# Introduction to probability and random variables

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# Recap of the previous class

# Recap: Probability space

- Review: Set theory and elementary concepts in probability (Bayes theorem, total probability law, Axioms of probability etc)
- A probability space consists of three elements $(\Omega, \mathcal{F}, P)$:
  1. Sample space $\Omega$: Set of all possible outcomes
  2. Event space $\mathcal{F}$: Collection of sets outcomes in $\Omega$ such that
     - ★ Contains both empty set $\phi$ and $\Omega$
     - ★ Closed under complement: If $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$
     - ★ Closed under countable union: If $A_1, A_2, \ldots \in \mathcal{F}$ then $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$
  3. Probability measure $P(.)$: Function from event space $\mathcal{F}$ to $[0, 1]$
- **Think: Probability space does capture "essential" model of a random experiment!**

# Motivation to random variables

# Example of rolling two dice

- Example of rolling two dice where we are interested in the sum of two dice.
- Suppose $X =$ sum of two dice. Then we have

$$\Omega = \Big\{(1,1),(1,2),\ldots,(1,6) \\ (2,1),(2,2),\ldots,(2,6) \\ \vdots \\ (6,1),(6,2),\ldots,(6,6)\Big\} \quad \xrightarrow{X} \quad \Omega' = \big\{2,3,\ldots,12\big\}$$

- Suppose $\mathcal{F}$ and $\mathcal{F}'$ are power sets of $\Omega$ and $\Omega'$ respectively.
- Can you write down $P'$?
- We have a map $X$ given by

$$X : (\Omega, \mathcal{F}, P) \to (\Omega', \mathcal{F}', P')$$

- For our application, it is more convenient to work with $(\Omega', \mathcal{F}', P')$ than $(\Omega, \mathcal{F}, P)$.

# Example of choosing two real numbers from $[0, 1]$

- Choose any two real numbers from $[0, 1]$
  - What will be $\Omega$?
  - What will be $\mathcal{F}$? (to be discussed soon)
  - What will be $P$? (e.g. choose any number with uniform probability)
- We are interested only in addition of these two numbers.
  - What will be $\Omega'$?
  - What will be $\mathcal{F}'$?
  - What will be $P'$? (Will it be uniform? Yes/no?)
- We have a map $X$ given by

$$X : (\Omega, \mathcal{F}, P) \rightarrow (\Omega', \mathcal{F}', P')$$

- It is more convenient to work with $(\Omega', \mathcal{F}', P')$ than $(\Omega, \mathcal{F}, P)$.

# Motivation to random variables

- One can write down the event space $\mathcal{F}$, when $\Omega$ has finite entries.
- What to do when $\Omega$ has infinite (countable/uncountable) entries?
- Given a random experiment with associated $(\Omega, \mathcal{F}, P)$, it is sometimes difficult to deal directly with $\omega \in \Omega$. Example: Rolling a dice twice
- Depending on the experiment, $\Omega$ will be different.
- Can we come up with a general platform which is independent of the choice of a particular $\Omega$?
- It is desirable and convenient to study probability space and related advanced concepts using this general platform!

Answer: Random Variables (rv or RV)

# Definition of random variables

# Random variables

- Recall example of rolling two dice.
  - Suppose we are only interested in 'sum of two dice'.
  - It is convenient to consider the map $X : (\Omega, \mathcal{F}, P) \to (\Omega', \mathcal{F}', P')$
- For random variables, we consider "special" $\Omega', \mathcal{F}'$ and the corresponding induced probability measure $P'$.
  - $\Omega'$ will be the set of real numbers, denoted by $\mathbb{R}$.
  - $\mathcal{F}'$ will be Borel $\sigma$-algebra, denoted by $\mathcal{B}(\mathbb{R})$. (This is an advanced level topic. We will not go into the details).
  - $P'$ will be the corresponding induced probability measure, denoted by $P_X$.
- A random variable $X$ is a map given by

$$X : (\Omega, \mathcal{F}, P) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}), P_X)$$

(Map $X$ needs to satisfy some more conditions! To be discussed soon.)

# Borel $\sigma$-algebra

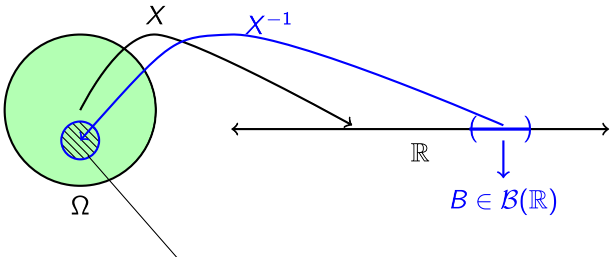
- Borel $\sigma$-algebra $\mathcal{B}(\mathbb{R})$:

  If $\Omega = \mathbb{R}$, then $\mathcal{B}(\mathbb{R})$ is the event space generated by open sets of the form $(a, b)$ where $a \leq b$ and $a, b \in \mathbb{R}$.

