

Prof: Prabhakar Bhimlapuram

Science - 1

Group - B

Marks :

Tutorial Quizzes

Endsem : 35 %

Midsem : 20 % $\pm 5\%$

Quiz : 10 + 10 %

In Class / Tut Quiz : 25 %

Science - 1

- Science explains everything on the time-scales, the length scales and the energy scales.

Time Scales : Climate Change (Decade - scale)
Bullet travel (Second - scale)
Molecular movement (Picosecond - scale)

Length Scales : High Energy Physics (Quark / lepton Scales)
Expansion of Universe (10^{29} m scale)

Energy Scale : Energy to lift a laptop
" " build a building

- There are a variety of things on these scales that we use in our daily life .

→ What is Science ? :

- Systematic investigation of natural phenomenon .

- Reproducibility is a key feature of Science . Any experiment must be reproducible . Even a single datapoint is sufficient to disprove a theory .

- We can use the data gathered from the experiment, to create a mathematical model, than can be used to predict things.

Mathematical modelling has enabled the development of sciences.

- Science will try to explain the behavior / reason behind the parameters present in the model, and also the limitations / inaccuracies in the model.



→ Newton's first law is not directly provable, since it requires highly ideal conditions. It is only true at asymptotical conditions.

→ Limitation of Newton's laws.

- Reductionism: Separation of the most important things and neglecting every other factor. ex: The radius of planets while observing their orbital behavior around the sun, through Kepler's planetary laws.

" A System is a sum of its parts " (Superposition-like)

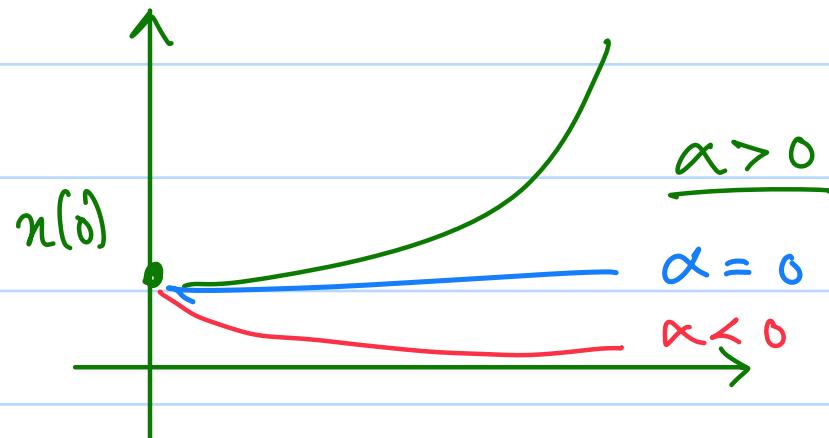
By decomposing a system into its pieces, we can figure out the interactions between the parts and also the most important parts of the system.

- Microscopics: The study behind the various interactions between the parts of a system.
- A reductionism model can be used to accurately predict the behavior of the model.
- Reductionism goes from microscopic behavior to macroscopic features.
- Even microscopic interactions follow laws, like conservation of energy.

→ Mathematics in Science:

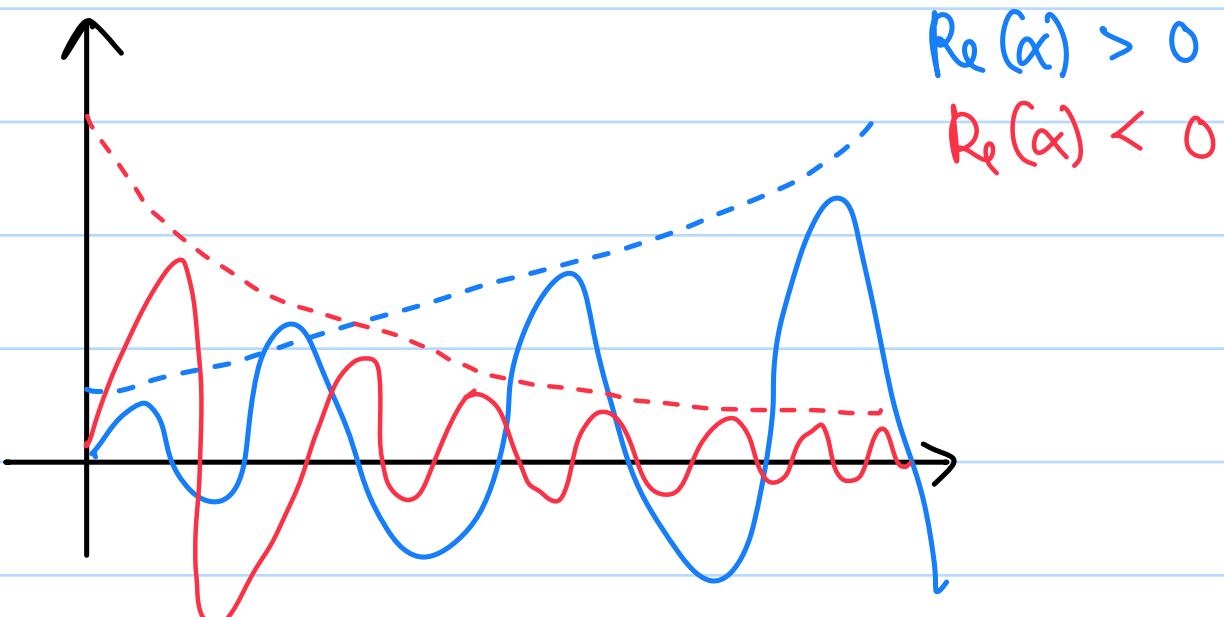
- Geometry and Linear Algebra
- Change and Calculus
- Chance, probability and statistics. → Uncertainty Principle

ex: The Behavior of a system with $n(t) = n(0) e^{\alpha t}$



In the above system, the value of the parameter can fundamentally change the behavior of the system.

If $\alpha \in \mathbb{C}$



→ The Prey-Predator Model :-

- x - No. of Rabbits

- y - No. of Tigers

- Parameters :

- α - Growth rate of Rabbits

- β - Effect of tigers on the rabbits (Rabbits are killed)

- γ - Death rate of Foxes

- δ - Effect of rabbit on foxes (Foxes can reproduce with nutrition)

$$\frac{dx}{dt} = \alpha x - \beta xy \quad \frac{dy}{dt} = -\gamma y + \delta xy$$

$$\Rightarrow \frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha x - \beta xy \\ -\gamma y + \delta xy \end{pmatrix} \rightarrow \begin{matrix} f \\ g \end{matrix}$$

$$\hookrightarrow \gamma \begin{pmatrix} x \\ y \end{pmatrix} = \nearrow$$

- The equilibrium points of the system,

$$\frac{dx}{dt} = 0$$

$$\frac{dy}{dt} = 0$$

$$\Rightarrow \alpha x - \beta ny = 0$$

$$\Rightarrow -\gamma y + \delta xy = 0$$

$$\Rightarrow n(\alpha - \beta y) = 0$$

$$\Rightarrow y(-\gamma + \delta x) = 0$$

$$\Rightarrow x = 0, y = \frac{\alpha}{\beta}$$

$$\Rightarrow y = 0, x = \frac{\gamma}{\delta}$$

$\Rightarrow (0,0), \left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$ are the equilibrium points.

$$f(x,y) \quad \begin{cases} SF_x \\ SF_y \end{cases}$$

Jacobian of the system,

$$\begin{pmatrix} \alpha x - \beta ny \\ -\gamma y + \delta xy \end{pmatrix} \xrightarrow{J} \begin{pmatrix} \alpha - \beta y & -\beta n \\ \delta y & -\gamma + \delta x \end{pmatrix} = J$$

$$J\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right) = \begin{pmatrix} 0 & -\beta\gamma/\delta \\ \delta\alpha/\beta & 0 \end{pmatrix} \rightarrow \gamma'$$

Let $(x_0, y_0) = \left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$ and define $\varepsilon = x - x_0, n = y - y_0$.

