Time Series Analysis

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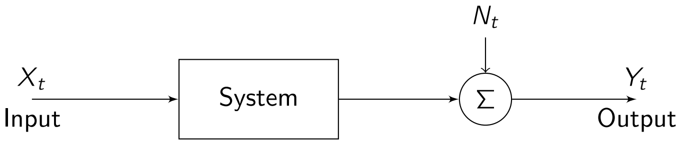
Outline of the lecture

- Chapter 9 – Multivariate time series

Multiple output models

Re-consider the univariate transfer function model:



$$Y_t = h(B)X_t + N_t$$

- What if there is a feedback from Y to X?

Closed Loop Models

$$Y_t = h_1(B)X_t + N_{1,t}$$
$$X_t = h_2(B)Y_t + N_{2,t}$$

Or:

$$\begin{pmatrix} 1 & -h_1(B) \\ -h_2(B) & 1 \end{pmatrix} \begin{pmatrix} Y_t \\ X_t \end{pmatrix} = \begin{pmatrix} N_{1,t} \\ N_{2,t} \end{pmatrix}$$

- Two inputs (N_1, N_2);
- Two outputs (Y, X);
- Four transfer functions from input to output.

We will look at them individually first.

Transfer from N_1, N_2 to Y:

$$Y = h_1(B)(N_2 + h_2(B)Y) + N_1$$

Z-domain:

$$Y(z) = H_1(z)(N_2(z) + H_2(z)Y(z)) + N_1(z)$$

solving for $Y(z)$:

$$Y(z) = \frac{1}{1 - H_1(z)H_2(z)} N_1(z) + \frac{H_1(z)}{1 - H_1(z)H_2(z)} N_2(z)$$

Transfer functions from N_1, N_2 to Y:

$$\frac{1}{1 - H_1(z)H_2(z)}$$

and

$$\frac{H_1(z)}{1 - H_1(z)H_2(z)}$$

Transfer from N_1, N_2 to X:

$$X = h_2(B)(N_1 + h_1(B)X) + N_2$$

Z-domain:

$$X(z) = H_2(z)(N_1(z) + H_1(z)X(z)) + N_2(z)$$

solving for $X(z)$:

$$X(z) = \frac{1}{1 - H_1(z)H_2(z)} N_2(z) + \frac{H_2(z)}{1 - H_1(z)H_2(z)} N_1(z)$$

Transfer functions from N_1, N_2 to X:

$$\frac{H_2(z)}{1 - H_1(z)H_2(z)}$$

and

$$\frac{1}{1 - H_1(z)H_2(z)}$$

Multivariate transfer function

Model equation:

$$\begin{pmatrix} 1 & -h_1(B) \\ -h_2(B) & 1 \end{pmatrix} \begin{pmatrix} Y_t \\ X_t \end{pmatrix} = \begin{pmatrix} N_{1,t} \\ N_{2,t} \end{pmatrix}$$

Model equation in Z-domain:

$$\begin{pmatrix} 1 & -H_1(z) \\ -H_2(z) & 1 \end{pmatrix} \begin{pmatrix} Y(z) \\ X(z) \end{pmatrix} = \begin{pmatrix} N_1(z) \\ N_2(z) \end{pmatrix}$$

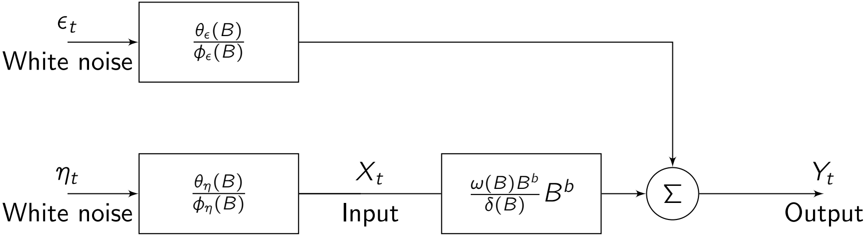
Thus,

$$\begin{pmatrix} Y(z) \\ X(z) \end{pmatrix} = \frac{1}{1 - H_1(z)H_2(z)} \begin{pmatrix} 1 & H_1(z) \\ H_2(z) & 1 \end{pmatrix} \begin{pmatrix} N_1(z) \\ N_2(z) \end{pmatrix}$$

Multivariate transfer function:

$$\begin{pmatrix} \frac{1}{1-H_1(z)H_2(z)} & \frac{H_1(z)}{1-H_1(z)H_2(z)} \\ \frac{H_2(z)}{1-H_1(z)H_2(z)} & \frac{1}{1-H_1(z)H_2(z)} \end{pmatrix}$$

Univ. Transfer function models with ARMA input



$$Y_t = \frac{\omega(B)}{\delta(B)} B^b X_t + \frac{\theta_\varepsilon(B)}{\varphi_\varepsilon(B)} \varepsilon_t$$

$$X_t = \frac{\theta_\eta(B)}{\varphi_\eta(B)} \eta_t$$

we require $\{\varepsilon_t\}$ and $\{\eta_t\}$ to be mutually uncorrelated.

