Exercises 10.1

1. Using Picard's method, solve $dy/dx = -xy$ with $x_0 = 0$, $y_0 = 1$ up to the third approximation.
2. Employ Picard 's method to obtain, correct to four places of decimals the, solution of the differential equation $dy/dx = x^2 + y^2$ for $x = 0.4$, given that $y = 0$ when $x = 0$.
3. Obtain Picard's second approximate solution of the initial value problem
$$y' = x^2/(y^2 + 1),\ y(0)=0.$$
4. Find an approximate value of y when $x = 0.1$, if $dy/dx = x - y^2$ and $y = 1$ at $x = 0$, using
   (*a*) Picard's method (*b*) Taylor's series.
5. Solve $y' = x + y$ given $y(1) = 0$. Find $y(1.1)$ and $y(1.2)$ by Taylor's method. Compare the result with its exact value.
6. Using Taylor's series method, compute $y(0.2)$ to three places of decimals from $\dfrac{dy}{dx} = 1 - 2xy$ given that $y(0) = 0$.
7. Evaluate $y(0.1)$ correct to six places of decimals by Taylor's series method if $y\,(x)$ satisfies
$$y' = xy + 1,\ y(0) = 1.$$
8. Solve $y' = y^2 + x$, $y(0) = 1$ using Taylor's series method and compute $y(0.1)$ and $y(0.2)$.
9. Evaluate $y(0.1)$ correct to four decimal places using Taylor's series methods if $dy/dx = x^2 + y^2$, $y(0) = 1$.
10. Using Taylor series method, find $y(0.1)$ correct to three decimal places given that $dy/dx = e^x - y^2$, $y(0) = 1$

10.4 Euler's Method

Consider the equation $\dfrac{dy}{dx} =$ ˘˘ (1)

given that $y(x_0) = y_0$.Its curve of solution through $P(x_0, y_0)$is shown dotted in Figure.10.1. Now we have to find the ordinate of any other point Q on this curve.

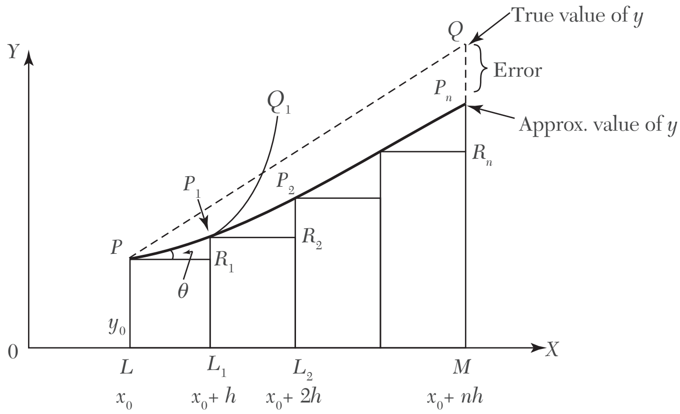

FIGURE 10.1

Let us divide LM into n sub-intervals each of width h at $L_1, L_2 \cdots$ so that h is quite small

In the interval LL_1, we approximate the curve by the tangent at P. If the ordinate through L_1 meets this tangent in $P_1(x_0 + h, y_1)$, then

$$y_1 = L_1P_1 = LP + R_1P_1 = y_0 + PR_1 \tan \theta$$
$$= y_0 + h\left(\frac{dy}{dx}\right)_p = y_0 + hf(x_0, y_0)$$

Let P_1Q_1 be the curve of solution of (1) through P_1 and let its tangent at P_1 meet the ordinate through L_2 in $P_2(x_0 + 2h, y_2)$. Then

$$y_2 = y_1 + hf(x_0 + h, y_1) \tag{1}$$

Repeating this process n times, we finally reach on an approximation MP_n of MQ given by

$$y_n = y_{n-1} + hf(x_0 + \overline{n-1}h, y_{n-1})$$

This is *Euler's method* of finding an approximate solution of (1).

NOTE **Obs.** *In Euler's method, we approximate the curve of solution by the tangent in each interval, i.e., by a sequence of short lines. Unless h is small, the error is bound to be quite significant. This sequence of lines may also deviate considerably from the curve of solution. As such, the method is very slow and hence there is a modification of this method which is given in the next section.*

EXAMPLE 10.8

Using Euler's method, find an approximate value of y corresponding to $x = 1$, given that $dy/dx = x + y$ and $y = 1$ when $x = 0$.

Solution:

We take $n = 10$ and $h = 0.1$ which is sufficiently small. The various calculations are arranged as follows:

x	y	$x + y = dy/dx$	Old y + 0.1 (dy/dx) = *new y*
0.0	1.00	1.00	1.00 + 0.1 (1.00) = 1.10
0.1	1.10	1.20	1.10 + 0.1 (1.20) = 1.22
0.2	1.22	1.42	1.22 + 0.1 (1.42) = 1.36
0.3	1.36	1.66	1.36 + 0.1 (1.66) = 1.53
0.4	1.53	1.93	1.53 + 0.1 (1.93) = 1.72
0.5	1.72	2.22	1.72 + 0.1 (2.22) = 1.94
0.6	1.94	2.54	1.94 + 0.1 (2.54) = 2.19
0.7	2.19	2.89	2.19 + 0.1 (2.89) = 2.48
0.8	2.48	3.29	2.48 + 0.1 (3.29) = 2.81
0.9	2.81	3.71	2.81 + 0.1 (3.71) = 3.18
1.0	3.18		

Thus the required approximate value of $y = 3.18$.

