Signal Processing

Prof. Santosh Nannuru

Quiz 1 – 10
Quiz 2 – 10
Assn – 20
Mid – 25
Final – 35

Theory = 100

Submission – 15
Mid Exam – 10
Project / End Exam – 15

Lab = 40

Textbooks:

Signals & Systems by Oppenheim
DSP, JG Proakis

Addn: Principles of SP, BP Lathi
DSP : Sanjit K Mitra

A > 130
F < 60

Labs:

MATLAB Based
Github Submission

Tutorial: 10:30 – 11:30 Saturday

Prof: Santosh Nannuru

Signal Processing

→ Overview of NeSS :-

- Time variant - function of time
- Signals can be a function of any variable, need not be time always.
- Multi-Dimensional signals - images and videos.
- Anything that carries information is a signal.
- The course focuses more on discrete time signals, since most digital devices work in a discrete manner.
- System process signals, $x(t) \longrightarrow \text{System}(H) \longrightarrow y(t)$.
- LTI system - Linear and Time-Invariant Systems.
- Any circuit with only R, L, C will always form an LTI system, since R, L, C are all linear components.

Diodes and Transistors form non-linear systems.

- In any LTI system, the output $y(t)$ of a signal $x(t)$ is,

$$\boxed{y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau} \rightarrow \underline{\text{Convolution}}$$

$h(t)$ = impulse response of the system.

- <u>Laplace Transform</u> :-

Representation of the signal in the s / complex frequency domain.

- Poles of the system function determine the behaviour / properties of the system.

- ROC - Region in the complex s - plane where the Laplace transform converges. Is always a vertical strip, defined by the location of poles.

- <u>Fourier Series</u> :-

- Any periodic signal can be represented as a sum of sines and cosines.

→ Course Topics:-

- Fourier Analysis, Z transform
- Sampling and Quantization - DSP
- Computational Aspect - fast Fourier Transform
- Manipulating Signals - filter design.

→ Fourier Analysis:-

- Periodic and Continuous-time - Fourier Series Rep.
- Aperiodic and Continuous - time - Fourier Transform
- Aperiodic and Discrete - time - Discrete Time Fourier Transform
- Finite Length and Discrete - time - DFT

  Faster → FFT

- Fourier Series Representation:

- Representation of any periodic signal as a sum of sinusoidal harmonics.

$$x(t) = b_0 + \sum_{k \in \mathbb{N}} \left(a_k \sin(k\omega_0 t) + b_k \cos(k\omega_0 t) \right)$$

ω_0 – Angular frequency of the signal. $\left(2\pi/T\right)$

$$x(t) = \sum_{k \in \mathbb{Z}} c_k e^{jk\omega_0 t}$$

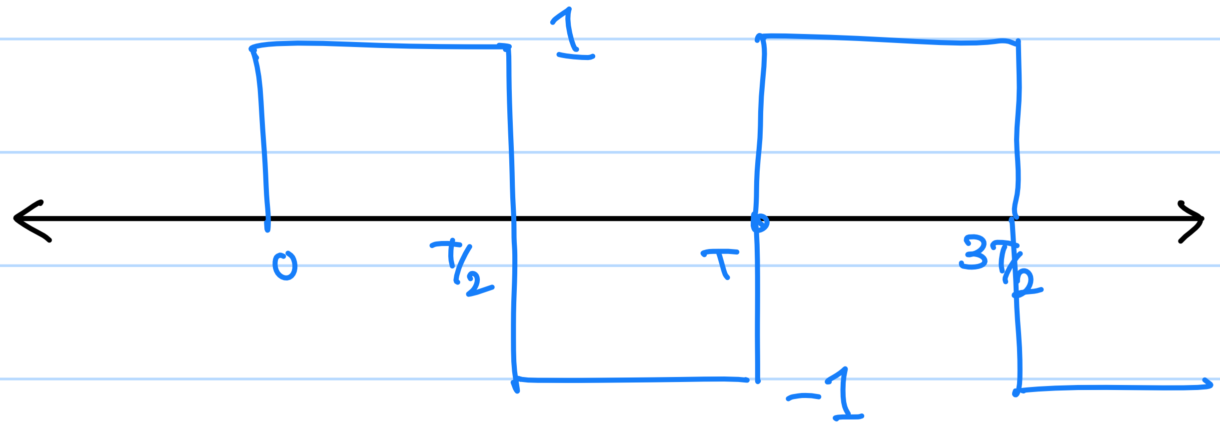
- Analysis : Computing the coefficients of the given signal
  Synthesis : Given the signal, computing the coefficients.

