**7.** Find a root of the following equations correct to three decimal places by the secant method:

($i$) $x^3 + x^2 + x + 7 = 0$ ($ii$) $x - e^{-x} = 0$

($iii$) $x \log_{10} x = 1.9$.

**8.** Use the iteration method to find a root of the equations to four decimal places:

($i$) $x^3 + x^2 - 100 = 0$ ($ii$) $x^3 - 9x + 1 = 0$

($iii$) $x = \dfrac{1}{2} + \sin x$ ($iv$) $\tan x = x$

($v$) $e^x = 5x$ ($vi$) $2^x - x - 3 = 0$ which lies between $(-3, -2)$.

**9.** Evaluate $\sqrt{30}$ by ($i$) secant method ($ii$) iteration method correct to four decimal places.

**10.** Find the root of the equation $2x = \cos x + 3$ correct to three decimal places using ($i$) iteration method, ($ii$) Aitken's $\Delta^2$ method.

**11.** Find the real root of the equation

$$x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \frac{x^9}{216} - \frac{x^{11}}{1320} + \ldots = 0.443$$

correct to three decimal places using iteration method

## 2.12 Newton-Raphson Method

Let $x_0$ be an approximate root of the equation $f(x) = 0$. If $x_1 = x_0 + h$ be the exact root, then $f(x_1) = 0$.

$\therefore$ Expanding $f(x_0 + h)$ by Taylor's series $f(x_0) + hf'(x_0) + \dfrac{h^2}{2!} f''(x_0) + \cdots = 0$

Since $h$ is small, neglecting $h^2$ and higher powers of $h$, we get $f(x_0) + h f'(x_0) = 0$

or
$$h = -\frac{f(x_0)}{f'(x_0)} \tag{1}$$

$\therefore$ A closer approximation to the root is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Similarly starting with $x_1$, a still better approximation $x_2$ is given by

$$x_2 = x_1 - \frac{f(x_0)}{f'(x_1)}.$$

In general, $X_{n+1} = X_n - \frac{f(X_n)}{f'(X_n)}$ $(n = 0, 1, 2....)$ (2)

which is known as the *Newton-Raphson formula* or *Newton's iteration formula*.

**NOTE**

**Obs.** *1. Newton's method is useful in cases of large values of $f'(x)$ i.e., when the graph of $f(x)$ while crossing the x-axis is nearly vertical.*

*For if $f'(x)$ is small in the vicinity of the root, then by (1), h will be large and the computation of the root is slow or may not be possible. Thus this method is not suitable in those cases where the graph of $f(x)$ is nearly horizontal while crossing the x-axis.*

**Obs. 2.** *Geometrical interpretation. Let $x_0$ be a point near the root $\alpha$ of the equation $f(x) = 0$ (Figure 2.8). Then the equation of the tangent at $A_0[x_0, f(x_0)]$ is*

$$y - f(x_0) = f'(x_0)\,(x - x_0).$$

*It cuts the x-axis at* $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

*which is a first approximation to the root $\alpha$. If $A_1$ is the point corresponding to $x_1$ on the curve, then the tangent at $A_1$ will cut the x-axis at $x_2$ which is nearer to $\alpha$ and is, therefore, a second approximation to the root. Repeating this process, we approach the root $\alpha$ quite rapidly. Hence the method consists in replacing*

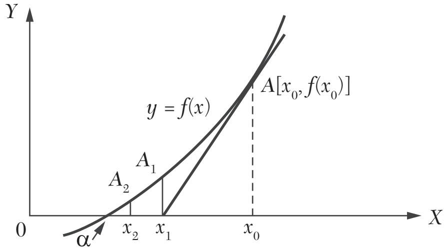


*FIGURE 2.8*

*the part of the curve between the point $A_0$ and the x-axis by means of the tangent to the curve at $A_0$.*

**Obs. 3.** *Newton's method is generally used to improve the result obtained by other methods. It is applicable to the solution of both algebraic and transcendental equations.*

**Convergence of Newton-Raphson Method.** Newton's formula converges provided the initial approximation $x_0$ is chosen sufficiently close to the root.

If it is not near the root, the procedure may lead to an endless cycle. A bad initial choice will lead one astray. *Thus a proper choice of the initial guess is very important for the success of Newton's method.*

Comparing (2) with the relation $xn_{+1} = \phi(xn)$ of the iteration method, we get

$$\phi(x_n) = x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

In general, $\phi(x) = x - \frac{f(x)}{f'(x)}$ which gives $\phi'(x) = \frac{f(x)f''(x)}{[f'(x)]^2}$

Since the iteration method (Section 2.10) converges if $|\phi'(x)| < 1$

$\therefore$ Newton's formula will converge if $|f(x)f''(x)| < |f'(x)|^2$ in the interval considered. Assuming $f(x)$, $f'(x)$ and $f''(x)$ to be continuous, we can select a small interval in the vicinity of the root $\alpha$, in which the above condition is satisfied. Hence the result.

