7.11 Interpolation with Unequal Intervals

The various interpolation formulae derived so far possess the disadvantage of being applicable only to equally spaced values of the argument. It is, therefore, desirable to develop interpolation formulae for unequally spaced values of x. Now we shall study two such formulae:

(*i*) Lagrange's interpolation formula

(*ii*) Newton's general interpolation formula with divided differences.

7.12 Lagrange's Interpolation Formula

If $y = f(x)$ takes the value $y_0, y_1, \ldots\ldots, y_n$ corresponding to $x = x_0, x_1, \cdots, x_n$, then

$$f(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}y_0 + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)}y_1 + \cdots + \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}y_n \quad (1)$$

This is known as *Lagrange's interpolation formula for unequal intervals*.

Proof: Let $y = f(x)$ be a function which takes the values $(x_0, y_0), (x_1, y_1), \cdots, (x_n, y_n)$. Since there are $n + 1$ pairs of values of x and y, we can represent $f(x)$ by a polynomial in x of degree n. Let this polynomial be of the form

$$y = f(x) = a_0(x-x_1)(x-x_2)\ldots(x-x_n) + a_1(x-x_0)(x-x_2)\cdots(x-x_n)$$

$$+a_2(x-x_0)(x-x_1)(x-x_3)\cdots(x-x_n) + \cdots + a_n(x-x_0)(x-x_1)\cdots(x-x_{n-1}) \quad (2)$$

Putting $x = x_0$, $y = y_0$, in (2), we get

$$y_0 = a_0(x_0-x_1)(x-x_2)\cdots(x-x_n)$$

$$a_0 = y_0 / [(x-x_1)(x-x_2)\cdots(x-x_n)]$$

Similarly putting $x = x_1$, $y = y_1$ in (2), we have

$$a_1 = y_1 / [(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)]$$

Proceeding the same way, we find $a_2, a_3 \ldots\ldots a_n$.

Substituting the values of $a_0, a_1, \cdots, a_n$ in (2), we get (1)

NOTE

Obs. *Lagrange's interpolation formula (1) for n points is a polynomial of degree (n – 1) which is known as the Lagrangian polynomial and is very simple to implement on a computer.*

This formula can also be used to split the given function into partial fractions.

For on dividing both sides of (1) by $(x-x_0)(x-x_1)\cdots(x-x_n)$, *we get*

$$\frac{f(x)}{(x-x_0)(x-x_1)\cdots(x-x_n)}=\frac{y_0}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}\cdot\frac{1}{(x-x_0)}$$
$$+\frac{y_1}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)}\cdot\frac{1}{(x-x_1)}+\cdots$$
$$+\frac{y_n}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}\cdot\frac{1}{(x-x_n)}$$

EXAMPLE 7.17

Given the values

x:	5	7	11	13	17
f(*x*):	150	392	1452	2366	5202

evaluate *f*(9), using *Lagrange's formula*

Solution:

(*i*) Here $x_0 = 5, x_1 = 7, x_2 = 11, x_3 = 13, x_4 = 17$

and $y_0 = 150, y_1 = 392, y_2 = 1452, y_3 = 2366, y_4 = 5202.$

Putting $x = 9$ and substituting the above values in Lagrange's formula, we get

$$f(9)=\frac{(9-7)(9-11)(9-13)(9-17)}{(5-7)(5-11)(5-13)(5-17)}\times 150+\frac{(9-5)(9-11)(9-13)(9-17)}{(7-5)(7-11)(7-13)(7-17)}\times 392$$
$$+\frac{(9-5)(9-7)(9-13)(9-17)}{(11-5)(11-7)(11-13)(11-17)}\times 1452$$
$$+\frac{(9-5)(9-7)(9-11)(9-17)}{(13-5)(13-7)(13-11)(13-17)}\times 2366$$
$$+\frac{(9-5)(9-7)(9-11)(9-13)}{(17-5)(17-7)(17-11)(17-13)}\times 5202$$
$$=-\frac{50}{3}+\frac{3136}{15}+\frac{3872}{3}+\frac{2366}{3}+\frac{578}{5}=810$$