$$a_k = \frac{2}{T} \int_T x(t) \sin(k\omega_0 t)\, dt \quad , \quad k \neq 0$$

$$b_k = \frac{2}{T} \int_T x(t) \cos(k\omega_0 t)\, dt$$

$$c_k = \frac{1}{T} \int_T x(t)\, e^{-jk\omega_0 t}\, dt$$

Compute the FS Coeffs for a Sq. wave.



$$x(t) = \begin{cases} 1 \ , & t \in (0, T/2) \\ -1 \ , & t \in (T/2, T) \end{cases}$$

$$c_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\left[\int_0^{T/2} e^{-jk\omega_0 t}\, dt - \int_{T/2}^{T} e^{-jk\omega_0 t}\, dt\right]$$

$$= \frac{1}{T}\left(\frac{-1}{k\omega_0}\right)\left[\left[e^{-jk\omega_0 t}\right]_0^{T/2} - \left[e^{-jk\omega_0 t}\right]_{T/2}^{T}\right]$$

$$= \frac{-1}{k\cancel{T}\,\frac{2\pi}{\cancel{T}}}\left(\left[e^{-jk\left(\frac{2\pi}{T}\right)\left(\frac{T}{2}\right)} - e^{-jk\omega_0(0)}\right] - \left[e^{-jk\left(\frac{2\pi}{T}\right)T} - e^{-jk\left(\frac{2\pi}{T}\right)\left(\frac{T}{2}\right)}\right]\right)$$

$$= \frac{-1}{k2\pi}\left[\left(e^{-jk\pi} - 1\right) - \left(e^{-jk2\pi} - e^{-jk\pi}\right)\right]$$

$$= \frac{-1}{k2\pi}\left(2e^{-jk\pi} - 1 - e^{-jk2\pi}\right)$$

$$= \frac{-1}{k2\pi}\left(2e^{-jk\pi} - 2\right)$$

$$= \frac{-1}{k\pi}\left(e^{-jk\pi} - 1\right) \longrightarrow \text{0 for even } k$$

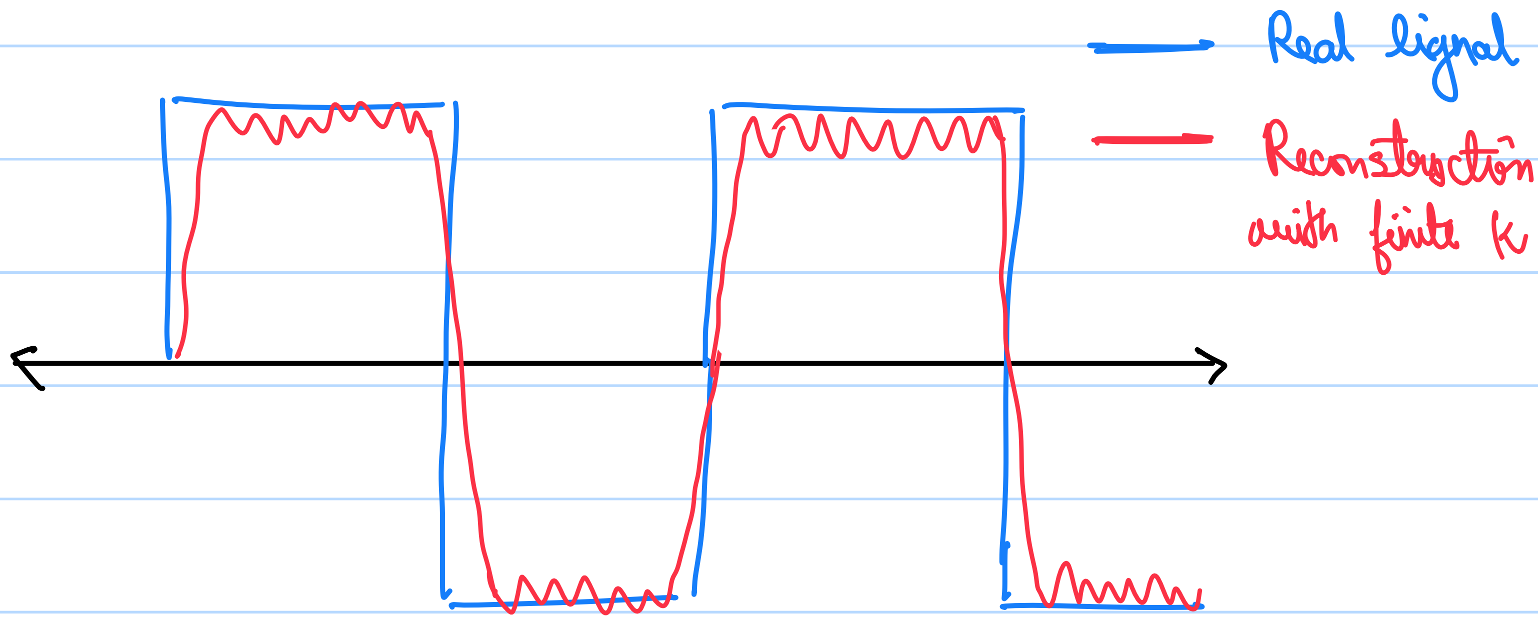
$$= \frac{1 - \cos k\pi}{k\pi} \quad \forall \text{ odd } k$$