Around the equilibrium point, the function can be approximated as,

$$\frac{d}{dt} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} = J \Big|_{(x_0, y_0)} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + \begin{pmatrix} \overset{0}{(x-x_0)} \\ \overset{0}{(y-y_0)} \end{pmatrix}$$

$$\Rightarrow \frac{d}{dt} \begin{pmatrix} \varepsilon \\ n \end{pmatrix} = \begin{pmatrix} 0 & -\beta\gamma/\delta \\ \delta\alpha/\beta & 0 \end{pmatrix} \begin{pmatrix} \varepsilon \\ n \end{pmatrix}$$

$$\approx \frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -\beta y/\delta \\ \delta x/\beta & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \text{near the equilibrium of } \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

Note: The approximation of the function around (x_0, y_0) is done using the result,

$$\begin{aligned} h(x, y) &= h(x_0, y_0) + \left[\frac{\partial h}{\partial x} \Big|_{(x_0, y_0)} (x - x_0) + \frac{\partial h}{\partial y} \Big|_{(x_0, y_0)} (y - y_0) \right] \\ &= h(x_0, y_0) + \left[\begin{pmatrix} \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{pmatrix} \Big|_{(x_0, y_0)} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} \right] \\ &\quad \text{is from the Jacobian} \end{aligned}$$

Linearization

All analysis can be done near $(0, 0)$.

$$\text{Jacobian near } (0, 0), \quad J \Big|_{(0, 0)} = \begin{pmatrix} \alpha & 0 \\ 0 & -\delta \end{pmatrix}$$

$$\Rightarrow \frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & -\delta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad dt = \frac{dn}{\alpha n}$$

$$\Rightarrow \frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha n \\ -\delta y \end{pmatrix} \rightarrow \begin{aligned} \frac{dn}{dt} &= \alpha n \\ \frac{dy}{dt} &= -\delta y \end{aligned}$$

$$\Rightarrow x = c_1 e^{\alpha t}, \quad y = c_2 e^{-\delta t}$$

$$\Rightarrow x = x(0) e^{\alpha t}, \quad y = y(0) e^{-\delta t}$$

For the other Eq. point, $\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$,

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -\beta\gamma/\delta \\ \delta\alpha/\beta & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \frac{d\bar{x}}{dt} = \bar{J}\bar{x}$$

The general solution of such a system is of the form,

$$\bar{x}(t) = c_1 e^{\lambda_1 t} \bar{v}_1 + c_2 e^{\lambda_2 t} \bar{v}_2$$

where, \bar{v}_1, \bar{v}_2 are the Eigenvectors of J , λ_1, λ_2 are their respective Eigenvalues.

The Eigenvalues of J are,

$$\det \begin{pmatrix} -\lambda & -\beta\gamma/\delta \\ \delta\alpha/\beta & -\lambda \end{pmatrix} = 0$$

$$= \lambda^2 - \left(\frac{-\beta\gamma}{\delta}\right) \left(\frac{\delta\alpha}{\beta}\right) = 0$$

$$= \lambda^2 - (-\alpha\gamma) = 0$$

$$= \lambda^2 = -\alpha\gamma$$

$$= \lambda = \pm i\sqrt{\alpha\gamma}$$

for $\lambda_1 = i\sqrt{\alpha\gamma}$,

$$\begin{pmatrix} -i\sqrt{\alpha\gamma} & -\frac{\beta\gamma}{s} \\ \frac{s\alpha}{\beta} & -i\sqrt{\alpha\gamma} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow -x_1 i\sqrt{\alpha\gamma} - \frac{\beta\gamma}{s} x_2 = 0$$

$$\Rightarrow x_1 i\sqrt{\alpha\gamma} = -\frac{\beta\gamma}{s} x_2$$

$$x_1 = -\frac{\beta\gamma}{i s \sqrt{\alpha\gamma}} x_2 = -\frac{\beta}{i s \sqrt{\alpha}} x_2$$

$$\frac{s\alpha}{\beta} x_1 - i\sqrt{\alpha\gamma} x_2 = 0$$

$$= x_1 = \frac{i\sqrt{\alpha\gamma}\beta}{s\alpha} x_2 = \frac{i\beta}{s} \sqrt{\frac{\gamma}{\alpha}} x_2$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{i\beta}{s} \sqrt{\frac{\gamma}{\alpha}} \\ 1 \end{pmatrix} = \vec{v}_1$$

For $\lambda_2 = -i\sqrt{\alpha\gamma}$

$$\begin{pmatrix} i\sqrt{\alpha\gamma} & -\frac{\beta\gamma}{s} \\ \frac{s\alpha}{\beta} & i\sqrt{\alpha\gamma} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$= i\sqrt{\alpha\gamma} x_1 - \frac{\beta\gamma}{s} x_2 \Rightarrow x_1 = \frac{\beta}{is} \sqrt{\frac{\gamma}{\alpha}} = \frac{-i\beta}{s} \sqrt{\frac{\gamma}{\alpha}}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -i\beta/s \sqrt{\gamma/\alpha} \\ 1 \end{pmatrix} = \vec{v}_2$$

$$\Rightarrow \vec{r}(t) = c_1 e^{i\sqrt{\alpha\gamma}t} \begin{pmatrix} i\beta/s \sqrt{\gamma/\alpha} \\ 1 \end{pmatrix} + c_2 e^{-i\sqrt{\alpha\gamma}t} \begin{pmatrix} -i\beta/s \sqrt{\gamma/\alpha} \\ 1 \end{pmatrix}$$

$$= \bar{r}(t) = c_1 \left(\cos(\sqrt{\alpha\gamma}t) + i \sin(\sqrt{\alpha\gamma}t) \right) \begin{pmatrix} \frac{i\beta\sqrt{r}}{s\sqrt{\alpha}} \\ 1 \end{pmatrix} \\ + c_2 \left(\cos(\sqrt{\alpha\gamma}t) - i \sin(\sqrt{\alpha\gamma}t) \right) \begin{pmatrix} -\frac{i\beta\sqrt{r}}{s\sqrt{\alpha}} \\ 1 \end{pmatrix}$$

$$\Rightarrow \bar{r}(t) = \left[c_1 \left(\frac{i\beta\sqrt{r}}{s\sqrt{\alpha}} \right) + c_2 \left(\frac{-i\beta\sqrt{r}}{s\sqrt{\alpha}} \right) \right] \cos(\sqrt{\alpha\gamma}t) \\ + \left[c_1 \left(\frac{i\beta\sqrt{r}}{s\sqrt{\alpha}} \right) + c_2 \left(\frac{-i\beta\sqrt{r}}{s\sqrt{\alpha}} \right) \right] i \sin(\sqrt{\alpha\gamma}t)$$

→ Common Differential Equations :-

- $\frac{dx}{dt} = \alpha x \Rightarrow x = c_1 e^{\alpha t}$

- $\frac{\delta^2 f}{\delta t^2} = c^2 \frac{\delta^2 f}{\delta x^2} \rightarrow \text{Diffusion Equation}$

