Hence $x = 1, y = 2, z = -1$.

Gauss elimination method. In this method, the unknowns are eliminated successively and the system is reduced to an upper triangular system from which the unknowns are found by back substitution. The method is quite general and is well-adapted for computer operations. Here we shall explain it by considering a system of three equations for the sake of clarity.

Consider the equations

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \qquad (1)$$

Step I. To eliminate x from the second and third equations.

Assuming $a_1 \neq 0$, we eliminate x from the second equation by subtracting (a_2/a_1) times the first equation from the second equation. Similarly we eliminate x from the third equation by eliminating (a_3/a_1) times the first equation from the third equation. We thus, get the new system

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$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ b_2'y + c_2'z &= d_2' \\ b_3'y + c_3'z &= d_3' \end{aligned}\right\} \qquad (2)$$

Here the first equation is called the *pivotal equation* and a_1 is called the *first pivot*.

Step II. To eliminate y from third equation in (2).

Assuming $b_2' \neq 0$, we eliminate y from the third equation of (2), by subtracting $(b_3' b_2')$ multiplied by times the second equation from the third equation. We thus, get the new system

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ b_2'y + c_2'z &= d_2' \\ c_3''z &= d_3'' \end{aligned}\right\} \qquad (3)$$

Here the second equation is the *pivotal equation* and b'_2 is the *new pivot*.

Step III. To evaluate the unknowns.

The values of x, y, z are found from the reduced system (3) by back substitution.

NOTE **Obs. 1.** *On writing the given equations as*

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \text{ i.e., } AX = D,$$

this *method consists in* transforming the coefficient matrix A to the upper triangular matrix *by elementary row transformations only.*

2. *Clearly the method will fail if any one of the pivots a_1, b'_2, or c''_3 becomes zero. In such cases, we rewrite the equations in a different order so that the pivots are non-zero.*

3. *Partial and complete pivoting. In the first step, the numerically largest coefficient of x is chosen from all the equations and brought as the first pivot by interchanging the first equation with the equation having the largest coefficient of x. In the second step, the numerically largest coefficient of y is chosen from the remaining equations (leaving the first equation) and brought as the second pivot by interchanging the second equation with the equation having the largest coefficient of y. This process is continued until we arrive at the equation with the single variable. This modified procedure is called partial pivoting.*

If we are not keen about the elimination of x, y, z in a specified order, then we can choose at each stage the numerically largest coefficient of the entire matrix of coefficients. This requires not only an interchange of equations but also an interchange of the position of the variables. This method of elimination is called *complete pivoting*. It is more complicated and does not appreciably improve the accuracy.

EXAMPLE 3.18

Apply Gauss elimination method to solve the equations $x + 4y - z = -5$; $x + y - 6z = -12$; $3x - y - z = 4$.

Solution:

		Check sum	
We have	$x + 4y - z = -5$	-1	(*i*)
	$x + y - 6z = -12$	-16	(*ii*)
	$3x - y - z = 4$	5	(*iii*)

Step I. To eliminate x, operate (*ii*) – (*i*) and (*iii*) – 3(*i*):

	Check sum	
$-3y - 5z = -7$	-15	(*iv*)
$-13y + 2z = 19$	8	(*v*)

Step II. To eliminate y, operate (*v*) $-\frac{13}{3}$ (*iv*):

	Check sum	
$\frac{71}{3}z = \frac{148}{3}$	73	(*vi*)

Step III. By back-substitution, we get

From (*vi*): $z = \frac{148}{71} = 2.0845$

From (*iv*): $y = \frac{7}{3} - \frac{5}{3}\left(\frac{148}{71}\right) = -\frac{81}{71} = -1.1408$

From (*i*): $x = -5 - 4\left(-\frac{81}{71}\right) + \left(\frac{148}{71}\right) = \frac{117}{71} = 1.6479$

Hence, $x = 1.6479$, $y = -1.1408$, $z = 2.0845$.

Note. A useful check is provided by noting the sum of the coefficients and terms on the right, operating on those numbers as on the equations and checking that the derived equations have the correct sum.

