$$i.e., \quad y_p = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots \tag{1}$$

It is called *Newton's backward interpolation formula* as (1) contains y_n and backward differences of y_n

NOTE

Obs. *This formula is used for interpolating the values of y near the end of a set of tabulated values and also for extrapolating values of y a little ahead (to the right) of* y_n

EXAMPLE 7.1

The table gives the distance in nautical miles of the visible horizon for the given heights in feet above the earth's surface:

x = height:	100	150	200	250	300	350	400
y = distance:	10.63	13.03	15.04	16.81	18.42	19.90	21.27

Find the values of y when

(*i*) $x = 160$ *ft*. (*ii*) $x = 410$.

Solution:

The difference table is as under:

x	y	Δ	Δ^2	Δ^3	Δ^4
100	10.63				
		2.40			
150	**13.03**		– 0.39		
		2.01		0.15	
200	15.04		**– 0.24**		– 0.07
		1.77		**0.08**	
250	16.81		– 0.16		**– 0.05**
		1.61		0.03	
300	18.42		– 0.13		– 0.01
		1.48		**0.02**	
350	19.90		– **0.11**		
		1.37			
400	**21.27**				

(*i*) If we take $x_0 = 160$, then $y_0 = 13.03$, $\Delta y_0 = 2.01$, $\Delta^2 y_0 = -0.24$, $\Delta^3 = 0.08$, $\Delta^4 y_0 = -0.05$

Since $x = 160$ and $h = 50$, $\therefore\ p = \frac{x - x_0}{h} = \frac{10}{50} = 0.2$

$\therefore$ Using Newton's forward interpolation formula, we get

$$y_{218} = y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \cdots$$

$y_{160} = 13.03 + 0.402 + 0.192 + 0.0384 + 0.00168 = 13.46$ nautical miles

(*ii*) Since $x = 410$ is near the end of the table, we use Newton's backward interpolation formula.

$\therefore$ Taking $x_n = 400$, $p = \frac{x - x_n}{h} = \frac{10}{50} = 0.2$

Using the line of backward difference

$y_n = 21.27$, $\nabla y_n = 1.37$, $\nabla^2 y_n = -0.11$, $\nabla^3 y_n = 0.02$ etc.

$\therefore$ Newton's backward formula gives

$$y_{410} = y_{400} + p\nabla y_{400} + \frac{p(p+1)}{2!}\nabla^2 y_{400} + \frac{p(p+1)(p+2)}{3!}\Delta^3 y_{400} + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_{400} + \cdots$$

$$= 21.27 + 0.2(1.37) + \frac{0.2(1.2)}{2!}(-0.11) + \frac{0.2(1.2)(2.2)}{3!}(0.02) + \frac{0.2(1.2)(2.2)(3.2)}{4!}(-0.01)$$

$$= 21.27 + 0.274 - 0.0132 + 0.0018 - 0.0007$$

$= 21.53$ nautical miles

EXAMPLE 7.2

From the following table, estimate the number of students who obtained marks between 40 and 45:

Marks:	30—40	40—50	50—60	60—70	70—80
No. of students:	31	42	51	35	31

Solution:

First we prepare the cumulative frequency table, as follows:

Marks less than (x):	40	50	60	70	80
No. of students (y_x):	31	73	124	159	190

Now the difference table is

x	y_x	Δy_x	$\Delta^2 y_x$	$\Delta^3 y_x$	$\Delta 4yx$
40	**31**				
		42			
50	73		**9**		
		51		– **25**	
60	124		**– 16**		**37**
		35		**12**	
70	159		**– 4**		
		31			
80	190				

We shall find y_{45}, *i.e., the* number of students with marks less than 45.

Taking $x_0 = 40$, $x = 45$, we have

$$p = \frac{x - x_0}{h} = \frac{5}{10} = 0.5 \qquad [\because h = 10]$$

∴ Using Newton's forward interpolation formula, we get

$$y_{45} = y_{40} + p\Delta y_{40} + \frac{p(p-1)}{2!}\Delta^2 y_{40} + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_{40} + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_{40}$$

$$= 31 + 0.5 \times 42 + \frac{(0.5)(-0.5)}{2} \times 9 + \frac{(0.5)(-0.5)(-15)}{6} \times (-25) + \frac{(0.5)(-0.5)(-15)(-2.5)}{24} \times 37$$

$= 31 + 21 - 1.125 - 1.5625 - 1.4453$

$= 47.87$, on simplification.

