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Teaching Session
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Assignment R:-

Let $\{x_t\}$ be a zero-mean, unit-variance stationary process with autocorrelation function $\rho_h$. Suppose that $\mu_t$ is a non-constant function and that $\sigma_t$ is a positive-valued non-constant function. The observed series is formed as

$$Y_t = \mu_t + \sigma_t \cdot X_t$$

(a) Find the mean and covariance function for the process $\{Y_t\}$

Mean:-

$$E[Y_t] = E[\mu_t] + E[\sigma_t \cdot X_t]$$
$$= \mu_t + \sigma_t \cdot E[X_t]$$
$$= \mu_t + 0$$
$$= \mu_t$$

Autocovariance:-

$$\text{cov}(Y_s, Y_t) = \text{cov}((\mu_s + \sigma_s \cdot X_s), (\mu_t + \sigma_t \cdot X_t))$$
$$= E[(\mu_s + \sigma_s X_s - E[\mu_s + \sigma_s X_s]) \cdot (\mu_t + \sigma_t \cdot X_t) - E[\mu_t + \sigma_t \cdot X_t]]$$

We know that $E[\mu_t + \sigma_t \cdot X_t] = \mu_t$ $= E[(\mu_s + \sigma_s \cdot X_s - \mu_s) \cdot (\mu_t + \sigma_t X_t) - \mu_t]$

$$= (\sigma_s \cdot X_s)(\sigma_t \cdot X_t)$$
$$= \sigma_s \cdot \sigma_t \cdot E[X_s \cdot X_t]$$

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We know that $E[X_s] = 0$ so we can add that in the above eq.

$$= \sigma_s \cdot \sigma_t \cdot [E[(X_s - E[X_s])(X_t - E[X_t])]]$$

$$= \sigma_s \cdot \sigma_t \cdot \text{cov}(X_s \cdot X_t) \quad \text{—} \quad 1$$

Now, given auto correlation function $\rho_h$

or assume $s = t - h$

$\therefore$ auto covariance $= \sigma_t \cdot \sigma_{t-h} \cdot \text{cov}(X_t, X_{t-h}) \quad \text{—} \quad 2$

We know: $$\rho(t, t-h) = \frac{\gamma(t, t-h)}{\sqrt{(\gamma(t,t) \cdot \gamma(t-h, t-h))}}$$

Given:

$$\gamma(t,t) = \text{var}(X_t) = 1$$

$$\therefore \quad \rho(t, t-h) = \gamma(t, t-h)$$

Using this in equation 2 we get

autocovariance $= \sigma_t \cdot \sigma_{t-h} \cdot \rho_h$

(3)

(b) Show that the auto correlation function for the $\{y_t\}$ process depends only on the time lag. Is the $\{y_t\}$ process stationary?

Autocorrelation function

$$\rho(s,t) = \frac{\gamma(s,t)}{\sqrt{\gamma(s,s)\cdot\gamma(t,t)}}$$

$$\rho(s,t) = \frac{\gamma(Y_s, Y_t)}{\sqrt{var(Y_s)\, var(Y_t)}}$$

From solution (a) we know that $\gamma(s,t) = cov(Y_s, Y_t) = \sigma_s \cdot \sigma_t \cdot \rho_h$

assume $s = t-h$

$$\gamma(t, t-h) = cov(Y_t, Y_{t-h}) = \sigma_t \cdot \sigma_{t-h} \cdot \sigma_h$$

$$var(Y_t) = var(\mu_t + \sigma_t \cdot X_t)$$

$$= \sigma_t^2 \cdot var(X_t)$$

$$= \sigma_t^2$$

$$\therefore var(Y_{t-h}) = \sigma_{t-h}^2$$

$$\therefore \rho(t, t-h) = \frac{\sigma_t \cdot \sigma_{t-h} \cdot \rho_h}{\sqrt{\sigma_t^2 \cdot \sigma_{t-h}^2}} = \rho_h$$

with this, we can say that the auto correlation function depends only on the lag.

④

ⓒ Is it possible to have a time series with a constant mean and with $cov(Y_t, Y_{t+h})$ free of $t$ but with $\{Y_t\}$ not stationary?