Otherwise: We have $\begin{bmatrix} 1 & 4 & -1 \\ 1 & 1 & -6 \\ 3 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -12 \\ 4 \end{bmatrix}$

Operate $R_2 - R_1$ and $R_3 - 3R_1$, $\begin{bmatrix} 1 & 4 & -1 \\ 0 & -3 & -5 \\ 0 & -13 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -7 \\ 19 \end{bmatrix}$

Operate $R_3 - \frac{13}{3}R_2$, $\begin{bmatrix} 1 & 4 & -1 \\ 0 & -3 & -5 \\ 0 & 0 & 71/3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -7 \\ 148/3 \end{bmatrix}$

Thus, we have $z = 148/71 = 2.0845$,

$3y = 7 - 5z = 7 - 10.4225 = -3.4225$, *i.e.*, $y = -1.1408$

and $x = -5 - 4y + z = -5 + 4(1.1408) + 2.0845 = 1.6479$

Hence $x = 1.6479$, $y = -1.1408$, $z = 2.0845$.

EXAMPLE 3.19

Solve $10x - 7y + 3z + 5u = 6$, $-6x + 8y - z - 4u = 5$, $3x + y + 4y + 11u = 2$, $5x - 9y - 2z + 4u = 7$ by the Gauss elimination method.

Solution:

		Check sum	
We have	$10x - 7y + 3z + 5u = 6$	17	(*i*)
	$-6x + 8y - z - 4u = 5$	2	(*ii*)
	$3x + y + 4z + 11u = 2$	21	(*iii*)
	$5x - 9y - 2z + 4u = 7$	5	(*iv*)

Step I. To eliminate x, operate

$$\left[(ii) - \left(\frac{-6}{10}\right)(i),\right], \left[(iii) - \frac{3}{10}(i)\right], \left[(iv) - \frac{5}{10}(i)\right]:$$

	Check sum	
$3.8y + 0.8z - u = 8.6$	12.2	(*v*)
$3.1y + 3.1z + 9.5u = 0.2$	15.9	(*vi*)
$-5.5y - 3.5z + 1.5u = 4$	-3.5	(*vii*)

Step II. To eliminate y, operate $\left[(vi) - \frac{3.1}{3.8}(v)\right], \left[(vii) - \left(-\frac{5.5}{3.8}\right)(v)\right]:$

$2.4473684z + 10.315789u = -6.8157895$ (*viii*)

$-2.3421053z + 0.0526315u = 16.447368$ (*ix*)

Step III. To eliminate z, operate $\left[(ix) - \left(\frac{-2.3421053}{2.4473684}\right)(viii)\right]:$

$9.9249319u = 9.9245977$

Step IV. By back-substitution, we get

$$u = 1, z = -7, y = 4 \text{ and } x = 5.$$

EXAMPLE 3.20

Using the Gauss elimination method, solve the equations: $x + 2y + 3z - u = 10$, $2x + 3y - 3z - u = 1$, $2x - y + 2z + 3u = 7$, $3x + 2y - 4z + 3u = 2$.

Solution:

We have $\begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 3 & -3 & -1 \\ 2 & -1 & 2 & 3 \\ 3 & 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 10 \\ 1 \\ 7 \\ 2 \end{bmatrix}$

Operate $R_2 - 2R_1$, $R3 - 2R_1$, $R_4 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -1 & -9 & 1 \\ 0 & -5 & -4 & 5 \\ 0 & -4 & -13 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} -10 \\ -19 \\ -13 \\ -28 \end{bmatrix}$$

Operate $R_3 - 5R_2$, $R_4 - 4R_2$ $\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -1 & -9 & 1 \\ 0 & 0 & 41 & 0 \\ 0 & 0 & 23 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 10 \\ -19 \\ 82 \\ 48 \end{bmatrix}$

Thus, we have $41z = 82$, *i.e.*, $z = 2$.

$$23z + 2u = 48, \text{ i.e., } 46 + 2u = 48, \qquad \therefore u = 1$$

$$-y - 9z + u = -19, \text{ i.e., } -y - 18 + 1 = -19, \qquad \therefore y = 2$$

$$x + 2y + 3z - u = 10, \text{ i.e., } x + 4 + 6 - 1 = 10, \qquad \therefore x = 1$$

Hence $x = 1, y = 2, z = 2, u = 1$.

Gauss-Jordan method. This is a modification of the Gauss elimination method. In this method, elimination of unknowns is performed not in the equations below but in the equations above also, ultimately reducing the system to a diagonal matrix form, *i.e., each equation involving only one unknown. From these equations, the unknowns x, y, z can be obtained readily.*

Thus in this method, the labor of back-substitution for finding the unknowns is saved at the cost of additional calculations.