Notes
Coupled Differential equation

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1 INTRODUCTION

The TMM wavefunction ansatz,

$$\Psi(x,t) = \psi_L(t)\bar{L}(x) + \psi_R(t)\bar{R}(x), \tag{1}$$

with $\psi_{L,R}(t) = \sqrt{N_{L,R}}\, e^{i\theta_{L,R}(t)}$ and a constant total number of particles $N_L + N_R = |\psi_L|^2 + |\psi_R|^2 = 2N_{sol}$. Replacing 1 in the GPE with the quintic term as shown below

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + g_{1D}|\Psi(x,t)|^2 - g_2|\Psi(x,t)|^4\right]\Psi(x,t), \tag{2}$$

we get the following set of non-linear two-mode dynamical equations '

$$i\hbar\frac{\partial}{\partial t}\psi_L(t) = (E_L + U_L^{2B}N_L + U_L^{3B}N_L^2)\psi_L - K\psi_R \tag{3}$$

$$i\hbar\frac{\partial}{\partial t}\psi_R(t) = (E_R + U_R^{2B}N_R + U_R^{3B}N_R^2)\psi_R - K\psi_L \tag{4}$$

where,

$$\begin{aligned}
E_{L,R} &= \int \Big[\frac{\hbar^2}{2m}|\nabla\bar{L}(x)|^2\Big]dx = \omega, \\
U_{L,R}^{2B} &= g_{1D}\int |\bar{L}(x)|^4 dx = \chi, \\
U_{L,R}^{3B} &= -g_2\int |\bar{L}(x)|^6 dx = \eta, \\
K &= -\int \Big[\frac{\hbar^2}{2m}(\nabla\bar{L}(x)\nabla\bar{R}(x))\Big]dx = T.
\end{aligned} \tag{5}$$

Hence Eqns 3 and 4 can be written as

$$i\hbar\frac{\partial}{\partial t}\psi_L(t) = (\omega + \chi N_L + \eta N_L^2)\psi_L - T\psi_R \tag{6}$$

$$i\hbar\frac{\partial}{\partial t}\psi_R(t) = (\omega + \chi N_R + \eta N_R^2)\psi_R - T\psi_L \tag{7}$$

The population imbalance and inter-well phase difference are defined as:

$$\begin{aligned}
z &= \frac{N_L - N_R}{2N_{sol}} \\
\varphi &= \theta_R - \theta_L
\end{aligned} \tag{8}$$

2 Result

Hence now we have a coupled differential equation that describes the system

$$\begin{aligned} \dot{z} &= -2K\sqrt{1-z^2}\sin[\varphi] \\ \dot{\varphi} &= \frac{z}{\sqrt{1-z^2}}\cos[\varphi]2K + \Lambda z. \end{aligned} \tag{9}$$

where,

$$\Lambda = \frac{1}{2}2N_{sol}\left((U_L^{2B} + U_R^{2B}) + (U_L^{3B} + U_R^{3B})2N_{sol}\right) \tag{10}$$

Thus,

$$\Lambda = \frac{1}{2}2N_{sol}\left((2\chi) + (2\eta)2N_{sol}\right) \tag{11}$$