RA Assignment - 3

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Q1. Given: A_n is connected $\forall n \in \mathbb{N}$. $A_n \cap A_{n+1} \neq \phi$ $\forall n$.
To Prove: $\bigcup_n A_n$ is connected

Define $B_k = \bigcup_{k+1} A_i$ — ①

Let $k = 1$,
$$B_1 = A_1 \cup A_2$$

Since A_1 and A_2 are connected and $A_1 \cap A_2 \neq \phi$ (Given), to prove that $A_1 \cup A_2$ is connected,

Assume $A_1 \cup A_2$ is disconnected, ie, $\exists$ open G_1, G_2 st $G_1 \cap G_2 = \phi$, such that,
i) $A_1 \cup A_2 \subseteq G_1 \cup G_2$
ii) $(A_1 \cup A_2) \cap G_1 \neq \phi$
iii) $(A_1 \cup A_2) \cap G_2 \neq \phi$

$A_1 \cup A_2 \subseteq G_1 \cup G_2 \Rightarrow A_1 \subseteq G_1 \cup G_2$ & $A_2 \subseteq G_1 \cup G_2$
$(A_1 \cup A_2) \cap G_1 \neq \phi \Rightarrow A_1 \cap G_1 \neq \phi$ or $A_2 \cap G_1 \neq \phi$
$(A_1 \cup A_2) \cap G_2 \neq \phi \Rightarrow A_1 \cap G_2 \neq \phi$ or $A_2 \cap G_2 \neq \phi$

Since $A_1 \cap A_2 \neq \phi$, $\exists$ x st $x \in A_1$ & $x \in A_2$

If $A_1 \cap G_1 \neq \phi$ & $A_1 \cap G_2 \neq \phi$ is true, we get that A_1 is disconnected, which is a contradiction. Illly for A_2 as well.

If $A_1 \cap G_1 \neq \phi$ and $A_2 \cap G_2 \neq \phi$ or $A_2 \cap G_1 \neq \phi$ and $A_1 \cap G_2 \neq \phi$ is true, we get that A_1 is entirely within G_2 and A_2 is entirely within G_1 (~~and~~ or vice versa).

But since $G_1 \cap G_2 = \phi$, $\Rightarrow$ $A_1 \cap A_2 = \phi$
(Contradiction).

$\therefore$ The union of 2 connected sets is connected if their intersection is non-empty. —— ②

Using ②,

$B_1 = A_1 \cup A_2$ is connected

Assume B_K is connected, then for B_{K+1}

$$B_{K+1} = B_K \cup A_{K+2}$$

Again using ②, ~~since~~ $\Rightarrow$ B_{K+1} is connected.

$\therefore$ By mathematical induction,

B_K is connected $\forall\ K \in N$.

$\therefore$ The union of finite no. of connected sets is connected.

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Q2. Let A be a connected set.

$int(A)$ is the ~~d~~ interior of set A, $\bar{A}$ is the closure of set A

$\bar{A} = A \cup A'$ (A' = set of limit points of A).

Assume $\bar{A}$ is not connected.

$\therefore \exists\ G_1, G_2$ that are open sets such that $G_1 \cap G_2 = \phi$ and,

i) $\bar{A} \subseteq G_1 \cup G_2$

ii) $\bar{A} \cap G_1 \neq \phi$ and $\bar{A} \cap G_2 \neq \phi$

However, since A is connected and $A \subseteq \bar{A}$,

$A \cap G_1 \neq \phi$ or $A \cap G_2 \neq \phi$ (otherwise A would be disconnected).

Let $A \cap G_1 \neq \phi$ be true, ie. A is entirely within G_2.

Since $A \subseteq G_2$, $A' \subset \overline{G_2}$ and since $G_1 \cap G_2 = \phi$, the limit points of G_2 cannot be in G_1.

