Signal Processing - Assignment - 3

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Q1. CTFT:-

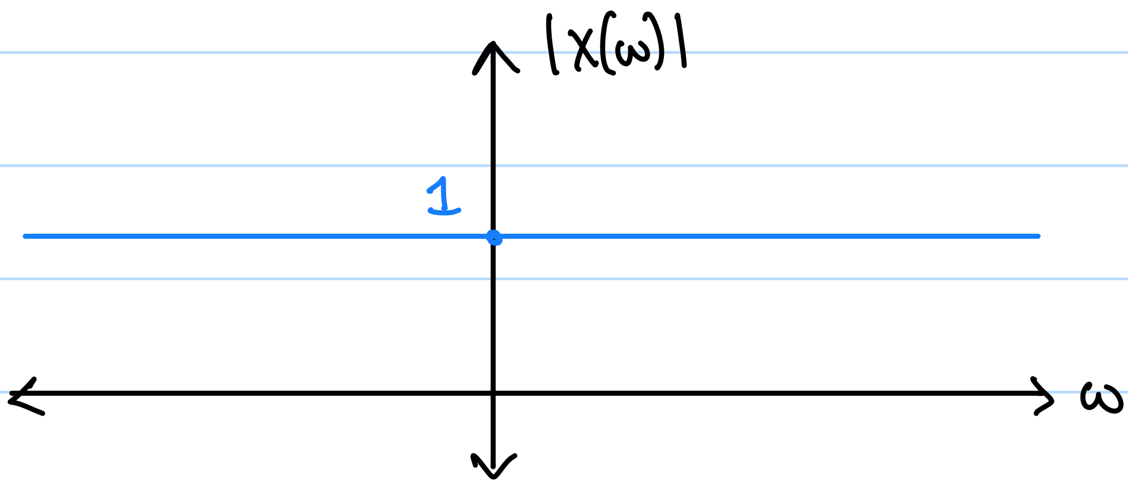
1. a) $x(t) = \delta(t - t_0)$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} \delta(t-t_0)\, e^{-j\omega t}\, dt$$

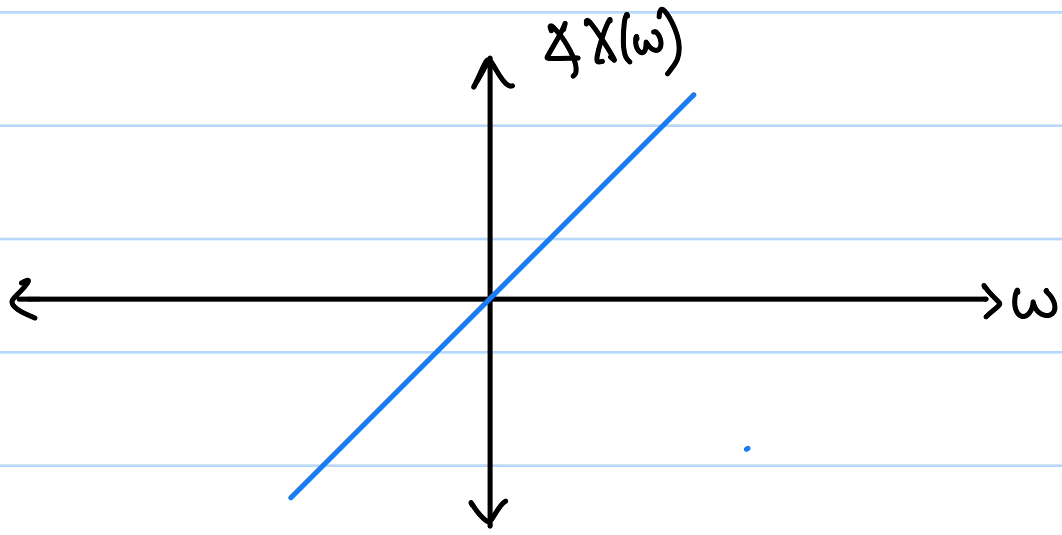
$$X(\omega) = e^{-j\omega t_0}$$

$$|X(\omega)| = 1 \quad , \quad \measuredangle X(\omega) = -\omega t_0$$

$|X(\omega)|$:



$\measuredangle X(\omega)$



b) $x(t) = \begin{cases} 1, & |t| < T_0 \\ 0, & |t| \geq T_0 \end{cases}$

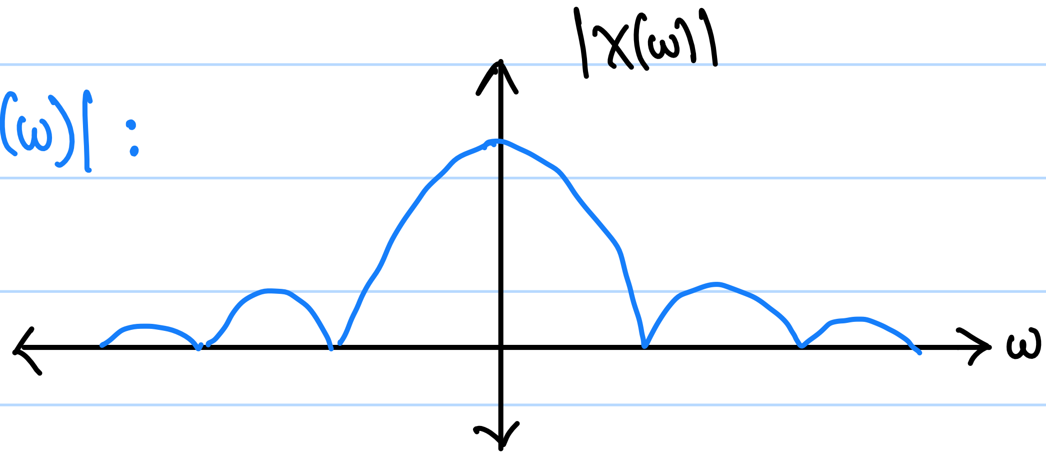
$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{-T_0}^{T_0} e^{-j\omega t} dt$$

$$= \left(\frac{-1}{j\omega}\right)\left[e^{-j\omega T_0} - e^{j\omega T_0}\right] = \left(\frac{-1}{j\omega}\right)\left(-2j\sin(\omega T_0)\right)$$

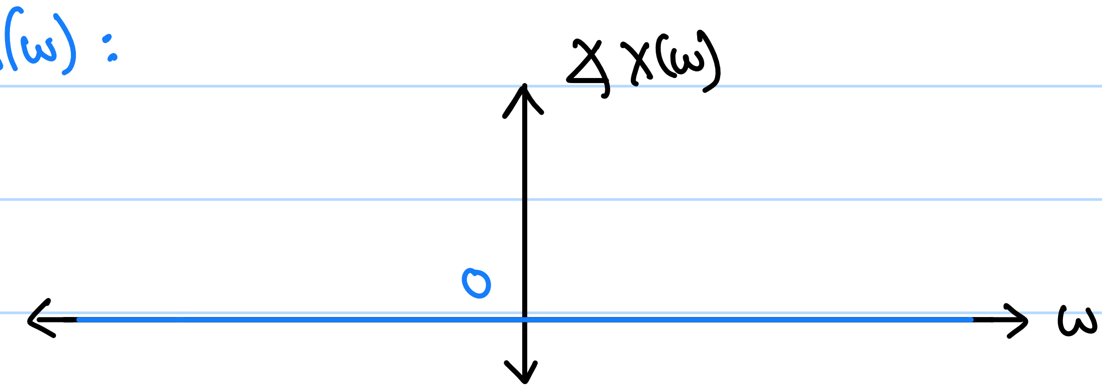
$$= \frac{2\sin(\omega T_0)}{\omega}$$

$$|X(\omega)| = 2\left|\frac{\sin(\omega T_0)}{\omega}\right| \qquad \measuredangle X(\omega) = \tan^{-1}\left(\frac{0}{\frac{2}{\omega}\sin(\omega T_0)}\right) = \underline{\underline{0}}$$

$|X(\omega)|$:



$\measuredangle X(\omega)$:



c) $x(t) = e^{-\alpha t} u(t)$, $\alpha \in \mathbb{C}$

Let $\alpha = a + jb$, $a, b \in \mathbb{R}$, $a > 0$, $b \neq 0$.

$\Rightarrow \quad x(t) = e^{-at} e^{-jbt} u(t)$

$$X(\omega) = \int_{-\infty}^{\infty} e^{-at} e^{-jbt} e^{-j\omega t} u(t) dt = \int_{-\infty}^{\infty} e^{-(a+j(b+\omega))t} u(t)\, dt$$