- $\mathcal{B}(\mathbb{R})$ contains unions and intersections of the intervals of the form

$$[a, b]$$
$$[a, b)$$
$$(a, \infty)$$
$$[a, \infty)$$
$$(-\infty, b]$$
$$(-\infty, b)$$

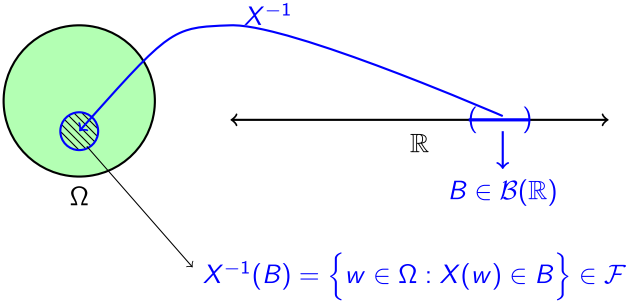
# Random variables




$$X^{-1}(B) := \Big\{ w \in \Omega : X(w) \in B \Big\} \in \mathcal{F}$$

- $\Omega \xrightarrow{X} \mathbb{R}$, $\mathcal{F} \xrightarrow{X} \mathcal{B}(\mathbb{R})$, and $P(.) \xrightarrow{X} P_X(.)$
- Care must be taken such that the events you consider in the new event space $\mathcal{B}(\mathbb{R})$ are also valid events included in $\mathcal{F}$.
- $X^{-1}(B)$ is called as the preimage or the inverse image of $B$.

# Definition of a random variables




A random variable $X$ is a map $X : (\Omega, \mathcal{F}, P) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}), P_X)$ such that for each $B \in \mathcal{B}(\mathbb{R})$, the inverse image $X^{-1}(B) := \{w \in \Omega : X(w) \in B\}$ satisfies

$$X^{-1}(B) \in \mathcal{F} \text{ and}$$
$$P_X(B) = \Pr(w \in \Omega : X(w) \in B)$$

# Examples

# Definition of a random variable: Example-1

- Definition of a random variable:

A rv $X$ is a map $X : (\Omega, \mathcal{F}, P) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}), P_X)$ s.t. for each $B \in \mathcal{B}(\mathbb{R})$, the inverse image $X^{-1}(B) := \{w \in \Omega : X(w) \in B\}$ satisfies

$$X^{-1}(B) \in \mathcal{F} \text{ and } P_X(B) = \Pr(w \in \Omega : X(w) \in B)$$

- Example: Consider the following map corresponding to two coin tosses.

$$HH \xrightarrow{X} 0,\ HT \xrightarrow{X} 1,\ TH \xrightarrow{X} 2,\ TT \xrightarrow{X} 3$$

- For which of the following event spaces can the map $X$ defined above will be a valid random variable and why?
  - $\mathcal{F} = \big\{\phi, \Omega, \{HH \cup TT\}, \{HT \cup TH\}\big\}$
  - $\mathcal{F}$ is a power set of $\Omega$

# Definition of a random variable: Example-2 (Homework)

- Definition of a random variable:

> A rv $X$ is a map $X : (\Omega, \mathcal{F}, P) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}), P_X)$ s.t. for each $B \in \mathcal{B}(\mathbb{R})$, the inverse image $X^{-1}(B) := \{w \in \Omega : X(w) \in B\}$ satisfies
>
> $$X^{-1}(B) \in \mathcal{F} \;\text{ and }\; P_X(B) = \Pr(w \in \Omega : X(w) \in B)$$

- Example: Consider the following map corresponding to three coin tosses.

$$HHH \xrightarrow{X} 0,\; HHT \xrightarrow{X} 1,\; HTH \xrightarrow{X} 2,\; HTT \xrightarrow{X} 3$$
$$THH \xrightarrow{X} 4,\; THT \xrightarrow{X} 5,\; TTH \xrightarrow{X} 6,\; TTT \xrightarrow{X} 7$$

- For which of the following event spaces can the map $X$ defined above will be a valid random variable and why?
  - $\mathcal{F} = \big\{\phi, \Omega, \{HHH, HTT, THT, TTH\}, \{HHT, HTH, THH, TTT\}\big\}$
  - $\mathcal{F}$ is a power set of $\Omega$

