LA Assignment 5

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Q1. If V and W are 2 vector spaces and $T: V \to W$ is a linear transformation and V is finite dimensional, then prove that

$$\text{rank}(T) + \text{nullity}(T) = \dim(V)$$

We know that,

Rank (T) = Dimension of Range of T
Nullity (T) = Dimension of Null Space of T

Zero vector of W.

The null space of T is defined as $\{\bar{\alpha} \in V \mid T\bar{\alpha} = \bar{0}_W\}$

Let nullity $(T) = l$, $\dim(V) = n$

Let $\{\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3, \ldots\ldots, \bar{\alpha}_n\}$ be a basis in V

which also shows that range (T) is a subspace of W.

$\{T\bar{\alpha}_1, T\bar{\alpha}_2, \ldots, T\bar{\alpha}_n\}$ will span the range of T, since the linear combination of the vectors $\{\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_n\}$ span V (definition of basis).

Since the null space of T is a subspace of V, $\exists$ l vectors in $\{\bar{\alpha}_i\}$ st they form the basis of the null space.

Let this set be $\{\bar{\alpha}_1, \bar{\alpha}_2, \ldots \bar{\alpha}_l\}$

$$T\bar{\alpha}_i = \bar{0}_w \quad \forall\ i \leq l \quad \text{(By definition of null space)}$$

Since the basis of V is $\{\bar{\alpha}_1, \bar{\alpha}_2, \ldots \bar{\alpha}_l, \bar{\alpha}_{l+1}, \bar{\alpha}_{l+2} \cdots \bar{\alpha}_n\}$

The set $\{T\bar{\alpha}_i \mid i \leq n\}$ will span the range of T, since the basis $\{\bar{\alpha}_i\}$ can be used to describe any vector in V.

Also w.r.t $T\bar{\alpha}_i = \bar{0}_w\ \forall\ i \leq l$, the set $\{T\bar{\alpha}_{l+1}, T\bar{\alpha}_{l+2}, \ldots T\bar{\alpha}_n\}$ will be non-zero (if any of these are zero, the pre-image will be a part of the null space of T)

Also,

$$\sum_{i=l+1}^{n} c_i T\bar{\alpha}_i = T \sum_{i=l+1}^{n} c_i \bar{\alpha}_i = \bar{0}_w$$

$$\Rightarrow \sum_{i=l+1}^{n} c_i \bar{\alpha}_i = \bar{0}_v \quad \text{(Defn of Linear Transformation)}$$

$$\Rightarrow c_i = 0 \quad \forall\ i \in [l+1, n]$$

Since the set of $\{\bar{\alpha}_i\}$ is a basis in V and any subset of a basis is linearly independent.

$$\therefore \sum_{i=l+1}^{n} c_i T\bar{\alpha}_i = \bar{0}_w \Rightarrow c_i = 0 \quad \forall\ i \in [l+1, n]$$

$\Rightarrow \{T\bar{\alpha}_i \mid i \in [l+1, n]\}$ is linearly independent.

Since $T\bar{\alpha}_i = 0 \ \forall \ i \leq l$, the set $\{T\bar{\alpha}_i \mid i \in [l+1, n]\}$ will also span the range of T.

$\therefore \{T\bar{\alpha}_i \mid i \in [l+1, n]\}$ is a basis of the range of T

$\Rightarrow \dim(\text{range}(T)) = n - l$

$\Rightarrow \text{rank}(T) = \dim(V) - \text{nullity}(T)$

$\Rightarrow \underline{\text{rank}(T) + \text{nullity}(T) = \dim(V)}$

To prove that the null space of T is a subspace of V, we need to show that $\forall \ \bar{\alpha}, \bar{\beta} \in \text{null}(T)$, $c\bar{\alpha} + \bar{\beta} \in \text{null}(T) \ \forall \ c \in F$.

$\bar{\alpha}, \bar{\beta} \in \text{null}(T) \Rightarrow T\bar{\alpha} = \bar{0}_W \ \& \ T\bar{\beta} = \bar{0}_W$

$T(c\bar{\alpha} + \bar{\beta}) = cT\bar{\alpha} + T\bar{\beta} = c \cdot \bar{0}_W + \bar{0}_W = \bar{0}_W$

$\Rightarrow \underline{c\bar{\alpha} + \bar{\beta} \in \text{null}(T)}$

$\therefore$ Null space of T is a valid subspace of V.

