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Time Series: Teaching session III

Sridhar Adhikarla.

Assignment 1

Proove the Kalman filtering recursion for the following state space model with initial prior on the state

$$f(z_1) = N(z_1; m_0, P_0) \quad \text{where } e_t \sim N(0, Q_t) \text{ and } v_t \sim N(0, R_t)$$

$$z_t = A_{t-1} \cdot z_{t-1} + e_t$$

$$x_t = C_t z_t + v_t$$

(1) Particularly show that given $f(z_t \mid x_{1:t}) = N(z_t; m_{t|t}, P_{t|t})$, the predicted density $f(z_{t+1} \mid x_{1:t})$ is given by

$$f(z_{t+1} \mid x_{1:t}) = N(z_{t+1}; A_t m_{t|t}, A_t P_{t|t} A_t^T + Q_{t+1})$$

(2) Also show that given $f(z_t \mid x_{1:t-1}) = N(z_t; m_{t|t-1}, P_{t|t-1})$, the observation updated density $f(z_t \mid x_{1:t})$ is given by

$$f(z_t \mid x_{1:t}) = N(z_t; m_{t|t}, P_{t|t})$$

where:

$$m_{t|t} = m_{t|t-1} + K_t(x_t - C_t \cdot m_{t|t-1})$$

$$P_{t|t} = (I - K_t \cdot C_t) P_{t|t-1}$$

$$K_t = P_{t|t-1} \cdot C_t^T (C_t \cdot P_{t|t-1} C_t^T + R_t)^{-1}$$

③

Given

$\boxed{Q1}$ $f(z_t \mid x_{1:t}) = N(z_t; m_{t|t}, P_{t|t})$

Also from the transition model of the state space model we have..

$$z_t = A_{t-1} \cdot z_{t-1} + e_t \qquad e_t \sim N(0, Q_t)$$

$$\therefore f(z_{t+1} \mid z_t) = N(z_{t+1}; A_t \cdot z_t, Q_{t+1})$$

Joint distribution

$$f(z_{t+1}, z_t \mid x_{1:t}) = f(z_t \mid x_{1:t}) \cdot f(z_{t+1} \mid z_t)$$

$$= N(z_t; m_{t|t}, P_{t|t}) \times N(z_{t+1}; A_t \cdot z_t, Q_{t+1})$$

$$= N\left(\begin{bmatrix} z_t \\ z_{t+1} \end{bmatrix}; \begin{bmatrix} m_{t|t} \\ A_t \cdot m_{t|t} \end{bmatrix}; \begin{bmatrix} P_{t|t} & \cdots \\ \cdots & A_t \cdot P_{t|t} \cdot A_t^T + Q_{t+1} \end{bmatrix} \right)$$

Marginalizing for z_{t+1} we get..

$$\left\{ f(z_{t+1} \mid x_{1:t}) = N(z_{t+1}; A_t \cdot m_{t|t}, A_t \cdot P_{t|t} \cdot A_t^T + Q_{t+1}) \right\}$$

③

$\boxed{Q-2}$ Given

$$f(z_t \mid x_{1:t-1}) = N(z_t; m_{t|t-1}, P_{t|t-1})$$

$$f(z_t \mid x_{1:t}) = \frac{f(z_t, x_t \mid x_{1:t-1})}{f(x_t \mid x_{1:t-1})}$$

$$= \frac{f(z_t \mid x_{1:t-1}) \cdot f(x_t \mid z_t, x_{1:t-1})}{f(x_t \mid x_{1:t-1})}$$

$$= \frac{f(z_t \mid x_{1:t-1}) \cdot f(x_t \mid z_t)}{f(x_t \mid x_{1:t-1})}$$

$$\propto f(z_t \mid x_{1:t-1}) \cdot \underbrace{f(x_t \mid z_t)}_{\text{likelihood function}\ldots}$$

$$\propto N(z_t; m_{t|t-1}, P_{t|t-1}) \cdot \times N(x_t; C_t \cdot z_t, R_t)$$

$$\propto N\left(\begin{bmatrix} z_t \\ x_t \end{bmatrix}; \begin{bmatrix} m_{t|t-1} \\ C_t \cdot m_{t|t-1} \end{bmatrix}, \begin{bmatrix} P_{t|t-1} & P_{t|t-1} \cdot C_t^T \\ C_t \cdot P_{t|t-1} & C_t \cdot P_{t|t-1} C_t^T + R_t \end{bmatrix} \right)$$

using conditioning rules of normal distribution

$$f(z_t \mid x_{1:t}) = N\Big(z_t; \overbrace{m_{t|t-1} + \underbrace{P_{t|t-1} \cdot C_t^T (C_t \cdot P_{t|t-1} \cdot C_t^T + R_t)^{-1}}_{K_t} \cdot (x_t - C_t m_{t|t-1})}^{m_{t|t}},$$

$$\underbrace{P_{t|t-1} - P_{t|t-1} \cdot C_t^T (C_t \cdot P_{t|t-1} \cdot C_t^T + R_t)^{-1} \cdot (C_t \cdot P_{t|t-1})}_{P_{t|t}}\Big)$$

$$\therefore \boxed{f(z_t \mid x_{1:t}) = N(z_t; m_{t|t}, P_{t|t})}$$