EXAMPLE 7.18

Find the polynomial $f(x)$ by using Lagrange's formula and hence find $f(3)$ for

x:	0	1	2	5
$f(x)$:	2	3	12	147

Solution:

Here $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 5$

and $y_0 = 2, y_1 = 3, y_2 = 12, y_3 = 147.$

Lagrange's formula is

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$
$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$
$$= \frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)}(2) + \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)}(3)$$
$$+\frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)}(12) + \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)}(147)$$

Hence $f(x) = x^3 + x^2 - x + 2$

$\therefore \quad f(3) = 27 + 9 - 3 + 2 = 35$

EXAMPLE 7.19

A curve passes through the points (0, 18), (1, 10), (3, –18) and (6, 90). Find the slope of the curve at $x = 2$.

Solution:

Here $x_0 = 0, x_1 = 1, x_2 = 3, x_3 = 6$ and $y_0 = 18, y_1 = 10, y_2 = -18, y_3 = 90.$

Since the values of x are unequally spaced, we use the Lagrange's formula:

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$
$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$= \frac{(x-1)(x-3)(x-6)}{(0-1)(0-3)(0-6)}(18) + \frac{(x-0)(x-3)(x-6)}{(1-0)(1-3)(1-6)}(10)$$

$$+ \frac{(x-0)(x-1)(x-6)}{(3-0)(3-1)(3-6)}(-18) + \frac{(x-0)(x-1)(x-3)}{(6-0)(6-1)(6-3)}(90)$$

$$= (-x^3 + 10x^2 - 27x + 18) + (x^3 - 9x^2 + 18x)$$

$$+ (x^3 - 7x^2 + 6x) + (x^3 - 4x^2 + 3x)$$

i.e., $y = 2x^3 - 10x^2 + 18$

Thus the slope of the curve at $x = 2 = \left(\frac{dy}{dx}\right)_{x-2}$

$$= (6x^2 - 20x)_{x=2} = -16$$

EXAMPLE 7.20

Using Lagrange's formula, express the function $\dfrac{3x^2 + x + 1}{(x-1)(x-2)(x-3)}$ as a sum of partial fractions.

Solution:

Let us evaluate $y = 3x^2 + x + 1$ for $x = 1$, $x = 2$ and $x = 3$

These values are

x:	$x_0 = 1$	$x_1 = 2$	$x_2 = 3$
y:	$y_0 = 5$	$y_1 = 15$	$y_2 = 31$

Lagrange's formula is

$$y = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}y_2$$

Substituting the above values, we get

$$y = \frac{(x-2)(x-3)}{(1-2)(1-3)}(5) + \frac{(x-1)(x-3)}{(2-1)(2-3)}(15) + \frac{(x-1)(x-2)}{(3-1)(3-2)}(31)$$

$$= 2.5\,(x-2)\,(x-3) - 15\,(x-1)\,(x-3) + 15.5\,(x-1)\,(x-2)$$

Thus $$\frac{3x^2 + x + 1}{(x-1)(x-2)(x-3)} = \frac{2.5\,(x-2)\,(x-3) - 15(x-1)\,(x-3) + 15.5\,(x-1)\,(x-2)}{(x-1)(x-2)(x-3)}$$

$$= \frac{25}{x-1} - \frac{15}{x-2} + \frac{15.5}{x-3}$$

EXAMPLE 7.21

Find the missing term in the following table using interpolation:

x:	0	1	2	3	4
y:	1	3	9	...	81

Solution:

Since the given data is unevenly spaced, therefore we use Lagrange's interpolation formula:

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$
$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