$$= \int_0^{\infty} e^{-(a+j(b+\omega))t} \, dt$$

$$= \frac{1}{-(a+j(b+\omega))} [1-0]$$

$$\Rightarrow \quad X(\omega) = \frac{-1}{a+j(b+\omega)}$$

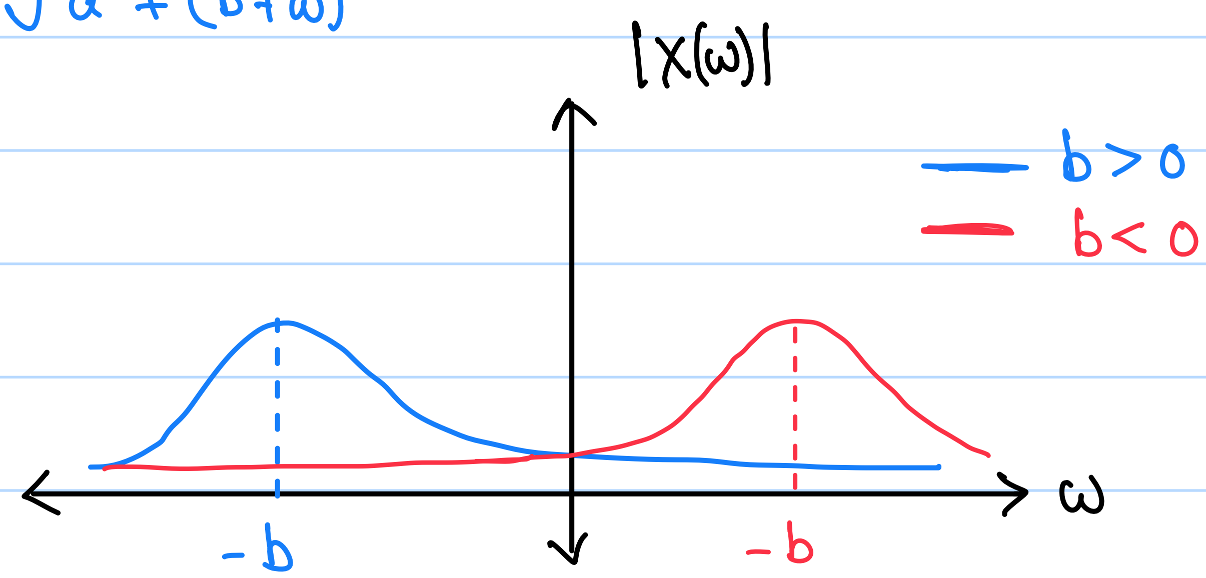
$$= \frac{-1}{a+j(b+\omega)} \times \frac{a-j(b+\omega)}{a-j(b+\omega)}$$

$$= \frac{j(b+\omega)-a}{a^2+(b+\omega)^2}$$

$$|X(\omega)| = \frac{1}{a^2+(b+\omega)^2} \sqrt{a^2+(b+\omega)^2}$$

$$|X(\omega)| = \frac{1}{\sqrt{a^2+(b+\omega)^2}}$$

$|X(\omega)|$:



d) $x(t) = e^{-\alpha|t|}, \quad \alpha \in \mathbb{R}$ and $\alpha > 0$,

$$X(\omega) = \int_{-\infty}^{\infty} e^{-\alpha|t|} e^{-j\omega t} \, dt$$

$$= \int_{0}^{\infty} e^{-\alpha t} e^{-j\omega t} \, dt + \int_{-\infty}^{0} e^{\alpha t} e^{-j\omega t} \, dt$$

$$= \int_{0}^{\infty} e^{-(\alpha + j\omega)t} \, dt + \int_{-\infty}^{0} e^{(\alpha - j\omega)t} \, dt$$

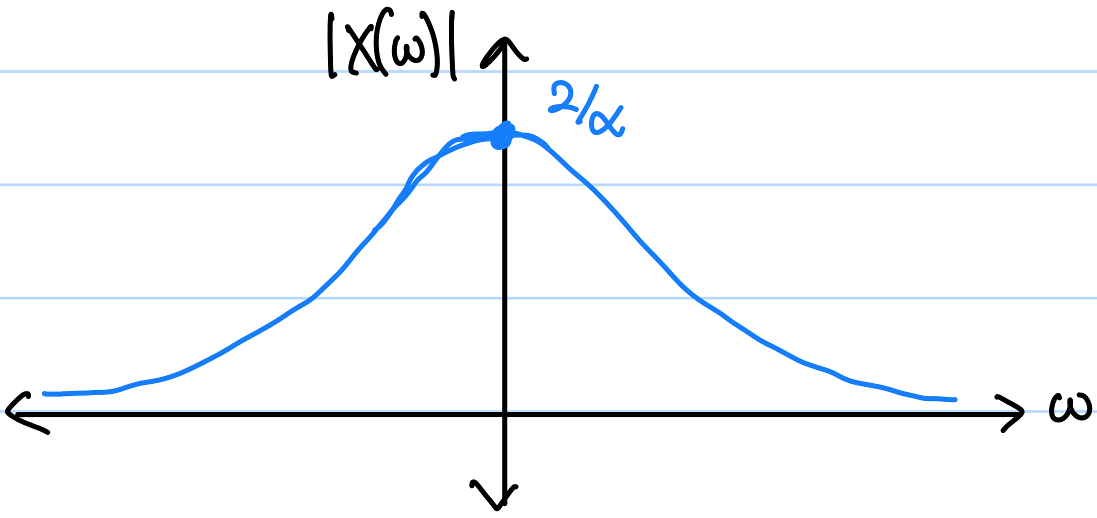
$$= \frac{-1}{\alpha + j\omega} [0 - 1] + \frac{1}{\alpha - j\omega} [1 - 0]$$