NOTE **Obs.** *In Example 10.1(Obs.), we obtained the true values of y from its exact solution to be 3.44 where as by Euler's method $y = 3.18$ and by Picard's method $y = 3.434$. In the above solution, had we chosen $n = 20$, the accuracy would have been considerably increased but at the expense of double the labor of computation. Euler's method is no doubt very simple but cannot be considered as one of the best.*

EXAMPLE 10.9

Given $\dfrac{dy}{dx} = \dfrac{y - x}{y + x}$ with initial condition $y = 1$ at $x = 0$; find y for $x = 0.1$ by Euler's method.

Solution:

We divide the interval (0, 0.1) in to five steps, *i.e.*, we take $n = 5$ and $h = 0.02$. The various calculations are arranged as follows:

x	y	dy/dx	Old$y$ + 0.02 (dy/dx) = new y
0.00	1.0000	1.0000	1.0000 + 0.02(1.0000) = 1.0200
0.02	1.0200	0.9615	1.0200 + 0.02(0.9615) = 1.0392
0.04	1.0392	0.926	1.0392 + 0.02(0.926) = 1.0577
0.06	1.0577	0.893	1.0577 + 0.02(0.893) = 1.0756
0.08	1.0756	0.862	1.0756 + 0.02(0.862) = 1.0928
0.10	1.0928		

Hence the required approximate value of $y = 1.0928$.

10.5 Modified Euler's Method

In Euler's method, the curve of solution in the interval LL_1 is approximated by the tangent at P (Figure 10.1) such that at P_1, we have

$$y_1 = y_0 + h\,f(x_0, y_0) \tag{1}$$

Then the slope of the curve of solution through P_1

$$[\text{i.e., } (dy/dx)P_1 = f(x_0 + h, y_1)]$$

is computed and the tangent at P_1 to P_1Q_1 is drawn meeting the ordinate through L_2 in

$$P_2(x_0 + 2h, y_2).$$

Now we find a better approximation $y_1^{(1)}$ of $y(x_0 + h)$ by taking the slope of the curve as the mean of the slopes of the tangents at P and P_1, *i.e.*,

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_0 + h, y_1)]$$

As the slope of the tangent at $P1$ is not known, we take y_1 as found in (1) by Euler's method and insert it on R.H.S. of (2) to obtain the first modified value $y_1(1)$

Again (2) is applied and we find a still better value $y_{1(2)}$ corresponding to L_1 as

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_0 + h, y_1^{(1)})]$$

We repeat this step, until two consecutive values of y agree. This is then taken as the starting point for the next interval L_1L_2.

Once y_1 is obtained to a desired degree of accuracy, y corresponding to L_2 is found from (1).

$$y_2 = y_1 + hf(x_0 + h, y_1)$$

and a better approximation $y_2^{(1)}$ is obtained from (2)

$$y_2^{(1)} = y_1 + \frac{h}{2}[f(x_0 + h, y_1) + f(x_0 + 2h, y_2)]$$

We repeat this step until y_2 becomes stationary. Then we proceed to calculate y_3 as above and so on.

This is the *modified Euler's method* which gives great improvement in accuracy over the original method.

EXAMPLE 10.10

Using modified Euler's method, find an approximate value of y when $x = 0.3$, given that $dy/dx = x + y$ and $y = 1$ when $x = 0$.

Solution:

The various calculations are arranged as follows taking $h = 0.1$:

x	$x + y = y'$	*Mean slope*	*Old* y + 0.1 (*mean slope*) = *new* y
0.0	0 + 1	—	1.00 + 0.1 (1.00) = 1.10
0.1	0.1 + 1.1	$\frac{1}{2}(1+1.2)$	1.00 + 0.1 (1.1) = 1.11
0.1	0.1 + 1.11	$\frac{1}{2}(1+1.21)$	1.00 + 0.1 (1.105) = 1.1105
0.1	0.1 + 1.1105	$\frac{1}{2}(1+1.2105)$	1.00 + 0.1 (1.1052) = 1.1105
Since the last two values are equal, we take $y(0.1) = 1.1105$.			
0.1	1.2105	—	1.1105 + 0.1 (1.2105) = 1.2316
0.2	0.2 + 1.2316	$\frac{1}{2}(1.12105+1.4316)$	1.1105 + 0.1 (1.3211) = 1.2426
0.2	0.2 + 1.2426	$\frac{1}{2}(1.2105+1.4426)$	1.1105 + 0.1 (1.3266) = 1.2432
0.2	0.2 + 1.2432	$\frac{1}{2}(1.2105+1.4432)$	1.1105 + 0.1 (1.3268) = 1.2432
Since the last two values are equal, we take $y(0.2) = 1.2432$.			
0.2	1.4432	—	1.2432 + 0.1 (1.4432) = 1.3875
0.3	0.3 + 1.3875	$\frac{1}{2}(1.4432+1.6875)$	1.2432 + 0.1 (1.5654) = 1.3997
0.3	0.3 + 1.3997	$\frac{1}{2}(1.4432+1.6997)$	1.2432 + 0.1 (1.5715) = 1.4003
0.3	0.3 + 1.4003	$\frac{1}{2}(1.4432+1.7003)$	1.2432 + 0.1 (1.5718) = 1.4004
0.3	0.3 + 1.4004	$\frac{1}{2}(1.4432+1.7004)$	1.2432 + 0.1 (1.5718) = 1.4004
Since the last two values are equal, we take $y(0.3) = 1.4004$.			

