(Co)Limits in Functor Categories

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Recap

- In the previous lecture, we introduced the notion of (co)limits
  - Pullbacks (limits) and pushouts (colimits)
  - Equalizers (limits) and coequalizers (colimits
- We saw that these constructions were applicable in a variety of settings
  - Audio-video synchronization is an example of a pullback
  - Limits in the category of database tables correspond to joins

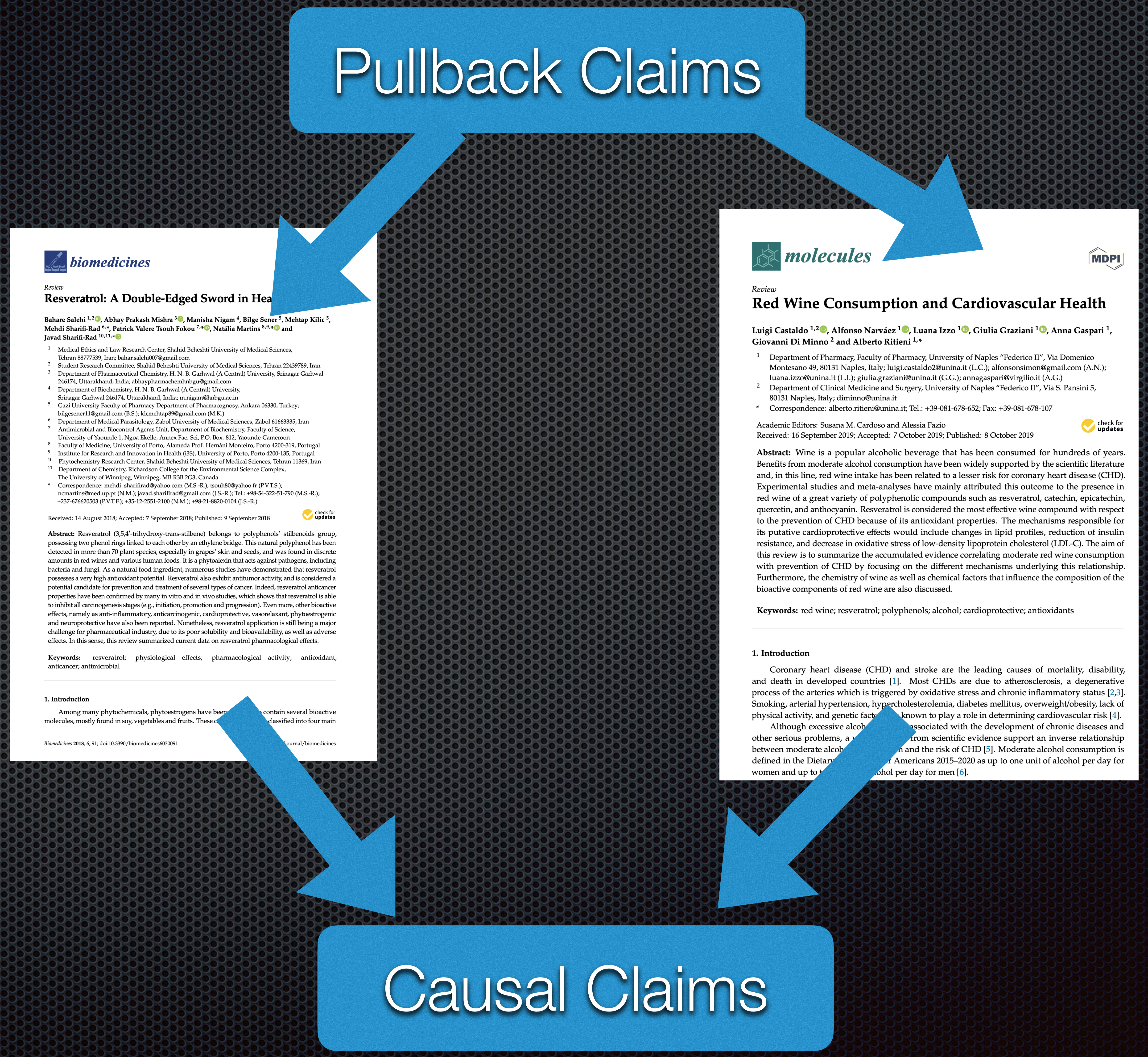
Functor Categories $C(-, x)$

- Yoneda embeddings create nice functor categories
- All limits and colimits exist
- Subobject classifiers can be constructed
- Exponentials can be defined
- Internal intuitionistic logic for reasoning

Does red wine have health benefits?

Numerous studies have investigated the health effects of drinking red wine

Integrating studies using pullback



Soft pullback reconciliation: resveratrol $\Rightarrow$ mitochondrial biogenesis

Soft pullback consensus ($A \times_C B$):

rel	src	dst	score_joint
INCREASES	resveratrol	mitochondrial biogenesis . . .	6.892

Witness evidence (top focus patches):

Atlas A: Cardiovascular Health (PMC6804046)

- **25.15** (r=3) Antioxidant effects of resveratrol on cellular aging
- **8.86** (r=2) Attenuation of cardiac fibrosis via TGF-β/Smad signaling inhibition
- **8.42** (r=2) Resveratrol and polyphenol-mediated cardioprotective pathways in red wine
- **8.42** (r=3) Enhancement of endothelial function and nitric oxide bioavailability

Atlas B: Resveratrol review (PMC6164842)

- **26.07** (r=2) Anti-inflammatory effects of resveratrol
- **25.70** (r=1) Resveratrol
- **25.63** (r=2) Resveratrol pharmacological effects
- **25.26** (r=2) Anti-inflammatory effects and cytokine modulation

Takeaway: Soft pullback glues two atlases and returns both the aligned claim and auditable provenance.

Soft pullback reconciliation: resveratrol $\Rightarrow$ endothelial function

Soft pullback consensus ($A \times_C B$):

rel	src	dst	score_joint
INCREASES	resveratrol	endothelial function . . .	4.305

Witness evidence (top focus patches):

Atlas A: Cardiovascular Health (PMC6804046)

- **8.86** (r=2) Attenuation of cardiac fibrosis via TGF-β/Smad signaling inhibition
- **8.42** (r=2) Resveratrol and polyphenol-mediated cardioprotective pathways in red wine
- **8.42** (r=3) Enhancement of endothelial function and nitric oxide bioavailability
- **7.98** (r=3) Resveratrol's anti-inflammatory effects through NF-κB and TNF-α suppression