$$= \frac{1}{\alpha + j\omega} + \frac{1}{\alpha - j\omega}$$

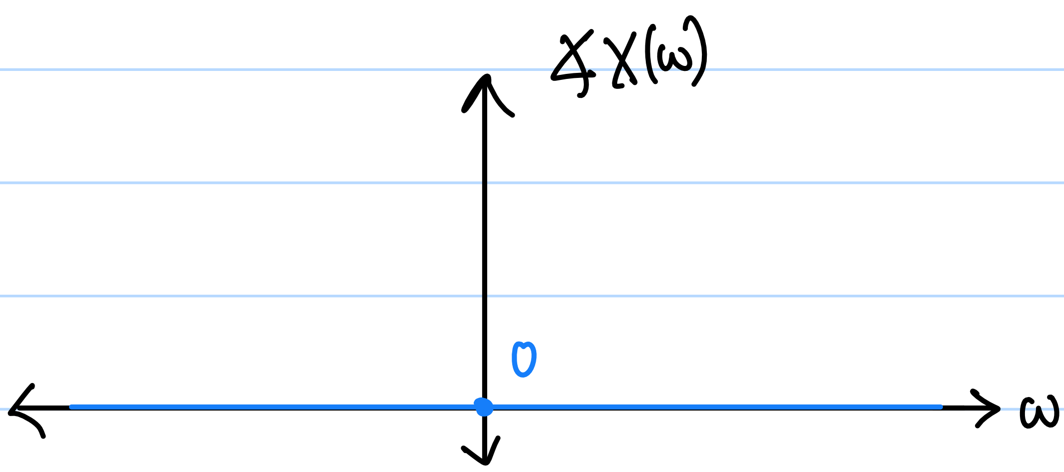
$$= \frac{\alpha - j\omega + \alpha + j\omega}{\alpha^2 + \omega^2}$$

$$\Rightarrow |X(\omega)| = \frac{2\alpha}{\alpha^2 + \omega^2} \qquad \measuredangle X(\omega) = \tan^{-1}\left(\frac{0}{2\alpha/\alpha^2+\omega^2}\right) = \underline{\underline{0}}$$

$|X(\omega)|$:



$\measuredangle X(\omega)$:



e) $$x(t) = \frac{\sin^2(\pi t)\cos(\pi t)}{\pi t^2}$$

$$= \left(\frac{\sin(\pi t)}{\pi t}\right)^2 \cos(\pi t)$$

Wkt $$\frac{\sin(\pi t)}{\pi t} \xrightarrow{CTFT} rect(\omega)$$

$$\cos(\pi t) \xrightarrow{CTFT} \frac{1}{2}\delta\left(\omega - \frac{1}{2}\right) + \frac{1}{2}\delta\left(\omega + \frac{1}{2}\right)$$

$$\Rightarrow X(\omega) = rect(\omega) * rect(\omega) * \frac{1}{2}\left(\delta\left(\omega - \frac{1}{2}\right) + \delta\left(\omega + \frac{1}{2}\right)\right)$$

$$= rect(\omega) * \frac{1}{2}\left(rect\left(\omega - \frac{1}{2}\right) + rect\left(\omega + \frac{1}{2}\right)\right)$$

$$= \frac{1}{2}\left(rect(\omega) * rect\left(\omega - \frac{1}{2}\right) + rect(\omega) * rect\left(\omega + \frac{1}{2}\right)\right)$$

$$\underbrace{rect(\omega) * rect\left(\omega - \frac{1}{2}\right)}_{X_1(\omega)} = \int_{-\infty}^{\infty} rect(\alpha)\, rect\left(\omega - \frac{1}{2} - \alpha\right) d\alpha$$

$$= \int_{-1/2}^{1/2} rect\left(\omega - \alpha - \frac{1}{2}\right) d\alpha$$

$$rect\left(\omega - \alpha - \frac{1}{2}\right) = \begin{cases} 1, & -\frac{1}{2} \leq \omega - \alpha - \frac{1}{2} \leq \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1, & 0 \leq \omega - \alpha \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1, & \omega - 1 \leq \alpha \leq \omega \\ 0, & \text{otherwise} \end{cases}$$

Case 1: $\omega < -\frac{1}{2}$, $X_1(\omega) = 0$

Case 2: $\omega \in [-\frac{1}{2}, \frac{1}{2}]$,

$$X_1(\omega) = \int_{-1/2}^{\omega} d\alpha = \omega + \frac{1}{2}$$

Case 3: $\omega > \frac{1}{2}$ and $\omega - 1 \in [-\frac{1}{2}, \frac{1}{2}]$

$$X_1(\omega) = \int_{\omega-1}^{1/2} d\alpha = \frac{1}{2} - \omega + 1 = \frac{3}{2} - \omega$$

Case 4: $\omega - 1 > \frac{1}{2}$, $X_1(\omega) = 0$

$$\Rightarrow X_1(\omega) = \begin{cases} \omega + \frac{1}{2}, & \omega \in [-\frac{1}{2}, \frac{1}{2}) \\ \frac{3}{2} - \omega, & \omega \in [\frac{1}{2}, \frac{3}{2}] \\ 0, & \text{otherwise} \end{cases}$$

$$\underbrace{\text{rect}(\omega) * \text{rect}(\omega + \tfrac{1}{2})}_{X_2(\omega)} = \int_{-\infty}^{\infty} \text{rect}(\alpha)\,\text{rect}(\omega + \tfrac{1}{2} - \alpha)\, d\alpha$$

$$= \int_{-1/2}^{1/2} \text{rect}(\omega + \tfrac{1}{2} - \alpha)\, d\alpha$$

$$\text{rect}(\omega + \tfrac{1}{2} - \alpha) = \begin{cases} 1, & -\frac{1}{2} \le \omega + \frac{1}{2} - \alpha \le \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1, & -1 \le \omega - \alpha \le 0 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1, & \omega \le \alpha \le \omega + 1 \\ 0, & \text{otherwise} \end{cases}$$

