*Seventh iteration* $x_1 = 0.9939$, $x_2 = 1.9975$, $x_3 = 2.9976$, $x_4 = -0.0025$

*Eighth iteration* $x_1 = 0.999$, $x_2 = 1.999$, $x_3 = 2.999$, $x_4 = -0.001$

*Ninth iteration* $x_1 = 0.9996$, $x_2 = 1.9996$, $x_3 = 2.9996$, $x_4 = -0.004$

*Tenth iteration* $x_1 = 0.9998$, $x_2 = 1.9998$, $x_3 = 2.9998$, $x_4 = -0.0001$

Hence $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, $x_4 = 0$.

**Gauss-Seidal iteration method.** This is a modification of Jacobi's method. As before the system of equations:

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \qquad (1)$$

is written as

$$\left.\begin{aligned} x &= \frac{1}{a_1}(d_1 - b_1y - c_1z) \\ y &= \frac{1}{b_2}(d_2 - a_2x - c_2z) \\ z &= \frac{1}{c_3}(d_3 - a_3x - b_3y) \end{aligned}\right\} \qquad (2)$$

Here also we start with the initial approximations $x_0$, $y_0$, $z_0$ for $x$, $y$, $z$, respectively which may each be taken as zero. Substituting $y = y_0$, $z = z_0$ in the first of the equations (2), we get

$$x_1 = \frac{1}{a_1}(d_1 - b_1y_0 - c_1z_0)$$

Then putting $x = x_1$, $z = z_0$ in the second of the equations (2), we have

$$y_1 = \frac{1}{b_2}(d_2 - a_2x_1 - c_2z_0)$$

Next substituting $x = x_1$, $y = y_1$ in the third of the equations (2), we obtain

$$z_1 = \frac{1}{c_3}(d_3 - a_3x_1 - b_3y_1)$$

and so on, *i.e.*, as soon as a new approximation for an unknown is found, it is immediately used in the next step.

This process of iteration is repeated until the values of $x$, $y$, $z$ are obtained to a desired degree of accuracy.

NOTE

**Obs. 1.** *Since the most recent approximations of the unknowns are used while proceeding to the next step, the convergence in the Gauss-Seidal method is twice as fast as in Jacobi's method.*

**2.** *Jacobi and Gauss-Seidal methods converge for any choice of the initial approximations if in each equation of the system, the absolute value of the largest co-efficient is almost equal to or is at least one equation greater than the sum of the absolute values of all the remaining coefficients.*

**EXAMPLE 3.28**

Apply the Gauss-Seidal iteration method to solve the equations $20x + y - 2z = 17$; $3x + 20y - z = -18$; $2x - 3y + 20z = 25$. (cf. Example 3.25)

**Solution:**

We write the given equations in the form

$$x = \frac{1}{20}(17 - y + 2z) \qquad (i)$$

$$y = \frac{1}{20}(-18 - 3x + z) \qquad (ii)$$

$$z = \frac{1}{20}(25 - 2x + 3y) \qquad (iii)$$

*First iteration*

Putting $y = y_0$, $z = z_0$ in $(i)$, we get $\quad x_1 = \frac{1}{2}(17 - y_0 + 2z_0) = 0.8500$

Putting $x = x_1$, $z = z_0$ in $(ii)$, we have $\quad y_1 = \frac{1}{20}(-18 - 3x_1 + z_0) = -1.0275$

Putting $x = x_1$, $y = y_1$ in $(iii)$, we obtain $\quad z_1 = \frac{1}{20}(25 - 2x_1 + 3y_1) = 1.0109$

*Second iteration*

Putting $y = y_1$, $z = z_1$ in $(i)$, we get $\quad x_2 = \frac{1}{20}(17 - y_1 + 2z_1) = 1.0025$

Putting $x = x_2$, $z = z_1$ in $(ii)$, we obtain $\quad y_2 = \frac{1}{20}(-18 - 3x_2 + z_1) = -0.9998$

Putting $x = x_2$, $y = y_2$ in $(iii)$, we get $\quad z_2 = \frac{1}{20}(25 - 2x_2 + 3y_2) = 0.9998$

*Third iteration*, we get

$$x_3 = \frac{1}{20}(17 - y_2 + 2z_2) = 1.0000$$

$$y_3 = \frac{1}{20}(-18 - 3x_3 + z_2) = -1.0000$$

$$z_3 = \frac{1}{20}(25 - 2x_3 + 3y_3) = 1.0000$$

The values in the second and third iterations being practically the same, we can stop.