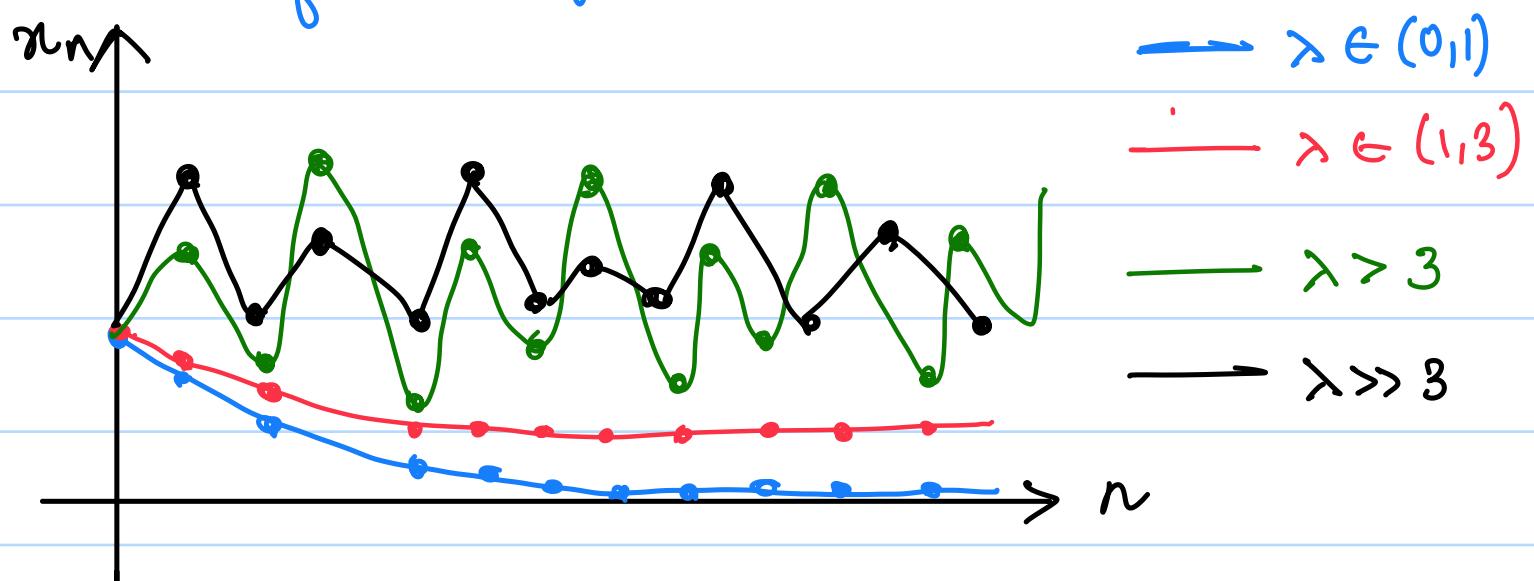
$$\frac{\delta F}{\delta t} = c^2 \frac{\delta^2 F}{\delta x^2} \longleftrightarrow \text{Quantum Mechanics}$$

→ Logistics Mapping :-

- A sequence such that,

$$x_{n+1} = \lambda x_n (1 - x_n), \quad \forall n \in \mathbb{N}$$

- Analyzing the terms of the sequence,



Therefore, depending on the value of λ , the same initial value can result in very different sequences, and for some λ , small differences in initial value can result in large differences in the final value.

- Therefore, this sequence is said to exhibit chaotic behavior.

→ Classical Mechanics :-

- Scalar Field: Each point in space is associated with a scalar quantity. ex: Temperature
- Vector Field: Each point in space is associated with a vector of some direction and magnitude. ex: Air Speeds.
- Gradient: The change of a quantity within its field.

$$\nabla f(x_0, y_0, z_0) = \left[\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y} \quad \frac{\partial f}{\partial z} \right]$$

- Divergence: Scalar that measures the outflowness of a vector field at a point.

Laws Of Motion :

- 1) In an inertial frame, an object undergoes uniform rectilinear motion in the absence of external force.
- 2) Rate of change of momentum is equal to the net force on the object.
- 3) Reaction force is equal and opposite to the action force.

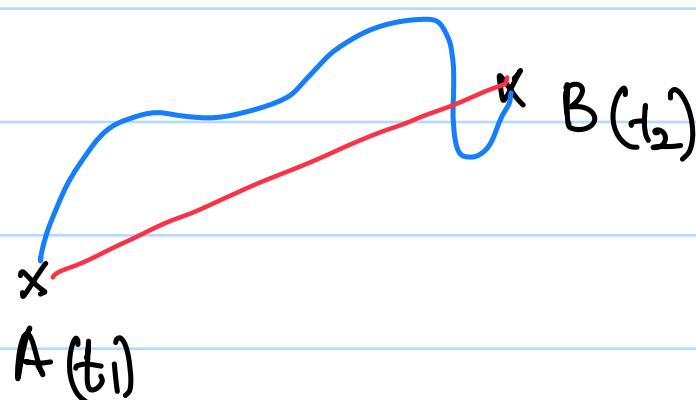
- Work: The work done by the application of an external force is given by,

$$W = \int_{t_1}^{t_2} \bar{F}(t) d\bar{r}(t) \quad \text{infinitesimal } F_x$$

Using the 2nd Law,

$$W = \int_{t_1}^{t_2} \frac{d\bar{p}}{dt} v(t) dt = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

$\Rightarrow W = KE_2 - KE_1 \rightarrow$ Work is just change in kinetic energy.



Different paths may have different amounts of work done.
(No. difference in conservative fields)

In vectorial form, $KE = \frac{1}{2}m(\bar{v} \cdot \bar{v})$

- Conservative Forces: Forces that can be modelled as a gradient of a scalar field, i.e.,

$$\bar{F}(\bar{r}) = -\nabla U(\bar{r})$$

$$\Rightarrow W = \int \bar{F}(\bar{r}) d\bar{r} = - (U_2 - U_1)$$

- The scalar quantity (field $U(\bar{r})$) is the potential energy due to the force.

- Since Work now only depends on PE at 1 and 2, it does not depend on the exact path travelled between 1 and 2.

$$W = K_2 - K_1 = - (U_2 - U_1)$$

$$\Rightarrow K_2 + U_2 = K_1 + U_1 \rightarrow \text{Law of Conservation of Mechanical Energy}$$

- Conservation of Linear Momentum :-

- For an object, if $\bar{F} \cdot \bar{s} = 0$ (\bar{F} is zero or \perp dir to \bar{s})

$$\bar{F} \cdot \bar{s} = 0 \Rightarrow \frac{d\bar{p}}{dt} \cdot \bar{s} = 0 \Rightarrow \frac{d(\bar{p} \cdot \bar{s})}{dt} = 0 \quad (\text{Assuming } \bar{s} \text{ is const})$$

$\Rightarrow \bar{p} \cdot \bar{s}$ is constant \Rightarrow Momentum in the direction of s is conserved

∴ If force in a direction is zero, momentum is conserved in that direction.

- Torque and Conservation of Angular Momentum :-

$$\bullet \text{Angular momentum } \bar{L} = \bar{r} \times \bar{p}$$

$$\bullet \text{Torque } \bar{\tau} = \frac{d}{dt} \bar{L} = \frac{d}{dt} \bar{r} \times \bar{p} = \frac{d\bar{r}}{dt} \times \bar{p} + \bar{r} \times \frac{d\bar{p}}{dt}$$

$$\bar{\tau} = \cancel{\bar{r} \times \bar{p}}^0 + \bar{r} \times \bar{F}$$

$$\Rightarrow \bar{\tau} = \bar{r} \times \bar{F}$$

- ## • Conservation of Angular Momentum,

If $\bar{T} \cdot \bar{s} = 0$, then $\bar{L} \cdot \bar{s}$ is constant.