Case 1 : $\omega + 1 \leq \frac{-1}{2}$, $X_2(\omega) = 0$

Case 2: $\omega \leq \frac{-1}{2}$ and $\omega + 1 \in \left[\frac{-1}{2}, \frac{1}{2}\right)$

$$X_2(\omega) = \int_{-1/2}^{\omega+1} d\alpha = \omega + \frac{1}{2}$$

Case 3: $\omega + 1 > \frac{1}{2}$ and $\omega \in \left[\frac{-1}{2}, \frac{1}{2}\right)$

$$X_2(\omega) = \int_{\omega}^{1/2} d\alpha = \frac{1}{2} - \omega$$

Case 4: $\omega > \frac{1}{2}$, $X_2(\omega) = 0$

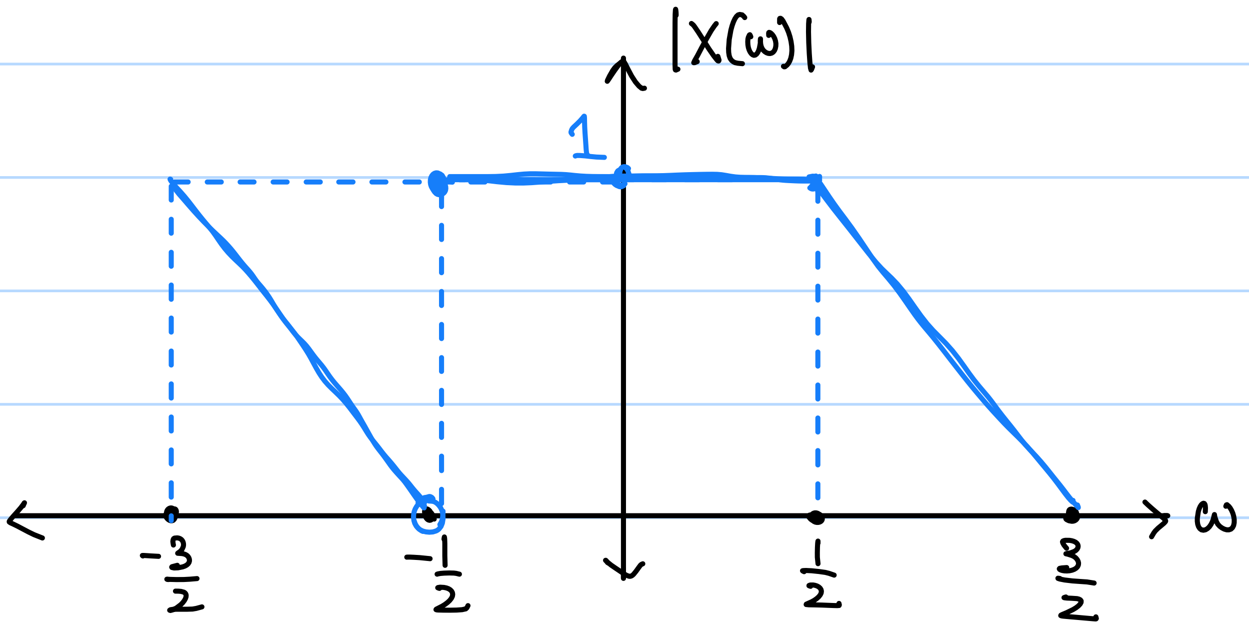
$$\Rightarrow \quad X_2(\omega) = \begin{cases} \frac{1}{2} + \omega, & \omega \in \left[\frac{-3}{2}, \frac{-1}{2}\right) \\ \frac{1}{2} - \omega, & \omega \in \left[\frac{-1}{2}, \frac{1}{2}\right] \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow \quad X(\omega) = \begin{cases} \frac{1}{2} + \omega, & \omega \in \left[\frac{-3}{2}, \frac{-1}{2}\right) \\ \frac{1}{2} - \omega + \frac{1}{2} + \omega, & \omega \in \left[\frac{-1}{2}, \frac{1}{2}\right) \\ \frac{3}{2} - \omega, & \omega \in \left[\frac{1}{2}, \frac{3}{2}\right] \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow \quad X(\omega) = \begin{cases} \frac{1}{2} + \omega, & \omega \in \left[\frac{-3}{2}, \frac{-1}{2}\right) \\ 1, & \omega \in \left[\frac{-1}{2}, \frac{1}{2}\right) \\ \frac{3}{2} - \omega, & \omega \in \left[\frac{1}{2}, \frac{3}{2}\right] \\ 0, & \text{otherwise} \end{cases}$$

$$|X(\omega)| = \begin{cases} \left|\frac{1}{2}+\omega\right|, & \omega \in \left[\frac{-3}{2}, \frac{-1}{2}\right) \\ 1, & \omega \in \left[\frac{-1}{2}, \frac{1}{2}\right) \\ \frac{3}{2}-\omega, & \omega \in \left[\frac{1}{2}, \frac{3}{2}\right] \\ 0, & \text{otherwise} \end{cases} \qquad \measuredangle X(\omega) = 0$$

$|X(\omega)|$:



2. <u>To Prove:</u> If $x(t)$ is real and odd, then $X(\omega)$ is purely imaginary.

