Weakest Precondition Calculus

19CSE205 : PROGRAM REASONING

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Contents

Can proof constructions be automated?

In the last lecture we saw the process of carrying out program proofs.

Source/Model annotated with specification $\longrightarrow$ Verification system $\longrightarrow$
- YES, if able to prove.
- NO, if able to disprove.
- NO RESPONSE

So, how does the verfication system construct proofs? Is it possible to automate the proof construction?

In the last lecture we saw the process of carrying out program proofs.

Source/Model
annotated with $\longrightarrow$ Verification $\longrightarrow$ - YES, if able to prove.
specification   system   - NO, if able to disprove.
                                                                    - NO RESPONSE

So, how does the verfication system construct proofs? Is it possible to automate the proof construction?

- Dikjstra first provided a procedural approach to realize this objective in the form of Weakest Precondition calculus (topic of this lecture).

- This was followed by several improvisation and new techniques such as model checking, design by contract, satisfiability modulo theories, etc.

- The verification process demands lot of computing power. With the advancements in hardware speed, several tools are now available to construct proofs in an automated way.

Dijkstra has made phenomenal contributions to computer science. He was one of the reasons for computer science to become a separate discipline. He received the ACM Turing Award in the year 1972.

A subset of his works you will study in B.Tech

- Formal specification and verification
  - Weakest precondition calculus
- Concurrency control
  - Banker's algorithm to prevent deadlock
  - Semaphore: A synchronization mechanism
  - Dining Philosophers Problem
- Algorithms
  - Single source shortest path algorithm

Must read: https://en.wikipedia.org/wiki/Edsger_W._Dijkstra.

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Weakest precondition calculus is a deductive system, proposed by Dijkstra, that provides an algorithmic solution to perform symbolic execution on program statements in the backward direction in order to deduce predicates that will guarantee a given postcondition.

Predicate	An expression that evaluates to either true or false.
Postcondition	A predicate that evaluates to true after execution a statement/block.
Precondition	A predicate which when true before execution of a statement/block ensures postcondion is true after execution of that statement/block.
Symbolic execution	An analysis technique that uses symbolic values to variables in an attempt to identify different execution paths that a program takes.
Deductive system	A system that uses axioms and rules of inference to prove theorems.

A weakest precondition P for a statement S given a postcondition Q is written as $P = wp(S, Q)$

A skip statement refers to a blank statement.

- A skip statement does not change the program state.
- More specifically, it doesn't affect the postcondition.

Consider the example below

P

S   skip

Q   x > 0

A skip statement refers to a blank statement.

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Consider the example below

P	$x > 0$	
S	skip	$wp(skip, Q) = Q$
Q	$x > 0$	

1. Skip

A skip statement refers to a blank statement.

- A skip statement does not change the program state.
- More specifically, it doesn't affect the postcondition.

Consider the example below

P	$x > 0$	
S	skip	$wp(skip,Q) = Q$
Q	$x > 0$	

$$wp(skip,x>0) = x>0$$

Each statement can be viewed as a predicate transformer that turns a precondition to a postcondition. $wp(S,Q)$ does the reverse transformation.

A assignment statement is of the form value = expression.

- An assignment statement changes the program state.
- It results in the variable on the left hand side to change.

Consider the example below

P

S $x = y + 5$

Q $x > 7$

A assignment statement is of the form value = expression.

- An assignment statement changes the program state.
- It results in the variable on the left hand side to change.

Consider the example below

P	$y > 2$	
S	$x = y + 5$	$wp(x{=}E,Q) = Q[x{\leftarrow}E] = P$
Q	$x > 7$	

2. Assignment

A assignment statement is of the form value = expression.

- An assignment statement changes the program state.
- It results in the variable on the left hand side to change.

Consider the example below

P   y > 2

S   x = y + 5   wp(x=E,Q) = Q[x←E] = P

Q   x > 7

wp(x=y+5,x>7) = y+5>7 = y>2

A postcondition can have many preconditions. For the above example y>2, y>3, y>100 will all ensure the postcondition x>7. Among them y> 2 is the least constraining condition and hence it is the weakest precondition.

A sequence denotes a block of statements.

- A sequence usually results in change of state more than once.

P

S_1 y = z * 2

Q

S_2 x = y + 5

R   x > 7

A sequence denotes a block of statements.

- A sequence usually results in change of state more than once.

P

S_1 y = z * 2

Q   y > 2

S_2 x = y + 5

R   x > 7

$$wp(S_2,R) = wp(x{=}E_2,R)$$
$$= R[x{\leftarrow}E_2] \qquad = Q$$

A sequence denotes a block of statements.