Here we have $x_0 = 0 \quad x_1 = 1 \quad x_2 = 2 \quad x_3 = 4$

$y_0 = 1 \quad y_1 = 3 \quad y_2 = 9 \quad y_3 = 81$

$$\therefore \quad y = \frac{(x-1)(x-2)(x-4)}{(0-1)(0-2)(0-4)}(1) + \frac{(x-0)(x-2)(x-4)}{(1-0)(1-2)(1-4)}(3)$$
$$+\frac{(x-0)(x-1)(x-4)}{(2-0)(2-1)(2-4)}(9) + \frac{(x-0)(x-1)(x-2)}{(4-0)(4-1)(4-2)}(81)$$

When $x = 3$, then

$$\therefore \quad y = \frac{(3-1)(3-2)(3-4)}{-8} + 3(3-2)(3-4) + \frac{3(3-1)(3-4)(9)}{-4} +$$
$$+\frac{3(3-1)(3-2)}{24}(81) = \frac{1}{4} - 3 + \frac{27}{2} + \frac{81}{24} = 31$$

Hence the missing term for $x = 3$ is $y = 31$.

EXAMPLE 7.22

Find the distance moved by a particle and its acceleration at the end of 4 seconds, if the time verses velocity data is as follows:

t:	0	1	3	4
v:	21	15	12	10

Solution:

Since the values of t are not equispaced, we use Lagrange's formula:

$$v=\frac{(t-t_1)(t-t_2)(t-t_3)}{(t_0-t_1)(t_0-t_2)(t_0-t_3)}v_0+\frac{(t-t_0)(t-t_2)(t-t_3)}{(t_1-t_0)(t_1-t_2)(t_1-t_3)}v_1$$

$$+\frac{(t-t_0)(t-t_1)(t-t_3)}{(t_1-t_0)(t_1-t_2)(t_1-t_3)}v_2+\frac{(t-t_0)(t-t_1)(t-t_3)}{(t_1-t_0)(t_1-t_2)(t_1-t_3)}v_3$$

$$i.e.\ ,\ v=\frac{(t-1)(t-3)(t-4)}{(-1)(-2)(-4)}(21)+\frac{t(t-3)(t-4)}{(1)(-2)(-3)}(15)$$

$$+\frac{t(t-1)(t-4)}{(3)(2)(-1)}(12)+\frac{t(t-1)(t-3)}{(4)(3)(1)}(10)$$

$$i.e.,\ v=\frac{1}{12}(-5t^3+38t^2-105^t+252)$$

$$\therefore \text{ Distance moved } s=\int_0^4 vdt=\int_0^4(-5t^3+38t^2-105^t+252) \qquad \left[\because v=\frac{ds}{dt}\right]$$

$$=\frac{1}{12}\left(-\frac{5t^4}{4}+\frac{38t^3}{3}-\frac{105t^2}{2}+252t\right)_0^4$$

$$=\frac{1}{12}\left(-320+\frac{2432}{3}-840+1008\right)=54.9$$

Also acceleration $=\dfrac{dv}{dt}=\dfrac{1}{2}(-15t2+76t-105+0)$

Hence acceleration at $(t=4)=\dfrac{1}{2}(-15\pm+76(4)-105)=-3.4$

Exercises 7.3

1. Use Lagrange's interpolation formula to find the value of y when $x = 10$, if the following values of x and y are given:

x:	5	6	9	11
y:	12	13	14	16

2. The following table gives the viscosity of oil as a function of temperature. Use Lagrange's formula to find the viscosity of oil at a temperature of 140°.

Temp°:	110	130	160	190
Viscosity:	10.8	8.1	5.5	4.8

3. Given $\log_{10}654 = 2.8156$, $\log_{10} 658 = 2.8182$, $\log_{10} 659 = 2.8189$, $\log_{10}661 = 2.8202$, find by using Lagrange's formula, the value of $\log_{10} 656$.

4. The following are the measurements T made on a curve recorded by oscilograph representing a change of current I due to a change in the conditions of an electric current.

