CHAPTER 6

FINITE DIFFERENCES

Chapter Objectives

- Introduction
- Finite differences
- Differences of a polynomial
- Factorial notation
- Reciprocal factorial function
- Inverse operator of Δ
- Effect of an error on a difference table
- Other difference operators
- Relations between the operators
- To find one or more missing terms
- Application to summation of series
- Objective type of questions

6.1 Introduction

The calculus of finite differences deals with the changes that take place in the value of the function (dependent variable), *due to finite changes in the independent variable*. Through this, we also study the relations that exist between the values assumed by the function, whenever the independent variable changes by finite jumps whether equal or unequal. On the other hand, in infinitesimal calculus, we study those changes of the function which occur when the independent variable changes continuously in a given interval. In this chapter, we shall study the variations in the function when the independent variable changes by equal intervals.

6.2 Finite Differences

Suppose that the function $y = f(x)$ is tabulated for the equally spaced values $x = x_0, x_0 + h, x_0 + 2h, \ldots, x_0 + nh$ giving $y = y_0, y_1, y_2, \cdots, y_n$. To determine the values of $f(x)$ or $f'(x)$ for some intermediate values of x, the following three types of differences are found useful:

Forward differences. The differences $y_1 - y_0, y_2 - y_1, \cdots, y_n - y_{n-1}$ when denoted by $\Delta y_0, \Delta y_1, \cdots, \Delta y_{n-1}$ respectively are called the *first forward differences* where Δ is the *forward difference operator*. Thus the first forward differences are $\Delta y_r = y_{r+1} - y_r$.

Similarly these second forward differences are defined by $\Delta^2 y_r = \Delta y_{r+1} - \Delta y_r$.

In general, $\Delta^p y_r = \Delta^{p-1} y_{r+1} - \Delta^{p-1} y_r$ defines the pth forward differences. These differences are systematically set out in Table 6.1.

In a difference table, x is called the *argument* and y the *function* or the *entry*. y_0, the first entry, is called the *leading term* and $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0$ etc. are called the *leading differences*.

TABLE 6.1 Forward Difference Table

Value of x	Value of y	1st diff.	2nd diff.	3rd diff.	4th diff.	5th diff.
x_0	y_0					
		Δy_0				
$x_0 + h$	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
$x_0 + 2h$	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
$x_0 + 3h$	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		
$x_0 + 4h$	y_4		$\Delta^2 y_3$			
		Δy_4				
$x_0 + 5h$	y_5					

NOTE

Obs. 1. *Any higher order forward difference can be expressed in terms of the entries.*

We have $D^2y_0 = Dy_1 - Dy_0 = (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0$

$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0 = (y_3 - 2y_2 + y_1) - (y_2 - 2y_1 + y_0)$$
$$= y_3 - 3y_2 + 3y_1 - y_0$$
$$\Delta^4 y0 = \Delta^3 y_1 - \Delta^3 y_0$$
$$= (y_4 - 3y_3 + 3y_2 - y_1) - (y_3 - 3y_2 + 3y_1 - y_0)$$
$$= y_4 - 4y_3 + 6y_2 - 4y_1 + y_0$$

The coefficients occurring on the right-hand side being the binomial coefficients, we have in general,

$$\Delta^n y_0 = y_n - {}^nc_1\, y_{n-1} + {}^nc_2\, y_{n-2} - \cdots + (-1)^n\, y_0.$$

Obs.2.*The operator* Δ *obeys the distributive, commutative, and index laws*

i.e., (*i*) $\Delta[f(x) \pm \phi(x)] = \Delta f(x) \pm \Delta\phi(x)$.

(*ii*) $\Delta[cf(x)] = c\Delta f(x)$,$c$ being a constant.

(*iii*) $\Delta m\ \Delta n\ f(x) = \Delta m + n\ f(x)$, m and n being positive integers. In view of (*i*) and (*ii*), Δ is a *linear operator*.

But $\Delta[f(x).\phi(x)] \neq f(x).\Delta\phi\ (x)$.

Backward differences. The differences $y_1 - y_0$, $y_2 - y_1, \cdots, y_n - y_{n-1}$ when denoted by $\nabla y_1, \nabla y_2, \cdots, \nabla y_n$ respectively, are called the *first backward differences* where Δ

TABLE 6.2 Backward Difference Table

Value of x	Value of y	1st diff.	2nd diff.	3rd diff.	4th diff.	5th diff.
x_0	y_0					
		∇y_1				
$x_0 + h$	y_1		$\nabla^2 y_2$			
		∇y_2		$\nabla^3 y_3$		
$x_0 + 2h$	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$	
		∇y_3		$\nabla^3 y_4$		$\nabla^5 y_5$
$x_0 + 3h$	y_3		$\nabla^2 y_4$		$\nabla^4 y_5$	
		Δy_4		$\nabla^3 y_5$		
$x_0 + 4h$	y_4		$\nabla^2 y_5$			
		∇y_5				
$x_0 + 5h$	y_5					

is the *backward difference operator*. Similarly we define higher order backward differences. Thus we have $\nabla y_r = y_r - y_{r-1}$, $\nabla^2 y_r = \nabla y_r - \nabla y_{r-1}$, $\nabla^3 y_r = \nabla^2 y_r - \nabla^2{}_{r-1}$, etc.

These differences are exhibited in the Table 6.2.

Central differences. Sometimes it is convenient to employ another system of differences known as *central differences*. In this system, the *central difference operator* δ is defined by the relations:

$$y_1 - y_0 = \delta y_{1/2},\ y_2 - y_1 = \delta y_{3/2},\ \cdots,\ y_n - y_{n-1} = \delta y_{n-1/2}$$

Similarly, higher order central differences are defined as

$$\delta y_{3/2} - \delta y_{1/2} = \delta^2 y_1,\ \delta y_{5/2} - \delta y_{3/2} = \delta^2 y_2,\ \cdots, \delta^2 y_2 - \delta^2 y_1 = \delta^3 y_{3/2} \text{ and so on.}$$

These differences are shown in Table 6.3.

TABLE 6.3 Central Difference Table

Value of x	Value of y	1st diff.	2nd diff.	3rd diff.	4th diff.	5th diff.
x_0	y_0					
		$\delta y_{1/2}$				
$x_0 + h$	y_1		$\delta^2 y_1$			
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$		
$x_0 + 2h$	y_2		$\delta^2 y_2$		$\delta^4 y_2$	
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$		$\delta^5 y_{5/2}$
$x_0 + 3h$	y_3		$\delta^2 y_3$		$\delta^4 y_3$	
		$\delta y_{7/2}$				
$x_0 + 4h$	y_4		$\delta^2 y_4$	$\delta^3 y_{7/2}$		
		$\delta y_{9/2}$				
$x_0 + 5h$	y_5					

We see from this table that the central differences on the same horizontal line have the same suffix. Also the differences of odd order are known only for half values of the suffix and those of even order for only integral values of the suffix.

It is often required to find the mean of adjacent values in the same column of differences. We denote this mean by μ.

Thus $\mu\delta y_1 = \frac{1}{2}(\delta y_{1/2} + \delta y_{3/2}), \mu\delta^2 y_{3/2} = \frac{1}{2}(\delta^2 y_1 + \delta^2 y_2),$ etc.

NOTE

Obs. *The reader should note that it is only the notation which changes and not the differences. e.g.*

$$y_1 - y_0 = \nabla y_0 = \Delta y_1 = \delta y_{1/2}.$$

Of all the formulae, those involving central differences are most useful in practice as the coefficients in such formulae decrease much more rapidly.

EXAMPLE 6.1

Evaluate (*i*) $\Delta \tan^{-1}x$ (*ii*) $\Delta(e^x \log 2x)$ (*iii*) $\Delta(x^2/\cos 2x)$ (*iv*) $\Delta({}^nC_{r+1})$.

Solution:

(*i*)
$$\Delta\tan^{-1}x = \tan^{-1}(x+h) - \tan^{-1}x$$
$$= \tan^{-1}\left\{\frac{x+h-x}{1+(x+h)x}\right\} = \tan^{-1}\left\{\frac{h}{1+hx+x^2}\right\}$$

(*ii*)
$$\Delta(e^x \log 2x) = e^{x+h}\log 2\ (x+h) - e^x \log 2x$$
$$= e^{x+h}\log 2\ (x+h) - e^{x+h}\log 2x + e^{x+h}\log 2x - e^x \log 2x$$
$$= e^{x+h}\log\frac{x+h}{x}(e^{x+h} - e^x)\log 2x$$
$$= e^x\left[e^h \log\left(1+\frac{h}{x}\right) + \left(e^h - 1\right)\log 2x\right]$$

(*iii*)
$$\Delta\left(\frac{x^2}{\cos 2x}\right) = \frac{(x+h)^2}{\cos 2(x+h)} - \frac{x^2}{\cos 2x}$$
$$= \frac{(x+h)^2\cos 2x - x^2\cos 2(x+h)}{\cos 2(x+h)\cos 2x}$$
$$= \frac{\left[(x+h)^2 - x^2\right]\cos 2x + x^2\left[\cos 2x - \cos 2(x+h)\right]}{\cos 2(x+h)\cos 2x}$$
$$= \frac{(2hx+h^2)\cos 2x + 2x^2\ \sin(h)\sin(2x+h)}{\cos 2(x+h)\cos 2x}$$

(*iv*)
$$\Delta({}^nC_{r+1}) = {}^{n+1}C_{r+1} - {}^nC_{r+1}$$
$$= \frac{(n+1)!}{(r+1)!(n-r)!} - \frac{(n)!}{(r+1)!(n-r-1)!}$$
$$= \frac{(n)!}{(r+1)!(n-r-1)!}\left(\frac{n+1}{n-r} - 1\right)$$

$$=\frac{(n)!}{(r+1)!(n-r-1)!}\frac{(r+1)}{(n-r)}=\frac{n!}{r!(n-r)!}={}^{n}c_{r}$$

EXAMPLE 6.2

Evaluate $(i)\ \Delta^2\left(\frac{5x+12}{x^2+5x+16}\right)$ $\quad (ii)\ \Delta^2\cos 2x$