\therefore Angular momentum is conserved along the axis of zero torque.

$$\bullet \quad E = T(\bar{v}(t)) + U(\bar{r}(t), t)$$

$$\frac{dT}{dt} = m\bar{v} \frac{d\bar{v}}{dt} = m\bar{v} \cdot \bar{a} = \bar{V} \cdot \bar{F}$$

$$\frac{dU(\vec{r}, t)}{dt} = \frac{\partial U}{\partial \vec{r}} \frac{d\vec{r}}{dt} + \frac{\partial U}{\partial t} = \vec{v} \cdot \nabla U + \frac{\partial U}{\partial t}$$

$$\Rightarrow \frac{dE}{dt} = \bar{v} (\bar{F} + \cancel{\Delta U}) + \frac{\partial U}{\partial t}$$

$$\Rightarrow \frac{dE}{dt} = \frac{\partial U}{\partial t} \Rightarrow \text{If PE is constant, E is conserved}$$

- ## o Galilean Relativity :-

Let there be 2 inertial frames 'K' and 'K'
 and K_r is at relative velocity \bar{v} wrt K. Let an object be
 moving with velocity \bar{J} and \bar{v}_r , ie

$$\overline{c_1} = \overline{c_1} - \overline{v}$$

$$\Rightarrow \frac{d\bar{V}_I}{dt} = \frac{d(\bar{U} - \bar{V})}{dt} = \frac{d\bar{U}}{dt} \quad \bar{V} \text{ is const}$$

$$\Rightarrow \bar{F}_I = \bar{F}$$

\therefore All mechanical laws are the same in any inertial frame.

\rightarrow Simple Harmonic Motion :-

- Hooke's Law : $F(x) = -kx \Rightarrow m \frac{d^2x}{dt^2} = -kx$

To solve the DE,

$$m \frac{d^2x}{dt^2} = -kx$$

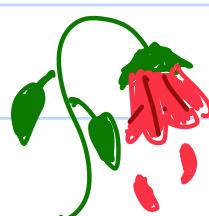
$$\text{Take } p = m \frac{dx}{dt} \Rightarrow m \frac{d^2x}{dt^2} = \frac{dp}{dt}$$

$$\Rightarrow \frac{dp}{dt} = -kx \Rightarrow dp = -kx dt$$

$$\therefore p = [-kxt]_0^t = -kxl$$

$$\frac{dx}{dt} = \frac{p}{m}$$

\Rightarrow



Is too tuff :(

→ Limitations of Newtonian Mechanics :-

- Galilean relativity violated by EM waves.
- Stability of atoms cannot be explained by Newtonian mechanics.

→ Special Theory of Relativity :-

The postulates of STR are as follows,

- i) The laws of physical phenomena are the same in all inertial reference frames.
- ii) The velocity of light (in free space) is a universal constant, independent of any relative motion.

◦ Galilean Invariance :-

Consider a frame K and another frame K' moving at a speed of v_r along the x-axis.

- Let (x, y, z) be coordinates in K and (x', y', z') in K'. The transformation of coordinates is given by,

$$x' = x - v_r t$$

$$y' = y$$

$$z' = z$$

$$t' = t$$

- The above transformation is termed as a Galilean Transformation.
- Length is conserved in a Galilean transformation,

$$ds^2 = \sum dx_i^2 = \sum dx_i'^2 = ds'^2$$

Newton's laws are also conserved,

$$F = ma = ma' = F'$$

- Velocity in K' is given by,

$$v' = v - v_r$$

That means the velocity of light in K' is $c - v_r \neq c$, in violation of STR.

- Lorentz Transformation :-

- If a light pulse is emitted from the common origin of the systems K and K₁ when they are coincident, then,

$$x = ct$$

$$x_1 = c t_1$$

$$\text{Let } x_1 = \gamma(x - v_r t) \Rightarrow x = \gamma'(x_1 + v_r t_1)$$

Since the laws of physics must be the same in K and K',
 $\gamma = \gamma'$.

$$\Rightarrow x = \gamma(v(x - v_r t) + v_r t_1)$$

$$\Rightarrow x = \gamma^2(x - v_r t) + \gamma v_r t_1$$

$$\Rightarrow x = x\gamma^2 - \gamma^2 v_r t + \gamma v_r t_1$$

$$\Rightarrow \gamma v_r t_1 = x - x\gamma^2 + \gamma^2 v_r t \\ = x(1 - \gamma^2) + \gamma^2 v_r t$$

$$\Rightarrow t_1 = \frac{x}{v_r} (1 - \gamma^2) + rt$$

$$= t_1 = \frac{x}{v_r} \left(\frac{1 - \gamma^2}{\gamma} \right) + rt \quad \text{--- (1)}$$

$$x = ct \Rightarrow x' = \gamma(ct - v_r t) - \gamma(c - v_r)t$$

$$\Rightarrow t_1 = \frac{ct}{v_r} \left(\frac{1 - \gamma^2}{\gamma} \right) + rt$$

$$x_1 = ct_1 \Rightarrow c = \frac{x_1}{t_1} = \frac{\gamma(c - v_r)t}{\frac{ct}{v_r} \left(\frac{1 - \gamma^2}{\gamma} \right) + rt}$$

$$\Rightarrow c = \frac{\gamma(c - v_r)}{\frac{c}{v_r} \left(\frac{1 - \gamma^2}{\gamma} \right) + \gamma}$$

$$\Rightarrow \gamma(c - v_r) = \frac{c^2}{v_r} \left(\frac{1 - \gamma^2}{\gamma} \right) + cr$$

$$\Rightarrow \gamma c - \gamma v_r = \frac{c^2}{v_r} \left(\frac{1 - r^2}{r} \right) + \gamma$$

$$= \gamma v_r = \frac{c^2}{v_r} \left(\frac{r^2 - 1}{r} \right)$$

$$= \frac{\gamma v_r}{c^2} = \frac{1}{v_r} \left(\frac{r^2 - 1}{r} \right) \quad \textcircled{2}$$

Substitute \textcircled{2} in \textcircled{1},

$$\Rightarrow t_1 = \gamma \left(-\frac{\gamma v_r}{c^2} \right) + rt$$

$$= t_1 = \gamma \left(t - \frac{v_r}{c^2} \gamma \right)$$

$$\Rightarrow \gamma = \frac{1}{\sqrt{1 - v_r^2/c^2}}$$

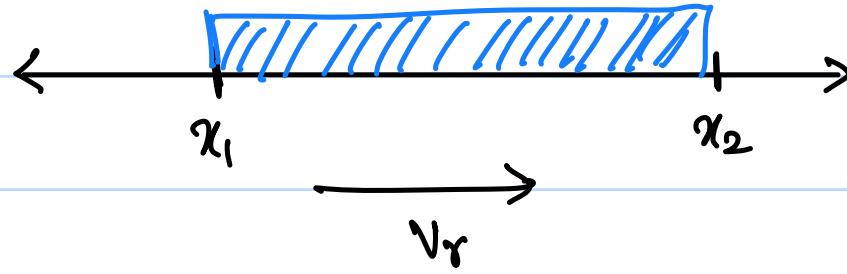
With this, the transformation can be described as,

$$x' = \frac{x - vt}{\sqrt{1 - v_r^2/c^2}}$$

$$t' = \frac{t - v_r x/c^2}{\sqrt{1 - v_r^2/c^2}} = \gamma \left(t - \frac{v_r x}{c^2} \right)$$

The above transformation is described as the Lorentz transformation.

- Length Contraction:



- An observer is moving velocity v_r towards x_2 , from x_1 . He measures the length of the above rod ($x_2 - x_1$) at a time instant t .

The length of the rod in the observer's frame is,

$$l' = x'_2 - x'_1$$

$$\begin{aligned} &= \gamma(x_2 - v_r t_2) - \gamma(x_1 - v_r t_1) \\ &= \gamma(x_2 - x_1 - v_r(t_2 - t_1)) \end{aligned}$$

$$\text{Since } t'_2 = t'_1$$

$$\Rightarrow \gamma\left(t_2 - \frac{v_r x_2}{c^2}\right) = \gamma\left(t_1 - \frac{v_r x_1}{c^2}\right)$$

$$t_2 - t_1 = \frac{v_r}{c^2}(x_2 - x_1)$$

$$\Rightarrow l' = \gamma\left(x_2 - x_1 - \frac{v_r^2}{c^2}(x_2 - x_1)\right)$$

$$= \frac{1}{\sqrt{1 - \frac{v_r^2}{c^2}}} (x_2 - x_1) \left(1 - \frac{v_r^2}{c^2}\right)$$

$$= (x_2 - x_1) \sqrt{1 - \frac{v_r^2}{c^2}}$$

$$\Rightarrow \underline{l'} = l/\gamma \Rightarrow l' < l \quad \sqrt{1 - \frac{v_r^2}{c^2}} < 1$$

$\hookrightarrow \gamma > 1$

This is termed as length contraction.

