Signal Processing - Assignment 5

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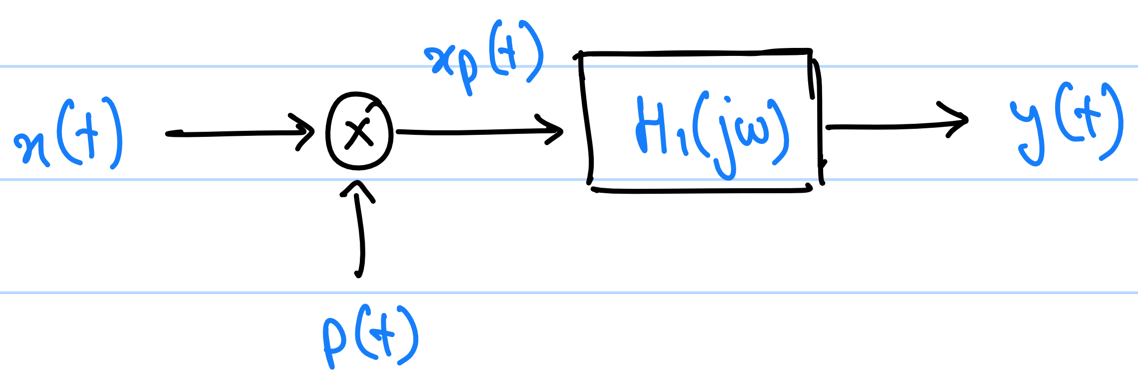
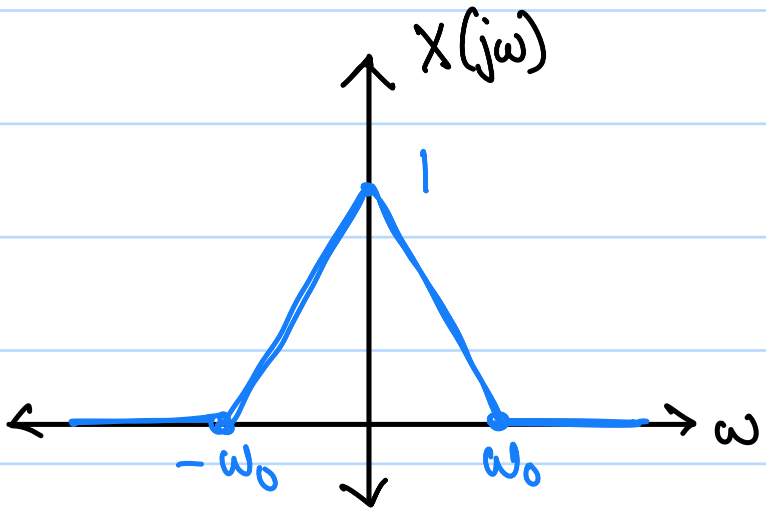
Q5.



i/p: $X(j\omega)$:



a) $p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$, $T = \frac{2\pi}{3\omega_0} \Rightarrow \omega_p = 3\omega_0$

$$X(j\omega) = \begin{cases} 1 + \frac{\omega}{\omega_0}, & \omega \in [-\omega_0, 0) \\ 1 - \frac{\omega}{\omega_0}, & \omega \in [0, \omega_0] \\ 0, & \text{otherwise} \end{cases}$$

$$x_p(t) = x(t)\,p(t) \Rightarrow X_p(\omega) = X(\omega) * P(\omega)$$

$$P(\omega) = \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_p) = \sum_{n=-\infty}^{\infty} \delta(\omega - 3n\omega_0)$$