Univ. Transfer function models with ARMA input continued...

$$
\begin{aligned}
Y_t &= \frac{\omega(B)}{\delta(B)} B^b X_t + \frac{\theta_\varepsilon(B)}{\varphi_\varepsilon(B)} \varepsilon_t \\
X_t &= \frac{\theta_\eta(B)}{\varphi_\eta(B)} \eta_t
\end{aligned}
$$

which leads to:

$$
\begin{aligned}
\delta(B)\varphi_\varepsilon(B) Y_t &= \varphi_\varepsilon(B)\omega(B) B^b X_t + \delta(B)\theta_\varepsilon(B)\varepsilon_t \\
\varphi_\eta(B) X_t &= \theta_\eta(B)\eta_t
\end{aligned}
$$

The term including X_t on the RHS is moved to the LHS:

$$
\begin{aligned}
\delta(B)\varphi_\varepsilon(B) Y_t - \varphi_\varepsilon(B)\omega(B) B^b X_t &= \delta(B)\theta_\varepsilon(B)\varepsilon_t \\
\varphi_\eta(B) X_t &= \theta_\eta(B)\eta_t
\end{aligned}
$$

This can be written in matrix notation...

Univ. Transfer function models with ARMA input continued...

from previous slide:

$$\begin{aligned} \delta(B)\varphi_\varepsilon(B)Y_t - \varphi_\varepsilon(B)\omega(B)B^bX_t &= \delta(B)\theta_\varepsilon(B)\varepsilon_t \\ \varphi_\eta(B)X_t &= \theta_\eta(B)\eta_t \end{aligned}$$

Is equivalent to

$$\begin{bmatrix} \delta(B)\varphi_\varepsilon(B) & -\varphi_\varepsilon(B)\omega(B)B^b \\ 0 & \varphi_\eta(B) \end{bmatrix} \begin{bmatrix} Y_t \\ X_t \end{bmatrix} = \begin{bmatrix} \delta(B)\theta_\varepsilon(B) & 0 \\ 0 & \theta_\eta(B) \end{bmatrix} \begin{bmatrix} \varepsilon_t \\ \eta_t \end{bmatrix}$$

For multivariate ARMA-models in general:

- Replace the off diagonal zeroes by polynomials in B.
- This introduces feedback from Y to X or reverse
- Non-zero correlation between ε_t and η_t

Multivariate ARMA models

- The model can be written
$$\boldsymbol{\phi}(B)(\boldsymbol{Y}_t - \boldsymbol{c}) = \boldsymbol{\theta}(B)\boldsymbol{\epsilon}_t$$
- The individual time series may have been transformed and differenced
- The variance-covariance matrix of the multivariate white noise process $\{\boldsymbol{\epsilon}_t\}$ is denoted $\boldsymbol{\Sigma}$.
- The matrices $\boldsymbol{\phi}(B)$ and $\boldsymbol{\theta}(B)$ have elements which are polynomials in the backshift operator
- The diagonal elements have leading terms of unity
- The off-diagonal elements have leading terms of zero (i.e. they normally start in B)

Air pollution in cities NO and NO_2

$$\begin{bmatrix} X_{1,t} \\ X_{2,t} \end{bmatrix} = \begin{bmatrix} 0.9 & -0.1 \\ 0.4 & 0.8 \end{bmatrix} \begin{bmatrix} X_{1,t-1} \\ X_{2,t-1} \end{bmatrix} + \begin{bmatrix} \xi_{1,t} \\ \xi_{2,t} \end{bmatrix}, \quad \boldsymbol{\Sigma} = \begin{bmatrix} 30 & 21 \\ 21 & 23 \end{bmatrix}$$

Matrix formulation:

$$\boldsymbol{X}_t - \begin{bmatrix} 0.9 & -0.1 \\ 0.4 & 0.8 \end{bmatrix} \boldsymbol{X}_{t-1} = \boldsymbol{\xi}_t \quad \text{or} \quad \boldsymbol{X}_t - \boldsymbol{\phi}_1 \boldsymbol{X}_{t-1} = \boldsymbol{\xi}_t$$

Matrix formulation using the backshift operator:

$$\begin{bmatrix} 1 - 0.9B & 0.1B \\ -0.4B & 1 - 0.8B \end{bmatrix} \boldsymbol{X}_t = \boldsymbol{\xi}_t \quad \text{or} \quad \boldsymbol{\phi}(B) \boldsymbol{X}_t = \boldsymbol{\xi}_t$$