Newtons method converges conditionally while the regula-falsi method always converges. However when the Newton-Raphson method converges, it converges faster and is preferred.

**Newton's method has a quadratic convergence.**

Suppose $x_n$ differs from the root $\alpha$ by a small quantity $\varepsilon_n$ so that

$x_0 = \alpha + \varepsilon_n$ and $x_{n+1} = \alpha + \varepsilon_{n+1}$.

Then (2) becomes

$$\alpha + \varepsilon_{n+1} = \alpha + \varepsilon_n - \frac{f(\alpha + \varepsilon_n)}{f'(\alpha + \varepsilon_n)}$$

$$i.e., \ \varepsilon_{n+1} = \varepsilon_n - \frac{f(\alpha + \varepsilon_n)}{f'(\alpha + \varepsilon_n)}$$

$$= \varepsilon_n - \frac{f(\alpha) + \varepsilon_n f'(\alpha) + \frac{1}{2!}\varepsilon_n 2 f''(\alpha) + \cdots}{f'(\alpha) + \varepsilon_n f'(\alpha) + \cdots} \text{ by Taylor's expansion}$$

$$= \varepsilon_n - \frac{\varepsilon_n f'(\alpha) + \frac{1}{2}\varepsilon_n 2 f''(\alpha) + \cdots}{f'(\alpha) + \varepsilon_n f''(\alpha) + \cdots} = \frac{\varepsilon_n 2}{2}\frac{f''(\alpha)}{f'(\alpha)}. \quad [\because f(\alpha) = 0]$$

This shows that the subsequent error at each step is proportional to the square of the previous error and as such the convergence is quadratic. Thus the Newton-Raphson method has second order convergence.

## EXAMPLE 2.30

Find the positive root of $x^4 - x = 10$ correct to three decimal places, using the Newton-Raphson method.

**Solution:**

Let $f(x) = x4 - x - 10$

so that $f(1) = -10 = -$ ve, $f(2) = 16 - 2 - 10 = 4 = +$ ve.

$\therefore$ A root of $f(x) = 0$ lies between 1 and 2.

Let us take $x_0 = 2$

Also $f'(x) = 4x^3 - 1$

Newton-Raphson's formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Putting $n = 0$, the first approximation $x_1$ is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{f(2)}{f'(2)}$$

$$= 2 - \frac{4}{4 \times 2^3 - 1} = 2 - \frac{4}{31} = 1.871$$

Putting $n = 1$ in $(i)$, the second approximation is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.871 - \frac{f(1.871)}{f'(1.871)}$$

$$= 1.871 - \frac{(1.871)^4 - (1.871) - 10}{4(1.871)^3 - 1}$$

$$= 1.871 - \frac{0.3835}{25.199} = 1.856$$

Putting $n = 2$ in $(ii)$, the third approximation is

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.856 - \frac{(1.856)^4 - (1.856) - 10}{4(1.856)^3 - 1}$$

$$= 1.856 - \frac{0.010}{24.574} = 1.856$$

Here $x_2 = x_3$. Hence the desired root is 1.856 correct to three decimal places.

**EXAMPLE 2.31**

Find by Newton's method, the real root of the equation $3x = \cos x + 1$, correct to four decimal places.

**Solution:**

Let $f(x) = 3x - \cos x - 1$

$f(0) = -2 = -\text{ve}, f(1) = 3 - 0.5403 - 1 = 1.4597 = +\text{ve}.$

So a root of $f(x) = 0$ lies between 0 and 1. It is nearer to 1. Let us take $x_0 = 0.6$.

Also $\quad f'(x) = 3 + \sin x$

$\therefore$ Newton's iteration formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{3x_n - \cos x_n - 1}{3 + \sin x_n}$$

$$= \frac{x_n \sin x_n + \cos x_n + 1}{3 + \sin x_n} \qquad (i)$$

Putting $n = 0$, the first approximation $x_1$ is given by

$$x_1 = \frac{x_0 \sin x_0 + \cos x_0 + 1}{3 + \sin x_0} = \frac{(0.6)\sin(0.6) + \cos(0.6) + 1}{3 + \sin(0.6)}$$

$$= \frac{0.6 \times 0.5729 + 0.82533 + 1}{3 + 0.5729} = 0.6071$$

Putting $n = 1$ in $(i)$, the second approximation is

$$x_2 = \frac{x_1 \sin x_1 + \cos x_1 + 1}{3 + \sin x_1} = \frac{0.6071 \sin(0.6071) + \cos(0.6071) + 1}{3 + \sin(0.6071)}$$

$$= \frac{0.6071 \times 0.57049 + 0.8213 + 1}{3 + 0.57049} = 0.6071$$

Here $x_1 = x_2$. Hence the desired root is 0.6071 correct to four decimal places.