- A sequence usually results in change of state more than once.

P $z > 1$

S_1 $y = z * 2$

Q $y > 2$

S_2 $x = y + 5$

R $x > 7$

$$wp(S_1;S_2,R) = wp(S_1,wp(S_2,R))$$
$$= wp(y=E_1,Q)$$
$$= Q[y \leftarrow E_1] \quad = P$$

$$wp(S_2,R) = wp(x=E_2,R)$$
$$= R[x \leftarrow E_2] \quad = Q$$

$$wp(y=z*2;x=y+5,x>7) = wp(y=z*2,wp(x=y+5,x>7)) = wp(y=z*2,y+5>7)$$
$$= wp(y=z*2,y>2) = z*2>2 = z>1$$

3. Sequence

A sequence denotes a block of statements.

- A sequence usually results in change of state more than once.

P	$z > 1$	
S_1	$y = z * 2$	
Q	$y > 2$	
S_2	$x = y + 5$	
R	$x > 7$	

$$wp(S_1;S_2,R) = wp(S_1,wp(S_2,R))$$
$$= wp(y=E_1,Q)$$
$$= Q[y \leftarrow E_1] \quad = P$$

$$wp(S_2,R) = wp(x=E_2,R)$$
$$= R[x \leftarrow E_2] \quad = Q$$

$$wp(y=z*2;x=y+5,x>7) = wp(y=z*2,wp(x=y+5,x>7)) = wp(y=z*2,y+5>7)$$
$$= wp(y=z*2,y>2) = z*2>2 = z>1$$

Generalization: $wp(S_1;..;S_n,Q) = wp(S_1;..;S_{n-1},P_{n-1}) \ldots = wp(S_1,P_1) = P$
where $P_n = R$, $P_{i-1} = wp(S_i,P_i)$ and $P_0 = P$

A conditional is a control point that results in alternative execution paths. This has some subtle issues. We will do it in two steps.

Example: If it rains then I will play chess else I will shop.

A prelude to conditional branching

A conditional is a control point that results in alternative execution paths. This has some subtle issues. We will do it in two steps.

Example: If it rains then I will play chess else I will shop.

1. It rains ⇒ I will play chess

   It doesn't rain ⇒ I will shop

A conditional is a control point that results in alternative execution paths. This has some subtle issues. We will do it in two steps.

Example: If it rains then I will play chess else I will shop.

1. It rains $\Rightarrow$ I will play chess

   $\underline{\text{AND / OR?}}$

   It doesn't rain $\Rightarrow$ I will shop

Generalizing, if B then S_1 else S_2

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A conditional is a control point that results in alternative execution paths. This has some subtle issues. We will do it in two steps.

Example: If it rains then I will play chess else I will shop.

① It rains $\Rightarrow$ I will play chess

AND

It doesn't rain $\Rightarrow$ I will shop

Generalizing, if B then S_1 else S_2

① $B \Rightarrow S_1 \land \neg B \Rightarrow S_2$

A conditional is a control point that results in alternative execution paths. This has some subtle issues. We will do it in two steps.

Example: If it rains then I will play chess else I will shop.

1. It rains $\Rightarrow$ I will play chess

AND

It doesn't rain $\Rightarrow$ I will shop

2. It rains $\wedge$ I play chess

Its doesn't rain $\wedge$ I shop

Generalizing, if B then S_1 else S_2

1. $B \Rightarrow S_1 \wedge \neg B \Rightarrow S_2$

A conditional is a control point that results in alternative execution paths. This has some subtle issues. We will do it in two steps.

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   AND / OR?

   Its doesn't rain $\wedge$ I shop

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2. It rains $\wedge$ I play chess

    OR

    Its doesn't rain $\wedge$ I shop

Generalizing, if B then S_1 else S_2

1. $B \Rightarrow S_1 \wedge \neg B \Rightarrow S_2$

A conditional is a control point that results in alternative execution paths. This has some subtle issues. We will do it in two steps.

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   AND

   It doesn't rain $\Rightarrow$ I will shop

2. It rains $\wedge$ I play chess

   OR

   Its doesn't rain $\wedge$ I shop

Generalizing, if B then S_1 else S_2

1. $B \Rightarrow S_1 \wedge \neg B \Rightarrow S_2$
2. $B \wedge S_1 \vee \neg B \wedge S_2$

A conditional is a control point that results in alternative execution paths. This has some subtle issues. We will do it in two steps.

Example: If it rains then I will play chess else I will shop.