- Time Dilation :-

- A clock, at a fixed position x , ticks at 2 time instants t_1 and t_2 . An observer moving in frame K' , with relative velocity v_r measures the time interval.

$$\Delta t' = t_2' - t_1'$$

$$\Delta t' = \gamma \left(t_2 - \frac{v_r}{c^2} x_2 \right) - \gamma \left(t_1 - \frac{v_r}{c^2} x_1 \right)$$

$$= \gamma \left(t_2 - t_1 - \frac{v_r}{c^2} (x_2 - x_1) \right)$$

$$= \gamma (\Delta t)$$

$$\Rightarrow \underline{\Delta t'} = \gamma \Delta t \Rightarrow \Delta t' > \Delta t$$

This phenomenon is termed as time dilation.

- Relativistic Doppler Effect :-

Consider a source of light and a receiver, with approach velocity v .

Let the receiver be in a frame K and the source in K' .

In time Δt , the source emits n waves. The distance between the front and back is,

$$l = c\Delta t - v\Delta t$$

$$\Rightarrow \lambda = \frac{c\Delta t - v\Delta t}{n}$$

$$\Rightarrow f = \frac{nc}{c\Delta t - v\Delta t}$$

Since the source emits with frequency f_0 ,

$$n = f_0 \Delta t' = f_0 \frac{\Delta t}{\gamma} \quad [\text{time dilation}]$$

$$\Rightarrow f = \frac{f_0 c \Delta t}{\gamma (c \Delta t - v \Delta t)}$$

$$= \frac{1}{\gamma \left(1 - \frac{v}{c}\right)} f_0$$

$$f = \frac{\sqrt{1 - v^2/c^2}}{1 - v/c} f_0$$

If the source and receiver are separating with velocity v ,

$$l = c\Delta t + v\Delta t$$

$$\Rightarrow \lambda = \frac{c\Delta t + v\Delta t}{n}$$

$$\Rightarrow f = \frac{nc}{c\Delta t + v\Delta t}$$

$$n = f_0 \Delta t' = f_0 \frac{\Delta t}{\gamma}$$

$$\Rightarrow f = \frac{c\Delta t}{\gamma(c\Delta t + v\Delta t)} f_0$$

$$\Rightarrow f = \frac{\sqrt{1 - v^2/c^2}}{1 + v/c} f_0$$

Note:

Lorentz transformation for velocities,

Let K and K' be 2 inertial frames with relative velocity v_r , with an object moving at velocities v and v' in the frames respectively.

$$v' = \frac{dx'}{dt'} = \frac{\frac{dx'}{dt}}{\frac{dt'}{dt}}$$

Wkt,

$$x' = \gamma(x - v_r t) \quad t' = \gamma(t - \frac{v_r x}{c^2})$$

γ is a constant w.r.t t .

$$\Rightarrow \frac{dx'}{dt} = \gamma \left(\frac{dx}{dt} - v_r \right) \quad \frac{dt'}{dt} = \gamma \left(1 - \frac{v_r}{c^2} \frac{dx}{dt} \right)$$
$$= \gamma (v - v_r) \quad = \gamma \left(1 - \frac{v_r v}{c^2} \right)$$

$$\Rightarrow v' = \frac{\gamma (v - v_r)}{\gamma \left(1 - \frac{v_r v}{c^2} \right)} = \frac{v - v_r}{1 - \frac{v_r v}{c^2}}$$

Verification by taking $v = c$

$$v' = \frac{c - v_r}{1 - \frac{v_r c}{c^2}} = \frac{c - v_r}{\frac{(c - v_r)}{c}} = \underline{\underline{c}}$$

$\therefore v = v'$ if $v = c$. STR is not violated.

o Proper Time:-

Time as measured in the observer's frame of reference, with respect to events that occur at the same time.

- Proper length :-

Length as measured in the observer's frame of reference, with respect to an object that is at rest in his frame.

- Events that happen at different times will undergo length contraction between them.

Note: An event is defined with both space and time.

- The Twin Paradox :-

- If there are 2 twins, one on Earth and one on an interstellar voyage travelling near the speed of light, the space travelling twin will age much slower than the twin on Earth.
- The paradox is that, since with respect to the traveller, the twin on Earth is the one moving, the Earth twin must be the younger one, due to symmetry.
- The solution, is that since the traveller is going away, and then coming back to Earth, he is actually in 2 different reference frames. (One while separation, one while approach)
- To explain this further, we can use space time diagrams.

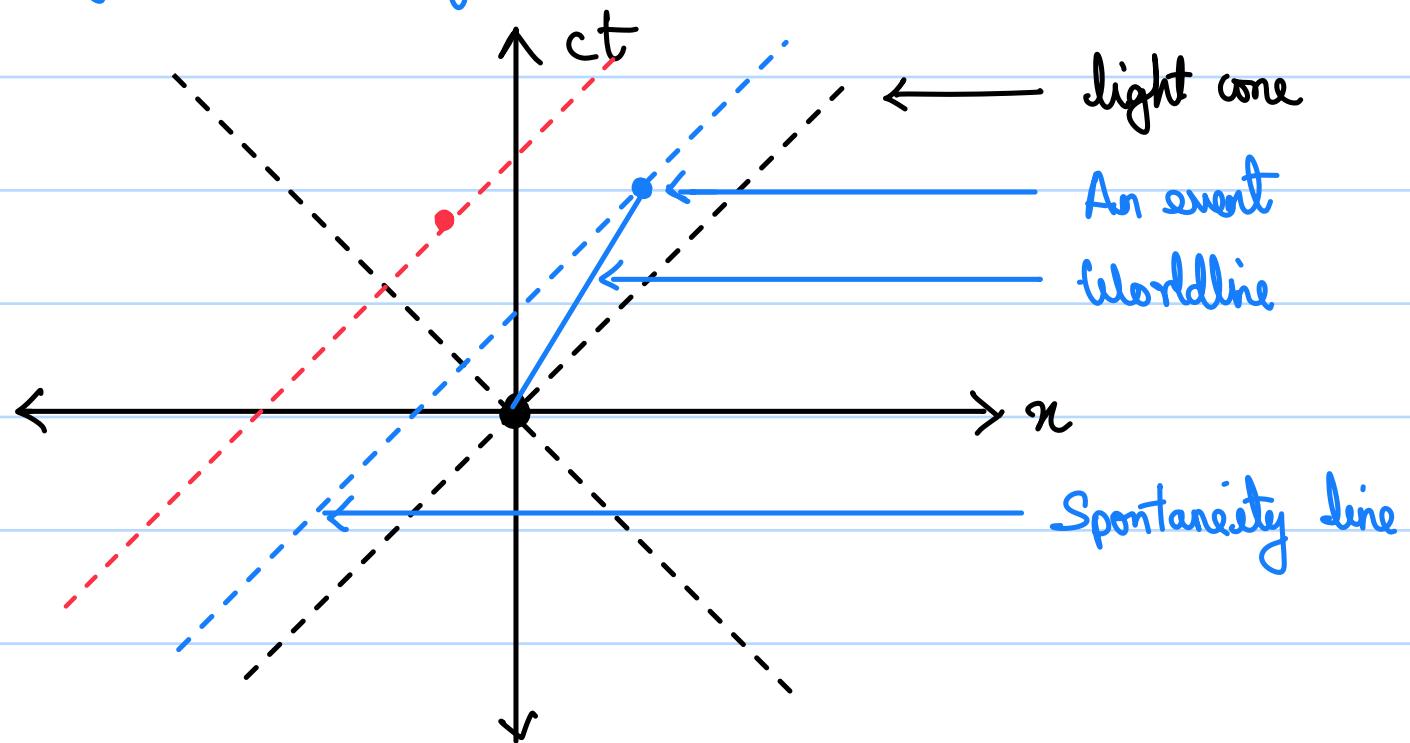
Spacetime Diagrams (Minkowski Geometry) :-

- In Minkowski Geometry, the spacetime interval between 2 events is defined as,

$$ds^2 = dx^2 + dy^2 + dz^2 - (cdt)^2$$

- A spacetime diagram is as follows, (on one spatial dimension x)

(ignore $-x$ for now)



Event : A point on the spacetime diagram

Worldline : The path travelled by an event, is the history of an object through space and time.