$(iii)\ \Delta^2(ab^{cx})$ $\quad (iv)\Delta^n(e^x)$

Interval of differencing being unity

Solution:

$$(i){:}\ \Delta^2\left(\frac{5x+12}{x2+5x+16}\right)=\Delta^2\left\{\frac{5x+12}{(x+2)(x+3)}\right\}=\Delta^2\left\{\frac{2}{x+2}+\frac{3}{x+3}\right\}$$

$$=\Delta\left\{\Delta\left(\frac{2}{x+2}\right)+\Delta\left(\frac{3}{x+3}\right)\right\}$$

$$=\Delta\left\{2\left(\frac{1}{x+3}-\frac{1}{x+2}\right)+3\left(\frac{1}{x+4}-\frac{1}{x+3}\right)\right\}$$

$$=-2\Delta\left\{\frac{1}{(x+2)(x+3)}\right\}-3\Delta\left\{\frac{1}{(x+3)(x+4)}\right\}$$

$$=-2\Delta\left\{\frac{1}{(x+3)(x+4)}-\frac{1}{(x+2)(x+3)}\right\}$$

$$-3\left\{\frac{1}{(x+4)(x+5)}-\frac{1}{(x+3)(x+4)}\right\}$$

$$=\frac{4}{(x+2)(x+3)(x+4)}+\frac{6}{(x+3)(x+4)(x+5)}$$

$$=\frac{2(5x+16)}{(x+2)(x+3)(x+4)(x+5)}$$

$$(ii)\ \Delta^2\cos 2x=\Delta\{\cos 2(x+h)-\cos 2x\}$$

$$=\Delta\cos 2(x+h)-\Delta\cos 2x$$

$$=[\cos 2(x+2h)-\cos 2(x+h)]-[\cos 2(x+h)-\cos 2x]$$

$$=-2\sin(2x+3h)\sin h+2\sin(2x+h)\sin h$$

$$=-2\sin h[\sin(2x+3h)-\sin(2x+h)]$$

$$= -2\sinh[2\cos(2x+2h)\sinh] = -4\sin^2 h\cos(2x+2h).$$

(*iii*) $\Delta(ab^{cx}) = a\Delta(b^{cx}) = a[b^{c(x+1)} - b^{cx}] = ab^{cx}(b^c - 1)$

$$\Delta^2(ab^{cx}) = \Delta[\Delta(ab^{cx})] = a(b^c - 1)\Delta(b^{cx})$$

$$= a(b^c - 1)\,(b^{c(x+1)} - b^{cx}) = a[b^c - 1]^2 b^{cx}$$

(*iv*) $\Delta e^x = e^{x+1} - e^x = (e-1)e^x$

$$\Delta^2 e^x = \Delta(\Delta e^x) = \Delta[(e-1)e^x]$$

$$= (e-1)\Delta e^x = (e-1)(e-1)e^x = (e-1)^2 e^x$$

Similarly $\Delta^3 e^x = (e-1)^3 e^x, \Delta^4 e^x = (e-1)^4 e^x, \quad \cdots$

and $\Delta^n e^x = (e-1)^n e^x.$

EXAMPLE 6.3

If $y = a(3)^x + b(-2)^x$ and $h = 1$, prove that $(\Delta^2 + \Delta - 6)y = 0$.

Solution:

We have $y = a(3)^x + b(-2)^x$

$\therefore \quad \Delta y = [a(3)^{x+1} + b(-2)^{x+1}] - [a(3)^x + b(-2)^x]$

$$= 2a(3)^x - 3b(-2)^x$$

and $\quad \Delta^2 y = [2a(3)^{x+1} - 3b(-2)^{x+1}] - [2a(3)^x - 3b(-2)^x]$

$$= 4a(3)^x + 9b(-2)^x$$

Hence$(\Delta^2 + \Delta - 6)y = [4a(3)^x + 9b(-2)^x] + (2a(3)^x - 3b(-2)^x]$
$- 6[a(3)^x + b(-2)^x] = 0$

EXAMPLE 6.4

Find the missing y_x values from the first differences provided:

y_x	0	—	—	—	—	—
Δy_x	0	1	2	4	7	11

Solution:

Let the missing values be y_1, y_2, y_3, y_4, y_5. Then we have

y_x	0	y_1	y_2	y_3	y_4	y_5
Δy_x	0	1	2	4	7	11

$\therefore \qquad y_1 - 0 = 1,\ y_2 - y_1 = 2,\ y_3 - y_2 = 4,\ y_4 - y_3 = 7,\ y_5 - y_4 = 11$

i.e., $\qquad y_1 = 1,\ y_2 = 2 + y_1 = 3,\ y_3 = 4 + y_2 = 7,\ y_4 = 7 + y_3 = 14,$

$y_5 = 11 + y_4 = 25.$

6.3 Differences of A Polynomial

The nth differences of a polynomial of the nth degree are constant and all higher order differences are zero.

Let the polynomial of the nth degree in x, be

$$f(x) = ax^n + bx^{n-1} + cx^{n-2} + \cdots + kx + l$$

$$\therefore \quad \Delta f(x) = f(x+h) - f(x)$$

$$= a[(x+h)^n - x^n] + b[(x+h)^{n-1} - x^{n-1}] + \cdots + kh$$

$$= anhx^{n-1} + b'x^{n-2} + c'x^{n-3} + \cdots + k'x + l' \qquad (1)$$

where $b', c', \cdots, l'$ are the new constant co–efficients.

Thus the first differences of a polynomial of the nth degree is a polynomial of degree $(n - 1)$.

Similarly

$$\Delta^2 f(x) = \Delta[f(x+h) - f(x)] = \Delta f(x+h) - \Delta f(x)$$

$$= anh[(x+h)^{n-1} - x^{n-1}] + b'[(x+h)^{n-2} - x^{n-2}] + \cdots + k'h$$

$$= an(n-1)h^2x^{n-2} + b''x^{n-3} + c''x^{n-4} + \cdots + k'', \qquad \text{by}(1)$$

$\therefore$ The second differences represent a polynomial of degree $(n - 2)$

Continuing this process, for the nth differences we get a polynomial of degree zero *i.e.*

$$\Delta^n f(x) = an(n-1)(n-2)\ldots 1.h^n = an\,!h^n \qquad (2)$$

which is a constant. Hence the $(n + 1)$th and higher differences of a polynomial of nth degree

will be zero.

NOTE

Obs. *The converse of this theorem is also true, i.e., if the nth differences of a function tabulated at equally spaced intervals are constant, the function is a polynomial of degree n. This* *fact is important in numeric alanalysis as it enables us to approximate a function by a polynomial of nth degree, if it s nth order differences become nearly constant.*

EXAMPLE 6.5

Evaluate $\Delta^{10}[(1-ax)(1-bx^2)(1-cx^3)(1-cx^3)(1-dx^4)]$.

Solution:

$$\Delta^{10}[(1-ax)(1-bx^2)(1-cx^3)(1-dx^4)] = \Delta^{10}[abcdx^{10} + (\ )x^9 + (\ )x^8 + \ldots + 1]$$

$$= abcd\ \Delta^{10}(x^{10}) \qquad [\therefore\ \Delta^{10}(x^n) = 0 \text{ for } n<10]$$

$$= abcd(10!) \qquad [\text{by}(2)\text{above}]$$

Exercises 6.1

1. Write forward difference table if

x:	10	20	30	40
y:	1.1	2.0	4.4	7.9

2. Construct the table of differences for the data below:

x:	0	1	2	3	4
$f(x)$:	1.0	1.5	2.2	3.1	4.6

Evaluate $\Delta^3 f(2)$.

3. If $u_0 = 3$, $u_1 = 12$, $u_2 = 81$, $u_3 = 2000$, $u_4 = 100$, calculate $\Delta^4 u_0$.

4. Show that $\Delta^3 y_i = y_{i+3} - 3y_{i+2} + 3y_{i+1} - yi$.

5. If $y = x^3 + x^2 - 2x + 1$, evaluate the values of y for $x = 0,1, 2, 3, 4, 5$ from the difference table. Find the value of y at $x = 6$ by extending the table and verify that same value is obtained by substitution.

6. Form a table of differences for the function $f(x) = x^3 + 5x - 7$ for $x = -1, 0, 1, 2, 3, 4, 5$.

Continue the table to obtain $f(6)$.

7. Extend the following table to two more terms on either side by constructing the difference table.

x:	–0.2	0.0	0.2	0.4	0.6	0.8	1.0
y:	2.6	3.0	3.4	4.28	7.08	14.2	29.0.

8. Show that

(i) $\Delta\left[\dfrac{1}{f(x)}\right] = \dfrac{-\Delta f(x)}{f(x)f(x+1)}$ (ii) $\Delta \log f(x) = \log\left\{1 + \dfrac{\Delta f(x)}{f(x)}\right\}$

9. Evaluate (taking interval of differencing as unity)

(i) $\Delta(x + \cos x)$ (ii) $\Delta \tan^{-1}\left(\dfrac{n-1}{n}\right)$

(iii) $\Delta\ (e^{3x}\log 2x)$ (iv) $\Delta(2^x/x!)$

10. Evaluate:

(i) $\Delta^2 \cos 3x$ (ii) $\Delta^2\left(\dfrac{1}{x^2+5x+6}\right)$

(iii) $\Delta n(e^{2x+3})$ (iv) $\Delta^n\left(\dfrac{1}{x}\right)$

(v) $\Delta^n \sin\ (ax + b)$

11. If $f(x) = e^{ax+b}$, show that its leading differences form a geometric progression

12. Prove that

(i) $y_3 = y_2 + \Delta y_1 + \Delta^2 y_0 + \Delta^3 y_0$ (ii) $\Delta^2 y_8 = y_8 - 2y_7 + y_6$

(iii) $\delta^2 y_5 = y_6 - 2y_5 + y_4$.

13. Evaluate:

(i) $\Delta^4[(1-x)(1-2x)(1-3x)(1-4x)],(h=1)$.

(ii) $\Delta^{10}[(1-x)(1-2x^2)(1-3x^3)(1-4x^4)]$, if the interval of differencing is 2.

6.3 Factorial Notation

A product of the form $x(x-1)(x-2)\cdots(x-r+1)$ is denoted by $[x]^r$ and is called a **factorial.**

In particular $[x] = x, [x]^2 = x(x-1), [x]^3 = x(x-1)(x-2)$, etc.

