Yoneda lemma and AGI

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Yoneda lemma

Learn to look at any object,
not as it is, but as it varies

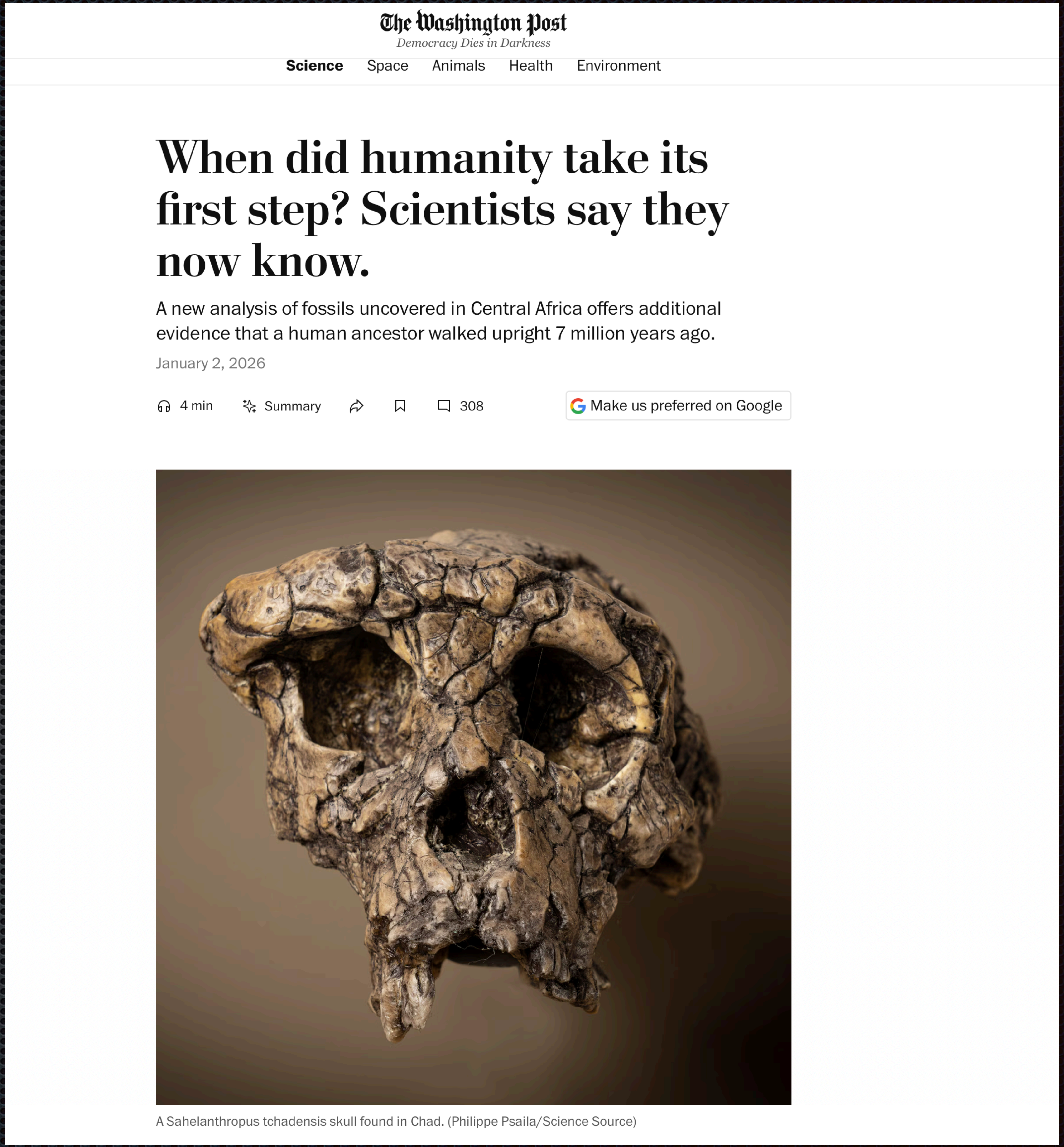
$$x \mapsto \mathscr{C}(-, x)$$

Databases and the Yoneda Lemma

- Databases are everywhere
  - Every organization stores its knowledge using databases
  - What is a database, categorically?
- Databases are set-valued functors!

cSQL: Building causal databases from documents

cSQL builds on Democritus, a categorical ML framework for learning large causal models from large language models



The Washington Post
Democracy Dies in Darkness

Science Space Animals Health Environment

When did humanity take its first step? Scientists say they now know.

A new analysis of fossils uncovered in Central Africa offers additional evidence that a human ancestor walked upright 7 million years ago.

January 2, 2026

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A Sahelanthropus tchadensis skull found in Chad. (Philippe Psaila/Science Source)

[Mahadevan, cSQL: Mapping Documents into Causal Databases, Arxiv 2026]

LLM Summarization

It is common now to use LLMs to summarize documents

AI Overview

Summary is AI-generated, newsroom-reviewed.

A study published in Science Advances suggests Sahelanthropus tchadensis, a 7-million-year-old ancestor, walked upright, indicating early bipedalism in human evolution. The analysis of limb bones, particularly the femoral tubercle, supports this claim. However, the debate continues as some scientists argue the fossils are too damaged to confirm bipedalism. Further discoveries are needed to resolve the controversy.

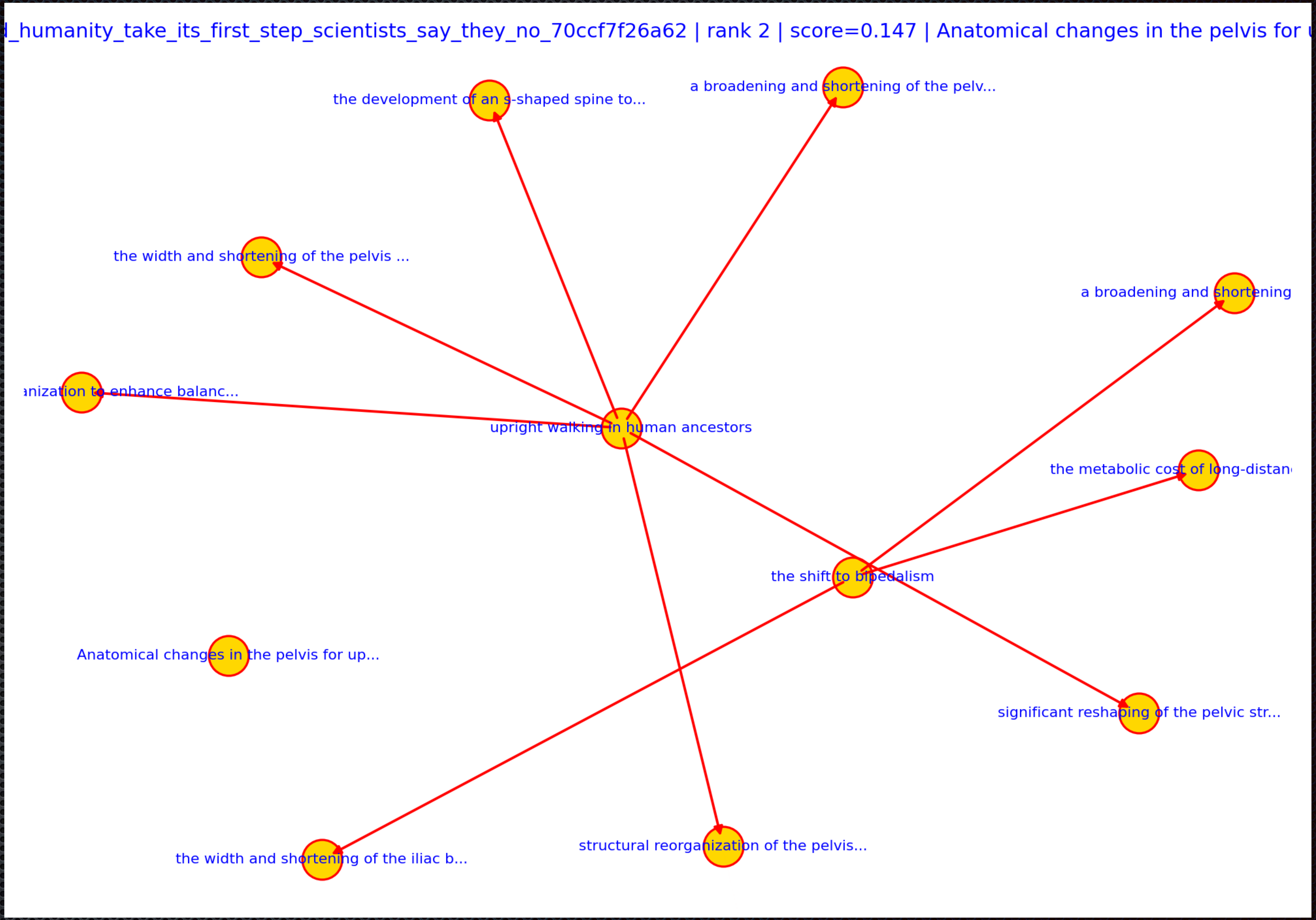
Read the full article for more on:

- The significance of the femoral tubercle in determining bipedalism.
- Why some scientists remain skeptical about the study's conclusions.
- Future plans for fossil hunting in Chad's Djurab Desert.