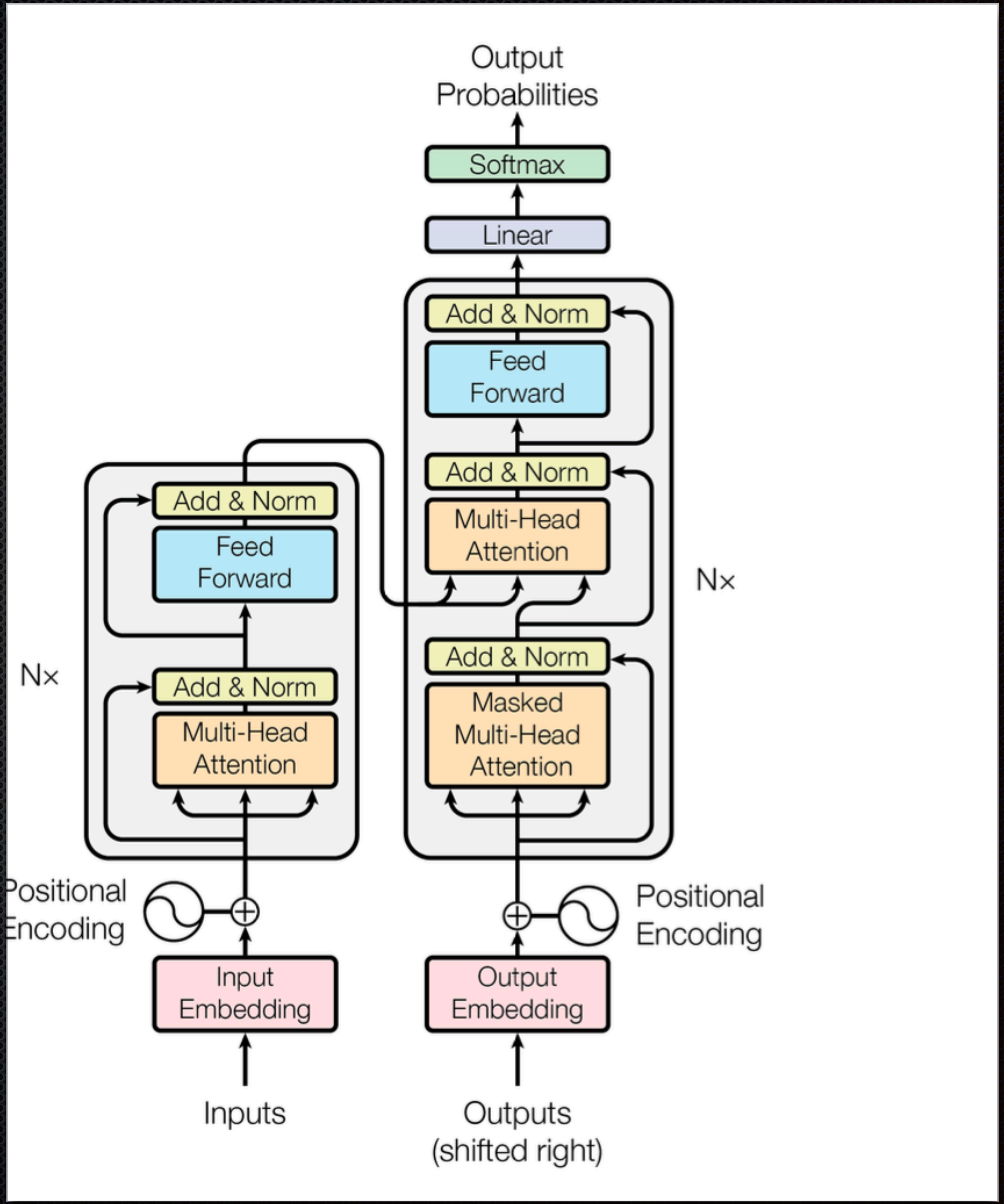
Atlas B: Resveratrol review (PMC6164842)

- **25.63** (r=2) Resveratrol pharmacological effects
- **24.82** (r=3) Age-related inflammation (inflammaging) and resveratrol's anti-inflammatory efficacy
- **24.78** (r=2) Improvement in insulin sensitivity and glucose metabolism
- **24.37** (r=3) Cardioprotective effects and endothelial function improvement

Takeaway: Soft pullback recovers a shared mechanism and provides auditable witnesses from both atlases

Transformers

A Transformer is a deep learning architecture that implements a sequence-to-sequence function

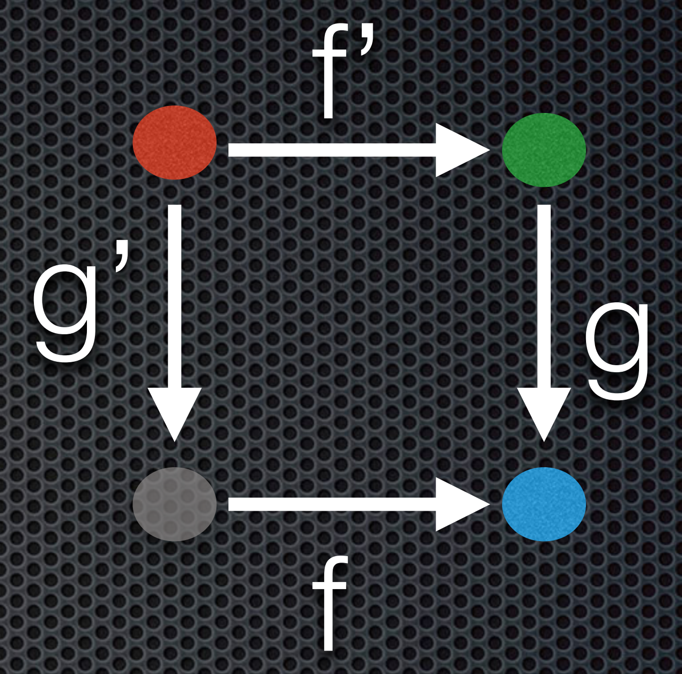


Functor Categories and AGI

- We can define a category of Transformers as a functor category, where each object is a sequence-to-sequence function
- We want to show that such Transformer categories must possess all (co)limits
- This implies many interesting new constructions exist for Transformer models that have not yet been explored in practice!

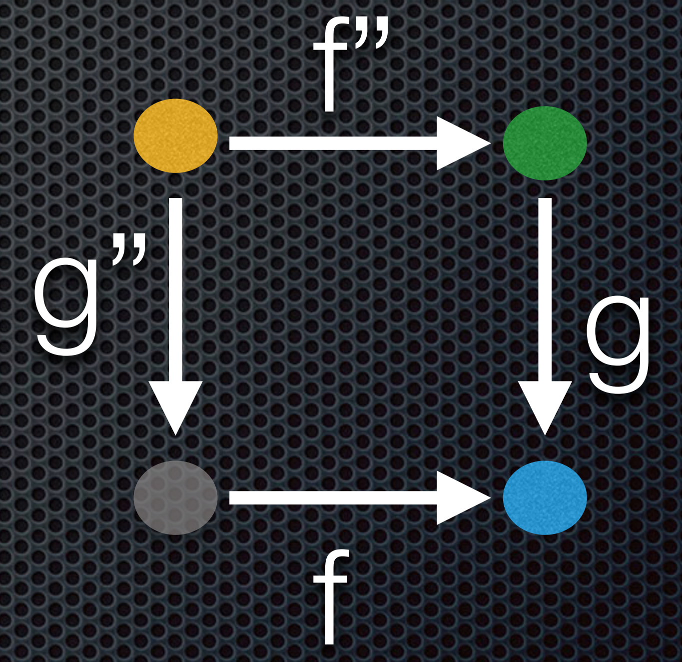
[Mahadevan, Topos Theory for Generative AI and LLMs, Arxiv 2025]

Pullback



In the category of sets, pullbacks always exist

The pullback of f and g is simply the set of ordered pairs (x,y) such that $f(x) = g(y)$



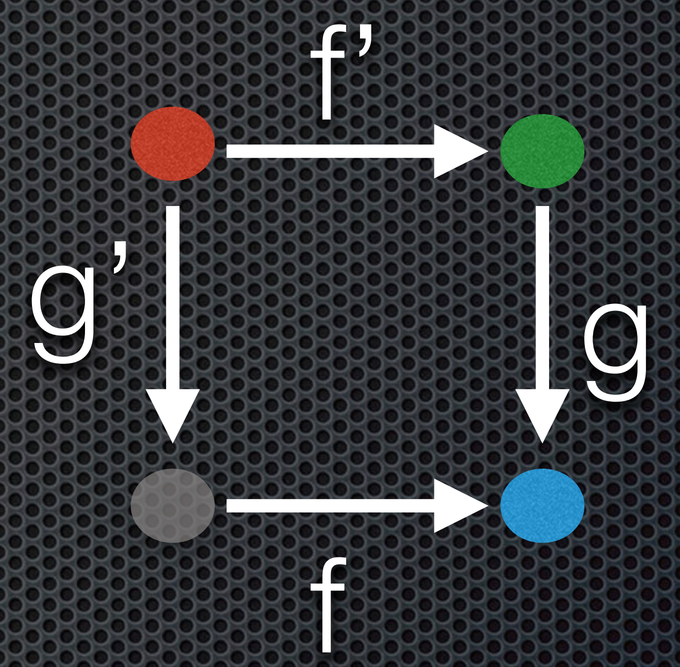
$f'' = f'\,h$

$g'' = g'\,h$

Functor Category

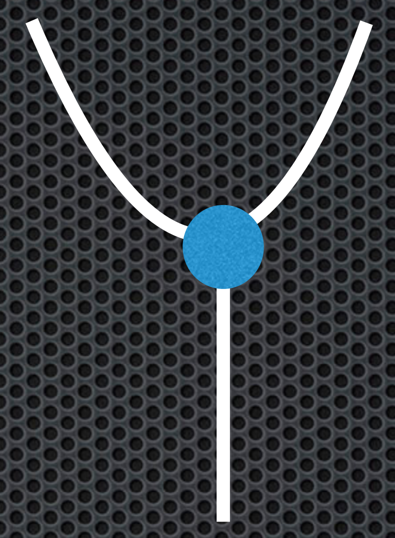
If each object in the category is a functor, pullbacks may or may not exist