- **Partial Reconstruction :-**

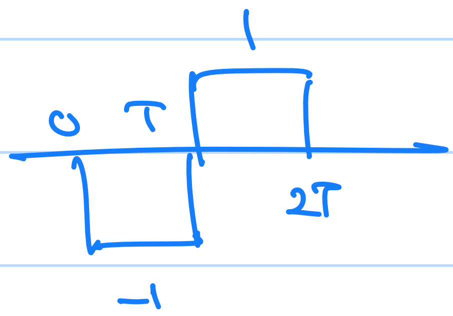
- Not all the coefficients are known to us, so only an approximation of the signal is constructed.

- Reconstruction Error $e(t) = x(t) - \hat{x}(t)$

- Discontinuous signals will have an infinite number of non-zero coefficients.



FS Coeffs of Sq. Wave Continuation,



$$a_k = \frac{2}{2T}\int_{2T} x(t)\sin(k\omega_0 t)\,dt$$

$$= \frac{1}{T}\left[\int_0^T -\sin(k\omega_0 t)\,dt + \int_T^{2T} \sin(k\omega_0 t)\,dt\right]$$

$$= \frac{1}{T}\left(\frac{1}{k\omega_0}\right)\left[\left[\cos(k\omega_0 t)\right]_0^T + \left[-\cos(k\omega_0 t)\right]_T^{2T}\right]$$

$$= \frac{1}{k\pi}\left[\left(\cos\left(k\frac{2\pi}{2T}(T)\right) - 1\right) + \left(\cos\left(k\left(\frac{2\pi}{2T}\right)T\right) - \cos\left(k\left(\frac{2\pi}{2T}\right)2T\right)\right)\right]$$

$$= \frac{1}{k\pi}\left[(\cos k\pi - 1) + (\cos k\pi - 1)\right]$$

$$a_k = \frac{2(\cos k\pi - 1)}{k\pi}$$

$$b_k = \frac{2}{2T}\int_{2T} x(t)\cos(k\omega_0 t)\, dt$$

$$= \frac{1}{T}\left[\int_0^T -\cos(k\omega_0 t)\, dt + \int_T^{2T}\cos(k\omega_0 t)\, dt\right]\left(\frac{1}{k\omega_0}\right)$$

$$= \frac{1}{T}\left[\left(-\sin(k\omega_0 t)\right)_0^T + \left(\sin(k\omega_0 t)\right)_T^{2T}\right]\left(\frac{1}{k\omega_0}\right)$$

$$= \frac{1}{T}\left[\left(0 - \cancel{\sin(k\pi)}^{\,0}\right) + \left(\cancel{\sin k\frac{2\pi}{2T}2T}^{\,0} - \cancel{\sin k\frac{2\pi}{2T}T}^{\,0}\right)\right]$$

$$= \underline{\underline{0}} \quad \forall\, k$$

- **Note:** If $a_k \in \mathbb{R}$ $\forall\, k$, then the signal is Even.

→ **Discrete-Time Signals :-** 4/8/25

- Represented by $x[n]$, where $n \in \mathbb{Z}$, n is an index where time duration is the sampling interval.