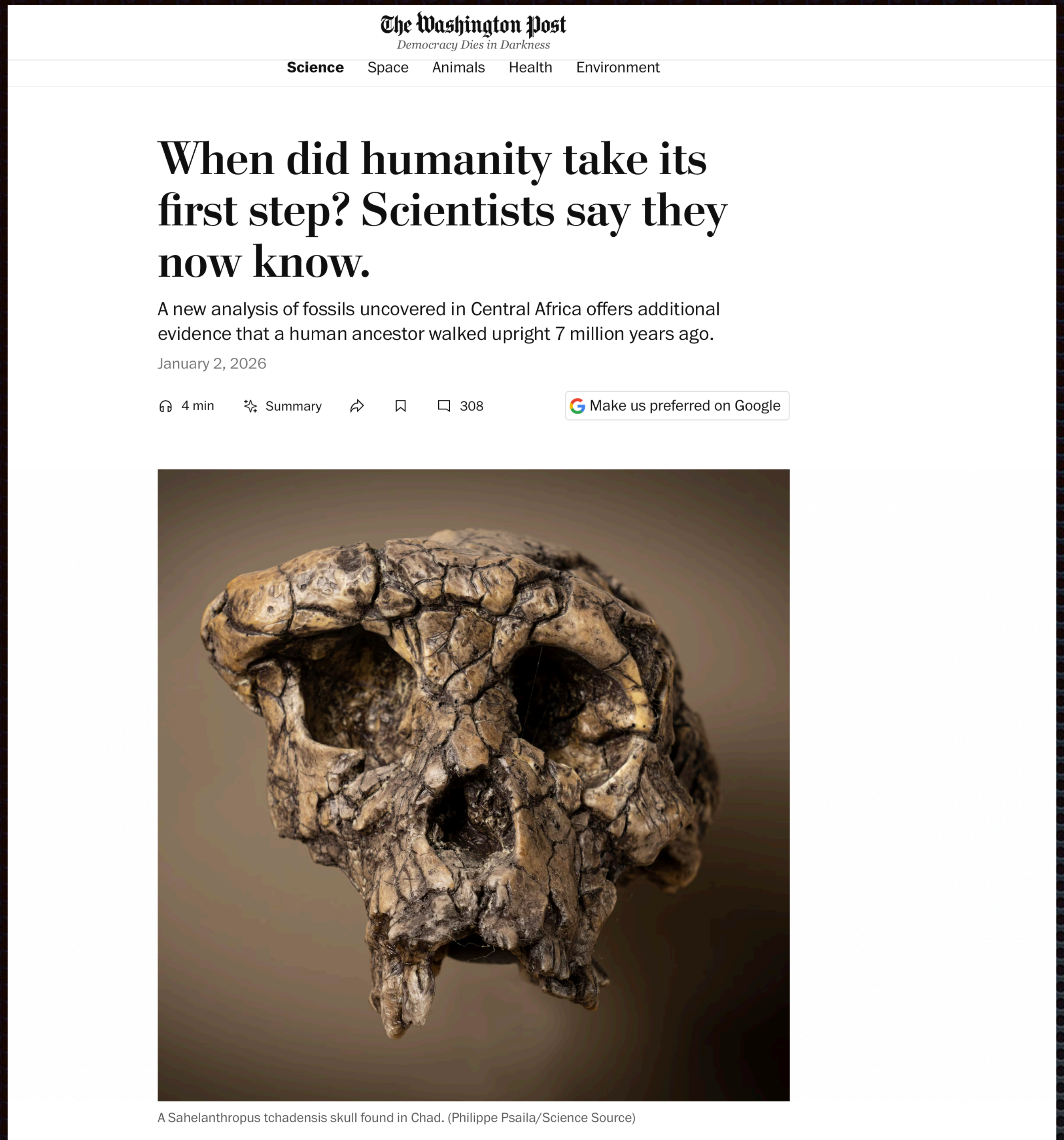
Did our AI help? Share your thoughts.