Hence $y(0.3) = 1.4004$ approximately.

NOTE **Obs.** *In Example 10.8, we obtained the approximate value of y for x = 0.3 to be 1.53 whereas by the modified Euler's method the corresponding value is 1.4003 which is nearer its true value 1.3997, obtained from its exact solution $y = 2ex - x - 1$ by putting x = 0.3.*

EXAMPLE 10.11

Using the modified Euler's method, find $y(0.2)$ and $y(0.4)$ given

$$y' = y + e^x, \; y(0) = 0.$$

Solution:

We have $y' = y + ex = f(x, y)$; $x = 0$, $y = 0$ and $h = 0.2$

The various calculations are arranged as under:

To calculate $y(0.2)$:

x	$y + ex = y'$	*Mean slope*	*Old y + h (Mean slope) = new y*
0.0	1	—	$0 + 0.2\,(1) = 0.2$
0.2	$0.2 + e^{0.2} = 1.4214$	$\frac{1}{2}(1 + 1.4214) = 1.2107$	$0 + 0.2\,(1.2107) = 0.2421$
0.2	$0.2421 + e^{0.2} = 1.4635$	$\frac{1}{2}(1 + 1.4635) = 1.2317$	$0 + 0.2\,(1.2317) = 0.2463$
0.2	$0.2463 + e^{0.2} = 1.4677$	$\frac{1}{2}(1 + 1.4677) = 1.2338$	$0 + 0.2\,(1.2338) = 0.2468$
0.2	$0.2468 + e^{0.2} = 1.4682$	$\frac{1}{2}(1 + 1.4682) = 1.2341$	$0 + 0.2\,(1.2341) = 0.2468$

Since the last two values of y are equal, we take $y\,(0.2) = 0.2468$.

To calculate $y(0.4)$:

x	$y + ex$	*Mean slope*	*Oldy + 0.2 (mean slope) new y*
0.2	$0.2468 + e^{0.2} = 1.4682$	—	$0.2468 + 0.2\,(1.4682) = 0.5404$
0.4	$0.5404 + e^{0.4} = 2.0322$	$\frac{1}{2}(1.4682 + 2.0322) = 1.7502$	$0.2468 + 0.2\,(1.7502) = 0.5968$

x	$y + ex$	*Mean slope*	*Oldy* + 0.2 (*mean slope*) *new y*
0.4	$0.5968 + e^{0.4} = 2.0887$	$\frac{1}{2}(1.4682 + 2.0887)$ $= 1.7784$	0.2468 + 0.2 (1.7784) = 0.6025
0.4	$0.6025 + e^{0.4} = 2.0943$	$\frac{1}{2}(1.4682 + 2.0943)$ $= 1.78125$	0.2468 + 0.2 (1.78125) = 0.6030
0.4	$0.6030 + e^{0.4} = 2.0949$	$\frac{1}{2}(1.4682 + 2.0949)$ $= 1.7815$	0.2468 + 0.2 (1.7815) = 0.6031
0.4	$0.6031 + e^{0.4} = 2.0949$	$\frac{1}{2}(1.4682 + 2.0949)$ $= 1.7816$	0.2468 + 0.2 (1.7815) = 0.6031

Since the last two value of y are equal, we take $y(0.4) = 0.6031$

Hence $y(0.2) = 0.2468$ an d $y(0.4) = 0.6031$ approximately.

EXAMPLE 10.12

Solve the following by Euler's modified method:

$$\frac{dy}{dx} = \log(x + y), y(0) = 2$$

at $x = 1.2$ and 1.4 with $h = 0.2$.