If each object in these diagrams is a functor, the arrows are natural transformations

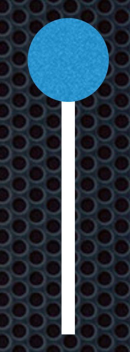


Markov Category

A Markov category is a symmetric monoidal category where each object can be "copied" or "deleted"



"copy"



"delete"

[Tobias Fritz, A Synthetic Approach to Probability…, 2020]

A synthetic approach to Markov kernels, conditional independence and theorems on sufficient statistics

Tobias Fritz

Abstract. We develop *Markov categories* as a framework for synthetic probability and statistics, following work of Golubtsov as well as Cho and Jacobs. This means that we treat the following concepts in purely abstract categorical terms: conditioning and disintegration; various versions of conditional independence and its standard properties; conditional products; almost surely; sufficient statistics; versions of theorems on sufficient statistics due to Fisher–Neyman, Basu, and Bahadur.

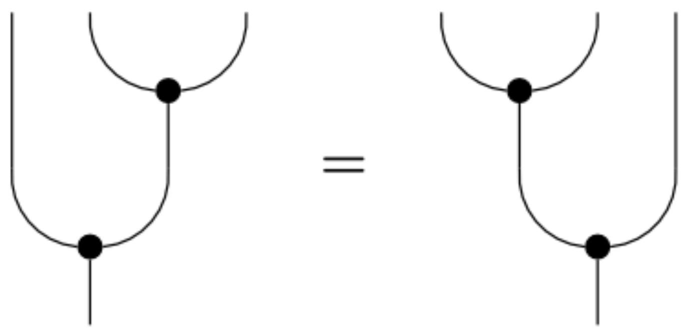
Besides the conceptual clarity offered by our categorical setup, its main advantage is that it provides a uniform treatment of various types of probability theory, including discrete probability theory, measure-theoretic probability with general measurable spaces, Gaussian probability, stochastic processes of either of these kinds, and many others.

Contents

2.1. Definition. *A* Markov category C *is a symmetric monoidal category in which every object $X \in \mathsf{C}$ is equipped with a commutative comonoid structure given by a comultiplication $\mathsf{copy}_X : X \to X \otimes X$ and a counit $\mathsf{del}_X : X \to I$, depicted in string diagrams as*

(2.1)

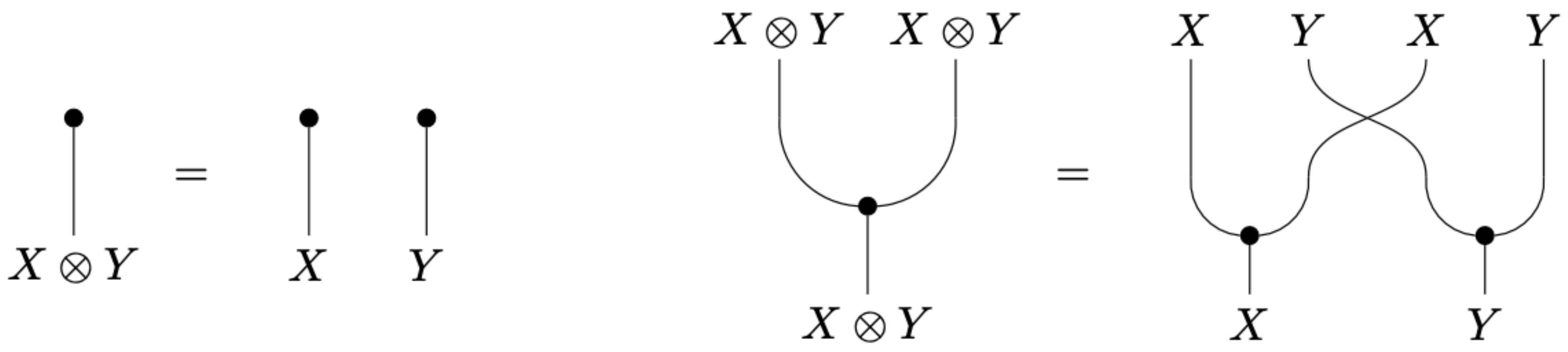
and satisfying the commutative comonoid equations,



(2.2)

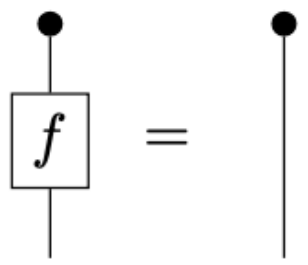
(2.3)

as well as compatibility with the monoidal structure,



(2.4)

and naturality of del, *which means that*

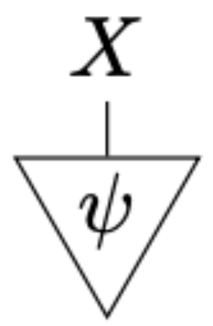


(2.5)

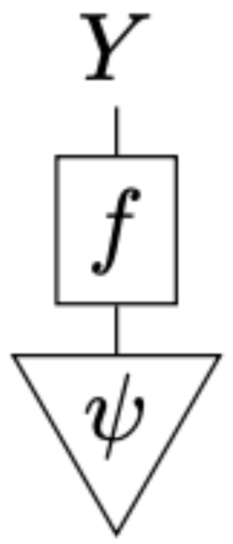
for every morphism f.

Probabilistic Reasoning with Diagrams

$\psi : I \to X$

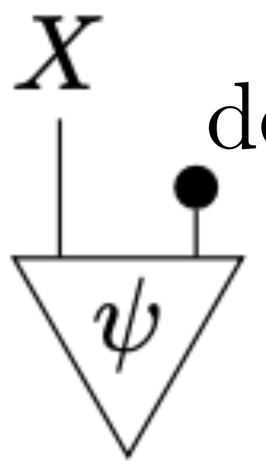


Distribution



$f : X \to Y$

Random Variable



$\mathrm{del}_Y : Y \to I$

$\psi : I \to X \otimes Y$

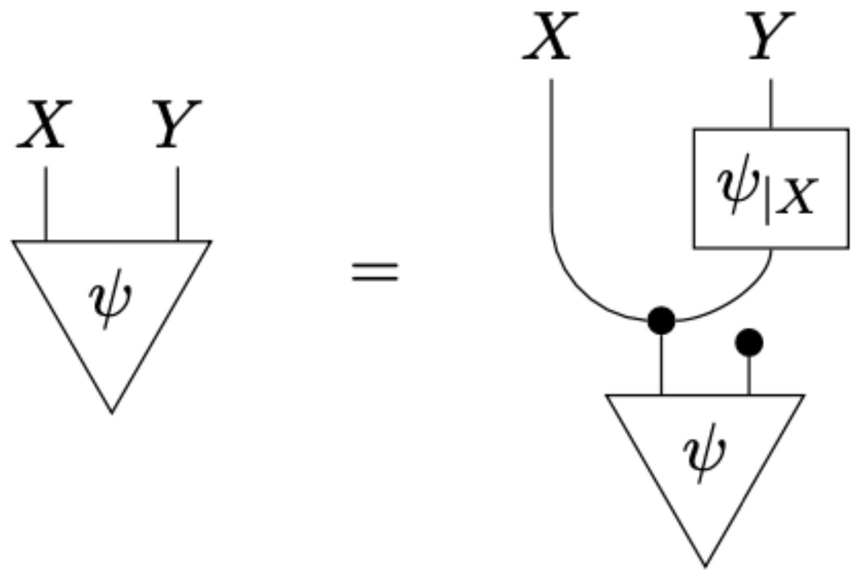
Marginalization

Replaces the conventional ``measure-theoretic” foundation of probability

[Fritz, Adv. in Math, 2020]