**EXAMPLE 2.32**

Using Newton's iterative method, find the real root of $x \log_{10} x = 1.2$ correct to five decimal places.

**Solution:**

Let $f(x) = x \log_{10} x - 1.2$

$f(1) = -1.2 = -$ ve, $f(2) = 2 \log_{10} 2 - 1.2 = 0.59794 = -$ ve

and $f(3) = 3 \,_{\log}10\ 3 - 1.2 = 1.4314 - 1.2 = 0.23136 = +$ ve.

So a root of $f(x) = 0$ lies between 2 and 3. Let us take $x_0 = 2$.

Also $f'(x) = \log_{10} x + x.\frac{1}{x}\log_{10} e = \log_{10} x + 0.43429$

$\therefore$ Newton's iteration formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = \frac{0.43429x_n + 12}{\log 10x_n + 0.43429} \qquad (i)$$

Putting $n = 0$, the first approximation is

$$x_1 = \frac{0.43429 \times x_0 + 12}{\log_{10} x_0 + 0.43429} = \frac{0.43429 \times 2 + 12}{\log_{10} 2 + 0.43429}$$

$$= \frac{0.86858 \times 12}{0.30103 + 0.43429} = 2.81$$

Similarly putting $n = 1, 2, 3, 4$ in $(i)$, we get

$$x_2 = \frac{0.43429 \times 2.81 + 1.2}{\log_{10} 2.81 + 0.43429} = 2.741$$

$$x3 = \frac{0.43429 \times 2.741 + 1.2}{\log_{10} 2.741 + 0.43429} = 2.74064$$

$$x4 = \frac{0.43429 \times 2.741 + 1.2}{\log_{10} 2.74064 + 0.43429} = 2.74065$$

$$x5 = \frac{0.43429 \times 2.74065 + 1.2}{\log_{10} 2.74065 + 0.43429} = 2.74065$$

Here $x_4 = x_5$. Hence the required root is 2.74065 correct to five decimal places.

## 2.13 Some Deductions From Newton-Raphson Formula

We can derive the following useful results from the Newton's iteration formula:

(1) Iterative formula to find $1/N$ is $x_{n+1} = x_n(2 - Nx_n)$

(2) Iterative formula to find $\sqrt{N}$ is $x_{n+1} = \frac{1}{2}(x_n + N/x_n)$

(3) Iterative formula to find $1/\sqrt{N}$ is $x_{n+1} = \frac{1}{2}(x_n + 1/Nx_n)$

(4) Iterative formula to find $\sqrt[k]{N}$ is $x_{n+1} = \frac{1}{k}[(k-1)x_n + N/x_n^{k-1})]$

*Proofs.* (1) Let $x = 1/N$ or $1/x - N = 0$

Taking $f(x) = 1/x - N$, we have $f'(x) = -x{-}2$.

Then Newton's formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{(1/x_n - N)}{-x_n - 2} = x_n + \left(\frac{1}{x_n} - N\right)x_n 2.$$

$$= x_n + x_n - Nx_n{}^2 = x_n(2 - Nx_n)$$

(2) Let $x = \sqrt{N}$ or $x^2 - N = 0$.

Taking $f(x) = x^2 - N$, we have $f'(x) = 2x$.

Then Newton's formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n{}^2 - N}{2x_n} = \frac{1}{2}(x_n N / x_n)$$

(3) Let $x = \frac{1}{\sqrt{N}}$ or $x^2 - \frac{1}{N} = 0$

Taking $f(x) = x^2 - 1/N$, we have $f'(x) = 2x$.

Then Newton's formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n{}^2 - 1/N}{2x_n} = \frac{1}{2}\left(x_n + \frac{1}{Nx_n}\right)$$

(4) Let $x = \sqrt[k]{N}$ or $x^k - N = 0$

Taking $f(x) = x^k - N$, we have $f'(x) = kx^{k-1}$

Then Newton's formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n \frac{x_n{}^h - N}{kx_n{}^{k-1}} = \frac{1}{k}\left[(k-1)x_n + \frac{N}{x_n{}^{k-1}}\right].$$

**EXAMPLE 2.33**

*Evaluate the following (correct to four decimal places) by Newton's iteration method:*

(*i*) 1/31 (*ii*) $\sqrt{5}$ (*iii*) $1/\sqrt{14}$ (*iv*) 24 3

(*v*) $(30)^{-1/5}$.

