Time Series Analysis

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Outline of the lecture

- Regression based methods, 2nd part:
  - Global trend models (Sec. 3.4)
- R examples

Trend models

- Linear regression model
- Functions of time are taken as the independent variables

$$Y_{N+j} = f^T(j)\boldsymbol{\theta} + \varepsilon_{N+j}$$

Linear trend - A motivation

- Observations for $t = 1, \ldots, N$. Naive formulation of the model: $Y_t = \phi_0 + \phi_1\, t + \varepsilon_t$
- If we want to forecast Y_{N+j} given information up to N we use $\widehat{Y}_{N+j|N} = \hat{\phi}_0 + \hat{\phi}_1\,(N + j)$
- However, for on-line applications $N + j$ can be arbitrary large
- The problem arise because ϕ_0 and ϕ_1 are defined w.r.t. the origin 0

- Defining the parameters w.r.t. the origin N we obtain the model: $Y_t = \theta_0 + \theta_1\,(t - N) + \varepsilon_t$
- Using this formulation we get: $\widehat{Y}_{N+j|N} = \hat{\theta}_0 + \hat{\theta}_1\, j$
- Same model, different parameterisation.

Linear trend in a general setting

- The general trend model:
$$Y_{N+j} = \boldsymbol{f}^T(j)\boldsymbol{\theta} + \varepsilon_{N+j}$$
- The linear trend model is obtained when: $\boldsymbol{f}(j) = \begin{pmatrix} 1 \\ j \end{pmatrix}$
- It follows that for $N+1+j$:
$$Y_{N+1+j} = \begin{pmatrix} 1 \\ j+1 \end{pmatrix}^T \boldsymbol{\theta} + \varepsilon_{N+1+j} = \left(\underbrace{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}}_{\boldsymbol{L}} \begin{pmatrix} 1 \\ j \end{pmatrix} \right)^T \boldsymbol{\theta} + \varepsilon_{N+1+j}$$
- The 2×2 matrix $\boldsymbol{L}$ defines the transition from $\boldsymbol{f}(j)$ to $\boldsymbol{f}(j+1)$

Trend models in general

- Model: $Y_{N+j} = \boldsymbol{f}^T(j)\boldsymbol{\theta} + \varepsilon_{N+j}$
- Requirement: $\boldsymbol{f}(j+1) = \boldsymbol{L}\boldsymbol{f}(j)$
- Initial value: $\boldsymbol{f}(0)$
- In Section 3.4 some trend models which fulfill the requirement above are listed.
  - Constant mean: $Y_{N+j} = \theta_0 + \varepsilon_{N+j}$
  - Linear trend: $Y_{N+j} = \theta_0 + \theta_1 j + \varepsilon_{N+j}$
  - Quadratic trend: $Y_{N+j} = \theta_0 + \theta_1 j + \theta_2 \frac{j^2}{2} + \varepsilon_{n+j}$
  - k'th order polynomial trend: $Y_{n+j} = \theta_0 + \theta_1 j + \theta_2 \frac{j^2}{2} + \cdots + \theta_k \frac{j^k}{k!} + \varepsilon_{N+j}$
  - Harmonic model with the period p: $Y_{N+j} = \theta_0 + \theta_1 \sin \frac{2\pi}{p} j + \theta_2 \cos \frac{2\pi}{p} j + \varepsilon_{N+j}$

Estimation

- Model equations written for all observations $Y_1, \dots, Y_N$

$$
\begin{array}{ccc}
\boldsymbol{Y_N} = & \boldsymbol{x}_N \boldsymbol{\theta}_N + & \boldsymbol{\varepsilon} \\
\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_N \end{bmatrix} = & \begin{bmatrix} \boldsymbol{f}^T(-N+1) \\ \boldsymbol{f}^T(-N+2) \\ \vdots \\ \boldsymbol{f}^T(0) \end{bmatrix} \boldsymbol{\theta}_N + & \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_N \end{bmatrix}
\end{array}
$$

- OLS-estimates:

$$\widehat{\boldsymbol{\theta}}_N = (\boldsymbol{x}_N^T \boldsymbol{x}_N)^{-1} \boldsymbol{x}_N^T \boldsymbol{Y}_N = \boldsymbol{F}_N^{-1} \boldsymbol{h}_N$$

$$\boldsymbol{F}_N = \boldsymbol{x}_N^T \boldsymbol{x}_N = \sum_{j=0}^{N-1} \boldsymbol{f}(-j) \boldsymbol{f}^T(-j)$$

$$\boldsymbol{h}_N = \boldsymbol{x}_N^T \boldsymbol{Y} = \sum_{j=0}^{N-1} \boldsymbol{f}(-j)\, Y_{N-j}$$