String Diagrams: Markov Categories

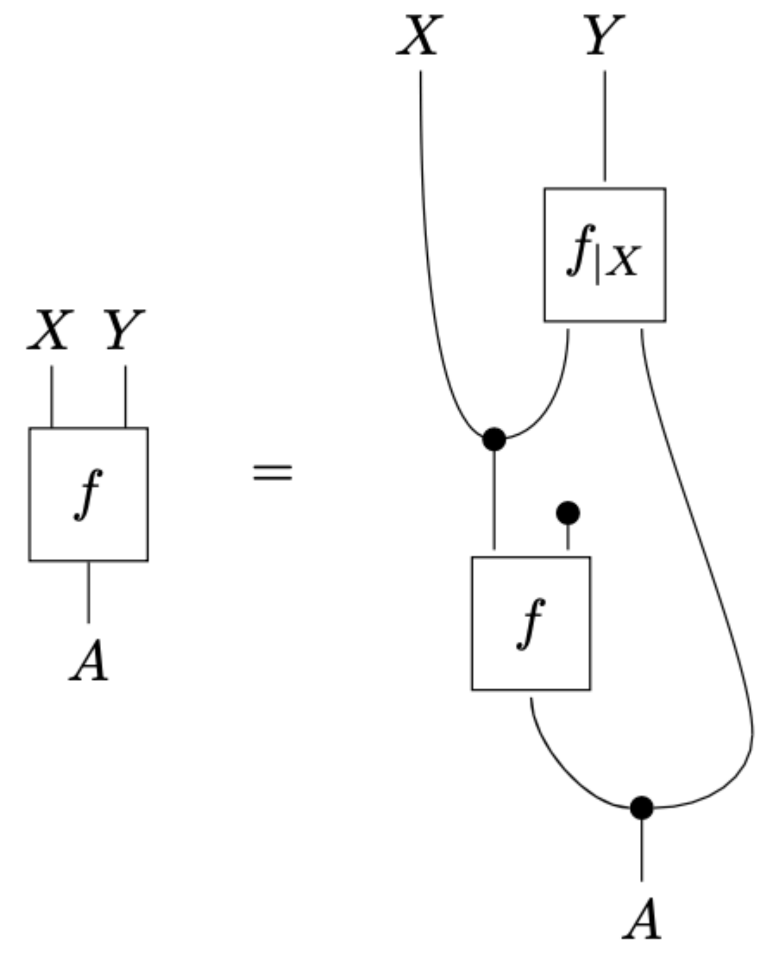
$\psi : I \to X \otimes Y$



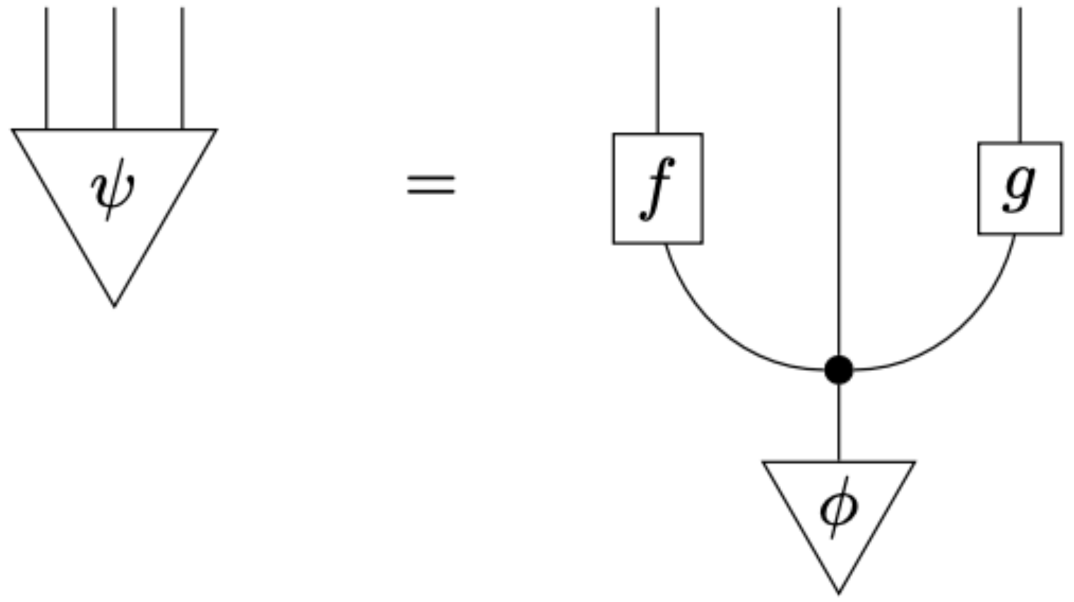
$\psi_{|X} : X \to Y$

Conditional Distribution

Generalizes P(X,Y) = P(Y|X) P(X)



