EC5.102 Information and Communication **Spring 2020**

Lecture 7: Continuous Random Variables, Mean and Variance

Instructor: Dr. Lalitha Vadlamani

We redefine the random variables for quick reference. Given an experiment and the corresponding set of possible outcomes (the sample space), a random variable associates a particular number with each outcome (See Fig. 7.1). Mathematically, a random variable is a real-valued function of the experimental outcome. Note that random variable is not a variable but a function.

$$X : \Omega \to \mathbb{R}.$$

X is a function such that preimage of every half interval $(-\infty, x]$ under X is an event in the event space $\mathcal{F}$. Since we can talk about probabilities of events, hence we can also talk about probabilities of half intervals for all $x \in \mathbb{R}$. This results in cumulative distribution function of a r.v.

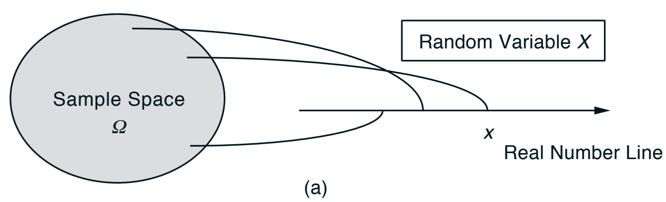

Figure 7.1: Illustrating a sample space and a random variable.

7.1 Continuous Random Variables

A random variable X is called continuous if its probability law can be described in terms of a nonnegative function f_X , called the probability density function of X, or pdf for short, which satisfies

$$P(X \in B) = \int_B f_X(x)dx \tag{7.1}$$

for every subset B of the real line. In particular, the probability that the value of X falls within a interval $[a, b]$ is given by

$$P(a \leq X \leq b) = \int_a^b f_X(x)dx \tag{7.2}$$

Also note that based on the above definition, $P(X = a) = \int_a^a f_X(x) = 0$. Thus, for a continuous random variable, the probability of the random variable taking any specific value is zero (unlike the discrete random variable).

A function $f_X(.)$ is a valid pdf if

- $f_X(x) \geq 0$, for every x

- $$\int_{-\infty}^{-\infty} f_X(x)dx = 1, \tag{7.3}$$

  i.e., the area under the graph of pdf must be equal to 1.

To interpret the pdf, note that for an interval $[x, x+\delta]$ with very small δ, we have

$$P([x, x+\delta]) = \int_{x}^{x+\delta} f_X(t)dt \approx f_X(x).\delta \tag{7.4}$$

Even though pdf is used to calculate event probabilities, $f_X(x)$ is not the probability of any particular event. In particular, it is not restricted to be less than or equal to one.

7.1.1 Relation between Cumulative Distribution Function and Probability Density Function

Cumulative distribution function (cdf) can be expressed in terms of probability density function (pdf) as follows:

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(x)dx$$

Pdf can be expressed in terms of cdf as follows:

$$f_X(x) = \frac{dF_X(x)}{dx}$$

7.1.2 Uniform Random Variable

A gambler spins a wheel of fortune, continuous calibrated between 0 and 1, and observes the resulting number. Assuming that all subintervals of $[0, 1]$ of the same length are equally likely, this experiment can be modelled in terms of a random variable X with pdf

$$f_X(x) = \begin{cases} 1 & 0 \leq x \leq 1, \\ 0 & \text{otherwise} \end{cases}. \tag{7.5}$$

More generally, we can consider a random variable X that takes values in an interval $[a, b]$, and again assume that all subintervals of the same length are equally likely. We refer to this random variable as uniform random variable. Its pdf has the form

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b, \\ 0 & \text{otherwise} \end{cases}. \tag{7.6}$$

Example 7.1. *Quantization discretizes a signal in its amplitude as well and performed after the sampling process. Consider a quantizer which does the following operation:*

$$Q(x) = \delta round\left[\frac{x}{\Delta}\right].$$

The difference of the actual value and the quantized value $x - Q(x)$ is referred to as quantization error (or quantization noise), which is modelled as a uniform random variable in the range $[-\frac{\Delta}{2}, \frac{\Delta}{2}]$.

