CHAPTER 8

NUMERICAL DIFFERENTIATION AND INTEGRATION

Chapter Objectives

- Numerical differentiation
- Formulae for derivatives
- Maxima and minima of a tabulated function
- Numerical integration
- Quadrature formulae
- Errors in quadrature formulae
- Romberg's method
- Euler-Maclaurin formula
- Method of undetermined coefficients
- Gaussian integration
- Numerical double integration
- Objective type of questions

8.1 Numerical Differentiation

It is the process of calculating the value of the derivative of a function at some assigned value of x from the given set of values (x_i, y_i). To compute dy/dx, we first replace the exact relation $y = f(x)$ by the best interpolating polynomial $y = \phi(x)$ and then differentiate the latter as many times as we desire. The choice of the interpolation formula to be used, will depend on the assigned value of x at which dy/dx is desired.

If the values of x are equispaced and dy/dx is required near the beginning of the table, we employ Newton's forward formula. If it is required near the end of the table, we use Newton's backward formula. For values near the middle of the table, dy/dx is calculated by means of Stirling's or Bessel's formula. If the values of x are not equispaced, we use Lagrange's formula or Newton's divided difference formula to represent the function.

Hence corresponding to each of the interpolation formulae, we can derive a formula for finding the derivative.

NOTE

Obs. *While using these formulae, it must be observed that the table of values defines the function at these points only and does not completely define the function and the function may not be differentiable at all. As such, the process of numerical differentiation should be used only if the tabulated values are such that the differences of some order are constants. Otherwise, errors are bound to creep in which go on increasing as derivatives of higher order are found. This is due to the fact that the difference between $f(x)$ and the approximating polynomial $\phi(x)$ may be small at the data points but $f'(x) - \phi'(x)$ may be large.*

8.2 Formulae for Derivatives

Consider the function $y = f(x)$ which is tabulated for the values $x_i (= x_0 + ih)$, $i = 0, 1, 2, \ldots n$.

Derivatives using Newton's forward difference formula

Newton's forward interpolation formula (p. 274) is

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots$$

Differentiating both sides w.r.t. p, we have

$$\frac{dy}{dp} = \Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2 - 6p + 2}{3!}\Delta^3 y_0 + \ldots$$

Since $p = \dfrac{(x - x_0)}{h}$

Therefore $\dfrac{dp}{dx} = \dfrac{1}{h}$

Now $\dfrac{dy}{ds} = \dfrac{dy}{dp} \cdot \dfrac{dp}{dx} = \dfrac{1}{h}\left[\Delta y_0 + \dfrac{2p-1}{2!}\Delta^2 y_0 + \dfrac{3p^2-6p+2}{3!}\Delta^3 y_0\right.$

$$\left. + \frac{4p^3-18p^2+22p-6}{4!}\Delta^4 y_0 + \cdots\right] \quad (1)$$

At $x = x_0$, $p = 0$. Hence putting $p = 0$,

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \frac{1}{6}\Delta^6 y_0 + \ldots\right] \quad (2)$$

Again differentiating (1) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dp}\left(\frac{dy}{dp}\right)\frac{dp}{dx}$$

$$= \frac{1}{h}\left[\frac{2}{2!}\Delta^2 y_0 + \frac{6p-6}{3!}\Delta^3 y_0 + \frac{12p^2-36p^2-36p+22}{4!}\Delta^4 y_0 + \cdots\right]\frac{1}{h}$$

Putting $p = 0$, we obtain

$$\left(\frac{d^2y}{dx^2}\right) = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 y_0 + \cdots\right] \quad (3)$$

Similarly $\left(\dfrac{d^3y}{dx^3}\right) = \dfrac{1}{h^3}\left[\Delta^3 y_0 - \dfrac{3}{2}\Delta^4 y_0 + \ldots\right]$

Otherwise: We know that $1 + \Delta = E = e^{hD}$

$$\therefore \quad hD = \log(1+\Delta) = \Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \cdots$$

or $\quad D = \dfrac{1}{h}\left[\Delta - \dfrac{1}{2}\Delta^2 + \dfrac{1}{3}\Delta^3 - \dfrac{1}{4}\Delta^4 + \cdots\right]$

and $\quad D^2 = \dfrac{1}{h^2}\left[\Delta - \dfrac{1}{2}\Delta^2 + \dfrac{1}{3}\Delta^3 - \dfrac{1}{4}\Delta^4 + \cdots\right]^2 = \dfrac{1}{h^2}\left[\Delta^2 - \Delta^3 + \dfrac{11}{12}\Delta^4 + \cdots\right]$

and $\quad D^3 = \dfrac{1}{h^2}\left[\Delta^3 - \dfrac{3}{2}\Delta^4 + \cdots\right]$