ℓ-step prediction

- Prediction:

$$\widehat{Y}_{N+\ell|N} = \boldsymbol{f}^{T}(\ell)\widehat{\boldsymbol{\theta}}_N$$

- Variance of the prediction error:

$$V[Y_{N+\ell} - \widehat{Y}_{N+\ell|N}] = \sigma^2 \left[1 + \boldsymbol{f}^{T}(\ell)\boldsymbol{F}_N^{-1}\boldsymbol{f}(\ell)\right]$$

- $100(1-\alpha)\%$ prediction interval:

$$\begin{aligned} &\widehat{Y}_{N+\ell|N} \pm \mathrm{t}_{\alpha/2}(N-p)\sqrt{V[e_N(\ell)]} \\ =&\widehat{Y}_{N+\ell|N} \pm \mathrm{t}_{\alpha/2}(N-p)\widehat{\sigma}\sqrt{1 + \boldsymbol{f}^{T}(\ell)\boldsymbol{F}_N^{-1}\boldsymbol{f}(\ell)} \end{aligned}$$

where $\hat{\sigma}^2 = \boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon}/(N-p)$ (p is the number of estimated parameters)

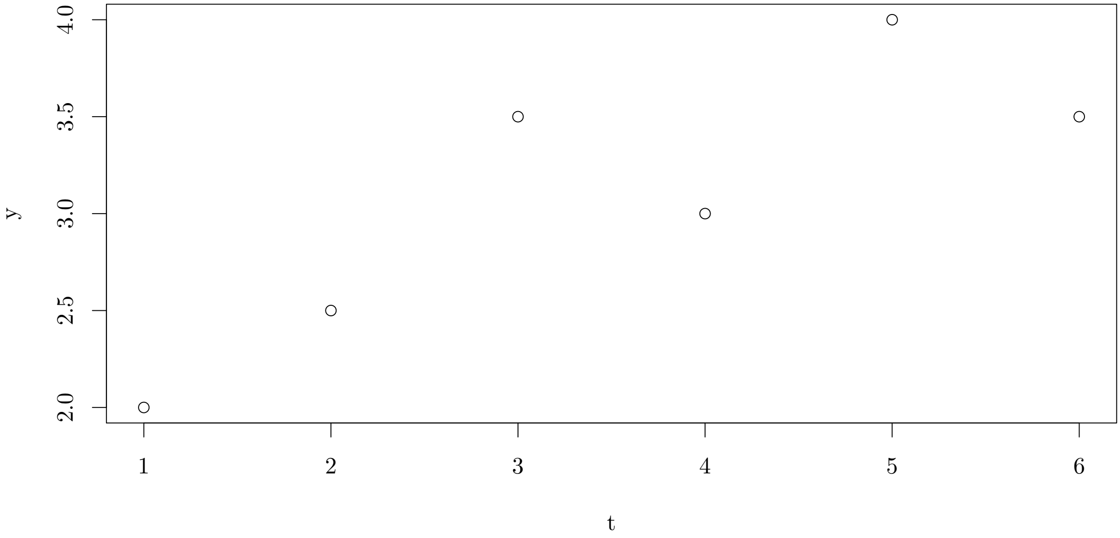
Updating the estimates when Y_{N+1} is available

- Task:
  - Going from estimates based on $t = 1, \ldots, N$, i.e. $\widehat{\boldsymbol{\theta}}_N$ to
  - estimates based on $t = 1, \ldots, N, N+1$, i.e. $\widehat{\boldsymbol{\theta}}_{N+1}$
  - without redoing everything...
- Solution:

$$
\begin{aligned}
\boldsymbol{F}_{N+1} &= \boldsymbol{F}_N + \boldsymbol{f}(-N)\boldsymbol{f}^T(-N) \\
\boldsymbol{h}_{N+1} &= \boldsymbol{L}^{-1}\boldsymbol{h}_N + \boldsymbol{f}(0)\,Y_{N+1} \\
\widehat{\boldsymbol{\theta}}_{N+1} &= \boldsymbol{F}_{N+1}^{-1}\boldsymbol{h}_{N+1}
\end{aligned}
$$

Local Trend Models - an Example

6 observations ($N = 6$):



Global Linear Trend:

$$Y_{N+j} = \theta_0 + \theta_1 j + \varepsilon_{N+j} \Rightarrow \boldsymbol{f}(j) = \begin{pmatrix} 1 & j \end{pmatrix}^T$$