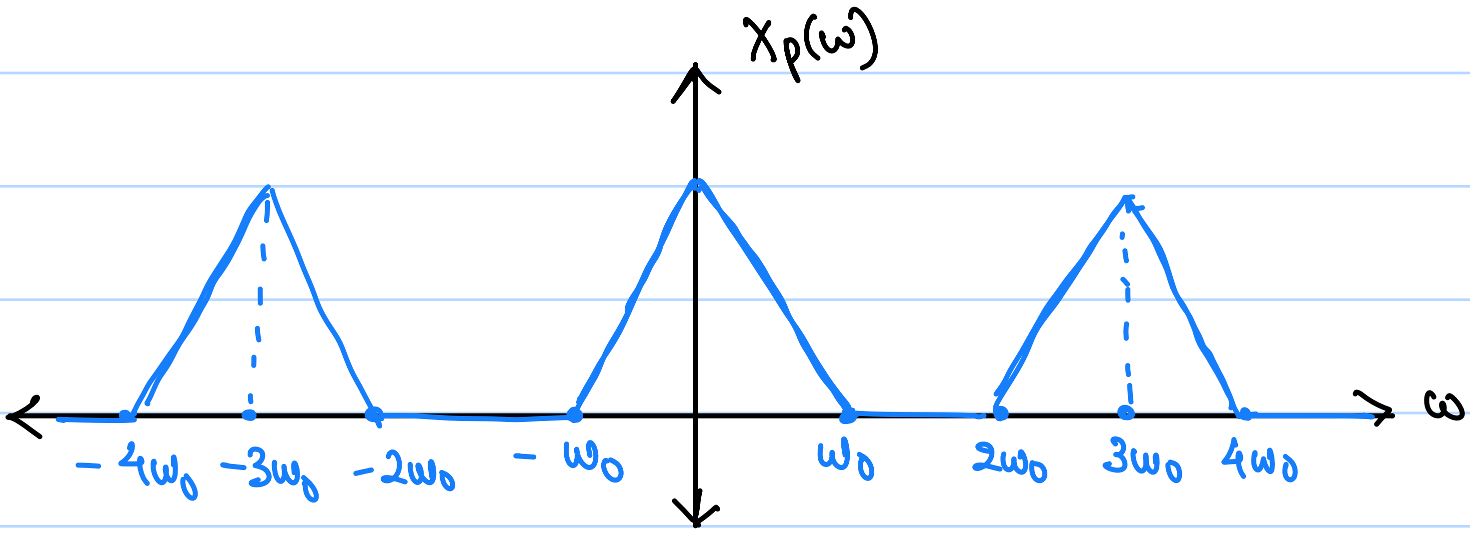
$$\Rightarrow X_p(\omega) = \int_{-\infty}^{\infty} P(\alpha)\, X(\omega-\alpha)\, d\alpha$$

$$= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(\alpha - 3n\omega_0)\, X(\omega-\alpha)\, d\alpha$$

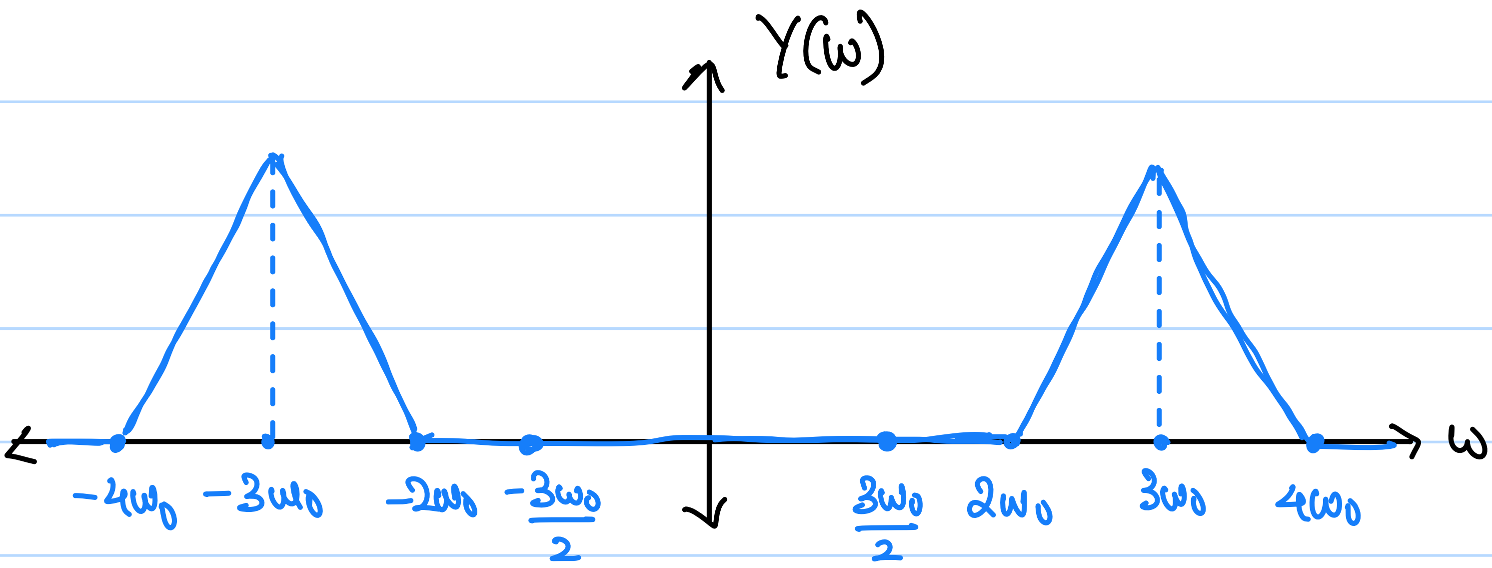
$$= \sum_{n=-\infty}^{\infty} X(\omega - 3n\omega_0)$$