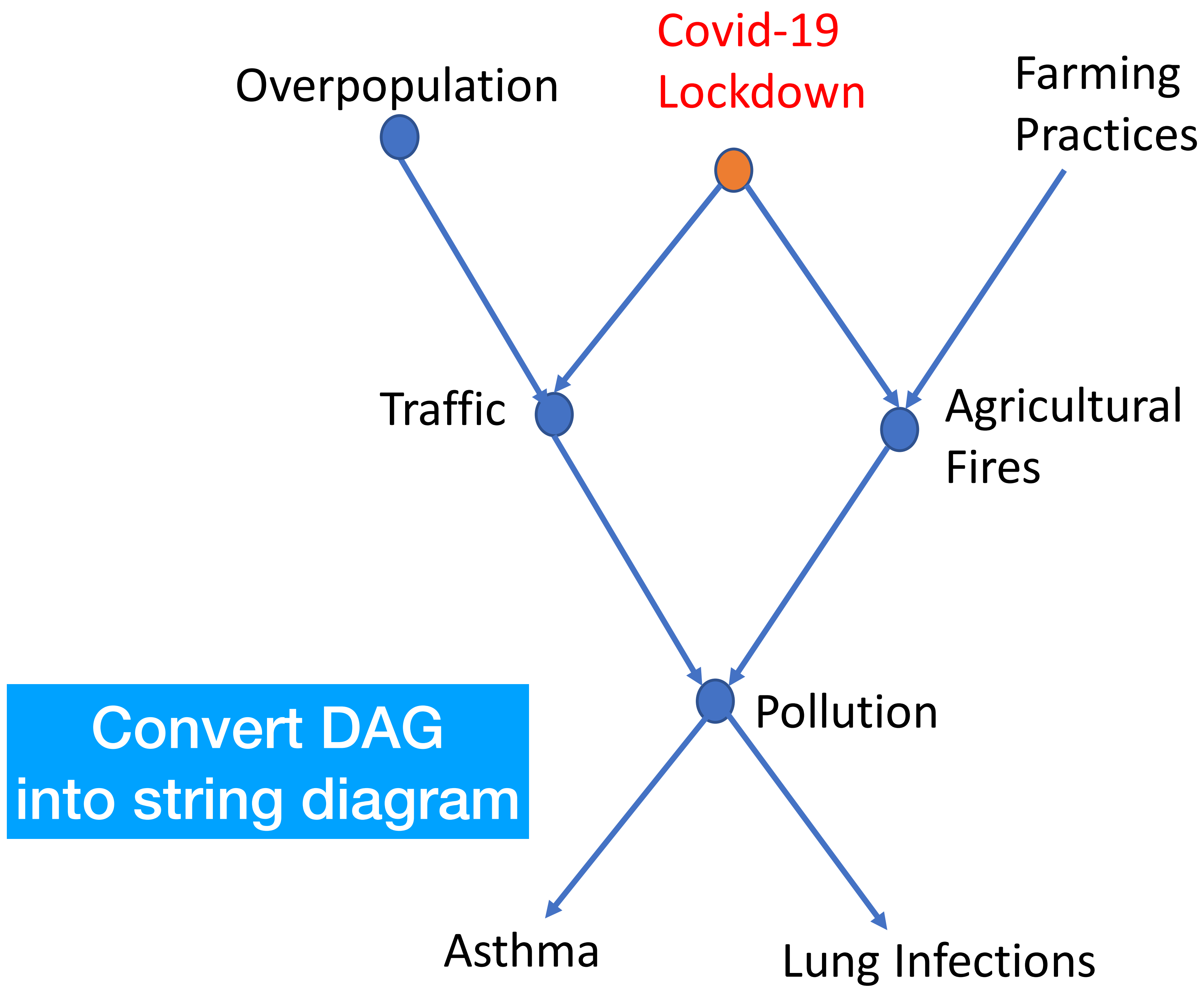
Disintegration:Bayes Rule

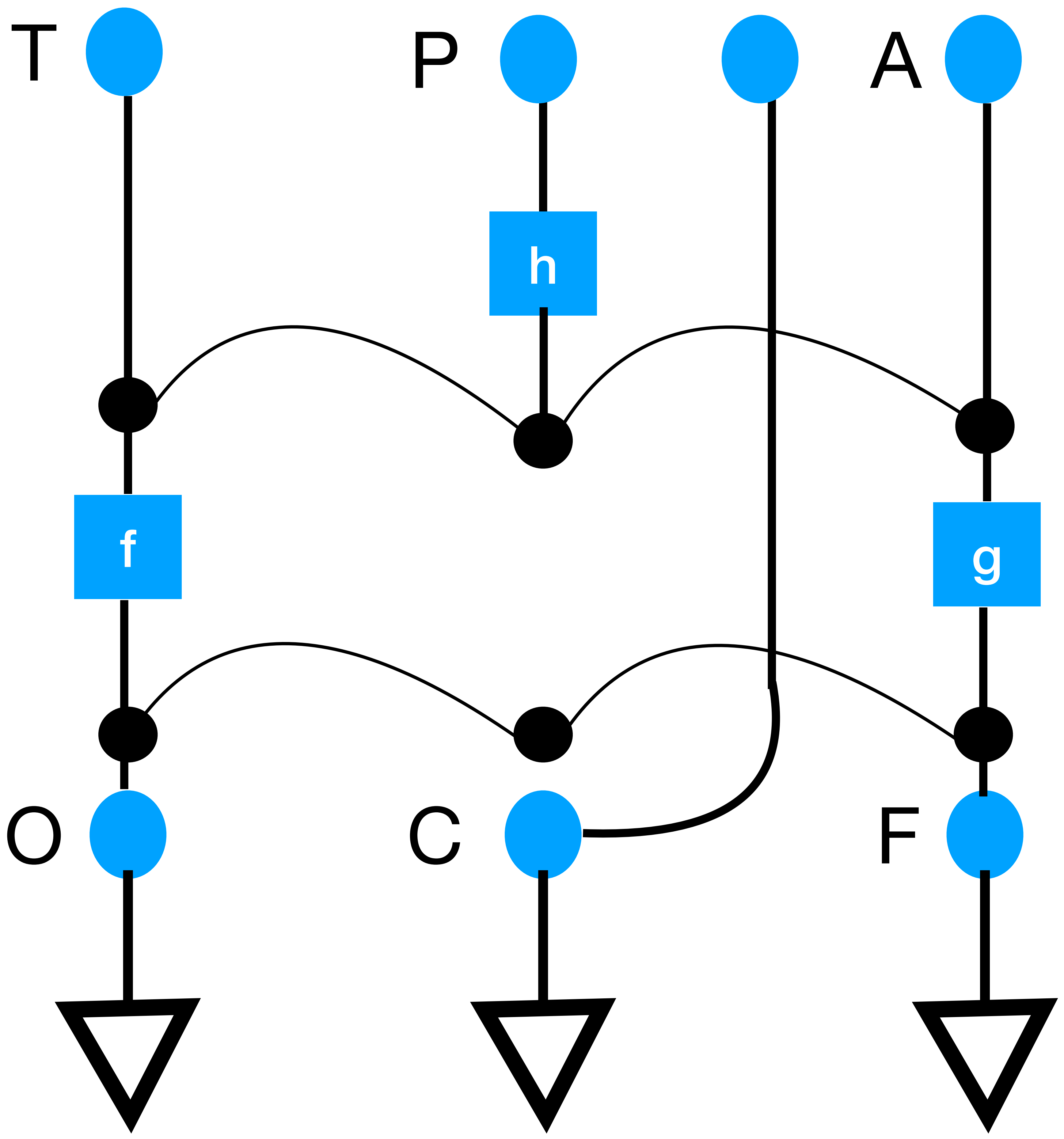


Conditional independence

P(X, W, Y) = P(X | W) P(Y | W)