The number of students with marks less than 45 is 47.87, i.e., 48. But the number of students with marks less than 40 is 31.

Hence the number of students getting marks between 40 and 45 = 48 – 31 = 17.

EXAMPLE 7.3.

Find the cubic polynomial which takes the following values:

x:	0	1	2	3
$f(x)$:	1	2	1	10

Hence or otherwise evaluate $f(4)$.

Solution:

The difference table is

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	**1**			
		1		
1	2		**– 2**	
		– 1		**12**
2	1		**10**	
		9		
3	**10**			

We take $x_0 = 0$ and $p = \dfrac{x-0}{h} = x$ $\quad [\because h = 1]$

$\therefore$ Using Newton's forward interpolation formula, we get

$$f(x) = f(0) + \frac{x}{1}\Delta f(0) + \frac{x(x-1)}{1.2}\Delta^2 f(0) + \frac{x(x-1)(x-2)}{1.2.3}\Delta^3 f(0)$$

$$= 1 + x(1) + \frac{x(x-1)}{2}(-2) + \frac{x(x-1)(x-2)}{6}(12)$$

$$= 2x^3 - 7x^2 + 6x + 1$$

which is the required polynomial.

To compute $f(4)$, we take $x_n = 3$, $x = 4$ so that $p = \dfrac{x - x_n}{h} = 1$ $\quad [\because h = 1]$

NOTE **Obs.** *Using Newton's backward interpolation formula, we get*

$$f(4) = f(3) + p\nabla f(3) + \frac{p(p+1)}{1.2}\nabla^2 f(3) + \frac{p(p+1)(p+2)}{1.2.3}\nabla^3 f(3)$$

$$= 10 + 9 + 10 + 12 = 41$$

which is the same value as that obtained by substituting $x = 4$ in the cubic polynomial above.

The above example shows that if a tabulated function is a polynomial, then interpolation and extrapolation give the same values.

EXAMPLE 7.4

Using Newton's backward difference formula, construct an interpolating polynomial of degree 3 for the data: $f(-0.75) = -0.0718125$, $f(-0.5) = -0.02475$, $f(-0.25) = 0.3349375$, $f(0) = 1.10100$. Hence find $f(-1/3)$.

Solution:

The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
– 0.75	– 0.0718125			
		0.0470625		
– 0.50	– 0.02475		0.312625	
		0.3596875		**0.09375**
– 0.25	0.3349375		**0.400375**	
		0.7660625		
0	**1.10100**			

We use Newton's backward difference formula

$$y(x) = y_3 + \frac{p}{1!}\nabla y_3 + \frac{p(p+1)}{2!}\nabla^2 y_3 + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_3$$

taking $\quad x_3 = 0, p = \dfrac{x-0}{h} = \dfrac{x}{0.25} = 4x \qquad [\because h = 0.25]$

$$y(x) = 1.10100 + 4x(0.7660625) + \frac{4x(4x+1)}{2}(0.400375) + \frac{4x(4x+1)(4x+2)}{6}(0.09375)$$

$$= 1.101 + 3.06425x + 3.251x^2 + 0.81275x + x^3 + 0.75x^2 + 0.125x$$

$$= x^3 + 4.001x^2 + 4.002x + 1.101$$

Put $\quad x = -\dfrac{1}{3}$, so that

$$y\left(-\frac{1}{3}\right)=\left(-\frac{1}{3}\right)^3+4.001\left(-\frac{1}{3}\right)^2+4.002\left(-\frac{1}{3}\right)+1.101$$
$$= 0.1745$$

EXAMPLE 7.5

In the table below, the values of y are consecutive terms of a series of which 23.6 is the 6th term. Find the first and tenth terms of the series:

x:	3	4	5	6	7	8	9
y:	4.8	8.4	14.5	23.6	36.2	52.8	73.9

Solution:

The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
3	4.8				
		3.6			
4	8.4		2.5		
		6.1		0.5	
5	14.5		3.0		0
		9.1		0.5	
6	23.6		3.5		0
		12.6		0.5	
7	36.2		4.0		0
		16.6		0.5	
8	52.8		4.5		
		21.1			
9	73.9				

