The problem of numerical integration, like that of numerical differentiation, is solved by representing $f(x)$ by an interpolation formula and then integrating it between the given limits. In this way, we can derive quadrature formulae for approximate integration of a function defined by a set of numerical values only.

8.5 Newton-Cotes Quadrature Formula

Let $\quad I = \int_a^b f(x)dx$

where $f(x)$ takes the values $y_0, y_1, y_2, \cdots y_n$ for $x = x_0, x_1, x_2, \cdots x_n$.

Let us divide the interval (a, b) into n sub-intervals of width h so that $x_0 = a, x_1 = x_0 + h, x_2 = x_0 + 2h, \cdots .x_n = x_0 + nh = b$. Then

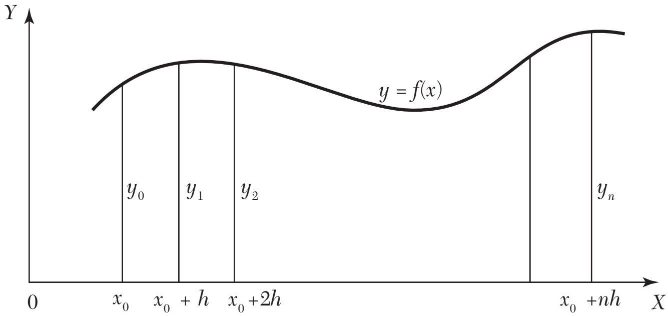

FIGURE 8.1

$$I = \int_{x_0}^{x_{0+nh}} f(x)dx = h\int_0^n f(x_0 + rh)dr, \text{ Putting } x = x_0 + rh, dx = hdr$$

$$= h\int_0^n \Big[y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0$$

$$+ \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 y_0 + \frac{r(r-1)(r-20(r-3)(r-4)}{5!}\Delta^5 y_0$$

$$\left. + \frac{r(r-1)(r-2)(r-3)(r-4)(r-5)}{6!}\Delta^6 y_0 + \ldots\right]dr$$

[by Newton's forward interpolation formula]

Integrating term by term, we obtain

$$\int_{x_0}^{x_0+nh} f(x)dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0\right.$$

$$+\left(\frac{n^4}{5}-\frac{3n^3}{2}+\frac{11n^2}{3}-3n\right)\frac{\Delta^4 y_0}{4!}$$
$$\left(+\frac{n^5}{6}-2n^4+\frac{34n^3}{4}-\frac{50n^2}{3}+12n\right)\frac{\Delta^5 y_0}{5!} \tag{1}$$
$$\left.+\left(\frac{n^6}{7}-\frac{15n^5}{6}+17n^4-\frac{225n^3}{4}+\frac{274n^2}{3}-60n\right)\frac{\Delta^6 y_0}{6!}+\ldots\right]$$

This is known as *Newton-Cotes quadrature formula*. From this general formula, we deduce the following important quadrature rules by taking $n = 1, 2, 3, \ldots$

I. Trapezoidal rule. Putting $n = 1$ in (1) and taking the curve through (x_0, y_0) and (x_1, y_1) as a straight line (Figure 8.2) *i.e.*, a polynomial of first order so that differences of order higher than first become zero, we get

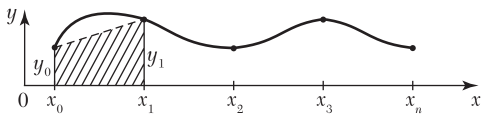

FIGURE 8.2

$$\int_{x_0}^{x_0+h} f(x)dx = h\left(y_0+\frac{1}{2}\Delta y_0\right)=\frac{h}{2}(y_0+y_1)$$

Similarly $$\int_{x_0+h}^{x_0+2h} f(x)dx = h\left(y_1+\frac{1}{2}\Delta y_1\right)=\frac{h}{2}(y_1+y_2)$$

....

$$\int_{x_0+(n-1)}^{x_0+nh} f(x)dx = \frac{h}{2}(y_{n-1}+yn)$$

Adding these n integrals, we obtain

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{2}[(y_0+y_n)+2(y_1+y_2+\cdots+y_{n\text{-}1})] \tag{2}$$

This is known as the *trapezoidal rule*.

NOTE **Obs.** *The area of each strip (trapezium) is found separately. Then the area under the curve and the ordinates at x_0 and x_n is approximately equal to the sum of the areas of the n trapeziums.*

II. Simpson's one-third rule. Putting $n = 2$ in (1) above and taking the curve through (x_0, y_0), (x_1, y_1), and (x_2, y_2) as a parabola (Figure 8.3), *i.e.*, a polynomial of the second order so that differences of order higher than the second vanish, we get

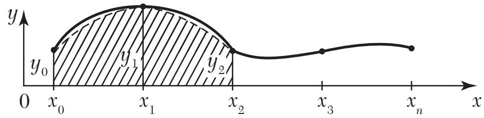

FIGURE 8.3

$$\int_{x_0}^{x_0+2h} f(x)dx = 2h(y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0) = \frac{h}{3}(y_0 + 4y_1 + y_2)$$

Similarly $\int_{x_0+2h}^{x_0+4h} f(x)dx = \frac{h}{3}(y_2 + 4y_3 + y_4)$

....

$\int_{x_0+(n-2)h}^{x_0+nh} f(x)dx = \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$, n being even.

Adding all these integrals, we have when n is even

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{3}[(y_0+y_n)+4(y_1+y_3+\ldots+y_{n-1})+2(y_2+y_4+\ldots y_{n-2})] \quad (3)$$

This is known as the *Simpson's one-third rule* or simply *Simpson's rule* and is most commonly used.

NOTE

Obs. *While applying (3), the given interval must be divided into an even number of equal subintervals, since we find the area of two strips at a time.*

III. Simpson's three-eighth rule. Putting $n = 3$ in (1) above and taking the curve through (x_i, y_i): $i = 0, 1, 2, 3$ as a polynomial of the third order (Figure 8.4) so that differences above the third order vanish, we get

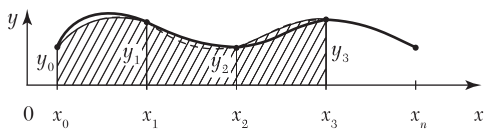

FIGURE 8.4

$$\int_{x_0}^{x_0+3h} f(x)dx = 3h\left(y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\right)$$

$$= \frac{3h}{8}(y_0 + 3y_1 + 3y_2 + y_3)$$

Similarly,

$$\int_{x_0+3h}^{x_0+5h} f(x)dx = \frac{3h}{8}(y_3 + 3y_4 + 3y_5 + y_6) \text{ and so on.}$$

Adding all such expressions from x_0 to $x_0 + nh$, where n is a multiple of 3, we obtain

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \cdots + y_{n-1}) + 2(y_3 + y_6 + \cdots + y_{n-3})] \quad (4)$$

NOTE **Obs.** *While applying (4), the number of sub-intervals should be taken as a multiple of 3.*

IV. Boole's rule. Putting $n = 4$ in (1) above and taking the curve (x_i, y_i), $i = 0, 1, 2, 3, 4$ as a polynomial of the fourth order (Figure 8.5) and neglecting all differences above the fourth, we obtain

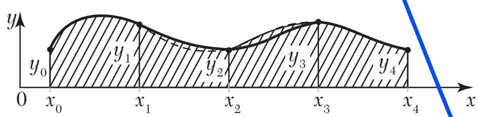

FIGURE 8.5

$$\int_{x_0}^{x_0+4h} f(x)dx = 4h\left(y_0 + 2\Delta y_0 + \frac{5}{3}\Delta^2 y_0 = \frac{2}{3}\Delta^3 y_0 + \frac{7}{90}\Delta^4 y_0\right)$$

$$= \frac{2h}{45}(7y_0 = 32y_1 + 12y_2 + 32y_{33} + 7y_4)$$

Similarly $\int_{x_0+4h}^{x_0+8h} f(x)dx = \frac{2h}{45}(7y_4 + 32y_5 + 12y_6 + 32y_7 + 7y_8)$ and so on.

Adding all these integrals from x_0 to $x_0 + nh$, where n is a multiple of 4, we get

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{2h}{45}(7y_0 + 32y_1 + 12y_2 + 32y_3 + 14y_4 + 32y_5 + 12y_6 + 32y_7 + 14y_8 + \cdots \quad (5)$$

This is known as *Boole's rule*.

