

RA Assignment 1

Sricharan Vinoth Kumar

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Question 1

It is given that

$$a_n - a_{n-2} \rightarrow 0 \text{ as } n \rightarrow \infty \quad (1)$$

Therefore, it is implied that the odd-numbered terms and the even-numbered terms of the sequence are very close to each other after a certain number of terms. That is, for any $\epsilon > 0$,

$$|a_n - a_{n-2}| < \epsilon \quad \forall n \geq N_0, N_0 \in \mathbb{N} \quad (2)$$

If we take $\{e_n\}$ and $\{o_n\}$ to be the sequence of even numbered terms and odd numbered terms of $\{a_n\}$, using the above result, we can state that $\{e_n\}$ and $\{o_n\}$ are Cauchy sequences.

Therefore, assuming that the matrix space to be closed, these sequences are convergent.

Since these sequences are convergent, they must be bounded. Let the bounds of $\{e_n\}$ and $\{o_n\}$ be $|E|$ and $|O|$. E and O will be finite numbers.

So,

$$b_n = \frac{a_n - a_{n-1}}{n} \leq \frac{|E| + |O|}{n} \quad (3)$$

Since $\frac{|E|+|O|}{n} \rightarrow 0$ (as the numerator is finite), by the inequality,

$$b_n \rightarrow 0 \quad (4)$$

Question 2

Since n is large, $\frac{k}{n^2}$ will be a small number. Therefore, using the binomial approximation, we get,

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\sqrt{1 + \frac{k}{n^2}} - 1 \right) &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(1 + \frac{k}{2n^2} - 1 \right) \\
 &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\frac{k}{2n^2} \right) \\
 &= \lim_{n \rightarrow \infty} \left(\frac{1}{2n^2} \right) \left(\sum_{k=1}^n k \right) \\
 &= \lim_{n \rightarrow \infty} \left(\frac{1}{2n^2} \right) \left(\frac{n(n+1)}{2} \right) \\
 &= \lim_{n \rightarrow \infty} \left(\frac{1}{4} \right) \left(\frac{n+1}{n} \right) \\
 &= \lim_{n \rightarrow \infty} \left(\frac{1}{4} \right) \left(1 + \frac{1}{n} \right) \\
 &= \frac{1}{4} \\
 \therefore \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\sqrt{1 + \frac{k}{n^2}} - 1 \right) &= \frac{1}{4}
 \end{aligned}$$

Question 3

It is given that $a_n \geq 1$ and,

$$\left(a_n + \frac{1}{a_n} \right) \text{ is convergent to (let's say) } L \tag{1}$$

By AM-GM inequality, we get,

$$\left(a_n + \frac{1}{a_n} \right) \geq 2 \tag{2}$$

$$\implies L \geq 2$$

Assume $a_n \rightarrow A$ such that $A \notin \mathbb{R}$, that is, a_n is divergent. Then,

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \left(a_n + \frac{1}{a_n} \right) &= \lim_{n \rightarrow \infty} a_n + \frac{1}{\lim_{n \rightarrow \infty} a_n} \\
 \implies L &= A + \frac{1}{A} \\
 \implies A^2 + 1 - LA &= 0
 \end{aligned} \tag{3}$$

In equation (3), we get a quadratic equation in A . The discriminant of this equation, let's say, D must be such that $D < 0$. for our assumption to hold.

$$D = L^2 - 4(1)(1) \tag{4}$$

$$\text{W.k.t } L \geq 2 \implies L^2 \geq 4$$

$$\implies D \geq 0 \tag{5}$$

But this is a contradiction with what was stated previously. Therefore our assumption is incorrect

$$\begin{aligned}\therefore A &\in \mathbb{R} \\ \therefore a_n &\text{ is convergent}\end{aligned}$$

Question 4

We know that

$$\left(1 + \frac{1}{n}\right)^n \rightarrow e \text{ as } n \rightarrow \infty \quad (1)$$

Let $\{x_n\}$ be any sequence such that, $\{x_n\} \rightarrow \infty$ as $n \rightarrow \infty$, ie, $\{x_n\}$ is a divergent sequence. Also, let us define a function $f: \mathbb{R} \rightarrow \mathbb{R}$ as

$$f(x) = \left(1 + \frac{1}{x}\right)^x \quad (2)$$

Using equation (1), we can see that,

$$\{f(x_n)\} \rightarrow e \text{ as } n \rightarrow \infty \quad (3)$$

Since $\{x_n\}$ is any divergent sequence, using the sequential criterion theorem of limits, we can state that,

$$\lim_{x \rightarrow \infty} f(x) = e \quad (4)$$

$$\therefore \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e \quad (5)$$

Question 5

$$\lim_{x \rightarrow 0} \frac{(1+x)^a - 1}{x} = a$$

Using Binomial Theorem, we get,

$$\begin{aligned}(1+x)^a &= 1^a + ax + \frac{a(a-1)}{2}x^2 + \dots + x^a \\ \Rightarrow (1+x)^a &= 1 + ax + \frac{a(a-1)}{2}x^2 + \dots + x^a \\ \Rightarrow (1+x)^a - 1 &= ax + \frac{a(a-1)}{2}x^2 + \dots + x^a \\ \Rightarrow \frac{(1+x)^a - 1}{x} &= a + \frac{a(a-1)}{2}x + \dots + x^{a-1} \\ \therefore \lim_{x \rightarrow 0} \frac{(1+x)^a - 1}{x} &= a\end{aligned}$$