Now applying the above identities to y_0, we get

Dy_0 i.e., $\left(\dfrac{dy}{dx}\right)_{x_0} = \dfrac{1}{h}\Delta y_0 - \dfrac{1}{2}\left[\Delta^2 y_0 \dfrac{1}{3}\Delta^3 y_0 - \dfrac{1}{4}\Delta^4 y_0 + \dfrac{1}{5}\Delta^5 y_0 - \dfrac{1}{6}\Delta^6 y_0 + \ldots\right]$

$$\left(\frac{d^2y}{dx^2}\right) = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 y_0 - \ldots\right]$$

and $$\left(\frac{d^3y}{dx^3}\right) = \frac{1}{h^3}\left[\Delta^2 y_0 - \frac{3}{2}\Delta^4 y_0 + \ldots\right]$$

which are the same as (2), (3), and (4), respectively.

Derivatives using Newton's backward difference formula

Newton's backward interpolation formula (p. 274) is

$$y = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots$$

Differentiating both sides w.r.t. p, we get

$$\frac{dy}{dp} = \nabla y_n + \frac{2p+1}{2!}\nabla^2 y_n + \frac{3p^2+6p+2}{3!}\nabla^3 y_n + \cdots$$

Since $p = \dfrac{x - x_n}{h}$, therefore. $\dfrac{dp}{dx} = \dfrac{1}{h}$

Now $$\frac{dy}{dx} = \frac{dy}{dp}\cdot\frac{dp}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{2p+1}{2!}\nabla^2 y_n + \frac{3p^3+6p+2}{3!}\nabla^3 y_n + \cdots\right] \quad (5)$$

At $x = x_n$, $p = 0$. Hence putting $p = 0$, we get

$$\left(\frac{dy}{dx}\right)_{x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \frac{1}{5}\nabla^5 y_n + \frac{1}{6}\nabla^6 y_n + \cdots\right] \quad (6)$$

Again differentiating (5) w.r.t. x, we have

$$\frac{d^2y}{dx^2} = \frac{d}{dp}\left(\frac{dy}{dx}\right)\frac{dp}{dx3}$$

$$= \frac{1}{h^2}\left[\nabla^2 y_n + \frac{6p+6}{3!}\nabla^3 y_n + \frac{6p^2+18p+11}{12}\nabla^4 y_n + \cdots\right]$$

Putting $p = 0$, we obtain

$$\left(\frac{d^2y}{dx^2}\right) = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \frac{137}{180}\nabla^6 y_n + \cdots\right] \quad (7)$$

Similarly, $$\left(\frac{d^3y}{dx^3}\right)_{x_n} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \cdots\right] \quad (8)$$

Otherwise: We know that $1 - \nabla = E^{-1} = e^{-hD}$

$$\therefore \qquad -hD = \log(1-\nabla) = -[\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \frac{1}{3}\nabla^4 + \cdots]$$

or
$$D = \frac{1}{h}\left[\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \frac{1}{4}\nabla^4 + \ldots\right]$$

$$\therefore \qquad D^2 = \frac{1}{h^2}\left[\nabla + \frac{1}{2}\nabla^2 + \frac{1}{2}\nabla^3 + \cdots\right]^2 = \frac{1}{h^2}\left[\nabla^2 + \nabla^3 + \frac{11}{12}\nabla^4 + \cdots\right]$$

Similarly, $D^3 = \frac{1}{h^3}\left[\nabla^3 + \frac{3}{2}\nabla^4 + \cdots\right]$

Applying these identities to y_n, we get

$$Dy_n \text{ i.e., } \left(\frac{dy}{dx}\right)_{x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{2}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n \frac{1}{5}\nabla^5 y_n + \frac{1}{6}\nabla^6 y_n + \cdots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \frac{137}{180}\nabla^6 y_n + \cdots\right]$$

and
$$\left(\frac{d^3y}{dx^3}\right)_{x_n} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \cdots\right]$$

which are the same as (6), (7), and (8).

