Natural Transformations and the Yoneda lemma: Categorical AGI

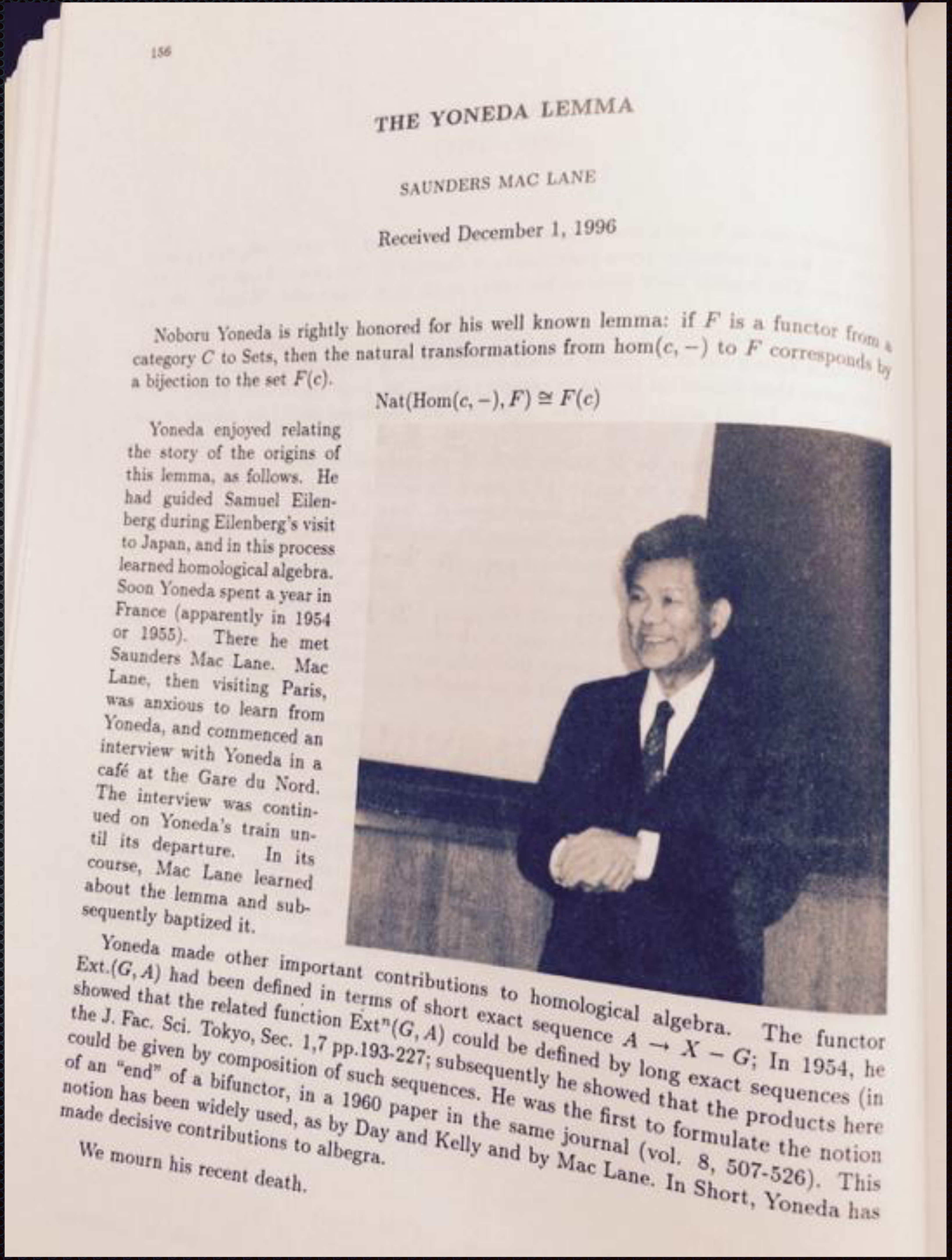
Sridhar Mahadevan, Adobe Research and U.Mass, Amherst

“The Yoneda lemma is arguably the most important result in category theory.”