Question 6

The summation $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is known as the Basel Problem. Using Euler's approach to this problem.

$$\sin(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \quad (1)$$

$$\Rightarrow \frac{\sin(x)}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots \dots \dots (2)$$

In equation (2), we can see that the coefficient of $x^2 = \frac{-1}{3!}$. Using the Weiestrauss Factorization Theorem to write $\frac{\sin(x)}{x}$ as a linear product of its roots, we get,

$$\frac{\sin(x)}{x} = \left(1 - \frac{x}{\pi}\right) \left(1 + \frac{x}{\pi}\right) \left(1 - \frac{x}{2\pi}\right) \left(1 + \frac{x}{2\pi}\right) \dots \dots \dots (3)$$

If we take all the x^2 terms of equation (3), we get the x^2 coefficient of the equation to be,

$$-\left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2}\right) = -\frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} (4)$$

By comparing equation (4) with the result of equation (2), we get,

$$\begin{aligned} -\frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} &= \frac{-1}{3!} \\ \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} &= \frac{\pi^2}{6} \end{aligned} (5)$$

$$\begin{aligned} \frac{\pi^2}{6} &= \frac{9.8696\dots}{6} = \frac{29.6088\dots}{18} \\ \Rightarrow \frac{29}{18} &< \frac{\pi^2}{6} < \frac{30}{18} \end{aligned} (6)$$

$$\therefore \frac{29}{18} < \sum_{n=1}^{\infty} \frac{1}{n^2} < \frac{31}{18}$$

Question 7

$$\begin{aligned} n^4 + n^2 + 1 &= n^4 + 2n^2 + 1 - n^2 \\ &= (n^2 + 1)^2 - n^2 \\ &= (n^2 + 1 - n)(n^2 + 1 + n) \\ \Rightarrow \sum_{n=1}^{\infty} \frac{n}{n^4 + n^2 + 1} &= \sum_{n=1}^{\infty} \frac{n}{(n^2 + 1 - n)(n^2 + 1 + n)} \end{aligned}$$

Using Partial Fractions Method,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{n}{(n^2 + 1 - n)(n^2 + 1 + n)} &= \sum_{n=1}^{\infty} \frac{\frac{1}{2}((n^2 + 1 - n) - (n^2 + 1 + n))}{(n^2 + 1 - n)(n^2 + 1 + n)} \\ &= \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{1}{n^2 + 1 - n} - \frac{1}{n^2 + 1 + n} \right) \end{aligned}$$

To simplify the expression, take,

$$\begin{aligned} \{a_n\} &= \frac{1}{n^2 + 1 - n} \\ \{b_n\} &= \frac{1}{n^2 + 1 + n} \end{aligned}$$

We can observe that,

$$\begin{aligned} a_{n+1} &= \frac{1}{(n+1)^2 + 1 - (n+1)} \\ &= \frac{1}{n^2 + 1 + n} \\ &= b_n \end{aligned}$$

Therefore, the given summation simplifies to,

$$\begin{aligned} \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{1}{n^2 + 1 - n} - \frac{1}{n^2 + 1 + n} \right) &= \frac{1}{2} (a_1 - a_2 + a_2 - a_3 \dots) \\ &= \frac{1}{2} (a_1) = \frac{1}{2} \\ \Rightarrow \sum_{n=1}^{\infty} \frac{n}{n^4 + n^2 + 1} &= \frac{1}{2} \end{aligned}$$

Question 8

1

To test the convergence of $\sum_{n=1}^{\infty} \frac{a^n}{(n!)^{\frac{1}{n}}}$, we can use the Ratio test,

$$\lim_{n \rightarrow \infty} \frac{\frac{a^{n+1}}{((n+1)!)^{\frac{1}{n+1}}}}{\frac{a^n}{(n!)^{\frac{1}{n}}}} \quad (\text{Let it be } L) \quad (0)$$

Simplyfying the expression, we get

$$\lim_{n \rightarrow \infty} \frac{a(n!)^{\frac{1}{n}}}{(n+1)!^{\frac{1}{n+1}}} \quad (0)$$

As n becomes very large, the ratio between $(n!)^{\frac{1}{n}}$ and $(n+1)!^{\frac{1}{n+1}}$ will tend to 1, as the change in value of these terms with increasing n, reduces. Therefore we get $L = a$ For the series to be convergent, $L < 1$.

Therefore, the given series is convergent for $0 < a < 1$ and divergent otherwise

2

To observe the behavior of the term $\left(1 + \frac{1}{n}\right)^n$, for large n, we know from Q4 that

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e \quad (0)$$

Therefore, for large terms, the behavior of the series is same as $e \sum_{n=1}^{\infty} a^n$

Since the series is a Geometric Series, we can say that the series is convergent if $a < 1$ If $a = 1$, the sequence tends to e when n becomes large, which means that the series will not converge. \therefore The given series is convergent for $0 < a < 1$ and divergent otherwise