In general $[x]^n = x(x-1)(x-2)\ .....(x-n+1)$

If the interval of differencing is h, then $[x]^n = x(x-h)(x-2h)\cdots(x-\overline{n-1}h)$ which is called a **factorial polynomial** or **function** of degree n.

The factorial notation is of special utility in the theory of finite differences. It helps in finding the successive differences of a polynomial directly by simple rule of differentiation.

To show that $\Delta n[x]n = n!$ and $\Delta n + 1[x]n = 0$

We have

$$\Delta[x]^n = [x+h]^n - [x]^n$$

$$= (x+h)(x+h-h)(x+h-2h)\cdots(x+h-\overline{n-1}h) - x(x-h)(x-2h)\cdots(x-\overline{n-1}h) \qquad (i)$$

$$= x(x-h)\cdots(x-\overline{n-2}h)[x+h-(x-nh+h)]$$

$$= nh[x]^{n-1}$$

Similarly $\Delta^2[x]^n = \Delta\{nh[x]^{n-1}\} = nh\Delta[x]^{n-1}$

Replacing n by $n-1$ in (i),

we get $\Delta^2[x]^n = nh.(n-1)h[x]^{n-2} = n(n-1)h^2[x]^{n-2}$

Proceeding in this way, we obtain $\Delta^{n-1}[x]^n = n(n-1)\cdots 2h^{n-1}x$

$$\therefore \quad \Delta^n[x]^n = n(n-1)\cdots 2.h^{n-1}\Delta x$$

$$= n(n-1)\cdots 2.1.h^{n-1}(x+h-x)$$

$$= n!h^n \qquad (ii)$$

Also $\Delta^{n+1}[x]^n = n!h^n - n!h^n = 0$

In particular, when $h = 1$,we have

$$\Delta[x]^n = n[x]^{n-1} \quad \text{and} \quad \Delta^n[x]^n = n! \qquad (iii)$$

Similarly $\Delta^r[ax+b]^n = n(n-1)\cdots(n-r+1)a^r h^r[ax+b]^{n-r}$

Thus we have an important result:

$$\Delta[x]^n = n[x]^{n-1};\ \Delta[ax+b]^n = an[ax+b]^{n-1} \qquad (iv)$$

i.e., the result of differencing $[x]^n$ is analogous to that of differentiating x^n.

NOTE

Obs.1. *As it is easier to find Δx[x]n than Δr xn, xn must always be expressed as a factorial polynomial before finding Δx.*

Obs.2. *Every polynomial of degree n can be expressed as a factorial polynomial of the same degree and vice versa.*

EXAMPLE 6.6

Express $y = 2x^3 - 3x^2 + 3x - 10$ in factorial notation and hence show that $\Delta^3 y = 12$.

Solution:

First method: Let $y = A[x]^3 + B[x]^2 + C[x] + D$.

Using the method of synthetic division (p.29), we divide by x, $x - 1$, $x - 2$, etc. successively. Then

	x^3	x^2	x	
1	2	-3	3	$-10 = D$
	—	2	-1	
2	2	-1	$2 = C$	
	—	4		
3	2	$3 = B$		
	—			
	$2 = A$			

Hence $\qquad y = 2[x]^3 + 3[x]^2 + 2[x] - 10$

$\therefore \qquad \Delta_y = 2 \times 3[x]^2 + 3 \times 2[x] + 2$

$\qquad \Delta^2 y = 6 \times 2[x] + 6$

$\Delta^3 y = 12$, which shows that the third differences of y are constant, as they should be.

NOTE **Obs.** *The coefficient of the highest power of x remains unchanged while transforming a polynomial to factorial notation.*

Second method (*Direct method*):

Let $\qquad y = 2x^3 - 3x^2 + 3x - 10$

$\qquad = 2x(x-1)(x-2) + Bx(x-1) + Cx + D$

Putting $\qquad x = 0, -10 = D.$

Putting $\qquad x = 1, 2 - 3 + 3 - 10 = C + D$

$\therefore \qquad C = -8 - D = -8 + 10 = 2$

Putting $\qquad x = 2, 16 - 12 + 6 - 10 = 2B + 2C + D$

$$\therefore \qquad B = \frac{1}{2}(-2C - D) = \frac{1}{2}(-4 + 10) = 3$$

Hence $\qquad y = 2x(x-1)(x-2) + 3x(x-1) + 2x - 10$

$\qquad = 2[x]^3 + 3[x]^2 + 2[x] - 10$

$\therefore \qquad \Delta_y = 2 \times 3[x]^2 + 3 \times 2[x] + 2, \Delta^2 y = 6 \times 2[x] + 6, \Delta^3 y = 12.$

EXAMPLE 6.7

Express $u = x^4 - 12x^3 + 24x^2 - 30x + 9$ and its successive differences in factorial notation. Hence show that $\Delta^5 u = 0$.

Solution:

Let $u = A[x]^4 + B[x]^3 + C[x]^2 + D[x] + E$.

Using the method of synthetic division, we divide by x, $x - 1$, $x - 2$, $x - 3$ successively.

Then

	x^4	x^3	x^2	x	
1	1	-12	24	-30	$9(=E)$
	0	1	-11	13	
2	1	-11	13	$-17\ (=D)$	
	0	2	-18		
3	1	-9	$-5(=C)$		
	0	3			
	$1(=A)$	$-6(=B)$			

Hence $\quad u = [x]^4 - 6[x]^3 - 5[x]^2 - 17[x] + 9$

$\therefore \quad \Delta u = 4[x]^3 - 18[x]^2 - 10[x] - 17$

$$\Delta^2 u = 12[x]^2 - 36[x] - 10$$

$$\Delta^3 u = 24[x] - 36$$

$$\Delta^4 u = 24 \quad \text{and} \quad \Delta^5 u = 0.$$

EXAMPLE 6.8

If $f(x) = (2x + 1)(2x + 3)(2x + 5)\cdots(2x + 15)$, find the value of $\Delta^4 f(x)$

Solution:

We have $f(x) = 2^8\left(x+\frac{15}{2}\right)\left(x+\frac{13}{2}\right)\cdots\left(x+\frac{3}{2}\right)\left(x+\frac{1}{2}\right)$ [There are 8 factors]

$$= 2^8\left[x+\frac{15}{2}\right]^8$$

$\therefore \quad \Delta f(x) = 2^8 \times 8\left[x+\frac{15}{2}\right]^7; \Delta^2 f(x) = 2^8 \times 8 \times 7\left[x+\frac{15}{2}\right]^6$

$$\Delta^3 f(x) = 2^8 \times 56 \times 6\left[x + \frac{15}{2}\right]^5$$

$$\Delta^3 f(x) = 2^8 \times 336 \times 5\left[x + \frac{15}{2}\right]^4$$

$$= 2^8 \times 1680\left(x + \frac{15}{2}\right)\left(x + \frac{13}{2}\right)\left(x + \frac{11}{2}\right)\left(x + \frac{9}{2}\right)$$

$$= 26880\ (2x + 9)\ (2x + 11)\ (2x + 13)\ (2x + 15)$$

6.5 Reciprocal Factorial Function

The function $\frac{1}{(x+1)(x+2)\cdots(x+n)}$ is denote d by $[x]^{-n}$ and is called a **reciprocal factorial function.**

If the interval of differencing is h, then

$$[x]^{-n} = \frac{1}{(x+h)(x+2h)\cdots(x+nh)}$$

Which is called a reciprocal factorial function of order n

Differences of

$$[x]^{-n} = [x+h]^{-n} - [x]^{-n}$$

$$= \frac{1}{(x+2h)(x+3h)\ldots(x+\overline{n+1}h)} - \frac{1}{(x+h)(x+2h)\ldots(x+nh)}$$

$$= \frac{1}{(x+2h)(x+3h)\ldots(x+\overline{n+1}h)}(x+h-[x+\overline{n+1}h])$$

$$= -nh\ [x]^{-(n+1)} \qquad (i)$$

Similarly $\Delta^2[x]^{-n} = (-1)^2 n(n+1)h^2[x]^{-(n+2)}$

In general, $\Delta^x[x]^{-h} = (-1)^r n(n+1)\ldots(n+r+1)h^r[x]^{-(n+r)}$ (ii)

In particular when $h = 1, \Delta^r[x]^{-n} = (-1)^r n(n+1)\ldots(n+r+1)(x)^{-(n+r)}$

Similarly $\Delta^r[ax+b]^{-n} = (-1)^r n(n+1)\ \ldots\ (n+r-1)a^r h^r[ax+b]^{-(n+r)}$ (iii)

Thus we have an important result:

$$\Delta[x]^{-n} = -n[x]^{-(n+1)}\ ; \Delta[ax\ +\ b]^{-n} = -na[ax+b]^{-(n+1)} \qquad (iv)$$

6.6 Inverse Operator of Δ

The process of finding y_x when Δy_x is given is known as inverse finite difference operation.

i.e., If $\Delta y_x = u_x$ then $y_x = \Delta^{-1} u_x$

The symbol Δ^{-1} or $1/\Delta$ is called the inverse of the operator Δ.

Thus we have two important results

$$\Delta^{-1}[x]^n = \frac{[x]^{n+1}}{n+1}; \Delta^{-1}[ax + b]^n = \frac{[ax+b]^{n+1}}{a(n+1)}$$

$$\Delta^{-1}[x]^{-n} = \frac{[x]^{-(n+1)}}{-n+1}; \Delta^{-1}[ax + b]^{-n} = \frac{[ax+b]^{-(n+1)}}{a(-n+1)}$$

i.e., $\Delta - 1$ is analogous to $D - 1$or integration in calculus.

EXAMPLE 6.9

Obtain the function whose first difference is $2x^3 - 3x^2 + 3x - 10$.

Solution:

Let $f(x)$ be the function whose first difference is given.