- **Unit Impulse :**

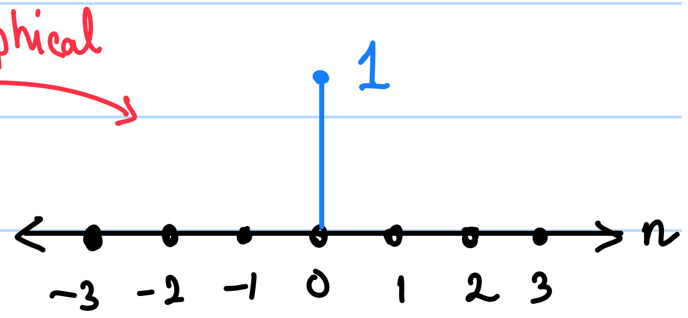
- In continuous time,

$$\delta(t) = 0 \;\forall\, t \neq 0,\ \text{s.t.} \int_{-\infty}^{\infty}\delta(t)\, dt = 1$$

- In discrete time,

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

Graphical

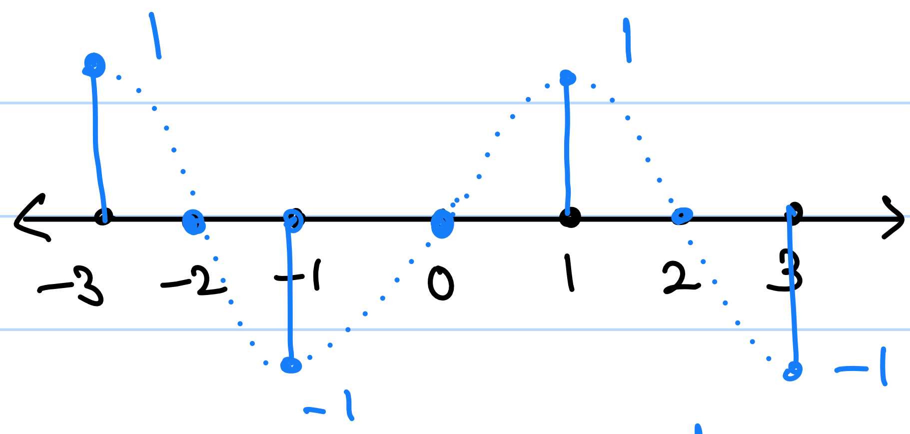


- **Unit Step:** $u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$

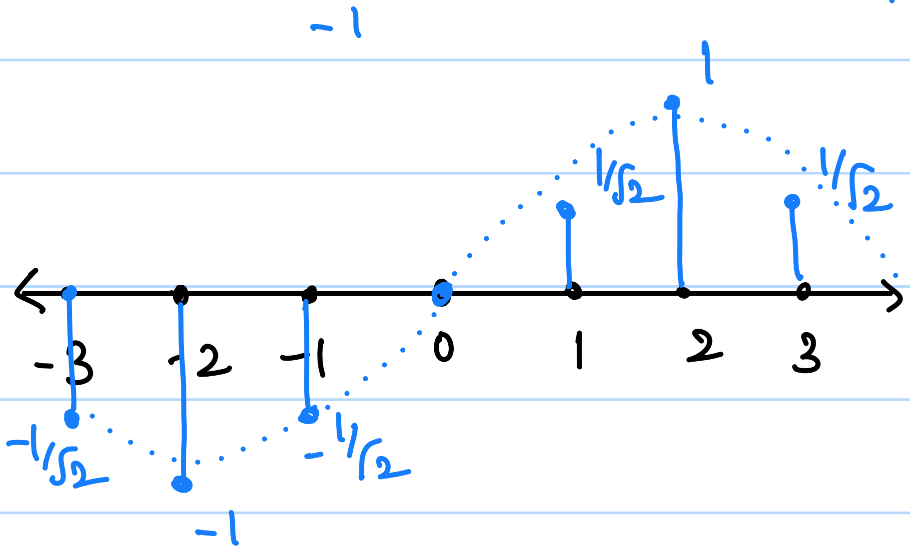
- **Exponential signal:** $x[n] = a^n, \ a \in \mathbb{R}$

Example: Plot $x[n] = \sin(\omega_0 n)$ for $\omega_0 = \frac{\pi}{2}, \frac{\pi}{4}$

1)



2)



- A signal that is periodic in continuous time may not be always be periodic in discrete time, due to the sampling frequency. ($x[n] = \sin(n)$)

