Real analysis
Assignment 2 solutions
Due: 9 November 2024 before 11:59 pm

1. (5 points) Find an example of a sequence of real numbers satisfying each set of properties:

    1. Cauchy but not monotone
    2. Monotone but not Cauchy
    3. Bounded but not Cauchy

    Solution:

    1. (2 marks) $a_n = (-1)^n \frac{1}{n}$.
    2. (1 mark) $a_n = n$.
    3. (2 marks) $a_n = (-1)^n$.

2. (5 points) Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \frac{x^3}{1+x^2}$. Show that f is continuous on $\mathbb{R}$. Is f uniformly continuous on $\mathbb{R}$?

    Solution:

    To simplify the inequalities a bit, we write

    $\frac{x^3}{1+x^2} = x - \frac{x}{1+x^2}$ (1 mark)
    For $x, y \in \mathbb{R}$, we have

    $$\begin{aligned} |f(x) - f(y)| &= |x - y - \frac{x}{1+x^2} + \frac{y}{1+y^2}| \\ &\leq |x-y| + |\frac{x}{1+x^2} - \frac{y}{1+y^2}| \end{aligned} \tag{1}$$

    (1 mark)

    Using the inequality $2|xy| \leq x^2 + y^2$, we get

    $$\begin{aligned} \frac{x}{1+x^2} - \frac{y}{1+y^2} &= \frac{x - y + xy^2 - x^2y}{|(1+x^2)(1+y^2)|} \\ &\leq \frac{x - y + xy^2 - x^2y}{|(1+x^2)(1+y^2)|} \\ &\leq \frac{x - y + xy^2 - x^2y}{|(1+x^2)(1+y^2)|} \end{aligned} \tag{2}$$

    (2 marks)

    It follows that $|f(x) - f(y)| \leq 2|x - y|$ for all $x, y \in \mathbb{R}$. Therefore f is Lipschitz continuous on $\mathbb{R}$, which implies that it is uni-formly continuous (take $\delta = \delta/2$). (1 mark)

3. (5 points) Let (a_n) and (b_n) be bounded sequences of real numbers. Define a sequence (c_n) by $c_n = a_n b_n$. Show that if $\limsup a_n$ and $\limsup b_n$ are negative, then $\limsup c_n = \liminf(a_n) \cdot \liminf(b_n)$.

    Solution:

We know that for any nonempty set $A \subset R$, we have $\sup\{-a|a \in A\} = -\inf A$. (1 mark)

Suppose $\limsup a_n$ and $\limsup b_n$ are negative. Then there exists $N \in \mathbb{N}$ such that $a_n < 0$ and $b_n < 0$ for $n \geq N$. Then $\{-a_m|m \geq n\}, \{-b_m|m \geq n\}$, and $\{c_m|m \geq n\}$ are sets of nonnegative numbers, for $n \geq N$. (2 marks)

Note that $\limsup c_n = \lim(\sup\{-a_m|m \geq n\}) \lim(\sup\{-b_m|m \geq n\}) = \lim(-\inf a_m|m \geq n) \lim(-\inf\{b_m|m \geq n\}) = (-\liminf a_n)(-\liminf b_n) = \liminf a_n \cdot \liminf b_n$. (2 marks)

4. (5 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and let $k \in \mathbb{R}$. Prove that the set $f^{-1}(k)$ is closed.

Solution:

Let $M = \mathbb{R} \setminus k$. Then M is open. (1 mark)

Since f is continuous, $f^{-1}(M) = \mathbb{R} \setminus f^{-1}(k)$ is open. (3 mark)

This implies that $f^{-1}(k)$ is closed. (1 mark)

5. (10 points) Let X be a metric space. Then show the following

   1. Any subset of a nowhere dense set is nowhere dense.
   2. The union of finitely many nowhere dense sets is nowhere dense.
   3. The closure of a nowhere dense set is nowhere dense.
   4. If X has no isolated points, then every finite set is nowhere dense.

Solution:

(a) and (c) are follow from the definition and the elementary properties of closure and interior. (2 marks)

To prove (b), it suffices to consider a pair of nowhere dense sets A_1 and A_2, and prove that their union is nowhere dense. It is convenient to pass to complements, and prove that the intersection of two dense open sets V_1 and V_2 is dense and open. (2 marks)

Then $V_1 \cap V_2$ is open, so let us prove that it is dense. Now, a subset is dense iff every nonempty open set intersects it. So fix any nonempty open set $U \subseteq X$. Then $U_1 = U \cap V_1$ is open and nonempty. And by the same reasoning, $U_2 = U_1 \cap V_2 = U \cap (V_1 \cap V_2)$ is open and nonempty as well. Since U was an arbitrary nonempty open set, we have proven that $V_1 \cap V_2$ is dense. (4 marks)

To prove (d), it suffices to note that a one-point set $\{x\}$ is open if and only if x is an isolated point of X, then use (b). (2 marks)

6. (10 points) Let (a_n) be a sequence. Let (b_n) be a nondecreasing convergent sequence of positive numbers such that $|a_{n+1} - a_n| \leq b_{n+1} - b_n$. Show that (a_n) is a Cauchy sequence.

Solution:

Observe that

$$\begin{aligned} |a_{n+m} - a_n| &\leq \sum_{j=1}^{m} |a_{n+j} - a_{n+j-1}| \\ &\leq \sum_{j=1}^{m} |b_{n+j} - b_{n+j-1}| \\ &= b_{n+m} - b_n \\ &= |b_{n+m} - b_n| \end{aligned} \tag{3}$$

(7 marks)

Since $\{b_n\}$ is Cauchy, given $\epsilon > 0$, choose $N \in \mathbb{N}$ so that $|b_{n+m} - b_n| < \epsilon$ for all $n \geq N$ and $m \in \mathbb{N}$. It then follows that $|a_{n+m} - a_n| < \epsilon$ for all such m, n which proves that $\{a_n\}$ is Cauchy. (3 marks)

7. (10 points) If f is a continuous mapping of a metric space X into a metric space Y , prove that $f(\overline{E}) \subseteq \overline{f(E)}$ for every set $E \subseteq X$. (Here $\overline{A}$ denotes the closure of set A.)

Solution:

For every $x \in E$, $f(x) \in f(E) \subseteq \overline{f(E)}$, hence $x \in f^{-1}(\overline{f(E)})$. Thus $E \subseteq f^{-1}(\overline{f(E)})$. (4 marks)

The last set must be closed as the preimage of the closed set $\overline{f(E)}$. (2 marks)

Hence it also contains $\overline{E}$. So, $\overline{E} \subset f^{-1}(\overline{f(E)})$, which implies $f(\overline{E}) \subseteq f(f^{-1}(\overline{f(E)}) \subseteq \overline{f(E)}$. (4 marks)