We first express $\Delta f(x)$ as a factorial polynomial. Referring to Example 6.6, we have

$$\Delta f(x) = 2[x]^3 + 3[x]^2 + 2[x] - 10$$

$$\therefore \qquad f(x) = \Delta^{-1}\{2[x]^3 + 3[x]^2 + 2[x] - 10\}$$

$$= 2\frac{[x]^4}{4} + 3\frac{[x]^3}{3} + 2\frac{[x]^2}{2} - 10[x]^1$$

$$= \frac{1}{2}(x-1)(x-2)(x-3) + x(x-1)(x-2) + x(x-1) - 10x$$

$$= \frac{1}{2}x^4 - 2x^3 + \frac{7}{2}x^2 - 12x$$

EXAMPLE 6.10

If $y = \dfrac{1}{(3x+1)(3x+4)(3x+7)}$ evaluate $\Delta^2 y$. Also find $\Delta^{-1} y$.

Solution:

(*i*) We have $y = \dfrac{1}{3^3}\dfrac{1}{(x+1/3)(x+4/3)(x+7/3)} = \dfrac{1}{27}\left[x-\dfrac{2}{3}\right]^{3}$

$$\Delta y = \frac{1}{27}(-3)\left[x-2/3\right]^{-4}$$

$$\Delta^2 y = \frac{1}{27}(-3)(-4)\left[x-2/3\right]^{-5}$$

$$= \frac{4}{9} - \frac{1}{\left(x+1/3\right)\left(x+4/3\right)\left(x+7/3\right)\left(x+10/3\right)\left(x+13/3\right)}$$

$$= \frac{108}{(3x+1)(3x+4)(3x+7)(3x+10)(3x+13)}$$

(*ii*)

$$y = \frac{1}{27}\left[x-2/3\right]^{-3}$$

$$\Delta^{-1} y = \frac{1}{27}\frac{\left[x-2/3\right]^{-3}}{-2} = -\frac{1}{54}\frac{1}{\left(x+1/3\right)\left(x+4/3\right)}$$

$$= -\frac{1}{6}\frac{1}{(3x+1)(3x+4)}$$

6.7 Effect of an Error on a Difference Table

Suppose there is an error ε in the entry y_5 of a table. As higher differences are formed, this error spreads out and is considerably magnified. Let us see, how it effects the difference table.

The below table shows that:

(*i*) The error increases with the order of differences.

(*ii*) The coefficients of ε's in any column are the binomial coefficients of $(1-\varepsilon)^n$. Thus the errors in the fourth difference column are ε, -4ε, 6ε, -4ε, ε.

(*iii*) The algebraic sum of the errors in any difference column is zero.

(*iv*) The maximum error in each column, occurs opposite to the entry containing the error, *i.e.*, $.y_5$.

The above facts enable us to detect errors in a difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0	y_0				
		Δy_0			
x_1	y_1		$\Delta^2 y_0$		
		Δy_1		$\Delta^3 y_0$	
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$
		Δy_2		$\Delta^3 y_1$	
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1 + \varepsilon$
		Δy_3		$\Delta^3 y_2 + \varepsilon$	
x_4	y_4		$\Delta_2 y_3 + \varepsilon$		$\Delta^4 y_2 - 4\varepsilon$
		$\Delta y_4 + \varepsilon$		$\Delta^3 y_3 - 3\varepsilon$	
x_5	$y_5 + \varepsilon$		$\Delta^2 y_4 - 2\varepsilon$		$\Delta^4 y_3 + 6\varepsilon$
		$\Delta y_5 - \varepsilon$		$\Delta^3 y_4 + 3\varepsilon$	
x_6	y_6		$\Delta^2 y_5 + \varepsilon$		$\Delta^4 y_4 - 4\varepsilon$
		Δy_6		$\Delta^3 y_5 - \varepsilon$	
x_7	y_7		$\Delta^2 y_6$		$\Delta^4 y_5 + \varepsilon$
		Δy_7		$\Delta^3 y_6$	
x_8	y_8		$\Delta^2 y_7$		
		Δy_8			
x_9	y_9				

EXAMPLE 6.11

One entry in the following table is incorrect and y is a cubic polynomial in x. Use the difference table to locate and correct the error.

x:	0	1	2	3	4	5	6	7
y:	1	–1	1	–1	1	—	—	—

Solution:

The difference table is as under:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	25			
		– 4		
1	21		1	
		– 3		2
2	18		3	
		0		6
3	18		9	
		9		0
4	27		9	
		18		4
5	45		13	
		31		3
6	76		16	
		47		
7	123			

y being a polynomial of the third degree, $\Delta^3 y$ must be constant, *i.e.*, .the same. The sum of the third differences being 15, each entry under $\Delta^3 y$ must be 15/5, *i.e.*, 3. Thus the four entries under $\Delta^3 y$ are in error which can be written as

$$3 + (-1),\ 3 - 3(-1),\ 3 + 3(-1), 3 - (-1)$$

Taking $\varepsilon = -1$, we find that the entry corresponding to $x = 3$ is in error.

$$\therefore \qquad y + \varepsilon = 18$$

Thus the true value of $y = 18 - \varepsilon = 18 - (-1) = 19$.

EXAMPLE 6.12

Assuming that the following values of y belong to a polynomial of degree 4, compute the next three values:

x:	0	1	2	3	4	5	6	7
y:	1	–1	1	–1	1	—	—	—

Solution:

We construct the difference table from the given data.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	$y_0 = 1$				
		-2			
1	$y_1 = -1$		4		
		2		-8	
2	$y_2 = 1$		-4		16
	-2			8	
3	$y_3 = -1$		4		16
	2			$\Delta^3 y_2$	
4	$y_4 = 1$		$\Delta^2 y_3$		16
		Δy_4		$\Delta^3 y_3$	
5	y_5		$\Delta^2 y_4$		16
	Δy_5			$\Delta^3 y_4$	
6	Y_6		$\Delta^2 y_5$		
	Δy_6				
7	y_7				

Since the values of y belong to a polynomial of degree 4, the fourth differences must be constant. But $\Delta^4 y = 16$.

$\therefore$ The other fourth order differences must also be 16. Thus,

$$\Delta^4 y_1 = 16 = \Delta^3 y_2 - \Delta^3 y_1$$

i.e., $$\Delta^3 y_2 = \Delta^3 y_1 + \Delta^4 y_1 = 8 + 16 = 24$$

$$\Delta^2 y_3 = \Delta^2 y_2 + \Delta^3 y_2 = 4 + 24 = 28$$

$$\Delta y_4 = \Delta y_3 + \Delta^2 y_3 = 2 + 28 = 30$$

and $$y_5 = y_4 + \Delta y_4 = 1 + 30 = 31$$

Similarly starting with $\Delta^4 y_2 = 16$,

we get $\Delta^3 y_3 = 40$, $\Delta^2 y_4 = 68$, $\Delta y_5 = 98$, $y_6 = 129$.

Starting with $\Delta^4 y_3 = 16$,

we obtain $\Delta^3 y_4 = 56$, $\Delta^2 y_5 = 124$, $\Delta y_6 = 222$, $y_7 = 351$.

Exercises 6.2

1. Express $x^3 - 2x^2 + x - 1$into factorial polynomial. Hence show that $\Delta^4 f(x) = 0$.
2. Express $3x^4 - 4x^3 + 6x^2 + 2x + 1$ as a factorial polynomial and find differences of all orders.
3. Find the first and second differences of $x^4 - 6x^3 + 11x^2 - 5x + 8$ with $h = 1$. Show that the fourth difference is constant.
4. Obtain the function whose first difference is (*i*) $2x^3 + 3x^2 - 5x + 4$. (*ii*) $x^4 - 5x^3 + 3x + 4$.
5. Show that $\Delta[x(x+1)(x+2)(x+3)] = 4(x+1)(x+2)(x+3)$.
6. Find $\Delta^4 f(x)$when $f(x) = (2x+1)(2x+3)(2x+5)\ldots(2x+19)$.
7. If $y = \dfrac{1}{(4x+1)(4x+5)(4x+9)}$, find $\Delta^2 y$ and $\Delta^{-1} y$.
8. Givenlog100 = 2, log101 = 2.0043, log103 = 2.0128, log104 = 2.0170, find log 102.
9. Find the first term of the series whose second and subsequent terms are 8, 3, 0, – 1,0.

10. Write down the polynomial of lowest degree which satisfies the following set of numbers: 0, 7, 26, 63, 124, 215, 342, 511

6.8 Other Difference Operators

We have already introduced the operators Δ, $\Delta_{,}$ and δ. Besides these, there are the operators E and μ, which we define below:

Shift operator E *is the operation of increasing the argument x by h so that* $E\,f(x) = f(x + h)$, $E^2 f(x) = f(x + 2h)$, $E^3 f(x) = f(x + 3h)$ etc.

The inverse operator $E^{-1}$is defined by $E^{-1}f(x) = f(x - h)$

If y_x is the function $f(x)$, then $Ey_x = y_{x+h}$,$E^{-1}y_x = y_{x-h}$,$E^n y_x = y_{x+nh}$, where n may be any real number.

Averaging operator μis defined by the equation $\mu y_x = \frac{1}{2}\left(y_{x+\frac{1}{2}}h + y_{x-\frac{1}{2}h}\right)$.

NOTE **Obs.** *In the difference calculus E is regarded as the fundamental operator and* Δ,∇, δ, μ *can be expressed in terms of E.*

6.9 Relations Between the Operators

We shall now establish the following identities:

(*i*) $\Delta = \mathbf{E} - \mathbf{1}$ (*ii*) $\Delta = \mathbf{1} - \mathbf{E}^{-1}$

(*iii*) $\delta = \mathbf{E}^{1/2} - \mathbf{E}^{-1/2}$ (*iv*) $\mu = \frac{1}{2}(E^{1/2} + E^{-1/2})$

(*v*) $\Delta = \mathbf{E}\Delta = \Delta\mathbf{E} = \delta\mathbf{E}1/2$ (*vi*) $\mathbf{E} = \mathbf{e}^{\mathbf{hD}}$.

Proofs.(*i*)$\Delta y_x = y_{x+h} - y_x = Ey_x - y_x = (E - 1)y_x$

This shows that the operators Δ and E are connected by the symbolic relation

$\Delta = E - 1$ or $\mathbf{E} = 1 + \Delta$.