$\Rightarrow \bar{A} \subseteq G_2$ ~~...~~ ($A \subseteq G_2$ and $A' \subseteq G_2$), ie $\bar{A}$ is entirely within G_2.

But this is a contradiction, since $G_1 \cap G_2 = \phi$ ~~but~~ and $\bar{A} \cap G_1 \neq \phi$.

$\therefore \bar{A}$ is connected.

For int(A),

~~Define a set A as $[0,1] \cup \{2\}$,~~

~~$int(A) = (0,1)$~~

Let A be $\{(x,y) : x^2 + y^2 \leq 1\}$
Let B be $\{(x,y) : (x-2)^2 + (y^2) \leq 1\}$

The point (1,0) is a boundary point of both A and B.

$\Rightarrow (1,0) \in A \cup B$, ~~and by coordinate geometry~~

If A, B are connected, $\therefore A \cup B$ is also connected.

To prove that A is connected,

~~Assume A is disconnected, ie, $\exists$ open G_1, G_2 st~~
~~$G_1 \cap G_2 = \phi$ and,~~
~~i) $A \subseteq G_1 \cup G_2$~~
~~ii) $A \cap G_1 \neq \phi$ and $A \cap G_2 \neq \phi$~~

By geometry, since a circle is a convex shape,
$\forall x, y \in A$, $\exists$ a path between x and y that is entirely within A.

~~Since~~ $\Rightarrow$ A is path connected
$\Rightarrow$ A is connected.

Similar proof can be stated for B as well.

$\Rightarrow$ A $\cup$ B is connected.

$\text{int}(A \cup B) = \{(x,y) : x^2+y^2 < 1 \text{ or } (x-2)^2+y^2 < 1\}$

By coordinate geometry, we can see that (1,0) is the only common point between A and B.

$\Rightarrow$ $x^2+y^2<1$ and $(x-2)^2+y^2<1$ represent 2 disjoint sets as they do not have any common points.

Since int (A $\cup$ B) is the union of 2 disjoint open sets, clearly, int (A $\cup$ B) is disconnected.

$\therefore$ int (A) may or may not be connected if A is connected

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Q3. ~~Given~~ A_i] To Prove: The union of finite no. of compact sets is compact.

Let $\{A_i\}$ be a collection of compact sets $\forall$ $i \in \mathbb{N}$.

Define $B_k = \bigcup_{m=1}^{k+1} A_m$

i) $k=1$,

$B_1 = A_1 \cup A_2$

Let C_1 be a open cover of A_1. By definition of compact sets. C_1 has a finite open subcover. Ully C_2

~~Claim~~:

(2) ~~$C_1 \cup C_2$ is a cover of $A_1 \cup$~~ A_2,

$\Rightarrow$ $A_1 \subseteq$ (union of sets in C_1)

$A_2 \subseteq$ (union of sets in C_2)

$A_1 \cup A_2 \subseteq$ (union of sets in C_1) $\cup$ (union of sets in C_2)

$\Rightarrow$ $A_1 \cup A_2 \subseteq$ (union of sets in $C_1 \cup C_2$)

$\therefore$ $C_1 \cup C_2$ is a open cover of $A_1 \cup A_2$.

Since both C_1 and C_2 have finite sub covers, using the same logic as before, $\exists$ a finite sub cover in C_1 and C_2.

$\Rightarrow$ $A_1 \cup A_2$ is compact. $\boxed{\therefore \text{Union of 2 compact sets is compact}}$ (1)

$\Rightarrow$ B_1 is compact

ii) Assume B_K is compact, for B_{K+1}

$$B_{K+1} = B_K \cup A_{K+2}$$

By (2), since B_K and A_{K+2} are compact, B_{K+1} is also compact.

$\therefore$ By induction, B_n is compact $\forall\, n \in \mathbb{N}$.

$\therefore$ The union of finite no. of compact sets is compact.