–Emily Riehl, Category Theory in Context, Chapter 2 (page 50)

Gare du Nord

The Yoneda lemma came to life in this iconic train station in Paris about 70 years ago



THE YONEDA LEMMA

SAUNDERS MAC LANE

Received December 1, 1996

Noboru Yoneda is rightly honored for his well known lemma: if F is a functor from a category C to Sets, then the natural transformations from $\hom(c,-)$ to F corresponds by a bijection to the set $F(c)$.

$$\mathrm{Nat}(\mathrm{Hom}(c,-),F) \cong F(c)$$

Yoneda enjoyed relating the story of the origins of this lemma, as follows. He had guided Samuel Eilenberg during Eilenberg's visit to Japan, and in this process learned homological algebra. Soon Yoneda spent a year in France (apparently in 1954 or 1955). There he met Saunders Mac Lane. Mac Lane, then visiting Paris, was anxious to learn from Yoneda, and commenced an interview with Yoneda in a café at the Gare du Nord. The interview was continued on Yoneda's train until its departure. In its course, Mac Lane learned about the lemma and subsequently baptized it.

Yoneda made other important contributions to homological algebra. The functor $\mathrm{Ext}.(G,A)$ had been defined in terms of short exact sequence $A \to X - G$; In 1954, he showed that the related function $\mathrm{Ext}^n(G,A)$ could be defined by long exact sequences (in the J. Fac. Sci. Tokyo, Sec. 1,7 pp.193-227; subsequently he showed that the products here could be given by composition of such sequences. He was the first to formulate the notion of an "end" of a bifunctor, in a 1960 paper in the same journal (vol. 8, 507-526). This notion has been widely used, as by Day and Kelly and by Mac Lane. In Short, Yoneda has made decisive contributions to albegra.

We mourn his recent death.

Yoneda lemma revolutionized 20th century math

- Alexander Grothendieck is considered by many the most influential mathematician of the 20th century
- He transformed algebraic geometry by heavily utilizing the Yoneda Lemma
- This allowed him to define spaces by their "functor of points," treating geometric objects as functors, thus shifting focus to internal, structural, and categorical perspectives rather than just set-theoretic points.

Yoneda's lemma as an identification of form and function: the case study of polynomials

25 August, 2023 in expository, math.CT, math.RA | Tags: polynomials, Yoneda lemma | by Terence Tao

As someone who had a relatively light graduate education in algebra, the import of Yoneda's lemma in category theory has always eluded me somewhat; the statement and proof are simple enough, but definitely have the "abstract nonsense" flavor that one often ascribes to this part of mathematics, and I struggled to connect it to the more grounded forms of intuition, such as those based on concrete examples, that I was more comfortable with. There is a popular MathOverflow post devoted to this question, with many answers that were helpful to me, but I still felt vaguely dissatisfied. However, recently when pondering the very concrete concept of a polynomial, I managed to accidentally stumble upon a special case of Yoneda's lemma in action, which clarified this lemma conceptually for me. In the end it was a very simple observation (and would be extremely pedestrian to anyone who works in an algebraic field of mathematics), but as I found this helpful to a non-algebraist such as myself, and I thought I would share it here in case others similarly find it helpful.

Noboru Yoneda

“Categories”

$$x \mapsto \mathcal{C}(x,-) : \mathcal{C} \to \mathbf{Sets}^{\mathcal{C}}$$

Yoneda Embedding

Terence Tao

“Polynomials,…”

Birds and Frogs

Freeman Dyson

Some mathematicians are birds, others are frogs. Birds fly high in the air and survey broad vistas of mathematics out to the far horizon. They delight in concepts that unify our thinking and bring together diverse problems from different parts of the landscape. Frogs live in the mud below and see only the flowers that grow nearby. They delight in the details of particular objects, and they solve problems one at a time. I happen to be a frog, but many of my best friends are birds. The main theme of my talk tonight is this. Mathematics needs both birds and frogs. Mathematics is rich and beautiful because birds give it broad visions and frogs give it intricate details. Mathematics is both great art and important science, because it combines generality of concepts with depth of structures. It is stupid to claim that birds are better than frogs because they see farther, or that frogs are better than birds because they see deeper. The world of mathematics is both broad and deep, and we need birds and frogs working together to explore it.