7.2 Mean and Variance

The expected value or mean $E(X)$ of a discrete random variable with pmf $p_X(.)$ is defined as

$$E(X) = \sum_{x \in \mathcal{X}} x p_X(x) \tag{7.7}$$

The expected value or mean $E(X)$ of a continuous random variable with pdf $f_X(.)$ is defined as

$$E(X) = \int_{-\infty}^{\infty} x f_X(x) dx \tag{7.8}$$

If X is a random variable, then $f(X) = f \circ X$ is also a random variable, which is a composition of functions X and f. The mean of function of a random variable is given by

$$E(f(X)) = \sum_{x \in \mathcal{X}} f(x) p_X(x).$$

The variance $V(X)$ of a discrete random variable with pmf $p_X(.)$ is defined as

$$V(X) = E[(X - E(X))^2] = \sum_{x \in \mathcal{X}} (x - \mu)^2 p_X(x), \tag{7.9}$$

Where $\mu = E(X)$.

The variance $V(X)$ of a continuous random variable with pff $f_X(.)$ is defined as

$$V(X) = E[(X - E(X))^2] = \int_{-\infty}^{\infty} (x - \mu)^2 f_X(x) dx, \tag{7.10}$$

Where $\mu = E(X)$.

We will show below that variance of a r.v. can also be expressed as follows:

$$V(X) = E(X^2) - (E(X))^2. \tag{7.11}$$

$$\begin{aligned}
V(X) &= E[(X - E(X))^2] \\
&= \sum_{x \in \mathcal{X}} (x - E(X))^2 p_X(x) \\
&= \sum_{x \in \mathcal{X}} (x - (\sum_{x \in \mathcal{X}} x p_X(x)))^2 p_X(x) \\
&= \sum_{x \in \mathcal{X}} x^2 p_X(x) - 2(\sum_{x \in \mathcal{X}} x p_X(x))^2 + (\sum_{x \in \mathcal{X}} x p_X(x))^2 \\
&= E(X^2) - (E(X))^2.
\end{aligned}$$

7.2.1 Mean and Variance of uniform random variable

Mean:

$$
\begin{aligned}
E[X] &= \int_{-\infty}^{\infty} x f_X(x) dx && (7.12)\\
&= \int_a^b x.\frac{1}{b-a} dx && (7.13)\\
&= \frac{a+b}{2}. && (7.14)
\end{aligned}
$$

To calculate variance, we first calculate the second moment as

$$
\begin{aligned}
E[X^2] &= \int_a^b \frac{x^2}{b-a} dx && (7.15)\\
&= \frac{b^3-a^3}{3(b-a)} && (7.16)\\
&= \frac{a^2+ab+b^2}{3}. && (7.17)
\end{aligned}
$$

Variance is then calculated as

$$
var(X) = E[X^2] - (E[X])^2 = \frac{a^2+ab+b^2}{3} - \frac{(a+b)^2}{4} = \frac{(b-a)^2}{12}. \tag{7.18}
$$

7.2.2 Exponential Random Variable

An exponential random variable has pdf of the form

$$
f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0, \\ 0 & \text{otherwise} \end{cases}. \tag{7.19}
$$

Where λ is a positive parameter characterising the pdf. This is a valid pdf because

$$
\int_{-\infty}^{\infty} f_X(x) = \int_0^{\infty} \lambda e^{-\lambda x} dx = -e^{-\lambda x}\Big|_0^{\infty} = 1. \tag{7.20}
$$

An exponential random variable can be a very good model for the amount of time until a piece of equipment breaks down, until a light bulb burns out, or until an accident occurs. It is also used to model job completion times when servers are running jobs.