Derivatives using Stirling's central difference formula

Stirling's formula (p. 289) is

$$y_p = y_0 + \frac{p}{1!}\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}\Delta^2 y_{-1}$$
$$+ \frac{p(p^2-1^2)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{p^2(p^2-1^2)}{4!}\Delta^4 y_{-2} + \cdots$$

Differentiating both sides w.r.t. p, we get

$$\frac{dy}{dp} = \left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{2p}{2!}\Delta^2 y_{-1} + \frac{3p^2-1}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right)$$
$$+ \frac{4p^3 - 2p}{4!}\Delta^4 y_{-2} + \cdots$$

Since $\quad p = \frac{x - x_0}{h}, \therefore \frac{dp}{dx} = \frac{1}{h}.$

Now $\dfrac{dy}{dx} = \dfrac{dy}{dp}\cdot\dfrac{dp}{dx} = \dfrac{1}{h}\left[\left(\dfrac{\Delta y_0 + \Delta y_{-1}}{26}\right) + p\Delta^2 y_{-1} + \dfrac{3p^2-1}{6}\left(\dfrac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right)\right.$

$$\left. + \frac{2p^3-p}{12}\Delta^4 y_{-2} + \ldots\right]$$

At $x = x_0$, $p = 0$. Hence putting $p = 0$, we get

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2} - \frac{1}{6}\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} + \frac{1}{30}\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2} + \cdots\right] \quad (9)$$

Similarly $\left(\dfrac{d^2y}{dx^2}\right) = \dfrac{1}{h^2}\left[\Delta^2 y_{-1} - \dfrac{1}{12}\Delta^4 y_{-2} + \dfrac{1}{90}\Delta^6 y_{-3} - \cdots\right]$ (10)

Derivatives using Bessel's central difference formula

Bessel's formula (p. 290) is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \left(p - \frac{1}{2}\right)\frac{p(p-1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{4p^3 - 6p^2 - 2p + 2}{4!}\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \ldots$$

Since $p = \dfrac{x - x_0}{h}$, $\therefore \dfrac{dp}{dx} = \dfrac{1}{h}$

Now $\dfrac{dy}{dx} = \dfrac{dy}{dp}\cdot\dfrac{dp}{dx} = \dfrac{1}{h}\left[\Delta y_0 + \dfrac{2p-1}{2!}\dfrac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right.$

$$\left. + \frac{3p^2 - 2p + \frac{1}{2}}{3!}\Delta^3 y_{-1} + \frac{4p^3 - 6p^2 - 2p + 2}{4!}\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \cdots\right]$$

At $x = x_0$, $p = 0$. Hence putting $p = 0$, we get

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \frac{1}{12}\Delta^3 y_{-1}\right. \quad (11)$$

$$\left. + \frac{1}{12}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right) + \cdots\right]$$

Similarly

$$\left(\frac{d^2y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) - \frac{1}{2}\Delta^3 y_{-1} - \frac{1}{12}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right) + \cdots\right] \quad (12)$$

Derivatives using unequally spaced values of argument

(*i*) *Lagranges's interpolation formula* is

$$f(x)=\frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}f(x_0)+\frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)}f(x_1)+\cdots$$

Differentiating both sides w.r.t. x, we get $f'(x)$.

(*ii*) *Newton's divided difference formula* is

$$f(x)=f(x_0)+(x-x_0)\Delta f(x_0)+(x-x_0)(x-x_1)\Delta^2 f(x_0)+(x-x_0)(x-x_1)(x-x_2)\Delta^3 f(x_0)+\cdots$$

Differentiating both sides w.r.t. x, we obtain

$$f'(x)=\Delta f(x0)+[2x-(x_0+x_1)]\,\Delta^2 f(x_0)+[3x^2-2x(x_0+x_1+x_2)+(x_0x_1+x_1x_2+x_2x_3)]\,\Delta^3 f(x_0)+\cdots$$

EXAMPLE 8.1

Given that

x:	1.0	1.1	1.2	1.3	1.4	1.5	1.6
y:	7.989	8.403	8.781	9.129	9.451	9.750	10.031

find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at (*a*) $x = 1.1$ (*b*) $x = 1.6$.

Solution:

(a) The difference table is:

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
1.0	7.989						
		0.414					
1.1	8.403		– 0.036				
		0.378		0.006			
1.2	8.781		– 0.030		– 0.002		
		0.348		0.004		0.001	
1.3	9.129		– 0.026		– 0.001		0.002
		0.322		0.003		0.003	
1.4	9.451		– 0.023		0.002		

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
		0.299		0.005			
1.5	9.750		– 0.018				
		0.281					
1.6	10.031						

We have

$$\left(\frac{dy}{dx}\right)_{x0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 + \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \frac{1}{6}\Delta^6 y_0 + \cdots\right] \quad (i)$$

and $$\left(\frac{d^2y}{dx^2}\right)_{x0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 y_0 - \cdots\right] \quad (ii)$$

Here $h = 0.1$, $x_0 = 1.1$, $\Delta y_0 = 0.378$, $\Delta^2 y_0 = -0.03$ etc.