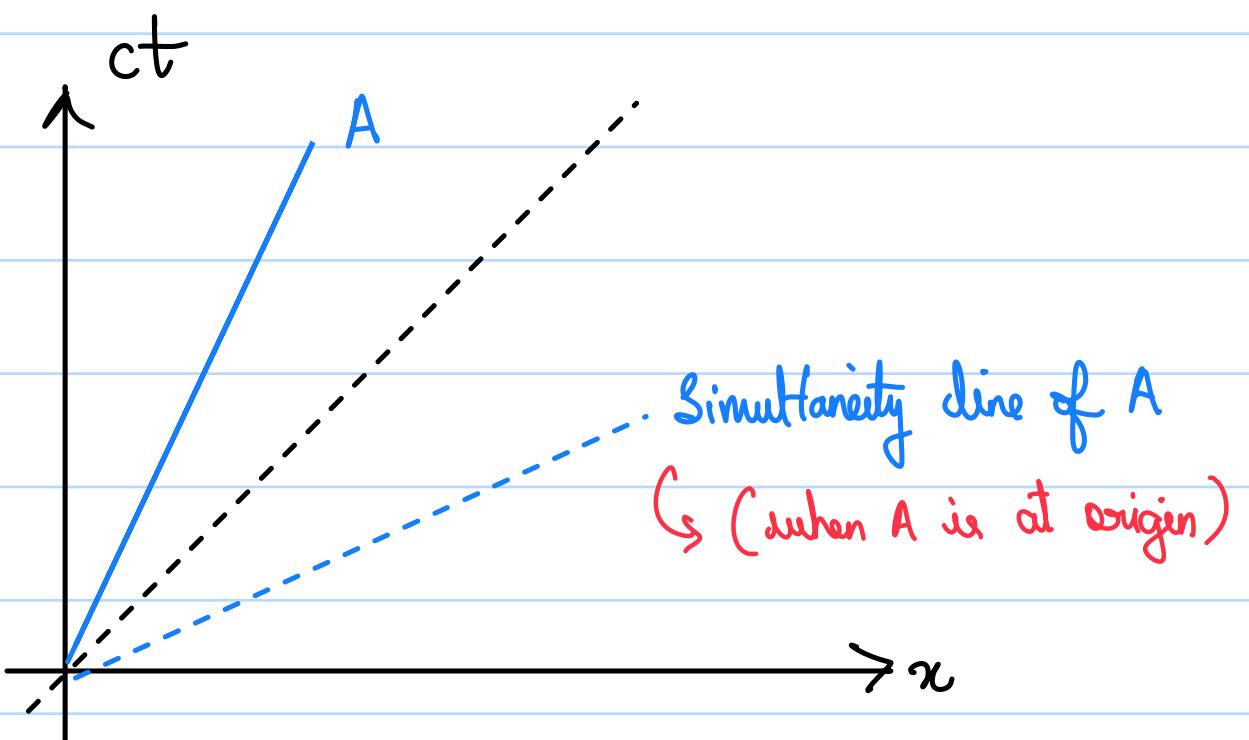
Simultaneity line : Set of all events that are simultaneous with the event 1 in the observer's frame (if observer is at rest || to light cone)

Light cone : Speed of light representation (worldlines cannot cross light cone, as nothing can cross the speed of light). Also, the set of events that are simultaneous to origin.

- The slope of the simultaneity line depends on the velocity of the object being analysed in the observer frame. Higher the speed, higher the slope, but cannot cross $m = 1$ or -1 (speed of light)

$\uparrow \rightarrow$ direction
- Length contraction and time dilation distort the object's spacetime diagram, smearing up the spontaneity line.

Let A represent a moving observer w.r.t ground. The ground's space-time diagram is as follows.



- To determine the simultaneity line of A, let velocity of A be v .

Applying Lorentz Transformation,

$$x' = \gamma(x - vt)$$

$$t' = \gamma(t - \frac{vx}{c^2})$$

Simultaneity means $t' = \text{const}$ (let it be k).

The slope of a worldline is given by $\frac{d(ct)}{dx}$.

$$\kappa = \gamma \left(t - \frac{vx}{c^2} \right)$$

$$\Rightarrow t - \frac{v}{c^2}x = \frac{\kappa}{\gamma}$$

$$\Rightarrow ct = \frac{v}{c}x + \frac{ck}{\gamma}$$

\therefore Slope of ct vs $x = \underline{\frac{v}{c}}$.

\therefore The simultaneity line of an object at (ct_0, x_0) moving at velocity v with respect to the observer is

$$(t - t_0) = \frac{v}{c} (x - x_0)$$

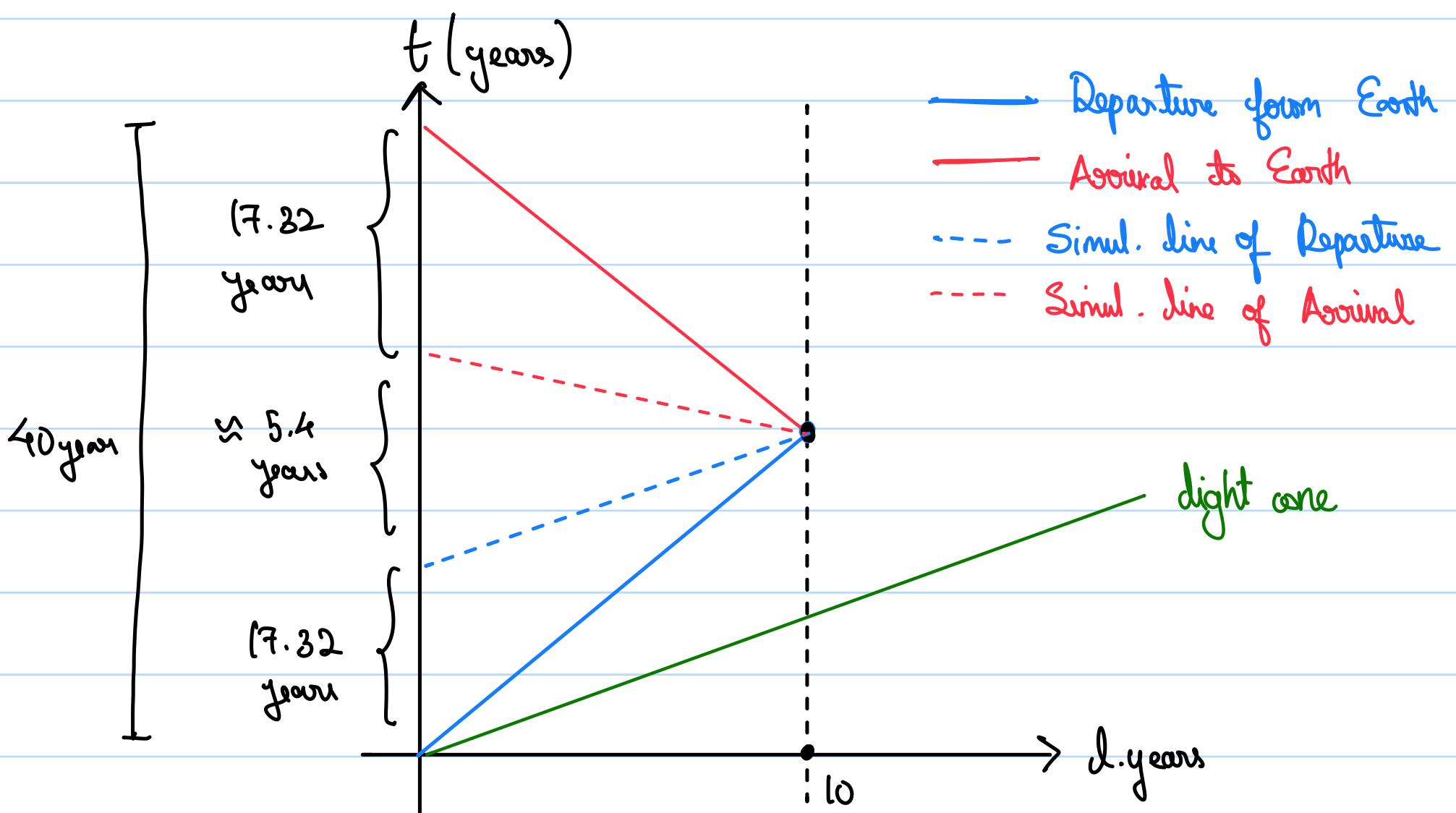
Verification: If $v = 0$, slope = 0, ie, set of all points (x, t) such that $t = t_0$. This is clearly correct.