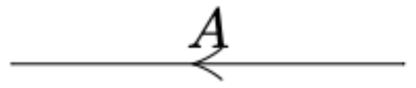
Causal Models as String Diagrams

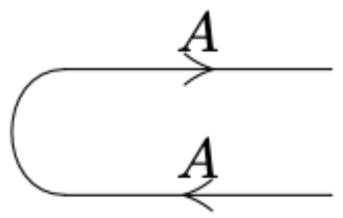


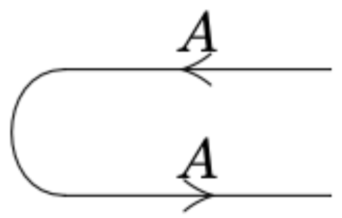


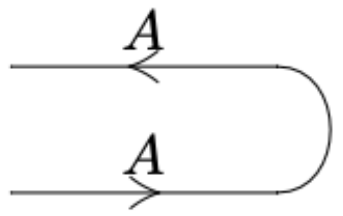
Exogenous variables

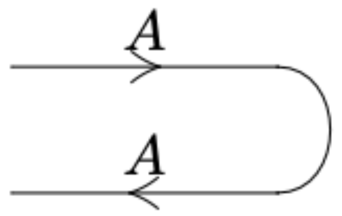
Rich Family of String Diagrams

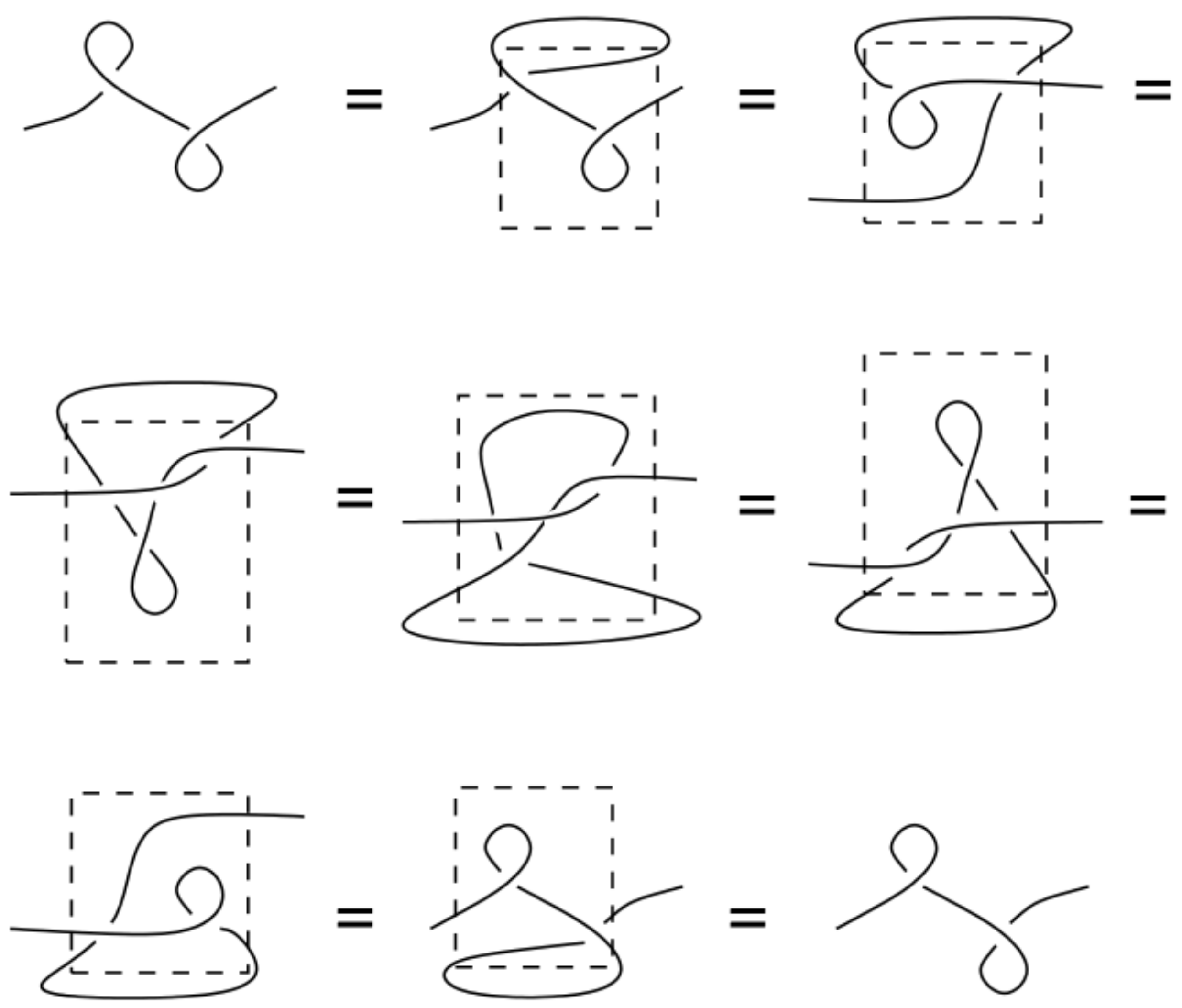
Dual $A^*, {}^*A$ 

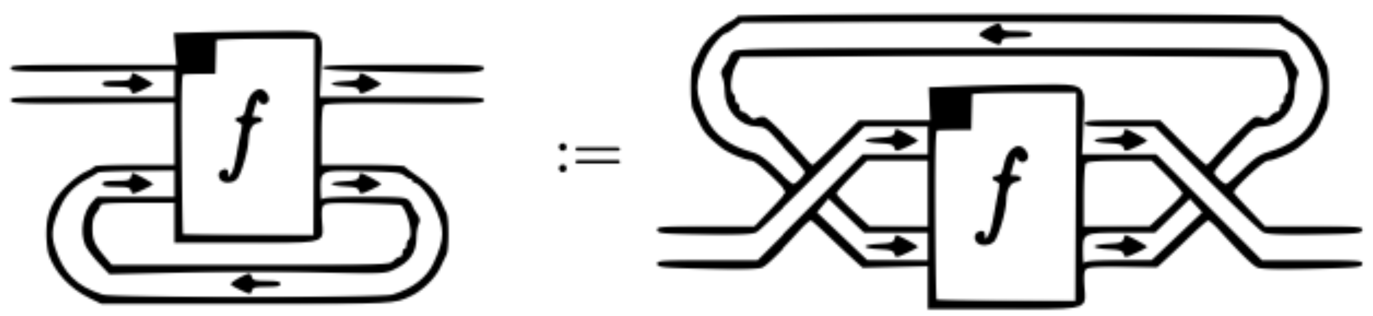
Unit $\eta_A : I \to A^* \otimes A$ 

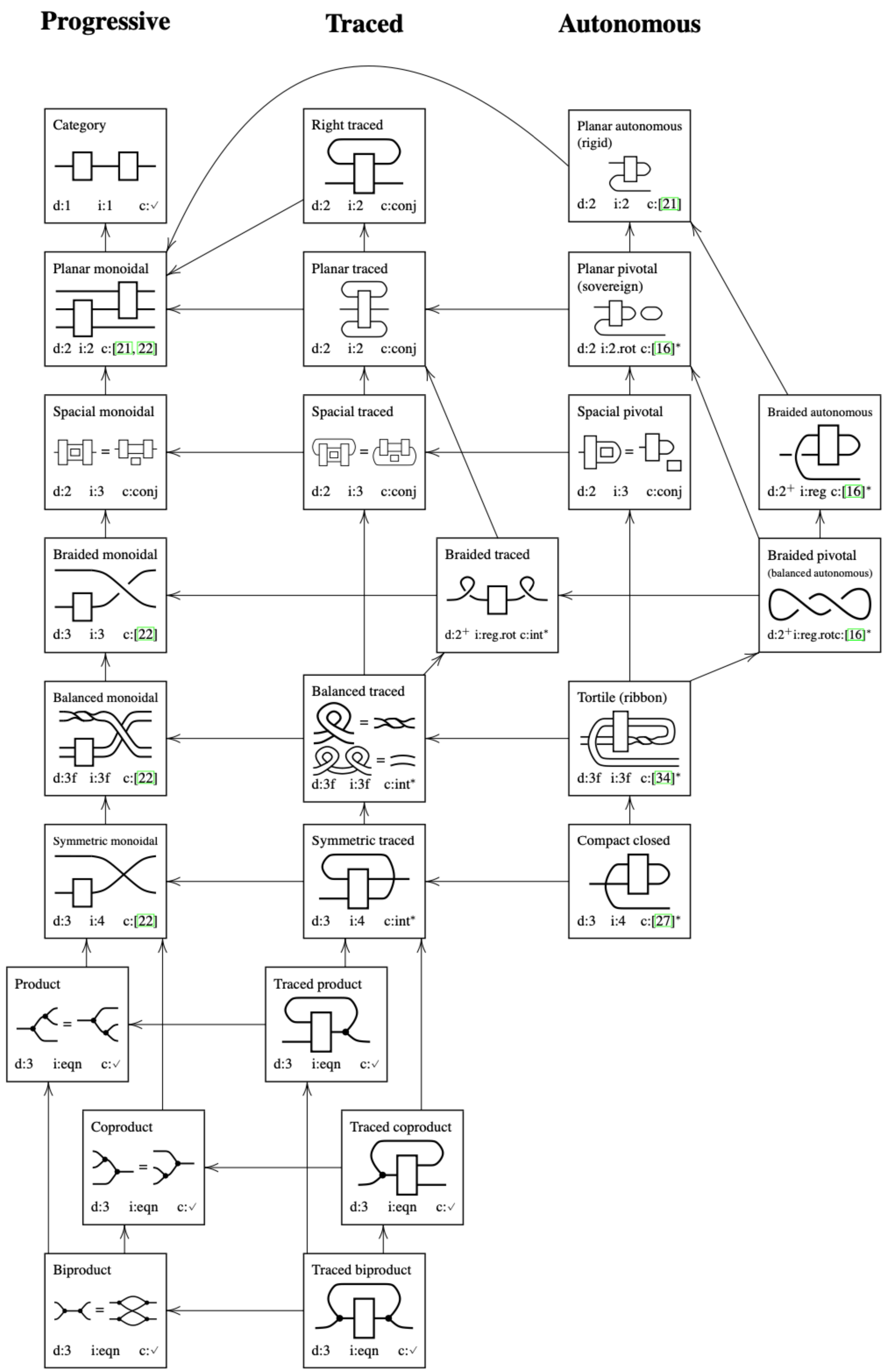
$\eta'_A : I \to A \otimes {}^*A$ 

Counit $\epsilon_A : A \otimes A^* \to I$ 

$\epsilon'_A : {}^*A \otimes A \to I$ 







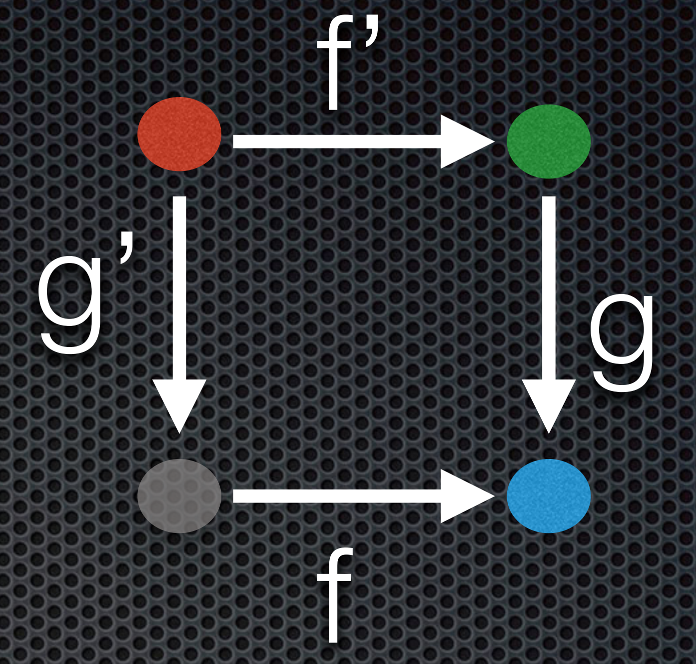
[Selinger, A survey of graphical languages for monoidal categories, Arxiv, 2020]

