Real analysis
Assignment - 1

1. Given a sequence $\{a_n\}$, such that $a_n - a_{n-2} \to 0$ as $n \to \infty$, Show that the sequence $b_n = \frac{a_n - a_{n-1}}{n}$ converges to 0.

For $\varepsilon > 0$, we have $|a_n - a_{n-2}| < \varepsilon \ \forall n \geq n_0$

Ans:

$$a_n - a_{n-1} = (a_n - a_{n-2}) - (a_{n-1} - a_{n-3})$$
$$+ (a_{n-2} - a_{n-4}) - (a_{n-3} - a_{n-5})$$
$$+ \cdots + \{(a_{n_0+2} - a_{n_0}) - (a_{n_0+1} - a_{n_0-1})\}$$

[Check]

Therefore by using triangle inequality,

$$|a_n - a_{n-1}| \leq |a_n - a_{n-2}| + |a_{n-1} - a_{n-3}| \cdots$$
$$+ |a_{n_0+2} - a_{n_0}| + |a_{n_0+1} - a_{n_0-1}|$$
$$= (n - n_0)\varepsilon + |a_{n_0+1} - a_{n_0-1}| .$$

Therefore $\left|\frac{a_n - a_{n-1}}{n}\right| < \varepsilon' \quad \forall n \geq n_0$.

$\varepsilon' \downarrow$ finite number

$\square$

2. Prove that

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(\sqrt{1+\frac{k}{n^2}} - 1 \right) = \frac{1}{4}$$

Ans: First observe that $\forall x > -$

$$\frac{x}{2+x} \leqslant \sqrt{1+x} - 1 \leqslant \frac{x}{2}$$

Set $x = \frac{k}{n^2}$

we have

$$\frac{k}{2n^2+k} \leqslant \sqrt{1+\frac{k}{n^2}} - 1 \leqslant \frac{k}{2n^2}$$

Therefore

$$\sum_{k=1}^{n} \frac{k}{2n^2+k} \leqslant S_n \leqslant \frac{1}{2n^2} \sum_{k=1}^{n} k .$$

$\downarrow$

Desired sequence.

We have. $\frac{1}{2n^2} \sum_{k=1}^{n} k = \frac{n(n+1)}{4n^2} \to \frac{1}{4}$

as $n \to \infty$

On the otherhand.

$$\lim_{n\to\infty}\left\{ \frac{1}{2n^2}\sum_{k=1}^{n} k - \sum_{k=1}^{n} \frac{k}{2n^2+k} \right\}.$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^2}{2n^2(2n^2+k)} \qquad \cancel{\leq \sum_{k=1}^{n} \frac{k^2}{4n^4}} \; \cancel{= \frac{1}{k}}$$

$$< \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^2}{4n^4}$$

$$= \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{24n^4}$$

~~[Check]~~.

Now. $\lim_{n\to\infty} \dfrac{n(n+1)(2n+1)}{24n^4} = 0$ [Check].

Therefore $\lim_{n\to\infty} \sum_{k=1}^{n} \dfrac{k}{2n^2+K} = \lim_{n\to\infty} \dfrac{1}{2n^2} \sum_{k=1}^{n} K = \dfrac{1}{4}$

Now use Squeeze theorem.

we have $\lim_{n\to\infty} S_n = \frac{1}{4}$

$\square$

3. Consider a sequence $\{a_n\}$, such that $a_n \geqslant 1 \ \forall n \in \mathbb{N}$ and $a_n + \frac{1}{a_n}$ converge. Then prove that a_n also converges.

Ans: Let $a = \liminf\limits_{n \to \infty} a_n$ [limit infimum]

$A = \limsup\limits_{n \to \infty} a_n$ [limit supremum].

If $A \to \infty$ then $a_n + \frac{1}{a_n} \geqslant a_n$

and therefore $a_n + \frac{1}{a_n}$ is not bounded, which is a contradiction [Since $a_n + \frac{1}{a_n}$ converges].

So A is finite.

So, if $(a_{n_k})_{k \geqslant 1}$ be a subsequence such that $(a_{n_k})_{k \geqslant 1} \to A$ as $k \to \infty$ and $(a_{m_k})_{k \geqslant 1}$ is a subsequence such that

$(a_{m_k})_{k \geqslant 1} \to a$ as $k \to \infty$.

but $A + \frac{1}{A} = a + \frac{1}{a}$ since $a_n + \frac{1}{a_n}$ converge

Thus $(A - a)(Aa - 1) = 0$

so either $A = a$ or $A = \frac{1}{a}$.

but this is a contradiction since $A > a \geqslant 1$.

Therefore $\{a_n\}$ converges $\square$.

