Notes

Finding the coupled differential equations that explain the dynamics of a double-well potential using the GPE with the quintic term

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1 INTRODUCTION

The double well potential system can be described by the following wavefunction ansatz,

$$\Psi(x,t) = \psi_1(t)\Phi_1(x) + \psi_2(t)\Phi_2(x), \tag{1}$$

with $\psi_{1,2}(t) = \sqrt{N_{1,2}}\, e^{i\theta_{1,2}(t)}$ and a constant total number of particles $N_1 + N_2 = |\psi_1|^2 + |\psi_2|^2 = N_T$. Also Φ_1 and Φ_2 are the ground state solutions for isolated traps.
Replacing 1 in the GPE with the quintic term as shown below

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + g_0|\Psi(x,t)|^2 - g_{3B}|\Psi(x,t)|^4 \right]\Psi(x,t), \tag{2}$$

we get the following set of non-linear two-mode dynamical equations '

$$i\hbar\frac{\partial}{\partial t}\psi_1(t) = (E_1 + U_1^{2B}N_1 + U_1^{3B}N_1^2)\psi_1 - K\psi_2 \tag{3}$$

$$i\hbar\frac{\partial}{\partial t}\psi_2(t) = (E_1 + U_2^{2B}N_2 + U_2^{3B}N_2^2)\psi_2 - K\psi_1 \tag{4}$$

where,

$$\begin{aligned} E_{1,2} &= \int \Big[\frac{\hbar^2}{2m}|\nabla\Phi_{1,2}|^2\Big]dx, \\ U_{1,2}^{2B} &= g_0\int |\Phi_{1.2}|^4 dx, \\ U_{1,2}^{3B} &= -g_{3B}\int |\Phi_{1,2}|^6 dx, \\ K = -\int \Big[\frac{\hbar^2}{2m}&(\nabla\Phi_1\nabla\Phi_2) + \Phi_1 V_{ext}\Phi_2\Big]dx. \end{aligned} \tag{5}$$

The population imbalance and inter-well phase difference are defined as:

$$\begin{aligned} z &= \frac{N_1 - N_2}{N_T} \\ \phi &= \theta_2 - \theta_1 \end{aligned} \tag{6}$$

2 First Differential Equation

Multiplying 3 with $\psi_1^*(t)$ and 4 with $\psi_2^*(t)$, and taking $\hbar = 1$we get

$$i\psi_1^*(t)\frac{\partial}{\partial t}\psi_1(t) = (E_1 + U_1^{2B}N_1 + U_1^{3B}N_1^2)|\psi_1|^2 - K\psi_2\psi_1^*, \tag{7}$$

$$i\psi_2^*(t)\frac{\partial}{\partial t}\psi_2(t) = (E_2 + U_2^{2B}N_2 + U_2^{3B}N_2^2)|\psi_2|^2 - K\psi_1\psi_2^*. \tag{8}$$

Taking Complex conjugate of equations 7 and 8, we get

$$-i\psi_1(t)\frac{\partial}{\partial t}\psi_1^*(t) = (E_1 + U_1^{2B}N_1 + U_1^{3B}N_1^2)|\psi_1|^2 - K\psi_1\psi_2^*, \tag{9}$$

$$-i\psi_2(t)\frac{\partial}{\partial t}\psi_2^*(t) = (E_2 + U_2^{2B}N_2 + U_2^{3B}N_2^2)|\psi_2|^2 - K\psi_2\psi_1^*. \tag{10}$$

Performing 7-9 and 8-10, we get

$$i\left[\psi_1^*(t)\frac{\partial}{\partial t}\psi_1(t) + \psi_1(t)\frac{\partial}{\partial t}\psi_1^*(t)\right] = K(\psi_2^*\psi_1 - \psi_2\psi_1^*), \tag{11}$$

$$i\left[\psi_1^*(t)\frac{\partial}{\partial t}\psi_2(t) + \psi_2(t)\frac{\partial}{\partial t}\psi_2^*(t)\right] = K(\psi_1^*\psi_2 - \psi_1\psi_2^*). \tag{12}$$

11 and 12 can be further written as:

$$i\left[\frac{\partial}{\partial t}|\psi_1(t)|^2\right] = K(\psi_2^*\psi_1 - \psi_2\psi_1^*), \tag{13}$$

$$i\left[\frac{\partial}{\partial t}|\psi_2(t)|^2)\right] = K(\psi_1^*\psi_2 - \psi_1\psi_2^*). \tag{14}$$