Substituting these values in (*i*) and (*ii*), we get

$$\left(\frac{dy}{dx}\right) = \frac{1}{0.1}\left[0.378 - \frac{1}{2}(-0.03) + \frac{1}{3}(0.004) - \frac{1}{4}(-0.001) + \frac{1}{5}(0.003)\right] = 3.952$$

$$\left(\frac{d^2y}{dx^2}\right) = \frac{1}{(0.1)^2}\left[-0.03 - (0.004) + \frac{11}{12}(-0.001) - \frac{5}{6}(0.003)\right] = -3.74$$

(*b*) We use the above difference table and the backward difference operator ∇ instead of Δ.

$$\left(\frac{dy}{dx}\right)_{x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{5}\nabla^5 y_n + \frac{1}{6}\nabla^6 y_n + \cdots\right] \quad (i)$$

and $$\left(\frac{d^2y}{dx^2}\right)_{xn} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \frac{137}{180}\nabla^6 y_n + \cdots\right] \quad (ii)$$

Here $h = 0.1$, $x_n = 1.6$, $\nabla y_n = 0.281$, $\nabla^2 y_n = -0.018$ etc.

Putting these values in (*i*) and (*ii*), we get

$$\left(\frac{dy}{dx}\right)_{1.6} = \frac{1}{0.1}\left[0.281 + \frac{1}{2}(-0.018) + \frac{1}{3}(0.05) + \frac{1}{4}(0.002) + \frac{1}{5}(0.003) + \frac{1}{6}(0.002)\right] = 2.75$$

$$\left(\frac{d^2y}{dx^2}\right)_{1.6} = \frac{1}{(0.1)^2}\left[-0.018+0.005+\frac{11}{12}(0.002)+\frac{5}{6}(0.003)\right.$$
$$\left.+\frac{137}{180}(0.002)\right] = -0.715.$$

EXAMPLE 8.2

The following data gives the velocity of a particle for twenty seconds at an interval of five seconds. Find the initial acceleration using the entire data:

Time t (sec):	0	5	10	15	20
Velocity v(m/sec): 0	3	14	69	228	

Solution:

The difference table is:

t	v	Δv	$\Delta^2 v$	$\Delta^3 v$	$\Delta^4 v$
0	0				
3					
5	3		8		
11		36			
10	14		44		24
55		60			
15	69		104		
159					
20	228				

An initial acceleration *i.e.*, $\left(\frac{dv}{dt}\right)$ at $t = 0$ is required, we use Newton's forward formula:

$$\left(\frac{dv}{dt}\right)_{t=0} = \frac{1}{h}\left(\Delta v_0 - \frac{1}{2}\Delta^2 v_0 + \frac{1}{3}\Delta^3 v_0 - \frac{1}{4}\Delta^4 v_0 + \cdots\right)$$

$$\therefore \quad \left(\frac{dv}{dt}\right)_{t=0} = \frac{1}{5}\left[3 - \frac{1}{1}(8) + \frac{1}{3}(36) - \frac{1}{4}(24)\right]$$

$$= \frac{1}{5}(3-4+12-6) = 1$$

Hence the initial acceleration is 1 m/sec^2.

EXAMPLE 8.3

Find the value of cos (1.74) from the following table:

x:	1.7	1.74	1.78	1.82	1.86
sin x:	0.9916	0.9857	0.9781	0.9691	0.9584

Solution:

Let $y = f(x) = \sin x$. so that $f'(x) = \cos x$.

The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.7	0.9916				
		– 0.0059			
1.74	0.9857		– 0.0017		
		– 0.0076		0.0003	
1.78	0.9781		– 0.0014		– 0.0006
		– 0.0090		– 0.0003	
1.82	0.9691		– 0.0017		
		– 0.0107			
1.84	0.9584				

Since we require $f'(1.74)$, we use Newton's forward formula

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 = \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right] \qquad (i)$$

Here $h = 0.04$, $x_0 = 1.7$, $\Delta y_0 = -\,0.0059$, $\Delta^2 y_0 = -\,0.0017$ etc.

