Time Series Analysis

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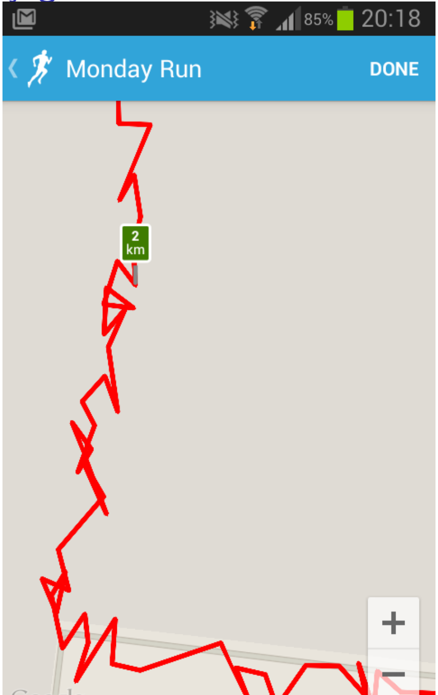
Where did he actually go?



Outline of the lecture

State space models, 2nd part:

- ARMA-models on state space form, Sec. 10.4
- Example: Random walk with measurement noise
- The Kalman filter when some observations are missing
- ML-estimates in state space models, Sec. 10.6
- Time-varying systems
- Example AR(1) through measurement noise

Cursory material:

- Signal extraction, Sec. 10.4.1
- Time series with missing observations, Sec. 10.5

The linear stochastic state space model

$$\begin{aligned} \text{System equation:} \quad & \boldsymbol{X}_t = \boldsymbol{A}\boldsymbol{X}_{t-1} + \boldsymbol{B}\boldsymbol{u}_{t-1} + \boldsymbol{e}_{1,t} \\ \text{Observation equation:} \quad & \boldsymbol{Y}_t = \boldsymbol{C}\boldsymbol{X}_t + \boldsymbol{e}_{2,t} \end{aligned}$$

- $\boldsymbol{X}$: State vector
- $\boldsymbol{Y}$: Observation vector
- $\boldsymbol{u}$: Input vector
- $\boldsymbol{e}_1$: System noise
- $\boldsymbol{e}_2$: Observation noise

- $\dim(X_t) = m$ is called the order of the system
- $\{\boldsymbol{e}_{1,t}\}$ and $\{\boldsymbol{e}_{2,t}\}$ mutually independent white noise
- $V[\boldsymbol{e}_1] = \boldsymbol{\Sigma}_1$, $V[\boldsymbol{e}_2] = \boldsymbol{\Sigma}_2$
- $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{\Sigma}_1$, and $\boldsymbol{\Sigma}_2$ are **known** matrices

- The state vector contains all information available for future evaluation; the state vector is a *Markov process*.

The ARMA(p, q) model as a state space model

$$Y_t + \phi_1 Y_{t-1} + \cdots + \phi_p Y_{t-p} = \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q}$$

State space form:

$$\begin{aligned} \boldsymbol{X}_t &= \boldsymbol{A}\boldsymbol{X}_{t-1} + \boldsymbol{G}\boldsymbol{\varepsilon}_t \\ \boldsymbol{Y}_t &= \boldsymbol{C}\boldsymbol{X}_t \end{aligned}$$

Or:

$$\boldsymbol{X}_t = \begin{bmatrix} -\phi_1 & 1 & 0 & \cdots & 0 \\ -\phi_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\phi_{d-1} & 0 & 0 & 0 & 1 \\ -\phi_d & 0 & 0 & \cdots & 0 \end{bmatrix} \boldsymbol{X}_{t-1} + \begin{pmatrix} 1 \\ \theta_1 \\ \vdots \\ \theta_{d-1} \end{pmatrix} \boldsymbol{\varepsilon}_t$$

$C = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$

where $d = max(p, q+1)$ and any extra parameter is fixed to zero.

For multivariate processes, just plug in matrices, and use I in stead of 1.