- **Discrete Time Periodic Signals:-**

$x[n]$ is periodic if,

$$x[n+P] = x[n] \ \forall n, \text{ for some } P \in \mathbb{Z}$$

- **Complex Exponentials :-**

$$x[n] = e^{sn}, \quad s \in \mathbb{C}$$, i.e. $s = \sigma + j\omega$, $\sigma, \omega \in \mathbb{R}$

$$\Rightarrow \quad x[n] = e^{\sigma n}\left(\cos(\omega n) + j\sin(\omega n)\right)$$

Note: Sinusoids are periodic if $\omega_0 = \frac{2\pi}{P}$ for some $P \in \mathbb{Z}$

- **Even and Odd:**

Even : $x[n] = x[-n] \quad \forall\, n \in \mathbb{Z}$

Odd : $x[n] = -x[-n] \quad \forall\, n \in \mathbb{Z}$

→ **Energy of a Discrete Time Signal :-**

$$E = \sum_{n \in \mathbb{Z}} |x[n]|^2.$$

It may not relate to any physical quantity / property of the signal.

$$\text{Power} = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

Energy Signals : Signals that carry finite energy, and hence, zero power

Power Signals: Signals that carry finite power.

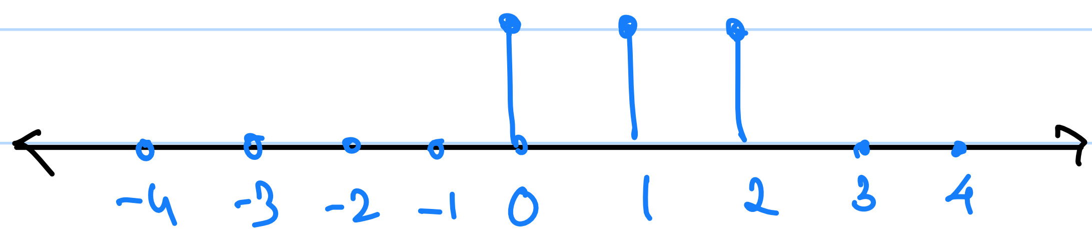
→ Time axis Manipulation for Discrete time Signals :-

- $x[n-n_0]$ – Shifts right by n_0
  $x[n+n_0]$ – Shifts left by n_0

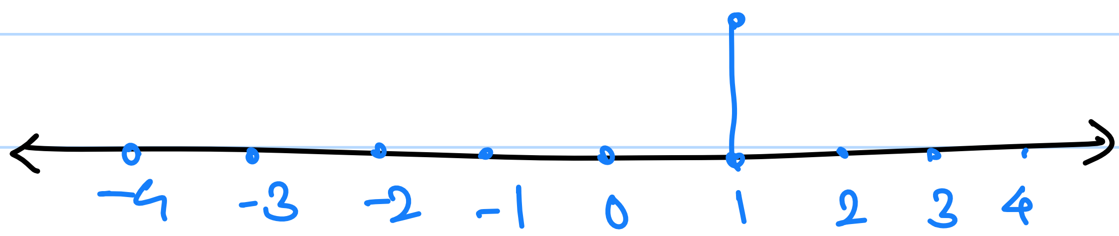
- $x[kn]$ – Only every k^{th} sample is taken

Example: $x[n] = u[n] - u[n-3]$
$y[n] = x[2n-1] = ?$

$x[n]$



$x[2n-1]$



- Also, due to the nature of discrete signals, $\omega_0 = \pi$ is the highest possible angular frequency for a sinusoid.

Beyond $\omega_0 = \pi$, the signal will be the same as those of a lower frequency, due to the properties of sinusoids.

$\cos(\pi n) = (-1)^n \longrightarrow$ Oscillation b/w 1 and -1 at the freq of sampling

$$\sin((\pi+\phi)n) = \sin(\pi n + \phi n)$$
$$= \sin \pi n \cos \phi n + \cos \pi n \sin \phi n$$
$$= (-1)^n \sin \phi n \longrightarrow \text{freq} < \pi$$

$$\cos((\pi+\phi)n) = \cos(\pi n)\cos(\phi n) - \sin(\pi n)\sin(\phi n)$$
$$= (-1)^n \cos \phi n \longrightarrow \text{freq} < \pi$$

- Signal Representation By Unit Impulses :-

$$x[n] = \sum_{m \in \mathbb{Z}} x[m]\,\delta[n-m] \quad \forall\, n \in \mathbb{Z}$$

Since $\delta[n-m] = 0 \;\forall\; n \neq m$, all the other value of the signal are nullified at each n.