Causal model of bipedalism

Democritus builds causal models from documents



[Mahadevan, Learning Large Causal Models from Large Language Models, Arxiv, 2025]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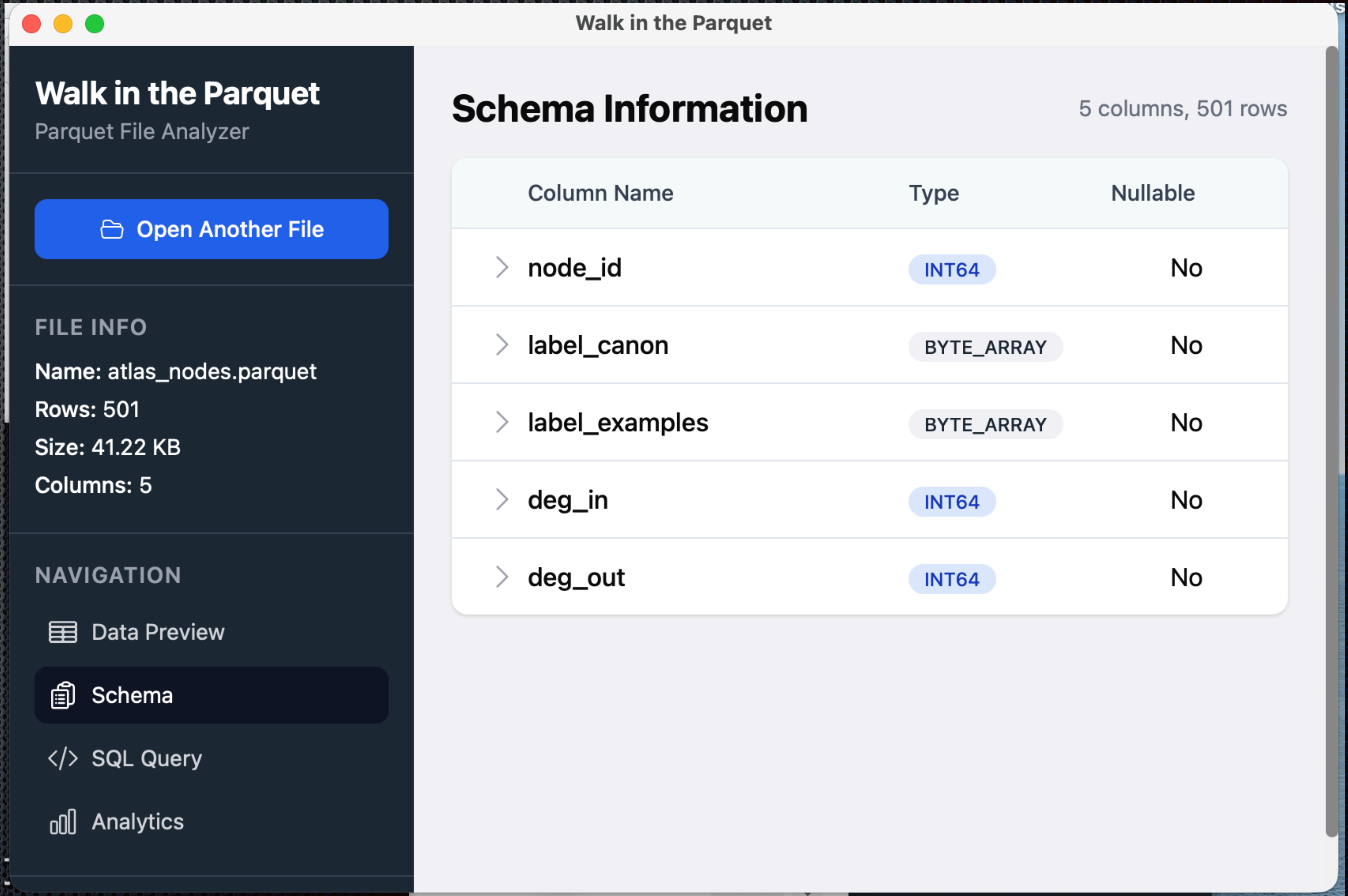
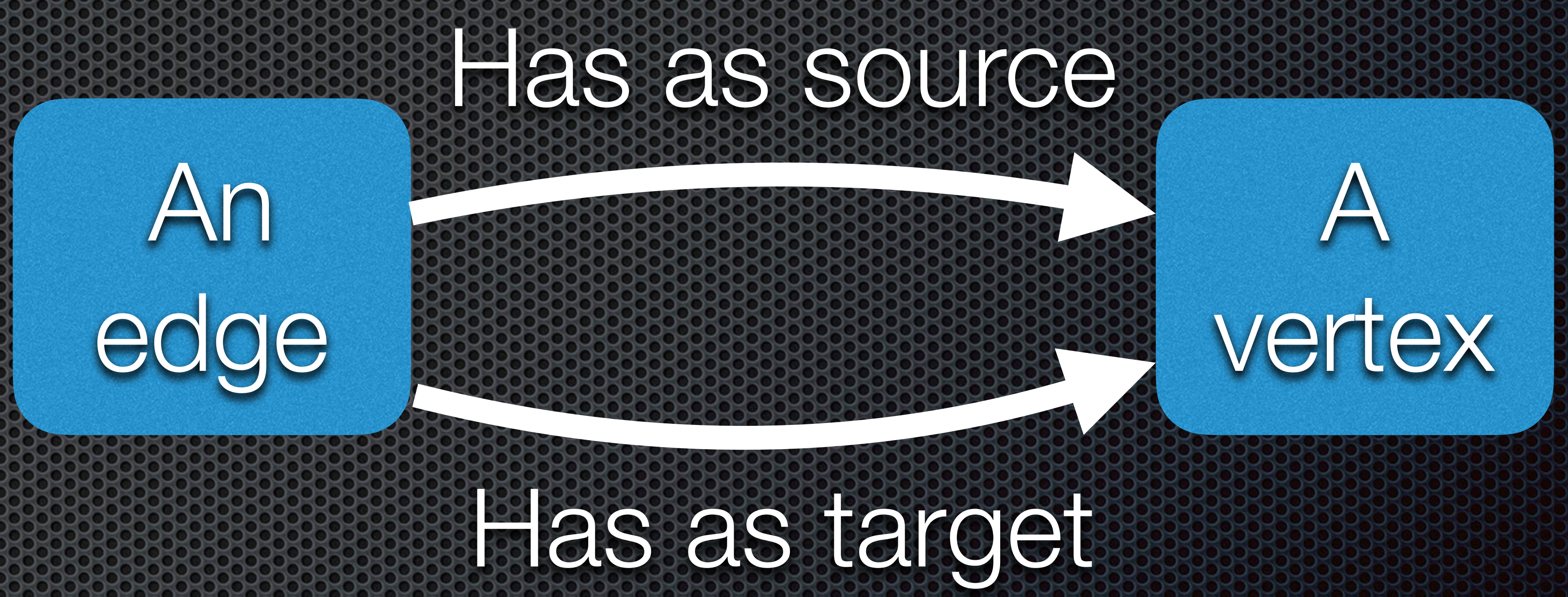
cSQL Relational Database



Database Schemas and instances



A database schema is a "small" category $\mathscr{C}$

A database instance is a functor $F : \mathscr{C} \to \mathbf{Set}$

Schemas are like graphs with constraints

Database Schemas as Categories

Schema Information

10 columns, 717 rows

Column Name	Type	Nullable
edge_id	INT64	No
doc_id	BYTE_ARRAY	No
lcm_instance_id	BYTE_ARRAY	No
focus	BYTE_ARRAY	No
radius	INT64	No
lcm_file	BYTE_ARRAY	No
lcm_json	BYTE_ARRAY	No
score	DOUBLE	No
score_raw	DOUBLE	No
coupling	DOUBLE	No

Tables are Set-Valued Functors

Data Preview

Showing 1 to 20 of 717 rows

EDGE_ID	DOC_ID	LCM_INSTANCE_ID	FOCUS	RADIUS	LCM_FILE
10262796833405321000	0001_when_did_humanity_take_its_first_step_scientists_say_they_no_70ccf7f26a62	e672245fd8850cae2e07633f5a37a64bfad0051a	Fossil evidence from Australopithecus afarensis (e.g., Lucy)	3	fossil_evidence_from_australopithecus_afarensis_eg_lucy_r3_m40.json
7422821195646369000	0001_when_did_humanity_take_its_first_step_scientists_say_they_no_70ccf7f26a62	e672245fd8850cae2e07633f5a37a64bfad0051a	Fossil evidence from Australopithecus afarensis (e.g., Lucy)	3	fossil_evidence_from_australopithecus_afarensis_eg_lucy_r3_m40.json
10157939426482416000	0001_when_did_humanity_take_its_first_step_scientists_say_they_no_70ccf7f26a62	e672245fd8850cae2e07633f5a37a64bfad0051a	Fossil evidence from Australopithecus afarensis (e.g., Lucy)	3	fossil_evidence_from_australopithecus_afarensis_eg_lucy_r3_m40.json
2071862364600755700	0001_when_did_humanity_take_its_first_step_scientists_say_they_no_70ccf7f26a62	e672245fd8850cae2e07633f5a37a64bfad0051a	Fossil evidence from Australopithecus afarensis (e.g., Lucy)	3	fossil_evidence_from_australopithecus_afarensis_eg_lucy_r3_m40.json
5914198021085193000	0001_when_did_humanity_take_its_first_step_scientists_say_they_no_70ccf7f26a62	e672245fd8850cae2e07633f5a37a64bfad0051a	Fossil evidence from Australopithecus afarensis (e.g., Lucy)	3	fossil_evidence_from_australopithecus_afarensis_eg_lucy_r3_m40.json
3002299461243261000	0001_when_did_humanity_take_its_first_step_scientists_say_they_no_70ccf7f26a62	e672245fd8850cae2e07633f5a37a64bfad0051a	Fossil evidence from Australopithecus afarensis (e.g., Lucy)	3	fossil_evidence_from_australopithecus_afarensis_eg_lucy_r3_m40.json
6229395304377507000	0001_when_did_humanity_take_its_first_step_scientists_say_they_no_70ccf7f26a62	e672245fd8850cae2e07633f5a37a64bfad0051a	Fossil evidence from Australopithecus afarensis (e.g., Lucy)	3	fossil_evidence_from_australopithecus_afarensis_eg_lucy_r3_m40.json
11410190103457808000	0001_when_did_humanity_take_its_first_step_scientists_say_they_no_70ccf7f26a62	e672245fd8850cae2e07633f5a37a64bfad0051a	Fossil evidence from Australopithecus afarensis (e.g., Lucy)	3	fossil_evidence_from_australopithecus_afarensis_eg_lucy_r3_m40.json
14783753344765247000	0001_when_did_humanity_take_its_first_step_scientists_say_they_no_70ccf7f26a62	e672245fd8850cae2e07633f5a37a64bfad0051a	Fossil evidence from Australopithecus afarensis (e.g., Lucy)	3	fossil_evidence_from_australopithecus_afarensis_eg_lucy_r3_m40.json
4003074693340940300	0001_when_did_humanity_take_its_first_step_scientists_say_they_no_70ccf7f26a62	e672245fd8850cae2e07633f5a37a64bfad0051a	Fossil evidence from Australopithecus afarensis (e.g., Lucy)	3	fossil_evidence_from_australopithecus_afarensis_eg_lucy_r3_m40.json
8935735324631167000	0001_when_did_humanity_take_its_first_step_scientists_say_they_no_70ccf7f26a62	e672245fd8850cae2e07633f5a37a64bfad0051a	Fossil evidence from Australopithecus afarensis (e.g., Lucy)	3	fossil_evidence_from_australopithecus_afarensis_eg_lucy_r3_m40.json