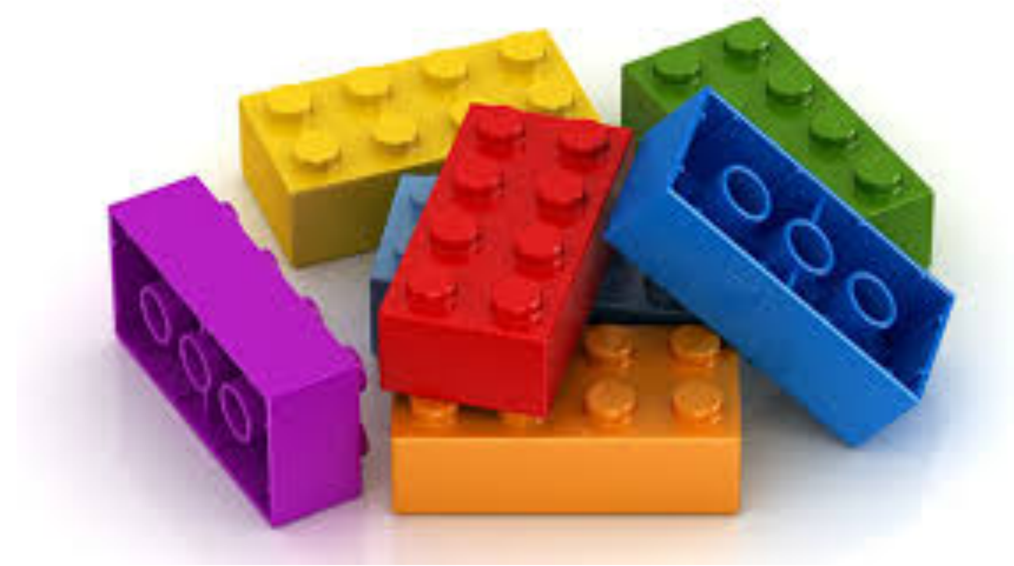
This talk is called the Einstein lecture, and I am grateful to the American Mathematical Society for inviting me to do honor to Albert Einstein. Einstein was not a mathematician, but a physicist who had mixed feelings about mathematics. On the one hand, he had enormous respect for the power of mathematics to describe the workings of nature, and he had an instinct for mathematical beauty which led him onto the right track to find nature’s laws. On the other hand, he had no interest in pure mathematics, and he had no technical skill as a mathematician. In his later years he hired younger colleagues with the title of assistants to do mathematical calculations for him. His way of thinking was physical rather than mathematical. He was supreme among physicists as a bird who saw further than others. I will not talk about Einstein since I have nothing new to say.

Francis Bacon and René Descartes

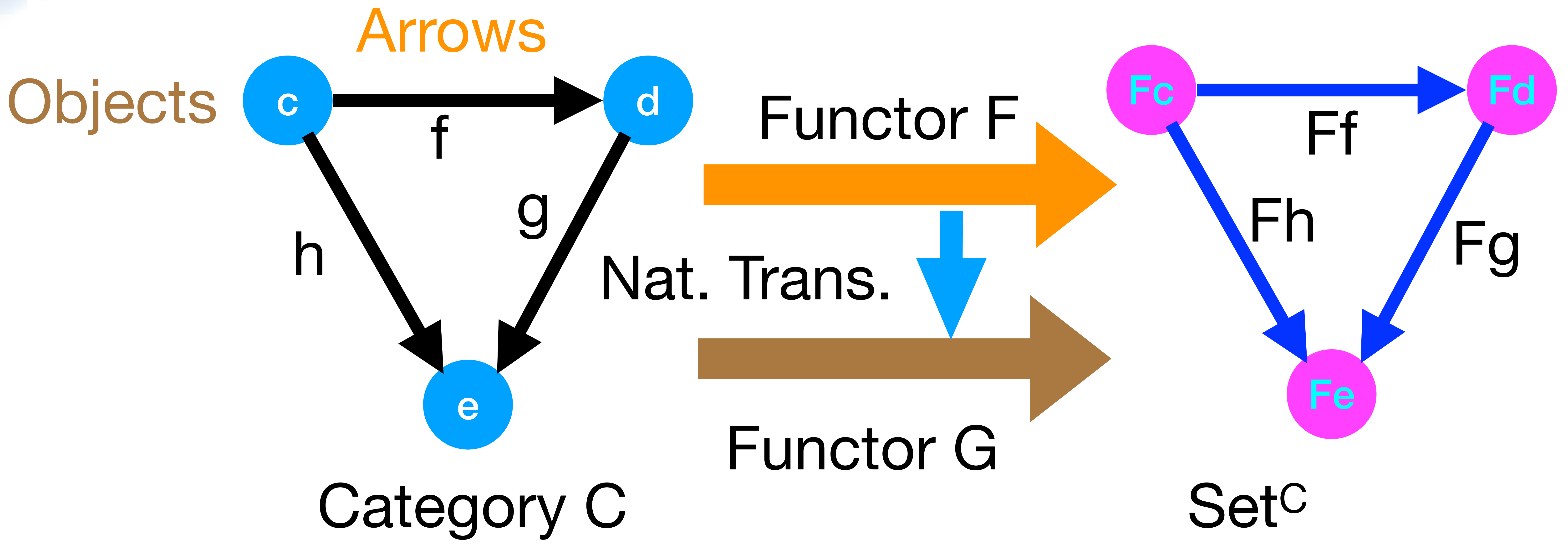
At the beginning of the seventeenth century, two great philosophers, Francis Bacon in England and René Descartes in France, proclaimed the birth of modern science. Descartes was a bird, and Bacon was a frog. Each of them described his vision of the future. Their visions were very different. Bacon said, “All depends on keeping the eye steadily fixed on the facts of nature.” Descartes said, “I think, therefore I am.” According to Bacon, scientists should travel over the earth collecting facts, until the accumulated facts reveal how Nature works. The scientists will then induce from the facts the laws that Nature obeys. According to Descartes, scientists should stay at home and deduce the laws of Nature by pure thought. In order to deduce the laws correctly, the scientists will need only the rules of logic and knowledge of the existence of God. For four hundred years since Bacon and Descartes led the way, science has raced ahead by following both paths simultaneously. Neither Baconian empiricism nor Cartesian dogmatism has the power to elucidate Nature’s secrets by itself, but both together have been amazingly successful. For four hundred years English scientists have tended to be Baconian and French scientists Cartesian. Faraday and Darwin and Rutherford were Baconians; Pascal and Laplace and Poincaré were Cartesians. Science was greatly enriched by the cross-fertilization of the two contrasting cultures. Both cultures were always at work in both countries. Newton was at heart a Cartesian, using