Substituting these values in (i), we get

$$\left(\frac{dy}{dx}\right)_{1.74} = \frac{1}{0.04}\left[0.0059 - \frac{1}{2}(-0.0017) + \frac{1}{3}(0.003) - \frac{1}{4}(-0.0006)\right]$$

$$= \frac{1}{0.04}(0.007) = 0.175$$

Hence cos (1.74) = 0.175

EXAMPLE 8.4

A slider in a machine moves along a fixed straight rod. Its distance x cm. along the rod is given below for various values of the time t seconds. Find the velocity of the slider and its acceleration when $t = 0.3$ second.

$t =$	0	0.1	0.2	0.3	0.4	0.5	0.6
$x =$	30.13	31.62	32.87	33.64	33.95	33.81	33.24

Solution:

The difference table is:

T	x	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
0	30.13						
		1.49					
0.1	31.62		– 0.24				
		1.25		– 0.24			
0.2	32.87		– 0.48		0.26		
		0.77		0.02		– 0.27	
0.3	33.64		– 0.46		– 0.01		0.29
		0.31		0.01		0.02	
0.4	33.95		– 0.45		0.01		
		– 0.14		0.02			
0.5	33.81		– 0.43				
		– 0.57					
0.6	33.24						

As the derivatives are required near the middle of the table, we use Stirling's formulae:

$$\left(\frac{dx}{dt}\right)_{t_0} = \frac{1}{h}\left(\frac{\Delta x_0 + \Delta x_{-1}}{2}\right) - \frac{1}{6}\left(\frac{\Delta^3 x_{-1} + \Delta^3 x_{-2}}{2}\right) + \frac{1}{30}\left(\frac{\Delta^5{}_{x-2} + \Delta^5 x_{-3}}{2}\right) + \quad (i)$$

$$\left(\frac{d^2x}{dt^2}\right) = \frac{1}{h^2}\left[\Delta^2 x_{-1} - \frac{1}{12}\Delta^4 x_{-2} + \frac{1}{90}\Delta^6 x_{-3} \cdots\right] \quad (ii)$$

Here $h = 0.1$, $t_0 = 0.3$, $\Delta x_0 = 0.31$, $\Delta x_{-1} = 0.77$, $\Delta^2 x_{-1} = -0.46$ etc.

Putting these values in (*i*) and (*ii*), we get

$$\left(\frac{dt}{dx}\right)_{0.3} = \frac{1}{0.1}\left[\frac{0.31+0.77}{2} - \frac{1}{6}\left(\frac{0.01+0.02}{2}\right) + \frac{1}{30}\left(\frac{0.02-0.27}{2}\right) - \cdots\right] = 5.33$$

$$\left(\frac{d^2x}{dt^2}\right)_{0.3} = \frac{1}{(0.1)^2}\left[-0.46 - \frac{1}{12}(-0.01) + \frac{1}{90}(0.29) - \cdots\right] = -45.6$$

Hence the required velocity is 5.33 cm/sec and acceleration is – 45.6 cm/sec^2.

EXAMPLE 8.5

The elevation above a datum line of seven points of a road are given below:

x:	0	300	600	900	1200	1500	1800
y:	135	149	157	183	201	205	193

Find the gradient of the road at the middle point.

Solution:

Here $h = 300$, $x_0 = 0$, $y_0 = 135$, we require the gradient dy/dx at $x = 900$.

The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	135					
		14				
300	149		– 6			
		8		24		
600	157		18		– 50	
		26		– 26		70
900	183		– 8		20	
		18		– 6		– 16
1200	201		– 14		4	
		4		– 2		
1500	205		– 16			
		– 12				
1800	193					

Using Stirling's formula for the first derivative [(9) p. 000], we get

$$y'(x_0) = \frac{1}{h}\left[\left(\frac{\Delta y_0 + \Delta y_{-1}}{26}\right) - \frac{1}{6}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{1}{30}\left(\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right)\right]$$

$$= \frac{1}{300}\left[\frac{1}{2}(18 + 26) - \frac{1}{12}(-6 - 26) + \frac{1}{60}(-16 + 70)\right]$$

$$= \frac{1}{300}(22 + 2.666 + 0.9) = 0.085$$

Hence the gradient of the road at the middle point is 0.085.