Lifting Diagrams

$$\begin{tikzcd} W \arrow[r, "p"] \arrow[d, "m"'] & I \arrow[d, "\pi"] \\ R \arrow[r, "n"'] & S. \end{tikzcd}$$

$$\begin{array}{ll} \text{SELECT} & * \\ \text{FROM} & R \xrightarrow{n} S \\ \text{WHERE} & R \xleftarrow{m} W \xrightarrow{p} I \end{array}$$

Database instance: $\pi : I \to S$

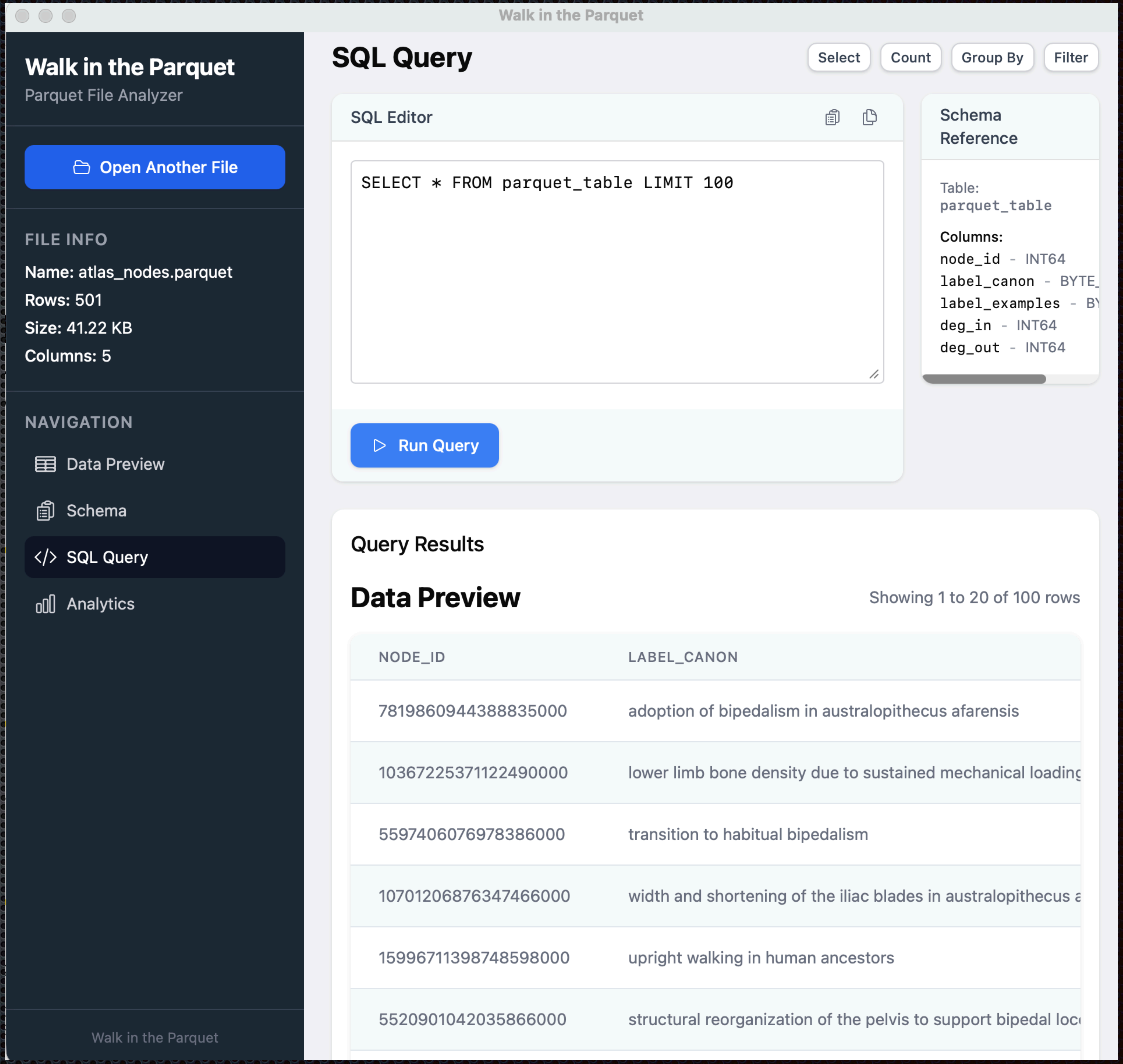
Queries are functors: $m : W \to R$

Database queries can be categorically formalized as solving lifting diagrams

Figure source: [Spivak, Database Queries and Constraints via Lifting Problems]

cSQL Queries

We can run causal queries and get a deeper insight into the origins of bipedalism



SQL using lifting diagrams

$$\begin{tikzcd} W \arrow[r, "p"] \arrow[d, "m"'] & I \arrow[d, "\pi"] \\ R \arrow[r, "n"'] \arrow[ur, "\ell"] & S. \end{tikzcd}$$

The result of computing an answer using SQL can be viewed as solving a “lifting diagram”

Database instance: $\pi : I \to S$

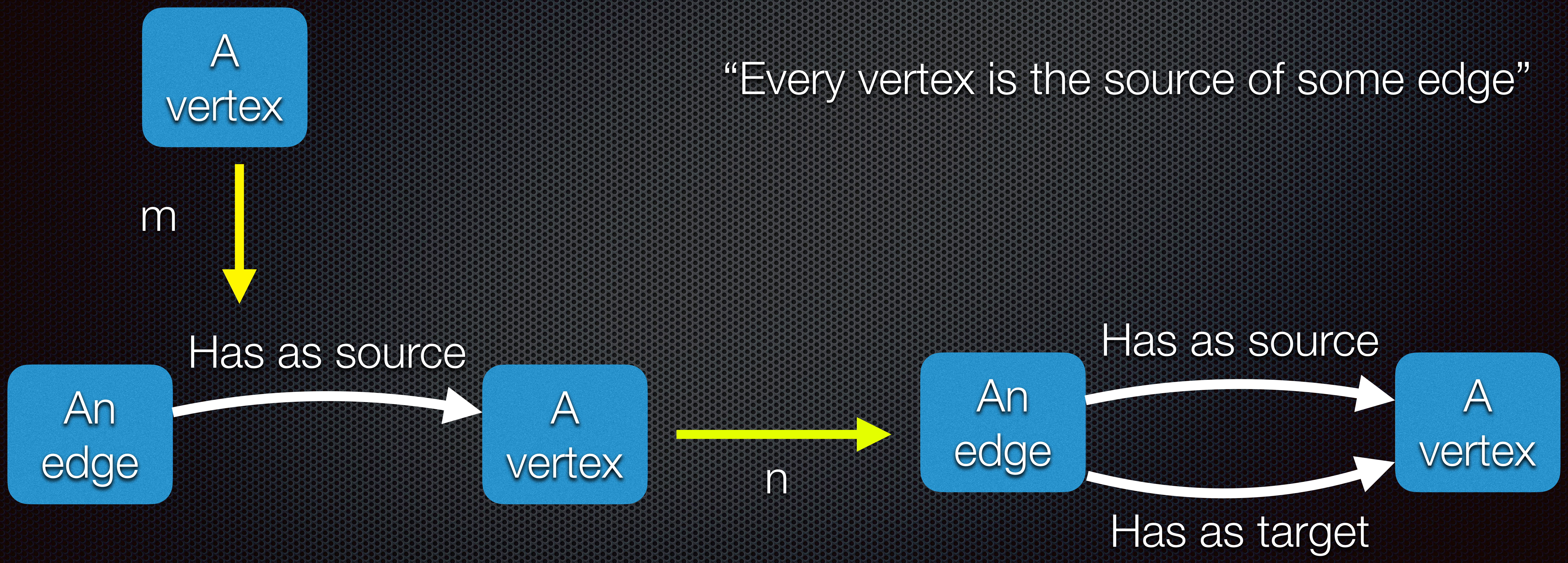
Queries are functors: $m : W \to R$

Results are lifts: $l : R \to I$

Figure source: [Spivak, Database Queries and Constraints via Lifting Problems]

Constraining graphs with lifting diagrams

“Every vertex is the source of some edge”



cSQL query example

SQL Query

Select Count Group By Filter

SQL Editor

```
SELECT edge_id, COUNT(*) as count
FROM parquet_table
GROUP BY edge_id
ORDER BY count DESC
LIMIT 5
```