$$\Rightarrow X_p(\omega) = \begin{cases} 1 + \dfrac{\omega - 3n\omega_0}{\omega_0}, & \omega \in [-\omega_0 + 3n\omega_0, 3n\omega_0) \\ 1 - \dfrac{\omega - 3n\omega_0}{\omega_0}, & \omega \in [3n\omega_0, \omega_0 + 3n\omega_0] \\ 0, & \text{otherwise} \end{cases}$$

$$X_p(\omega) = \begin{cases} 1 + \dfrac{\omega}{\omega_0} - 3n, & \omega \in [(3n-1)\omega_0, 3n\omega_0) \\ 1 - \dfrac{\omega}{\omega_0} + 3n, & \omega \in [3n\omega_0, (3n+1)\omega_0] \\ 0, & \text{otherwise} \end{cases}$$



b) Passing $X_p(\omega)$ through a high pass filter of $\omega_c = \frac{3\omega_0}{2}$, we get,



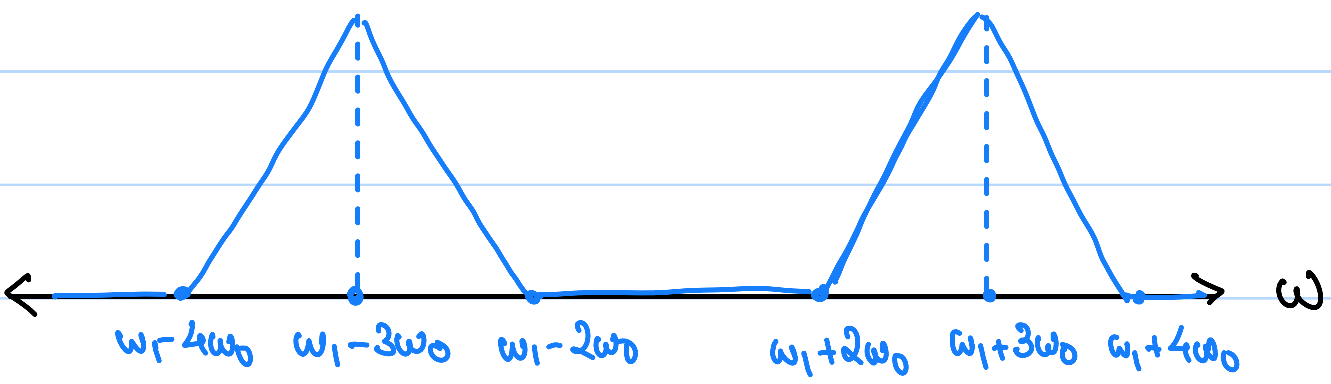
c) $c(t) = e^{-j\omega_1 t}$, $y_c(t) = y(t) \cdot c(t)$