Solution:

The various calculations are arranged as follows:

x	$\log(x + y) = y'$	*Mean slope*	*Old y* + 0.2 (*mean slope*) = *new y*
0.0	log (0 + 2)	—	2 + 0.2(0.301) = 2.0602
0.2	log (0.2 + 2.0602)	$\frac{1}{2}(0.310 = 0.3541)$	2 + 0.2 (0.3276) = 2.0655
0.2	log (0.2 + 2.0655)	$\frac{1}{2}(0.301 + 0.3552)$	2 + 0.2 (0.3281) = 2.0656
0.2	0.3552	—	2.0656 + 0.2 (0.3552) = 2.1366
0.4	log (0.4 + 2.1366)	$\frac{1}{2}(0.3552 + 0.4042)$	2.0656 + 0.2 (0.3797) = 2.1415
0.4	log (0.4 + 2.1415)	$\frac{1}{2}(0.3552 + 0.4051)$	2.0656 + 0.2 (0.3801) = 2.1416
0.4	0.4051	—	2.1416+ 0.2 (0.4051) = 2.2226
0.6	log (0.6 + 2.2226)	$\frac{1}{2}(0.4051 + 0.4506)$	2.1416 + 0.2 (0.4279) = 2.2272
0.6	log (0.6 + 2.2272)	$\frac{1}{2}(0.4051 + 0.4514)$	2.1416 + 0.2 (0.4282) = 2.2272

x	$\log(x+y) = y'$	*Mean slope*	*Old* $y + 0.2$ (*mean slope*) = *new* y
0.6	0.4514	—	2.2272 + 0.2 (0.4514) = 2.3175
0.8	log (0.8 + 2.3175)	$\frac{1}{2}(0.4514 + 0.4938)$	2.2272 + 0.2 (0.4726) = 2.3217
0.8	log (0.8 + 2.3217)	$\frac{1}{2}(0.4514 + 0.4943)$	2.2272 + 0.2 (0.4727) = 2.3217
0.8	0.4943	—	2.3217 + 0.2 (0.4943) = 2.4206
1.0	log (1 + 2.4206)	$\frac{1}{2}0.4943 + 0.5341)$	2.3217 + 0.2 (0.5142) = 2.4245
1.0	log (1 + 2.4245)	$\frac{1}{2}(0.4943 + 0.5346)$	2.3217 + 0.2 (0.5144) = 2.4245
1.0	0.5346	—	2.4245 + 0.2 (0.5346) = 2.5314
1.2	log (1.2 + 2.5314)	$\frac{1}{2}(0.5346 + 0.5719)$	2.4245 + 0.2 (0.5532) = 2.5351
1.2	log (1.2 + 2.5351)	$\frac{1}{2}(0.5346 + 0.5723)$	2.4245 + 0.2 (0.5534) = 2.5351
1.2	0.5723	—	2.5351 + 0.2 (0.5723) = 2.6496
1.4	log (1.4 + 2.6496)	$\frac{1}{2}(0.5723 + 0.6074)$	2.5351 + 0.2 (0.5898) = 2.6531
1.4	log (1.4 + 2.6531)	$\frac{1}{2}(0.5723 + 0.6078)$	2.5351 + 0.2 (0.5900) = 2.6531

Hence $y(1.2) = 2.5351$ an d $y(1.4) = 2.6531$ approximately.

EXAMPLE 10.13

Using Euler's modified method, obtain a solution of the equation

$$dy/dx = x + \left|\sqrt{y}\right|$$

with initial conditions $y = 1$ at $x = 0$, for the range 0 £ x £ 0.6 in steps of 0.2.

Solution:

The various calculations are arranged as follows:

x	$x + \left\|\sqrt{y}\right\| = y'$	*Mean slope*	*Old* $y + 0.2$ (*mean slope*) = *new* y
0.0	0 + 1 = 1	—	1 + 0.2 (1) = 1.2
0.2	$0.2 + \left\|\sqrt{(1.2)}\right\|$ = 1.2954	$\frac{1}{2}(1 + 1.2954)$ = 1.1477	1 + 0.2 (1.1477) = 1.2295

x	$x+\left\|\sqrt{y}\right\|=y'$	*Mean slope*	*Old y + 0.2 (mean slope) = new y*
0.2	$0.2+\left\|\sqrt{(1.2295)}\right\|$ = 1.3088	$\frac{1}{2}(1+1.3088)$ = 1.1544	1 + 0.2 (1.1544) = 1.2309
0.2	$0.2+\left\|\sqrt{(1.2309)}\right\|$ = 1.3094	$\frac{1}{2}(1+1.3094)$ = 1.1547	1 + 0.2 (1.1547) = 1.2309
0.2	1.3094	—	1.2309 + 0.2 (1.3094) = 1.4927
0.4	$0.4+\left\|\sqrt{(1.4927)}\right\|$ = 1.6218	$\frac{1}{2}(1.3094+1.6218)$ = 1.4654	1.2309 + 0.2 (1.4654) = 1.5240
0.4	$0.4+\left\|\sqrt{(1.524)}\right\|$ = 1.6345	$\frac{1}{2}(1.3094+1.6345)$ = 1.4718	1.2309 + 0.2 (1.4718) = 1.5253
0.4	$0.4+\left\|\sqrt{(1.5253)}\right\|$ = 1.6350	$\frac{1}{2}(1.3094+1.6350)$ = 1.4721	1.2309 + 0.2 (1.4721) = 1.5253
0.4	1.6350	—	1.5253 + 0.2 (1.635) = 1.8523
0.6	$0.6+\left\|\sqrt{(1.8523)}\right\|$ = 1.9610	$\frac{1}{2}(1.635+1.961)$ = 1.798	1.5253 + 0.2 (1.798) = 1.8849
0.6	$0.6+\left\|\sqrt{(1.8849)}\right\|$ = 1.9729	$\frac{1}{2}(1.635+1.9729)$ = 1.8040	1.5253 + 0.2 (1.804) = 1.8861
0.6	$0.6+\left\|\sqrt{(1.8861)}\right\|$ = 1.9734	$\frac{1}{2}(1.635+1.9734)$ = 1.8042	1.5253 + 0.2 (1.8042) = 1.8861

Hence $y(0.6) = 1.8861$ approximately.