Q4. Given: $f: A \mapsto \mathbb{R}$ is continuous, $K \subseteq A$ is compact.
To Prove: $f(K)$ is also compact.

Since $f: A \mapsto \mathbb{R}$ is continuous in A, and $K \subseteq A$, we can say that $f: K \mapsto \mathbb{R}$ is also ~~con~~ continuous.

Let $\{B_i : i \in \mathbb{N}\}$ be an open cover of K. $\exists$ a finite subcover of $\{B_i\}$ since K is compact.

$$\Rightarrow K \subseteq \bigcup_{i=1}^{n} B_i$$

$$\Rightarrow f(K) \subseteq \bigcup_{i=1}^{n} f(B_i)$$

$\Rightarrow \{f(B_i) : i \in \mathbb{N}\}$ is ~~an open~~ a cover of $f(K)$

Also $\{f(B_i)\}$ is an open cover since,

$$B_i \text{ is open } \forall i \in \mathbb{N} \Rightarrow f(B_i) \text{ is open } \forall i \in \mathbb{N}$$

(Since f is continuous)

Let $\{C_i : i \in \mathbb{N}\}$ be a finite subcover of K. By the same logic as above $\{f(C_i) : i \in \mathbb{N}\}$ is a finite subcover of $f(K)$.

$\therefore \forall$ open covers of $f(K)$, $\exists$ a finite subcover.

$\Rightarrow \therefore f(K)$ is compact.

Q8. <u>To find</u> :- A open cover of $(0,1)$, which has no finite subcover.

Define a collection of sets as, $\{(0,0.2), (0.1,0.3), (0.2,0.4), (0.3,0.5), \ldots\ldots (0.8,1)\}$, i.e.

$$C = \{(0.1n, 0.1n+0.2), \; n = 0,1,2,\ldots,8\}$$

Clearly C is an open cover of $(0,1)$, since all the elements of C are open sets, and $\bigcup_i C_i = (0,1)$.

But any subset of this cover is not a subcover of $(0,1)$, since

$$C' = \{(0.1n, 0.1n+0.2), \; n \in \ldots\}$$
$$0 \leq n \leq 8$$

is the general form of all subsets of the cover.

If $n > 0$, $0.1n > 0$

If $n < 8$, ~~$0.1n + 0.2 < 0.8 + 0.2$~~

$\Rightarrow 0.1n < 0.8$

$\Rightarrow 0.1n + 0.2 < 1$

$\therefore \bigcup C' \subseteq (0,1)$

$\therefore \nexists$ a subcover of C that is finite.

$\therefore$ <u>C fits the example asked.</u>

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Q6. To find :- Pointwise limit of $\{f_n(x)\}$, where $f_n(x) = x^n \ \forall n \in \mathbb{N}$. on the closed segment $[0,1]$.

Pointwise limit of $\{f_n(x)\}$ in E is $f(x) = \lim_{n\to\infty} f_n(x)$, such that $\forall x \in E$, $\forall \varepsilon > 0$ $\exists N_0 \in \mathbb{N}$ st $\forall n > N_0$,

$$|f_n(x) - f(x)| < \varepsilon$$

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n$$

Since $x \in [0,1]$

$$\Rightarrow f(x) = \begin{cases} 1, & x = 1 \\ 0, & x \in [0,1) \end{cases}$$

For uniform convergence,

$f_n(x)$ is said to uniformly converge to $f(x)$ if $\forall \varepsilon > 0$, $\exists n \in N_0$ st $\forall n > N_0$,

$$|f_n(x) - f(x)| < \varepsilon, \ \forall x \in E$$

If $x_0 \in [0,1)$,

$$|f_n(x) - f(x)| = |f_n(x)| < \varepsilon$$

$$= |x^n| < \varepsilon = x^n < \varepsilon, \ \forall \varepsilon > 0, \ \forall x \in [0,1)$$

Assume $\exists N_0 \in \mathbb{N}$ st the above statement is true (i.e. assume f_n is uniform convergent).