$$C(\omega) = \delta(\omega - \omega_1)$$

$$\Rightarrow \quad Y_c(\omega) = Y(\omega) * C(\omega)$$

$$= Y(\omega) * \delta(\omega - \omega_1)$$

$$Y_c(\omega) = Y(\omega - \omega_1)$$



d) For the reconstructed signal to match the original signal,

$$\omega_1 - 3n\omega_0 = 0 \quad \forall \; n \in \mathbb{Z} - \{0\}$$

$$\Rightarrow \quad \omega_1 = 3n\omega_0 \quad \forall \; n \in \mathbb{Z} - \{0\}$$

e) The system S_2 will act as the inverse for S_1 for any arbitrary i/p signal, only if the bandwidth of the signal is less than $3\omega_0/2$, since the cutoff frequency of the low pass filter in S_2 is $3\omega_0/2$ and if the bandwidth of the input signal is greater than $3\omega_0/2$, the frequency components greater than $3\omega_0/2$ would be nullified. Also, if the signal bandwidth is $> 3\omega_0/2$, aliasing may occur in the sampling step, since the sampling freq $3\omega_0$ is lower than the Nyquist frequency.

Q6. $x[n] = \dfrac{\sin(\omega_0 n)}{\pi n}, \quad \omega_0 \in (0, 2\pi)$

a) <u>To Prove:</u> $x[n-n_1] * x[n-n_2] = x[n-(n_1+n_2)]$

$$\frac{\sin(\omega_0 n)}{\pi n} \xrightarrow{DTFT} \text{rect}\left(\frac{\omega}{2\omega_0}\right)$$

$$\frac{\sin(\omega_0(n-n_1))}{\pi(n-n_1)} \xrightarrow{DTFT} \text{rect}\left(\frac{\omega}{2\omega_0}\right) e^{-j\omega n_1}$$

$$\frac{\sin(\omega_0(n-n_2))}{\pi(n-n_2)} \xrightarrow{DTFT} \text{rect}\left(\frac{\omega}{2\omega_0}\right) e^{-j\omega n_2}$$

$[x(n-n_0) \xrightarrow{DTFT} X(\omega)e^{-j\omega n_0}]$

$$\frac{\sin(\omega_0(n-n_1))}{\pi(n-n_1)} * \frac{\sin(\omega_0(n-n_2))}{\pi(n-n_2)} \xrightarrow{DTFT} \text{rect}\left(\frac{\omega}{2\omega_0}\right)^2 e^{-j\omega(n_1+n_2)}$$

$$= \text{rect}\left(\frac{\omega}{2\omega_0}\right) e^{-j\omega(n_1+n_2)}$$

$$\text{rect}\left(\frac{\omega}{2\omega_0}\right) e^{-j\omega(n_1+n_2)} \xrightarrow{DTFT^{-1}} \frac{\sin(\omega_0(n-(n_1+n_2)))}{\pi(n-(n_1+n_2))} = x[n-(n_1+n_2)]$$

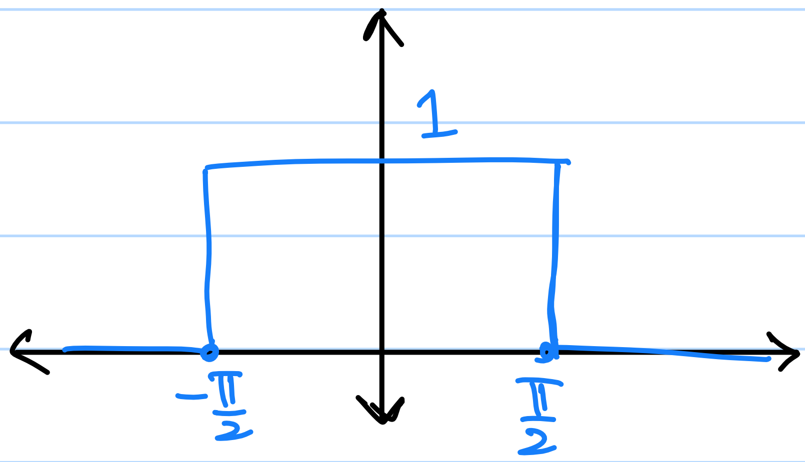
$\therefore \; x[n-n_1] * x[n-n_2] = x[n-(n_1+n_2)]$ Hence Proved.