Random walk with measurement noise

Consider the state space model

$$
\begin{aligned}
X_t &= X_{t-1} + \eta_t \\
Y_t &= X_t + \varepsilon_t
\end{aligned}
$$

where $\{\eta_t\}$ and $\{\varepsilon_t\}$ are white noise processes with $\eta_t \sim \mathcal{N}(0, \sigma_\eta^2)$ and $\varepsilon_t \sim \mathcal{N}(0, \sigma_\varepsilon^2)$.

- $\{X_t\}$ is a random walk, that is not directly observed.
- The observations, $\{Y_t\}$, are influenced by measurement noise.
- What is the ARIMA structure of the Y process?
- Hint:

$$\nabla Y_t = \nabla X_t + \nabla \varepsilon_t = \eta_t + \varepsilon_t - \varepsilon_{t-1}$$

Random walk with measurement noise II

$$\nabla Y_t = \eta_t + \varepsilon_t - \varepsilon_{t-1}$$

ACF for ∇Y_t:

$$\rho(k) = \begin{cases} 1 & k = 0 \\ -\sigma_\varepsilon^2/(\sigma_\eta^2 + 2\sigma_\varepsilon^2) & k = 1 \\ 0 & k > 1 \end{cases}$$

This is the ACF of an $MA(1)$ process; thus, Y is $IMA(1,1)$
Alternative formulation:

$$\nabla Y_t = \xi_t + \theta_1 \xi_{t-1}\,, \quad \theta_1 < 0\,,$$

where ξ is white noise with variance σ_ξ^2.

Random walk with measurement noise III

Parameter relations in the two formulations, found by equaling the ACF expressions:

$$
\begin{aligned}
(1+\theta_1^2)\sigma_\xi^2 =& \sigma_\eta^2 + 2\sigma_\varepsilon^2 \\
\theta_1\sigma_\xi^2 =& -\sigma_\varepsilon^2
\end{aligned}
$$

- The ARMA process coefficients for the MA-parts are covariance parameters in the State Space formulation.
- The ARMA representation may be used to derive estimates for Σ_1, Σ_2.

The linear stochastic state space model

$$\text{System equation:} \quad \boldsymbol{X}_t = \boldsymbol{A}\boldsymbol{X}_{t-1} + \boldsymbol{B}\boldsymbol{u}_{t-1} + \boldsymbol{e}_{1,t}$$
$$\text{Observation equation:} \quad \boldsymbol{Y}_t = \boldsymbol{C}\boldsymbol{X}_t + \boldsymbol{e}_{2,t}$$

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- $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{\Sigma}_1$, and $\boldsymbol{\Sigma}_2$ are **known** matrices

The Kalman filter

Initialization

$$\widehat{\boldsymbol{X}}_{1|0} = E\left[\boldsymbol{X}_1\right] = \boldsymbol{\mu}_0$$
$$\boldsymbol{\Sigma}^{xx}_{1|0} = V\left[\boldsymbol{X}_1\right] = \boldsymbol{V}_0$$
$$\boldsymbol{\Sigma}^{yy}_{1|0} = \boldsymbol{C}\boldsymbol{\Sigma}^{xx}_{1|0}\boldsymbol{C}^T + \boldsymbol{\Sigma}_2$$

For: $t = 1, 2, 3, \ldots$

Reconstruction:

$$\boxed{\begin{aligned} \boldsymbol{K}_t &= \boldsymbol{\Sigma}^{xx}_{t|t-1}\boldsymbol{C}^T\left(\boldsymbol{\Sigma}^{yy}_{t|t-1}\right)^{-1} \\ \widehat{\boldsymbol{X}}_{t|t} &= \widehat{\boldsymbol{X}}_{t|t-1} + \boldsymbol{K}_t\left(\boldsymbol{Y}_t - \boldsymbol{C}\widehat{\boldsymbol{X}}_{t|t-1}\right) \\ \boldsymbol{\Sigma}^{xx}_{t|t} &= \boldsymbol{\Sigma}^{xx}_{t|t-1} - \boldsymbol{K}_t\boldsymbol{\Sigma}^{yy}_{t|t-1}\boldsymbol{K}_t^T \end{aligned}}$$