NOTE **Obs.** *While applying (5), the number of sub-intervals should be taken as a multiple of 4.*

V. Weddle's rule. Putting $n = 6$ in (1) above and neglecting all differences above the sixth, we obtain

$$\int_{x_0}^{x_0+6h} f(x)dx = 6h\Big(y_0 + 3\Delta y_0 + \frac{9}{2}\Delta^2 y_0 + 4\Delta^3 y_0 + \frac{123}{60}\Delta^4 y_0 + \frac{11}{20}\Delta^5 x_0 + \frac{1}{6}\cdot\frac{41}{140}\Delta^6 y_0\Big)$$

If we replace $\frac{41}{140}\Delta^6 y_0$ by $\frac{3}{10}\Delta^6 y_0$, the error made will be negligible.

$$\therefore \qquad \int_{x_0}^{x_0+6h} f(x)dx = \frac{3h}{10}(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6)$$

Similarly

$$\int_{x_0+6h}^{x_0+12h} f(x)dx = \frac{3h}{10}(y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12}) \text{ and so on.}$$

Adding all these integrals from x_0 to $x_0 + nh$, where n is a multiple of 6, we get

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{3h}{10}(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 + 5y_7 + y_8 + \cdots) \quad (6)$$

This is known as *Weddle's rule*.

NOTE

Obs. *While applying (6), the number of sub-intervals should be taken as a multiple of 6. Weddle's rule is generally more accurate than any of the others. Of the two Simpson rules, the 1/3 rule is better.*

EXAMPLE 8.10

Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by using

(*i*) Trapezoidal rule,

(*ii*) Simpson's 1/3 rule,

(*iii*) Simpson's 3/8 rule,

~~(*iv*) Weddle's rule and compare the results with its actual value.~~

Solution:

Divide the interval (0, 6) into six parts each of width $h = 1$. The values of $f(x) = \frac{1}{1+x^2}$ are given below:

x	0	1	2	3	4	5	6
$f(x)$	1	0.5	0.2	0.1	0.0588	0.0385	0.027
$= y$	y_0	y_1	y_2	y_3	y_4	y_5	y_6

(*i*) By Trapezoidal rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{2}[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$= \frac{1}{2}[(1 + 0.027) + 2(0.5 + 0.2 + 0.1 + 0.0588 + 0.0385)] = 1.4108.$$

(*ii*) By Simpson's 1/3 rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{3}[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4$$

$$= \frac{1}{3}[(1 + 0.027) + 4(0.5 + 0.1 + 0.0385) + 2(0.2 + 0.0588)] = 1.3662.$$

(*iii*) By Simpson's 3/8 rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{8}[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3}{8}[(1 + 0.027) + 3(0.5 + 0.2 + 0.0588 + 0.0385) + 2(0.1)] = 1.3571$$

(*iv*) By Weddle's rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6]$$

$$= 0.3[1 + 5(0.5) + 0.2 + 6(0.1) + 0.0588 + 5(0.0385) + 0.027] = 1.3735$$

Also $\int_0^6 \frac{dx}{1+x^2} = \left|\tan^{-1} x\right|_0^6 = \tan^{-1} 6 = 1.4056$

This shows that the value of the integral found by Weddle's rule is the nearest to the actual value followed by its value given by Simpson's 1/3 rule.

EXAMPLE 8.11

Evaluate the integral $\int_0^1 \frac{x^2}{1+x^3}dx$ using Simpson's 1/3 rule. Compare the error with the exact value.

Solution:

Let us divide the interval (0, 1) into 4 equal parts so that h = 0.25. Taking $y = \frac{x^2}{(1+x^3)}$, we have

x:	0	0.25	0.50	0.75	1.00
y:	0	0.06153	0.22222	0.39560	0.5
	y_0	y_1	y_2	y_3	y_4

By Simpson's 1/3 rule, we have

$$\int_0^1 \frac{x^2}{1+x^3}dx = \frac{h}{3}[(y_0+y_4)+2(y_2)+4(y_1+y_3)]$$

$$= \frac{0.25}{3}[(0+0.5)+2(0.22222)+4(0.06153+0.3956)]$$

$$= \frac{0.25}{3}[0.5+0.44444+1.82852] = 0.23108$$

Also $\int_0^1 \frac{x^2}{1+x^3}dx = \frac{1}{3}\left|\log(1+x^3)\right|_0^1 = \frac{1}{3}\log e2 = 0.23108$

Thus the error = 0.23108 – 0.23105 = – 0.00003.

EXAMPLE 8.12

Use the Trapezoidal rule to estimate the integral $\int_0^2 ex^2 dx$ taking the number 10 intervals.

Solution:

Let $y = ex^2$, $h = 0.2$ and $n = 10$.

The values of x and y are as follows:

x:	0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
y:	1	1.0408	1.1735	1.4333	1.8964	2.1782	4.2206	7.0993	12.9358	25.5337	54.5981
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}

By the Trapezoidal rule, we have

$$\int_0^1 e^{x2}dx = \frac{h}{2}[(y_0+y_{10})+2(y_1+y_2+y_3+y_4+y_5+y_6+y_7+y_8+y_9)]$$

$$= \frac{0.2}{2}[(1+54.5981)+2(1.0408+1.1735+1.8964+2.1782$$

$$+4.2206+7.0993+12.9358+25.5337)]$$

Hence $\int_0^2 e^{x^2}dx = 17.0621.$

EXAMPLE 8.13

Use Simpson's 1/3rd rule to find $\int_0^{0.6} e^{-x2} dx$ by taking seven ordinates.

Solution:

Divide the interval (0, 0.6) into six parts each of width $h = 0.1$. The values of $y = f(x) = e^{-x^2}$ are given below:

x	0	0.1	0.2	0.3	0.4	0.5	0.6
x^2	0	0.01	0.04	0.09	0.16	0.25	0.36
y	1	0.9900	0.9608	0.9139	0.8521	0.7788	0.6977
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's 1/3rd rule, we have

$$\begin{aligned}\int_0^{0.6} e^{-x^2} dx &= \frac{h}{3}[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)] \\ &= \frac{0.1}{3}[(1 + 0.6977) + 4(0.99 + 0.9139 + 0.7788) + 2(0.9608 + 0.8521)] \\ &= \frac{0.1}{3}[1.6977 + 10.7308 + 3.6258] = \frac{0.1}{3}(16.0543) = 0.5351.\end{aligned}$$

EXAMPLE 8.14

Compute the value of $\int_{0.2}^{1.4} (\sin x - \log x + e^x) dx$ using Simpson's 3/8 rule.

Solution:

Let $y = \sin x - \log e\, x + e^x$ and $h = 0.2$, $n = 6$.

The values of y are as given below:

x:	0.2	0.4	0.6	0.8	1.0	1.2	1.4
y:	3.0295	2.7975	2.8976	3.1660	3.5597	4.0698	4.4042
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's 3/8 rule, we have

$$\begin{aligned}\int_{0.2}^{1.4} y dx &= \frac{3h}{8}[(y_0 + y_6) + 2(y_3) + 3(y_1 + y_2 + y_4 + y_5)] \\ &= \frac{3}{8}(0.2)[7.7336 + 2(3.1660) + 3(13.3247)] = 4.053\end{aligned}$$

Hence $\int_{0.2}^{1.4} (\sin x - \log e^x + e^x)dx = 4.053.$

NOTE **Obs. Applications of Simpson's rule.** *If the various ordinates in Section 8.5 represent equispaced cross-sectional areas, then Simpson's rule gives the volume of the solid. As such, Simpson's rule is very useful to civil engineers for calculating the amount of earth that must be moved to fill a depression or make a dam. Similarly if the ordinates denote velocities at equal intervals of time, the Simpson's rule gives the distance travelled. The following Examples illustrate these applications.*

EXAMPLE 8.15

The velocity v(km/min) of a moped which starts from rest, is given at fixed intervals of time t (min) as follows:

t:	2	4	6	8	10	12	14	16	18	20
v:	10	18	25	29	32	20	11	5	2	0

Estimate approximately the distance covered in twenty minutes.

Solution:

If s (km) be the distance covered in t (min), then $\frac{ds}{dt} = v$

$$\therefore \qquad |s|_{t=0}^{20} = \int_0^{20} v\,dt = \frac{h}{3}[X + 4.0 + 2.E], \text{ by Simpson's rule}$$

Here $h = 2$, $v_0 = 0$, $v_1 = 10$, $v_2 = 18$, $v_3 = 25$ etc.

$$\therefore \qquad X = v_0 + v_{10} = 0 + 0 = 0$$

$$O = v_1 + v_3 + v_5 + v_7 + v_9 = 10 + 25 + 32 + 11 + 2 = 80$$

$$E = v_2 + v_4 + v_6 + v_8 = 18 + 29 + 20 + 5 = 72$$

Hence the required distance $= |s|_{t=0}^{20} = \frac{2}{3}(0 + 4 \times 80 + 2 \times 72) = 309.33$ km.