**Solution:**

(*i*) Taking $N = 31$, the above formula (1) becomes

$$x_{n+1} = x_n(2 - 31x_n)$$

Since an approximate value of $1/31 = 0.03$, we take $x_0 = 0.03$.

Then $x_1 = x_0(2 - 31x_0) = 0.03(2 - 31 \times 0.03) = 0.0321$

$x_2 = x_1(2 - 31x_1) = 0.0321(2 - 31 \times 0.0321) = 0.032257$

$x_3 = x_2(2 - 31x_2) = 0.032257(2 - 31 \times 0.032257) = 0.03226$

Since $x_2 = x_3$ upto four decimal places, we have $1/31 = 0.0323$.

(*ii*) Taking $N = 5$, the above formula (2), becomes $x_{n+1} = \frac{1}{2}(x_n + 5/x_n)$.

Since an approximate value of $\sqrt{5} = 2,\ = 2$, we take $x_0 = 2$.

Then $x_1 = \frac{1}{2}(x_0 + 5/x_0) = \frac{1}{2}(2 + 5/2) = 2.25$

$x_2 = \frac{1}{2}(x_1 + 5/x_1) = 2.2361$

$x_3 = \frac{1}{2}(x_2 + 5/x_2) = 2.2361$

Since $x_2 = x_3$ upto four decimal places, we have $\sqrt{5} = 2.2361$.

(*iii*) Taking $N = 14$, the above formula (3), becomes $x_{n+1} = \frac{1}{2}[x_n + 1/(14x_n)]$

Since an approximate value of $1/\sqrt{14} = 1/\sqrt{16} = \frac{1}{4} = 0.25$, we take $x_0 = 0.25$,

$$\text{Then } x_1 = \frac{1}{2}[x_0 + (14x_0)^{-1}] = \frac{1}{2}\left[0.25 + (14 \times 0.25)^{-1}\right] = 0.26785$$

$$x_2 = \frac{1}{2}\left[x_1 + (14x_1)^{-1}\right] = \frac{1}{2}\left[0.26785 + (14 \times 0.26785)^{-1}\right] = 0.2672618$$

$$x_3 = \frac{1}{2}\left[x_2 + (14x_2)^{-1}\right] = \frac{1}{2}\left[0.2672618 + (14 \times 0.2672618)^{-1}\right] = 0.2672612$$

Since $x_2 = x_3$ upto four decimal places, we take $1/\sqrt{14} = 0.2673$.

(*iv*) Taking $N = 24$ and $k = 3$, the above formula (4) becomes

$$X_{n+1} = \frac{1}{3}\left[2X_n + 24/X_n^{\ 2}\right]$$

Since an approximate value of $(24)^{1/3} = (27)^{1/3} = 3$, we take $x_0 = 3$.

$$\text{Then} \quad X_1 = \frac{1}{3}\left(2X_0 + 24X_0^{\ 2}\right) = \frac{1}{3}\left(6 + 24/9\right) = 2.88889$$

$$X_2 = \frac{1}{3}\left(2X_1 + 24/X_1^{\ 2}\right) = \frac{1}{3}\left[(2 \times 2.88889) + 24/(2.88889)^2\right] = 2.88451$$

$$X_3 = \frac{1}{3}\left(2X_2 + 24/X_2^{\ 2}\right) = \frac{1}{3}\left[2 \times 2.88451 + 24/(2.88451)^2\right] = 2.8845$$

Since $X_2 = X_3$ up to four decimal places, we take $(24)^{1/3} = 2.8845$.

(*v*) Taking $N = 30$ and $k = -5$, the above formula (4) becomes

$$X_{n+1} = \frac{1}{-5}\left(6X_n + 30/X_n^{-6}\right) = \frac{X_n}{5}\left(6 - 30X_n^{\ 5}\right)$$

Since an approximate value of $(30)^{-1/5} = (32)^{-1/5} = 1/2$, we take $x_0 = 1/2$

$$\text{Then} \quad X_1 = \frac{X_0}{5}\left(6 - 30X_0^{\ 5}\right) = \frac{1}{10}\left(6 - 30/2^5\right) = 0.506495$$

$$X_2 = \frac{X_1}{5}(6 - 30x_1^{\ 5}) = \frac{0.50625}{5}[6 - 30(0.50625)^5] = 0.506495$$

$$X_3 = \frac{X_2}{5}(6 - 30X_2^{\ 5}) = \frac{0.506495}{5}[6 - 30(0.506495)^5] = 0.506496$$

Since $x_2 = x_3$ up to four decimal places, we take $(30)^{-1/5} = 0.5065$.