T:	1.2	2.0	2.5	3.0
I:	1.36	0.58	0.34	0.20

Using Lagrange's formula, find I and $T = 1.6$.

5. Using Lagrange's interpolation, calculate the profit in the year 2000 from the following data:

Year:	1997	1999	2001	2002
Profit in Lakhs of Rs:	43	65	159	248

6. Use Lagrange's formula to find thee form of $f(x)$, given

x:	0	2	3	6
$f(x)$:	648	704	729	792

7. If $y(1) = -3$, $y(3) = 9$, $y(4) = 30$, $y(6) = 132$, fin d the Lagrange's interpolation polynomial that takes the same values as y at the given point s.

8. Given $f(0) = -18$, $f(1) = 0$, $f(3) = 0$, $f(5) = -248$, $f(6) = 0$, $f(9) = 13104$, find $f(x)$.

9. Find the missing term in the following table using interpolation

x:	1	2	4	5	6
y:	14	15	5	...	9

10. Using Lagrange's formula, express the function $\dfrac{x^2+x-3}{x^3-2x^2-x+2}$ as a sum of partial fractions.

11. Using Lagrange's formula, express the function $\dfrac{x^2+6x-1}{(x^2-1)(x-4)(x-6)}$ as a sum of partial fractions.

[***Hint***. Tabulate the values of $f(x) = x^2 + 6x - 1$ for $\underline{x} = -1, 1, 4, 6$ and apply Lagrange's formula.]

12. Using ***Lagrange's formula***, prove that

$$y_o = \frac{1}{2}(y_1 + y_{-1}) = \frac{1}{8}\left\{\frac{1}{2}(y_3 + y_1) - \frac{1}{2}(y_{-1} + y_{-3})\right\}.$$

[***Hint***: Here $x_0 = -3$, $x_1 = -1$, $x_2 = 1$, $x_3 = 3$.]

7.13 Divided Differences

The Lagrange's formula has the drawback that if another interpolation value were inserted, then the interpolation coefficients are required to be recalculated. This labor of recomputing the interpolation coefficients is saved by using Newton's general interpolation formula which employs what are called "**divided differences**." Before deriving this formula, we shall first define these differences.

If $(x_0,y_0),(x_1,y_1),(x_2,y_2),\cdots$ be given points, then the *first divided difference* for the arguments x_0, x_1 is defined by the relation $[x_0, x_1]$ or $\underset{x_1}{\Delta} y_0 = \dfrac{y_1 - y_0}{x_1 - x_0}$

Similarly $[x_1, x_2]$ or $\underset{x_2}{\Delta} y_0 = \dfrac{y_2 - y_1}{x_2 - x_1}$ and $[x_2, x_3]$ or $\underset{x_3}{\Delta} y_0 = \dfrac{y_3 - y_2}{x_3 - x_2}$

The *second divided difference* for x_0, x_1, x_2 is defined as

$[x_0, x_1, x_2]$ or $\underset{x_1,x_2}{\Delta^2} = \dfrac{[x_1,x_2]-[x_0,x_1]}{x_2 - x_0}$

The *third divided difference* for x_0, x_1, x_2, x_3 is defined as

$[x_0, x_1, x_2, x_3]$ or $\underset{x_1,x_2,x_3}{\Delta^3} y_0 = \dfrac{[x_1,x_2,x_3]-[x_0,x_1,x_2]}{x_2 - x_0}$

Properties of Divided Differences

I. The divided differences are symmetrical in their arguments, i.e,. independent of the order of the arguments. For it is easy to write

$$[x_0,x_1]=\frac{y_0}{x_0 - x_1}+\frac{y_1}{x_1 - x_0}=[x_1,x_0],[x_0,x_1,x_2]$$

$$=\frac{y_0}{(x_0 - x_1)(x_0 - x_2)}+\frac{y_1}{(x_1 - x_0)(x_1 - x_2)}+\frac{y_2}{(x_2 - x_0)(x_2 - x_1)}$$