To find the first term, use Newton's forward interpolation formula with $x_0 = 3$, $x = 1$, $h = 1$, and $p = -2$. We have

$$y(1)=4.8+\frac{(-2)}{1}\times 3.6+\frac{(-2)(-3)}{1.2}\times 2.5+\frac{(-2)(-3)(-4)}{1.2.3}\times 0.5=3.1$$

To obtain the tenth term, u se Newton's backward interpolation formula with $x_n = 9$, $x = 10$, $h = 1$, and $p = 1$.This gives

$$y(10)=73.9+\frac{1}{1}\times 21.1+\frac{1(2)}{1.2}\times 4.5+\frac{1(2)(3)}{1.2.3}\times 0.5=100$$

EXAMPLE 7.6

Using Newton's forward interpolation formula show

$$\sum n^3 = \left\{\frac{n(n+1)}{2}\right\}^2$$

Solution:

If $s_n = sn^3$, then $s_{n+1} = \Sigma(n+1)^3$

$$\therefore \qquad \Delta s_n = s_{n+1} - s_n = \sum(n+1)^3 - \sum n^3 = (n+1)^3$$

Then $\quad \Delta^2 s_n = \Delta s_{n+1} - \Delta s_n = (n+2)^3 - (n+1)^3 = 3n^2 + 9n + 7$

$$\Delta^3 s_n = \Delta^2 s_{n+1} - \Delta^2 s_n$$

$$= \left[3(n+1)^2 + 9(n+1) + 7\right] - \left(3n^2 + 9n + 7\right) = 6n + 12$$

$$\Delta^4 s_n = \Delta^3 s_{n+1} - \Delta^3 s_n = [6(n+1) + 12] - [6n + 12] = 6$$

and $\quad \Delta^5 s_n = \Delta^5 s_n = \ldots\ldots = 0$

Since the first term of the given series is 1, therefore taking $n = 1$, $s_1 = 1$, $\Delta s_1 = 8$, $\Delta^2 s_1 = 19$, $\Delta^3 s_1 = 18$, $\Delta^4 s_1 = 6$.

Substituting these in the Newton's for war d interpolation formula, *i.e.*,

$$s = s + (n-1)\Delta s_1 + \frac{(n-1)(n-2)}{2!}\Delta^2 s_1 + \frac{(n-1)(n-2)(n-3)}{3!}\Delta^3 s_1$$
$$+ \frac{(n-1)(n-2)(n-3)(n-4)}{4!}\Delta^4 s_1$$

$$sn = 1 + 8(n-1) + \frac{19}{2}(n-1)(n-2) + 3(n-1)(n-2)(n-3)$$

$$+ \frac{1}{4}(n-1)(n-2)(n-3)(n-4) = \frac{1}{4}(n^4 + 2n^3 + n^2) = \left\{\frac{n(n+1)}{2}\right\}^2$$

Exercises 7.1

1. Using Newton 's forward formula, fin d the value of $f(1.6)$, if

x:	1	1.4	1.8	2.2
$f(x)$:	3.49	4.82	5.96	6.5

2. From the following table find y when $x = 1.85$ an d 2.4 by Newton's interpolation formula:

x:	1.7	1.8	1.9	2.0	2.1	2.2	2.3
$y = e^x$:	5.474	6.050	6.686	7.389	8.166	9.025	9.974

3. Express the value of θ in terms of x using the following data:

x:	40	50	60	70	80	90
θ:	184	204	226	250	276	304

Also find θ at $x = 43$.

4. Given $\sin 45° = 0.7071$, $\sin 50° = 0.7660$, $\sin 55° = 0.8192$, $\sin 60° = 0.8660$, find sin 52° using Newton's forward formula.

5. From the following table:

x:	0.1	0.2	0.3	0.4	0.5	0.6
$f(x)$:	2.68	3.04	3.38	3.68	3.96	4.21

find $f(0.7)$ approximately.

6. The area A of a circle of diameter d is given for the following values:

d:	80	85	90	95	100
A:	5026	5674	6362	7088	7854

Calculate the area of a circle of diameter 105

7. From the following table:

$x°$:	10	20	30	40	50	60	70	80
$\cos x$:	0.9848	0.9397	0.8660	0.7660	0.6428	0.5000	0.3420	0.1737

Calculate cos 25° and cos 73° using the Gregory-1 Newton formula.