Run Query

Schema Reference

Table: `parquet_table`

Columns:
- `edge_id` - INT64
- `doc_id` - BYTE_ARRAY
- `lcm_instance_id` - BYTE_ARRAY
- `focus` - BYTE_ARRAY
- `radius` - INT64
- `lcm_file` - BYTE_ARRAY
- `lcm_json` - BYTE_ARRAY
- `score` - DOUBLE
- `score_raw` - DOUBLE
- `coupling` - DOUBLE

Query Results

Data Preview

Showing 1 to 5 of 5 rows

EDGE_ID	COUNT
13639463363777124000	15
3557567388432306700	13
13513761877943288000	13
7685251784901854000	13
14783753344765247000	13

Grothendieck Category of Elements

- With every set-valued functor, we can define a new category of elements

$$\mathrm{Ob}\left(\int F\right) = \{(s, x) \,|\, s \in \mathrm{Ob}\mathscr{C}), x \in F(s)\}$$

$$\mathrm{Hom}_{\int F}((s, x), (s', x')) = \{f : s \to s' \,|\, F(f)(x) = x'\}$$

GCE Projection Functor

- Associated with the Grothendieck Category of Elements (GCE) is a projection functor that maps GCE back to the original category

$$\pi_F : \int F \to \mathscr{C}$$

$$\pi_F((s,x)) = s, \;\; \pi_F(f : (s,x) \to (s',x')) = f : s \to s'$$

Universal Properties

- A universal property for an object c in a category C is defined as
  - A contravariant functor F with a representation C(-, c) ~ F
  - A covariant functor F with a representation C(c, -) ~ F

Yoneda Lemma and Universal Properties

- An initial object c in a category C is one such that there is a unique arrow from c to every other object
- A terminal object c in a category C is one such that there is a unique arrow into c from every other object
- A covariant set-valued functor is representable if and only if its category of elements has an initial object
- A contravariant set-valued functor is representable if and only if its category of elements has a terminal object

Yoneda lemma & Topos Causal Models

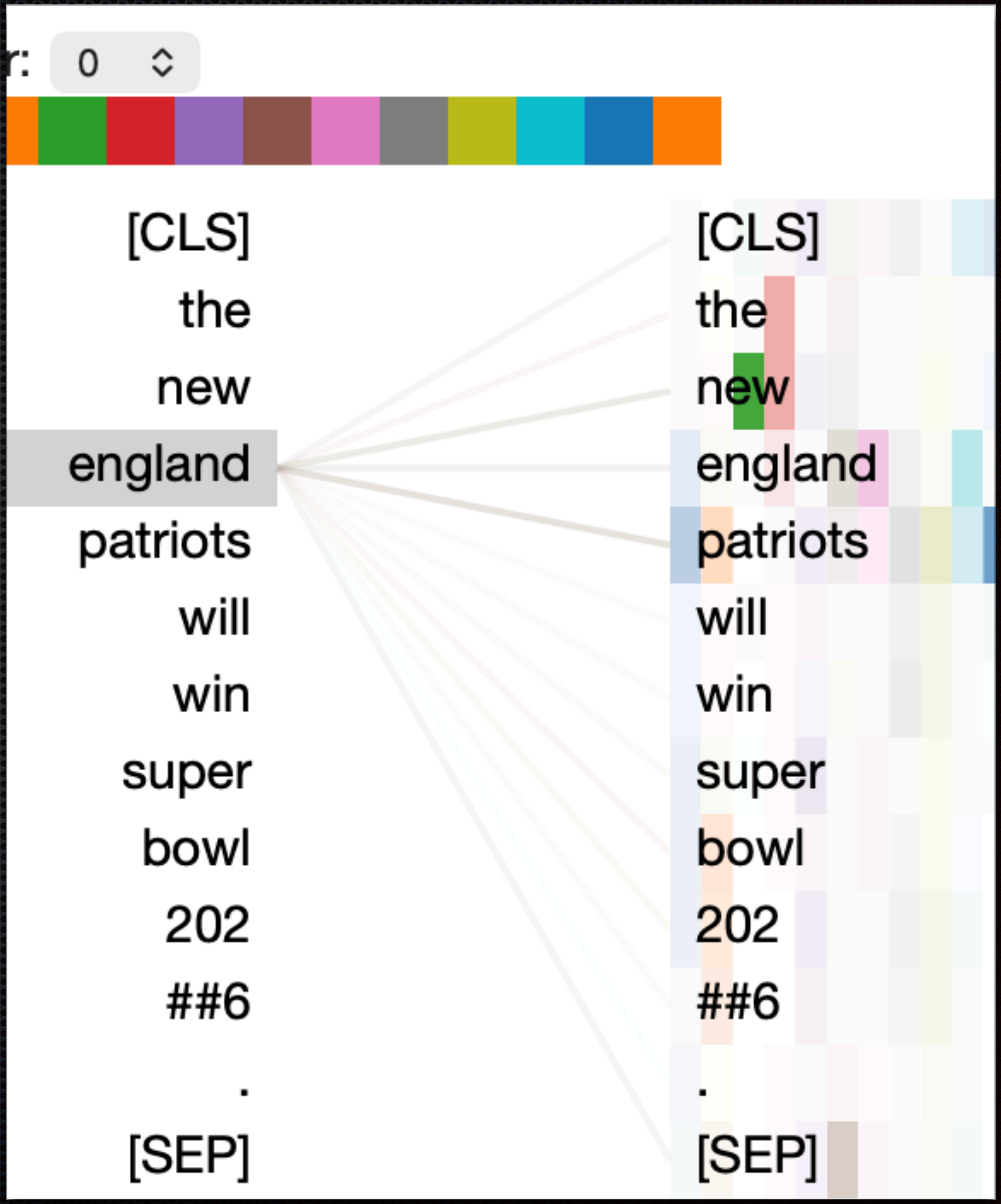
- A sieve is a "subobject" of the Yoneda embedding C(-, x)
  - Here, we are dealing with a category of functor objects C(-,x)
- A Grothendieck topology is a function J that assigns to each object x of C a collection J(x) of sieves
  - J(x) contains the maximum sieve J(x) = {f | codomain(f) = x}
  - J(x) is "closed" under composition

Transformers and the Yoneda lemma

- A Transformer captures the meaning of a word through “self-attention”
- Consider the two sentences:
  - New England Patriots will win Super Bowl 2026
  - New England winters are cold and long