EXAMPLE 8.6

Using Bessel's formula, find $f'(7.5)$ from the following table:

x:	7.47	7.48	7.49	7.50	7.51	7.52	7.53
$f(x)$:	0.193	0.195	0.198	0.201	0.203	0.206	0.208

Solution:

Taking $x_0 = 7.50$, $h = 0.1$, we have $p = \dfrac{x - x_0}{h} = \dfrac{x - 7.50}{0.01}$

The difference table is

x	p	y_p	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
7.47	– 3	0.193						
0.002								
7.48	– 2	0.195		0.001				
0.003		– 0.001						
7.49	– 1	0.198		0.000		0.000		
0.003		– 0.001		0.003				
7.50	0	0.201		– 0.001		0.003		– 0.01
0.002		0.002		– 0.007				
7.51	1	0.203		0.001		– 0.004		
0.003		– 0.002						
7.52	2	0.206		– 0.001				
0.002								
7.53	3	0.208						

Using Bessel's formula for the first derivative [(11) p. 000], we get

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{4}(\Delta^2 y_{-1} + \Delta^2 y_0) + \frac{1}{12}\Delta^3 y_{-1} + \frac{1}{24}(\Delta^4 y_{-2} + \Delta^4 y_{-1}) - \frac{1}{120}\Delta^5 y_{-2} - \frac{1}{240}(\Delta^6 y_{-3} + \Delta^6 y_{-2})\right]$$

$$\left(\frac{dy}{dx}\right)_{7.5} = \frac{1}{0.01}\left[0.002 - \frac{1}{4}(-0.001 + 0.001) + \frac{1}{12}(0.002) + \frac{1}{24}(-0.004 + 0.003) - \frac{1}{120}(-0.007) - \frac{1}{240}(-0.010 + 0)\right]$$

$[\because \Delta^6 y_{-2} = 0]$

$= 0.2 + 0 + 0.01666 – 0.0416 + 0.00583 + 0.00416 = 0.223.$

EXAMPLE 8.7

Find $f'(10)$ from the following data:

x:	3	5	11	27	34
$f(x)$:	– 13	23	899	17315	35606

Solution:

As the values of x are not equispaced, we shall use Newton's divided difference formula. The divided difference table is

x	$f(x)$	1st div diff.	2nd div. diff.	3rd div. diff.	4th div. diff.
3	– 13				
		18			
5	23		16		
		146		0.998	
11	899		39.96		0.0002
		1025		1.003	
27	17315		69.04		
		2613			
34	35606				

Fifth differences being zero, Newton's divided difference formula for the first derivative (p. 274), we get

$$f'(x) = f(x_0, x_1) + (2x - x_0 - x_1)f(x_0, x_1, x_2)$$
$$+ [3x^2 - 2x(x_0 + x_1 + x_2) + x_0x_1 + x_1x_2 + x_2x_0)] \times f(x_0, x_1, x_2, x_3)$$
$$+ [4x^3 - 3x^2(x_0 + x_1 + x_2 + x_3) + 2x(x_0x_1 + x_1x_2 + x_2x_3 + x_3x_0 + x_1x_3 + x_0x_2)$$
$$- (x_0x_1x_2 + x_1x_2x_3 + x_2x_3x_0 + x_0x_1x_3)]\, f(x_0, x_1, x_2, x_3, x_4)$$

Putting $x_0 = 3$, $x_1 = 5$, $x_2 = 11$, $x_3 = 27$ and $x = 10$, we obtain

$$f'(0) = 18 + 12 \times 16 + 23 \times 0.998 - 426 \times 0.0002 = 232.869.$$

8.3 Maxima and Minima of a Tabulated Function

Newton's forward interpolation formula is

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots$$

Differentiating it w.r.t. p, we get

$$\frac{dy}{dp} = \Delta y_0 + \frac{2p-1}{2}\Delta^2 y_0 + \frac{3p^2-6p+2}{6}\Delta^3 y_0 + \cdots \tag{1}$$

For maxima or minima, $dy/dp = 0$. Hence equating the right-hand side of (1) to zero and retaining only up to third differences, we obtain

$$\Delta y_0 + \frac{2p-1}{2}\Delta^2 y_0 + \frac{3p^2-6p+2}{6}\Delta^3 y_0 = 0$$

i.e., $$\left(\frac{1}{2}\Delta^3 y_0\right)p^2 + \left(\Delta^2 y_0 - \Delta^3 y_0\right)p + (\Delta y_0 - \frac{1}{2}\Delta^2 y_0 \frac{1}{3}\Delta^3 y_0) = 0.$$

Substituting the values of Δy_0, $\Delta^2 y_0$, $\Delta^3 y_0$ from the difference table, we solve this quadratic for p. Then the corresponding values of x are given by $x = x_0 + p_h$ at which y is maximum or minimum.