Marginalization is not projection

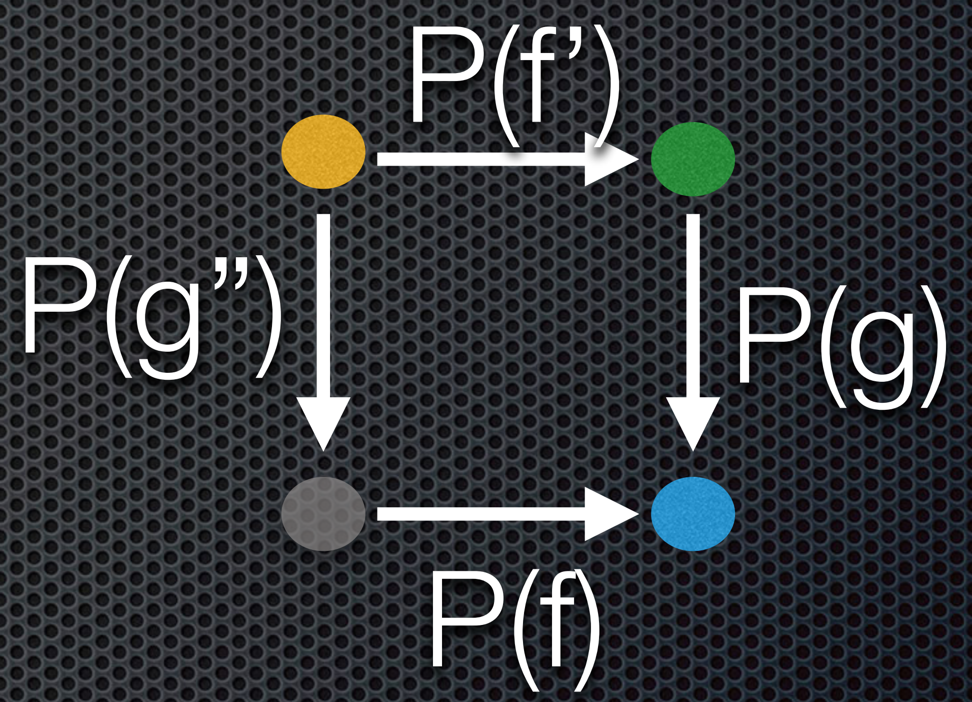
- Let f: I -> X*Y be a joint distribution on the product object X*Y
- Since any object can be deleted, we get
  - I -> X*Y -> X (marginalization on Y)
  - I -> X*Y -> Y (marginalization on X)
- However, a joint distribution does not uniquely determine marginal distributions
- So, marginalization is not like a projection (such as Cartesian products of sets)

CoPresheaves

P: C -> Set



Recall from the Yoneda Lemma that for any object x in category C, its embedding C(x,-) is a set-valued functor (also called a copresheaf)



If each object in these diagrams is a set-valued functor, pullbacks always exist

Pullbacks are computed “point-wise” in this case

P(f”) = P(f’) P(h)

Pg”) = P(g’) P(h)

Pullbacks in Transformer Category

- We want to prove that in a category of sequence-to-sequence functions, pullbacks exist
- We reduce the problem to pullback in the category of sets

$$
\begin{tikzcd}
\bullet \arrow[r, "f"] \arrow[d, "T_1"'] & \bullet \arrow[d, "T_2"] \\
\bullet \arrow[r, "g"] & \bullet
\end{tikzcd}
$$

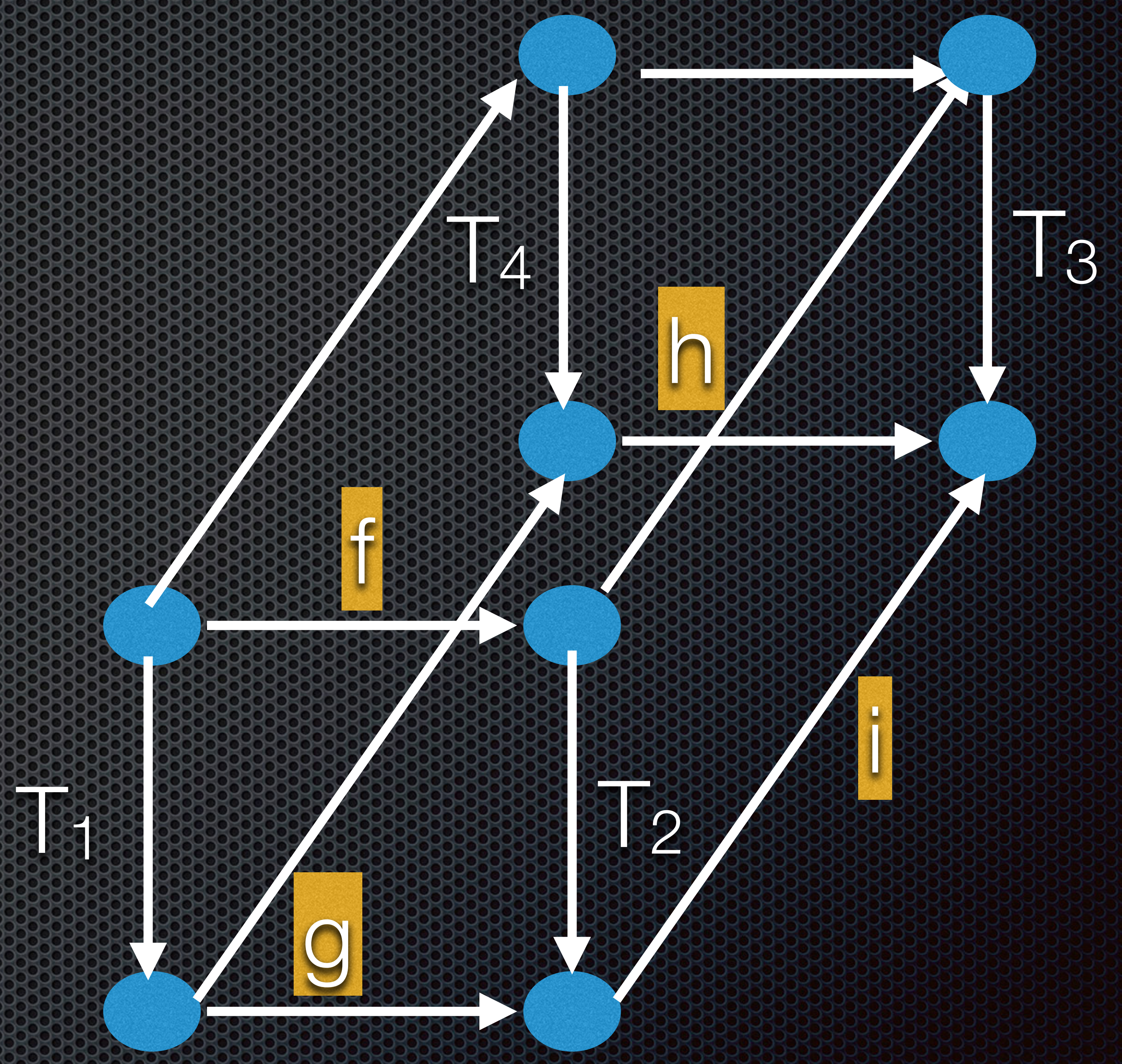
Each object T_i is a Transformer

Arrows are (f,g) that make diagrams

commutative

Why Pullbacks exist in Transformer Categories

- We can construct a “pullback Transformer” model given any two Transformer models
- Problem: fill in the missing arrows using pullbacks of functions in the category of sets