Prediction:

$$\boxed{\begin{aligned} \widehat{\boldsymbol{X}}_{t+1|t} &= \boldsymbol{A}\widehat{\boldsymbol{X}}_{t|t} + \boldsymbol{B}\boldsymbol{u}_t \\ \boldsymbol{\Sigma}^{xx}_{t+1|t} &= \boldsymbol{A}\boldsymbol{\Sigma}^{xx}_{t|t}\boldsymbol{A}^T + \boldsymbol{\Sigma}_1 \\ \boldsymbol{\Sigma}^{yy}_{t+1|t} &= \boldsymbol{C}\boldsymbol{\Sigma}^{xx}_{t+1|t}\boldsymbol{C}^T + \boldsymbol{\Sigma}_2 \end{aligned}}$$

- What happens if the observation Y_t is missing for some t?

Estimation in ARMA(p, q)-models using the KF

- Using the Kalman filter we can get the mean and variance of the one-step predictions of the observations:

$$\widehat{\boldsymbol{Y}}_{t+1|t} = \boldsymbol{C}\widehat{\boldsymbol{X}}_{t+1|t}$$
$$\boldsymbol{\Sigma}^{yy}_{t+1|t} = \boldsymbol{C}\boldsymbol{\Sigma}^{xx}_{t+1|t}\boldsymbol{C}^T + \boldsymbol{\Sigma}_2$$

- The Kalman filter can handle missing observations
- An ARMA(p, q)-model can be written as a state space model

- This gives us a way of calculating ML-estimates in the ARMA(p, q)-model even when some observations are missing.

The ARMA(p, q) model as a state space model

$$Y_t + \phi_1 Y_{t-1} + \cdots + \phi_p Y_{t-p} = \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q}$$

State space form:

$$\begin{aligned} \boldsymbol{X}_t &= \boldsymbol{A}\boldsymbol{X}_{t-1} + \boldsymbol{\varepsilon}_t \\ \boldsymbol{Y}_t &= \boldsymbol{C}\boldsymbol{X}_t \end{aligned}$$

For $p \geq q$:

$$\boldsymbol{X}_t = \begin{bmatrix} -\phi_1 & 1 & 0 & \cdots & 0 \\ -\phi_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\phi_{d-1} & 0 & 0 & 0 & 1 \\ -\phi_d & 0 & 0 & \cdots & 0 \end{bmatrix} \boldsymbol{X}_{t-1} + \begin{pmatrix} 1 \\ \theta_1 \\ \vdots \\ \theta_{d-1} \end{pmatrix} \boldsymbol{\varepsilon}_t$$

$C = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$
where $d = max(p, q+1)$ and any extra parameter is fixed to zero.

ML-estimates in state space models

$$\boldsymbol{X}_t = \boldsymbol{A}\boldsymbol{X}_{t-1} + \boldsymbol{G}\boldsymbol{e}_{1,t}$$
$$\boldsymbol{Y}_t = \boldsymbol{C}\boldsymbol{X}_t + \boldsymbol{e}_{2,t}$$

- $\{\boldsymbol{e}_{1,t}\}$ and $\{\boldsymbol{e}_{2,t}\}$ are mutually uncorrelated normally distributed white noise
- $V(\boldsymbol{e}_{1,t}) = \Sigma_1$ and $V(\boldsymbol{e}_{2,t}) = \Sigma_2$
- For ARMA(p, q)-models we have $\boldsymbol{A}$, $\boldsymbol{C}$, and $\boldsymbol{G}$ as stated on the previous slide. Furthermore, $\boldsymbol{e}_{1,t} = \varepsilon_t$, $\boldsymbol{\Sigma}_1 = \sigma_\varepsilon^2$, and $\boldsymbol{\Sigma}_2 = 0$