- Solving the Twin Paradox using Spacetime Paradox :-

- Let the speed of the traveller be $0.5c$, ie, 50% of the speed of light, and let his destination be 10 light years away.

- Time taken to travel = $\frac{10}{0.5} = 20$ years } In Earth frame
 \Rightarrow Total time = 40 years

Drawing the spacetime diagram on the Earth frame,



- Here, we see why the switching of reference frames causes the difference in aging.
- The simultaneity line of the travelling twin changes when he changes direction. Therefore, there is a "line jump" with respect to Earth for the traveller. The aging of the Earth twin happens in that jump.
- The Earth twin ages 40 years. To calculate the age of the traveller twin, we can use length contraction to determine the duration of the trip as per him,

$$l' = l/\gamma = 10 \sqrt{1 - \frac{(0.5c)^2}{c^2}}$$

$$= 10 \sqrt{0.75} \approx 8.66 \text{ light years}$$

$$t = \frac{8.66}{0.5} = 17.32 \text{ years.}$$

$$\text{Total } t = 34.64$$

\therefore The traveller twin will age 34.64 years.

\Rightarrow The "time jump" is ≈ 5.4 years.

- The travelling twin does not "feel" the time jump, but the Earth twin does, hence causing the difference in aging.
- Since the time jump is +ve for all speeds $< c$, the Earth twin will always be older than the travelling twin.

 Converse: If speed is $> c$, Earth twin will be younger than traveller twin (Yay Time Travel !!!)

• Spacetime Invariance :

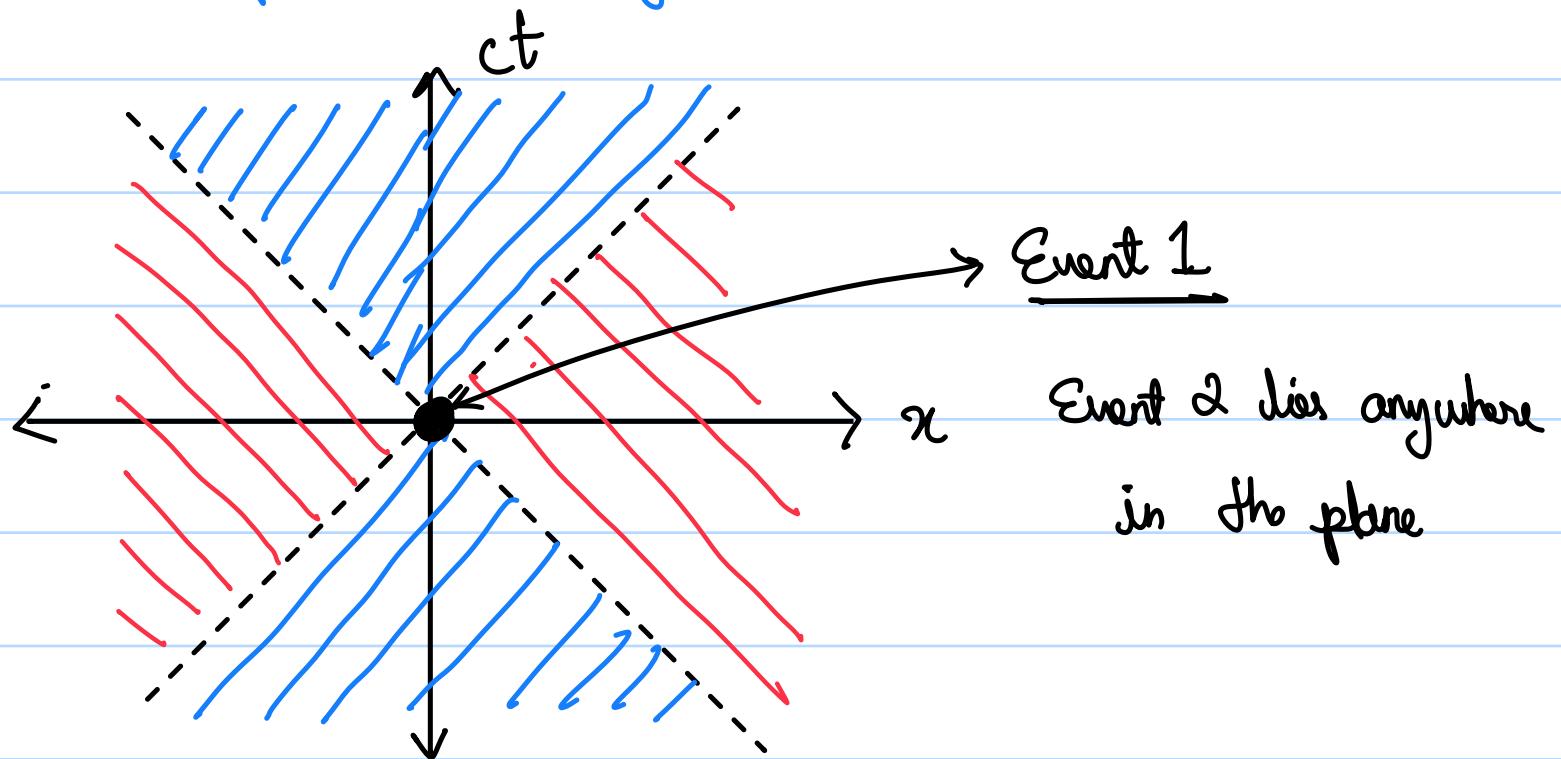
- The spacetime interval between 2 events is,

$$ds^2 = dx^2 + dy^2 + dz^2 - (cdt)^2$$

- Since our examples only have x coordinate and time,

$$ds^2 = dx^2 - (c dt)^2$$

- In the below spacetime diagram,



We know that the slope of any line is,

$$m = \frac{\Delta(ct)}{\Delta(x)} = \frac{ct}{x} \text{ at Event 2, but since need to prove for gen. case}$$

$$\Rightarrow \Delta(ct) = m \Delta(x)$$

If $m = 1 \rightarrow$ The light cone,

$$\Delta(ct) = \Delta x \Rightarrow \Delta(ct)^2 = \Delta x^2$$

$$\Rightarrow \underline{\Delta s^2 = 0}$$

Any event with $\Delta s^2 = 0$ is simultaneous to the origin event. They are termed as lightlike events.