Freeman Dyson is an emeritus professor in the School of Natural Sciences, Institute for Advanced Study, Princeton, NJ. His email address is dyson@ias.edu.

This article is a written version of his AMS Einstein Lecture, which was to have been given in October 2008 but which unfortunately had to be canceled.



Yoneda lemma & AGI

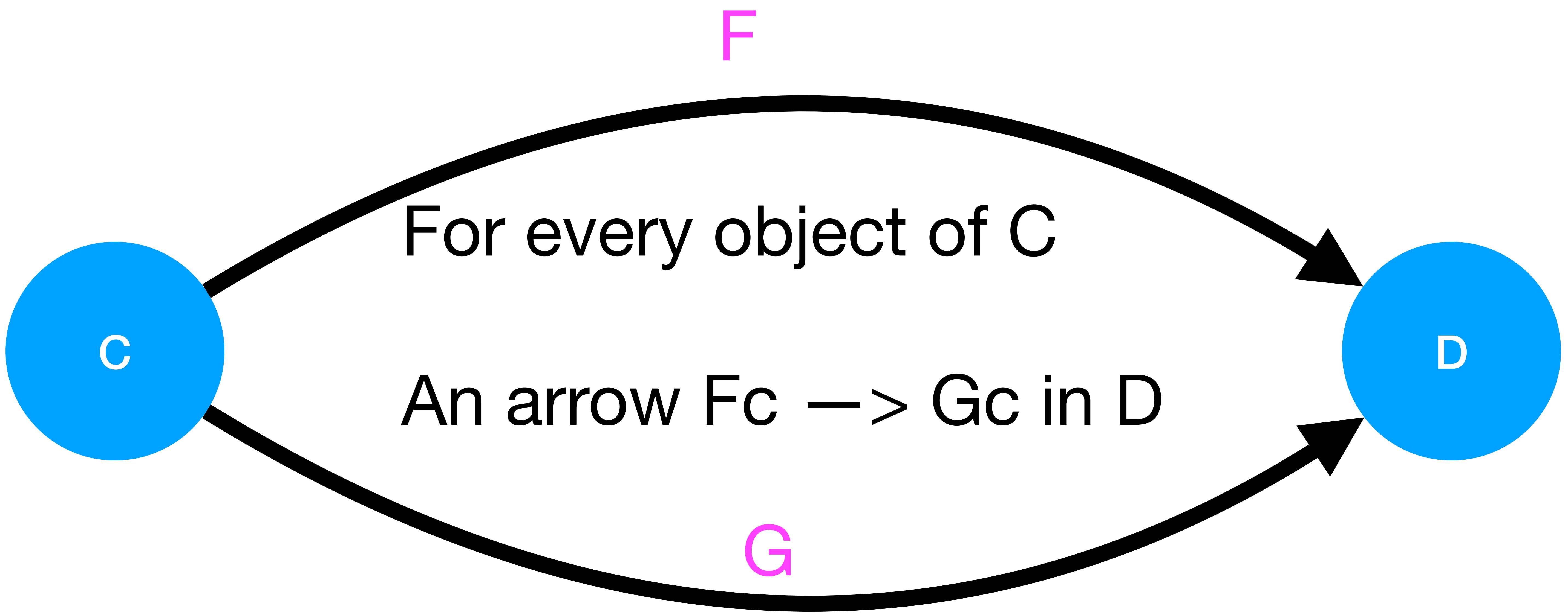


Yoneda Embedding of C

From causal inference, deep learning and RL

to consciousness!