To prove that the range of T is a valid subspace of W, we need to show that $\forall \ \bar{\alpha}, \bar{\beta} \in \text{range}(T)$, $c\bar{\alpha} + \bar{\beta} \in \text{range}(T) \ \forall \ c \in F$

$$\bar{\alpha}, \bar{\beta} \in \text{range}(T) \Rightarrow \exists \ \bar{\alpha}_0, \bar{\beta}_0 \in V \text{ st } T(\bar{\alpha}_0) = \bar{\alpha}$$
$$T(\bar{\beta}_0) = \bar{\beta}$$

Since V is a valid vector space, if $\bar{\alpha}_0, \bar{\beta}_0 \in V$, then $c\bar{\alpha}_0 + \bar{\beta}_0 \in V$, $\forall c \in F$.

$$T(c\bar{\alpha}_0 + \bar{\beta}_0) = cT\bar{\alpha}_0 + T\bar{\beta}_0 = c\bar{\alpha} + \bar{\beta}$$

$\therefore \quad c\bar{\alpha} + \bar{\beta} \in \text{range}(T)$

$\Rightarrow$ Range (T) is a valid subspace of W.

Q2. Prove that the set of all linear transformations b/w 2 vector spaces V and W, form a vector space.

Let the set of all possible linear transformations b/w V and W be represented as $L(V, W)$.

Since any vector space needs a zero vector, let us define a zero vector in $L(V, W)$ as Z, st,

$$Z \in L(V, W) \ \& \ Z\bar{\alpha} = \bar{0}_W \quad \forall \ \bar{\alpha} \in V$$

Also define vector addition and scalar multiplication as:

1) Addition: If $T_1 : V \to W$ and $T_2 : V \to W$ are 2 linear transformations, then the transformation $(T_1 + T_2)$ is defined as,

$$(T_1+T_2)\bar{\alpha} = T_1\bar{\alpha} + T_2\bar{\alpha} \quad \forall\ \bar{\alpha} \in V.$$

2) Scalar Multiplication : If $T: V \to W$ is a linear transformation and c is a scalar in the field of V, W, then the linear transformation cT is defined as,

$$(cT)\bar{\alpha} = T.(c\bar{\alpha}) \quad \forall\ \bar{\alpha} \in V$$

For addition,

1) (T_1+T_2) is also a valid linear transformation, so Closure is followed since $T_1+T_2 \in L(V,W)\ \forall\, T_1, T_2 \in L(V,W)$

2) $(T_1+T_2)\bar{\alpha} = T_1\bar{\alpha} + T_2\bar{\alpha} = T_2\bar{\alpha} + T_1\bar{\alpha} = (T_2+T_1)\bar{\alpha}$,

so Commutativity is followed. ↳ Commutativity in $\underline{\underline{W}}$

since $T_1\bar{\alpha}, T_2\bar{\alpha} \in W$

3) $(T_1+Z)\bar{\alpha} = T_1\bar{\alpha} + Z\bar{\alpha} = T_1\bar{\alpha} + \bar{0}_W$

$= T_1\bar{\alpha}$ ↳ $\bar{0}_W$ is additive identity in W

So Z is the additive identity

4) $(T_1+(T_2+T_3))\bar{\alpha} = T_1\bar{\alpha} + (T_2+T_3)\bar{\alpha}$

$= T_1\bar{\alpha} + T_2\bar{\alpha} + T_3\bar{\alpha}$

$= (T_1+T_2)\bar{\alpha} + T_3\bar{\alpha}$

$= ((T_1+T_2)+T_3)\bar{\alpha}$

So, Associativity is followed.

5) $(T-T)\bar{\alpha} = T\bar{\alpha} - T\bar{\alpha} = \bar{0}_W = Z\bar{\alpha}$

$\Rightarrow (T-T) = Z$ (Additive Inverse exists $\forall\ T \in L(V,W)$)

For scalar multiplication,

1) $(1 \cdot T)\bar{\alpha} = T(1 \cdot \bar{\alpha}) = T\bar{\alpha}$,

Since 1 is scalar multiplicative identity in V, It is also a scalar multiplicative identity in $L(V,W)$.