EXAMPLE 8.8

From the table below, for what value of x, y is minimum? Also find this value of y.

x:	3	4	5	6	7	8
y:	0.205	0.240	0.259	0.262	0.250	0.224

Solution:

The difference table is

x	y	Δ	Δ^2	Δ^3
3	0.205			
		0.035		
4	0.240		– 0.016	
		0.019		0.000
5	0.259		– 0.016	
		0.003		0.001
6	0.262		– 0.015	
		– 0.012		0.001
7	0.250		– 0.014	
		– 0.026		
8	0.224			

Taking $x_0 = 3$, we have $y_0 = 0.205$, $\Delta y_0 = 0.035$, $\Delta^2 y_0 = -0.016$ and $\Delta^3 y_0 = 0$.

$\therefore$ Newton's forward difference formula gives

$$y = 0.205 + p(0.035) + \frac{p(p-1)}{2}(-0.016) \qquad (i)$$

Differentiating it w.r.t. p, we have

$$\frac{dy}{dp} = 0.035 + \frac{29-1}{2}(-0.016)$$

For y to be minimum, $dy/dp = 0$

$\therefore$ $0.035 - 0.008(2p - 1) = 0$

which gives $p = 2.6875$

$\therefore$ $x = x_0 + ph = 3 + 2.6875 \times 1 = 5.6875.$

Hence y is minimum when $x = 5.6875$.

Putting $p = 2.6875$ in (i), the minimum value of y

$$= 0.205 + 2.6875 \times 0.035 + \frac{1}{2}(2.6875 \times 1.6875)(-0.016) = 0.2628.$$

EXAMPLE 8.9

Find the maximum and minimum value of y from the following data:

x:	– 2	– 1	0	1	2	3	4
y:	2	– 0.25	0	– 0.25	2	15.75	56

Solution:

The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
– 2	2					
		– 2.25				
– 1	– 0.25		2.5			
		0.25		– 3		
0	0		– 0.5		6	
		– 0.25		3		0
1	– 0.25		2.5		6	

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
		2.25		9		0
2	2		11.5		6	
		13.75		15		
3	15.75		26.5			
		40.25				
4	56					

Taking $x_0 = 0$, we have $y_0 = 0$, $\Delta y_0 = -0.25$, $\Delta^2 y_0 = 2.5$, $\Delta^3 y_0 = 9$, $\Delta^4 y_0 = 6$.

Newton's forward difference formula for the first derivative gives

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 - \frac{(2p-1)}{2!}\Delta^2 y_0 + \frac{3p^2 - 6p + 2}{3!}\Delta^3 y_0 - \frac{4p^3 - 18p^2 + 22p - 6}{4!}\Delta^4 y_0 - \cdots\right]$$

$$= \frac{1}{1} - 0.25 + \frac{2x-1}{2}(2.5) + \frac{1}{6}(3x^2 - 6x + 2)(9) + \frac{1}{24}(4x^3 - 18x^2 + 22x - 6)(6)$$

$$= \frac{1}{2}[-0.25 + 2.5x - 1.25 + 4.5x^2 - 9x + 3 + x^3 - 4.5x^2 + 5.5x - 1.5 = x^3 - x$$

For y to be maximum or minimum, $\dfrac{dy}{dx} = 0$ i.e., $x^3 - x = 0$

i.e., $x = 0, 1, -1$

Now $$\frac{d^2y}{dx^2} = 3x^2 - 1 = -\text{ve for } x = 0$$

$$= +\text{ve for } x = 1$$

$$= +\text{ve for } x = -1.$$

Since $$y = y_0 + x\Delta y_0 + \frac{x(x-1)}{2!}\Delta^2 y_0 + \cdots, y(0) = 0$$

Thus y is maximum for $x = 0$, and maximum value $= y(0) = 0$.

Also y is minimum for $x = 1$ and minimum value $= y(0) = -0.25$

Exercises 8.1

1. Find y' (0) and y (0) from the following table:

x: 0		1	2	3	4	5
y: 4		8	15	7	6	2

2. Find the first, second and third derivatives of $f(x)$ at $x = 1.5$ if

x:	1.5	2.0	2.5	3.0	3.5	4.0
$f(x)$:	3.375	7.000	13.625	24.000	38.875	59.000

3. Find the first and second derivatives of the function tabulated below, at the point $x = 1.1$:

x:	1.0	1.2	1.4	1.6	1.8	2.0
$f(x)$:	0	0.128	0.544	1.296	2.432	4.00

4. Given the following table of values of x and y

x:	1.00	1.05	1.10	1.15	1.20	1.25	1.30
y:	1.000	1.025	1.049	1.072	1.095	1.118	1.140

find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at (*a*) $x = 1.05$. (*b*) $x = 1.25$ (*c*) $x = 1.15$.