NOTE **Obs.** *These relations imply that the effect of operator E on yx is the same as that of the operators* $(1 + \Delta)$ *on yx.The operator's E and* Δ *do not have any existence as separate entities.*

(*ii*) $\Delta y_x = y_x - y_{x-h} = y_x - E^{-1}y_x = (1 - E^{-1})y_x$

$\therefore$ $\Delta = 1 - E^{-1}$

(*iii*) $\delta y_x = y_{x+\frac{1}{2}h} + y_{x-\frac{1}{2}h} = E^{1/2}y_x - E^{-1/2}y_x = \left(E^{1/2} - E^{-1/2}\right)y_x$

$\delta = E^{1/2} - E^{-1/2}$

(*iv*) $\mu y_x = \frac{1}{2}\left(y_{x+\frac{1}{2}h} + y_{x-\frac{1}{2}h}\right) = \frac{1}{2}\left(E^{\frac{1}{2}}y_x - E^{-\frac{1}{2}}y_x\right) = \frac{1}{2}\left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)y_x$

$\therefore \quad \mu = \frac{1}{2}\left(E^{1/2} - E^{-1/2}\right)$

(*v*) $E\Delta y_x = E(y_x - y_{x-h}) = Ey_x - Ey_{x-h} = y_{x+h} - y_x = \Delta y_x$

$\therefore \quad E\Delta = \Delta$

$\Delta Ey_x = \Delta y_{x+h} = y_{x+h} - y_x = \Delta y_x$

$\therefore \quad \Delta E = \Delta$

$$\delta E^{1/2} yx = \delta y_{x+\frac{1}{2}h} = y_{x+\frac{1}{2}h} + \frac{1}{2}h - y_{x+\frac{1}{2}h} - \frac{1}{2}h - \frac{1}{2}h = y_{x+h} - y_x = \Delta y_x$$

$\therefore \quad \delta E^{1/2} = \Delta$

Hence $\Delta = E\Delta = \Delta E = \delta E^{1/2}$

(vi) $\quad Ef(x) = f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots$

[by Taylor's series]

$$= f(x) + hDf(x) + \frac{h^2}{2!}D^2 f(x) + \ldots.$$

$$= \left(1 + hD + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + \cdots\right)f(x) = e^{hD}f(x)$$

$\therefore \quad E = e^{hD}$

Cor. = $\quad E = 1 + \Delta = e^{hD}$

NOTE *A table showing the symbolic relations between the various operators is given below for ready reference To prove such relations between the operators, always express each operator in terms of the fundamental operator E.*

Relations between the various operators

In terms of	E	Δ	∇	δ	hD
E	—	$\Delta + 1$	$(1-\nabla)^{-1}$	$1+\frac{1}{2}\delta^2 + \Delta\sqrt{(1+\delta^2/4)}$	e^{hD}
Δ	$E-1$	—	$(1-\nabla)^{-1}-1$	$\frac{1}{2}\delta^2 + \Delta\sqrt{(1+\delta^2/4)}$	$e^{hD}-1$
∇	$1-E-1$	$1-(1+\Delta)^{-1}$	—	$-\frac{1}{2}\delta^2 + \Delta\sqrt{(1+\Delta^2/4)}$	$1-e^{-hD}$
σ	$E^{1/2}-E^{-1/2}$	$\Delta(1+\Delta)^{-1/2}$	$\Delta(1-\Delta)^{-1/2}$	—	$2\sinh(hD/2)$
μ	$\frac{1}{2}(E^{1/2}+E^{-1/2})$	$(1+\Delta/2)(1+\Delta)^{-1/2}$	$(1+\Delta/2)(1+\Delta)^{-1/2}$	$\sqrt{(1+\Delta^2/4)}$	$\cosh(hD/2)$
hD	$\log E$	$\log(1+\Delta)$	$\log(1-\nabla)^{-1}$	$2\sinh^{-1}(\delta/2)$	—

EXAMPLE 6.13

Prove that $e^x = \left(\frac{\Delta^2}{E}\right)e^x \cdot \frac{Ee^x}{\Delta^2 e^x}$, ,the interval of differencing being h.

EXAMPLE 6.14

Prove with the usual notations, that

(*i*) $hD = \log(1+\Delta) = -\log(1-\nabla) = \sin h^{-1}(\mu\delta)$

(*ii*) $(E^{1/2}+E^{-1/2})(1+\Delta)^{1/2} = 2+\Delta$

(*iii*) $\Delta - \nabla = \Delta\nabla = \delta^2$

(*iv*) $\Delta^3 y^2 = \Delta^3 y_5$.

Solution:

(*i*) We know that $e^{hD} = E = 1+\Delta$

$\therefore$ $hD = \log(1+\Delta)$

Also $hD = \log E = -\log(E^{-1}) = -\log(1-\nabla)$ $[\therefore E^{-1} = 1-\nabla]$

We have prove that $\mu = \frac{1}{2}\left(E^{1/2}+E^{1/2}\right)$

and $\delta = E^{1/2} - E^{1/2}$

$$\mu\delta = \frac{1}{2}\left(E^{1/2}+E^{1/2}\right)\left(E^{1/2}-E^{1/2}\right)$$

$$= \frac{1}{2}(E - E^{-1}) = \frac{1}{2}(e^{hD} - e^{-hD}) = \sinh(hD)$$

i.e., $$hD = \sinh^{-1}(\mu\delta)$$

Hence $hD = \log(1 + \Delta) = -\log(1 - \Delta) = \sinh - 1(\mu\delta)$.

(*ii*) $(E^{1/2} + E^{-1/2})(1 + \Delta)^{1/2}$

$$= (E^{1/2} + E^{-1/2})E^{1/2} = E + 1 = 1 + \Delta + 1 = 2 + \Delta.$$

We know that $\quad \Delta = E - 1, \nabla = 1 - E^{-1}$ and $\Delta = E^{1/2} - E^{-1/2}$

$$\therefore \qquad \Delta - \nabla = E - 2 + E^{-1} = (E^{1/2} - E^{-1/2})^2 = \delta^2$$

Also $$\Delta\nabla = (E - 1)(1 - E^{-1}) = E + E^{-1} - 2$$

$$= (E^{1/2} - E^{-1/2})^2 = \delta^2.$$

Hence $$\Delta - \nabla = \Delta\nabla = \delta^2.$$

(*iv*) $$\Delta^3 y_2 = (E - 1)^3 y_2 \qquad [\because \Delta = E - 1]$$

$$= (E^3 - 3E^2 + 3E - 1)y_2$$

$$= y_5 - 3y_4 + 3y_3 - y_2 \qquad (1)$$

$$\Delta^3 y_5 = (1 - E^{-1})^3 y_5 \qquad [\because = 1 - E^{-1}]$$

$$= (1 - 3E^{-1} + 3E^{-2} - E^{-3})y_5$$

$$= y_5 - 3y_4 + 3y_3 - y_2 \qquad (2)$$

From (1) and (2), $$\Delta^3 y_2 = \Delta^3 y_5$$

EXAMPLE 6.15

Prove that

(*i*) $$\Delta = \frac{1}{2}\delta^2 + \delta\sqrt{\left(1 + \frac{\delta^2}{4}\right)}$$

(*ii*) $$1 + \delta^2\mu^2 = \left(1 + \frac{1}{2}\delta^2\right)$$

(*iii*) $$\mu = \frac{2 + \Delta}{2\sqrt{(1 + \Delta)}} + \sqrt{\left(1 + \frac{1}{4}\delta^2\right)}$$

Solution:

(i) $$\frac{1}{2}\delta^2+\delta\sqrt{\left(1+\frac{\delta^2}{4}\right)}$$

$$=\frac{1}{2}(E^{1/2}-E^{-1/2})^2+(E^{1/2}-E^{-1/2})\sqrt{[1+(E^{1/2}-E^{-1/2})^{2/4}]}$$

$$=\frac{1}{2}(E+E^{-1}-2)+(E^{1/2}-E^{-1/2})\sqrt{[(E+E^{-1}+2)/4]}$$

$$=(E+E^{-1}-2)+(E^{1/2}-E^{-1/2})(E^{1/2}+E^{-1/2})/2$$

$$=\frac{1}{2}[(E+E^{-1}-2)+(E-E^{-1})]=E-1=\Delta.$$

(*ii*) We know that $\Delta = E^{1/2} - E^{-1/2}$ and $\mu = (E^{1/2} + E^{-1/2})/2$.

$\therefore$ L.H.S. $= 1 + \delta^2\mu^2 = 1 + (E^{1/2} - E^{-1/2})^2(E^{1/2} + E^{-1/2})^2/4$

$$=\frac{1}{4}[4+(E-E^{-1})^2]=\frac{1}{4}(E^2+E^{-2}+2)=\frac{1}{4}(E+E^{-1})2$$

R.H.S. $$=\left(1+\frac{1}{2}\delta^2\right)^2=\left[1+\frac{1}{2}\left(E^{1/2}-E^{-1/2}\right)^2\right]^2=\left[1+\frac{1}{2}\left(E-E^{-1}-2\right)\right]^2$$

$$=-\left(E-E\quad\right)$$

Hence $$1+\delta^2\mu^2=\left(1+\frac{1}{2}\delta^2\right)^2$$

(*iii*) Since $\Delta=E-1, \delta=E^{1/2}-E^{-1/2}$ and $\mu=\frac{1}{2}\left(E^{1/2}-E^{-1/2}\right)$

$\therefore$ $$\frac{2+\Delta}{2\sqrt{1+\Delta}}=\frac{2+E+1}{2\sqrt{1+E-1}}=\frac{E+1}{2\sqrt{E}}$$

$$=\frac{1}{2}\left(E^{1/2}+E^{-1/2}\right)=\mu \qquad (1)$$

Also $$\sqrt{\left(1+\frac{1}{4}\delta^2\right)}=\sqrt{\left(1+\frac{1}{4}\left(E^{1/2}-E^{-1/2}\right)^2\right)}=\sqrt{\left(1+\frac{1}{4}\left(E+E^{-1}-2\right)\right)}$$

$$=\frac{1}{2}\sqrt{(E+E^{-1}+2)}=\frac{1}{2}(E^{1/2}+E^{-1/2})=\mu \qquad (2)$$

Hence from (1) and (2), we get

$$\mu = \frac{2+\Delta}{2\sqrt{1+\Delta}} = \sqrt{\left(1+\frac{1}{4}\delta^2\right)}$$