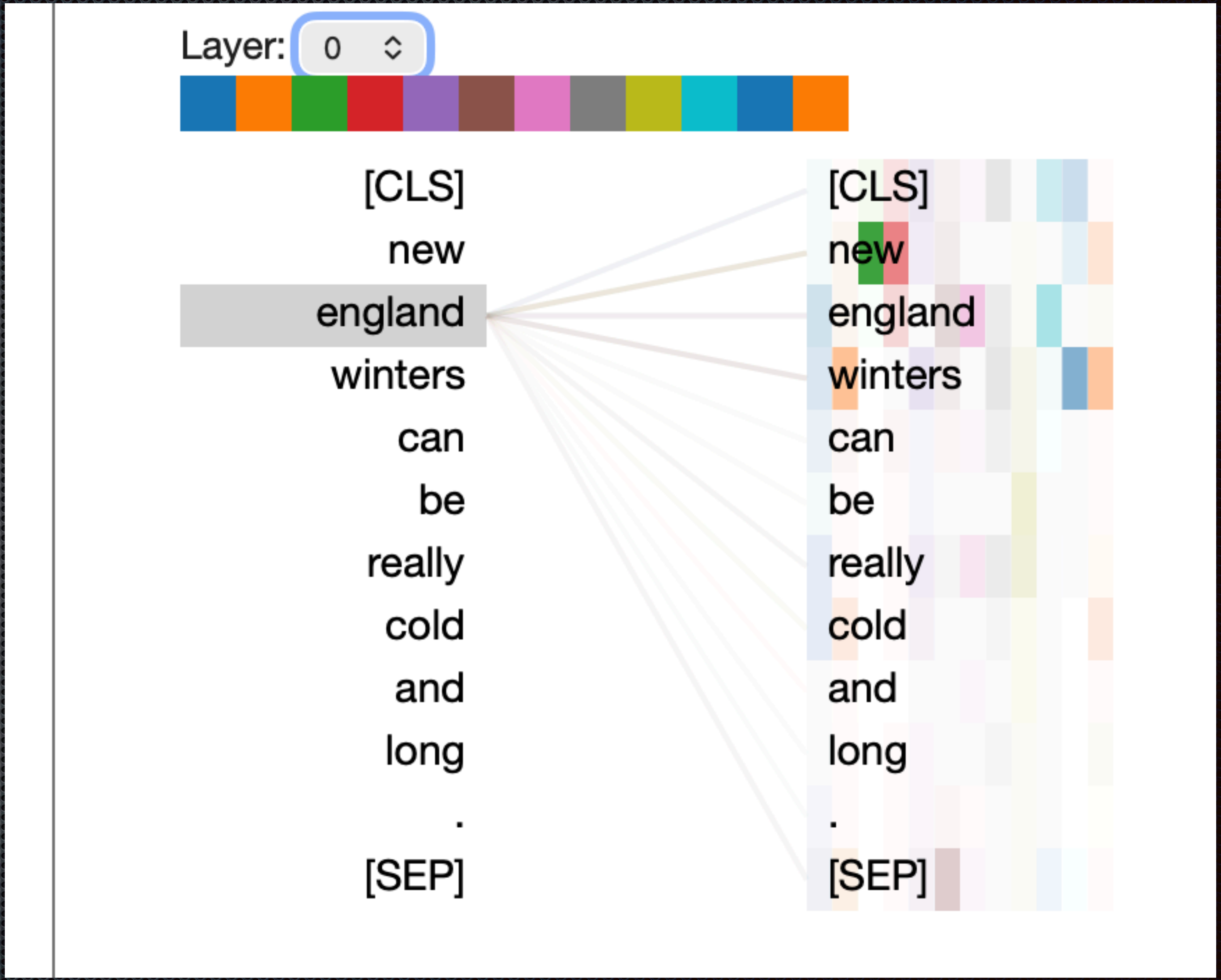
Self-attention

A Transformer models a word or a phrase in terms of its surrounding context

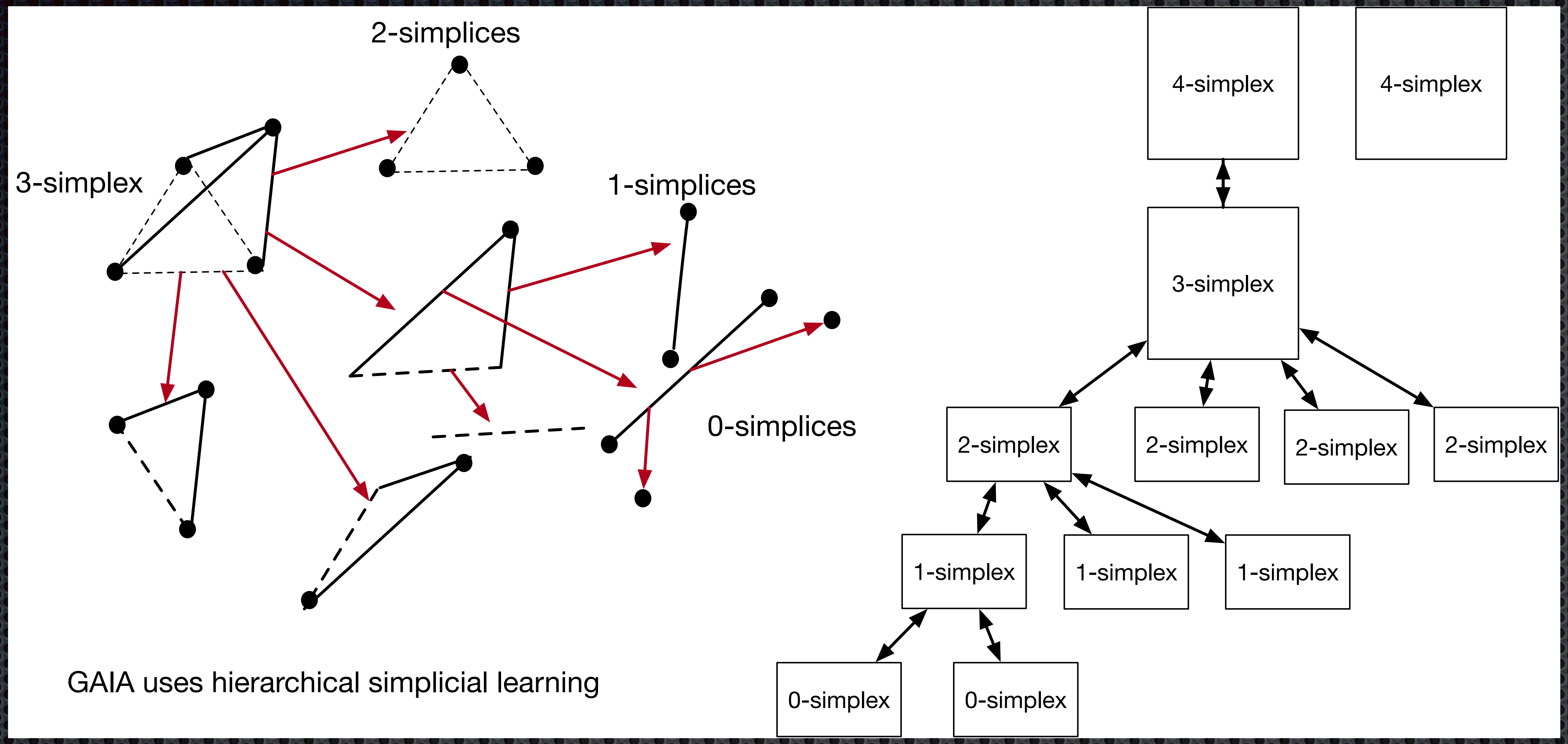


Self-attention

A Transformer models a word or a phrase in terms of its surrounding context



Why is the Yoneda lemma useful?



- The Yoneda lemma is central to the Geometric Transformer (Chapter 7)
  - GT models work by ``horn filling" gaps in simplicial sets
  - A simplicial set is a contravariant functor defined by Yoneda lemma

Yoneda lemma for enriched categories

- So far, we have assumed that the ``hom set” C(x,y) is a set
- In most AGI applications, C(x,y) is much more than a set
  - Vector space: Transformer models
  - Probability space: Causal models
  - Reward-enriched space: RL

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Theoretical Computer Science 193 (1998) 1–51

Theoretical Computer Science

Fundamental Study

Generalized metric spaces: Completion, topology, and powerdomains via the Yoneda embedding

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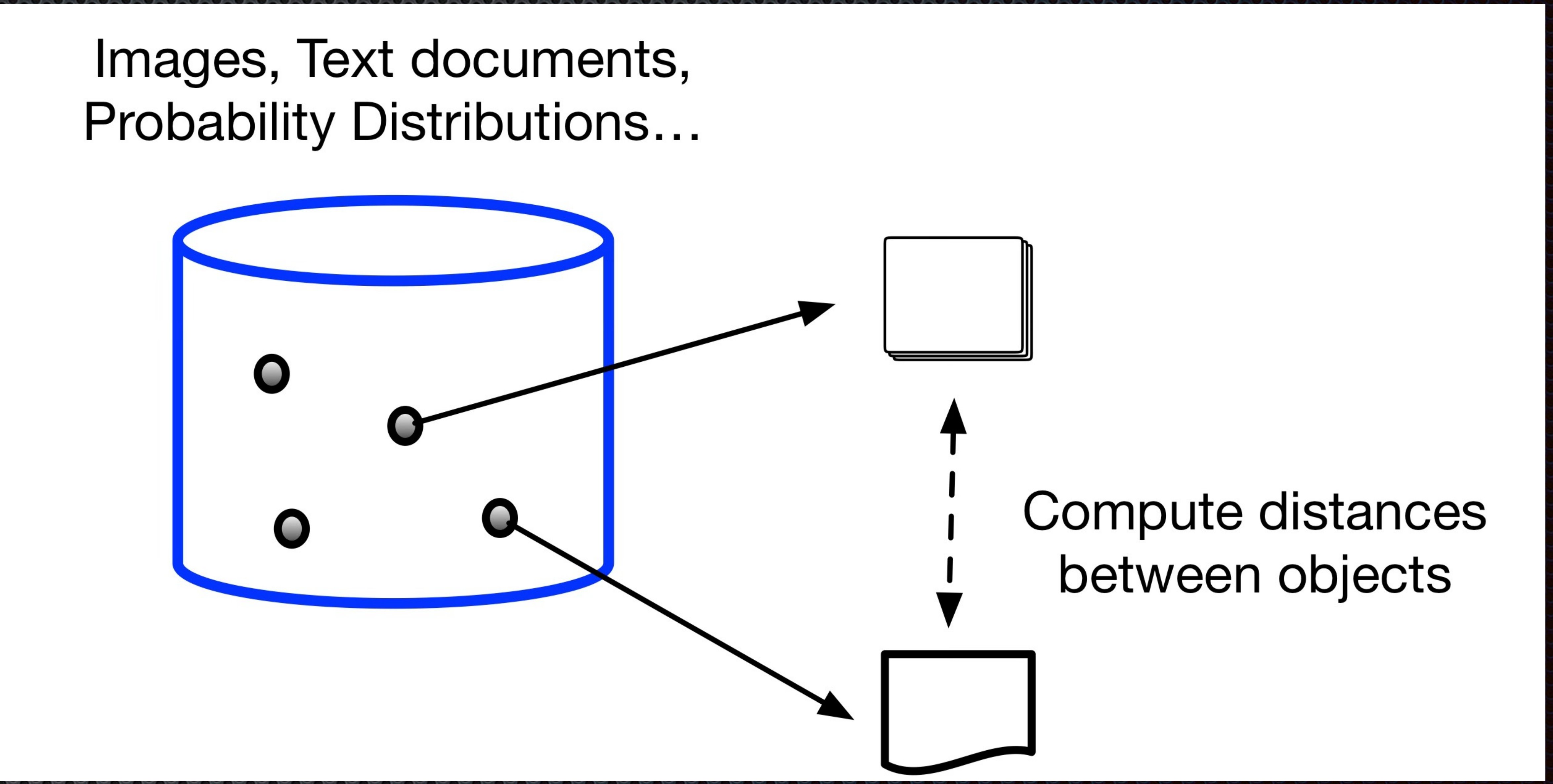
Abstract