Hence the solution is $x = 1, y = -1, z = 1$.

**EXAMPLE 3.29**

Solve the equations $27x + 6y - z = 85$; $x + y + 54z = 110$; $6x + 15y + 2z = 72$ by the Gauss-Jacobi method and the Gauss-Seidel method.

**Solution:**

Rewriting the given equations as

$$x = \frac{1}{27}(85 - 6y + z) \qquad (i)$$

$$y = \frac{1}{15}(72 - 6x - 2z) \qquad (ii)$$

$$z = \frac{1}{54}(110 - x - y) \qquad (iii)$$

(*a*) *Gauss-Jacobi's method*

We start from an approximation $x_0 = y_0 = z_0 = 0$

*First iteration*

$$x_1 = \frac{85}{27} = 3.148, y_1 = \frac{72}{15} = 4.8, z_1 = \frac{110}{54} = 2.037$$

*Second iteration*

$$x_2 = \frac{1}{27}(85 - 6y_1 + z_1) = 2.157$$

$$y_2 = \frac{1}{15}(72 - 6x_1 - 2z_1) = 3.269$$

$$z_2 = \frac{1}{54}(110 - x_1 - y_1) = 1.890$$

*Third iteration*

$$x_3 = \frac{1}{27}(85 - 6y_2 + 7z_2) = 2.492$$
$$y_3 = \frac{1}{15}(72 - 6x_2 - 2z_2) = 3.685$$
$$z_3 = \frac{1}{54}(110 - x_2 - y_2) = 1.937$$

*Fourth iteration*

$$x_4 = \frac{1}{27}(85 - 6y_3 + z_3) = 2.401$$
$$y_4 = \frac{1}{15}(72 - 6x_3 - 2z_3) = 3.545$$
$$z_4 = \frac{1}{54}(110 - x_3 - y_3) = 1.923$$

*Fifth iteration*

$$x_5 = \frac{1}{27}(85 - 6y_4 + z_4) = 2.432$$
$$y_5 = \frac{1}{15}(72 - 6x_4 - 2z_4) = 3.583$$
$$z_4 = \frac{1}{54}(110 - x_3 - y_3) = 1.927$$

Repeating this process, the successive iterations are:

$$x_6 = 2.423,\ y_6 = 3.570,\ z_6 = 1.926$$
$$x_7 = 2.426,\ y_7 = 3.574,\ z_7 = 1.926$$
$$x_8 = 2.425,\ y_8 = 3.573,\ z_8 = 1.926$$
$$x_9 = 2.426,\ y_9 = 3.573,\ z_9 = 1.926$$

Hence $x = 2.426$, $y = 3.573$, $z = 1.926$

(*b*) *Gauss-Seidal method*

*First iteration*

Putting $y = y_0 = 0$, $z = z_0 = 0$ in (*i*), $x_1 = \frac{1}{27}(85 - 6y_0 + z_0) = 3.14$

Putting $x = x_1$, $z = z_0$ in (*ii*), $y1 = \frac{1}{15}(72 - 6x_1 - 2z_0) = 3.541$

Putting $x = x_1$, $y = y_1$ in (*iii*), $z_1 = \frac{1}{54}(110 - x_1 - y_1) = 1.913$

*Second iteration*

$$x_2 = \frac{1}{27}(85 - 6y_1 + z_1) = 2.432$$
$$y2 = \frac{1}{15}(72 - 6x_2 - 2z_1) = 3.572$$
$$z_2 = \frac{1}{54}(110 - x_2 - y_2) = 1.926$$

*Third iteration*

$$x_3 = \frac{1}{27}(85 - 6y_2 + z_2) = 2.426$$
$$y_3 = \frac{1}{15}(72 - 6x_3 - 2z_2) = 3.573$$
$$z_3 = \frac{1}{54}(110 - x_3 - y_3) = 1.926$$

*Fourth iteration*

$$x_4 = \frac{1}{27}(85 - 6y_3 + z_3) = 2.426$$
$$y_4 = \frac{1}{15}(72 - 6x_4 - 2z_3) = 3.573$$
$$z_4 = \frac{1}{54}(110 - x_4 - y_4) = 1.926.$$

Hence $x = 2.426$, $y = 3.573$, $z = 1.926$.