Stationarity and Invertability

The multivariate ARMA process

$$\boldsymbol{\phi}(B)(\boldsymbol{Y}_t - \boldsymbol{c}) = \boldsymbol{\theta}(B)\boldsymbol{\epsilon}_t$$

is stationary if

$$\det(\boldsymbol{\phi}(z^{-1})) = 0 \quad \Rightarrow \quad |z| < 1$$

is invertible if

$$\det(\boldsymbol{\theta}(z^{-1})) = 0 \quad \Rightarrow \quad |z| < 1$$

Two formulations (centered data)

Either matrices with polynomials in B as elements:

$$\boldsymbol{\phi}(B)\boldsymbol{Y}_t = \boldsymbol{\theta}(B)\boldsymbol{\epsilon}_t$$

or without B, but with matrices as coefficients:

$$\boldsymbol{Y}_t + \boldsymbol{\phi}_1\boldsymbol{Y}_{t-1} + \ldots + \boldsymbol{\phi}_p\boldsymbol{Y}_{t-p} = \boldsymbol{\epsilon}_t + \boldsymbol{\theta}_1\boldsymbol{\epsilon}_{t-1} + \ldots + \boldsymbol{\theta}_q\boldsymbol{\epsilon}_{t-q}$$

Auto Covariance Matrix Functions

$$\boldsymbol{\Gamma}_k = E[(\boldsymbol{Y}_{t-k} - \boldsymbol{\mu}_Y)(\boldsymbol{Y}_t - \boldsymbol{\mu}_Y)^T] = \boldsymbol{\Gamma}_{-k}^T$$

Example for bivariate case $\boldsymbol{Y}_t = (Y_{1,t}\ Y_{2,t})^T$:

$$\boldsymbol{\Gamma}_k = \begin{bmatrix} \gamma_{11}(k) & \gamma_{12}(k) \\ \gamma_{21}(k) & \gamma_{22}(k) \end{bmatrix} = \begin{bmatrix} \gamma_{11}(k) & \gamma_{12}(k) \\ \gamma_{12}(-k) & \gamma_{22}(k) \end{bmatrix}$$

We can describe these by plotting

- each autocovariance or autocorrelation function for $k = 0, 1, 2, \ldots$ and
- each cross-covariance or cross-correlation function for $k = 0, \pm 1, \pm 2, \ldots$

The Theoretical Autocovariance Matrix Functions

Using the matrix coefficients $\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_p$ and $\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_q$, together with $\boldsymbol{\Sigma}$, the theoretical $\boldsymbol{\Gamma}_k$ can be calculated:

Pure Autoregressive Models: $\boldsymbol{\Gamma}_k$ is found from a multivariate version of Theorem 5.10 in the book, which leads to the Yule-Walker equations

Pure Moving Average Models: $\boldsymbol{\Gamma}_k$ is found from a multivariate version of (5.65) in the book

Autoregressive Moving Average Models: $\boldsymbol{\Gamma}_k$ is found multivariate versions of (5.100) and (5.101) in the book

- Examples can be found in the book. (Page 255++)

Autocorrelation for VAR Models

VAR: Vector Auto Regressive - Multivariate AR

$$
\begin{aligned}
\phi(B)Y =& \varepsilon \\
Y_t =& -\phi_1 Y_{t-1} - \cdots - \phi_p Y_{t-p} + \varepsilon_t \\
Y_t Y_t^T =& -Y_t Y_{t-1}^T \phi_1^T - \ldots - Y_t Y_{t-p}^T \phi_p^T + Y_t \varepsilon_t^T \\
\Gamma(0) =& -\Gamma(-1)\phi_1^T - \ldots - \Gamma(-p)\phi_p^T + \Sigma \\
=& -\phi_1 \Gamma(-1)^T - \ldots - \phi_p \Gamma(-p)^T + \Sigma \\
Y_{t-k} Y_t^T =& -Y_{t-k} Y_{t-1}^T \phi_1^T - \cdots - Y_{t-k} Y_{t-p}^T \phi_p^T + Y_{t-k} \varepsilon_t^T \\
\Gamma(k) =& -\Gamma(k-1)\phi_1^T - \ldots - \Gamma(k-p)\phi_p^T
\end{aligned}
$$

Multivariate Yule-Walker equations

Y is Vector $AR(k)$ ($VAR(k)$):

$$Y_t = \phi_1 Y_{t-1} + \cdots + \phi_k Y_{t-k} + \varepsilon_t$$

$$\begin{pmatrix} \Gamma(0) & \Gamma(1)^T & \cdots & \Gamma(k-1)^T \\ \Gamma(1) & \Gamma(0) & \cdots & \Gamma(k-2)^T \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma(k-1) & \Gamma(k-2) & \cdots & \Gamma(0) \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_k \end{pmatrix} = \begin{pmatrix} \Gamma(1) \\ \Gamma(2) \\ \vdots \\ \Gamma(k) \end{pmatrix}$$

