Assignment 2

(MA6.102) Probability and Random Processes, Monsoon 2025

Release date: 25 August 2025, Due date: 5 September 2025

Instructions

- Discussions with other students are not discouraged. However, all write-ups must be done individually with your own solutions.
- Any plagiarism when caught will be heavily penalised.
- Be clear and precise in your writing.

Problem 1. For each of the functions below, find the smallest σ-field on $\Omega = \{-2, -1, 0, 1, 2\}$ with respect to which the function is a random variable:
(a) $X(\omega) = \omega^2$,
(b) $X(\omega) = \omega + 1$.

Problem 2. Let $\Omega = [0, 1] \times [0, 1]$ be the unit square. For any (measurable) set $A \subseteq \Omega$, define $P(A)$ to be the area of A. For $\omega \in \Omega$, let $X(\omega)$ denote the distance from ω to the nearest edge of the square. Find the cumulative distribution function (CDF) F_X.

Problem 3. Since the extremal limits, non-decreasing property, and right-continuity completely characterize CDFs, determine which of the following given functions qualify as valid CDFs.
(a) $1 - (1 - F_X(x))^r, r \in \mathbb{N}$,
(b) $F_X(x) + (1 - F_X(x)) \log(1 - F_X(x))$.

Problem 4. Let N be a non-negative integer valued random random variable. Show that

$$\mathbb{E}[N] = \sum_{i=1}^{\infty} P(N \geq i).$$

Problem 5. Give an example of a non-constant random variable X such that $\mathbb{E}[\frac{1}{X}] = \frac{1}{\mathbb{E}[X]}$.

Problem 6. Show that, if X is a binomial or Poisson random variable, then the probability mass function (PMF) P_X has the property that $P_X(k-1)P_X(k+1) \leq P_X(k)^2$. Also, give an example of a PMF P_X such that $P_X(k)^2 = P_X(k-1)P_X(k+1)$.

Problem 7. Let $X \sim \text{Poisson}(\lambda)$, where λ is fixed but unknown. Let $\theta = \mathrm{e}^{-3\lambda}$, and suppose that we are interested in estimating θ based on the data. Since X is what we observe, our estimator is a function of X, call it $g(X)$. The bias of the estimator $g(X)$ is defined to be $\mathbb{E}[g(X)] - \theta$. An estimator is called unbiased if its bias is zero. Among the following estimators, determine which (if any) is unbiased:
(a) $g(X) = \mathrm{e}^{-3X}$.
(b) $g(X) = (-2)^X$.