Generalized metric spaces are a common generalization of preorders and ordinary metric spaces (Lawvere, 1973). Combining Lawvere's (1973) enriched-categorical and Smyth's (1988, 1991) topological view on generalized metric spaces, it is shown how to construct (1) completion, (2) two topologies, and (3) powerdomains for generalized metric spaces. Restricted to the special cases of preorders and ordinary metric spaces, these constructions yield, respectively: (1) chain completion and Cauchy completion; (2) the Alexandroff and the Scott topology, and the ε-ball topology; (3) lower, upper, and convex powerdomains, and the hyperspace of compact subsets. All constructions are formulated in terms of (a metric version of) the Yoneda (1954) embedding.

Metric Yoneda lemma

We now show how to apply the Yoneda lemma to generalized metric spaces



Metric spaces occur all over ML

Generalized metric space

- A gms is a set X together with a mapping $X(-,-) : X \times X \to [0,\infty]$
- A gms satisfies the following properties:
  - $X(x, x) = 0$
  - $X(x, z) \leq X(x, y) + X(y, z)$
- A gms does NOT satisfy other properties of standard metric spaces
  - Distances are symmetric
  - Distances are finite
  - Distances between two different objects cannot be 0

Examples: gms over Preorders

- Let us define a gms over a preordered set $(P, \leq)$
  - Reflexivity: $x \leq x$
  - Transitivity: $x \leq y, y \leq z \Rightarrow x \leq z$
  - The gms is defined as
    - If $x \leq y$, then $P(x, y) = 0$
    - If $x \nleq y$, then $P(x, y) = \infty$

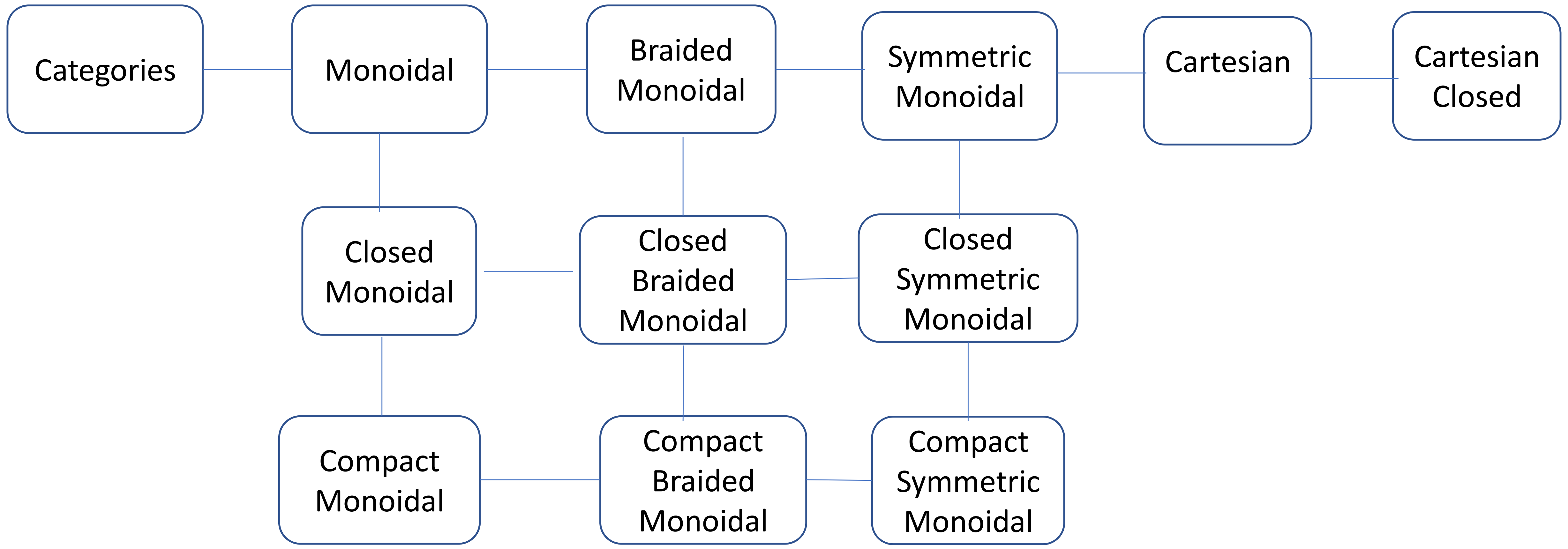
Example: gms over strings

- Consider the set of strings $\Sigma*$ over some alphabet Σ
- We can define a gms over the strings $\Sigma*$ as follows:
  - $\Sigma*(x, y) = 0$ if x is a prefix of y
  - $\Sigma*(x, y) = 2^{-n}$ otherwise where n is the longest common prefix

Example: gms over topological spaces

- We can define a gms over the power set $\mathscr{P}(X)$ of all subsets over a metric space as:
    - $\mathscr{P}(X)(V, W) = \inf\left(\epsilon > 0 \,|\, \forall v \in V, \exists w \in W \text{ s.t. } X(v, w) \leq \epsilon\right)$
- This distance is referred to as the non-symmetric Hausdorff distance

Monoidal Categories



- Equipped with a tensor product:
  - $\otimes : C \times C \to C$
  - Identity element 1: $1 \otimes c \simeq c \simeq c \otimes 1$
- Unit interval [0,1]: closed symmetric monoidal preorder
- $\mathscr{V}-$enriched monoidal category: $a, b \in C \Rightarrow C(a, b) \in \mathscr{V}$

Example: gms over distances

- Let us define a gms over the category $[0,\infty]$ of non-negative distances:
  - $[0,\infty](x, y) = 0$ if $x \geq y$
  - $[0,\infty](x, y) = y - x$ if $x < y$
- This category is complete and co-complete, symmetric monoidal, as well as compact and closed
  - Product of two elements is their max (or supremum)
  - Coproducts of two elements is their minimum (or infimum)
  - Monoidal product is defined as addition +

Metric Yoneda Lemma for gms

- We can construct "universal representers" in any gms by applying the Yoneda Lemma
- Let X be any gms. For any element $x \in X$
  - $X(-,x) : X^{op} \to [0,\infty] : y \mapsto X(y,x)$
- Let us define a category GMS with gms objects and arrows all non-expansive functions f
  - $Y(f(x), f(y)) \leq c \cdot X(x,y)$
  - where $c \in (0,1)$

