Assignment 5

(MA6.102) Probability and Random Processes, Monsoon 2025

Release date: 18 October 2025, Due date: 31 October 2025

INSTRUCTIONS

- Discussions with other students are not discouraged. However, all write-ups must be done individually with your own solutions.
- Any plagiarism when caught will be heavily penalised.
- Be clear and precise in your writing.

Problem 1. Let X and Y have the joint PDF $f_{X,Y}(x,y) = cx(y-x)\mathrm{e}^{-y}$, $0 \leq x \leq y < \infty$. Find $f_{X|Y}$ and $\mathbb{E}[X|Y]$.

Problem 2. Let X_1 and X_2 be independent and identically distributed random variables with the common PDF

$$f_X(x) = \frac{1}{x^2}, \text{ for } x > 1.$$

Define new random variables $Y_1 = X_1 + X_2$ and $Y_2 = \frac{X_2}{X_1+X_2}$. Find the joint PDF, explicitly stating the domain over which it is non-zero.

Problem 3. Let X_1 and X_2 be independent exponential random variables with parameter λ. Find the joint PDF of $Y_1 = X_1 + X_2$ and $Y_2 = \frac{X_1}{X_2}$ and check whether Y_1 and Y_2 are independent.

Problem 4. Suppose $Z = \max\{X, Y\}$ and $W = \min\{X, Y\}$, where X and Y are jointly continuous random variables with joint PDF $f_{X,Y}$. Express $f_{Z,W}$ in terms of $f_{X,Y}$.

Problem 5. Let X, Y, and Z be independent and uniformly distributed on $[0, 1]$. Find $P(XY < Z^2)$.

Problem 6. Let F_X and F_Y be strictly increasing CDFs of two random variables X and Y, respectively. Determine a function g such that the random variable $Z = g(X)$ has the same CDF as Y; i.e., $F_Z(y) = F_Y(y)$, $\forall y \in \mathbb{R}$.

Problem 7. (a) $\phi_X(t) = \mathbb{E}[\mathrm{e}^{itX}]$, where $i = \sqrt{-1}$, denotes the characteristic function of a random variable X. Is it true that if $\phi_X(2\pi) = 1$, then $P(X \in \mathbb{Z}) = 1$?
(b) Let X be an integer-valued random variable with PMF P_X. Show that

$$P_X(n) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{-itn} \phi_X(t)\, dt,\ n \in \mathbb{Z},$$

and $\phi_X(t) = \mathbb{E}[\mathrm{e}^{itX}]$ is the characteristic function of X.

Hint. Argue that $\int_0^{2\pi} \mathrm{e}^{it(k-n)}\, dt = \begin{cases} 0, & \text{if } k \neq n \\ 2\pi, & \text{if } k = n \end{cases}$, and assume that the integral and the summation can be interchanged.