$= [x_1, x_2, x_0]$ or $[x_2, x_0, x_1]$ and so on

II. The nth divided differences of a polynomial of the nth degree are constant.

Let the arguments be equally spaced so that

$$x_1 - x_0 = x_2 - x_1 = \cdots = x_n - x_{n-1} = h. \text{ Then}$$

$$[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y_0}{h}$$

$$[x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0 - x_1]}{x_2 - x_0} = \frac{1}{2h}\left\{\frac{\Delta y_1}{h} - \frac{\Delta y_0}{h}\right\}$$

$$= \frac{1}{2!h^2}\Delta^2 y_0 \text{ and in general, } [x_0, x_1, x_2, \ldots\ldots, x_n] = \frac{1}{n!h^n}\Delta^n y_0$$

If the tabulated function is a nth degree polynomial, then $\Delta^n y_0$ will be constant. Hence the nth divided differences will also be constant

III. The divided difference operator $\blacktriangle$ is linear

i.e.,
$$\blacktriangle\{au_x + bv_x\} = a\blacktriangle u_x + b\blacktriangle v_x$$

We have
$$\underset{x_1}{\blacktriangle}\left(au_{x_0} + bv_{x_0}\right) = \frac{\left(au_{x_1} + bv_{x_1}\right) - \left(au_{x_0} + bv_{x_0}\right)}{x_1 - x_0}$$
$$= a\left\{\frac{u_{x_1} - u_{x_0}}{x_1 - x_0}\right\} + b\left\{\frac{v_{x_1} - v_{x_0}}{x_1 - x_0}\right\}$$
$$= a\underset{x_1}{\blacktriangle} u_{x_0} + b\underset{x_0}{\blacktriangle} v_{x_0}$$

In general $\blacktriangle\left(au_x + bv_x\right) = a\blacktriangle u_x + b\blacktriangle v_x$ This property is also true for higher order differences.

7.14 Newton's Divided Difference Formula

Let $y_0, y_1, \cdots, y_n$ be the values of $y = f(x)$ corresponding to the arguments $x_0, x_1, \cdots, x_n$. Then from the definition of divided differences, we have

$$[x, x_0] = \frac{y - y_0}{x - x_0}$$

So that
$$y = y_0 + (x - x_0)[x, x_0]$$

Again
$$[x, x_0, x_1] = \frac{[x, x_0] - [x_0, x_1]}{x - x_1}$$

which gives $[x, x_0] = [x_0, x_1] + (x - x_1)[x, x_0, x_1]$

Substituting this value of $[x, x_0]$ in (1), we get

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x, x_0, x_1] \tag{2}$$

Also $$[x, x_0, x_1, x_2] = \frac{[x.x_{0,} x_1] - [x.x_{0,} x_2]}{x - x_2}$$

which gives $[x, x_0, x_1] = [x_0, x_1, x_2] + (x - x_2)[x, x_0, x_1, x_2]$

Substituting this value of $[x, x_0, x_1]$ in (2), we obtain

$$\begin{aligned} y = y_0 &+ (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] \\ &+ (x - x_0)(x - x_1)(x - x_2)[x, x_0, x_1, x_2] \end{aligned}$$

Proceeding in this manner, we get

$$\begin{aligned} y = y_0 &+ (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] \\ &+ (x - x_0)(x - x_1)\cdots(x - x_n)[x, x_0, x_1, \cdots x_n] \\ &+ (x - x_0)(x - x_1)(x - x_2)[x, x_0, x_1, x_2] + \cdots \end{aligned} \tag{3}$$

which is called *Newton's general interpolation formula with divided differences*.