b) $\omega_0 = \frac{\pi}{2}$,

Parseval's theorem : $\displaystyle\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|X(\omega)|^2 d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\text{rect}\left(\frac{\omega}{\pi}\right)\right|^2 d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}\text{rect}\left(\frac{\omega}{\pi}\right) d\omega$$

$\text{rect}\left(\frac{\omega}{\pi}\right)$:



$$\Rightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi}\text{rect}\left(\frac{\omega}{\pi}\right) d\omega = \frac{1}{2\pi}\left(1 \times 2\left(\frac{\pi}{2}\right)\right) = \frac{1}{2\pi}\pi = \frac{1}{2}$$

$$\Rightarrow \sum_{n\in\mathbb{Z}} |x[n]|^2 = \frac{1}{2}$$

$$\Rightarrow \sum_{n\in\mathbb{Z}} \left|\frac{\sin\left(\frac{\pi}{2}n\right)}{\pi n}\right|^2 = \frac{1}{2}$$

$$\Rightarrow \sum_{m\in\mathbb{Z}} \left|\frac{\sin\left(\frac{\pi}{2}(2m+1)\right)}{\pi(2m+1)}\right|^2 = \frac{1}{2}$$

$$= \sum_{m\in\mathbb{Z}} \frac{1}{\pi^2(2m+1)^2} = \frac{1}{2}$$

$$= \boxed{\sum_{m\in\mathbb{Z}} \left(\frac{1}{2m+1}\right)^2 = \frac{1}{2}\pi^2}$$ Hence Proved.

$$\boxed{\therefore \alpha = \frac{1}{2}}$$

Q7. $x(t) = \cos\left(\frac{\omega_s}{2}t + \phi\right)$.

$x_p(t)$: Sampling $x(t)$ at freq. ω_s

$$= \sum_{n \in \mathbb{Z}} x(nT_s)\, \delta(t - nT_s) \quad , \quad T = \frac{2\pi}{\omega_s}$$

a) $x(t) = \cos\phi \cos\left(\frac{\omega_s}{2}t\right) + g(t)$. $g(t) = ?$

$$x(t) = \cos\left(\frac{\omega_s}{2}t + \phi\right) = \cos\left(\frac{\omega_s}{2}t\right)\cos\phi - \sin\left(\frac{\omega_s}{2}t\right)\sin\phi$$

$$\therefore \quad g(t) = -\sin\left(\frac{\omega_s}{2}t\right)\sin\phi$$

b) $$g(nT) = -\sin\phi \sin\left(\frac{\omega_s}{2}nT\right)$$
$$= -\sin\phi \sin\left(\frac{\omega_s}{2} n \frac{2\pi}{\omega_s}\right)$$

$$= -\sin\phi \sin(\pi n)$$

$\sin(\pi n) = 0 \quad \forall \; n \in \mathbb{Z}$

$\Rightarrow$ $g(nT) = 0 \;\; \forall \; n \in \mathbb{Z}$ $\Rightarrow$ $x(nT) = \cos\left(\frac{\omega_s}{2}t\right)\cos\phi \;\; \forall \; n \in \mathbb{Z}$

c) <u>To Prove:</u> If $y(t)$ is the o/p of $x_p(t)$ passed through a low pass filter w/ $\omega_c = \omega_s/2$, then,