[Mahadevan, Topos Theory for Generative AI and LLMs, Arxiv 2025]

Properties of Transformer Categories

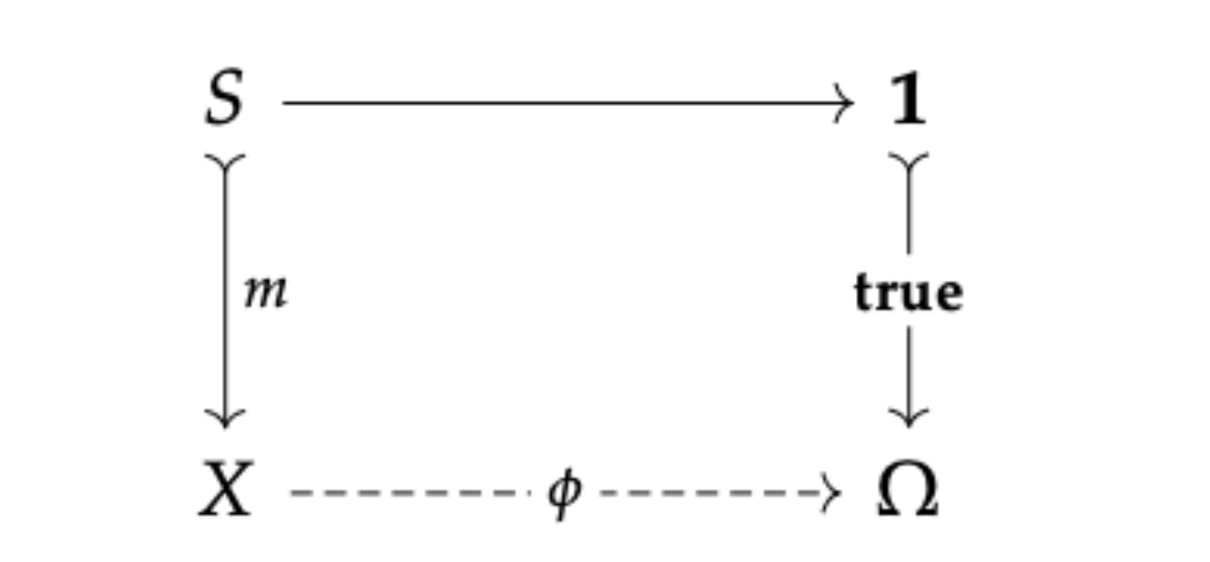
- All (co)limits exist, so it is (co)complete
- Subobject classifiers exist
- Exponential objects exist
- It forms a topos category

Subobject Classifier

- A subobject classifier for a category C is a C-object Ω and a C-arrow $\mathbf{true} : \mathbf{1} \to \Omega$ satisfying a pullback property

- Every "monic" arrow is classified by the subobject classifier

- A monic arrow f: a-> b is left cancellable, so that f g = f h implies g = h



Subobject Classifier in Sets

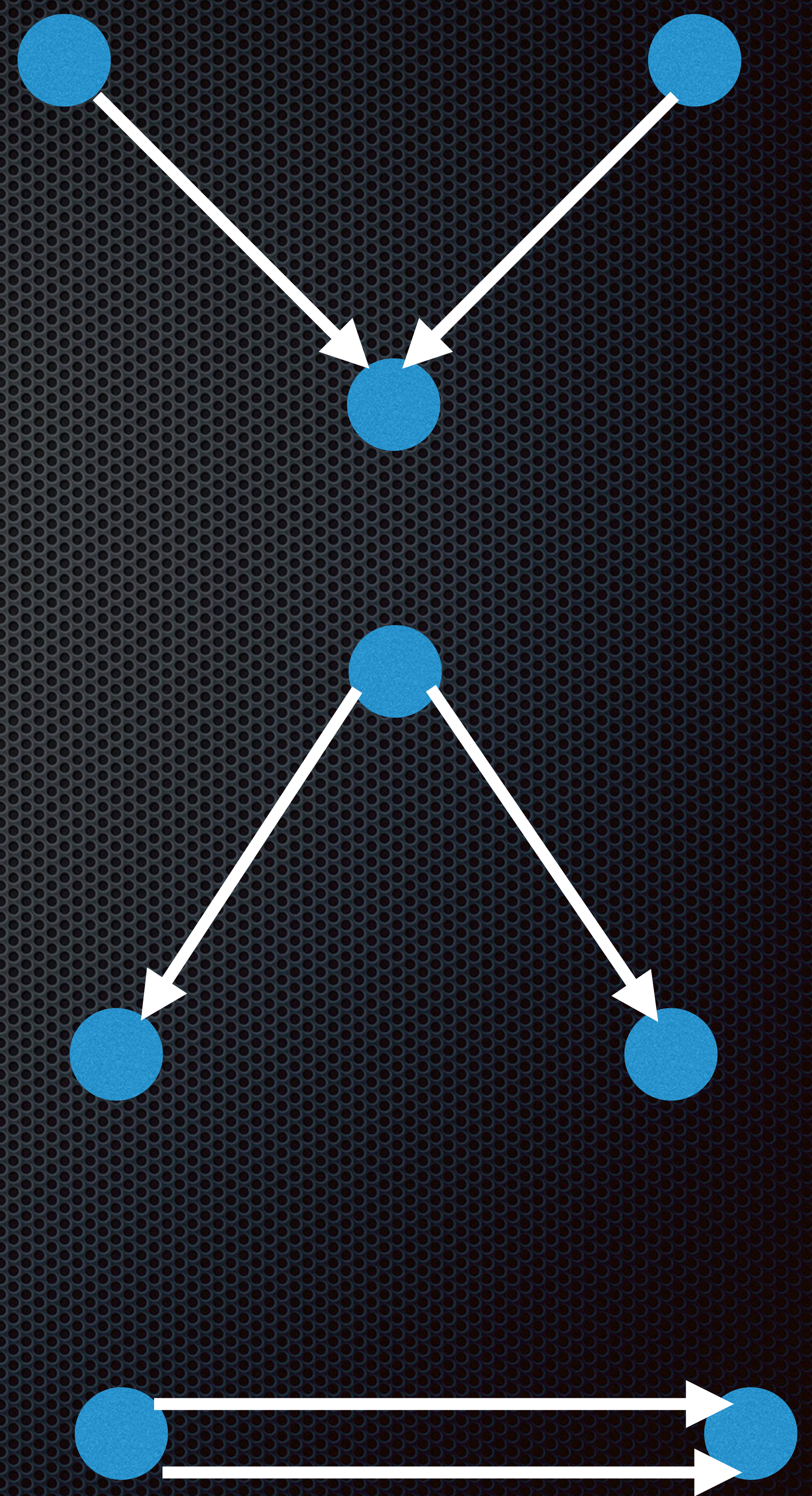
- A subobject classifier exists for the category of sets.
- The C-object Ω = {0,1}
- The C-arrow $\mathbf{true} : \mathbf{1} \rightarrow \Omega$ exists because the terminal object is a single element set
- Every “monic” arrow is classified by the subobject classifier, because for any subset S of a larger set X, the subobject classifier is defined but the induced boolean function mapping elements of X in S to 1, and other elements to 0

$$\begin{tikzcd}
S \arrow[r] \arrow[d, tail, "m"'] & \mathbf{1} \arrow[d, tail, "\mathbf{true}"] \\
X \arrow[r, dashed, "\phi"] & \Omega
\end{tikzcd}$$

Exponential Objects

- A category C has exponential objects if for each pair of objects c, d in C, there is a bijection
- $C(e \times d, c) \sim C(e, c^d)$

Novel Transformer Architectures



Category theory constructions like (co)limits suggest new architectures for Transformer models

Interpret each object on the right as a Transformer

Then the (co)limits give you new Transformer models!

Further Reading

- Chapter 3 in Riehl's text and my Categories for AI textbook
- Mahadevan, Topos Theory for Generative AI and LLMs, Arxiv 2025
- GitHub repo: experiment with various Transformer implementations
- Exercise: how would you implement a "pullback Transformer"?