Probability and Random Processes - Assignment 1

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Q1. Let Ω and $\mathcal{F}$ be the sample space and event space of some probability space.

Let A_1, A_2 be any 2 atoms in $\mathcal{F}$. Then, we can say that,

① $A_1, A_2 \neq \phi$, ② $\forall\ B \subseteq A_1$, $B = A_1$ or $B = \phi$
$\forall\ C \subseteq A_2$, $C = A_2$ or $C = \phi$

Assume, that A_1, A_2 are not disjoint, ie, $\exists\ x \in \Omega$ st $x \in A_1$ and $x \in A_2$, or $A_1 \cap A_2 \neq \phi$.

Let $B = \{x\}$. Clearly $B \subseteq A_1$ and $B \subseteq A_2$, and $B \neq \phi$.

Since $B \neq \phi$, by ②, we can say that $B = A_1$ and $B = A_2$

$\Rightarrow$ $A_1 = A_2$

$\Rightarrow$ If $A_1 \cap A_2 \neq \phi$, then $A_1 = A_2$

$\Rightarrow$ If $A_1 \neq A_2$, then $A_1 \cap A_2 = \phi$ [Negation of the prev. statement]

$\therefore$ Two distinct atoms are necessarily disjoint from each other.

Q2. $E_1, E_2, E_3 \dots E_n$ are mutually exclusive and exhaustive events in sample space Ω.

$E_i \cap E_j = \phi \;\; \forall \; i \neq j$ [Mutually Exclusive]

$\therefore$ Smallest σ-field: $\mathcal{F} = \left\{ \bigcup_{i \in I} E_i : I \subseteq \{1, 2, 3 \dots n\} \right\}$

To determine the validity of $\mathcal{F}$ as a σ field.

1) $\Omega \in F$.

If $I = \{1, 2, 3, \dots n\}$ [A set is a subset of itself]

$$\bigcup_{i \in I} E_i = E_1 \cup E_2 \dots \cup E_n = \Omega$$

$\therefore \Omega \in \mathcal{F}$

2) Closure under complement

Since each E_i is disjoint from another and they are collectively exhaustive, the complement of the union of some E_i's, is the union of all other E_i's, ie,

$$\left(\bigcup_{i \in I} E_i \right)^c = \bigcup_{i \notin I} E_i \text{ or } \bigcup_{i \in I^c} E_i$$

Since $I^c \subseteq \{1,2,3 \dots n\}$, $\bigcup_{i \in I^c} E_i \in \mathcal{F}$

$\therefore$ Closure under complement is satisfied.

3) Closure under Union.

Let $A_1, A_2, A_3 \dots$ be a sequence of events, such that

$$A_k = \bigcup_{i \in I_k} E_i, \quad I_k \subseteq \{1,2,3,\dots n\} \ \forall \ k \in \mathbb{N}$$

$$\bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} \left(\bigcup_{i \in I_k} E_i \right) = \bigcup_{i \in \bigcup_{k=1}^{\infty} I_k} E_i$$

Since $I_k \subseteq \{1,2,3 \dots n\} \ \forall \ k$, $\bigcup_{k=1}^{\infty} I_k \subseteq \{1,2,3 \dots n\}$

$\therefore \bigcup_{i \in \bigcup_{k=1}^{\infty} I_k} E_i \in \mathcal{F}$. $\Rightarrow$ Closure under Union is satisfied.

Therefore, the above mentioned σ-field is valid.

Q3. $P(A) = \frac{3}{4}$, $P(B) = \frac{1}{3}$

a) $P(A \cap B)$ is maximum when $B \subseteq A$. $[A \nsubseteq B$ as $P(A) > P(B)]$

$\Rightarrow P(A \cap B) = P(B) = \frac{1}{3}$

$P(A \cap B)$ is minimum when $A \cup B = \Omega$ [ie, exhaustive]

$$P(A\cup B) = P(A) + P(B) - P(A\cap B)$$

$$\Rightarrow P(A\cap B) = \left(\frac{3}{4} + \frac{1}{3}\right) - 1 = \frac{13}{12} - 1$$

$$\Rightarrow P(A\cap B) = 1/12$$

$$\Rightarrow \underline{\underline{1/12 \leq P(A\cap B) \leq 1/3}}$$

b) i) Maxima condition.

Let $\Omega = \{x, y, z\}$, $P(\{x\}) = \frac{4}{12} = \frac{1}{3}$, $P(\{y\}) = \frac{5}{12}$, $P(\{z\}) = \frac{3}{12} = \frac{1}{4}$. $P(A) = \sum_{k\in A} P(\{k\})$

Define $A = \{x, y\}$, $B = \{x\}$. $\Rightarrow B \subseteq A$

$$P(A) = \frac{4}{12} + \frac{5}{12} = \frac{9}{12} = \frac{3}{4}$$

$$P(B) = \frac{4}{12} = \frac{1}{3}$$

$$P(A\cap B) = \underline{\underline{P(B) = \frac{1}{3}}}$$

ii) Minima Condition

Let $\Omega = \{x, y, z\}$, $P(\{x\}) = \frac{1}{12}$, $P(\{y\}) = \frac{8}{12}$, $P(\{z\}) = \frac{3}{12}$

$P(A) = \sum_{k\in A} P(\{k\})$.