→ System :

$$x[n] \longrightarrow \boxed{\text{System}} \longrightarrow y[n]$$

- Delay :

$$y[n] = x[n-\Delta] \; . \; \Delta \in \mathbb{Z}$$

- Amplitude Scaling :

$$y[n] = \alpha x[n] \,, \quad \alpha \in \mathbb{C}$$

- Integrator / Summation :

$$y[n] = \sum_{m \in -\infty}^{n} x[m]$$

- **Difference:**

$$y[n] = x[n] - x[n-1]$$

Q. $x(t) = \cos(4\pi t)$ is sampled with sampling frequency below. Identify the frequency of the obtained DT signal.

f_s: 1 Hz, 2 Hz, 4 Hz, 8 Hz

$x(t) = \cos(4\pi t)$

In Sampling, $t \longrightarrow nT$, $T = \frac{1}{f_s}$

$\Rightarrow \quad x[n] = \cos\left(\frac{4\pi n}{f_s}\right)$, $2\pi f = \frac{\cancel{4}^{2}\pi}{f_s}$

$\Rightarrow \quad \underline{\underline{f = \frac{2}{f_s}}}$

→ **Properties of Systems:-**

- **Linearity:** If x_1 gives o/p y_1, x_2 gives o/p y_2 then,

$$\alpha x_1 + \beta x_2 \longrightarrow \alpha y_1 + \beta y_2 \, . \quad \alpha, \beta \in \mathbb{C} \text{ constants}$$

- **Time Invariant:** If $x[n]$ gives o/p $y[n]$, then,

$$x[n-n_0] \longrightarrow y[n-n_0] \quad \forall \, n_0 \in \mathbb{Z}$$

Also termed as <u>shift</u> invariant.

Example. Analyze time invariance of

1) $y[n] = x[n-\Delta]$, $\Delta \in \mathbb{Z}$

2) $y[n] = x^2[n]$

3) $y[n] = x[n] + x[n-1]$

4) $y[n] = n\,x[n]$

1) $y[n] = x[n-\Delta] \longrightarrow y[n-n_0] = x[n-n_0-\Delta]$

eq.

$x[n-n_0] \xrightarrow{\text{Sys}} x[n-n_0-\Delta]$ $\therefore$ Time Invariant

2) $y[n] = x^2[n] \longrightarrow y[n-n_0] = (x[n-n_0])^2$

eq.

$y_2[n] = (x[n-n_0])^2$ $\therefore$ Time Invariant

3) $y[n] = x[n] + x[n-1] \longrightarrow y[n-n_0] = x[n-n_0] + x[n-n_0-1]$

eq

$y_2[n] = x[n-n_0] + x[n-n_0-1]$ $\therefore$ Time Invariant

4) $y[n] = n\,x[n] \longrightarrow y[n-n_0] = (n-n_0)\,x[n-n_0]$

n. eq

$y_2[n] = n\,x[n-n_0]$ $\therefore$ Time Variant

- Causality: A system is said to be causal if the output $y[n]\ \forall\, n \in \mathbb{Z}$ should only depend on $x[m]$ for $m \leq n$.

- **Stability**: If $x[n] \leq B \ \forall\, n \in \mathbb{Z}$ for some $B \in \mathbb{C}$, then $y[n] \leq B_0 \ \forall\, n \in \mathbb{Z}$, for some $B_0 \in \mathbb{C}$. Bounded Input, Bounded Output.

→ **Linear-Time Systems**:-

- LTI systems are linear and time invariant.

- **Impulse Response**: Response of the system to the impulse signal $\delta[n]$. Denoted by $h[n]$.

- **Step Response**: Response of the system to the step signal $u[n]$. Denoted by $s[n]$

$$\alpha\,\delta[n-n_0] \xrightarrow{H} \alpha\, h[n-n_0]$$

- Since we can denote any signal in the form,

$$x[n] = \sum_m \underbrace{x[m]\,\delta[n-m]}$$

scaled and delayed impulses.

$$\Rightarrow \boxed{y[n] = \sum_m x[m]\, h[n-m]}$$

Applying the above 2 results.