2) cT is also a valid LT, so $cT \in L(V,W)\ \forall\, T \in L(V,W)$ and $c \in F$

3) $(c_1 + c_2)T\bar{\alpha} = T(c_1+c_2)\bar{\alpha} = T(c_1\bar{\alpha} + c_2\bar{\alpha})$

$= c_1T\bar{\alpha} + c_2T\bar{\alpha} = (c_1T + c_2T)\bar{\alpha}$

$\Rightarrow (c_1 + c_2)T = (c_1T + c_2T)\ \forall\, c_1, c_2 \in F,\ T \in L(V,W)$

4) $c\,(T_1 + T_2)\bar{\alpha} = (T_1 + T_2)(c\bar{\alpha}) = T_1c\bar{\alpha} + T_2c\bar{\alpha}$

$= (cT_1 + cT_2)\bar{\alpha}$

$\Rightarrow c(T_1 + T_2) = cT_1 + cT_2$

Therefore with the above defined addition and scalar multiplication operations, we have proved that $L(V,W)$ is a valid vector space.

Q3. Let V and W be 2 vector spaces over a field F st. $\dim(V) = n$, $\dim(W) = m$. Prove that $\dim(L(V,W)) = n.m$

Let $\{\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3 \dots \bar{\alpha}_n\}$ be a basis of V

Let $\{\bar{\beta}_1, \bar{\beta}_2, \bar{\beta}_3 \dots, \bar{\beta}_m\}$ be a basis of W.

Let $T: V \longrightarrow W$ be a linear transformation.

Since $\{\bar{\beta}_1, \bar{\beta}_2, \dots \bar{\beta}_m\}$ is a basis in W, any vector in W can be written as a linear combination of the vectors in $\{\bar{\beta}_1, \bar{\beta}_2 \dots\dots \bar{\beta}_m\}$

$\forall i \in \mathbb{N}$ & $i \in [1, n]$,

$$T(\bar{\alpha}_i) = c_{i1}\bar{\beta}_1 + c_{i2}\bar{\beta}_2 + \cdots c_{im}\bar{\beta}_m$$

Let $\bar{x} \in \bar{V}$

$\{\bar{\alpha}\}$ is a basis of V.

$$\bar{x} = d_1\bar{\alpha}_1 + d_2\bar{\alpha}_2 + \cdots d_n\alpha_n$$

$$\Rightarrow T(\bar{x}) = d_1 T(\bar{\alpha}_1) + d_2 T(\bar{\alpha}_2) + \cdots d_n T(\bar{\alpha}_n)$$

$$\begin{aligned} \Rightarrow T(\bar{x}) = \; & d_1(c_{11}\bar{\beta}_1 + c_{12}\bar{\beta}_2 \cdots + c_{1m}\bar{\beta}_m) \\ & + d_2(c_{21}\bar{\beta}_1 + c_{22}\bar{\beta}_2 \cdots + c_{2m}\bar{\beta}_m) \\ & \vdots \\ & + d_n(c_{n1}\bar{\beta}_1 + c_{n2}\bar{\beta}_2 \cdots + c_{nm}\bar{\beta}_m) \end{aligned}$$

$$= \sum_{i=1}^{n} d_i \sum_{j=1}^{m} c_{ij}\beta_j$$

Coordinate vector of $\bar{x}$ wrt $\{\bar{\alpha}\}$

$$T(\bar{x}) = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & & & \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$$

The above matrix multiplication will give us the coordinates of $T(\bar{x})$ in W with respect to the basis $\{\bar{\beta}_1, \bar{\beta}_2 \dots \bar{\beta}_m\}$

Since we know that the behaviour of a linear transformation is determined by how it acts on the basis of V, which is represented here by the coordinate vector $(d_1, d_2, d_3 \cdots d_n)$, the linear transformation itself can be written as,

$$T = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & \cdots & & c_{2n} \\ \vdots & & & \\ c_{m1} & \cdots & & c_{mn} \end{bmatrix} \longrightarrow m \times n \text{ matrix}$$

Since T is any general linear transform. in $L(V, W)$, we can conclude that any linear transformation b/w V and W is an $m \times n$ matrix

$\Rightarrow \quad \dim(L(V,W)) = m \times n$ (Since it is made up of $m \times n$ matrices $\Rightarrow$ each matrix has $m \cdot n$ independent elements)

$$\Rightarrow \quad \underline{\dim(L(V,W)) = \dim(V) \cdot \dim(W)}$$