1. It rains $\Rightarrow$ I will play chess

    AND

    It doesn't rain $\Rightarrow$ I will shop

2. It rains $\wedge$ I play chess

    OR

    Its doesn't rain $\wedge$ I shop

Generalizing, if B then S_1 else S_2

1. $B \Rightarrow S_1 \wedge \neg B \Rightarrow S_2$
2. $B \wedge S_1 \vee \neg B \wedge S_2$

wp (if B then S_1 else S_2, Q)

A conditional is a control point that results in alternative execution paths. This has some subtle issues. We will do it in two steps.

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$\qquad$ AND

It doesn't rain $\Rightarrow$ I will shop

② It rains $\wedge$ I play chess

$\qquad$ OR

Its doesn't rain $\wedge$ I shop

Generalizing, if B then S_1 else S_2

① $B \Rightarrow S_1 \wedge \neg B \Rightarrow S_2$

② $B \wedge S_1 \vee \neg B \wedge S_2$

wp (if B then S_1 else S_2,Q)

$= \quad B \Rightarrow wp(S_1,Q) \wedge \neg B \Rightarrow wp(S_2,Q)$

$= \quad B \wedge wp(S_1,Q) \vee \neg B \wedge wp(S_2,Q)$

A prelude to conditional branching

A conditional is a control point that results in alternative execution paths. This has some subtle issues. We will do it in two steps.

Example: If it rains then I will play chess else I will shop.

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It doesn't rain $\Rightarrow$ I will shop

2 It rains $\wedge$ I play chess

OR

Its doesn't rain $\wedge$ I shop

Generalizing, if B then S_1 else S_2

1 $B \Rightarrow S_1 \wedge \neg B \Rightarrow S_2$

2 $B \wedge S_1 \vee \neg B \wedge S_2$

wp (if B then S_1 else S_2,Q)

$= \quad B \Rightarrow wp(S_1,Q) \wedge \neg B \Rightarrow wp(S_2,Q)$

$= \quad B \wedge wp(S_1,Q) \vee \neg B \wedge wp(S_2,Q)$

B does not change state. Hence, wp(B,Q) is not required.

4. Conditional branching

We will use: if B then S_1 else S_2 = $B \wedge wp(S_1, Q) \vee \neg B \wedge wp(S_2, Q)$

P $\quad\quad\quad y > 1$

S_1
```
if y < 0 then
    x = y + 1
else
    x = y − 1
```
S_2

Q $\quad\quad\quad x > 0$

We will use: if B then S_1 else S_2 = B $\wedge$ wp(S_1,Q) $\vee$ $\neg$B $\wedge$ wp(S_2,Q)

P $\qquad\qquad$ $y > 1$

S_1

if $y < 0$ then
$\qquad$ $x = y + 1$
else
$\qquad$ $x = y - 1$

S_2

Q $\qquad\qquad$ $x > 0$

$\qquad$ wp(if y<0 then x=y+1 else x=y−1,x>0)

We will use: if B then S_1 else S_2 = B $\wedge$ wp(S_1,Q) $\vee$ $\neg$B $\wedge$ wp(S_2,Q)

P $y > 1$

S_1
S_2
```
if y < 0 then
    x = y + 1
else
    x = y - 1
```

Q $x > 0$

wp(if y<0 then x=y+1 else x=y−1,x>0)

          = y<0 $\wedge$ wp(x=y+1,x>0) $\vee$ $\neg$(y<0) $\wedge$ wp(x=y−1,x>0)

We will use: if B then S_1 else S_2 = B $\wedge$ wp(S_1,Q) $\vee$ $\neg$B $\wedge$ wp(S_2,Q)

P $y > 1$

S_1
```
if y < 0 then
    x = y + 1
else
    x = y − 1
```
S_2

Q $x > 0$

wp(if y<0 then x=y+1 else x=y−1,x>0)

= y<0 $\wedge$ wp(x=y+1,x>0) $\vee$ $\neg$(y<0) $\wedge$ wp(x=y−1,x>0)

= y<0 $\wedge$ y+1>0 $\vee$ y≥0 $\wedge$ y−1>0

We will use: if B then S_1 else S_2 = $B \wedge wp(S_1,Q) \vee \neg B \wedge wp(S_2,Q)$

P $y > 1$

S_1

```
if y < 0 then
    x = y + 1
else
    x = y − 1
```

S_2

Q $x > 0$

wp(if y<0 then x=y+1 else x=y−1,x>0)

$\quad = y<0 \wedge wp(x=y+1,x>0) \vee \neg(y<0) \wedge wp(x=y-1,x>0)$

$\quad = y<0 \wedge y+1>0 \qquad \vee \qquad y\geq0 \wedge y-1>0$

$\quad = y<0 \wedge y> -1 \qquad \vee \qquad y\geq0 \wedge y>1$

We will use: if B then S_1 else $S_2 = B \wedge wp(S_1,Q) \vee \neg B \wedge wp(S_2,Q)$