5. For the following values of x and y, find the first derivative at $x = 4$.

x:	1	2	4	8	10
y:	0	1	5	21	27

6. Find the derivative of $f(x)$ at $x = 0.4$ from the following table:

x:	0.1	0.2	0.3	0.4
$f(x)$:	1.10517	1.22140	1.34986	1.49182

7. From the following table, find the values of dy/dx and d^2y/dx^2 at $x = 2.03$.

x:	1.96	1.98	2.00	2.02	2.04
y:	0.7825	0.7739	0.7651 0.	7563	0.7473

8. Given sin 0° = 0.000, sin 10° = 0.1736, sin 20° = 0.3420, sin 30° = 0.5000, sin 40° = 0.6428,

(*a*) find the value of sin 23°,

(*b*) find the numerical value of cos x at $x = 10°$

(*c*) find the numerical value of d^2y/dx^2 at $x = 20°$ for $y = \sin x$.

9. The population of a certain town is given below. Find the rate of growth of the population in 1961 from the following table

Year:	1931	1941	1951	1961	1971
Population: (*in thousands*)	40.62	60.80	71.95	103.56	132.68

Estimate the population in the years 1976 and 2003. Also find the rate of growth of population in 1991.

10. The following data gives corresponding values of pressure and specific volume of a superheated steam.

v:	2	4	6	8	10
p:	105	42.7	25.3	16.	7 13

Find the rate of change of

(i) pressure with respect to volume when $v = 2$,

(ii) volume with respect to pressure when $p = 105$.

11. The table below reveals the velocity v of a body during the specified time t find its acceleration at $t = 1.1$?

t:	1.0	1.1	1.2	1.3	1.4
v:	43.1	47.7	52.1	56.4	60.8

12. The following table gives the velocity v of a particle at time t. Find its acceleration at $t = 2$.

t:	0	2	4	6	8	10	12
v:	4	6	16	34	60	94	131

13. A rod is rotating in a plane. The following table gives the angle θ (radians) through which the rod has turned for various values of the time t second.

t:	0	0.2	0.4	0.6	0.8	1.0	1.2
θ:	0	0.12	0.49	1.12	2.02	3.20	4.67

Calculate the angular velocity and the angular acceleration of the rod, when $t = 0.6$ second.

14. Find dy/dx at $x = 1$ from the following table by constructing a central difference table:

x:	0.7	0.8	0.9	1.0	1.1	1.2	1.3
y:	0.644218	0.717356 0	0.783327	0.841471	0.891207	0.932039	0.963558

15. Find the value of $f'(x)$ at $x = 0.04$ from the following table using Bessel's formula.

x:	0.01	0.02	0.03	0.04	0.05	0.06
$f(x)$:	0.1023	0.1047	0.1071	0.1096	0.1122	0.1148

16. If $y = f(x)$ and y_n denotes $f(x_0 + nh)$, prove that, if powers of h above h^6 are neglected.

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{3}{4h}\left[(y_1 - y_{-1}) - \frac{1}{5}(y_2 - y_{-2}) + \frac{1}{45}(y_3 - y_{-3})\right].$$

[***HINT*:** Differentiate Stiling's formula w.r.t. x, and put $x = 0$]

17. Find the value of $f'(8)$ from the table given below:

x:	6	7	9	12
$f(x)$:	1.556	1.690	1.908	2.158

18. Given the following pairs of values of x and y:

x:	1	2	4	8	10
y:	0	1	5	21	27

Determine numerically dy/dx at $x = 4$.

19. Find $f'(6)$ from the following data:

x:	0	2	3	4	7	8
$f(x)$:	4	26	58	112	466	922

20. Find the maximum and minimum value of y from the following table:

x:	0	1	2	3	4	5
y:	0	0.25	0	2.25	16	56.25

21. Using the following data, find x for which y is minimum and find this value of y.

x:	0.60	0.65	0.70	0.75
y:	0.6221	0.6155	0.6138	0.6170

22. Find the value of x for which $f(x)$ is maximum, using the table

x:	9	10	11	12	13	14
$f(x)$:	1330	1340	1320	1250	1120	930

Also find the maximum value of $f(x)$.

8.4 Numerical Integration

The process of evaluating a definite integral from a set of tabulated values of the integrand $f(x)$ is called *numerical integration*. This process when applied to a function of a single variable, is known as *quadrature*.