Exercises 10.2

1. Apply Euler's method to solve $y' = x + y$, $y(0) = 0$, choosing the step length = 0.2. (Carry out six steps).

2. Using Euler's method, find the approximate value of y when $x = 0.6$ of $dy/dx = 1 - 2xy$, given that $y = 0$ when $x = 0$ (take $h = 0.2$).

3. Using the simple Euler's method solve for y at $x = 0.1$ from $dy/dx = x + y + xy$, $y(0) = 1$, taking step size $h = 0.025$.

4. Solve $y' = 1 - y, y(0) = 0$
by the modified Euler's method and obtain y at $x = 0.1, 0.2, 0.3$

5. Given that $dy/dx = x^2 + y$ and $y(0) = 1$. Find an approximate value of $y(0.1)$, taking $h = 0.05$ by the modified Euler's method.

6. Given $y' = x + \sin y, y(0) = 1$. Compute $y(0.2)$ and $y(0.4)$ with $h = 0.2$ using Euler's modified method.

7. Given $\frac{dy}{dx} = \frac{y - x}{y + x}$ with boundary conditions $y = 1$ when $x = 0$, find approximately y for $x = 0.1$, by Euler's modified method (five steps)

8. Given that $dy / dx = 2 + \sqrt{(xy)}$ and $y = 1$ when $x = 1$. Find approximate value of y at $x = 2$ in steps of 0.2, using Euler's modified method.

10.6 Runge's Method*

Consider the differential equation, $\frac{dy}{dx} = f(x,y), y(x_0) = y_0$ (1)

Clearly the slope of the curve through $P(x_0, y_0)$ is $f(x_0, y_0)$ (Figure 10.2).

Integrating both sides of (1) from (x_0, y_0) to $(x_0 + h, y_0 + k)$, we have

$$\int_{y_0}^{y_0+k} dy = \int_{x_0}^{x_0+h} f(x,y)dx \qquad (2)$$

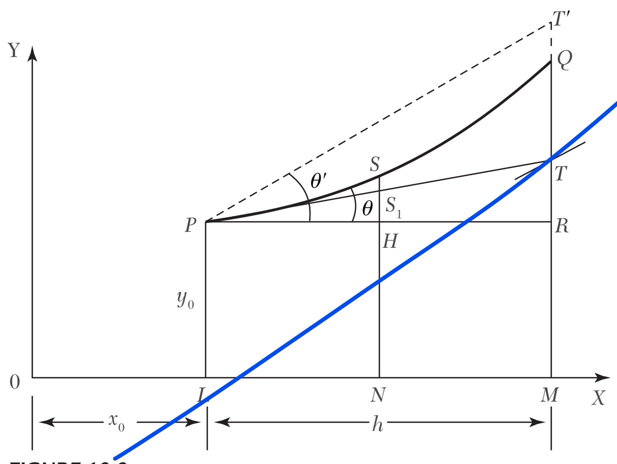

FIGURE 10.2

*Called after the German mathematician *Carl Runge* (1856-1927).

To evaluate the integral on the right, we take N as the mid-point of LM and find the values of $f(x, y)$ (*i.e.*, dy/dx) at the points $x_0, x_0 + h/2, x_0 + h$. For this purpose, we first determine the values of y at these points.

Let the ordinate through N cut the curve PQ in S and the tangent PT in S_1. The value of y_S is given by the point S_1

$$\therefore \qquad y_s = NS_1 = LP + HS_1 = y_0 + PH.\tan\theta$$

$$= y_0 + h(dy/dx)_p = y_0 + \frac{h}{2} f(x_0, y_0) \qquad (3)$$

Also $\qquad y_T = MT = LP + RT = y_0 + PR.\tan\theta = y_0 + hf(x_0 + y_0).$

Now the value of y_Q at $x_0 + h$ is given by the point T'' where the line through P draw with slope at $T(x_0 + h, y_T)$ meets MQ.

$\therefore$ Slope at $\; T = \tan\theta' = f(x_0 + h, y_T) = f[x_0 + h, y_0 + hf(x_0, y_0)]$

$$\therefore \qquad y_Q = R + RT = y_0 + PR\ .\tan\theta' = y_0 + hf[x_0 + h, y_0 + hf(x_0,\ y_0)] \qquad (4)$$

Thus the value of $f(x, y)$ at $P = f(x_0, y_0)$,

the value of $f(x, y)$ at $S = f(x_0 + h/2, y_S)$

and the value of $f(x, y)$ at $Q = (x_0 + h, y_Q)$

where y_S and y_Q are given by (3) and (4).