P $\qquad y > 1$

S_1

S_2

```
if y < 0 then
    x = y + 1
else
    x = y - 1
```

Q $\qquad x > 0$

$wp($if $y<0$ then $x=y+1$ else $x=y-1, x>0)$

$\qquad = y<0 \wedge wp(x=y+1, x>0) \vee \neg(y<0) \wedge wp(x=y-1, x>0)$

$\qquad = y<0 \wedge y+1>0 \qquad \vee \qquad y\geq0 \wedge y-1>0$

$\qquad = y<0 \wedge y> -1 \qquad \vee \qquad y\geq0 \wedge y>1$

$\qquad = $ FALSE $\qquad\qquad\quad \vee \qquad y>1$

We will use: if B then S_1 else S_2 = B ∧ wp(S_1,Q) ∨ ¬B ∧ wp(S_2,Q)

P $y > 1$

S_1

S_2

```
if y < 0 then
    x = y + 1
else
    x = y − 1
```

Q $x > 0$

wp(if y<0 then x=y+1 else x=y−1,x>0)

$\quad\quad$ = y<0 ∧ wp(x=y+1,x>0) ∨ ¬(y<0) ∧ wp(x=y−1,x>0)

$\quad\quad$ = y<0 ∧ y+1>0 $\quad\quad\quad$ ∨ $\quad\quad$ y≥0 ∧ y−1>0

$\quad\quad$ = y<0 ∧ y> −1 $\quad\quad\quad$ ∨ $\quad\quad$ y≥0 ∧ y>1

$\quad\quad$ = FALSE $\quad\quad\quad\quad\quad$ ∨ $\quad\quad$ y>1

$\quad\quad$ = y>1 $\quad\quad$ = P

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We will use: if B then S_1 else S_2 = B $\wedge$ wp(S_1,Q) $\vee$ $\neg$B $\wedge$ wp(S_2,Q)

P $y > 1$

S_1

S_2

```
if y < 0 then
    x = y + 1
else
    x = y − 1
```

What if we had
if..then without
the else part?

if $y < 0$ then
    $x = y + 1$

Q $x > 0$

wp(if y<0 then x=y+1 else x=y−1,x>0)

$= $ y<0 $\wedge$ wp(x=y+1,x>0) $\vee$ $\neg$(y<0) $\wedge$ wp(x=y−1,x>0)

$= $ y<0 $\wedge$ y+1>0 $\vee$ y≥0 $\wedge$ y−1>0

$= $ y<0 $\wedge$ y> −1 $\vee$ y≥0 $\wedge$ y>1

$= $ FALSE $\vee$ y>1

$= $ y>1 $= P$

We will use: if B then S_1 else S_2 = $B \wedge wp(S_1,Q) \vee \neg B \wedge wp(S_2,Q)$

P $y > 1$

S$_1$ if y < 0 then
             x = y + 1
S$_2$ else
             x = y − 1

What if we had
if..then without
the else part?

As good as $\longrightarrow$

if y < 0 then
    x = y + 1
else
    x = x

Q $x > 0$

wp(if y<0 then x=y+1 else x=y−1,x>0)

= y<0 ∧ wp(x=y+1,x>0) ∨ ¬(y<0) ∧ wp(x=y−1,x>0)

= y<0 ∧ y+1>0   ∨   y≥0 ∧ y−1>0

= y<0 ∧ y> −1   ∨   y≥0 ∧ y>1

= FALSE   ∨   y>1

= y>1   = P

To conclude, by combining skip, assignment, sequence and conditional branching statements we can construct a wide variety of programs.

- We can hence deduce weakest precondition P for a program given Q.

Steps to prove program correctness

- Let S be the program.
  Let P' be the input condition.
  Let Q' be the output condition.

  Theorem: Does P' $\Rightarrow$ Q'?

  Proof: Let postcondition Q = Q'

  1. Obtain P = wp(S,Q). i.e. P $\Rightarrow$ Q
  2. Check P' $\Rightarrow$ P
     Therefore P' $\Rightarrow$ P $\Rightarrow$ Q = Q'

Recall previous example:
Let S: z = y * 2; x = y + 5;
Let Q': x > 6

1. Given P': z < 0
   We obtained P: z > 1
   Does z < 0 $\Rightarrow$ z > 1? No

2. Given P': z > 5
   We obtained P: z > 1
   Does z > 5 $\Rightarrow$ z > 1? Yes

Note: We will cover loops later as it involves some intricacies.