7.15 Relation Between Divided and Forward Differences

If $(x_0, y_0), (x_1, y_1), (x_2, y_2), \cdots$ be the given points, then

$$[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0}$$

Also $$\Delta y_0 = y_1 - y_0$$

If $x_0, x_1, x_2, \cdots$ are equispaced, then $x_1 - x_0 = h$, so that

$$[x_0, x_1] = \frac{\Delta y_0}{h}$$

Similarly $$[x_1, x_2] = \frac{\Delta y_1}{h}$$

Now $$[x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0}$$

$$= \frac{\Delta y_1 / h - \Delta y_0 / h}{2h} \qquad [\because x_2 - x_0 = 2h]$$

$$= \frac{\Delta y_1 - \Delta y_0}{2h^2}$$

Thus $$[x_0, x_1, x_2] = \frac{\Delta^2 y_0}{2! h^2}$$

Similarly $[x_0, x_1, x_2] = \dfrac{\Delta^2 y_1}{2!h^2}$

$$\therefore [x_0, x_1, x_2, x_3] = \frac{\Delta^2 y_1/2h^2 - \Delta^2 y_0/2h^2}{x_3 - x_0} = \frac{\Delta^2 y_1 - \Delta^2 y_0}{2h^2(3)} \qquad [\because x_3 - x_0 = 3h]$$

Thus $[x_0, x_1, x_2, x_3] = \dfrac{\Delta^3 y_0}{3!h^3}$

In general, $[x_0, x_1, \cdots x_n] = \dfrac{\Delta^n y_0}{n!h^n}$

This is the relation between divided and forward differences.

EXAMPLE 7.23

Given the values

x:	5	7	11	13	17
$f(x)$:	150	392	1452	2366	5202

evaluate $f(9)$, using Newton's divided difference formula

Solution:

The divided differences table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
5	150	$\dfrac{392-150}{7-5} = 121$		
7	392		$\dfrac{265-121}{11-5} = 24$	
		$\dfrac{1452-392}{11-7} = 265$		$\dfrac{32-24}{13-5} = 1$
11	1452		$\dfrac{457-265}{13-7} = 32$	
		$\dfrac{2366-1452}{13-11} = 457$		$\dfrac{42-32}{17-7} = 1$
13	2366		$\dfrac{709-457}{17-11} = 42$	
		$\dfrac{5202-2366}{17-13} = 709$		
17	5202			

Taking $x = 9$ in the Newton's divided difference formula, we obtain

$$f(9) = 150 + (9-5) \times 121 + (9-5)(9-7) \times 24 + (9-5)(9-7)(9-11) \times 1$$

$$= 150 + 484 + 192 - 16 = 810.$$

EXAMPLE 7.24

Using Newton's divided differences formula, evaluate $f(8)$ and $f(15)$ given:

x:	4	5	7	10	11	13
$y = f(x)$:	48	100	294	900	1210	2028

Solution:

The divided differences table is

x	$f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
4	48				0
		52			
5	100		15		
		97		1	
7	294		21		0
		202		1	
10	900		27		0
		310		1	
11	1210		33		
		409			
13	2028				

Taking $x = 8$ in the Newton's divided difference formula, we obtain

$$f(8) = 48 + (8-4)\,52 + (8-4)\,(8-5)\,15 + (8-4)\,(8-5)\,(8-7)\,1$$

$$= \mathbf{448.}$$

Similarly $f(15) = 3150$.

EXAMPLE 7.25

Determine $f(x)$ as a polynomial in x for the following data:

x:	-4	-1	0	2	5
$y = f(x)$:	1245	33	5	9	1335

Solution:

The divided differences table is

x	$f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
– 4	1245				
		– 404			
– 1	33		94		
		– 28		– 14	
0	5		10		3
		2		13	
2	9		88		
		442			
5	1335				

Applying Newton's divided difference formula

$$f(x) = f(x_0) + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + \cdots$$

$$= 1245 + (x + 4)(-404) + (x + 4)(x + 1)(94)$$

$$+ (x + 4)(x + 1)(x - 0)(-14) + (x + 4)(x + 1)x(x - 2)(3)$$

$$= 3x^4 - 5x^2 + 6x^2 - 14x + 5$$

EXAMPLE 7.26

Using Newton's divided difference formula, find the missing value from the table:

x:	1	2	4	5	6
y:	14	15	5	...	9

Solution:

The divided difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1	14			
		$\frac{15-14}{2-1} = 1$		
2	15		$\frac{-5-1}{4-1} = -2$	
		$\frac{5-15}{4-2} = -5$		$\frac{7/4+2}{6-1} = \frac{3}{4}$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
4	5		$\frac{2+5}{6-2} = \frac{7}{4}$	
		$\frac{9-6}{6-4} = 2$		
6	9			

Newton's divided difference formula is

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2]$$
$$+ (x - x_0)(x - x_1)(x - x_2)[x_0, x_1, x_2, x_3] + \cdots$$
$$= 14 + (x - 1)(1) + (x - 1)(x - 2)(-2) + (x - 1)(x - 2)(x - 4)\frac{3}{4}$$

Putting $x = 5$, we get

$$y(5) = 14 + 4 + (4)(3)(-2) + (4)(3)(1)\frac{3}{4} = 3.$$

Hence missing value is 3

Exercises 7.4

1. Find the third divided difference with arguments 2, 4, 9, 10 of the function $f(x) = x^3 - 2x$.

2. Obtain the Newton 's divided difference interpolating polynomial and hence find $f(6)$:

x:	3	7	9	10
$f(x)$:	160	120	72	63

3. Using Newton's divided differences interpolation, find $u(3)$, given that $u(1) = -26$, $u(2) = 12$, $u(4) = 256$, $u(6) = 844$.

4. A thermocouple gives the following output for rise in temperature

Tem p (°C)	0	10	20	30	40	50
Output (m V)	0.0	0.4	0.8	1.2	1.6	2.0

Find the output of thermocouple for 37°C temperature using Newton's divided difference formula.

5. Using Newton 's divided difference interpolation, find the polynomial of the given data:

x:	-1	0	1	3
$f(x)$:	2	1	0	-1

6. For the following table, find $f(x)$ a s a polynomial in x using Newton's divided difference formula:

x:	5	6	9	11
$f(x)$:	12	13	14	16

7. Using the following data, find $f(x)$ a s a polynomial in x:

x:	– 1	0	3	6	7
$f(x)$:	3	– 6	39	822	1611

8. The observed values of a function are respectively 168, 120, 72, and 63 at the four positions 3, 7, 9, and 10 of the independent variable. What is the best estimate value of the function at the position 6?

9. Find the equation of the cubic curve which passes through the point s (4, – 43), (7, 83), (9, 327), and (12, 1053).

10. Find the missing term in the following table using Newton's divided difference formula.

x:	0	1	2	3	4
y:	1	3	9	...	81

7.16 Hermite's Interpolation Formula

This formula is similar to the Lagrange's interpolation formula. In Lagrange's method, the interpolating polynomial $P(x)$ agrees with $y(x)$ at the points $x_0, x_1, \ldots\ldots, x_n$, whereas in

Hermite's method $P(x)$ and $y(x)$ as well as $P'(x)$ and $y'(x)$ coincide at the $(n + 1)$ points, *i.e.*,

$$P(x_i) = y(x_i) \text{ and } P'(x_i) = y'(x_i);\ i = 0, 1, \ldots\ldots, n \tag{1}$$

As there are $2(n + 1)$ conditions in (1), $(2n + 2)$ coefficients are to be determined.

Therefore $P(x)$ is a polynomial of degree $(2n + 1)$.

We assume that $P(x)$ is expressible in the form

$$p(x) = \sum_{i=0}^{n} U_i(x) y(x_i) + \sum_{i=0}^{n} V_i(x) y'(x_i) \tag{2}$$

where $U_i(x)$ and $V_i(x)$ are polynomials in x of degree $(2n + 1)$. These are to be determined. Using the conditions (1), we get