7.2.3 Mean and Variance of exponential random variable

The mean of exponential random variable is calculated below:

$$
\begin{aligned}
E[X] &= \int_0^{\infty} x\lambda e^{-\lambda x} dx \\
&= (-xe^{-\lambda x})\Big|_0^{\infty} + \int_0^{\infty} e^{-\lambda x} dx \\
&= \frac{1}{\lambda}.
\end{aligned}
$$

To calculate the variance, we first calculate $E(X^2)$ as follows:

$$\begin{aligned} E[X^2] &= \int_0^\infty x^2 \lambda e^{-\lambda x} dx \\ &= (-x^2 e^{-\lambda x})\Big|_0^\infty + \int_0^\infty 2x e^{-\lambda x} dx \\ &= \frac{2}{\lambda^2}. \end{aligned}$$

Thus, we have

$$V(X) = E(X^2) - (E(X))^2 = \frac{1}{\lambda^2}$$

7.2.4 Gaussian Random Variable

A continuous random variable X is said to be normal or Gaussian if it has a pdf of the form

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}, \tag{7.21}$$

where μ and σ are two scalar parameters characterizing the PDF, with σ assumed non-negative. μ is the mean of Gaussian r.v. and σ^2 is the variance of Gaussian r.v.

Gaussian random variable is used to model receiver noise (thermal noise) in communication systems. This is because the noise is the effect of addition of a large number of independent random variables. Based on central limit theorem in probability theory, it is known the sum of a large number of independent (and identically distributed) random variables is a Gaussian random variable.

It can be verified that the following normalization property holds

$$\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-(x-\mu)^2/2\sigma^2} dx = 1 \tag{7.22}$$

Proof. Consider the proof for $\mu = 0$. Similar proof follows for $\mu \neq 0$ also.

$$LHS = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-x^2/2\sigma^2} dx \tag{7.23}$$

$$= \frac{2}{\sqrt{2\pi}\sigma} \int_0^{\infty} e^{-x^2/2\sigma^2} dx \tag{7.24}$$

By using change of variables $y = \frac{x}{\sigma}$, we get

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-y^2/2} dy. \tag{7.25}$$

Now consider

$$\left(\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy\right)^2 = \left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx\right)\left(\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy\right) \tag{7.26}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{-(x^2+y^2)}{2}} dx dy \tag{7.27}$$

By changing into polar co-ordinates i.e using $x = rcos\theta$, $y = rsin\theta$, we get

$$
\begin{aligned}
&= \int_0^{2\pi}\int_0^{\infty} re^{\frac{-r^2}{2}}drd\theta && (7.28)\\
&= -2\pi e^{\frac{-r^2}{2}}|_0^{\infty} && (7.29)\\
&= 2\pi && (7.30)
\end{aligned}
$$

From which, we get

$$
\begin{aligned}
\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}dy &= \sqrt{2\pi} && (7.31)\\
\int_{0}^{\infty} e^{-\frac{y^2}{2}}dy &= \sqrt{\frac{\pi}{2}} && (7.32)
\end{aligned}
$$

By substituting (7.32) in (7.25), we get LHS=1=RHS which completes the proof. □

7.2.5 Mean and Variance of Gaussian random variable

The mean and the variance can be calculated to be

$$E[X] = \mu, \quad var(X) = \sigma^2. \tag{7.33}$$

Note that the PDF is symmetric around μ, so its mean must be μ. The variance is given by

$$var(X) = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}(x-\mu)^2 e^{-(x-\mu)^2/2\sigma^2}dx. \tag{7.34}$$

Using the change of variables $y = (x-\mu)/\sigma$ and integration by parts, we have

$$
\begin{aligned}
var(X) &= \frac{\sigma^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} y^2 e^{-y^2/2}dy && (7.35)\\
&= \frac{\sigma^2}{\sqrt{2\pi}}(-ye^{-y^2/2})|_{-\infty}^{\infty} + \frac{\sigma^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-y^2/2}dy && (7.36)\\
&= \frac{\sigma^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-y^2/2}dy && (7.37)\\
&= \sigma^2. && (7.38)
\end{aligned}
$$

The last equality above is obtained by the fact

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-y^2/2}dy = 1. \tag{7.39}$$