$x[m]$ is a constant wrt n.

$$\Rightarrow \boxed{y[n] = x[n] * h[n]}$$

- **Properties of Convolution :** ||| to multiplication

1) Commutativity : $x[n] * y[n] = y[n] * x[n]$ Proved using change of variable

2) Distributive of Addition : $x[n] * (y[n] + z[n]) = x[n] * y[n] + x[n] * z[n]$

3) Associative : $(x[n] * y[n]) * z[n] = x[n] * (y[n] * z[n])$

- **Properties of $\delta[n]$ in Convolution :**

1) $x[n] * \delta[n] = x[n]$

2) $x[n] * \delta[n-\Delta] = x[n-\Delta] \quad \forall \Delta \in \mathbb{Z}$

3) $x[n] \times \delta[n-\Delta] = \underbrace{x[\Delta]\,\delta[n-\Delta]}_{\text{Scaled impulse}} \quad \forall \Delta \in \mathbb{Z}$

→ **Description of a System :-**

1) Working description : Describe the working of the system.

2) Mathematical description : Describe the mathematical function relating input and output.

3) Impulse Response (LTI) : Describe the impulse response of the LTI system. Output can be calculated through convolution.

- **Properties of a System through Impulse Response** :-

- **Causal Systems** : $h[n] = 0 \ \forall \ n \in \mathbb{Z}^-$ for a Causal system.

$$y[n] = x[n] * h[n]$$

$$= \sum_m x[m] h[n-m]$$

should be zero

$$= \sum_{m=-\infty}^{n} x[m] h[n-m] + \sum_{m=n+1}^{\infty} x[m] h[n-m]$$

$$\Rightarrow h[n-m] = 0 \ \forall \ m \geq n+1$$

$$m - n \geq 1$$

$$n - m \leq -1 \Rightarrow n - m < 0$$

$$\Rightarrow h[r] = 0 \ \forall \ r < 0$$

- **Stable Systems** : $\sum_n |h[n]|$ should be bounded for stability

$$|y[n]| = \left| \sum_m x[m] h[n-m] \right|$$

$$\Rightarrow |y[n]| \leq \sum_m |x[m]| |h[n-m]|$$ Δle inequality

$$\Rightarrow |y[n]| \leq \sum_m B |h[n-m]| = B \sum_k |h[k]|$$

$\therefore$ If $\sum |h[k]|$ is bounded, $|y[n]|$ becomes bounded. (Sufficient Condition)

Note:

If $A \Rightarrow B$, then A is <u>sufficient</u> for B
If $A \Leftarrow B$, then A is <u>Necessary</u> for B

$A \Leftrightarrow B$, A is Necessary and sufficient (iff)

Example: Derive the impulse response of $y[n] = x[n] + \alpha y[n-1]$

$$y[n] = x[n] + \alpha y[n-1]$$
$$= x[n] + \alpha\left(x[n-1] + \alpha y[n-2]\right)$$
$$= x[n] + \alpha\left(x[n-1] + \alpha\left(x[n-2] \cdots\cdots\right.\right.$$
$$= x[n] + \alpha x[n-1] + \alpha^2 x[n-2] + \alpha^3 x[n-3] + \cdots\cdots$$
$$= \sum_{i=0}^{n} \alpha^i x[n-i]$$

[Assuming initial rest, ie $y[n] = 0 \ \forall \ n < 0$]

$$h[n] = \sum_{i=0}^{n} \alpha^i \delta[n-i]$$

$$h[n] = \alpha^n$$

→ **Linear Constant Coefficient Difference Equations** :- (LCCDE)

- First order difference equation : $y[n] = x[n] - x[n-1]$
- Generic Difference Eqn : $y[n] = \sum_{p=0}^{n_1} a_p x[n-p] + \sum_{q=1}^{n_2} b_q y[n-q]$

- If $x[n] = e^{j\omega n}$ is the i/p signal for an LTI system H,

$$y[n] = h[n] * e^{j\omega n}$$

$$= \sum_{m=-\infty}^{\infty} h[m]\, e^{j\omega(n-m)}$$

$$= e^{j\omega n} \underbrace{\sum_{m} h[m]\, e^{-j\omega m}}_{C}$$

$$y[n] = C e^{j\omega n} = \underline{\underline{C x[n]}}$$

For any LTI system H, $e^{j\omega n}$ is an Eigensignal of H. The output waveform will be a scaled and phase-shifted input waveform.