VAR(1) representation of VARMA processes

Just as in the univariate case, ARMA models may be written as VAR(1) models through stacking:

$$Y_t + \phi_1 Y_{t-1} + \cdots + \phi_p Y_{t-p} = \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q}$$

may for $p \geq q + 1$ be written as

$$\begin{pmatrix} Z_{1,t} \\ Z_{2,t} \\ \cdots \\ Z_{p,t} \end{pmatrix} = \begin{pmatrix} -\phi_1 & I & 0 & \cdots & 0 \\ -\phi_2 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & I \\ -\phi_p & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} Z_{1,t-1} \\ Z_{2,t-1} \\ \cdots \\ Z_{p,t-1} \end{pmatrix} + \begin{pmatrix} I \\ \theta_1 \\ \vdots \\ \theta_{p-1} \end{pmatrix} \varepsilon_t$$

with $Z_{1,t} = Y_t$.

Identification using Autocovariance Matrix Functions

Sample Correlation Matrix Function; $\boldsymbol{R}_k$ near zero for pure moving average processes of order q when $k > q$

Sample Partial Correlation Matrix Function; $\boldsymbol{S}_k$ near zero for pure autoregressive processes of order p when $k > p$

Sample q-conditioned Partial Correlation Matrix Function; $\boldsymbol{S}_k(q)$ near zero for autoregressive moving average processes of order (p, q) when $k > p$ – can be used for univariate processes also. Not so useful in practice.

Identification using (multivariate) prewhitening

- Fit univariate models to each individual series
- Investigate the residuals as a multivariate time series
- The cross correlations can then be compared with $\pm 2/\sqrt{N}$

This is **not** the same form of prewhitening as in Chapter 8

The multivariate model $\boldsymbol{\phi}(B)Y_t = \boldsymbol{\theta}(B)\epsilon_t$ is equivalent to

$$\mathbf{diag}(\det(\boldsymbol{\phi}(B)))Y_t = \mathbf{adj}(\boldsymbol{\phi}(B))\boldsymbol{\theta}(B)\epsilon_t$$

Therefore the corresponding univariate models will have much higher order, so although this is often done in the literature: Don't take this approach!

Multivariate ARMA(p,q) processes (centered data)

- Matrices with polynomials in B as elements:

$$\boldsymbol{\phi}(B)\boldsymbol{Y}_t = \boldsymbol{\theta}(B)\boldsymbol{\epsilon}_t$$

  So the coefficients are now matrices:

$$\boldsymbol{Y}_t + \boldsymbol{\phi}_1\boldsymbol{Y}_{t-1} + \ldots + \boldsymbol{\phi}_p\boldsymbol{Y}_{t-p} = \boldsymbol{\epsilon}_t + \boldsymbol{\theta}_1\boldsymbol{\epsilon}_{t-1} + \ldots + \boldsymbol{\theta}_q\boldsymbol{\epsilon}_{t-q}$$

- In general, no analytic solution exits.
- Therefore, estimation algorithms (or numerical optimization) is necessary.

Estimation procedures

For multivariate ARX(p)

- Least squares estimation is possible

For multivariate ARMAX(p,q)

- The Spliid method (Henrik Spliid, 1983)
- Maximum likelihood

See the book for details.

Highlights

- Closed loop model as multivariate transfer function

$$\begin{pmatrix} 1 & -h_1(B) \\ -h_2(B) & 1 \end{pmatrix} \begin{pmatrix} Y_t \\ X_t \end{pmatrix} = \begin{pmatrix} N_{1,t} \\ N_{2,t} \end{pmatrix}$$

- Multivariate ARMA models

$$\boldsymbol{\phi}(B)(\boldsymbol{Y}_t - \boldsymbol{c}) = \boldsymbol{\theta}(B)\boldsymbol{\epsilon}_t$$

  is stationary if

$$\det(\boldsymbol{\phi}(z^{-1})) = 0 \quad \Rightarrow \quad |z| < 1$$

  is invertible if

$$\det(\boldsymbol{\theta}(z^{-1})) = 0 \quad \Rightarrow \quad |z| < 1$$

- Auto covariance matrix functions

$$\boldsymbol{\Gamma}_k = E[(\boldsymbol{Y}_{t-k} - \boldsymbol{\mu}_Y)(\boldsymbol{Y}_t - \boldsymbol{\mu}_Y)^T] = \boldsymbol{\Gamma}_{-k}^T$$

- All VARMA models can be written as VAR(1)