EXAMPLE 8.16

The velocity v of a particle at distance s from a point on its linear path is given by the following table:

s (m):	0	2.5	5.0	7.5	10.0	12.5	15.0	17.5	20.0
v (m/sec):	16	19	21	22	20	17	13	11	9

Estimate the time taken by the particle to traverse the distance of 20 meter, using Boole's rule.

Solution:

If t sec be the time taken to traverse a distance s (m) then $\dfrac{ds}{dt} = v$

or
$$\frac{dr}{ds} = \frac{1}{v} = y\,(\text{say}),$$

then
$$|t|_{s=0}^{s=20} = \int_0^{20} y\,ds$$

Here $h = 2.5$ and $n = 8$.

Also $y_0 = \dfrac{1}{16}, y_1 = \dfrac{1}{19}, y_2 = \dfrac{1}{4}, y_3 = \dfrac{1}{22}, y_4 = \dfrac{1}{20},$

$y_5 = \dfrac{1}{17}, y_6 = \dfrac{1}{13} y_7 = \dfrac{1}{11}, y_8 = \dfrac{1}{9}.$

$\therefore$ By Boole's Rules, we have

$$|t|_{s=0}^{s=20} = \int_0^{20} y\,ds = \frac{2h}{45}[7y_0 + 32y_1 + 312y_2 + 14y_3 + 32y_5 + 12y_6 + 32y_7 + 14y_8]$$

$$= \frac{2(2.5)}{45}\left[7\left(\frac{1}{16}\right) + 32\left(\frac{1}{19}\right) + 12\left(\frac{1}{21}\right) + 32\left(\frac{1}{22}\right) + 14\left(\frac{1}{20}\right) + 32\left(\frac{1}{17}\right) + 12\left(\frac{1}{3}\right) + 32\left(\frac{1}{11}\right) + 14\left(\frac{1}{9}\right)\right]$$

$$= \frac{1}{9}(12.11776) = 1.35$$

Hence the required time = 1.35 sec.

EXAMPLE 8.17

A solid of revolution is formed by rotating about the x-axis, the area between the x-axis, the lines $x = 0$ and $x = 1$ and a curve through the points with the following co-ordinates:

x:	0.00	0.25	0.50	0.75	1.00
y:	1.0000	0.9896	0.9589	0.9089	0.8415

Estimate the volume of the solid formed using Simpson's rule.

Solution:

Here $h = 0.25$, $y_0 = 1$, $y_1 = 0.9896$, $y_2 = 0.9589$ etc.

$\therefore$ Required volume of the solid generated

$$= \int_0^1 \pi y^2 dx = \pi \cdot \frac{h}{3}[(y_0{}^2 + y_4{}^2) + 4(y_1{}^2 + y_3{}^2) + 2y_2{}^2]$$

$$= \frac{0.25\pi}{3}[\{1 + 0.8415)^2\} + 4\{(0.9896)^2 + (0.9089)^2\} + 2(0.9589)^2]$$

$$= \frac{0.25 \times 3.1416}{3}[1.7081 + 7.2216 + 1.839]$$

$$= 0.2618(10.7687) = 2.8192.$$

Exercises 8.2

1. Use trapezoidal rule to evaluate $\int_0^1 x^3 dx$ considering five sub-intervals.

2. Evaluate $\int_0^1 \frac{dx}{1+x}$ applying

(*i*) Trapezoidal rule
(*ii*) Simpson's 1/3 rule
(*iii*) Simpson's 3/8 rule.

3. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using

(*i*) Trapezoidal rule taking h = 1/4.
(*ii*) Simpson's 1/3rd rule taking h = 1/4.
(*iii*) Simpson's 3/8th rule taking h = 1/6.
(*iv*) Weddle's rule taking h = 1/6.
Hence compute an approximate value of ϖ in each case.

4. Find an approximate value of log$e$ 5 by calculating to four decimal places, by Simpson's 1/3 rule, $\int_0^5 \frac{dx}{4x+5}$ dividing the range into ten equal parts.

5. Evaluate $\int_0^4 e^x dx$ by Simpson's rule, given that

$e = 2.72$, $e^2 = 7.39$, $e^3 = 20.09$, $e^4 = 54.6$

and compare it with the actual value.

6. Find $\int_0^6 \frac{e^x}{1+x} dx$ using Simpson's 1/3 rule.

7. Evaluate $\int_0^2 e^{-x^2} dx$, using Simpson's rule. (Take $h = 0.25$)

8. Evaluate using Simpson's 1/3 rule,

(i) $\int_0^{\pi} \sin x \, dx$, taking eleven ordinates.

(ii) $\int_0^{\pi/2} \sqrt{\cos\theta}\, d\theta$, taking nine ordinates.

9. Evaluate by Simpson's 3/8 rule:

(i) $\int_0^9 \frac{dx}{1+x^3}$

(ii) $\int_0^{\pi/2} \sin x \, dx$

(iii) $\int_0^{\pi/2} e^{\sin x} dx$

(iv) $\int_0^{\pi} \sqrt{(1+3\cos^2\theta)}\, d\theta$, using six ordinates

10. Given that

x:	4.0	4.2	4.4	4.6	4.8	5.0	5.2
$\log x$:	1.3863	1.4351	1.4816	1.5261	1.5686	1.6094	1.6487

evaluate $\int_4^{5.2} \log x dx$ by

(a) Trapezoidal rule

(b) Simpson's 1/3 rule,

(c) Simpson's 3/8 rule,

~~(d) Weddle's rule~~

Also find the error in each case.

~~**11.** Use Boole's five point formula to compute $\int_0^{\pi/2} \sqrt{(\sin x)}\, / dx$.~~

12. The table below shows the temperature $f(t)$ as a function of time:

t:	1	2	3	4	5	6	7
$f(t)$:	81	75	80	83	78	70	60

Using Simpson's $\frac{1}{3}$ rule to estimate $\int_1^7 f(t) dt$.

13. A curve is drawn to pass through the points given by the following table:

x:	1	1.5	2	2.5	3	3.5	4
y:	2	2.4	2.7	2.8	3	2.6	2.1

Estimate the area bounded by the curve, x-axis and the lines $x = 1$, $x = 4$.

14. A river is 80 feet wide. The depth d in feet at a distance x feet. from one bank is given by the following table:

x:	0	10	20	30	40	50	60	70	80
y:	0	4	7	9	12	15	14	8	3

15. Find approximately the area of the cross-section.

A curve is drawn to pass through the points given by the following table:

x:	1	1.5	2	2.5	3	3.5	4
y:	2	2.4	2.7	2.8	3	2.6	2.1

Using Weddle's rule, estimate the area bounded by the curve, the x-axis, and the lines $x = 1$, $x = 4$.

16. A curve is given by the table:

x:	0	1	2	3	4	5	6
y:	0	2	2.5	2.3	2 1.	7 1.	5

The x-coordinate of the C.G. of the area bounded by the curve, the end ordinates, and the x-axis is given by $A\bar{x} = \int_0^6 xy\,dx$, where A is the area. Find $\bar{x}$ by using Simpson's rule.

17. A body is in the form of a solid of revolution. The diameter D in cms of its sections at distances x cm. from one end are given below. Estimate the volume of the solid.

x:	0	2.5	5.0	7.5	10.0	12.5	15.0
D:	5	5.5	6.0	6.75	6.25	5.5	4.0

18. The velocity v of a particle at distance s from a point on its path is given by the table:

s ft:	0	10	20	30	40	50	60
v ft/sec:	47	58	64	65	61	52	38

Estimate the time taken to travel sixty feet by using Simpson's 1/3 rule. Compare the result with Simpson's 3/8 rule.

19. The following table gives the velocity v of a particle at time t:

t (seconds):	0	2	4	6	8	10	12
v (m/sec.):	4	6	16	34	60	94	136

Find the distance moved by the particle in twelve seconds and also the acceleration at $t = 2$ sec.

20. A rocket is launched from the ground. Its acceleration is registered during the first eighty seconds and is given in the table below. Using Simpson's 1/3 rule, find the velocity of the rocket at $t = 80$ seconds.

t (sec):	0	10	20	30	40	50	60	70	80
f (cm/sec2):	30	31.63	33.34	35.47	37.75	40.33	43.25	46.69	50.67

21. A reservoir discharging water through sluices at a depth h below the water surface has a surface area A for various values of h as given below:

h (ft.):	10	11	12	13	14
A (sq. ft.):	950	1070	1200	1350	1530

If t denotes time in minutes, the rate of fall of the surface is given by $dh/dt = -48\sqrt{h}/A$.