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$X^*(\omega) = \left(\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt\right)^* = \int_{-\infty}^{\infty} x(t)\, e^{j\omega t}\, dt$$

Take $v = -t$,

$$X^*(\omega) = \int_{\infty}^{-\infty} x(-v)\, e^{-j\omega v}\, (-dv)$$

$$= -\int_{-\infty}^{\infty} -\,x(-v)\, e^{-j\omega v}\, dv$$

Since $x(t)$ is odd, $x(t) = -x(-t) \quad \forall\, t$

$$\Rightarrow X^*(\omega) = -\int_{-\infty}^{\infty} x(v)\, e^{-j\omega v}\, dv$$

$$\Rightarrow \quad \overset{*}{X}(\omega) = -\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$\Rightarrow \quad \overset{*}{X}(\omega) = -X(\omega)$$

$\therefore$ $X(\omega)$ is purely imaginary.

Q2. Inverse CTFT :-

a) $$X_1(\omega) = \frac{1}{\lambda + j\omega}$$

wkt $$e^{-at}u(t) \xleftrightarrow{\text{CTFT}} \frac{1}{a+j\omega}$$

$$\therefore \quad \mathcal{F}^{-1}\left(\frac{1}{\lambda + j\omega}\right) = e^{-\lambda t}u(t)$$

b) $$X_2(\omega) = \delta(\omega + T_0) + \delta(\omega - T_0)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} [\delta(\omega + T_0) + \delta(\omega - T_0)]\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\left(e^{-jT_0 t} + e^{jT_0 t}\right)$$

$$x(t) = \frac{1}{\pi}\cos(T_0 t)$$

c) $$X(\omega) = X_1(\omega)\, X_2(\omega)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\omega)\, X_2(\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1(\tau) e^{-j\omega\tau} d\tau \int_{-\infty}^{\infty} x_2(\sigma) e^{-j\omega\sigma} d\sigma \; e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} x_1(\tau) x_2(\sigma)\left(\int_{-\infty}^{\infty} e^{j\omega(t-\sigma-\tau)} d\omega\right) d\tau\right) d\sigma$$

$$e^{j\omega(t-\sigma-\tau)} \xleftrightarrow{\text{CTFT}} 2\pi\delta(t)$$

$$x(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1(\tau) x_2(\sigma)\, \delta(t-\sigma-\tau)\, d\tau\, d\sigma$$

$$= \int_{-\infty}^{\infty} x_1(\sigma)\left(\int_{-\infty}^{\infty} x_2(\tau)\, \delta(t-\sigma-\tau)\, d\tau\right) d\sigma$$

$$= \int_{-\infty}^{\infty} x_1(\sigma)\, x_2(t-\sigma)\, d\sigma$$

$$= x_1(t) * x_2(t)$$

$$\Rightarrow \quad x(t) = x_1(t) * x_2(t)$$

$$= \int_{-\infty}^{\infty} \frac{\cos(T_0\tau)}{\pi}\, e^{-\lambda(t-\tau)}\, u(t-\tau)\, d\tau$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{t} \left(e^{-jT_0\tau} + e^{jT_0\tau}\right) e^{-\lambda(t-\tau)}\, d\tau$$

$$= \frac{e^{-\lambda t}}{2\pi}\int_{-\infty}^{t} e^{\tau(\lambda-jT_0)} + e^{\tau(\lambda+jT_0)}\, d\tau$$

$$= \frac{e^{-\lambda t}}{2\pi}\left(\left[\frac{e^{\tau(\lambda-jT_0)}}{\lambda-jT_0}\right]_{-\infty}^{t} + \left[\frac{e^{\tau(\lambda+jT_0)}}{\lambda+jT_0}\right]_{-\infty}^{t}\right)$$

$$= \frac{e^{-\lambda t}}{2\pi}\left(\frac{e^{t(\lambda-jT_0)}}{\lambda-jT_0} + \frac{e^{t(\lambda+jT_0)}}{\lambda+jT_0}\right)$$

$$x(t) = \frac{e^{-jT_0 t}}{2\pi(\lambda - jT_0)} + \frac{e^{jT_0 t}}{2\pi(\lambda + jT_0)}$$