1. Prove that $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$

Ans. Let n_x denote a unique integer, such that $n_x \leq x < n_x+1$

Therefore

$$\left(1+\frac{1}{x}\right)^x \leq \left(1+\frac{1}{n_x}\right)^{n_x+1}.$$

$$= \left(1+\frac{1}{n_x}\right)^{n_x}\left(1+\frac{1}{n_x}\right).$$

So $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x \leq \lim_{n_x\to\infty}\left(1+\frac{1}{n_x}\right)^{n_x} \cdot \lim_{n_x\to\infty}\left(1+\frac{1}{n_x}\right)$

The RHS converges to e.

also $\left(1+\frac{1}{x}\right)^x \geq \left(1+\frac{1}{n_x+1}\right)^{n_x}$.

$$= \left(1+\frac{1}{n_x+1}\right)^{n_x+1} \cdot \frac{1}{1+\frac{1}{n_x+1}}$$

~~Ag~~ Again the RHS converges to e.

Therefore, use Squeeze theorem to prove the rest. $\square$

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5. Let $a \in \mathbb{R}$. Prove that

$$\lim_{x \to 0} \frac{(1+x)^a - 1}{x} = a$$

Ans. $(1+x)^a - 1 = e^{a \ln(1+x)} - 1$

Use the expansion of ~~ln (1+x~~

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} \ldots$$

~~Since x is very small ($x \to 0$) we can ignore the higher order terms.~~

Now again use the expansion of e^x.

So, $(1+x)^a - 1 = 1 + ax + a$ (higher order of x) $- 1$

$= ax + a$ (x^2 and higher order) $= O(x^2)$) $- 1$

$= ax + O(x^2)$.

So $\dfrac{(1+x)^a - 1}{x} = a + \dfrac{O(x^2)}{x}$.

Hence $\lim_{x \to 0} \dfrac{(1+x)^a - 1}{x} = a$ $\square$.

6. Prove that $\dfrac{29}{18} < \sum_{n=1}^{\infty} \dfrac{1}{n^2} < \dfrac{31}{18}$.

Ans: We have

$$\frac{29}{18} = 1 + \frac{1}{4} + \frac{1}{9} + \sum_{n=4}^{\infty} \frac{1}{n(n+1)} < \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Again. $$1 + \frac{1}{4} + \frac{1}{9} + \sum_{n=4}^{\infty} \frac{1}{n(n-1)} > \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$= \frac{61}{36} < \frac{31}{18}.$$

Therefore $$\frac{29}{18} < \sum_{n=1}^{\infty} \frac{1}{n^2} < \frac{31}{18} \quad \square.$$

7. Evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^4+n^2+1}$

Ans. $$0 < \frac{n}{n^4+n^2+1} < \frac{1}{n^3}$$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ Converges. So $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^4+n^2+1}$ Converges.

Now, $$\frac{n}{n^4+n^2+1} = \frac{n}{(n^2+1)^2 - n^2} = \frac{n}{(n^2-n+1)(n^2+n+1)}$$

$$= \frac{1/2}{n^2-n+1} - \frac{1/2}{n^2+n+1}.$$

If $a_n = \dfrac{1/2}{n^2-n+1}$, then $a_{n+1} = \dfrac{1/2}{n^2+n+1}$

Therefore.

$$S_N = \sum_{n=1}^{N} \frac{n}{n^4+n^2+1}$$

$$= \sum_{n=1}^{N} (a_n - a_{n+1}).$$

$$= (a_1 - a_2) + (a_2 - a_3) + \cdots (a_{N-1} + a_N$$

$$= a_1 - a_N$$

$$= \frac{1}{2} - \frac{1/2}{N^2+N+1}$$

So.

$$\sum_{n=1}^{\infty} \frac{n}{n^4+n^2+1} = \lim_{N \to \infty} S_N = \frac{1}{2} \quad \text{(Check)}.$$

$\square$

8. a) $\sum_{n=1}^{\infty} \frac{a^n}{(n!)^{1/n}}$; $a > 0$

Ans. $\sqrt[n]{n!} \geq 1$ for all $n \in \mathbb{N}$.

Therefore $\frac{a^n}{\sqrt[n]{n!}} \leq a^n$

Thus by the first comparison test. the given series converges if $a < 1$.

Similarly $\frac{a^n}{n} \leq \frac{a^n}{\sqrt[n]{n}}$ $\forall$ $n \in \mathbb{N}$.

But the series $\sum_{n=1}^{\infty} \frac{a^n}{n}$ diverges for $a \geq 1$.

Therefore the given series diverges if $a \geq 1$.

8. b) $\sum_{n=1}^{\infty} a^n \left(1 + \frac{1}{n}\right)^n$; $a > 0$

Ans. Set $x_n = a^n\left(1 + \frac{1}{n}\right)^n$

Thus $\lim_{n\to\infty} \sqrt[n]{x_n} = a$.

Hence. the series converges if $a < 1$ and diverges ~~otherwise~~ (Root test). for $a > 1$.

$\boxed{\text{If } a = 1. \text{ then } x_n = \left(1+\frac{1}{n}\right)^n. \text{ So, the sum is } e.}$