Maximum Likelihood Estimates

- Let $\mathcal{Y}_{N^*}$ contain the available observations and let $\boldsymbol{\theta}$ contain the parameters of the model
- The likelihood function is the density of the random vector corresponding to the observations and given the set of parameters:

$$L\left(\boldsymbol{\theta}; \mathcal{Y}_{N^*}\right) = f\left(\mathcal{Y}_{N^*} | \boldsymbol{\theta}\right)$$

- The ML-estimates is found by selecting $\boldsymbol{\theta}$ so that the density function is as large as possible at the actual observations
- The random variables $\boldsymbol{Y}_{N^*} | \mathcal{Y}_{N^*-1}$ and $\mathcal{Y}_{N^*-1}$ are independent:

$$\begin{aligned} L\left(\boldsymbol{\theta}; \mathcal{Y}_{N^*}\right) = f\left(\mathcal{Y}_{N^*} | \boldsymbol{\theta}\right) &= f\left(\boldsymbol{Y}_{N^*} | \mathcal{Y}_{N^*-1}, \boldsymbol{\theta}\right) f\left(\mathcal{Y}_{N^*-1} | \boldsymbol{\theta}\right) \\ &= f\left(\boldsymbol{Y}_{N^*} | \mathcal{Y}_{N^*-1}, \boldsymbol{\theta}\right) f\left(\boldsymbol{Y}_{N^*-1} | \mathcal{Y}_{N^*-2}, \boldsymbol{\theta}\right) \cdots f\left(\boldsymbol{Y}_1 | \boldsymbol{\theta}\right) \end{aligned}$$

- The conditional densities can be found using the Kalman filter

MLE / KF – Prediction

- Assume that at time t we have:

$$\widehat{\boldsymbol{X}}_{t|t} = E\left[\boldsymbol{X}_t|\mathcal{Y}_t\right] \quad \text{and} \quad \boldsymbol{\Sigma}^{xx}_{t|t} = V\left[\boldsymbol{X}_t|\mathcal{Y}_t\right]$$

- Using the model we obtain predictions for time $t+1$:

$$\begin{aligned}
\widehat{\boldsymbol{X}}_{t+1|t} &= \boldsymbol{A}\widehat{\boldsymbol{X}}_{t|t} \\
\boldsymbol{\Sigma}^{xx}_{t+1|t} &= \boldsymbol{A}\boldsymbol{\Sigma}^{xx}_{t|t}\boldsymbol{A}^T + \boldsymbol{G}\boldsymbol{\Sigma}_1\boldsymbol{G}^T \\
\widehat{\boldsymbol{Y}}_{t+1|t} &= \boldsymbol{C}\widehat{\boldsymbol{X}}_{t+1|t} \\
\boldsymbol{\Sigma}^{yy}_{t+1|t} &= \boldsymbol{C}\boldsymbol{\Sigma}^{xx}_{t+1|t}\boldsymbol{C}^T + \boldsymbol{\Sigma}_2
\end{aligned}$$

- Due to the normality of the white noise process $f(\boldsymbol{Y}_{t+1}|\mathcal{Y}_t, \boldsymbol{\theta})$ is then the (multivariate) normal density (see Chapter 2) with mean $\widehat{\boldsymbol{Y}}_{t+1|t}$ and variance-covariance $\boldsymbol{\Sigma}^{yy}_{t+1|t}$ $(= \boldsymbol{R}_{t+1})$

MLE / KF – Reconstruction

At time $t+1$ there are two possibilities for the reconstruction part:

The observation $\boldsymbol{Y}_{t+1}$ is available:

We update the state estimate using the reconstruction step of the Kalman Filter:

$$\begin{aligned}
\boldsymbol{K}_{t+1} &= \boldsymbol{\Sigma}^{xx}_{t+1|t}\boldsymbol{C}^T\left(\boldsymbol{\Sigma}^{yy}_{t+1|t}\right)^{-1} \\
\widehat{\boldsymbol{X}}_{t+1|t+1} &= \widehat{\boldsymbol{X}}_{t+1|t} + \boldsymbol{K}_{t+1}\left(\boldsymbol{Y}_{t+1} - \widehat{\boldsymbol{Y}}_{t+1|t}\right) \\
\boldsymbol{\Sigma}^{xx}_{t+1|t+1} &= \boldsymbol{\Sigma}^{xx}_{t+1|t} - \boldsymbol{K}_{t+1}\boldsymbol{\Sigma}^{yy}_{t+1|t}\boldsymbol{K}^T_{t+1}
\end{aligned}$$

The observation $\boldsymbol{Y}_{t+1}$ is missing:

We get no new information and we use:

$$\begin{aligned}
\widehat{\boldsymbol{X}}_{t+1|t+1} &= \widehat{\boldsymbol{X}}_{t+1|t} \\
\boldsymbol{\Sigma}^{xx}_{t+1|t+1} &= \boldsymbol{\Sigma}^{xx}_{t+1|t}
\end{aligned}$$

MLE / KF – The likelihood function

- Using the prediction errors and variances

$$\widetilde{\boldsymbol{Y}}_i = \boldsymbol{Y}_i - \widehat{\boldsymbol{Y}}_{i|i-1}$$
$$\boldsymbol{R}_i = \boldsymbol{\Sigma}^{yy}_{i|i-1}$$

- The likelihood function can be expressed as

$$L\left(\boldsymbol{\theta}; \mathcal{Y}_{N^*}\right) = \prod_{i=1}^{N^*} \left[(2\pi)^m \det \boldsymbol{R}_i\right]^{-\frac{1}{2}} \exp\left[-\frac{1}{2}\widetilde{\boldsymbol{Y}}_i^T \boldsymbol{R}_i^{-1} \widetilde{\boldsymbol{Y}}_i\right]$$

- In practice optimization is based on

$$\log L\left(\boldsymbol{\theta}; \mathcal{Y}_{N^*}\right) = -\frac{1}{2} \sum_{i=1}^{N} \left(\log \det \boldsymbol{R}_i + \widetilde{\boldsymbol{Y}}_i^T \boldsymbol{R}_i^{-1} \widetilde{\boldsymbol{Y}}_i\right) + c$$

- The variance of the estimates can be approximated by the 2nd order derivatives of the log-likelihood.

MLE / KF IV – Initialization

- The only outstanding issue is "prediction" of $\boldsymbol{Y}_1$, i.e. calculation of $\widehat{\boldsymbol{Y}}_{1|0}$
- This can be done by setting $\widehat{\boldsymbol{X}}_{0|0} = \mathbf{0}$ and $\boldsymbol{\Sigma}_{0|0}^{xx} = \alpha \boldsymbol{I}$, where $\boldsymbol{I}$ is the identity matrix and α is a 'large' constant (we don't know what it is)
- Alternatively, we can fix the initial state $\widehat{\boldsymbol{X}}_{0|0}$ and set $\boldsymbol{\Sigma}_{0|0}^{xx} = \mathbf{0}$, whereby $\boldsymbol{\Sigma}_{1|0}^{xx} = \boldsymbol{G}\boldsymbol{\Sigma}_1\boldsymbol{G}^T$
- Or combinations thereof - recommended
- The important part is that the (un-)certainty of $\widehat{\boldsymbol{X}}_{0|0}$ is reflected in $\boldsymbol{\Sigma}_{0|0}^{xx}$.

Autocovariance functions with missing data

Define the observation indicator a as

$$a(t) = \begin{cases} 1 & \text{if } Y_t \text{ is observed;} \\ 0 & \text{if } Y_t \text{ is missing.} \end{cases}$$

Define similarly

$$C_a(k) = \frac{1}{N} \sum_{t=1}^{N-|k|} a(t)a(t+|k|)\,.$$

- a indicates which data points are present
- For large N, $C_a(k)$ measures the fraction of pairs of data k time steps away from each other that are observed.