Q3. Sampling :-

1\. a) $p(t) = \sum_{n\in\mathbb{Z}} \delta(t - nT_0)$

$$p(t) = \sum_{k\in\mathbb{Z}} a_k e^{jk\omega_0 t} \quad , \quad \omega_0 = \frac{2\pi}{T_0}$$

$$a_k = \frac{1}{T_0}\int_0^{T_0} p(t)\, e^{-jk\frac{2\pi}{T_0}t}\, dt$$

$$= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \sum_{n\in\mathbb{Z}} \delta(t - nT_0)\, e^{-jk\frac{2\pi}{T_0}t}\, dt$$

$$= \frac{1}{T_0}\int_0^{T_0} \delta(t)\, e^{-jk\frac{2\pi}{T_0}t}\, dt$$

$$= \frac{1}{T_0}\left(e^{-jk\frac{2\pi}{T_0}(0)}\right)$$

$$\Rightarrow \quad a_k = \frac{1}{T_0} \quad \forall \quad k\in\mathbb{Z}$$

b) $P(s) = \sum_{k\in\mathbb{Z}} a_k\, \delta(\omega - k\omega_0)$, a_k – FSR coeffs of $p(t)$

$$P(s) = \sum_{k\in\mathbb{Z}} \frac{1}{T_0}\, \delta\left(\omega - \frac{2\pi k}{T_0}\right)$$

2. $x_p(t) = x(t)\,p(t)$, $x(t) = \cos(\omega_0 t)$, $T = 1/3$

$$\Rightarrow p(t) = \sum_{n\in\mathbb{Z}} \delta\left(t - \frac{n}{3}\right)$$

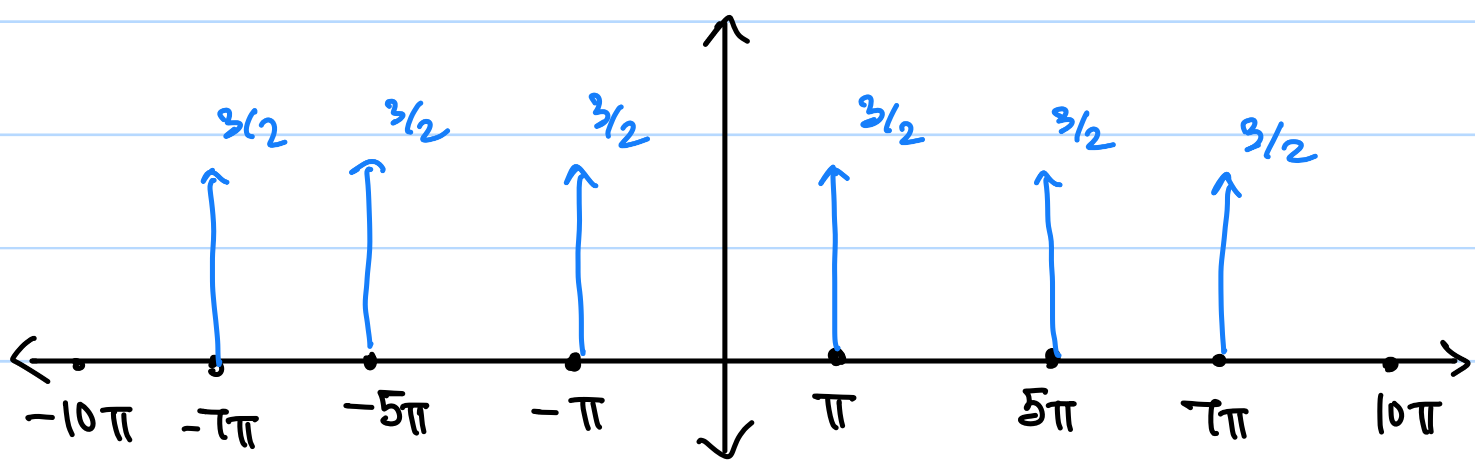
a) i) $\omega_0 = \pi$,

$$X_p(\omega) = X(\omega) * P(\omega)$$

$$= X(\omega) * \sum_{k\in\mathbb{Z}} \delta(\omega - 6\pi k)\cdot 3$$

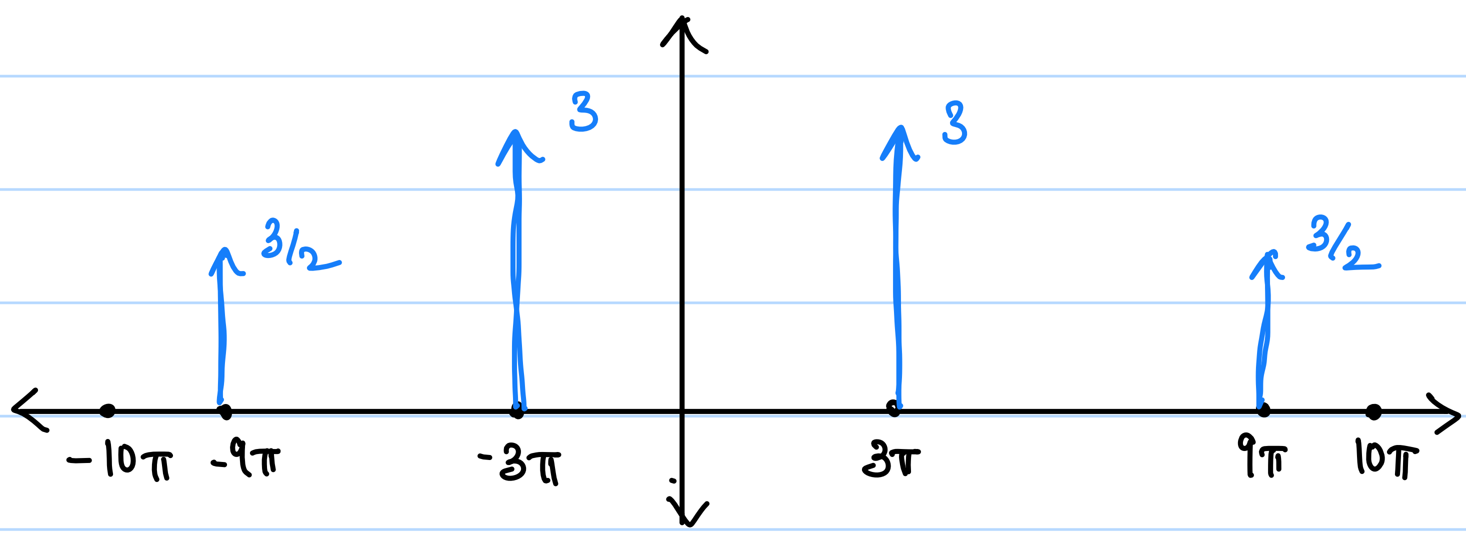
$$= 3\sum_{k\in\mathbb{Z}} X(\omega - 6\pi k)$$