Hence from (2), we obtain

$$k = \int_{x_0}^{x_0+h} f(x,y)dx = \frac{h}{6}\left[f_P + 4f_S + f_Q\right] \qquad \text{by Simpson's rule}$$

$$= \frac{h}{6}\left[f(x_0 + y_0) + f(x_0 + h/2, y_S) + (x_0 + h, y_Q\right]$$

Which gives a sufficiently accurate value of k and also $y = y_0 + k$

The repeated application of (5) gives the values of y for equi-spaced points.

Working rule *to solve* (1) *by Runge's method*:

Calculate successively

$$k_1 = hf(x_0,\ y_0),$$

$$k_2 = hf\left(x_0 + \frac{1}{2}hy_0 + \frac{1}{2}k_1\right)$$

$$k' = hf\left(x_0 + h, y_0 + k_1\right)$$

and $$k_3 = hf\left(x_0 + h, y_0 + k'\right)$$

Finally compute, $k = \frac{1}{6}(k_1 + 4k_2 + k_3)$

which gives the required approximate value as $y_1 = y_0 + k$.

(Note that k is the weighted mean of k_1, k_2, and k_3).

EXAMPLE 10.14

Apply Runge's method to find an approximate value of y when $x = 0.2$, given that $dy/dx = x + y$ and $y = 1$ when $x = 0$.

Solution:

Here we have $x_0 = 0$, $y_0 = 1$, $h = 0.2$, $f(x_0, y_0) = 1$

$$\therefore \quad k_1 = hf(x_0, y_0) = 0.2(1) = 0.200$$

$$k_2 = hf\left(x_0 + \frac{1}{2}hy_0 + \frac{1}{2}k_1\right) = 0.2f(0.1, 1.1) \quad 0.240$$

$$k' = hf\left(x_0 + h, y_0 + k_1\right) = 0.2f(0.2, 1.2) = 0.280$$

and $$k_3 = hf\left(x_0 + h, y_0 + k'\right) = 0.2f(0.1, 1.28) = 0.296$$

$$\therefore \quad k = \frac{1}{6}(k_1 + 4k_2 + k_3) = \frac{1}{6}(0.200 + 0.960 + 0.296) = 0.2426$$

Hence the required approximate value of y is 1.2426.

10.7 Runge-Kutta Method*

The Taylor's series method of solving differential equations numerically is restricted by the labor involved in finding the higher order derivatives. However, there is a class of methods known as Runge-Kutta methods which do not require the calculations of higher order derivatives and give greater accuracy. The Runge-Kutta formulae possess the advantage of requiring only the function values at some selected points. These methods agree with Taylor's series solution up to the term in h^r where r differs from method to method and is called the *order of that method*.

First order R-K method. We have seen that Euler's method (Section 10.4) gives

$$y_1 = y_0 + hf(x_0, y_0) = y_0 + hy_0' \qquad [\because y' = f(x, y)]$$

Expanding by Taylor's series

$$y_1 = y(x_0 + h) = y_0 + hy_0' + \frac{h^2}{2} y_0'' + \cdots$$

It follows that the Euler's method agrees with the Taylor's series solution upto the term in h.

Hence, *Euler's method is the Runge-Kutta method of the first order.*

Second order R-K method. The modified Euler's method gives

$$y_1 = y + \frac{h}{2}\left[f(x_0, y_0) + f(x_0 + h, y_1)\right] \tag{1}$$

Substituting $y_1 = y_0 + hf(x_0, y_0)$ on the right-hand side of (1), we obtain

$$y_1 = y_0 + \frac{h}{2}\left[f_0 + f(x_0 + h), y_0 + hf_0\right] \quad \text{where} \quad f_0 = (x_0, y_0) \tag{2}$$

Expanding L.H.S. by Taylor's series, we get

$$y_1 = y(x_0 + h) = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots \tag{3}$$

Expanding $f(x_0 + h, y_0 + hf_0)$ by Taylor's series for a function of two variables, (2) gives

$$y_1 = y_0 + \frac{h}{2}\left[f_0 + \left\{f_0 = (x_0, y_0) + h\left(\frac{\partial f}{\partial x}\right)_0 + hf_0\left(\frac{\partial f}{\partial y}\right)_0 + O(h^2)^{**}\right\}\right]$$

$$= y_0 + \frac{1}{2}\left[hf_0 + hf_0 + h^2\left\{\left(\frac{\partial f}{\partial x}\right)_0 + \left(\frac{\partial f}{\partial y}\right)_0\right\} + O(h^3)\right]$$

$$= y_0 + hf_0 + \frac{h^2}{2} f_0' + O(h^3) \qquad \left[\because \frac{df(x,y)}{dx} = \frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}\right]$$

$$= y_0 + hy_0' + \frac{h^2}{2!} y_0'' + O(h^3) \tag{4}$$

Comparing (3) and (4), it follows that the modified Euler's method agrees with the Taylor's series solution upto the term in h^2.

Hence the modified Euler's method is the Runge-Kutta method of the second order.