Metric Yoneda lemma

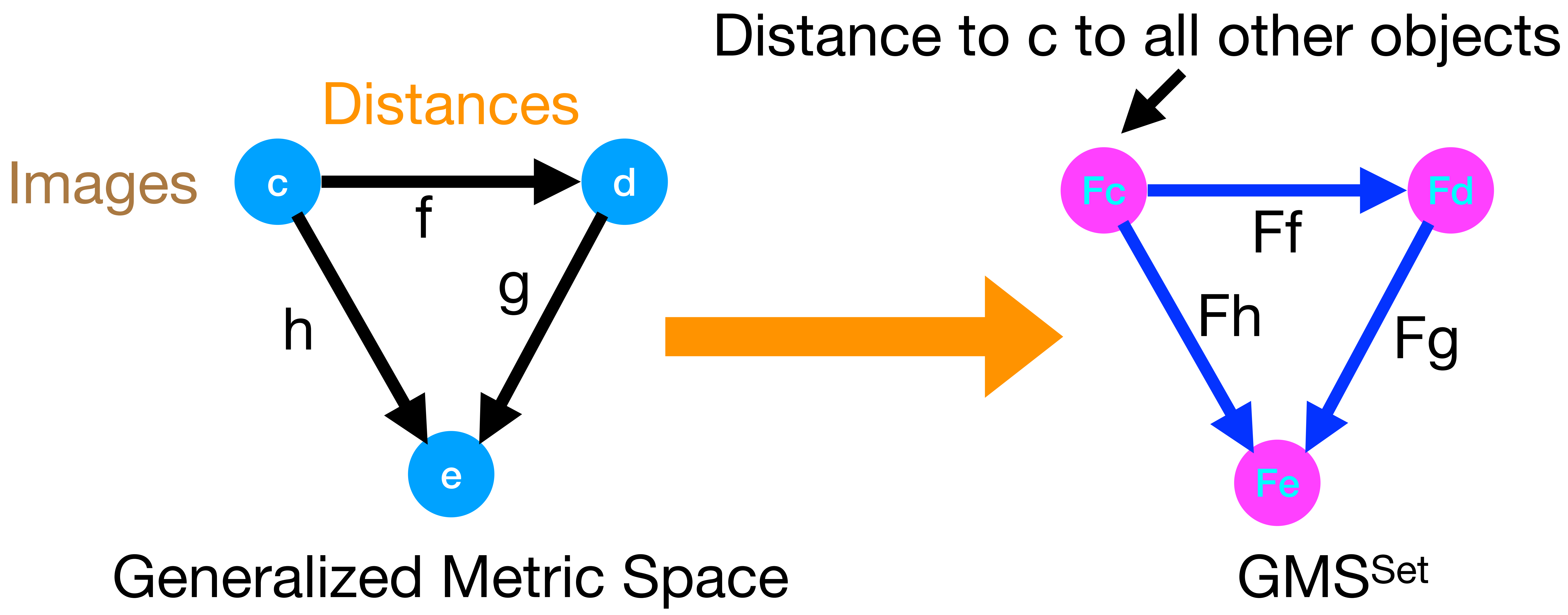
- Given a gms X, define its Yoneda embedding as the set of all non-expansive functions $\hat{X} = [0,\infty]^{X^{op}}$
- Metric Yoneda lemma: Given any gms X, for any x in X,

$$X(-,x) : X^{op} \to [0,\infty], y \mapsto X(y,x)$$

- For any non-expansive function $\phi(x) \in \hat{X}$

$$\hat{X}(X(-,x),\phi) = \phi(x)$$

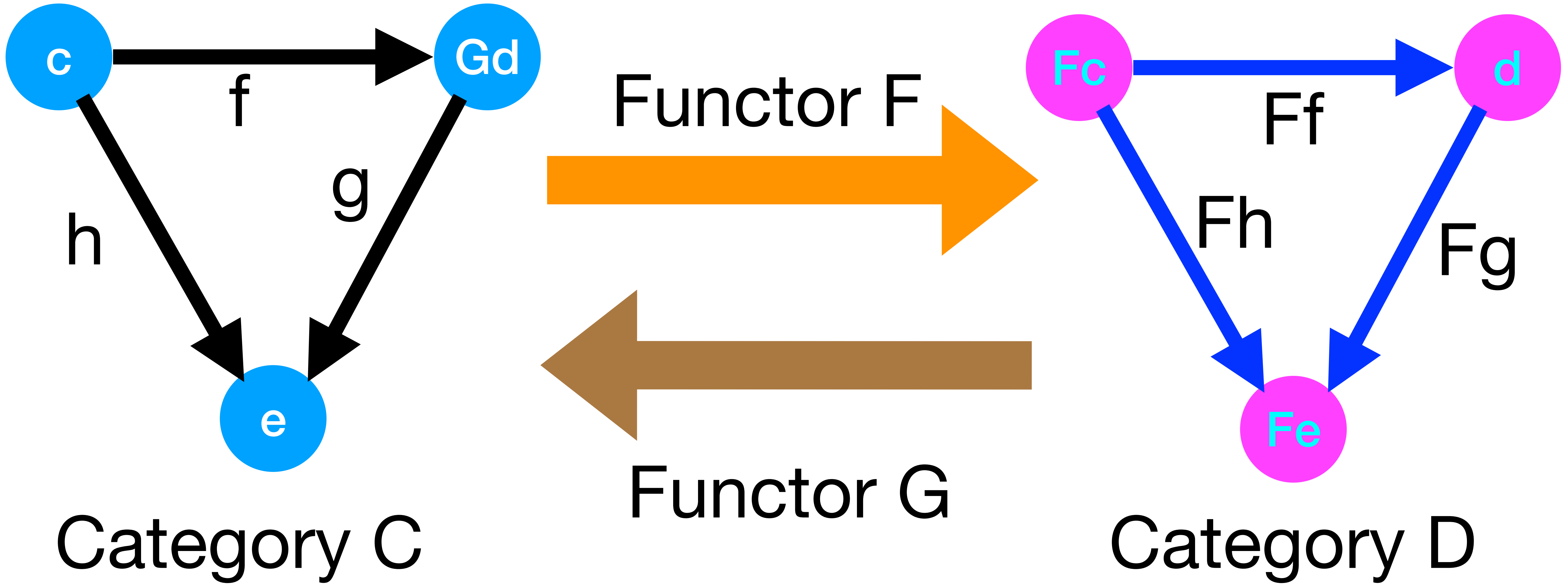
Yoneda embedding in metric spaces



Yoneda embeddings show how to construct generalized representers

Adjoint Functors

C(c,Gd) ~ D(Fc, d)



“Free” and “Forgetful” functors:

Sets —> Abelian Groups

Categories —> Graphs

Internal Hom and Closed Categories

- If C is a symmetric monoidal category, an internal hom in C is a functor

$$[\,-\,,-\,] : \mathscr{C}^{op} \times \mathscr{C} \to \mathscr{C}$$

such that for every object c in C, we have a pair of adjoint functors

$$((\,-\,) \times c \mapsto [c, -\,]) : \mathscr{C} \to \mathscr{C}$$

- Such categories are called closed monoidal categories

Exponential Objects

- A category C has “exponential objects” if and only if for each pair of objects c, d, we have the property $\mathscr{C}(e \times d, c) \simeq \mathscr{C}(e, c^d)$

We can define the exponential object for generalized metric spaces as

$$Y^X = \{f : X \to Y \,|\, f \text{ is non-expansive}\}$$

whose distance function is computed as

$$Y^X(f, g) = \sup\{Y(f(x), g(x)) \,|\, x \in X\}$$

Compact Closed Categories

- Let us define an "internal" Hom functor $[0,\infty](-,-)$ as simply the distance in $[0,\infty]$ as given previously
- The Yoneda embedding $[0,\infty](t,-)$ is ***right adjoint*** to $t+-$ for any $t \in [0,\infty]$
- **Theorem:** For all $r, s, t \in [0,\infty]$,
  - $t+s \geq r$ if and only if $s \geq [0,\infty](t,r)$

Metric Yoneda Lemma

- For any non-expansive function $\phi \in \hat{X}$
  - $\hat{X}(X(-,x), \phi) = \phi(x)$
- The Yoneda embedding is an *isometry!*
  - $\mathbf{y}(x) = X(-,x)$
  - $X(x,y) = \hat{X}(\mathbf{y}(x), \mathbf{y}(y)) = \hat{X}(X(-,x), X(-,y))$
- Recall we have made no assumptions about symmetry!

Further Reading

- Read Section 2.2 (Riehl's textbook), and Chapter 3 of my book
- Experiment with the Yoneda lemma micro-demo in GitHub repository
- Construct more examples of Natural Transformations and Yoneda lemma
  - Reinforcement learning
  - Natural language processing