Autocovariance functions with missing data II

The indicator, $a(t)$ is used to define and estimate of the mean of $\{Y_t\}$

$$\overline{\mu}_y = \frac{\sum_{t=1}^{N} a(t)Y_t}{\sum_{t=1}^{N} a(t)}$$

- $\overline{\mu}_y$ is the mean of the observed Y_t's.

And defining:

$$C_a^{\square}(k) = \frac{1}{N} \sum_{t=1}^{N-|k|} a(t)a(t+|k|)(Y_t - \overline{\mu}_y)(Y_{t+|k|} - \overline{\mu}_y)\,.$$

- Note: $C_a^{\square}(k)$ leave pairs out if one of the observations is missing.
- The sample autocovariance is estimated by:

$$C_{YY}(k) = \frac{C_a^{\square}(k)}{C_a(k)}$$

- These estimates are available with the `acf` function in R, by using
```
> acf(...,na.action=na.omit)
```

Time-varying systems

$$\begin{aligned} \text{System equation:} \quad \boldsymbol{X}_t &= \boldsymbol{A}_t \boldsymbol{X}_{t-1} + \boldsymbol{B}_t \boldsymbol{u}_{t-1} + \boldsymbol{e}_{1,t} \\ \text{Observation equation:} \quad \boldsymbol{Y}_t &= \boldsymbol{C}_t \boldsymbol{X}_t + \boldsymbol{e}_{2,t} \end{aligned}$$

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- $V[\boldsymbol{e}_{1,t}] = \boldsymbol{\Sigma}_{1,t}$, $V[\boldsymbol{e}_{2,t}] = \boldsymbol{\Sigma}_{2,t}$
- $\boldsymbol{A}_t$, $\boldsymbol{B}_t$, $\boldsymbol{C}_t$, $\boldsymbol{\Sigma}_{1,t}$, and $\boldsymbol{\Sigma}_{2,t}$ are **known** matrices at any point in time

The Kalman filter for time varying systems

Initialization

$$\widehat{\boldsymbol{X}}_{1|0} = E\left[\boldsymbol{X}_1\right] = \boldsymbol{\mu}_0$$
$$\boldsymbol{\Sigma}^{xx}_{1|0} = V\left[\boldsymbol{X}_1\right] = \boldsymbol{V}_0 \Rightarrow$$
$$\boldsymbol{\Sigma}^{yy}_{1|0} = \boldsymbol{C}_1 \boldsymbol{\Sigma}^{xx}_{1|0} \boldsymbol{C}_1^T + \boldsymbol{\Sigma}_{2,1}$$

For: $t = 1, 2, 3, \ldots$

Reconstruction:

$$\boxed{\begin{aligned} \boldsymbol{K}_t &= \boldsymbol{\Sigma}^{xx}_{t|t-1} \boldsymbol{C}_t^T \left(\boldsymbol{\Sigma}^{yy}_{t|t-1}\right)^{-1} \\ \widehat{\boldsymbol{X}}_{t|t} &= \widehat{\boldsymbol{X}}_{t|t-1} + \boldsymbol{K}_t \left(\boldsymbol{Y}_t - \boldsymbol{C}_t \widehat{\boldsymbol{X}}_{t|t-1}\right) \\ \boldsymbol{\Sigma}^{xx}_{t|t} &= \boldsymbol{\Sigma}^{xx}_{t|t-1} - \boldsymbol{K}_t \boldsymbol{\Sigma}^{yy}_{t|t-1} \boldsymbol{K}_t^T \end{aligned}}$$

Prediction:

$$\boxed{\begin{aligned} \widehat{\boldsymbol{X}}_{t+1|t} &= \boldsymbol{A}_{t+1} \widehat{\boldsymbol{X}}_{t|t} + \boldsymbol{B}_{t+1} \boldsymbol{u}_t \\ \boldsymbol{\Sigma}^{xx}_{t+1|t} &= \boldsymbol{A}_{t+1} \boldsymbol{\Sigma}^{xx}_{t|t} \boldsymbol{A}_{t+1}^T + \boldsymbol{\Sigma}_{1,t+1} \\ \boldsymbol{\Sigma}^{yy}_{t+1|t} &= \boldsymbol{C}_{t+1} \boldsymbol{\Sigma}^{xx}_{t+1|t} \boldsymbol{C}_{t+1}^T + \boldsymbol{\Sigma}_{2,t+1} \end{aligned}}$$