Hence,

$$i\left[\frac{\partial}{\partial t}N_1\right] = -K\sqrt{N_1N_2}\,2isin[\phi], \tag{15}$$

$$i\left[\frac{\partial}{\partial t}N_2\right] = K\sqrt{N_1N_2}\,2isin[\phi]. \tag{16}$$

Subtracting 16 from 15, dividing by $2KN_T$,using $\sqrt{N_1N_2} = \frac{N_T}{2}\sqrt{1-z^2}$, we get:

$$\dot{z} = -2K\sqrt{1-z^2}\,\sin[\phi] \tag{17}$$

3 Second Differential Equation

Doing the same analysis and calculations by multiplying 3 with $\psi_2^*(t)$ and 4 with $\psi_1^*(t)$, we get the following equation;

$$i\psi_2^*(t)\frac{\partial}{\partial t}\psi_1(t) = (E_1 + U_1^{2B}N_1 + U_1^{3B}N_1^2)\psi_1\psi_2^* - K|\psi_2|^2 \tag{18}$$

$$i\psi_1^*(t)\frac{\partial}{\partial t}\psi_2(t) = (E_2 + U_2^{2B}N_2 + U_2^{3B}N_2^2)\psi_2\psi_1^* - K|\psi_1|^2. \tag{19}$$

Taking Complex conjugate of equations 18 and 19, we get

$$-i\psi_2(t)\frac{\partial}{\partial t}\psi_1^*(t) = (E_1 + U_1^{2B}N_1 + U_1^{3B}N_1^2)\psi_1^*\psi_2 - K|\psi_2|^2, \tag{20}$$

$$-i\psi_1(t)\frac{\partial}{\partial t}\psi_2^*(t) = (E_2 + U_2^{2B}N_2 + U_2^{3B}N_2^2)\psi_2^*\psi_1 - K|\psi_1|^2. \tag{21}$$

Performing the operation, 18+20 and 19+21 we get,

$$i\left[\psi_2^*(t)\frac{\partial}{\partial t}\psi_1(t)-\psi_2(t)\frac{\partial}{\partial t}\psi_1^*(t)\right]=(E_1+U_1^{2B}N_1+U_1^{3B}N_1^2)(\psi_1\psi_2^*+\psi_1^*\psi_2)-2K(|\psi_2|^2) \tag{22}$$

$$i\left[\psi_1^*(t)\frac{\partial}{\partial t}\psi_2(t)-\psi_1(t)\frac{\partial}{\partial t}\psi_2^*(t)\right]=(E_2+U_2^{2B}N_2+U_2^{3B}N_2^2)(\psi_1\psi_2^*+\psi_1^*\psi_2)-2K(|\psi_1|^2). \tag{23}$$

Performing the operation 22-23, we get

$$\begin{aligned} i\left[\psi_2^*(t)\frac{\partial}{\partial t}\psi_1(t)-\psi_2(t)\frac{\partial}{\partial t}\psi_1^*(t)-\psi_1^*(t)\frac{\partial}{\partial t}\psi_2(t)+\psi_1(t)\frac{\partial}{\partial t}\psi_2^*(t)\right]&=(E_1+U_1^{2B}N_1+U_1^{3B}N_1^2-\\ E_2-U_2^{2B}N_2-U_2^{3B}N_2^2)(\psi_1\psi_2^*+\psi_1^*\psi_2)&+2K(|\psi_1|^2-|\psi_2|^2) \end{aligned} \tag{24}$$

Using the fact $\psi_{1,2}(t)=\sqrt{N_{1,2}}\,e^{i\theta_{1,2}(t)}$, we get

$$\begin{aligned} &\left[\sqrt{N_1N_2}\Big[\frac{\partial(\theta_2-\theta_1)}{\partial t}\Big](e^{i(\theta_1-\theta_2)}+e^{i(\theta_2-\theta_1)})\right]+\left[\frac{i}{2}(e^{i(\theta_1-\theta_2)}-e^{i(\theta_2-\theta_1)})\Big[\sqrt{\frac{N_2}{N_1}}\dot{N}_1+\sqrt{\frac{N_1}{N_2}}\dot{N}_2\Big]\right]=\\ &2K(N_1-N_2)+(E_1+U_1^{2B}N_1+U_1^{3B}N_1^2-E_2-U_2^{2B}N_2-U_2^{3B}N_2^2)\sqrt{N_1N_2}(e^{i(\theta_1-\theta_2)}+e^{i(\theta_2-\theta_1)}) \end{aligned} \tag{25}$$