# Definition of a random variable: Example-3

- General definition of a random variable:

A rv $X$ is a map $X : (\Omega, \mathcal{F}, P) \to (\Omega', \mathcal{F}', P')$ s.t. for each $f \in \mathcal{F}'$, the inverse image $X^{-1}(f) := \{w \in \Omega : X(w) \in f\}$ satisfies

$$X^{-1}(f) \in \mathcal{F} \text{ and } P_X(f) = \Pr(w \in \Omega : X(w) \in f)$$

- Example:
  - $\Omega = \{1, 2, 3, 4\}$ and $\Omega' = \{a, b, c\}$
  - $\mathcal{F}$ and $\mathcal{F}'$ are power sets of $\Omega$ and $\Omega'$ respectively.
  - Consider the following map.

$$X(1) = a, X(2) = b, X(3) = c, X(4) = a$$

  - Is $X : (\Omega, \mathcal{F}, P) \to (\Omega', \mathcal{F}', P')$ a random variable?

# Definition of a random variable: Example-4 (Homework)

- General definition of a random variable:

> A rv $X$ is a map $X : (\Omega, \mathcal{F}, P) \to (\Omega', \mathcal{F}', P')$ s.t. for each $f \in \mathcal{F}'$, the inverse image $X^{-1}(f) \coloneqq \{w \in \Omega : X(w) \in f\}$ satisfies
>
> $$X^{-1}(f) \in \mathcal{F} \text{ and } P_X(f) = \Pr(w \in \Omega : X(w) \in f)$$

- Example: Rolling two dice where we are interested in the sum of two dice.
  - Suppose $X$ denotes sum of two dice.
  - Write down $\Omega, P, \Omega', P'$
  - $\mathcal{F}$ = Power set of $\Omega$ and $\mathcal{F}'$ = Power set of $\Omega'$
  - Is $X : (\Omega, \mathcal{F}, P) \to (\Omega', \mathcal{F}', P')$ a random variable?

# Definition of a random variable: Example-5

- General definition of a random variable:

A rv $X$ is a map $X : (\Omega, \mathcal{F}, P) \rightarrow (\Omega', \mathcal{F}', P')$ s.t. for each $f \in \mathcal{F}'$, the inverse image $X^{-1}(f) \coloneqq \{w \in \Omega : X(w) \in f\}$ satisfies

$$X^{-1}(f) \in \mathcal{F} \text{ and } P_X(f) = \Pr(w \in \Omega : X(w) \in f)$$

- Example:
  - $\Omega = \{a, b, c\}$ and $\Omega' = \{0, 1\}$
  - $\mathcal{F} = \big\{\phi, \Omega, \{a\}, \{b \cup c\}\big\}$ and $\mathcal{F}'$ is power set of $\Omega'$.
  - Consider the following map.

$$X(a) = 1, \quad X(b) = 0, \quad X(c) = 0$$

  - Is $X : (\Omega, \mathcal{F}, P) \rightarrow (\Omega', \mathcal{F}', P')$ a random variable?
  - This is an indicator random variable.

# Recap till now

# Summary

- Motivation for random variables
- Definition of a random variable
- Examples

# Random variables

- Irrespective of which random experiment we are conducting, our probability space will be $(\mathbb{R}, \mathcal{B}(\mathbb{R}), P_X)$!
    - $\mathbb{R}$: Sample space
    - $\mathcal{B}(\mathbb{R})$: Event space
    - $P_X$: Probability measure (Note: $P_X : \mathcal{B}(\mathbb{R}) \to [0, 1]$)
- **Focus: Let $X$ be a random variable.**
    - $X$: Random variable (rv or RV or r.v. )
    - $\mathcal{X}$: "Support set" of rv $X$
    - $x$: "Realization" of rv $X$ (Note: $x \in \mathcal{X}$)

# Discrete random variables

# Discrete random variable: Example

- Support set of a rv: The set of values that it can take.
- A random variable is called "discrete" if its support set consists of finite or countable finite elements.
- Example of discrete rv: Tossing two coins
  - $\Omega = \{HH, HT, TH, TT\}$
  - Suppose $P[HH] = 0.2, P[HT] = 0.3, P[TH] = 0.35, P[TT] = 0.15$
  - Consider example map: In class
- For a discrete rv $X$, we shall next study:
  - Probability mass function (PMF or pmf) of a discrete rv
  - Cumulative distribution function (CDF or cdf) of a discrete rv