Example: An AR(1) and obs. noise

The heat transfer from a certain body to its surroundings is dominated by conduction. The temperature of the body is given by

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{1}{R}(T_{surr} - T)$$

Let the surrounding temperature be constantly 0. Then

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -\frac{1}{R}(T) = aT$$

The solution to the differential equation is

$$T = T_0 e^{at}$$

A discretization of this yields

$$T_{t+1} = e^{a\Delta t} \cdot T_t$$

We know in reality, such a process is influenced by noise:

$$T_{t+1} = -\phi \cdot T_t + e_t, \quad e_t \sim \mathcal{N}(0, \sigma_e^2)$$

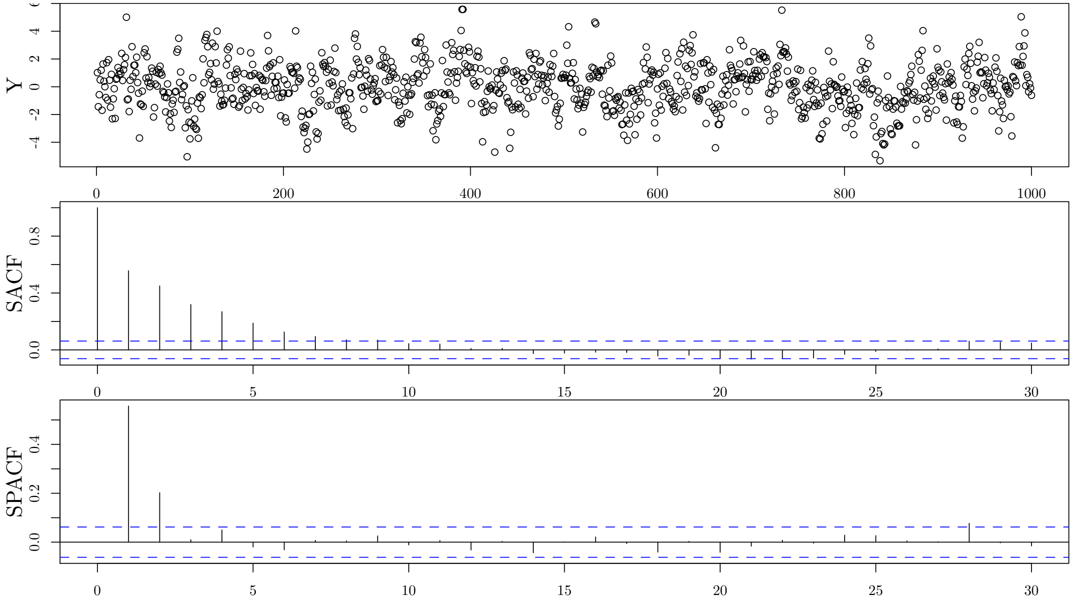
We measure the temperature at each time step:

$$Y_t = T_t + \eta_t, \quad \eta_t \sim \mathcal{N}(0, \sigma_\eta^2)$$

Example: An AR(1) and obs. noise – Simulation

```r
a <- 0.8
N <- 1000
X <- numeric(N)
X[1] <- 1
for (i in 2:N){
    X[i] <- a * X[i-1] + rnorm(1, sd=0.8)
}
Y <- X + rnorm(N, sd=1.0)
```

Example: An AR(1) and obs. noise – What process is this?



Highlights

- ARMA models on State space form
- Kalman filter
  - Handling missing values
  - Prediction
  - Maximum likelihood estimation of parameters
  - Comparing models using likelihood