

$$p_{X|Y}(x|y) = p_X(x)$$

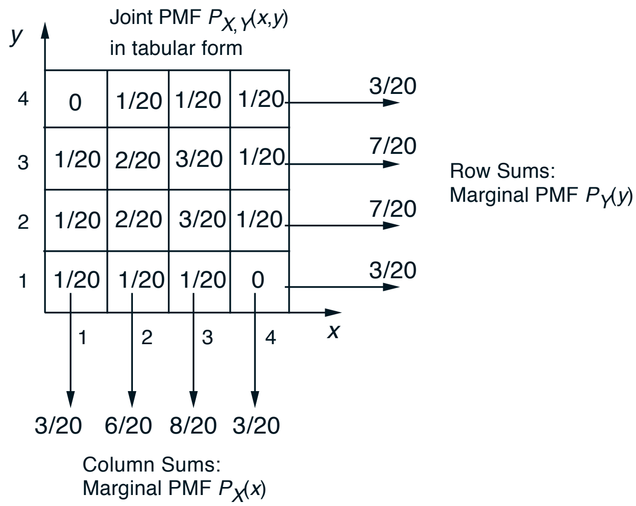

Figure 8.1: Illustrating a joint pmf of random variables X and Y. Marginal pmf p_X is obtained by taking the column sums and marginal pmf p_Y is obtained by taking the row sums.

Or equivalently $p_{X,Y}(x,y) = p_X(x)p_Y(y)$.

The conditional expectation of X given $Y = y$ is defined as

$$E[X|Y = y] = \sum_x x p_{X|Y}(x|y)$$

It can be verified that

$$E[X] = \sum_y p_Y(y)E[X|Y = y].$$

The above expression on the right $\sum_y p_Y(y)E[X|Y = y]$ is wrongly written in the class as $E[X|Y]$.

Example 8.1. *Assume that two r.v.s X and Y are independent and identically distributed (I.i.d.) as Ber(p) r.v.s Then we have the joint pmf of X and Y as*

$$p_{X,Y}(0,0) = p^2, p_{X,Y}(0,1) = p_{X,Y}(1,0) = p(1-p), p_{X,Y}(1,1) = (1-p)^2.$$

Example 8.2. *Consider four independent rolls of a 6-sided die. Let X be the number of $1's$ and Y be the number of $2's$ obtained. What is the joint pmf of X and Y? The marginal pdf p_Y is given by*

$$p_Y(y) = \binom{4}{y}\left(\frac{1}{6}\right)^y\left(\frac{5}{6}\right)^{4-y}, y = 0, 1, \ldots, 4.$$

To compute the conditional pdf $p_{X|Y}$, note that given $Y = y$, X is the number of $1s$ in the remaining $4 - y$ rolls, each of which can take the 5 values $1, 3, 4, 5, 6$ with equal probability $\frac{1}{5}$. Thus, the conditional pmf $p_{X|Y}$ is

$$p_{X|Y}(x|y) = \binom{4-y}{x}\left(\frac{1}{5}\right)^x\left(\frac{4}{5}\right)^{4-y-x},$$

For all x and y such that $x, y = 0, 1, \ldots 4$ and $0 \le x + y \le 4$. The joint pmf is given by

$$\begin{aligned}
p_{X,Y}(x,y) &= p_Y(y)p_{X|Y}(x|y) \\
&= \binom{4}{y}\left(\frac{1}{6}\right)^y\left(\frac{5}{6}\right)^{4-y}\binom{4-y}{x}\left(\frac{1}{5}\right)^x\left(\frac{4}{5}\right)^{4-y-x}.
\end{aligned}$$

Example 8.3. *Let's consider the joint pmf given in Figure 8.1. We would like to calculate* $E[X+Y]$*. We denote* $X+Y$ *as another random variable* Z *which takes the values* $\{2,3,4,5,6,7,8\}$ *with probabilities* $\{\frac{1}{20},\frac{2}{20},\frac{4}{20},\frac{5}{20},\frac{5}{20},\frac{2}{20},\frac{1}{20}\}$ *respectively. Hence, we have that*

$$E[X+Y] = 2\times\frac{1}{20} + 3\times\frac{2}{20} + 4\times\frac{4}{20} + 5\times\frac{5}{20} + 6\times\frac{5}{20} + 7\times\frac{2}{20} + 8\times\frac{1}{20}.$$