Dividing by $2KN_T$,using $\sqrt{N_1N_2}=\frac{N_T}{2}\sqrt{1-z^2}$,

$$\frac{\dot{\phi}\sqrt{1-z^2}\,cos[\phi]}{2K}+z\,sin^2[\phi]=z+\frac{1}{2K}\sqrt{1-z^2}\,cos[\phi]\,(E_1+U_1^{2B}N_1+U_1^{3B}N_1^2-E_2-U_2^{2B}N_2-U_2^{3B}N_2^2) \tag{26}$$

Dividing by $\frac{\sqrt{1-z^2}cos[\phi]}{2K}$, we get

$$\dot{\phi}=\frac{z}{\sqrt{1-z^2}}\,cos[\phi]2K+E_1-E_2+\boxed{U_1^{2B}N_1-U_2^{2B}N_2+U_1^{3B}N_1^2-U_2^{3B}N_2^2} \tag{27}$$

Now we will concentrate on the part of the equation that is in he box. We will try to rearrange it.

$$\frac{1}{2}\left[2U_1^{2B}N_1-2U_2^{2B}N_2+U_1^{2B}N_2-U_1^{2B}N_2+U_2^{2B}N_1-U_2^{2B}N_1\right]+\frac{1}{2}\left[2U_1^{3B}N_1^2-2U_2^{3B}N_2^2+U_1^{3B}N_2^2-U_1^{3B}N_2^2+U_2^{3B}N_1^2-U_2^{3B}N_1^2\right] \tag{28}$$

$$=\frac{1}{2}\left[(U_1^{2B}-U_2^{2B})(N_1+N_2)+(U_1^{2B}+U_2^{2B})(N_1-N_2)\right]+\frac{1}{2}\left[(U_1^{3B}-U_2^{3B})(N_1^2+N_2^2)+(U_1^{3B}+U_2^{3B})(N_1^2-N_2^2)\right] \tag{29}$$

$$=\frac{1}{2}\left[(U_1^{2B}-U_2^{2B})N_T+(U_1^{2B}+U_2^{2B})\,zN_T\right]+\frac{1}{2}\left[(U_1^{3B}-U_2^{3B})((N_1+N_2)^2-2N_1N_2)+(U_1^{3B}+U_2^{3B})(N_1-N_2)(N_1+N_2)\right] \tag{30}$$

$$=\frac{1}{2}\left[(U_1^{2B}-U_2^{2B})N_T+(U_1^{2B}+U_2^{2B})\,zN_T\right]+\frac{1}{2}\left[(U_1^{3B}-U_2^{3B})(N_T^2-2\frac{N_T^2}{4}(1-z^2))+(U_1^{3B}+U_2^{3B})\,zN_T^2\right] \tag{31}$$

Plugging 31 into the boxed part of the equation, we get:

$$\dot{\phi}=\frac{z}{\sqrt{1-z^2}}\cos[\phi]2K+\Lambda z+\Delta E-\beta(1-z^2), \tag{32}$$

where

$$\begin{aligned} \Delta E&=E_1-E_2+\left(\frac{U_1^{2B}-U_2^{2B}}{2}\right)N_T+\left(\frac{U_1^{3B}-U_2^{3B}}{2}\right)N_T^2\\ \Lambda&=\frac{1}{2}N_T\,((U_1^{2B}+U_2^{2B})+(U_1^{3B}+U_2^{3B})N_T)\\ \beta&=\frac{1}{4}(U_1^{3B}-U_2^{3B})N_T^2 \end{aligned} \tag{33}$$

4 Result

Hence now we have a coupled differential equation that describes the system

$$\begin{aligned} \dot{z} &= -2K\sqrt{1-z^2}\sin[\phi] \\ \dot{\phi} &= \frac{z}{\sqrt{1-z^2}}\cos[\phi]2K + \Lambda z + \Delta E - \beta(1-z^2). \end{aligned} \tag{34}$$

If we consider symmetric case both ΔE and β are zero. Then 13 becomes similar to the set of differential equations without the quintic term. The only difference is in the interaction strength Λ.

5 Equation used for Simulations

The equations used in the simulations:-

$$\begin{aligned} \dot{z} &= -2K\sqrt{1-z^2}\sin[\phi] \\ \dot{\phi} &= \frac{z}{\sqrt{1-z^2}}\cos[\phi]2K + \Lambda z. \end{aligned} \tag{35}$$

where,

$$\Lambda = \frac{1}{2}2N_{sol}\left((2\chi) + (2\eta)2N_{sol}\right) \tag{36}$$