2.11 Iteration Method

To find the roots of the equation $f(x) = 0$ (*i*)

by successive approximations, we rewrite (*i*) in the form $x = \phi(x)$ (*ii*)

The roots of (*i*) are the same as the points of intersection of the straight line $y = x$ and the curve representing $y = \phi(x)$.Figure 2.7 illustrates the working of the iteration method which provides a spiral solution.

Let $x = x_0$ be an initial approximation of the desired root α. Then the first approximation x_1 is given by $x_1 = \phi(x_0)$

Now treating x_1 as the initial value, the second approximation is $x_2 = \phi(x_1)$

Proceeding in this way, the nth approximation is given by $x_n = \phi(x_{n-1})$

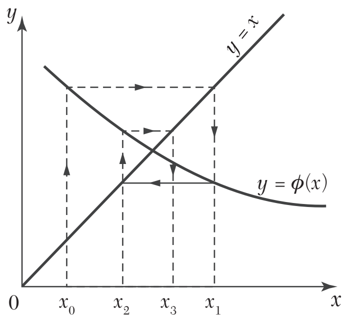

FIGURE 2.7

Sufficient condition for convergence of iterations. It is not certain whether the sequence of approximations $x_1, x_2, \ldots, x_n$ always converges to the same number which is a root of (1) or not. As such, we have to choose the initial approximation x_0 suitably so that the successive approximations $x_1, x_2, \ldots, x_n$ converge to the root α. The following theorem helps in making the right choice of x_0:

Theorem:

If (*i*) *α be a root of $f(x) = 0$ which is equivalent to $x = \phi(x)$,*

(*ii*) *I, be any interval containing the point $x = \alpha$,*

(*iii*) *$|\phi'(x)| < 1$ for all x in I,*

then the sequence of approximations $x_0, x_1, x_2, \ldots, x_n$ will converge to the root α provided the initial approximation x_0 is chosen in I.

Proof. Since α is a root of $x = \phi(x)$, we have $\alpha = \phi(\alpha)$

If x_{n-1} and xn be 2 successive approximations to α, we have $xn = \phi(x_{n-1})$

$$\therefore \qquad x_n - \alpha = \phi(x_{n-1}) - \phi(\alpha) \tag{i}$$

By mean value theorem, $\dfrac{\phi(x_{n-1}) - \phi(\alpha)}{x_{n-1} - \alpha} = \phi'(\xi)$ where $x_{n-1} < \xi < \alpha$

Hence (1) becomes $x_n - \alpha = (x_{n-1} - \alpha)\, \phi'(\alpha)$

If $|\, \phi'(x_i) \,| \le k < 1$ for all i, then

$$|\, x_n - \alpha \,| \le k \,|\, x_{n-1} - \alpha \,| \tag{2}$$

Similarly $\qquad |\, x_{n-1} - \alpha \,| \le k \,|\, x_{n-2} - \alpha \,|$

i.e., $\qquad |x_n - \alpha \,| \le k^2 \,|\, x_{n-2} - \alpha \,|$

Proceeding in this way, $|\, x_n - \alpha \,| \le k^n \,|\, x_0 - \alpha \,|$

As $n \to \infty$, the R.H.S. tends to zero, therefore, the sequence of approximations converges to the root α.

NOTE

Obs. *1. The smaller the value of $\phi'(x)$, the more rapid will be the convergence.*

2. This method of iteration is particularly useful for finding the real roots of an equation given in the form of an infinite series.

Acceleration of convergence. From (2), we have

$$|\, x_n - \alpha \,| \le k \,|\, x_{n-1} - \alpha \,|, k < 1.$$

It is clear from this relation that the iteration method is linearly convergent. *This slow rate of convergence can be improved by using the following method*:

Aitken's Δ^2 method. Let x_{i-1}, x_i, x_{i+1} be three successive approximations to the desired root α of the equation $x = \phi(x)$. Then we know that

$\alpha - x_i = k(\alpha - x_{i-1}),\ \alpha - x_{i+1} = k(\alpha - x_i)$

Dividing, we get $$\frac{\alpha - x_i}{\alpha - x_{i+1}} = \frac{\alpha - x_{i-1}}{\alpha - x_1}$$

