Time Series: Teaching session III

1

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Assignment 1

Proove the Kalman filtering recursion for the following state space model with initial prior on the state

$f(z_1) = N(z_1; m_0, P_0)$ where $e_t \sim N(0, Q_t)$ and $V_t \sim N(0, R_t)$

$$z_t = A_{t-1} \cdot z_{t-1} + e_t$$

$$x_t = C_t \, z_t + v_t$$

(1) Particularly show that given $f(z_t \mid x_{1:t}) = N(z_t : m_{t|t}, P_{t|t})$, the predicted density $f(z_{t+1} \mid x_{1:t})$ is given by

$$f(z_{t+1} \mid x_{1:t}) = N(z_{t+1}; A_t m_{t|t}, A_t \cdot P_{t|t} \cdot A_t^T + Q_{t+1})$$

(2) Also show that given $f(z_t \mid x_{1:t-1}) = N(z_t; m_{t|t-1}, P_{t|t-1})$, the observation updated density $f(z_t \mid x_{1:t})$ is given by

$$f(z_t \mid x_{1:t}) = N(z_t; m_{t|t}, P_{t|t})$$

where:

$$m_{t|t} = m_{t|t-1} + K_t (x_t - C_t \cdot m_{t|t-1})$$

$$P_{t|t} = (I - K_t \cdot C_t) \cdot P_{t|t-1}$$

$$K_t = P_{t|t-1} \cdot C_t^T (C_t \cdot P_{t|t-1} \, C_t^T + R_t)^{-1}$$

②

Answer:-

① $f(Z_{t+1} | X_{1:t}) = \int P(Z_{t+1} | X_{1:t})$

$= \int P(Z_{t+1} | Z_t) \cdot P(Z_t | X_{1:t}) \, dz_t$

$= \int N(Z_{t+1} | A_t \cdot Z_t, \, Q_{t+1}) \cdot N(Z_t | m_{t|t}, P_{t|t})$

using the properties of Normal density function given:-

$$f(z_t, z_{t+1}) = N\left(\begin{bmatrix} Z_t \\ Z_{t+1} \end{bmatrix}, \begin{bmatrix} m_{t|t} \\ A_t Z_t \end{bmatrix}, \begin{bmatrix} P_{t|t} & P_{t|t} \cdot A_t^T \\ A_t \cdot P_{t|t} & A_t \cdot P_{t|t} A_t^T + Q_{t+1} \end{bmatrix} \right)$$

$\therefore f(Z_{t+1} | X_{1:t}) = N(Z_{t+1}; A_t \cdot Z_t, A_t P_{t|t} \cdot A_t^T + Q_{t+1})$

②

$f(Z_t | X_{1:t-1}) = N(Z_t; m_{t|t-1}, P_{t|t-1})$

$$f(Z_t | X_{1:t}) = \frac{f(Z_t, X_t | X_{1:t-1})}{f(X_t | X_{1:t-1})}$$

$$= \frac{f(Z_t | X_{1:t-1}) \, f(X_t | Z_t, X_{1:t-1})}{f(X_t | X_{1:t-1})}$$

using conditional independence of X_t given Z_t on $X_{1:t-1}$

$$= \frac{f(Z_t | X_{1:t-1}) \cdot f(X_t | Z_t)}{f(X_t | X_{1:t-1})}$$

(3) From the state space equation we have

$$f(X_t | Z_t) = N(X_t ; C_t \cdot Z_t ; R_t)$$

given $\Rightarrow$ $f(Z_t | X_{1:t-1}) = N(Z_t ; m_{t|t-1} ; P_{t|t-1})$

$$\therefore f(Z_t | X_{1:t-1}) \cdot f(X_t | Z_t) = N(Z_t ; m_{t|t-1} ; P_{t|t-1}) \cdot N(X_t ; C_t \cdot Z_t ; R_t)$$

using the properties of normal density function.

$$= N\left(\begin{bmatrix} Z_t \\ X_t \end{bmatrix} ; \begin{bmatrix} m_{t|t-1} \\ C_t \; m_{t|t-1} \end{bmatrix} ; \begin{bmatrix} P_{t|t-1} & P_{t|t-1} C_t^T \\ C_t \cdot P_{t|t-1} & C_t \cdot P_{t|t-1} C_t^T + R_t \end{bmatrix} \right)$$