EXAMPLE 6.16

Prove that $\nabla y_{n+1} = h\left(1+\frac{1}{2}\nabla+\frac{5}{12}\nabla^2+\cdots\right)y'_n$

Solution:

We have $\nabla_{y+1} = y_{n+1} - y_n = (E-1)y_n$

$$= \left(e^{hD}-1\right)y_n = \left(1+hD+\frac{h^2D^2}{2!}+\frac{h^3D^3}{3!}+\cdots-1\right)y_n$$

$$= hD\left(1+\frac{hD}{2!}+\frac{h^2D^2}{3!}+\cdots\right)y_n$$

$$= h\left(1+\frac{hD}{2!}+\frac{h^2D^2}{3!}+\cdots\right)Dy_n$$

Since $\quad E^{-1} = 1-\nabla = e^{-hD}$,

$\therefore \quad hD = -\log(1-\nabla) = \nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \cdots$

$\therefore$

$$\nabla y_{n+1} = h\left\{1+\frac{1}{2}\left(\nabla+\frac{1}{2}\nabla^2+\frac{1}{3}\nabla^3+\cdots\right)\right.$$

$$\left.+\frac{1}{6}\left(\nabla+\frac{1}{2}\nabla^2+\frac{1}{3}\nabla^3+\cdots\right)+\cdots\right\}y'n$$

Hence $\nabla y_{n+1} = h\left(1+\frac{1}{2}\nabla+\frac{5}{12}\nabla^2+\cdots\right)y'_n$

6.10 To Find One or More Missing Terms

When one or more values of $y = f(x)$ corresponding to the equidistant values of x are missing, we can find these using any of the following two methods:

First method: We assume the missing term or terms as a, b etc. and form the difference table. Assuming the last difference as zero, we solve these equations for a, b. These give the missing term/terms.

Second method: If n entries of y are given, $f(x)$ can be represented by a$(n - 1)^{th}$ degree polynomial, *i.e.*, $\Delta n\ y = 0$. Since $\Delta = E - 1$, therefore $(E - 1)n\ y = 0$. Now expanding $(E - 1)n$ and substituting the given values, we obtain the missing term/terms.

EXAMPLE 6.17

Find the missing term is the table:

x:	2	3	4	5	6
y:	45.0	49.2	54.1	…	67.4

Solution:

Let the missing value be a. Then the difference table is as follows:

X	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
2	$45.0(=y_0)$				
		4.2			
3	$49.2(=y_1)$		0.7		
		4.9		$a - 59.7$	
4	$54.1(=y_2)$		$\alpha - 59.0$		$240.2 - 4a$
		$a - 54.1$		$180.5 - 3a$	
5	$a(=y_3)$		$121.5 - 2a$		
		$67.4 - \alpha$			
6	$67.4(=y_4)$				

We know that $\Delta^4 y = 0$, *i.e.*, $240.2 - 4a = 0$.

Hence $a = 60.05$.

Otherwise. As only four entries y_0, y_1, y_2, y_3 are given ,therefore $y = f(x)$ can be represented by a third degree polynomial.

$\therefore\ \Delta^3 y = \text{constant}$ or $\Delta^4 y = 0$, *i.e.*, $(E - 1)^4 y = 0$

i.e., $(E^4 - 4E^3 + 6E^2 - 4E + 1)\ y = 0$ or $y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0$

Let the missing entry y_3 be a so that

$$67.4 - 4a + 6(54.1) - 4(49.2) + 45 = 0 \quad \text{or} \quad -4a = -\ 240.2$$

Hence $a = 60.05$.

EXAMPLE 6.18

Find the missing values in the following data:

x:	45	50	55	60	65
y:	3.0	...	2.0	...	– 2.4

Solution:

Let the missing values be a, b. Then the difference table is as follows:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
45	$3(=y_0)$			
		$a-3$		
50	$a(=y_1)$		$5-2a$	
		$2-a$		$3a+b-9$
55	$2(=y_2)$		$b+a-4$	
		$b-2$		$3.6-a-36$
60	$b(=y_3)$		$-0.4-2b$	
		$-2.4-b$		
65	$-2.4\ (=y_4)$			

As only three entries y_0, y_2, y_4 are given, y can be represented by a second degree polynomial having third differences as zero.

$\therefore \qquad \Delta^3 y_0 = 0$ and $\Delta^3 y_1 = 0$

i.e., $\qquad 3a + b = 9,\ a + 3b = 3.6$

Solving these, we get $a = 2.925$, $b = 0.225$.

Otherwise. As only three entries $y_0 = 3$, $y_2 = 2$, $y_4 = -2.4$ are given, y can be represented by a second degree polynomial having third differences as zero.

$\therefore \qquad \Delta^3 y_0 = 0$ and $\Delta^3 y_1 = 0$

i.e., $\qquad (E-1)^3 y_0 = 0$ and $(E-1)^3 y_1 = 0$

i.e., $\qquad (E^3 - 3E^2 + 3E - 1)y_0 = 0;\ (E^3 - 3E^2 + 3E - 1).y_1 = 0$

or $\qquad y_3 - 3y_2 + 3y_1 - y_0 = 0;\ y_4 - 3y_3 + 3y_2 - y_1 = 0$

or $\qquad y_3 + 3y_1 = 9;\ 3y_3 + y_1 = 3.6$

Solving three, we get $y_1 = 2.925$, $y_2 = 0.225$.

EXAMPLE 6.19

The following table gives the values of y which is a polynomial of degree five. It is known that $f(3)$is in error. Correct the error.

x:	0	1	2	3	4	5	6
y:	1	2	33	254	1025	3126	7777

Solution:

Let the correct value of y when $x = 3$ be a. Then the difference table is as follows:

x:	y:	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
0	1						
		1					
1	2		30				
		31		$a - 94$			
2	3		$a - 64$		1216–4a		
	$a - 33$			1122–3a		–2320–10a	
3	A		1058–2a		–1104 + 6a		4880–20a
		1025–a		18 + 3a		2560 – 10a	
4	1025		1076 + a		1456 – 4a		
		2101		1474 – a			
5	3126		2550				
		4651					
6	7777						

Since y is a polynomial of fifth degree, the sixth difference $\Delta^6 y = 0$

i.e., $4880 - 20a = 0$

Hence $a = 244$.

Otherwise. As y is a polynomial of fifth degree, the sixth difference $\Delta^6 y = 0$

i.e., $(E - 1)^6 y = 0$

or $(E^6 - 6E^5 + 15E^4 - 20E^3 + 15E^2 - 6E + 1)y_0 = 0$

or $y_6 - 6y_5 + 15y_4 - 20y_3 + 15y_2 - 6y_1 + y_0 = 0$

i.e., $7777 - 6(3126) + 15(1025) + 20y_3 + 15\ (33) - 6\ (2) + 1 = 0$

$\therefore \quad 4880 = 20y_3 \qquad \therefore \quad y_3 = 244$

Hence the error = 254 – 244 = 10.

EXAMPLE 6.20

If $y_{10} = 3$, $y_{11} = 6$, $y_{12} = 11$, $y_{13} = 18$, $y_{14} = 27$, find y_4.

Solution:

Taking y_{14} as u_0, we are required to find y_4, *i.e.*, $.u_{-10}$. Then the difference table is

x	u	Δu	$\Delta^2 u$	$\Delta^3 u$
x_{-4}	$y_{10} = u_{-4} = 3$			
		3		
x_{-3}	$y_{11} = u_{-3} = 6$		2	
		5		0
x_{-2}	$y_{12} = u_{-2} = 11$		2	
		7		0
x_{-1}	$y_{13} = u_{-1} = 18$		2	
		9		
x_0	$y_{14} = u_0 = 27$			

Then $\quad y_4 = u_{-10} = (E^{-1})^{10} u_0 = (1 - \Delta)^{10} u_8$

$$= \left(1 - 10\nabla + \frac{10.9}{2}\nabla^2 - \frac{10.9.8}{1.2.3}\nabla^3 + \cdots\right) u_0$$

$$= u_0 - 10\Delta u_0 + 45\Delta^2 u_0 - 120\Delta_3 u_0$$

$$= 27 - 10 \times 9 + 45 \times 2 - 120 \times 0 = 27.$$

EXAMPLE 6.21

If y_x is a polynomial for which fifth difference is constant and $y_1 + y_7 = -784$, $y_2 + y_6 = 686$, $y_3 + y_5 = 1088$, find y_4.

Solution:

Starting with y_1 instead of y_0, we note that $\Delta^6 y_1 = 0$ [$\because$ $\Delta^5 y_1$ is constant]

i.e., $\quad (E - 1)^6 y_1 = (E^6 - 6E^5 + 15E^4 - 20E^3 + 15E^2 - 6E + 1) y_1 = 0$

$\therefore \quad y_7 - 6y_6 + 15y_5 - 20y_4 + 15y_3 - 6y_2 + y_1 = 0$

or $\quad (y_7 + y_1) - 6(y_6 + y_2) + 15(y_5 + y_3) - 20y_4 = 0$

i.e., $$y_4 = \frac{1}{20}\left[(y_1 + y_7) - 6(y_2 + y_6) + 15(y_3 + y_5)\right]$$

$$= \frac{1}{20}\left[-784 - 6(686) + 15(1088)\right] = 571$$

EXAMPLE 6.22

Using the method of separation of symbols, prove that

(i) $$u_1 x + u_2 x^2 + u_3 x^3 + \cdots = \frac{x}{1-x} u_1 + \left(\frac{x}{1-x}\right)^2 \Delta u_1 + \left(\frac{x}{1-x}\right)^3 \Delta u_1 + \cdots$$

(ii) $$u_0 + \frac{u_1 x}{1!} + \frac{u_2 x^2}{2!} + \frac{u_3 x^3}{3!} + \cdots = e^x \left(u_0 + x\Delta u_0 + \frac{x^2}{2!}\Delta^2 u_0 + \frac{x^3}{3!}\Delta^3 u_0 + \cdots\right)$$

Solution:

(i) L.H.S $= xu_1 + x^2 Eu_1 + x^3 Eu_1 + \cdots$

$$= x\left(1 + xE + x^2E^2 + \cdots\right)u^1 = x.\frac{1}{1-xE}u_1, \text{ taking sum of infinite G.P.}$$

$$= x\left[\frac{1}{1-x(1+\Delta)}\right]u_1 \qquad [\because E = 1+\Delta]$$

$$= x\left(\frac{1}{1-x-x\Delta}\right)u_1 = \frac{x}{1-x}\left(1-\frac{x\Delta}{1-x}\right)^{-1} u_1$$

$$= \frac{x}{1-x}\left(1+\frac{x\Delta}{1-x}+\frac{x^2\Delta^2}{(1-x)^2}\right)u_1$$

$$= \frac{x}{}u_1 + \frac{x^2}{(1-x)^2}\Delta u_1 + \frac{x^3}{(1-x)^3}\Delta\ u_1 + \quad = \text{R.H.S}$$

(ii) L.H.S. $= u_0 + \frac{x}{1!}Eu_0 + \frac{x^2}{2!}E^2u_0 + \frac{x^3}{3!}E^3u_0 + \cdots$

$$= \left(1 + \frac{xE}{1!} + \frac{x^2E^2}{2!} + \frac{x^3E^3}{3!} + \cdots\right)u_0 = e^{xE}u_0 = e^{x(1+\Delta)}u_0 = e^x.e^{x\Delta}u_0$$

$$= e^x\left(1 + \frac{x\Delta}{1!} + \frac{x^2\Delta^2}{2!} + \frac{x^3\Delta^3}{3!} + \cdots\right)u_0$$

$$= e^x\left(u_0 + \frac{x}{1!}\Delta u_0 + \frac{x^2}{2!}\Delta^2 u_0 + \frac{x^3}{3!}\Delta^3 u_0 + \cdots\right) = \text{R.H.S}$$

Exercises 6.3

1. Explain the difference between $\left(\frac{\Delta^2}{E}\right)u_x$ and $\left(\frac{\Delta^2 u_x}{Eu_x}\right)$.

2. Evaluate taking h as the interval of differencing:

(*i*) $\frac{\Delta^2}{E}\sin x$ (*ii*). $(\Delta+\nabla)^2\left(x^2+x\right), (h=1)$

(*iii*) $\frac{\Delta^2 x^3}{Ex^3}$ (*iv*) $\frac{\Delta^2}{E}\sin(x+h)+\frac{\Delta^2\sin(x+h)}{E\sin(x+h)}$

3. With the usual notations, show that

(*i*) $\Delta = 1-e^{-hD}$ (*ii*) $D=\frac{2}{h}\sinh^{-1}\left(\frac{\delta}{2}\right)$

(*iii*) $(1+\Delta)(1-\nabla)=1$ (*iv*) $\Delta\nabla=\nabla\Delta=\delta^2$

4. Prove that

(*i*) $\delta=\Delta(1+\Delta)^{-1/2}=\nabla(1-\nabla)^{-1/2}$

(*ii*) $\mu^2=1+\frac{\delta^2}{4}$ (*iii*) $\delta\left(E^{1/2}+E^{-1/2}\right)=\Delta E^{-1}+\Delta$

5. Show that

(*i*) $\Delta = \Delta E^{-1/2} = \Delta E^{1/2}$

(*ii*) $\mu\Delta=\frac{1}{2}(\Delta+\nabla)$ (*iii*) $1+\delta^2/2=\sqrt{(1+\delta^2\mu^2)}$

6. Show that

(*i*) $\Delta=\mu\delta+\frac{\delta^2}{2}$ (*ii*) $E^{1/2}=\left(1+\frac{\delta^2}{4}\right)^{1/2}+\frac{\delta^2}{2}$

(*iii*) $E^r=\left(\mu+\frac{1}{2}\delta\right)^{2r}$ (*iv*) $\mu=\frac{2+\Delta}{2\lambda(1+\Delta)}=\frac{2+\nabla}{2\lambda(1+\nabla)}$

7. Prove that

(*i*) $\Delta+\nabla=\frac{\Delta}{\nabla}-\frac{\nabla}{\Delta}$ (*ii*) $\nabla=\Delta E^{-1}=E^{-1}\Delta=1-E^{-1}$

(*iii*) $E=\sum_{i=0}^{\infty}\nabla_i$ (*iv*) $\nabla^2=h^2D^2-h^3D^3+\frac{7}{12}h^4D^4-\cdots$

8. Prove that $\delta^2 y_5 = y_6 - 2y_5 + y_4$.

9. Prove with usual notations, that

(i) $\nabla^r f_k = \Delta^r f_{k-r}$

(ii) $\Delta\left(f^2{}_k\right) = \left(f_k + f_k + 1\right)\Delta f_k$ (iii) $\sum_{k=0}^{n-1} \Delta^2 f_k = \Delta f_n - \Delta f_0$

10. Estimate the missing term in the following table:

x:	0	1	2	3	4
f(x)	1	3	9	–	81

11. Find the missing terms in the following table:

x:	1	1.5	2	2.5	3	3.5
y:	6	?	10	20	?	1.5

12. Find the missing values in the following table:

0	1	2	3	4	5	6
5	11	22	40	…	140	…

13. Estimate the production for 2004 and 2006 from the following data:

Year:	2001	2002	2003	2004	2005	2006	2007
Production:	200	200	260	…	350	…	430

14. If $U_{13} = 1, U_{14} = -3, U_{15} = -1, U_{16} = 13$, find U_8

15. Evaluate y_4 from the following data (stating the assumptions you make)

$$y_0 + y_8 = 1.9243,\ y_1 + y_7 = 1.9590$$

$$y_2 + y_6 = 1.9823,\ y_3 + y_5 = 1.9956$$

Using the method of separation of symbols, prove that

16. $u_0 + u_1 + u_2 + \cdots + u_n = {}^{n+1}C_1 u_0 + {}^{n+1}C_2 \Delta u_0 + {}^{n+1}C_3 \Delta^2 u_0 + \cdots + {}^{n+1}C_{n+1} \Delta^n u_0$

17. $\Delta^n u_x = u_{x+n} - {}^nC_1 u_{x+n-1} + {}^nC_2 u_{x+n-2} + \cdots + (-1)^n u_x$

18. $y_x = y_n - {}^{n-x}C_1 \Delta y_{n-1} + {}^{n-x}C_2 \Delta^2 y_{n-2} - \cdots + (-1)^{n-x} \Delta^{n-x} y_{n-(n-x)}$.

6.11 Application to Summation of Series

The calculus of finite differences is very useful for finding the sum of a given series. The inverse operator Δ^{-1}(Section 6.6) is especially useful to find the sum of a series. This is explained below:

If $u_r = \Delta y_r = y_{r+1} - y_r$

then $u_1 = y_2 - y_1, u_2 = y_3 - y_2 \cdots u_{n-1} = y_n - y_{n-1}, u_n = y_{n+1} - y_1$

$\therefore u_1 + u_2 + \ldots + u_n = y_{n+1} - y_1 = \Delta^{-1} \left| y_r \right|_{r=1}^{r=n+1}$

Thus $\sum_{r=1}^{n} u_r = \left| \Delta^{-1} u_r \right|_{r=1}^{r=n+1}$ $\qquad \because yr = \Delta^{-1} u_r]$

The method is best illustrated by the following examples

EXAMPLE 6.23

Find the sum to n terms of the series

(*i*) 2.3.4 + 3.4.5 + 4.5.6 + ⋯

(*ii*) $\dfrac{1}{3.4.5} + \dfrac{1}{4.5.6} + \dfrac{1}{5.6.7} + \cdots$

Solution:

(*i*) Let $\sum_{r=1}^{n} u_r = 2.3.4 + 3.4.5 + 4.5.6 + \cdots (n+1)(n+2)(n+3)$

$$\therefore \quad u_r = (r+1)(r+2)(r+3) = [r+3]^3$$

$$\therefore \quad \sum_{r=1}^{n} u_r = \left[\Delta^{-1} u_r\right]_{r=1}^{r=n+1}$$

$$= \frac{1}{4}\left\{[n+4]^4 - [4]^4\right\}$$

$$= \frac{1}{4}\{(n+4)(n+3)(n+2)(n+1) - 4.3.2.1\}$$

$$= \frac{1}{4}\{(n+4)(n+3)(n+2)(n+1) - 24\}$$

(*ii*) Let $\sum_{r=1}^{n} u_r = \dfrac{1}{3.4.5} + \dfrac{1}{4.5.6} + \dfrac{1}{5.6.7} + \cdots + \dfrac{1}{(n+2)(n+3)(n+4)}$

$$\therefore \quad u_r = \frac{1}{(r+2)(r+3)(r+4)} = [r+1]^{-3}$$

$$\sum_{r=1}^{n} u_r = \left\{\Delta^{-1} u_r\right\}_{r=1}^{r=n+1} = \left\{\Delta^{-1}[r+1]^{-3}\right\}_{r=1}^{r=n+1}$$

$$= \left\{\frac{[r+1]^{-2}}{-2}\right\}_{r=1}^{r=n+1} = -\frac{1}{2}\left\{[n+2]^{-2} - [2]^{-2}\right\}$$

$$= -\frac{1}{2}\left\{\frac{1}{(n+3)(n+4)} - \frac{1}{3.4}\right\} = \frac{1}{2}\left\{\frac{1}{12} - \frac{1}{(n+3)(n+4)}\right\}$$

EXAMPLE 6.24

Sum the following series $1^3 + 2^3 + 3^3 + \cdots + n^3$

Solution:

Denoting $1^3, 2^3, 3^3, \cdots, n^3$ by $u_0, u_1, u_2, \cdots$ respectively, the required sum

$$S = u_0 + u_1 + u_2 + \cdots + u_{n-1}$$

$$= \left(1 + E + E^2 + \cdots + E^{n-1}\right) \qquad \left[\because u_1 = Eu_0, u_2 = E^2 u_0\right]$$

$$= \frac{E^n - 1}{E - 1} u_0 = \frac{(1+\Delta) - 1}{\Delta} u_0$$

$$= \frac{1}{\Delta}\left[1 + n\Delta + \frac{n(n-1)}{2!}\Delta^2 + \frac{n(n-1)(n-2)}{3!}\Delta^3 + \ldots.\Delta^n - 1\right] u_0$$

$$= n + \frac{n(n-1)}{2!}\Delta\ u_0 + \frac{n(n-1)(n-2)}{3!}\Delta^2 u_0 + \ldots..$$