Whence $$\alpha = x_{i+1} - \frac{(x_{i+1} - x_i)^2}{x_{i+1} - 2x_i = x_{i-1}} \tag{3}$$

But in the sequence of successive approximations, we have

$$\Delta x_i = x_{i+1} - x_i$$

$$\Delta^2 x_i = \Delta(\Delta x_1) = \Delta(x_{i+1} - x_i) = \Delta x_{i+1} - \Delta x_i$$

$$= x_{i+2} - x_{i+1} - (x_{i+1} - x_i) = x_{i+2} - 2x_{i+1} + x_i$$

$$\therefore \quad \Delta^2 x_{i-1} = x_{i+1} - 2x_i + x_{i-1}$$

Hence (3) can be written as $\alpha = x_{i+1} - \dfrac{(\Delta x_i)^2}{\Delta^2 x_{i-1}}$ (4)

which yields successive approximations to the root α.

EXAMPLE 2.25

Find a real root of the equation $\cos x = 3x - 1$ correct to three decimal places using

(*i*) Iteration method

(*ii*) Aitken's Δ^2 method.

Solution:

(*i*) We have $f(x) = \cos x - 3x + 1 = 0$

$$f(0) = 2 = +\text{ ve and } f(\pi/2) = -3\pi/2 + 1 = -\text{ ve}$$

$\therefore$ A root lies between 0 and $\pi/2$.

Rewriting the given equation as $x = \dfrac{1}{3}(\cos x + 1) = \phi(x)$, we have

$$\phi'(x) = \frac{\sin x}{3} \text{ and } |\phi'(x)| = \frac{1}{3}|\sin x| < 1 \text{ in } (0, \pi/2).$$

Hence the iteration method can be applied and we start with $x0 = 0$. Then the successive approximations are,

$$x_1 = \phi(x0) = \frac{1}{3}(\cos 0 + 1) = 0.6667$$

$$x_2 = \phi(x1) = \frac{1}{3}(\cos 0.6667 + 1) = 0.5953$$

$$x_3 = \phi(x2) = \frac{1}{3}(\cos 0.5953 + 1) = 0.6093$$

$$x_4 = \phi(x3) = \frac{1}{3}(\cos 0.6093 + 1) = 0.6067$$

$$x_5 = \phi(x4) = \frac{1}{3}\ (\cos 0.6067 + 1) = 0.6072$$

$$x_6 = \phi(x5) = \frac{1}{3}(\cos 0.6072 + 1) = 0.6071$$

Hence x_5 and x_6 being almost the same, the root is 0.607 correct to three decimal places. (*ii*) We calculate x_1, x_2, x_3 as above. To use Aitken's method, we have

x	Δx	$\Delta^2 x$
$x_1 = 0.667$		
	-0.0714	
$x_2 = 0.5953$		0.0854
	0.014	
$x_3 = 0.6093$		

Hence $$x_4 = x_3 - \frac{(\Delta x_2)^2}{\Delta^2 x_1} = 0.6093 - \frac{(0.014)^2}{0.0854} = 0.607$$

which corresponds to six iterations in normal form.

Thus the required root is 0.607.

EXAMPLE 2.26

Using iteration method, find a root of the equation $x^3 + x^2 - 1 = 0$ correct to four decimal places.

Solution:

We have $f(x) = x^3 + x^2 - 1 = 0$

Since $f(0) = -1$ and $f(1) = 1$, a root lies between 0 and 1.

Rewriting the given equation as $x = (x + 1)–^{1/2} = \phi(x)$, we have $\phi'(x) = -\frac{1}{2}(x + 1)^{-3/2}$ and $|\phi'(x)| < 1$ for $x < 1$. Hence the iteration method can be applied. Starting with $x_0 = 0.75$, the successive approximations are

$$x_1 = \phi(x_0) = \frac{1}{\sqrt{(x_0+1)}} = 0.7559$$

$$x_2 = \phi(x_1) = \frac{1}{\sqrt{(0.7559+1)}} = 0.75466$$

$$x_3 = 0.75492,\ x_4 = 0.75487,\ x_5 = 0.75488$$

Hence x_4 and x_5 being almost the same, the root is 0.7548 correct to four decimal places.