Estimate the time taken for the water level to fall from fourteen to ten feet above the sluices.

8.6 Errors in Quadrature Formulae

The error in the quadrature formulae is given by

$$E = \int_a^b y\,dx - \int_a^b P(x)\,dx$$

where $P(x)$ is the polynomial representing the function $y = f(x)$, in the interval $[a, b]$.

Error in Trapezoidal rule. Expanding $y = f(x)$ around $x = x_0$ by Taylor's series, we get

$$y = y_0 + (x - x_0)y_0' + \frac{(x - x_0)^2}{2!}y_0'' + \ldots \tag{1}$$

$$\therefore \int_{x_0}^{x_0+h} y\,dx = \int_{x_0}^{x_0+h}\left[y_0 + (x - x_0)y_0' + \frac{(x - x_0)^2}{2!}y_0'' + \ldots\right]dx \tag{2}$$

$$= y_0h + \frac{h^2}{2!}y_0' + \frac{h^3}{3!}y_0'' + \ldots$$

Also A_1 = area of the first trapezium in the interval $[x_0, x_1] = \frac{1}{2}h(y_0 + y_1)$ (3)

Putting $x = x_0 + h$ and $y = y_1$ in (1), we get $y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \ldots$

Substituting this value of y_1 in (3), we get

$$A_1 = \frac{1}{2}h\left[y_0 + y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \ldots\right] \tag{4}$$

$$= hy_0 + \frac{h^2}{2}y_0' + \frac{h^3}{2 \cdot 2!}y_0'' + \ldots$$

$\therefore$ Error in the interval $[x_0, x1] = \int_{x_0}^{x_1} ydx - A_1$ [(2) – (4)]

$$= \frac{1}{3!} - \frac{1}{2.2!}h^3 y_0'' + \ldots = -\frac{h^3}{12}y_0'' + \ldots$$

i.e., Principal part of the error in $[x_0, x_1] = -\frac{h^3}{12}y_0''$

Hence the total error $E = -\frac{h^3}{12}[y_0'' + y_1'' + \ldots + y_{n-1}'']$

Assuming that $y''(X)$ is the largest of the n quantities $y_0'', y_1'', \ldots y_{n-1}''$ we obtain

$$E < -\frac{nh^3}{12}y''(X) = -\frac{(b-a)h^2}{12}y''(X) \qquad [\because nh = b - a \ \ldots(5)$$

Hence the error in the trapezoidal rule is of the order h^2.

Error in Simpson's 1/3 rule. Expanding $y = f(x)$ around $x = x_0$ by Taylor's series, we get (1).

$\therefore$ Over the first doubt strip, we get

$$\int_{x_0}^{x_2} ydx = \int_{x_0}^{x_0+2h}\left[y_0 + (x - x_0)y_0' + \frac{(x - x_0)^2}{2!}y_0'' + \ldots\right]dx \tag{6}$$

$$= 2hy_0 + \frac{4h^2}{2!}y_0' + \frac{8h^3}{3!}y_0'' + \frac{16h^4}{4!}y_0''' + \frac{32h^5}{5!}y_0^{iv} + \ldots$$

Also A_1 = area over the first doubt strip by Simpson's 1/3 rule

$$= \frac{1}{3}h(y_0 + 4y_1 + y_2) \tag{7}$$

Putting $x = x_0 + h$ and $y = y_1$ in (1), we get

$$y_1 = y_0 + hy_0{}' + \frac{h^2}{2!}y'' + \frac{h^3}{3!}y_0{}''' + \ldots$$

Again putting $x = x_0 + 2h$ and $y = y_2$ in (1), we have

$$y_2 = y_0 + 2hy_0{}' + \frac{4h^2}{2!}y_0{}'' + \frac{8h^3}{3!}y_0{}''' + \ldots$$

Substituting these values of y_1 and y_2 in (7), we get

$$A_1 = \frac{h}{3}\left[y_0 + 4\left(y_0 + hy_0{}' + \frac{h^2}{2!}y_0{}''' + \cdots\right) + y_0 + \left(2hy_0{}' + \frac{4h^2}{2!}y_0{}'' + \frac{8h^3}{3!}y_0{}''' + \cdots\right)\right]$$

$$= 2hy_0 + 2h^2y_0{}' + \frac{4h^3}{3}y_0{}'' + \frac{2h^2}{3}y_0{}''' + \frac{5h^5}{18}y_0{}^{iv} + \cdots \quad (8)$$

$\therefore$ Error in the interval $[x_0, x_2]$

$$= \int_{x_0}^{x_2} ydx - A_1 = \left(\frac{4}{5} - \frac{5}{18}\right)h^5 y_0{}^{iv} + \cdots \quad [(6)-(8)]$$

i.e., Principal part of the error in $[x_0, x_2]$

$$= \left(\frac{4}{15} - \frac{5}{18}\right)h^5 y_0{}^{iv} = -\frac{h^5}{90}y_0{}^{iv}$$

Similarly principal part of the error in $[x_2, x_4] = -\frac{h^5}{90}y_2{}^{iv}$ and so on.

Hence the total error $E = -\frac{h^5}{90}[y_0{}^{iv} + y_2{}^{iv} + \cdots + y^{iv}{}_{2(n-1)}]$

Assuming the $y^{iv}(X)$ is the largest of $y_0{}^{iv}, y_2{}^{iv}, \ldots, y^{iv}{}_{2n-2}$, we get

$$E < -\frac{nh^5}{90}y_0{}^{iv}(X) = -\frac{(b-a)h^4}{180}y^{iv}(X) \quad [\because 2nh = b-a \ldots(9)]$$

i.e., the error in Simpson's 1/3 -rule is of the order h^4.

Error in Simpson's 3/8 rule. Proceeding as above, here the principal part of the error in the interval $[x_0, x_3]$

$$= -\frac{3h^5}{80}y^{iv} \quad (10)$$

Error in Boole's rule. In this case, the principal part of the error in the interval

$$[x_0, x_4] = -\frac{8h^7}{945} y^{iv} \tag{11}$$

Error in Weddle's rule. In this case, principle part of the error in the interval

$$[x_0, x_6] = \frac{h^7}{140} y_0 iv \tag{12}$$

8.7 Romberg's Method

In Section 8.5, we have derived approximate quadrature formulae with the help of finite differences method. Romberg's method provides a simple modification to these quadrature formulae for finding their better approximations. As an illustration, let us improve upon the value of the integral

$$I = \int_a^b f(x)dx,$$

by the Trapezoidal rule. If I_1, I_2 are the values of I with sub-intervals of width h_1, h_2 and E_1, E_2 their corresponding errors, respectively, then

$$E_1 = -\frac{(b-a)h_1^{\,2}}{12} y''(X), E_2 = -\frac{(b-a)^2 h_2^{\,2}}{12} y''(\bar{X})$$

Since $y''(\bar{X})$ is also the largest value of $y''(x)$, we can reasonably assume that $y''(X)$ and $y''(\bar{X})$ are very nearly equal.

$$\therefore \qquad \frac{E_1}{E_2} = \frac{h_1^2}{h_2^2} \quad \text{or} \quad \frac{E_1}{E_2 - E_1} = \frac{h_1^2}{h_2^{\,2} - h_1^2} \tag{1}$$

Now since $I = I_1 + E_1 = I_2 + E_2$,

$$\therefore \qquad E_2 - E_1 = I_1 - I_2 \tag{2}$$

From (1) and (2), we have

$$E_1 = \frac{h_1^{\,2}}{h_2^{\,2} - h_1^{\,2}} (I_1 - I_2)$$

Hence $I = I_1 + E_1 = I_1 + \dfrac{h_1^{\,2}}{h_2^{\,2} - h_1^{\,2}}(I_1 - I_2)$ *i.e.*, $I = \dfrac{I_1 h_2^{\,2} - I_2 h_1^{\,2}}{h_2^{\,2} - h_1^{\,2}}$ (3)

which is a better approximation of I.