Now $\Delta u_0 = u_1 - u_0 = 2^3 - 1^3 = 7$, $\Delta^2 u_0 = u^2 - 2u_1 + u_0 = 3^3 - 2.2^3 + 1^3 = 12$,

$$\Delta^3 u_0 = u_3 - 3u_2 + 3u_1 - u_0 = 4^3 - 3.3^3 + 3.2^3 - 1^3 = 6$$

and $\Delta^4 u_0, \Delta^5 u_0, \cdots$ are all zero as $u_r = r^3$ is a polynomial of third degree

Hence $S = n + \frac{n(n-1)}{2}.7 + \frac{n(n-1)(n-2)}{6}12 + \frac{n(n-1)(n-2)(n-3)}{24}.6$

$$= \frac{n^2}{4}\left(n^2 + 2n + 1\right) + \left[\frac{n(n-1)}{2}\right]^2$$

EXAMPLE 6.25

Prove Montmort's theorem that

$$u_0+u_1x+u_2x^2+\cdots+\infty=\frac{u_0}{1-x}+\frac{x\Delta u_0}{(1-x)^2}+\frac{x^2\Delta^2u_0}{(1-x)^3}+\cdots+\infty$$

Hence find the sum of the series $1.2+2.3x+3.4x^2+\cdots+\infty$

Solution:

$$u_0+u_1x+u_2x^2+\cdots+\infty=\left(1+xE+x^2E^2+x^3E^3+\cdots+\infty\right)u_0$$

$$=\frac{1}{1-xE}u_0=\frac{1}{1-x(1+\Delta)}u_0$$

$$=\frac{1}{(1-x)-x\Delta}u_0=\frac{1}{(1-x)}\left(1-\frac{x}{1-x}\Delta\right)^{-1}u_0$$

$$==\frac{1}{1-x}\left\{1-\frac{x\Delta}{(1-x)}+\frac{x^2\Delta^2}{(1-x^2)}+\ldots\right\}u_0$$

$$=\frac{u_0}{(1-x)}+\frac{x}{(1-x)^2}\Delta u_0+\frac{x^2}{(1-x)^3}\Delta^2u_0+\cdots\infty$$

Now let us construct the difference table for the coefficients of the given series:

u	Δu	$\Delta^2 u$	$\Delta 3u$
$u_0=2$			
	4		
$u_1=6$		2	
	6		0
$u_2=12$	2		
	8		0
$u_3=20$		2	
	10		
$u_4=30$			

This shows that $u_0=2, \Delta u_0=4, \Delta^2u_0=2, \Delta^3u_0=\Delta^4u_0$ etc. all $=0$.

Thus $\quad 1.2+2.3x+3.4x^2+\cdots+\infty$

$$=u_0+u_1x+u_2x^2+\cdots+\infty$$

$$= \frac{u_0}{(1-x)} + \frac{x}{(1-x)^2}\Delta u_0 + \frac{x^2}{(1-x)^3}\Delta^2 u_0 + \cdots \infty$$

$$= \frac{2}{1-x} + \frac{4x}{(1-x)^2} + \frac{2x^2}{(1-x)^3} = \frac{2}{(1-x)^3}$$

Exercises 6.4

Using the method of finite differences, sum the following series:

1. $2.5 + 5.8 + 8.11 + 11.14 + \cdots n$ terms.

2. $1.2.3 + 2.3.4 + 3.4.5 + \cdots$.to n terms

3. $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \ldots$ to n terms

4. $\frac{1}{4.5.6} + \frac{1}{5.6.7} + \frac{1}{6.7.8} + \cdots$ to n terms

5. $1^2 + 2^2 + 3^2 + \cdots + n^2$

6. Show that $u_0 + \frac{u_1 x}{1!} + \frac{u_2 x^2}{2!} + \cdots + = e^x\left(= u_0 + \frac{x}{1!}\Delta u_0 + \frac{x^2}{2!}\Delta^2 u_0 + \cdots \right)$

Hence sum the series

$(i)\ 1^3 + \frac{2^3}{1!}x + \frac{3^3}{2!}x^2 + \frac{4^3}{3!} + x^3 + \cdots + \infty$

$(ii)\ 1 + \frac{4x}{1!} + \frac{10x^2}{2!} + \frac{20x^3}{3!} + \frac{35x^4}{4!} + \frac{56x^5}{5!} + \cdots + \infty$

7. Using Montmort's theorem find the sum of the series

$(i)\ 1.3 + 3.5x + 5.7x^2 + 7.9x^3 + \cdots + \infty$

$(ii)\ 1^2 + 2^2 x + 3^2 x^2 + 4^2 x^3 + \cdots + n^2$

8. show that $\sum u \; = nC_1 u_1 +^n C_2 \Delta u_1 +^n C_3 \Delta^2 u_1 + \quad + \Delta^{n-1} u_1 r =$

Hence evaluate $1^4 + 2^4 + 3^4 + \cdots + n^4$.

9. Sum the series $1.2\Delta xn - 2.3\Delta^2 x^n + 3.4\Delta^3 x^n - 4.5\Delta^4 x^n + \cdots$ to n terms

10. Show that $\Delta x^n - \frac{1}{2}\Delta^2 x^n + \frac{1.3}{2.4}\Delta^3 x^n + \frac{1.3.5}{2.4.6}\Delta^4 x^n + \cdots$ to n terms $=$

$$(x+1/2)^n - (x-1/2)^n$$

6.12 Objective Type of Questions

Exercises 6.5

Select the correct answer or fill up the blanks in the following questions:

1. $\Delta\nabla =$

(*a*) $\nabla\Delta$ (*b*) $\nabla + \Delta$ (*c*) $\nabla - \Delta$.

2. Which one of the following results is correct:

(*a*) $\Delta x^n = nx^{n-1}$ (*b*) $\Delta x^{(n)} = nx^{(n-1)}$

(*c*) $\Delta^n e^x = e^x$ (*d*) $\Delta \cos x = -\sin x$.

3. If $f(x) = 3x^3 - 2x^2 + 1$, then $\Delta^3 f(x) = \cdots$

4. The relationship between the operators E and D is$\cdots$.

5. The $(n+1)$th order difference of the nth degree polynomial is$\cdots$

6. If $y(x) = x(x-1)(x-2)$, then $\Delta y(x) = \cdots$. .

7. $x^3 - 2x^2 + x - 1$ in factorial form $= \cdots$...

8. Taking $h$as the interval of differencing, $\Delta^2 x^3 = \cdots$

9. In terms of $E, \Delta = \cdots$.

10. The form of the function tabulated at equally spaced intervals with sixth differences constant, is$\cdots$

11. If the interval of differencing is unity, then $\Delta^4[(1-x)(1-2x)(1-3x)] = \cdots$

12. Taking the interval of differencing as unity, the first difference of $x^4 - 3x^3 + 2x - 1$ is $\cdots$.

13. The missing values of y in the following data:

yx:	0	...	...	...	25
Δyx:	1	2	4	7	11,

are$\cdots$.

14. $\Delta^3[(1-x)(1-3x)(1-5x)] = \cdots$(interval of differencing being 1)

15. $\Delta \tan^{-1} x = \cdots$.

16. If $y = x^2 - 2x + 2$, taking interval of differencing as unity,$\Delta^2 y = \cdots$.

17. Relation between Δ and E is given by$\cdots$..

18. The kth difference of a polynomial of degree k is$\cdots$

19. $\Delta^r y_k$ in terms of backward differences = $\cdots$.

20. The value of $(\Delta^2/E)e^x = \cdots$.

21. The relation between the shift operator E and second order backward difference operator Δ^2 is$\cdots$

22. The value of $\Delta^n(e^x) = \cdots$(intervalofdifferencingbeing1).

23. Relationship between E, Δ and Δ is$\cdots$

24. If the fifth and higher order differences of a function vanish, then the function represents a polynomial of degree$\cdots$.

25. The value of $E^{-1}\Delta = \cdots$.

26. If $E^2 u_x = x^2$ and $h = 1$, then $u_x = \cdots$.

27. Given $y_0 = 2$, $y_1 = 4$, $y_2 = 8$, $y_4 = 32$,then $y_3 = \cdots$.

28. $y_0 = 1$, $y_1 = 5$, $y_2 = 8$, $y_3 = 3$, $y_4 = 7$, $y_5 = 0$, then $\Delta^5 y_6 =$

(*a*) 61 (*b*) – 62
(*c*) 62 (*d*) – 61.

29. Given x = 1 2 3

$f(x)$ = 3 815, then $\Delta^2 f(1) =$

(*a*) 3 (*b*) 4
(*c*) 2 (*d*) 1

30. $(E^{1/2} + E^{-1/2})(1+\Delta)^{1/2} =$

(*a*) $\Delta + 1$ (*b*) $\Delta - 1$
(*c*) $\Delta + 2$ (*d*) $\Delta - 2$.

31. Which one is incorrect?

(*a*) $E = 1 + \Delta$ (*b*) $\Delta(5) = 0$
(*c*) $\Delta(f_1 + f_2) = \Delta f_1 + \Delta f_2$ (*d*) $\Delta(f_1 . f_2) = \Delta f_1 + \Delta f_2$.

32. $\Delta - \nabla = \delta^2$. (True or False)

33. $\Delta + \nabla = E + E^{-1.}$ (True or False)

34. $E = e^{-hD}$. (True or False)

35. If $f(x) = e^x$, then $\Delta^6 e^x = (e^h - 1)^6 e^x$. (True or False)

36. $\Delta^n = \delta^n E^{n/2}$ (True or False)

37. $(1 + \Delta)(1 - \nabla) = 1$. (True or False)

38. With the usual notations, match the items on right hand side with those in left hand side:

(i) $E\nabla$	(a) $\frac{1}{2}(\Delta + \nabla)$
(ii) hD	(b) $\Delta - \nabla$
(iii) $\nabla\Delta$	(c) Δ
(iv) $\mu\delta$	(d) $-\log(1 - \nabla)$