## Exercises 2.5

**1.** Find by Newton-Raphson method, a root of the following equations correct to three decimal places:

(*i*) $x^3 - 3x + 1 = 0$ (*ii*) $x^3 - 2x - 5 = 0$

(*iii*) $x^3 - 5x + 3 = 0$ (*iv*) $3x^3 - 9x^2 + 8 = 0$.

**2.** Using Newton's iterative method, find a root of the following equations correct to four decimal places:

(*i*) $x^4 + x^3 - 7x^2 - x + 5 = 0$ which lies between 2 and 3.

(*ii*) $x^5 - 5x^2 + 3 = 0$.

**3.** Find the negative root of the equation $x^3 - 21x + 3500 = 0$ correct to 2 decimal places by Newton's method.

**4.** Using Newton-Raphson method, find a root of the following equations correct to three decimal places:

(*i*) $x^2 + 4 \sin x = 0$

(*ii*) $x \sin x + \cos x = 0$ or $x \tan x + 1 = 0$

(*iii*) $e^x = x^3 + \cos 25x$ which is near 4.5

(*iv*) $x \log_{10} x = 12.34$, start with $x_0 = 10$.

(*v*) $\cos x = xe^x$ (*vi*) $10^x + x - 4 = 0$.

**5.** The equation $2e^{-x} = \dfrac{1}{x+2} + \dfrac{1}{x+1}$ has two roots greater than – 1. Calculate these roots correct to five decimal places.

**6.** The bacteria concentration in a reservoir varies as $C = 4e^{-2t} + e^{-0.1t}$. Using the Newton Raphson (N.R.) method, calculate the time required for the bacteria concentration to be 0.5.

**7.** Use Newton's method to find the smallest root of the equation $e^x \sin x = 1$ to four decimal places.

**8.** The current $i$ in an electric circuit is given by $i = 10e^{-t} \sin 2\pi t$ where $t$ is in seconds. Using Newton's method, find the value of $t$ correct to three decimal places for $i = 2$ amp.

**9.** Find the iterative formulae for finding $\sqrt{N}, \sqrt[3]{N}$ where $N$ is a real number, using the Newton-Raphson formula.

Hence evaluate:

(*a*) $\sqrt{10}$

(*b*) $\sqrt{21}$

(*c*) the cube-root of 17 to three decimal places.

**10.** Develop an algorithm using the N.R. method, to find the fourth root of a positive number $N$ and hence find $\sqrt[4]{32}$

**11.** Evaluate the following (correct to three decimal places) by using the Newton-Raphson method.

(*i*) 1/18 (*ii*) $1/\sqrt{15}$ (*iii*) $(28)^{-1/4}$.

**12.** Obtain *Newton-Raphson extended formula*

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{1}{2}\frac{|f(x_0)|^2 f''(x_0)}{\{f'(x_0)\}^2}$$

*for the root of the equation* $f(x) = 0$.

Hence find the root of the equation $\cos x = xe^x$ correct to five decimal places.

**Solution:**

Expanding $f(x)$ in the neighborhood of $x_0$ by Taylor's series; we have

$0 = f(fx) = f(x_0 + \overline{x - x_0}) = f(x_0) + (x - x_0) f'(x_0)$ to first approximately.

Hence the first approximation to the root is given by

$$x_1 - x_0 = -f(x_0)/f'(x) \qquad (i)$$

Again by Taylor's series to the second approximation, we get

$$f(x_1) = f(x_0) + (x_1 - x_0) f'(x_0) + \frac{1}{2!}(x_1 - x_0)2 f''(x_0)$$

Since $x_1$ is an approximation to the root, $f(x_1) = 0$

$$\therefore \qquad f(x_0) + (x_1 - x_0) f'(x_0) + \frac{1}{2}(x_1 - x_0)^2 f''(x_0) = 0$$

or

$$x_1 - x_0 = -\frac{f(x_0)}{f'(x_0)} - \frac{1}{2}\left\{\frac{-f(x_0)}{f'(x_0)}\right\} f''(x_0) \qquad [\text{by } (i)]$$

whence follows the desired formula. [This is known as the **Chebyshev formula** of third order.]