$$= \frac{3}{2}\sum_{k\in\mathbb{Z}} \delta(\omega - \pi - 6\pi k) + \delta(\omega + \pi - 6\pi k)$$



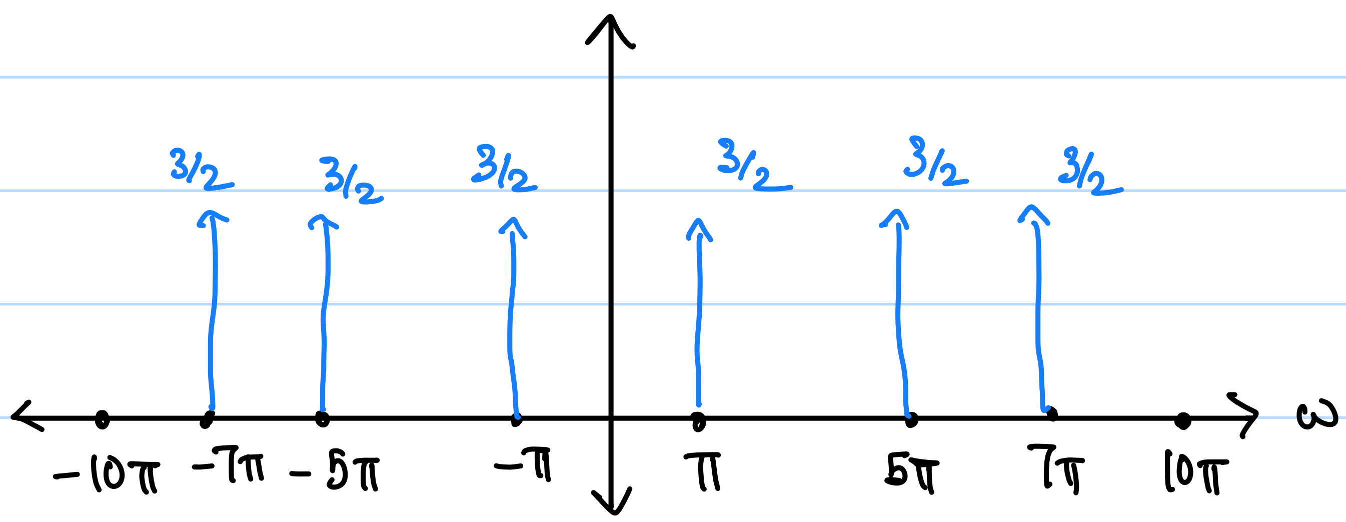
ii) $\omega_0 = 3\pi$,

$$X_p(\omega) = \frac{3}{2}\sum_{k\in\mathbb{Z}} \delta(\omega - 3\pi - 6\pi k) + \delta(\omega + 3\pi - 6\pi k)$$



iii) $\omega_0 = 5\pi$

$$X_p(\omega) = \frac{3}{2} \sum_{k \in \mathbb{Z}} \delta(\omega - 5\pi - 6\pi k) + \delta(\omega + 5\pi - 6\pi k)$$



b) The spectrum for $\omega_0 = \pi$ and $\omega_0 = 5\pi$ are the same.

This is because, since the sampling frequency is less than the Nyquist frequency for $\omega_0 = 5\pi$, the signal undergoes aliasing.

Aliased frequency $\omega_{alias} = |\omega_0 - k\omega_s|$, $k \in \mathbb{Z}$ st $\omega_{alias} \in [0, \frac{\omega_s}{2}]$

$$\omega_{alias} = |5\pi - 6\pi| = \underline{\pi}$$

Therefore due to aliasing, $\cos(5\pi t)$ becomes similar to $\cos(\pi t)$, making their spectrum the same.