** $O(h^2)$ means "terms containing second and higher powers of h" and is read as *order of* h^2.

$\therefore$ *The second order Runge-Kutta formula is*

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

Where $k_1 = hf(x_0, y_0)$ and $k_2 = hf(x_0 + h, y_0 + k)$

(*iii*) *Third order R-K method.* Similarly, it can be seen that Runge's method (Section 10.6) agrees with the Taylor's series solution upto the term in h^3.

As such, *Runge's method is the Runge-Kutta method of the third order.*

$\therefore$ *The third order Runge-Kutta formula is*

$$y_1 = y_0 + \frac{1}{6}(k_1 + 4k_2 + k_3)$$

Where, $k_1 = hf(x_0, y_0), k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)$

And $\quad k_3 = hf(x_0 + h, y_0 + k')$, where $k' = k_3 = hf(x_0 + h, y_0 + k_1)$.

(*iv*) *Fourth order R-K method.* This method is most commonly used and is often referred to as the *Runge-Kutta method* only.

Working rule *for finding the increment k of y corresponding to an increment h of x by Runge-Kutta method from*

$$\frac{dy}{dx} = f(x, y), y(x_0)$$

is as follows:

Calculate successively $k_1 = hf(x_0, y_0)$,

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right)$$

and $\quad k_4 = hf(x_0 + h, y_0 + k_3)$

Finally compute $\quad k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

which gives the required approximate value as $y_1 = y_0 + k$.

(Note that k is the weighted mean of k_1, k_2, k_3, and k_4).

NOTE **Obs.** *One of the advantages of these methods is that the operation is identical whether the differential equation is linear or non-linear.*

EXAMPLE 10.15

Apply the Runge-Kutta fourth order method to find an approximate value of y when $x = 0.2$ given that $dy/dx = x + y$ and $y = 1$ when $x = 0$.

Solution:

Here $x_0 = 0,\ y_0 = 1,\ h = 0.2,\ f(x_0, y_0) = 1$

$$\therefore \quad k_1 = hf(x_0, y_0) = 0.2 \times 1 = 0.2000$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = 0.2 \times f(0.1, 1.1) = 0.2400$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = 0.2 \times f(0.1, 1.12) = 0.2440$$

and $$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2 \times f(0.2, 1.244) = 0.2888$$

$$\therefore \quad k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.2000 + 0.4800 + 0.4880 + 0.2888)$$

$$= \frac{1}{6} \times (1.4568) = 0.2428$$

Hence the required approximate value of y is 1.2428.

EXAMPLE 10.16

Using the Runge-Kutta method of fourth order, solve $\dfrac{dy}{dx} = \dfrac{y^2 - x^2}{y^2 + x^2}$ with $y(0) = 1$ at $x = 0.2,\ 0.4$.

Solution:

We have $f(x, y) = \dfrac{y^2 - x^2}{y^2 + x^2}$

To find $y(0.2)$

Hence $x_0 = 0,\ y_0 = 1,\ h = 0.2$

$$k_1 = hf(x_0, y_0) = 0.2 f(0, 1) = 0.2000$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = 0.2 \times f(0.1, 1.1) = 0.19672$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = 0.2f(0.1, 1.09836) = 0.1967$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2f(0.2, 1.1967) = 0.1891$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}[0.2 + 2(0.19672) + 2(0.1967) + 0.1891] = 0.19599$$

Hence $y(0.2) = y_0 + k = 1.196$.

To find $y(0.4)$:

Here $x_1 = 0.2$, $y_1 = 1.196$, $h = 0.2$.

$$k_1 = hf(x_1, y_1) = 0.1891$$

$$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right) = 0.2f(0.3, 1.2906) = 0.1795$$

$$k_3 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right) = 0.2f(0.3, 1.2858) = 0.1793$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.2f(0.4, 1.3753) = 0.1688$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}[0.1891 + 2(0.1795) + 2(0.1793) + 0.1688] = 0.1792$$

Hence $y(0.4) = y_1 + k = 1.196 + 0.1792 = 1.3752$.

EXAMPLE 10.17

Apply the Runge-Kutta method to find the approximate value of y for $x = 0.2$, in steps of 0.1, if $dy/dx = x + y^2$, $y = 1$ where $x = 0$.

Solution:

Given $f(x, y) = x + y^2$.

Here we take $h = 0.1$ and carry out the calculations in two steps.

Step I. $x0 = 0$, $y0 = 1$, $h = 0.1$

$\therefore$ $$k_1 = hf(x_0, y_0) = 0.1f(0, 1) = 0.1000$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = 0.1f(0.05, 1.1) = 0.1152$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = 0.1f(0.05, 1.1152) = 0.1168$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1f(0.1, 1.1168) = 0.1347$$

$$\therefore \qquad k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.1000 + 0.2304 + 0.2336 + 0.1347) = 0.1165$$

giving $y(0.1) = y_0 + k = 1.1165$

Step II. $x_1 = x_0 + h = 0.1$, $y_1 = 1.1165$, $h = 0.1$

$$\therefore \qquad k_1 = hf(x_1, y_1) = 0.1f(0.1, 1.1165) = 0.1347$$

$$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right) = 0.1f(0.15, 1.1838) = 0.1551$$

$$k_3 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right) = 0.1f(0.15, 1.194) = 0.1576$$

$$k_4 = hf(x_1 + h, y_2 + k_3) = 0.1f(0.2, 1.1576) = 0.1823$$

$$\therefore \qquad k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.1571$$