Linear Model form:

$$\begin{pmatrix} 2.0 \\ 2.5 \\ 3.5 \\ 3.0 \\ 4.0 \\ 3.5 \end{pmatrix} = \begin{pmatrix} 1 & -5 \\ 1 & -4 \\ 1 & -3 \\ 1 & -2 \\ 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \theta_0 \\ \theta_1 \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{pmatrix} \Leftrightarrow \boldsymbol{y} = \boldsymbol{x}_6 \theta + \varepsilon$$

Global linear trend: Estimation

$$
\begin{aligned}
\boldsymbol{F}_6 &= \boldsymbol{x}_6^T \boldsymbol{x}_6 = \begin{pmatrix} 6 & -15 \\ -15 & 55 \end{pmatrix} \\
\boldsymbol{h}_6 &= \boldsymbol{x}_6^T \boldsymbol{y} = \begin{pmatrix} 18.5 \\ -40.5 \end{pmatrix} \\
\widehat{\boldsymbol{\theta}}_6 &= \boldsymbol{F}_6^{-1} \boldsymbol{h}_6 = \begin{pmatrix} 0.5238 & 0.1429 \\ 0.1429 & 0.0571 \end{pmatrix} \begin{pmatrix} 18.5 \\ -40.5 \end{pmatrix} = \begin{pmatrix} 3.905 \\ 0.329 \end{pmatrix}
\end{aligned}
$$

Global linear trend: Estimation with R

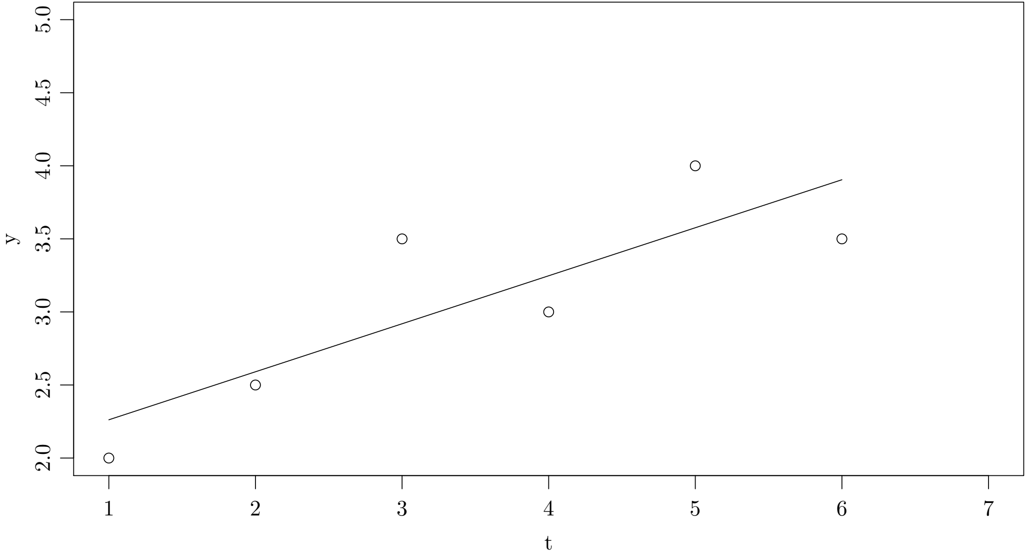
```
F6 <- t(x)%*%x
h6 <- t(x)%*%y
(th.hat6 <- solve(F6, h6))

##            [,1]
## [1,] 3.9047619
## [2,] 0.3285714

## check
(lm(y~0+x))

##
## Call:
## lm(formula = y ~ 0 + x)
##
## Coefficients:
##     x1      x2
## 3.9048  0.3286
```

Global linear trend: Estimation - global linear trend



Global linear trend: Prediction

Linear predictor:

$$\widehat{Y}_{6+\ell|6} = f(\ell)^T \widehat{\theta}_6 = 3.905 + 0.328\ell$$

LS-estimate for σ^2:

$$\begin{aligned}
\widehat{\sigma}^2 &= (\boldsymbol{y} - \boldsymbol{x}_6 \widehat{\theta}_6)^T (\boldsymbol{y} - \boldsymbol{x}_6 \widehat{\theta}_6)/(6-2) \\
&= \frac{(-0.262)^2 + 0.090^2 + 0.581^2 + (-0.248)^2 + 0.424^2 + (-0.405)^2}{4} \\
&= 0.453^2
\end{aligned}$$

