Assignment 3

(MA6.102) Probability and Random Processes, Monsoon 2025

Release date: 11 September 2025, Due date: 19 September 2025

INSTRUCTIONS

- Discussions with other students are not discouraged. However, all write-ups must be done individually with your own solutions.
- Any plagiarism when caught will be heavily penalised.
- Be clear and precise in your writing.

Problem 1. Let $\Omega = \{1, 2, 3, \dots\}$ be a sample space equipped with the σ-algebra of all subsets of Ω and a probability law P such that $P(\{\omega\}) = 2^{-\omega}$ for each $\omega = 1, 2, 3, \dots$. Consider the random variables $X(\omega) = \omega$ and $Y(\omega) = (-1)^{\omega}$. Find $\mathbb{E}[X \mid Y]$, i.e., express the random variable $\mathbb{E}[X \mid Y]$ as a function from Ω to $\mathbb{R}$.

Problem 2. For three discrete random variables X, Y, and Z, show that

(a) $\mathbb{E}[X|X] = X$.

(b) $\mathbb{E}[Xg(Y)|Y] = g(Y)\mathbb{E}[X|Y]$. From this, deduce that $\mathbb{E}[g(Y)|Y] = g(Y)$.

(c) $\mathbb{E}[\mathbb{E}[X|Y, Z]|Y] = \mathbb{E}[X|Y]$.

Problem 3. Consider a discrete integer-valued random variable Y with CDF

$$F_Y(y) = 1 - \frac{2}{(y+1)(y+2)}, \text{ for integer values } y \geq 0.$$

Let Z be another integer valued random variable with the conditional PMF

$$P_{Z|Y}(z|y) = \frac{1}{y^2}, \text{ for } 1 \leq z \leq y^2.$$

Find $\mathbb{E}[Z]$.

Problem 4. Let X and Y be discrete random variables with mean 0, variance 1, and covariance ρ. Show that $\mathbb{E}[\max\{X^2, Y^2\}] \leq 1 + \sqrt{1 - \rho^2}$.

Problem 5. A permutation on the numbers in $[1:n]$ can be represented as a function $\pi : [1:n] \to [1:n]$, where $\pi(i)$ is the position of i in the ordering given by the permutation. A fixed point of a permutation $\pi : [1:n] \to [1:n]$ is a value x for which $\pi(x) = x$. Let X be number of fixed points of a permutation chosen uniformly at random from all permutations. Find $\mathbb{E}[X]$ and $\mathrm{Var}(X)$.

Problem 6. Let $X_1, X_2, \dots, X_n$ be independent discrete random variables and let $X = X_1 + X_2 + \cdots + X_n$. Suppose that each X_i is a geometric random variable with parameter p_i, and that $p_1, p_2, \dots, p_n$ are chosen so that the mean of X is a given $\mu > 0$. Show that the variance of X is minimized if the p_i values are chosen to be all equal to $\frac{n}{\mu}$.

Problem 7. Let X_1, X_2, X_3 be independent random variables taking values in the positive integers and having PMFs given by $P_{X_i}(k) = (1-p_i)p_i^{k-1}$, for $k = 1, 2, \dots$, and $i = 1, 2, 3$. Compute $P(X_1 < X_2 < X_3)$ and $P(X_1 \leq X_2 \leq X_3)$.