To evaluate I systematically, we take $h_1 = h$ and $h_2 = \frac{1}{2}h$

so that (3) gives $I = \dfrac{I_1(h/2)^2 - I_2 h_2^{\,2}}{(h/2)^2 - h^2} = \dfrac{4I_2 - I_1}{3}$

i.e., $I(h,h/2) = \frac{1}{3}[4I(h/2) - I(h)]$ (4)

Now we use the trapezoidal rule several times successively halving h and apply (4) to each pair of values as per the following scheme:

$$\begin{array}{llll}
I(h) & & & \\
 & I(h, h/2) & & \\
I(h/2) & & I(h, h/2, h/4) & \\
 & I(h/2, h/4) & & I(h, h/2, h/4, h/8) \\
I(h/4) & & I(h/2, h/4, h/8) & \\
 & I(h/4, h/8) & & \\
I(h/8) & & &
\end{array}$$

The computation is continued until successive values are close to each other. This method is called *Richardson's deferred approach to the limit* and its systematic refinement is called *Romberg's method*.

EXAMPLE 8.18

Evaluate $\int_0^1 \frac{dx}{1+x}$ correct to three decimal places using Romberg's method. Hence find the value of $\log_e 2$.

Solution:

Taking $h = 0.5$, 0.25, and 0.125 successively, let us evaluate the given integral by the Trapezoidal rule.

(*i*) When $h = 0.5$, the values of $y = (1 + x)^{-1}$ are:

x:	0	0.5	1
y:	1	0.6666	0.5

$\therefore \quad I = \frac{0.5}{2}(1 + 0.5 + 2 \times 0.6666) = 0.7083.$

(*ii*) When $h = 0.25$, the values of $y = (1 + x)^{-1}$ are:

x:	0	0.25	0.5	0.75	1
y:	1	0.8	0.6666	0.5714	0.5

$$\therefore \qquad I=\frac{0.25}{2}[(1+0.5)+2(0.8+0.666+0.5714)]=0.697$$

(*iii*) When $h = 0.125$, the values of $y = (1 + x)^{-1}$ are:

x:	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
y:	1	0.8889	0.8 0.	7272	0.6667	0.6153	0.5714	0.5333	0.5

$$\therefore \qquad I=\frac{0.125}{2}[(1+0.5)+2(0.8889+0.8+0.7272+0.6667$$
$$+0.6513+0.5714+0.5333)]$$
$$=0.6941$$

Using Romberg's formulae, we obtain

$$I(h,h/2)=\frac{1}{3}[4I(h/2)-I(h)]=\frac{1}{3}[4\times 0.697-0.7083]=0.6932$$

$$I(h/2,h/4)=\frac{1}{3}[4I(h/4)-I(h/2)]=\frac{1}{3}[4\times 0.6941-0.697]=0.6931$$

$$I(h,h/2,h/4)=\frac{1}{3}[4I(h/2,h/4)-I(h,h/2)]=0.6931$$

Hence the value of the integral $\int_0^1 \frac{dx}{1+x}=0.693$ (*i*)

Also $\int_0^1 \frac{dx}{1+x}=\left|\log(1+x)\right|_0^1=\log 2$ (*ii*)

Hence from (*i*) and (*ii*), we have

$\log_e 2 = 0.693.$

EXAMPLE 8.19

Use Romberg's method to compute $\int_0^1 \frac{dx}{1+x^2}$ correct to four decimal places.

Solution:

We take $h = 0.5$, 0.25 and 0.125 successively and evaluate the given integral using the Trapezoidal rule.

(*i*) When $h = 0.5$, the values of $y = (1 + x^2)^{-1}$ are

x:	0	0.5	1.0
y:	1	0.8	0.5

$$\therefore \qquad I = \frac{0.5}{2}[1 + 2 \times 0.8 \times 0.5) = 0.775$$

(*ii*) When $h = 0.25$, the values of $y = (1 + x^2)^{-1}$ are

x:	0	0.25	0.5	0.75	1.0
y:	1	0.9412	0.8	0.64	0.5

$$\therefore I = \frac{0.25}{2}[1 + 2(0.9412 + 0.8 + 0.64) + 0.5] = 0.7828$$

(*iii*) When $h = 0.125$, we find that $I = 0.7848$

Thus we have

$I(h) = 0.7750$, $I(h/2) = 0.7828$, $I(h/4) = 0.7848$

Now using (4) above, we obtain

$$I(h,h/2) = \frac{1}{3}[4I(h/2) - I(h)] = \frac{1}{3}(3.1312 - 0.775) = 0.7854$$

$$I(h/2,h/4) = \frac{1}{3}[(4I(h/4) - I(h/2)] = \frac{1}{2}(3.1392 - 0.7828) = 0.7855$$

$$I(h,h/2,h/4) = \frac{1}{3}[4I(h/2,h/4) - I(h,h/2)] = \frac{1}{3}(3.142 - 0.7854) = 0.7855$$

$\therefore$ The table of these values is

0.7750		
	0.7854	
0.7828		0.7855
	0.7855	
0.7848		

Hence the value of the integral = 0.7855.

EXAMPLE 8.20

Evaluate the integral $\int_0^{0.5} \left(\frac{x}{\sin x}\right) dx$ using Romberg's method, correct to three decimal places.

Solution:

Taking $h = 0.25$, 0.125, 0.0625 successively, let us evaluate the given integral by using Simpson's 1/3 rule.

(i) When $h = 0.25$, the values of $y = \dfrac{x}{\sin x}$ are

x:	0	0.25	0.5
y:	1	1.0105	1.0429
y_0	y_1	y_2	

$\therefore$ By Simpson's rule,

$$I = \frac{h}{3}[(y_0 + y_2) + 4y_1] = \frac{0.25}{3}[(1 + 1.0429) + 1.0105]$$
$$= 0.5071$$

(ii) When $h = 0.125$, the values of y are

x:	0	0.125	0.25	0.375	0.5
y:	1	1.0026	1.0105	1.1003	1.0429
	y_0	y_1	y_2	y_3	y_4

$\therefore$ By Simpson's rule

$$I = \frac{h}{3}[(y_0 + y_4) + 4(y_1 + y_3) + 2y_2]$$
$$= \frac{0.125}{3}[(1 + 1.0429) + 4(1.0026 + 1.1003) + 2(1.0105)]$$
$$= 0.5198$$

(iii) When $h = 0.0625$, the values of y are

x:	0	0.0625	0.125	0.1875	0.25	0.3125	0.1875	0.4375	0.5
y:	1	0.0006	1.0026	1.0059	1.0157	1.0165	1.1003	1.0326	1.0429
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

$\therefore$ By Simpson's rule:

$$I = \frac{h}{3}[(y_0 + y_8) + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)]$$
$$= \frac{0.0625}{3}[(1 + 1.0429) + 4(1.0006 + 1.0059 + 1.0165 + 1.0326$$
$$+ 2(1.0026 + 1.0105 + 1.1003)]$$
$$= 0.510253$$

Using Romberg's formulae, we obtain

$$I(h,\frac{h}{2}) = \frac{1}{3}\left[4I\left(\frac{h}{2}\right) - I(h)\right] = 0.5241$$

$$I(\frac{h}{2},\frac{h}{4}) = \frac{1}{3}\left[4I\left(\frac{h}{4}\right) - I\left(\frac{h}{2}\right)\right] = 0.5070$$

$$I(h.\frac{h}{2},\frac{h}{4}) = \frac{1}{3}\left[4I\left(\frac{h}{2},\frac{h}{4}\right) - I\left(h,\frac{h}{2}\right)\right] = 0.5013$$

Hence $\int_0^{0.5}\left(\frac{x}{\sin x}\right)dx = 0.501$

8.8 Euler-Maclaurin Formula

Taking $\Delta F(x) = f(x)$, we define the inverse operator Δ^{-1} as

$$F(x) = \Delta^{-1} f(x) \quad (1)$$

Now $\qquad F(x_1) - F(x_0) = \Delta F(x_0) = f(x_0)$

Similarly, $\qquad F(x_2) - F(x_1) = f(x_1)$

....

$$F(x_n) - F(x_{n-1}) = f(x_{n-1})$$

Adding all these, we get

$$F(x_n) - F(x_0) = \sum_{i=0}^{n=1} f(x_i) \tag{2}$$

where $x_0, x_1, \ldots.., x_n$ are the $(n + 1)$ equispaced values of x with difference h.