Assuming constant mean $E[Y_t] = E[\mu + \sigma_t . X_t] = \mu$

$$cov(Y_s, Y_t) = cov((\mu + \sigma_s \cdot X_s), (\mu + \sigma_t \cdot X_t))$$

$$= E[(\mu + \sigma_s . X_s - \mu)(\mu + \sigma_t . X_t - \mu)]$$

Assuming $s = t-h$

$$= E[(\sigma_t X_t)(\sigma_{t-h} \cdot X_{t-h})]$$

$$= \sigma_t \cdot \sigma_{t-h} \; cov(X_t \cdot X_{t-h})$$

This is same as equation ① from exercise ⓐ

hence from the above proof.

$$\gamma(h) = \sigma_t \cdot \sigma_{t-h} \cdot \rho_h \quad \text{——} \quad ③$$

Even though the mean is constant, and autocorrelation depends only on lag, the autocovariance depends on time 't' and hence the process is not stationary.

(5)

Assignment 18

For each of the following ARMA models, find the roots of the AR and MA polynomials, identify the values of p and q for which they are ARMA(p, q) (be careful of parameter redundancy), determine wether they are causal and determine whether they are invertible. In each case, $w_t \sim wn(0,1)$

(a) $x_t - 3x_{t-1} = w_t + 2w_{t-1} - 8w_{t-2}$

$\Rightarrow x_t - 3Bx_t = w_t + 2Bw_t - 8B^2 w_t$

$\Rightarrow x_t(1-3B) = w_t(1+2B-8B^2)$

$\Rightarrow \phi(B)x_t = \theta(B)w_t$

roots of $\phi(B) \Rightarrow 1-3B = 0$

$\Rightarrow B = \frac{1}{3}$

The root is not greater than 1 hence not causal

roots of $\theta(B) \Rightarrow 1+2B-8B^2 = 0$

using $\Rightarrow \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ to find roots.

where $a = -8,\ b = 2,\ c = 1$

$\Rightarrow \frac{-2 \pm \sqrt{4+32}}{18} \Rightarrow = \frac{(-2 \pm 6)}{16} = 0.25, -0.5$

The roots do not lie outside the unit circle, $\therefore$ not invertible.

(6)

(i) $x_t - 2x_{t-1} + 2x_{t-2} = w_t - \frac{8}{9} \cdot w_{t-1}$

$$x_t - 2Bx_t + 2B^2 \cdot x_t = w_t - \frac{8}{9} B \cdot w_t$$

$$x_t (1 - 2B + 2B^2) = w_t (1 - \frac{8}{9} B)$$

$$\phi(B) x_t = \theta(B) w_t$$

roots of $\phi(B) \Rightarrow 1 - 2B + 2B^2 = 0$

using $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ where $a = 2, b = -2, c = 1$

$$= \frac{2 \pm \sqrt{4-8}}{4} = \frac{(2 \pm 2i)}{4} = (0.5 + 0.5i), (0.5 - 0.5i)$$

The roots are not greater than 1, hence not causal.

roots of $\theta(B) \Rightarrow 1 - \frac{8}{9} B = 0$

$$B = \frac{9}{8}$$

The root lies outside unit circle, $\therefore$ invertible.

⑦

ⓒ

$$x_t - 4x_{t-2} = w_t - w_{t-1} + 0.5\,w_{t-2}$$

$$x_t - 4B^2 x_t = w_t - B w_t + 0.5B^2 w_t$$

$$x_t (1-4B^2) = w_t (1-B+0.5B^2)$$

$$\phi(B)\, x_t = \theta(B) \cdot w_t$$

roots of $\phi(B) \Rightarrow 1-4B^2 = 0$

$$B = \pm \frac{1}{2}$$

Roots are not greater than 1, hence not causal

roots of $\theta(B)$ using $\dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

where $a = 0.5,\ b = -1,\ c = 1$

$$= \frac{1 \pm \sqrt{1-2}}{1} = \frac{1 \pm i}{1} = (1+i);(1-i)$$

The roots lie outside unit circle. Hence invertible.

(8)

(d)

$$x_t - \frac{9}{4} x_{t-1} - \frac{9}{4} x_{t-2} = w_t$$

$$x_t - \frac{9}{4} B x_t - \frac{9}{4} B^2 x_t = w_t$$

$$x_t \left(1 - \frac{9}{4} B - \frac{9}{4} B^2\right) = w_t$$

$$\phi(B) x_t = \theta(B) w_t$$

roots of $\phi(B)$ using $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

where $a = \frac{-9}{4}$, $b = \frac{-9}{4}$, $c = 1$

$$\frac{\frac{9}{4} \pm \sqrt{\left(\frac{-9}{4}\right)^2 + 9}}{2 \times \frac{-9}{4}} = 0.33 \; ; \; -1.33$$

The roots are not greater than 1, hence not causal.

roots of $\theta(B) = 1$

The root lie on unit circle. Hence invertible.