**NOTE** **Obs.** *We have seen that the convergence is quite fast in the Gauss-Seidal method as compared to the Gauss-Jacobi method.*

**EXAMPLE 3.30**

Apply the Gauss-Seidal iteration method to solve the equations: $10x_1 - 2x_2 - x_3 - x_4 = 3$; $-2x_1 + 10x_2 - x_3 - x_4 = 15$; $-x_1 - x_2 + 10x_3 + 2x_4 = 27$; $-x_1 - x_2 - 2x_3 + 10x_4 = -9$. (cf. Example 3.27)

**Solution:**

Rewriting the given equations as

$$x_1 = 0.3 + 0.2x_2 + 0.1x_3 + 0.1x_4 \qquad (i)$$

$$x_2 = 1.5 + 0.2x_1 + 0.1x_3 + 0.1x_4 \qquad (ii)$$
$$x_3 = 2.7 + 0.1x_1 + 0.1x_2 + 0.2x_4 \qquad (iii)$$
$$x_4 = -0.9 + 0.1x_1 + 0.1x_2 + 0.2x_3 \qquad (iv)$$

*First iteration*

Putting $x_2 = 0$, $x_3 = 0$, $x_4 = 0$ in $(i)$, we get $x_1 = 0.3$

Putting $x_1 = 0.3$, $x_3 = 0$, $x_4 = 0$ in $(ii)$, we obtain $x_2 = 1.56$

Putting $x_1 = 0.3$, $x_2 = 1.56$, $x_4 = 0$ in $(iii)$, we obtain $x_3 = 2.886$

Putting $x_1 = 0.3$, $x_2 = 1.56$, $x_3 = 2.886$ in $(iv)$, we get $x_4 = -0.1368$.

*Second iteration*

Putting $x_2 = 1.56$, $x_3 = 2.886$, $x_4 = -0.1368$ in $(i)$, we obtain $x_1 = 0.8869$

Putting $x_1 = 0.8869$, $x_3 = 2.886$, $x_4 = -0.1368$ in $(ii)$, we obtain $x_2 = 1.9523$

Putting $x_1 = 0.8869$, $x_2 = 1.9523$, $x_4 = -0.1368$ in $(iii)$, we have $x_3 = 2.9566$

Putting $x_1 = 0.8869$, $x_2 = 1.9523$, $x_3 = 2.9566$ in $(iv)$, we get $x_4 = -0.0248$.

*Third iteration*

Putting $x_2 = 1.9523$, $x_3 = 2.9566$, $x_4 = -0.0248$ in $(i)$, we obtain $x_1 = 0.9836$

Putting $x_1 = 0.9836$, $x_3 = 2.9566$, $x_4 = -0.0248$ in $(ii)$, we obtain $x_2 = 1.9899$

Putting $x_1 = 0.9836$, $x_2 = 1.9899$, $x_4 = -0.0248$ in $(iii)$, we get $x_3 = 2.9924$

Putting $x_1 = 0.9836$, $x_2 = 1.9899$, $x_3 = 2.9924$ in $(iv)$, we get $x_4 = -0.0042$.

*Fourth iteration.* Proceeding as above

$x_1 = 0.9968$, $x_2 = 1.9982$, $x_3 = 2.9987$, $x_4 = -0.0008$.

*Fifth iteration* is $x_1 = 0.9994$, $x_2 = 1.9997$, $x_3 = 2.9997$, $x_4 = -0.0001$.

*Sixth iteration* is $x_1 = 0.9999$, $x_2 = 1.9999$, $x_3 = 2.9999$, $x_4 = -0.0001$

Hence the solution is $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, $x_4 = 0$.

**(3) Relaxation method**3**.** Consider the equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

[3] This method was originally developed by R.V. Southwell in 1935, for application to structural engineering Exercises