Define $A = \{x, y\}$, $B = \{x, z\}$.

$$P(A) = \frac{1}{12} + \frac{8}{12} = \frac{9}{12} = \frac{3}{4}$$

$$P(B) = \frac{1}{12} + \frac{3}{12} = \frac{4}{12} = \frac{1}{3}$$

$$P(A \cap B) = P(\{x\}) = \frac{1}{12}$$

Q4. C1 – Heads on both faces
C2 – Tails on both faces
C3 – Head and Tail

To find: If the outcome is head, find the probability of the other side being tail, ie, $P(C_3|H) = ?$

$$P(C_3|H) = \frac{P(H|C_3)\,P(C_3)}{P(H)} \quad \text{[Bayes' Theorem]}$$

$$= \frac{P(H|C_3)\,P(C_3)}{P(H|C_1)\,P(C_1) + P(H|C_2)\,P(C_2) + P(H|C_3)\,P(C_3)}$$

$$= \frac{1/2 \times 1/3}{1 \times 1/3 + 0 \times 1/3 + 1/2 \times 1/3}$$

$$= \frac{1/6}{1/3 + 1/6}$$

$$= \frac{1/6}{3/6} = \frac{1}{3}$$

∴ Probability of choosing C_3 given that the outcome was H is $1/3$

Q5. $C = \{A_1, A_2, A_3 \ldots A_n\}$ are mutually independent events, ie,

$$P(A_i \cap A_j) = P(A_i) \cdot P(A_j) \ \forall\ i \neq j$$

To Prove: $(C \setminus \{A_i\}) \cup \{A_i^c\}$ is a collection of mutually independent events for any i.

$(C \setminus \{A_i\}) \cup \{A_i^c\} \longrightarrow$ Essentially, we replace A_i with A_i^c in the collection C.

For any A_j (st $j \neq i$) in C, we know that

$$P(A_j \cap A_i) = P(A_j)P(A_i)$$

$$P(A_j \cap A_i^c) = P(A_j) - P(A_j \cap A_i) \quad \left[A \cap B^c = A \setminus B = A \setminus (A \cap B) \right]$$

$$= P(A_j) - P(A_j)P(A_i) \quad \left[A_j \text{ is independent of } A_i \ \forall j \neq i \right]$$

$$= P(A_j)(1 - P(A_i))$$

$$= P(A_j)P(A_i^c)$$

$$\Rightarrow P(A_j \cap A_i^c) = P(A_j)P(A_i^c) \quad \forall j \neq i$$

$\therefore$ A_i^c is independent of all A_j st $j \neq i$.

$\therefore$ $(C \setminus \{A_i\}) \cup \{A_i^c\}$ is a collection of mutually independent sets.

Q6. Let A, B be the events of Alice and Bob saying that E occurred respectively.

To find: $P(E|A\cap B) = ?$

$$P(E|A\cap B) = \frac{P(A\cap B|E)\,P(E)}{P(A\cap B|E)\,P(E) + P(A\cap B|E^c)\,P(E^c)}$$ [Bayes' Theorem]

$$= \frac{\left(\frac{9}{10}\times\frac{9}{10}\right)\left(\frac{1}{1000}\right)}{\frac{9}{10}\times\frac{9}{10}\times\frac{1}{1000} + \frac{1}{10}\times\frac{1}{10}\times\frac{999}{1000}}$$

$$= \frac{81}{81+999} = \frac{81}{1080}$$

$$= \underline{\underline{0.075}}$$

∴ The probability that E actually occurred, given that both Alice and Bob said it occurred is $\underline{0.075}$.

Q7. The man starts with 200 ₹ and must reach 20,00,000 ₹. Each round he either wins or loses 1 ₹, with equal probability.

This represents a classical gambler's ruin problem.

Let the current amount of money be i, and the target be n. Let P_i represent the probability of winning with i as the starting amount.

We can say that,

$$P_i = \underbrace{\frac{1}{2} P_{i-1}}_{\substack{\text{loses current round} \\ \text{so must win with} \\ \text{amount } i-1}} + \underbrace{\frac{1}{2} P_{i+1}}_{\substack{\text{wins current round} \\ \text{so must win with} \\ \text{amount } i+1}} \quad \forall\ 0 < i < n \qquad (1)$$

Also, clearly, $P_0 = 0$ and $P_n = 1$ (Already lost or won the whole game, respectively)

Solving eq. (1),

$$P_i = \frac{1}{2} P_{i+1} + \frac{1}{2} P_{i-1}$$

$$P_i = \frac{1}{2}\left(\frac{1}{2} P_{i+2} + \frac{1}{2} P_i\right) + \frac{1}{2}\left(\frac{1}{2} P_i + \frac{1}{2} P_{i-2}\right)$$

$$= \frac{1}{4} P_{i+2} + \frac{1}{2} P_i + \frac{1}{4} P_{i-2}$$

$$\Rightarrow P_i = \frac{1}{2} P_{i+2} + \frac{1}{2} P_{i-2}$$

$$\Rightarrow P_i = \frac{1}{2} P_{i+n} + \frac{1}{2} P_{i-n} \quad \forall\ n \leq i$$

$$\Rightarrow P_{200} = \frac{1}{2} P_{200+200} + \frac{1}{2} P_{200-200} = \frac{1}{2} P_{400} + \cancel{\frac{1}{2} P_0}^{\,0}$$

$$\Rightarrow P_{200} = \frac{1}{2} P_{400}$$

|||ly $P_{400} = \frac{1}{2} P_{800} \Rightarrow P_{200} = \frac{1}{4} P_{800}$

$$\therefore \; P_{200} = \frac{1}{\frac{n}{200}} P_n = \underline{\underline{\frac{200}{n} P_n}}$$

$$n = 2000000$$

$$\Rightarrow P_{200} = \frac{\cancel{200}}{\cancel{2000000}} \cancel{P_{2000000}}^{\,1}$$

$$\Rightarrow P_{200} = \frac{1}{10000} = \underline{\underline{0.0001\ \%}}$$

$\therefore$ Probability of winning the game, starting at 200 ₹ and reaching 20,00,000 ₹, is 0.0001 %

$\Rightarrow$ Probability of Bankruptcy = 0.9999 %