From (1), we have

$$F(x) = \Delta^{-1} f(x) = (E+1)^{-1} f(x) = (e^{hD} - 1)^{-1} f(x) \qquad [\because E = ehD]$$

$$= \left[\left(1 + hD + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + \cdots\right) - 1\right]^{-1} f(x)$$

$$= (hD)^{-1}\left[1 + \frac{hD}{2!} + \frac{h^2D^2}{3!} + \cdots\right]^{-1} f(x) \tag{3}$$

$$= \frac{1}{h}D^{-1}\left[1 - \frac{hD}{32} + \frac{h^2D^2}{12} - \frac{h^4D^4}{720} + \cdots\right] f(x)$$

$$= \frac{1}{h}\int f(x)dx - \frac{1}{2}f(x) + \frac{h}{12}f'(x) - \frac{h^3}{720}f'''(x) + \cdots$$

Putting $x = x_n$ and $x = x_0$ in (3) and then subtracting, we get

$$F(x_n) - F(x_0) = \frac{1}{h}\int_{x_0}^{x_n} f(x)dx - \frac{1}{2}[f(x_n) - f(x_0)] + \frac{h}{12}[f'(x_n) - f'(x_0)] - \frac{h^3}{720}[f'''(x_n) - f'''(x_0)] + \cdots \quad (4)$$

$\therefore$ From (2) and (4), we have

$$\sum_{i=0}^{n-1} f(x_i) = \frac{1}{h}\int_{x_0}^{x_n} f(x)dx - \frac{1}{2}[f(x_n) - f(x_0)] + \frac{h}{12}[f'(x_n) - f'(x_0)] - \frac{h^3}{720}[f'''(x_n) - f'''(x_0)] + \cdots$$

$$i.e., \frac{1}{h}\int_{x_0}^{x_n} f(x)dx = \sum_{i=0}^{n=1} f(x_i) + \frac{1}{2}[f(x_n) - f(x_0)] - \frac{h}{12}[f'(x_n) - f'(x_0)] + \frac{h^3}{720}[f'''(x_n) - f'''(x_0)] + \cdots$$

$$= \frac{1}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)] - \frac{h}{12}[f'(x_n) - f'(x_0)] + \frac{h^3}{720}[f'''(x_{n)} - f'''(x_0)] + \cdots$$

Hence $$\int_{x_0}^{x_0 \_ nh} y\,dx = \frac{h}{2}[y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n] - \frac{h^2}{12}(y_n' - y_0{}') + \frac{h^4}{720}(y_n{}''' - y_0{}''') + \ldots \quad (5)$$

which is called the *Euler-Maclaurin formula*.

NOTE

Obs. *The first term on the right-hand side of (5) represents the approximate value of the integral obtained from trapezoidal rule and the other terms denote the successive corrections to this value. This formula is often used to find the sum of a series of the form*

$$y(x_0) + y(x_0 + h) + \ldots + y(x_0 + nh).$$

EXAMPLE 8.21

Using the Euler-Maclaurin formula, find the value of loge 2 from $\int_0^1 \frac{dx}{1+x}$

Solution:

Taking $y = \dfrac{1}{(1+x)}, x_0 = 0, n = 10, h = 0.1,$ we have

$$y' = -\frac{1}{(1+x)^2} \quad \text{and} \quad y''' = \frac{-6}{(1+x)^4}.$$

Then the Euler-Maclaurin formula gives

$$\int_0^1 \frac{dx}{1+x} = \frac{0.1}{2}\left[\frac{1}{1+0} + \frac{2}{1+0.1} + \frac{2}{1+0.2} + \frac{2}{1+0.3} + \frac{2}{1+0.1}\right.$$
$$\left. + \frac{2}{1+0.5} + \frac{2}{1+0.6} + \frac{2}{1+0.7} + \frac{2}{1+0.8} + \frac{2}{1+0.9} + \frac{1}{1+1}\right]$$
$$-\frac{(0.1)^2}{12}\left[\frac{-1}{(1+1)^2} - \frac{-1}{(1+0)^2}\right] + \frac{(0.1)^4}{720}\left[\frac{-6}{(1+1)^4} - \frac{-6}{(1+0)^4}\right]$$
$$= 0.693773 - 0.000625 + 0.000002 = 0.693149$$

Also $\displaystyle\int_0^1 \frac{dx}{1+x} = \left|\log(1+x)\right|_0^1 = \log e^2$

Hence $\log_e 2 = 0.693149$ approx.

EXAMPLE 8.22

Apply the Euler-Maclaurin formula to evaluate

$$\frac{1}{51^2} + \frac{1}{53^2} + \frac{1}{55^2} + \ldots + \frac{1}{99^2}$$

Solution:

Taking $y = \dfrac{1}{x^2}, x_0 = 51, h = 2, n = 24,$ we have $y' = \dfrac{-2}{x^3}, y''' = \dfrac{-24}{x^5}$

Then the Euler-Maclaurin formula gives

$$\int_{51}^{99} \frac{dx}{x^2} = \frac{2}{2}\left[\frac{1}{51^2} + \frac{2}{53^2} + \frac{2}{55^2} + \ldots + \frac{2}{97^2} + \frac{1}{99^2}\right]$$
$$-\frac{(2)^2}{12}\left[\frac{-2}{99^3} - \frac{-2}{51^3}\right] + \frac{(2)^4}{720}\left[\frac{-24}{99^5} - \frac{-24}{51^5}\right]$$

$$\therefore \quad \frac{1}{51^2} + \frac{1}{53^2} + \frac{1}{55^2} + \ldots + \frac{1}{99^2} = \frac{1}{2}\int_{51}^{99} \frac{dx}{x^2}$$
$$+\frac{1}{2}\left(\frac{1}{51^2} + \frac{1}{99^2}\right) + \frac{1}{3}\left(\frac{1}{51^3} - \frac{1}{99^3}\right) - \frac{8}{30}\left(\frac{1}{51^5} - \frac{1}{99^5}\right) + \ldots$$
$$= \frac{1}{2}\left|\frac{1}{x}\right|_{51}^{99} + 0.000243 + 0.0000022 - \ldots = 0.00499 \text{ approx.}$$

8.9 Method of Undetermined Coefficients

This method is based on imposing certain conditions on a preassigned formula involving certain unknown coefficients and then using these conditions for evaluating these unknown coefficients. Assuming the formula to be exact for the polynomials 1, x, ¼, x^n respectively and taking y_i for $y(x_i)$, we shall determine the unknown coefficients to derive the formulae.

Differentiation formulae. We first derive the two-term formula by assuming

$$y_0' = a_0 y_0 + a_1 y_1 \tag{1}$$

where the unknown constants a_0, a_1 are determined by making (1) exact for $y(x) = 1$ and x respectively.

So, putting $y(x) = 1$, x successively in (1), we get

$0 = a_0 + a_1$ and $1 = a_0 x_0 + a_1(x_0 + h)$

where $a_1 = 1/h$ and $a_0 = -1/h$.

Hence
$$y_0' = \frac{1}{h}(y_1 - y_0) \tag{2}$$

The three-term formula can be derived by taking

$$y_0' = a_{-1}y_{-1} + a_0 y_0 + a_1 y_1 \tag{3}$$

where the unknowns a_{-1}, a_0, a_1 are determined by making (3) exact for $y(x) = 1, x, x^2$, respectively.

$$\therefore \quad 0 = a_{-1} + a_0 + a_1$$
$$1 = a_{-1}(x_0 - h) + a_0 x_0 + a_1(x_0 + h)$$
and
$$2x_0 = a_{-1}(x_0 - h)^2 + a_0 x_0^2 + a_1(x_0 + h)^2.$$

To solve these equations, we shift the origin to x_0 *i.e.*, $x_0 = 0$. As such, y_0' being slope of the tangent to the curve $y = f(x)$ at $x = x_0$ remains unaltered. Thus the equations reduce to

$$a_{-1} + a_0 + a_1 = 0,$$
$$-a_{-1} + a_1 = 1/h \text{ and } a_{-1} + a_1 = 0,$$

giving $a_{-1} = -1/2h$, $a_0 = 0$, $a_1 = 1/2h$

Hence
$$y_0' = \frac{1}{2h}(y_1 - y_{-1}), \tag{4}$$

Similarly for second order derivative, taking

$$y_0'' = a_{-1}y_{-1} + a_0 y_0 + a_1 y_1$$

and making it exact for $y(x) = 1, x, x^2$ and putting $x_0 = 0$, we get

$$y_0'' = \frac{1}{h^2}(y_1 - 2y_0 + y_{-1}) \tag{5}$$

Integration formulae. *The two-term formula* is derived by assuming

$$\int_{x_0}^{x_0+h} y\,dx = a_0 y_0 + a_1 y_1 \tag{6}$$

where the unknowns $a0$, $a1$ are determined by making (6) exact for $y(x) = 1, x$ respectively.