→ <u>Discrete Time Fourier Transform</u> :-

- The Discrete Time Fourier Transform of a discrete signal $x[n]$ is defined as,

$$X(\omega) = \sum_{-\infty}^{\infty} x[n]\, e^{-j\omega n}$$ ⟶ Analysis Equation

Applying this in the Eigensignal concept,

$$y[n] = e^{j\omega_0 n} \sum_{m=-\infty}^{\infty} h[m]\, e^{-j\omega_0 m}$$

$$\Rightarrow y[n] = e^{j\omega_0 n} H(\omega_0)$$

$H(\omega) \rightarrow$ FT of $h[n]$, ie, system function

- For a discrete signal,

$$X[\omega + 2\pi] = X(\omega)$$

$X(\omega)$ is continuous in ω.

- **Inverse DTFT :-**

$$x[n] \xleftrightarrow{\text{DTFT}} X(\omega)$$

- We only need the values of $X(\omega)$ for $\omega \in [-\pi, \pi]$ or $[0, 2\pi]$, due to its periodicity.

- Inverse DTFT is done as follows,

$$X(\omega)\, e^{j\omega m} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}\, e^{j\omega m}$$

$$= \int_0^{2\pi} X(\omega)\, e^{j\omega m} d\omega = \int_0^{2\pi} \sum_{n=-\infty}^{\infty} x[n]\, e^{j\omega(m-n)} d\omega$$

$$= \sum_{n=-\infty}^{\infty} x[n] \int_0^{2\pi} e^{j\omega(m-n)} d\omega$$

$$\Rightarrow \boxed{x[m] = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X(\omega)\, e^{j\omega m} d\omega} \rightarrow \text{Synthesis Equation}$$

Examples: Compute DTFT of

1) $x[n] = a^n u[n] \quad |a| < 1$

$$X(\omega) = \sum x[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \frac{1}{1 - ae^{-j\omega}}$$ [Sum of ∞ GP]

2) $x[n] = \delta[n]$

$$X(\omega) = \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\omega n}$$

$$= e^{-j\omega(0)} = 1 \quad \forall \; \omega$$

3) $x[n] = \begin{cases} 1, & -M \leq n \leq M \\ 0, & \text{otherwise} \end{cases}$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = \sum_{n=-M}^{M} e^{-j\omega n}$$

$$= \frac{e^{j\omega M}(1 - e^{-j\omega 2M})}{1 - e^{-j\omega}}$$

$$X(\omega) = \frac{e^{j\omega M} - e^{-j\omega M}}{1 - e^{-j\omega}}$$

- If $X(\omega)$ is not periodic with period 2π, it cannot be a valid DTFT of a discrete signal.

Note: $\int_{-\infty}^{\infty} \delta(-t)\, x(-t)\, dt = x(0)$

Note 2: If one waveform in time domain gives another waveform in Fourier domain, the same waveform in Fourier domain gives the latter waveform in time domain, , ie, the shape of waveforms can be switched from time to Fourier and viceversa.

→ **Properties of DTFT:-**

1) **Linearity:**

$$x_1[n] \xleftrightarrow{DTFT} X_1(\omega)$$
$$x_2[n] \xleftrightarrow{DTFT} X_2(\omega)$$

$$\Rightarrow \alpha x_1[n] + \beta x_2[n] \xleftrightarrow{DTFT} \alpha X_1(\omega) + \beta X_2(\omega)$$

Can be proved by the linear nature of Integration.

2) **Time Shift:-**

$$x[n] \xleftrightarrow{DTFT} X(\omega)$$
$$\Rightarrow x[n-n_0] \xleftrightarrow{DTFT} X(\omega)e^{-j\omega n_0}$$

Proof:

$$X'(\omega) = \sum_n x[n-n_0]\, e^{-j\omega n}$$

Take $n - n_0 = m$

$$\Rightarrow X'(\omega) = \sum_m x[m]\, e^{-j\omega(m+n_0)}$$
$$= \sum x[m]\, e^{-j\omega m} \cdot e^{-j\omega n_0}$$
$$\Rightarrow \underline{X'(\omega) = X(\omega)\, e^{-j\omega n_0}}$$

Since $|e^{-j\omega n_0}| = 1$, the magnitude spectrum is unaffected. Only the phase spectrum changes.

3) Frequency Shift :-

$$X(\omega) \xleftrightarrow{DTFT} x[n]$$

$$\Rightarrow X(\omega - \omega_0) \xleftrightarrow{DTFT} x[n]e^{j\omega_0 n}$$

4) Symmetry:

$$x[n] \xleftrightarrow{DTFT} X(\omega)$$

real	$X(\omega) = X^*(-\omega)$
real and even	real and even
real and odd	imag. and odd.