8. A test performed on a *NPN* transistor gives the following result:

Base current f (mA)	0	0.01	0.02	0.03	0.04	0.05
Collector current I_C (mA)	0	1.2	2.5	3.6	4.3	5.34

Calculate (*i*) the value of the collector current for the base current of 0.005 mA.
(*ii*) the value of base current required for a collector correct of 4.0 mA.

9. Find $f(22)$ from the following data using Newton's backward formulae.

x:	20	25	30	35	40	45
$f(x)$:	354	332	291	260	231	204

10. Find the number of men getting wages between Rs. 10 and 15 from the following data:

Wages in Rs:	0—10	10—20	20—30	30—40
Frequency:	9	30	35	42

11. From the following data, estimate the number of persons having incomes between 2000 and 2500:

Income	Below 500	500–1000	1000–2000	2000–3000	3000–4000
No. of persons	6000	4250	3600	1500	650

12. Construct Newton's forward interpolation polynomial for the following data:

x:	4	6	8	10
y:	1	3	8	16

Hence evaluate y for $x = 5$.

13. Find the cubic polynomial which takes the following values:
$y(0) = 1$, $y(1) = 0$, $y(2) = 1$ and $y(3) = 10$.

Hence or otherwise, obtain $y(4)$.

14. Construct the difference table for the following data:

x:	0.1	0.3	0.5	0.7	0.9	1.1	1.3
$f(x)$:	0.003	0.067	0.148	0.248	0.370	0.518	0.697

Evaluate $f(0.6)$

15. Apply Newton's backward difference formula to the data below, to obtain a polynomial of degree 4 in x:

x:	1	2	3	4	5
y:	1	– 1	1	– 1	1

16. The following table gives the population of a town during the last six censuses. Estimate the increase in the population during the period from 1976 to 1978:

Year:	1941	1951	1961	1971	1981	1991
Population: (*in thousands*)	12	15	20	27	39	52

17. In the following table, the values of y are consecutive terms of a series of which 12.5 is the fifth term. Find the first and tenth terms of the series.

x:	3	4	5	6	7	8	9
y:	2.7	6.4	12.5	21.6	34.3	51.2	72.9

18. Using a polynomial of the third degree, complete the record given below of the export of a certain commodity during five years:

Year:	1989	1990	1991	1992	1993
Export: (*in tons*)	443	384	—	397	467

19. Given $u_1 = 40$, $u_3 = 45$, $u_5 = 54$, find u_2 and u_4.

20. If $u_{-1} = 10$, $u_1 = 8$, $u_2 = 10$, $u_4 = 50$, find u_0 and u_3.

21. Given $y_0 = 3$, $y_1 = 12$, $y_2 = 81$, $y_3 = 200$, $y_4 = 100$, $y_5 = 8$, without forming the difference table, find $\Delta^5 y0$.

7.4 Central Difference Interpolation Formulae

In the preceding sections, we derived Newton's forward and backward interpolation formulae which are applicable for interpolation near the beginning and end of tabulated values. Now we shall develop central difference formulae which are best suited for interpolation near the middle of the table.

If x takes the values $x_0 - 2h$, $x_0 - h$, x_0, $x_0 + h$, $x_0 + 2h$ and the corresponding values of $y = f(x)$ are y_{-2}, y_{-1}, y_0, y_1, y_2, then we can write the difference table in the two notations as follows:

x	y	*1st diff.*	*2nd diff.*	*3rd diff.*	*4th diff.*
$x_0 - 2h$	y_{-2}				
		$\Delta y-_2 \ (= \Delta y-_{3/2})$			
$x_0 - h$	y_{-1}		$\Delta^2 y-_2 (= \Delta^2 y_{-1})$		
		$\Delta y_{-1} \ (= \Delta y_{-1/2})$		$\Delta^3 y_{-2} \ (= \Delta^3 y_{-1/2})$	
x_0	y_0		$\Delta^2 y_{-1} \ (= \Delta^2 y_0)$		$\Delta^3 y_{-2} (= \Delta^4 y_0)$
		$\Delta y_0 \ (= \Delta y_{1/2})$		$\Delta^3 y_{-1} \ (= \Delta^3 y_{1/2})$	
$x_0 + h$	y_1		$\Delta^2 y_0 (= \Delta^2 y_1)$		
		$\Delta y_1 (= \Delta y_{3/2})$			
$x_0 + 2h$	y_2				