So putting $y(x) = 1, x$ successively in (6), we get

$$a_0 + a_1 = \int_{x_0}^{x_0+h} 1 \cdot dx = h$$

$$a_0 x_0 + a_1(x_0 + h) = \int_{x_0}^{x_0+h} x \cdot dx = \frac{1}{2}[(x_0 + h)^2 - x_0^2]$$

To solve these, we shift the origin to x_0 and take $x_0 = 0$.

∴ The above equations reduce to

$$a_0 + a_1 = h \text{ and } a_1 = \frac{1}{2}h, \text{whence } a_1 = \frac{1}{2}h, a_0 = \frac{1}{2}h$$

Hence $\int_{x_0}^{x_0+h} y\,dx = \frac{h}{2}(y_0 + y_1)$ which is *trapezoidal rule.* (7)

The three-term formula is derived by assuming

$$\int_{x_0-h}^{x_0+h} y\,dx = a_{-1}y_{-1} + a_0 y_0 + a_1 y_1 \tag{8}$$

where the unknowns a_{-1}, a_0, a_1 are determined by making (8) exact for $y(x) = 1, x, x^2$ respectively.

So putting $y = 1, x, x^2$ successively in (8), we obtain

$$a_{-1} + a_0 + a_1 = \int_{x_0-h}^{x_0+h} 1 \cdot dx = 2h$$

$$a_{-1}(x_0 - h) + a_0 x_0 + a(x_0 + h) = \int_{x_0-h}^{x_0+h} x\,dx = \frac{1}{2}[(x_0 + h)^2 - (x_0 - h)^2]$$

$$a_{-1}(x_0 - h)^2 + a_0 x_0{}^2 + a_1(x_0 + h)^2 = \int_{x_{0-h}}^{x_0+h} x^2 dx = \frac{1}{3}[(x_0 + h)^3 - (x_0 - h)^3]$$

To solve these equations, we shift the origin to x_0 and take $x_0 = 0$.

$\therefore$ The above equations reduce to

$a_{-1} + a_0 + a_1 = 2h,\ -a_{-1} + a_1 = 0$ and $a_{-1} + a_1 = \frac{2}{3}h$

Solving these, we get $a_{-1} = \frac{1}{3}h = a_1, a_0 = \frac{4}{3}h$

Hence $\int_{x_0-h}^{x_0+h} y\,dx = \frac{h}{3}(y_{-1} + 4y_0 + y_1)$ which is *Simpson's rule*. (9)

8.10 Gaussian Integration

So far the formulae derived for evaluation of $\int_a^b f(x)dx$, required the values of the function at equally spaced points of the interval. Gauss derived a formula which uses the same number of functional values but with different spacing and yields better accuracy.

Gauss formula is expressed as

$$\int_{-1}^{1} f(x)dx = w_1 f(x_1) + w_2 f(x_2) + \cdots + w_n f(x_n) = \sum_{i=1}^{n} w_i f(x_i) \tag{1}$$

where *wi* and *xi* are called the *weights* and *abscissae,* respectively. *The abscissae and weights are symmetrical with respect to the middle point of the interval*. There being $2n$ unknowns in (1), $2n$ relations between them are necessary so that the formula is exact for all polynomials of degree not exceeding $2n - 1$. Thus we consider

$$f(x) = c_0 + c_1 x + c_2 x + \cdots + c_{2n-1} x^{2n-1} \tag{2}$$

Then (1) gives

$$\int_{-1}^{1} f(x)dx = \int_{-1}^{1} (c_0 + c_1 x + c_2 x + \cdots + c_{2n-1} x^{2n-1})dx \tag{3}$$

$$= 2c_0 + \frac{2}{3}c_2 + \frac{2}{5}c_4 + \cdots$$

Putting $x = x_i$ in (2), we get

$$f(x_i) = c_0 + c_1 x_i + c_2 x_i^2 + c_3 x_i^3 + \ldots + c_{2n-1} x_i^{2n-1}$$

Substituting these values on the right hand side of (1), we obtain

$$\int_{-1}^{1} f(x)dx = w1(c_0 + c_1x_1 + c_2x_1^{\ 2} + c_3x_1^{\ 3} + \ldots c_{2n-1}x_1^{\ 2n-1}$$
$$+ w_2(c_0 + c_1x_2 + c_2x_2^{\ 2} + c_3x_2^{\ 3} + \ldots. + c_{2n-1}x_2^{\ 2n-1})$$
$$+ w_3(c_0 + c_1x_3 + c_2x_3^{\ 2} + c_3x_3^{\ 3} + \ldots. + c_{2n-1}\, x_3^{\ 2n-1})$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+ wn(c_0 + c_1xn + c_2xn^2 + c_3xn^3 + \ldots. + c_{2n-1}\, xn^{2n-1})$$
$$= c_0\,(w_1 + w_2 + w_3 + \cdots + w_n) + c_1\,(w_1x_1 + w_2x_2 + w_3x_3 + \cdots + w_n\, x_n)$$
$$+ c_2\,(w_1x_1^{\ 2} + w_2x_2^{\ 2} + w_3x_3^{\ 2} + \cdots + w_n\, x_n^{\ 2})$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+ c_{2n-1}\,(w_1x_1^{\ 2n-1} + w_2\, x_2^{\ 2n-1} + w_3\, x_3^{\ 2n-1} + \cdots + w_n\, x_n^{\ 2n-1}) \qquad (4)$$

But the equations (3) and (4) are identical for all values of c_i, hence comparing coefficients of c_i, we obtain $2n$ equations in $2n$ unknowns w_i and x_i $(i = 1, 2, \ldots\ldots, n)$.

$$\left.\begin{aligned} w_1 + w_2 + w_3 + \cdots + \ w_n &= 2 \\ w_1x_1 + w_2x_2 + w_3x_3 + \cdots + w_nx_n &= 0 \\ w_1x_1^{\ 2} + w_2x_2^{\ 2} + w_3x_3^{\ 2} + \cdots + w_nx_n^{\ 2} &= \frac{2}{3} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ w_1x_1^{\ 2n-1} + w_2x_2^{\ 2n-1} + w_3x_3^{\ 2n-1} + \cdots + w_nx_n^{\ 2n-1} &= \ 0 \end{aligned}\right\} \qquad (5)$$

The solution of the above equations is extremely complicated. It can however, be shown that x_i are the zeros of the $(n + 1)$th Legendre polynomial.

Gauss formula for n = 2 is

$$\int_{-1}^{1} f(x)dx = w_1 f(x_1) + w_2 f(x_2)$$

Then the equations (5) become

$$w_1 + w_2 = 2$$
$$w_1x_1 + w_2x_2 = 0$$
$$w_1x_1^{\ 2} + w_2x_2^{\ 2} = \frac{2}{3}$$
$$w_1x_1^{\ 3} + w_2x_2^{\ 3} = 0$$

Solving these equations, we obtain

$$w_1 = w_2 = 1,\ x_1 = -\,1/\sqrt{3} \text{ and } x_2 = 1/\sqrt{3}.$$

Thus *Gauss formula for n* = 2 is

$$\int_{-1}^{1} f(x)dx = f(-1\sqrt{3}) + f(1/\sqrt{3}) \tag{6}$$

which gives the correct value of the integral of $f(x)$ in the range (– 1, 1) for any function up to third order. Equation (6) is also known as **Gauss-Legendre formula.**

Gauss formula for n = 3 is

$$\int_{-1}^{1} f(x)dx = \frac{8}{9}f(0) + \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right] \tag{7}$$

which is exact for polynomials upto degree 5.

The abscissae xi and the weights wi in (*1*) *are tabulated for different values of n*. The following table lists the abscissae and weights for values of n from 2 to 5.

TABLE 8.1 Gauss integration: Abscissae and Weights

N	x_i	w_i
2	– 0.57735	1.0000
	0.57735	1.0000
3	– 0.7746	0 0.55555
	0.00000	0.88889
	0.77460	0.55555
4	– 0.86114	0.34785
	– 0.33998	0.65214
	0.33998	0.65214
	0.86114	0.34785
5	– 0.90618	0.23693
	– 0.53847	0.47863
	0.00000	0.56889
	0.53847	0.47863
	0.90618	0.23693

Gauss formula imposes a restriction on the limits of integration to be from – 1 to 1.

In general, the limits of the integral $\int_a^b f(x)dx$ are changed to – 1 to 1 by means of the transformation

$$x = \frac{1}{2}(b-1)u + \frac{1}{2}(b+a) \tag{8}$$

EXAMPLE 8.23

Evaluate $\int_{-1}^{1} \frac{dx}{1+x^2}$ using Gauss formula for $n = 2$ and $n = 3$.