If $m > 1 \rightarrow$ The Blue region

$$\Delta(ct) < \Delta(x)$$

$$\Rightarrow \Delta(ct)^2 < \Delta(x)^2$$

$$\Rightarrow \underline{\Delta s^2 > 0}$$

Any event which is causal to the origin event has $\Delta s^2 > 0$, but all such events with $\Delta s^2 > 0$ are not causal to origin. Such events with $\Delta s^2 > 0$ are termed as timelike events.

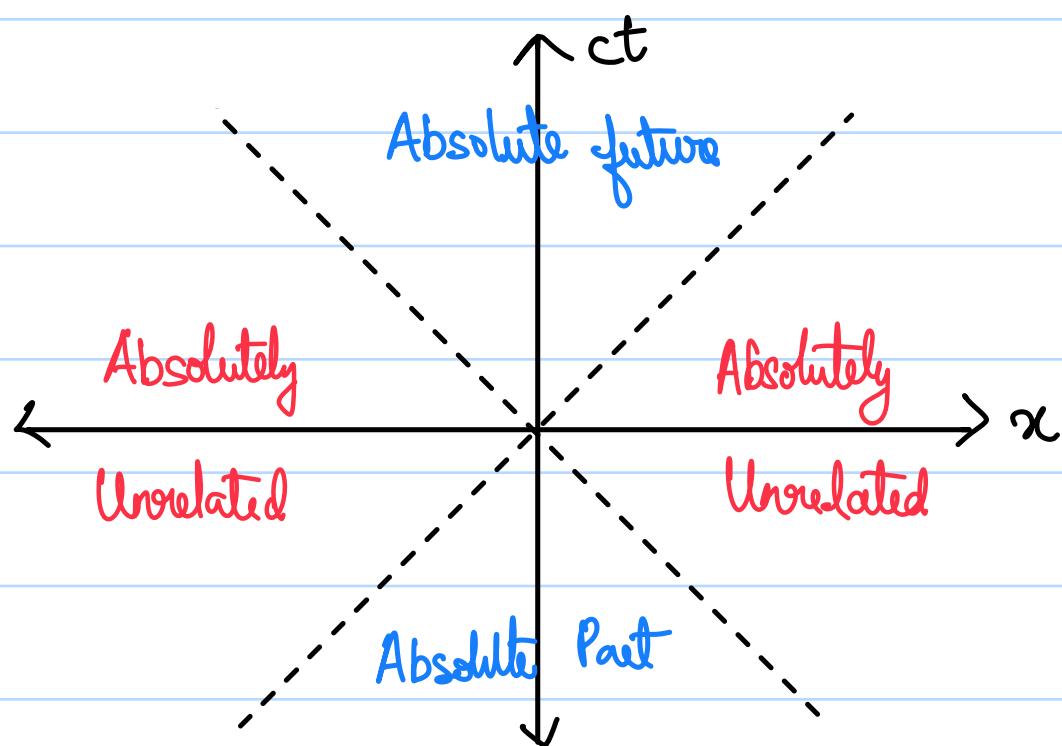
If $m < 1 \rightarrow$ The Red region

$$\Delta(ct)^2 > \Delta x^2$$

$$\Rightarrow \underline{\Delta s^2 < 0}$$

Events with $\Delta s^2 < 0$ with the origin, cannot be related to the origin. Such events are called space-like events.

- These results apply to any pair of 2 events, not only origin.

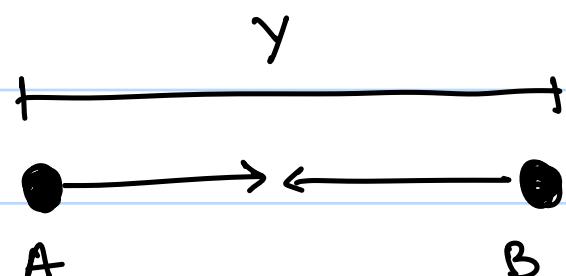


- The significance of these results lies in the fact that the spacetime interval is unaffected by the Lorentz transformation, i.e., Δs^2 is the same in all frames for any 2 events.

\therefore If 2 events are absolutely unrelated in a frame, they are absolutely unrelated in any frame

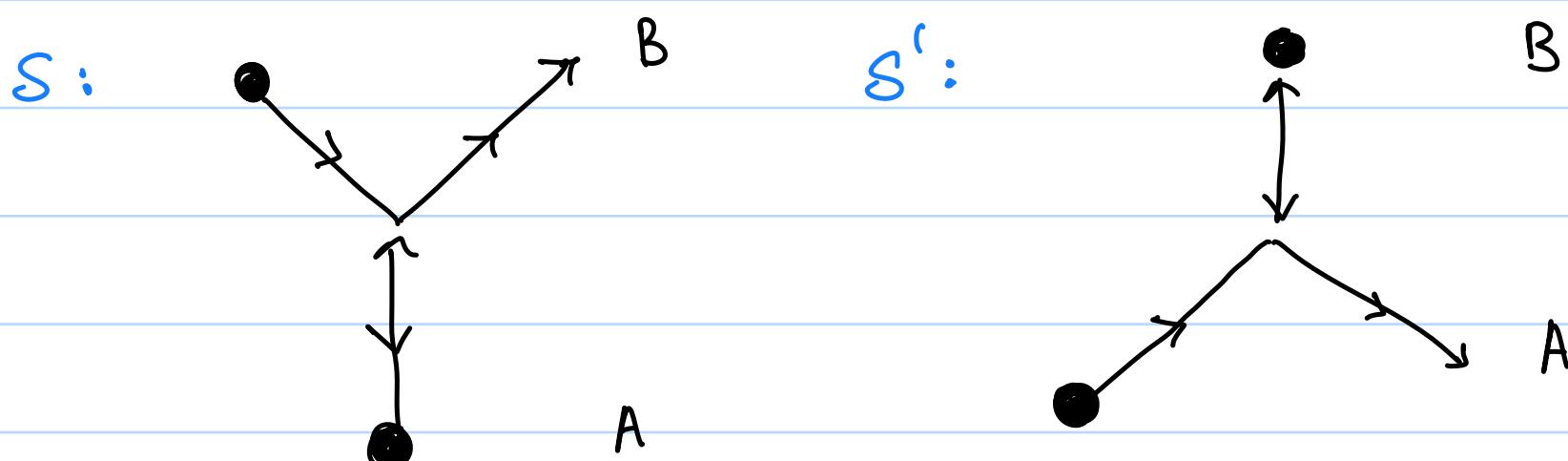
→ STR in the Real World :-

1) Relativistic Momentum :-



Let there be 2 balls A and B with velocities v_A, v_A' and v_B, v_B' , in the reference frames S and S' respectively, along the y-axis. Let S' move at v velocity along $+x$ w.r.t S.

If $v_A = v_B'$ and the balls are colliding, the collision will be,



$$\text{Time taken for A's round trip in } S = \frac{y}{v_A}$$

$$\text{" " " B's round trip in } S' = \frac{y}{v_B'} = \frac{y}{v_A} \quad \left. \right\} \text{ Let it be } T_0$$

In S, $v_B = \gamma/T$, T = Time for B's round trip in S

$$\Rightarrow T = \gamma T_0 \quad (T_0 \text{ is B's time in } S')$$

$$= T = \frac{T_0}{\sqrt{1 - v^2/c^2}}$$

$$\Rightarrow v_B = \frac{\gamma \sqrt{1 - v^2/c^2}}{T_0}$$

likewise $v_A = \frac{\gamma}{T_0}$

Using the classical definition of momentum, we get,

$$p_A = m_A v_A = m_A \left(\frac{\gamma}{T_0} \right)$$

$$p_B = m_B v_B = m_B \left(\sqrt{1 - v^2/c^2} \right) \left(\frac{\gamma}{T_0} \right)$$