Let $x = \left(\frac{1}{2}\right)^{\frac{1}{N_0+1}}$ and $\varepsilon = \frac{1}{4}$

$$x^n = \left(\frac{1}{2}\right)^{\frac{n}{N_0+1}}$$

$$\left(\frac{1}{2}\right)^{\frac{n}{N_0+1}} < \frac{1}{4} \quad \forall\ n > N_0$$

Take $n = N_0+1$,

$$\left(\frac{1}{2}\right)^{\frac{N_0+1}{N_0+1}} = \frac{1}{2} < \frac{1}{4} \longrightarrow \text{False, i.e., Contradiction}$$

$\therefore$ $f_n(x)$ is <u>not</u> uniform convergent.

Q7. <u>To Prove:</u> $\exists\, x \in \mathbb{Q}'$ st $x^n \in \mathbb{Q}'$ $\forall\ n \in \mathbb{N}$.

Define a set $A := \{(x,n) : x \in \mathbb{Q}',\ n \in \mathbb{N}$ st. n is the min possible no. for which $x^n \in \mathbb{Q}\}$.

Define a set $B := \{x^n : x \in \mathbb{Q}',\ n \in \mathbb{N},\ x^n \in \mathbb{Q}\}$.

Define a $f : A \longmapsto B$ st $f((x,n)) = x^n$.

f is a bijective function. To prove that:

For injectivity,

$$f(x_1, n_1) = f(x_2, n_2)$$

$$= x_1^{n_1} = x_2^{n_2}$$

Case 1: $x_1 = x_2$ and $n_1 \neq n_2$

$$x_1^{n_1} = x_1^{n_2}$$

$$\Rightarrow x_1^{n_1/n_2} = x_1$$

$$\Rightarrow \frac{n_1}{n_2} = 1$$

$$\Rightarrow n_1 = n_2 \longrightarrow \text{Not allowed}$$

$\therefore$ This case is not possible

Case 2: $x_1 \neq x_2$ and $n_1 = n_2$

$$= x_1^{n_1} = x_2^{n_1}$$

$$= x_1^{\frac{n_1}{n_1}} = x_2$$

$$= x_1 = x_2 \longrightarrow \text{Not allowed}$$

$\therefore$ This case is not possible.

$\therefore x_1^{n_1} = x_2^{n_2} \Rightarrow (x_1, n_1) = (x_2, n_2)$

$\therefore f((x_1, n_1)) = f((x_2, n_2)) \Rightarrow (x_1, n_1) = (x_2, n_2)$, ie, f is injective.

For surjectivity,

Since every element of B is of the form x^n, where $x \in \mathbb{Q}'$ and $n \in \mathbb{N}$, $\forall$ elements of B $\exists$ a corresponding element (x, n) of A such that $f(x,n) = x^n$.

$\therefore f(x,n)$ is surjective.

$\therefore f(x,n)$ is bijective.

Since every element of B is rational, $B \subseteq \mathbb{Q}$.

We know that $\mathbb{Q}$ is countably infinite.

Since $B \subseteq \mathbb{Q}$, B is also countably infinite.

Since $\exists$ a bijection between A and B, A is also countably infinite.

A is countably infinite $\Rightarrow$ There are countably infinite $x \in \mathbb{Q}'$ such that $x^n \in \mathbb{Q}$ ~~where $n \in \mathbb{N}$~~.
where $n \in \mathbb{N}$ such that the cond. is satisfied.

But, $\mathbb{Q}'$ is uncountably infinite set.

$\therefore \exists x \in \mathbb{Q}'$ such that $x^n \in \mathbb{Q}' \;\forall\, n \in \mathbb{N}$.

(Since the set of x that satisfy the prev. cond is countably finite and $\mathbb{Q}'$ is uncountably finite, there should be a number that $\in \mathbb{Q}'$ outside the set of x).