EXAMPLE 2.27

Apply iteration method to find the negative root of the equation $x^3 - 2x + 5 = 0$ correct to four decimal places.

Solution:

If α, β, γ are the roots of the given equation, then $-\alpha, -\beta, -\gamma$ are the roots of

$$(-x)^3 - 2(-x) + 5 = 0$$

$\therefore$ The negative root of the given equation is the positive root of

$$f(x) = x^3 - 2x - 5 = 0. \qquad (i)$$

Since $f(2) = -1$ and $f(3) = 16$, a root lies between 2 and 3.

Rewriting (i) as $x = (2x + 5)^{1/3} = \phi(x)$,

we have $\phi'(x) = \dfrac{1}{3}(2x+5)^{-2/3}. 2$ and $|\phi'(x)| < 1$ for $x < 3$.

$\therefore$ The iteration method can be applied:

Starting with $x_0 = 2$. The successive approximations are

$x_1 = \phi x_0) = (2x_0 + 5)^{1/3} = 2.08008$

$x_2 = \phi(x_1) = 2.09235,\qquad x_3 = 2.09422$

$x_4 = 2.09450,\qquad x_5 = 2.09454$

Since x_4 and x_5 being almost the same, the root of (i) is 2.0945 correct to four decimal places.

Hence the negative root of the given equation is -2.0945.

EXAMPLE 2.28

Find a real root of $2x - \log_{10} x = 7$ correct to four decimal places using the iteration method.

Solution:

We have $f(x) = 2x - \log_{10} x - 7$

$$f(3) = 6 - \log_{10}{}^{3} - 7 = 6 - 0.4771 - 7 = -1.4471$$

$$f(4) = 8 - \log_{10}{}^{4} - 7 = 8 - 0.602 - 7 = 0.398$$

$\therefore$ A root lies between 3 and 4.

Rewriting the given equation as $x = \frac{1}{2}(\log_{10} x + 7) = \phi(x)$, we have

$$\phi'(x) = \frac{1}{2}\left(\frac{1}{x}\log_{10} e\right)$$

$\therefore\ |\phi'(x)| < 1$ when $3 < x < 4$ [$\because \log_{10} e = 0.4343$]

Since $|f(4)| < |f(3)|$, the root is near to 4.

Hence the iteration method can be applied. Taking $x_0 = 3.6$, the successive approximations are

$$x_1 = \phi(x_0) = \frac{1}{2}(\log 10\ 3.6 + 7) = 3.77815$$

$$x_2 = \phi(x_1) = \frac{1}{2}(\log 10\ 3.77815 + 7) = 3.78863$$

$$x_3 = \phi(x_2) = \frac{1}{2}(\log 3.78863 + 7) = 3.78924$$

$$x_4 = \phi(x_3) = \frac{1}{2}(\log 3.78924 + 7) = 3.78927$$

Hence x_3 and x_4 being almost equal, the root is 3.7892 correct to four decimal places.

EXAMPLE 2.29

Find the smallest root of the equation

$$1 - \mathrm{x} + \frac{x^2}{(2!)^2} - \frac{x^3}{(3!)^2} + \frac{x^4}{(4!)^2} - \frac{x^5}{(5!)^2} + \cdots = 0$$

Solution:

Writing the given equation as

$$x = 1 + \frac{x^2}{(2!)^2} - \frac{x^3}{(3!)^2} + \frac{x^4}{(4!)^2} - \frac{x^5}{(5!)^2} + \cdots = \phi(x)$$

Omitting x^2 and higher powers of x, we get $x = 1$ approximately.

Taking $x_0 = 1$, we obtain

$$x_1 = \phi(x_0) = 1 + \frac{1}{(2!)^2} - \frac{1}{(3!)^2} + \frac{1}{(4!)^2} - \frac{1}{(5!)^2} + \cdots = 1.2239$$

$$x_2 = \phi(x_1) = 1 + \frac{(1.2239)^2}{(2!)^2} - \frac{(1.2239)^3}{(3!)^2} + \frac{(1.2239)^4}{(4!)^2} - \frac{(1.2239)^5}{(5!)^2} + \cdots = 1.3263$$