Hence $y(0.2) = y_1 + k = 1.2736$

EXAMPLE 10.18

Using the Runge-Kutta method of fourth order, solve for y at $x = 1.2$, 1.4

From $\dfrac{dy}{dx} = \dfrac{2xy + e^x}{x^2 + xe^x}$ given $x_0 = 1$, $y_0 = 0$

Solution:

We have $f(x, y) = \dfrac{2xy + e^x}{x^2 + xe^x}$

To find $y(1.2)$:

Here x_0=1, y_0=0, h=0.2

$$k_1 = hf(x_0, y_0) = 0.2\frac{0+e'}{1+e'} = 0.1462$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2\left\{\frac{2(1+0.1)(0+0.073)e^{1+0.1}}{(1+0.1)^2 + (1+0.1)e^{1+0.1}}\right\}$$

$$= 0.1402$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = 0.2\left\{\frac{2(1+0.1)(0+0.07)e^{1.1}}{(1+0.1)^2 + (1+0.1)e^{1.1}}\right\}$$

$$= 0.1399$$

$$k_4 = hf\left(x_0 + h, y_0 + k_3\right) = 0.2\left\{\frac{2(1.2)(0.1399)e^{1.2}}{(1.2)^2 + (1.2)e^{1.2}}\right\}$$

$$= 0.1348$$

and $$k = \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) = \frac{1}{6}[0.1462 + 0.2804 + 0.2798 + 0.1348]$$

$$= 0.1402$$

Hence $y(1.2) = y0 + k = 0 + 0.1402 = 0.1402$.

To find y (1.4):

Here $x_1 = 1.2, y_1 = 0.1402, h = 0.2$

$$k_1 = hf(x_1, y_1) = 0.2f(1.2, 0) = 0.1348$$

$$k_2 = hf\left(x_1 + h/2, y_1 + k_1/2\right) = 0.2f(1.3, 0.2076) = 0.1303$$

$$k_3 = hf\left(x_1 + h/2, y_1 + k_2/2\right) = 0.2f(1.3, 0.2053) = 0.1301$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.2f(1.3, 0.2703) = 0.1260$$

$$\therefore \quad k = \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

$$= \frac{1}{6}[0.1348 + 0.2606 + 0.2602 + 0.1260]$$

$$= 0.1303$$

Hence $y(1.4) = y_1 + k = 0.1402 + 0.1303 = 0.2705$.

Exercises 10.3

1. Use ~~Runge's method to approximate~~ y when $x = 1.1$, given that $y = 1.2$ when $x = 1$ and ~~$dy/dx = 3x$~~ $+ y^2$.

2. Using the Runge-Kutta method of order 4, find $y(0.2)$ given that $dy/dx = 3x + y^2$, $y(0) = 1$ taking $h = 0.1$.

3. Using the Runge-Kutta method of order 4, compute $y(0.2)$ and $y(0.4)$ from $10\,\dfrac{dy}{dx} = x^2 + y^2 \;\; y(0) = 1$, taking $h = 0.1$.

4. Use the Runge Kutta method to find y when $x = 1.2$ in steps of 0.1, given that $dy/dx = x^2 + y^2$ and $y(1) = 1.5$.

5. Given $dy/dx = x^3 + y$, $y(0) = 2$. Compute $y(0.2)$, $y(0.4)$, and $y(0.6)$ by the Runge-Kutta method of fourth order.

6. Find $y(0.1)$ and $y(0.2)$ using the Runge-Kutta fourth order formula, given that $y' = x^2 - y$ and $y(0) = 1$.

7. Using fourth order Runge-Kutta method, solve the following equation, taking each step of $h = 0.1$, given $y(0) = 3$. $dy/dx\ (4x/y - xy)$. Calculate y for $x = 0.1$ and 0.2.

8. Find by the Runge-Kutta method an approximate value of y for $x = 0.6$, given that $y = 0.41$ when $x = 0.4$ and $dy/dx = \sqrt{(x + y)}$

9. Using the Runge-Kutta method of order 4, find $y(0.2)$ for the equation $\dfrac{dy}{dx} = \dfrac{y - x}{y + x}$, , $y(0) = 1$. Take $h = 0.2$.

10. Using fourth order Runge-Kutta method, integrate $y' = -2x^3 + 12x^2 - 20x + 8.5$, using a step size of 0.5 and initial condition of $y = 1$ at $x = 0$.

11. Using the fourth order Runge-Kutta method, find y at $x = 0.1$ given that $dy/dx = 3e^x + 2y$, $y(0) = 0$ and $h = 0.1$.

12. Given that $dy/dx = (y^2 - 2x)/(y^2 + x)$ and $y = 1$ at $x = 0$, find y for $x = 0.1$, 0.2, 0.3, 0.4, and 0.5.