Solution:

(i) Gauss formula for $n = 2$ is

$$I = \int_{-1}^{1} \frac{dx}{1+x^2} = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) \text{ where } f(x) = \frac{1}{1+x^2}$$

$$\therefore I = \frac{1}{1+(-1\sqrt{3})^2} + \frac{1}{1+(1/\sqrt{3})^2} = \frac{3}{4} + \frac{3}{4} = 1.5.$$

(ii) Gauss formula for $n = 3$ is

$$I = \frac{8}{9} f(0) + \frac{5}{9}\left[f -\left(\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right] \text{ where } f(x) = \frac{1}{1+x^2}$$

$$\text{Thus } I = \frac{8}{9}(1) + \frac{5}{9}\left(\frac{5}{8} + \frac{5}{8}\right) = \frac{8}{9} + \frac{50}{72} = 1.5833.$$

EXAMPLE 8.24

Using the three-point Gaussian quadrature formula, evaluate $\int_{0}^{1} \frac{dx}{1+x}$

Solution:

We first change the limits (0, 1) to – 1 to 1 by (8) above, so that

$$x = \frac{1}{2}(1-0)u\frac{1}{2}(1+0) = \frac{1}{2}(u+1).$$

$$\therefore \qquad I = \int_{0}^{1} \frac{dx}{1+x} = \int_{-1}^{1} \frac{\frac{1}{2}du}{1+\frac{1}{2}(u+1)} = \int_{-1}^{1} \frac{du}{u+3}$$

Gauss-formula for $n = 3$ is

$$I = \frac{8}{9} f(0) + \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right) \text{ where } f(x) = \frac{1}{1+x^2}$$

$$\text{Thus} \qquad I = \frac{8}{9}\left(\frac{1}{3}\right) + \frac{5}{9}\left\{\frac{1}{\sqrt{(3/5)}+3} + \frac{1}{\sqrt{(3/5)}+3}\right\}$$

$$= \frac{8}{27} + \frac{25}{63} = 0.29629 + 0.39682 = 0.6931$$

Otherwise (*using the table*):

$$I = w_1 f(u_1) + w_2 f(u_2) + w_3 f(u_3) \text{ where } f(u_i) = \frac{1}{u_i + 3}$$

Using the abscissae and weights corresponding to $n = 3$ in the above table, we obtain

$$I = \frac{1}{3 - 0.7746}(0.555) + \frac{1}{3 - 0}(0.8889) + \frac{1}{3 + 0.7746}(0.555)$$

$$= 0.4497 \times 0.5555 + \frac{1}{3}(0.8889) + 0.2649 \times 0.5555 = 0.6931.$$

EXAMPLE 8.25

Evaluate $\int_0^2 \frac{x^2 + 2x + 1}{1 + (x+1)^4} dx$ by the Gaussian three-point formula.

Solution:

Changing the limits of integration 0 to 2 to – 1 to 1 by

$$x = \frac{1}{2}(b - a)u + \frac{1}{2}(b + a) = \frac{2 - 0}{2}u + \frac{2 + 0}{2} = u + 1$$

$$\therefore \quad I = \int_0^2 \frac{x^2 + 2x + 1}{1 + (x+1)^4} dx = \int_{-1}^1 \frac{(u+1)^2 + 2(u+1) + 1}{1 + (u + 1 + u)^4} du \qquad [\because dx = du]$$

$$= \int_{-1}^1 \frac{u^2 + 4u + 4}{(u+2)^4 + 1} du = \int_{-1}^1 f(u) du$$

$$= w_1 f(u_1) + w_2 f(u_2) + w_3 f(u_3) \text{ where } f(u_i) = \frac{u_i^2 + 4u_i + 4}{(u_i + 2)^4 + 1}$$

Now $f(0) = \frac{4}{2^4 + 1} = \frac{4}{17}$

$$f - \left(\frac{3}{\sqrt{5}}\right) = \frac{(-\sqrt{(3/5)} + 2)^2}{[-\sqrt{(3/5)} + 2]^4 + 1} = \frac{15016}{3.2548} = 0.4614$$

$$f\left(\sqrt{\frac{3}{5}}\right) = \frac{\sqrt{(3/5)} + 2}{[\sqrt{(3/5)} + 2]^4 + 1} = \frac{7.6984}{60.2652} = 0.1277$$

Using the three-point Gaussian formula, we have

$$I = \int_{-1}^{1} f(u)du = \frac{8}{9} f(0) + \frac{5}{9} f\left[\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right]$$

$$= \frac{8}{9}\left(\frac{4}{17}\right) + \frac{5}{9}[0.4614 + 0.1277] = 0.5365$$

Solution:

Changing the limits of integration (0.2 to 1.5) to (– 1, 1) by

$$x = \frac{1}{2}(b-a)u + \frac{1}{2}(b+a) = \frac{1}{2}(1.5-0.2)u + \frac{1}{2}(1.5+0.2)$$
$$= 0.65u + 0.85$$

$$\therefore \qquad I = \int_{0.2}^{1.5} e^{-x^2} dx = 0.65 \int_{-1}^{1} e^{-(0.65u+0.85)2} du = 0.65 \int_{-1}^{1} f(u)du$$

so that $f(u) = e^{-(0.65u\,+\,0.85)2}$

Now $f(0) = e^{-[0.65(0)\,+\,0.85]2} = 0.4855,$

$$f(-\sqrt{3/5}) = f(-0.7746) = e^{-[0.65(-0.7746)+0.85]^2} = 0.8869$$
$$f(\sqrt{3/5}) = f(0.7746) = e^{-[0.65(0.7746)+0.85]^2} = 0.1601.$$

Using the Gauss three-point formula, we have

$$I = \int \quad f(u)du = -f(0)[f(-\sqrt{3/5}) + f(\sqrt{3/5})]$$

$$= \frac{5}{9}(0.4855) + \frac{5}{9}[0.8869 + 0.1601] = 0.4316 + 0.5187 = 1.0133$$

Hence $\int_{0.2}^{1.5} e^{-x^2} dx = 0.65(1.0133) = 0.65865.$

Exercises 8.3

1. Obtain an estimate of the number of sub-intervals that should be chosen so as to guarantee that the error committed in evaluating $\int_1^2 dx/x$ by trapezoidal rule is less than 0.001.
2. Evaluate $\int_0^2 \frac{dx}{x^2+4}$ using the Romberg's method. Hence obtain an approximate value of π.

3. Apply Romberg's method to evaluate $\int_4^{5.2} \log x\, dx$, given that

x:	4.0	4.2	4.4	4.6	4.8	5.0	5.2
$\log_e x$:	1.3863	1.4351	1.4816	1.526	1.5686	1.6094	1.6486.

4. Using the Euler-Maclaurin formula, find the value of $\int_0^{\pi/2} \sin x\, dx$ correct to five decimal places.

5. Using the Euler-Maclaurin formula, prove that

$$(a)\ \sum_1^n x^2 = \frac{n(n+1)(2n+1)}{6} \qquad (b)\ \sum_1^n x^3 = \left\{\frac{n(n+1)}{2}\right\}^2$$

6. Apply the Euler-Maclaurin formula, to evaluate

$$(a)\ \frac{1}{400} + \frac{1}{402} + \frac{1}{404} + \cdots + \frac{1}{500}$$

$$(b)\ \frac{1}{(201)^2} + \frac{1}{(203)^2} + \frac{1}{(205)^2} + \ldots + \frac{1}{(299)^2}$$

7. Assuming that $\int_0^h y(x)dx = h(a_0 y_0 + a_1 y_1) + h^2(b_0 y_0' + b_1 y_1')$ derive the quadrature formula, using the method of undetermined coefficients.

8. Using the Gaussian two-point formula compute

$$(a)\ \int_{-2}^{2} e^{-x2} dx \qquad (b)\ \sum_1^n x^3 = \left\{\frac{n(n+1)}{2}\right\}^2$$

9. Using three point Gaussian quadrature formula, evaluate:

$$(a)\ (i) \int_2^5 \frac{1}{x} dx \qquad (b)\ \int_1^2 \frac{1}{1+x^3} dx\,.$$

10. Evaluate the following, integrals, using the Gauss three-point formula:

$$(a)\ \int_2^4 (1+x^4) dx \qquad (b)\ \int_3^5 \frac{4}{(2x^2)} dx$$

11. Using the four point Gauss formula, compute $\int_0^1 x dx$ correct to four decimal places.