Global linear trend: Prediction Error

$$\varepsilon_6(\ell) = Y_{6+\ell} - \widehat{Y}_{6+\ell|6}$$

$$\begin{aligned}
\widehat{\mathrm{Var}}(\varepsilon_6(\ell)) &= \widehat{\sigma}^2 \left(1 + \boldsymbol{f}^T(\ell)\boldsymbol{F}_6^{-1}\boldsymbol{f}(\ell)\right) \\
&= 0.453^2 \left(1 + \begin{pmatrix} 1 & \ell \end{pmatrix} \begin{pmatrix} 0.5238 & 0.1429 \\ 0.1429 & 0.0571 \end{pmatrix} \begin{pmatrix} 1 \\ \ell \end{pmatrix}\right) \\
&= 0.453^2(1.5238 + 0.2858\ell + 0.0571\ell^2)
\end{aligned}$$

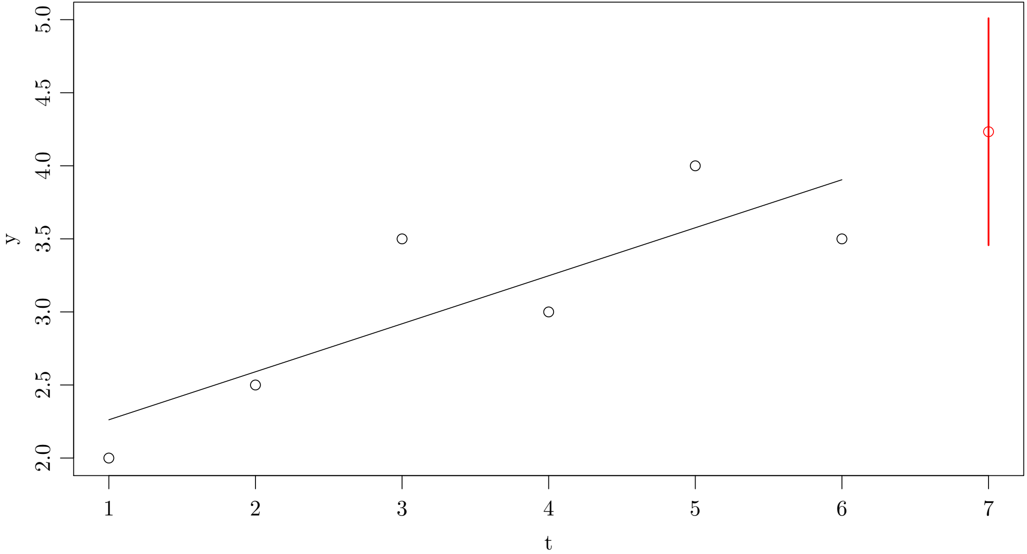
For example,

$$\widehat{Y}_{7|6} = 4.234 \text{ with } \widehat{\mathrm{Var}}(\varepsilon_6(1)) = 0.619^2.$$

90% prediction interval:

$$\widehat{Y}_{7|6} \pm t_{0.05}(6-2)\sqrt{\widehat{\mathrm{Var}}(\varepsilon_6(1))} = 4.234 \pm 1.320$$

Global linear trend: Estimation - global linear trend



Global linear trend: Updating the parameters

- New observation: $y_7 = 3.5$.

$$
\begin{aligned}
\boldsymbol{F}_7 &= \boldsymbol{F}_6 + \boldsymbol{f}(-6)\boldsymbol{f}^T(-6) \\
&= \begin{pmatrix} 6 & -15 \\ -15 & 55 \end{pmatrix} + \begin{pmatrix} 1 \\ -6 \end{pmatrix} \begin{pmatrix} 1 & -6 \end{pmatrix} = \begin{pmatrix} 7 & -21 \\ -21 & 91 \end{pmatrix}, \\
\boldsymbol{h}_7 &= L^{-1}\boldsymbol{h}_6 + \boldsymbol{f}(0)\boldsymbol{y}_7 \\
&= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 18.5 \\ -40.5 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} 3.5 = \begin{pmatrix} 22 \\ -59 \end{pmatrix}, \\
\widehat{\boldsymbol{\theta}}_7 &= \begin{pmatrix} 0.4643 & 0.1071 \\ 0.1071 & 0.0357 \end{pmatrix} \begin{pmatrix} 22 \\ -59 \end{pmatrix} = \begin{pmatrix} 3.896 \\ 0.250 \end{pmatrix}.
\end{aligned}
$$