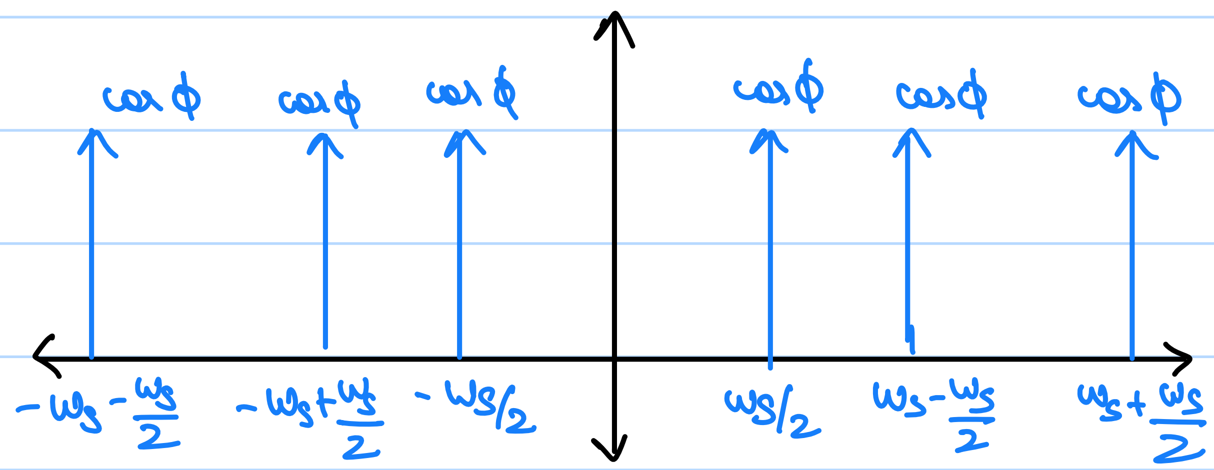
$$y(t) = \cos\phi \cos\left(\frac{\omega_s}{2}t\right)$$

$$x_p(t) = \sum_{n\in\mathbb{Z}} x(nT)\,\delta(t-nT)$$

$$x_p(t) = \sum \cos\left(\frac{\omega_s}{2}t\right)\cos\phi\;\delta(t-nT)$$

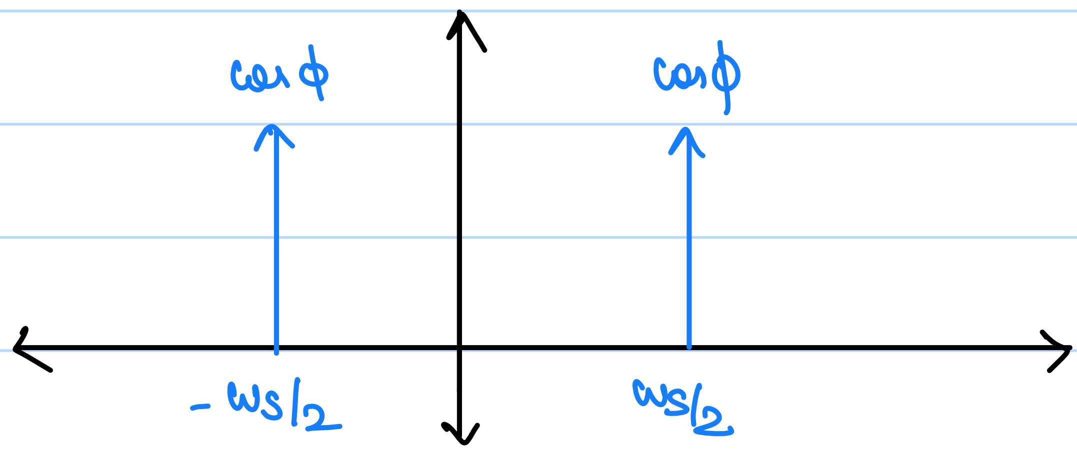
$$x_p(t) = \cos\phi \sum \cos\left(\frac{\omega_s}{2}t\right)\delta(t-nT)$$

$\Rightarrow$ $X_p(\omega)$:



Passing this through a LP filter of $\omega_c = \omega_s/2$,

$Y(\omega)$:



$\Rightarrow$ $\boxed{y(t) = \cos\phi\cos\left(\frac{\omega_s}{2}t\right)}$ Hence Proved.