Natural Transformations between Functors



$$\begin{tikzcd} Fc \arrow[r, "Ff"] \arrow[d, "\mu_c"'] & Fc' \arrow[d, "\mu_{c'}"] \\ Gc \arrow[r, "Gf"'] & Gc' \end{tikzcd}$$

Directed graphs as functors

Contravariant

Any particular graph is a functor F: C^{op} —> Set

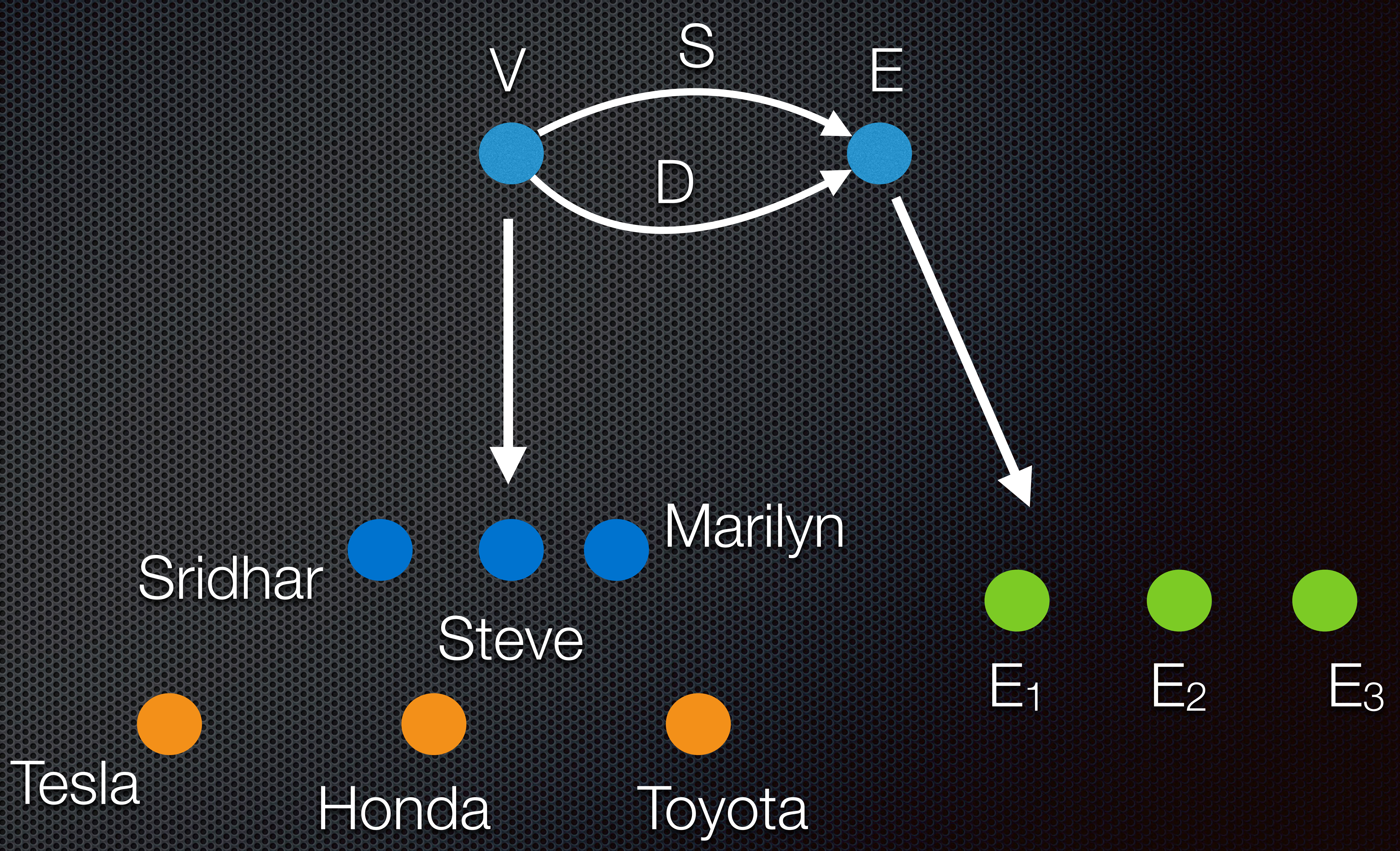
Covariant

Any particular graph is a functor F: C —> Set

Step 1: Make Graphs into Functors

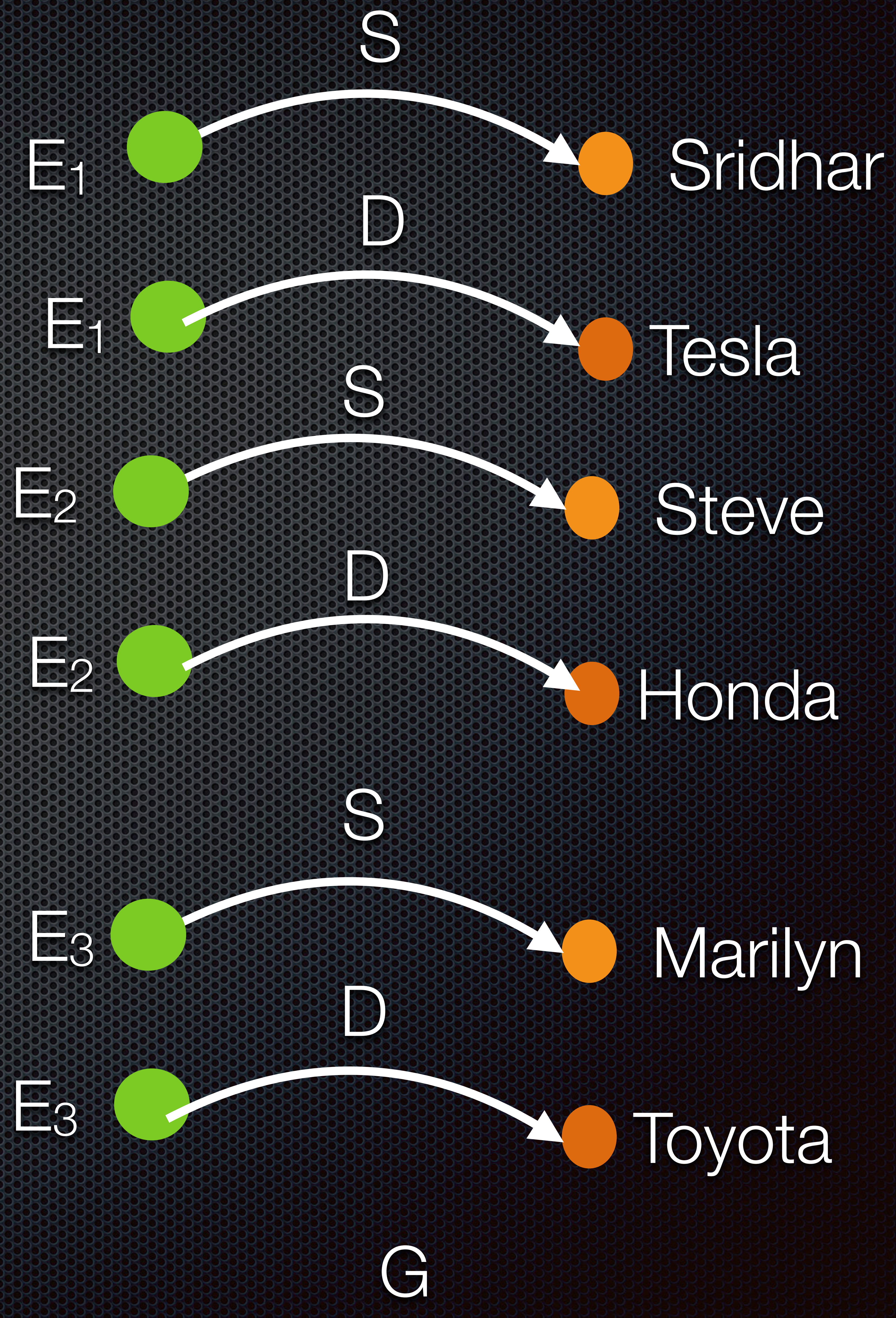
Define a graph as a functor from a category with two objects V and E to Sets

Graph G is a functor G: C^{op} —> Set



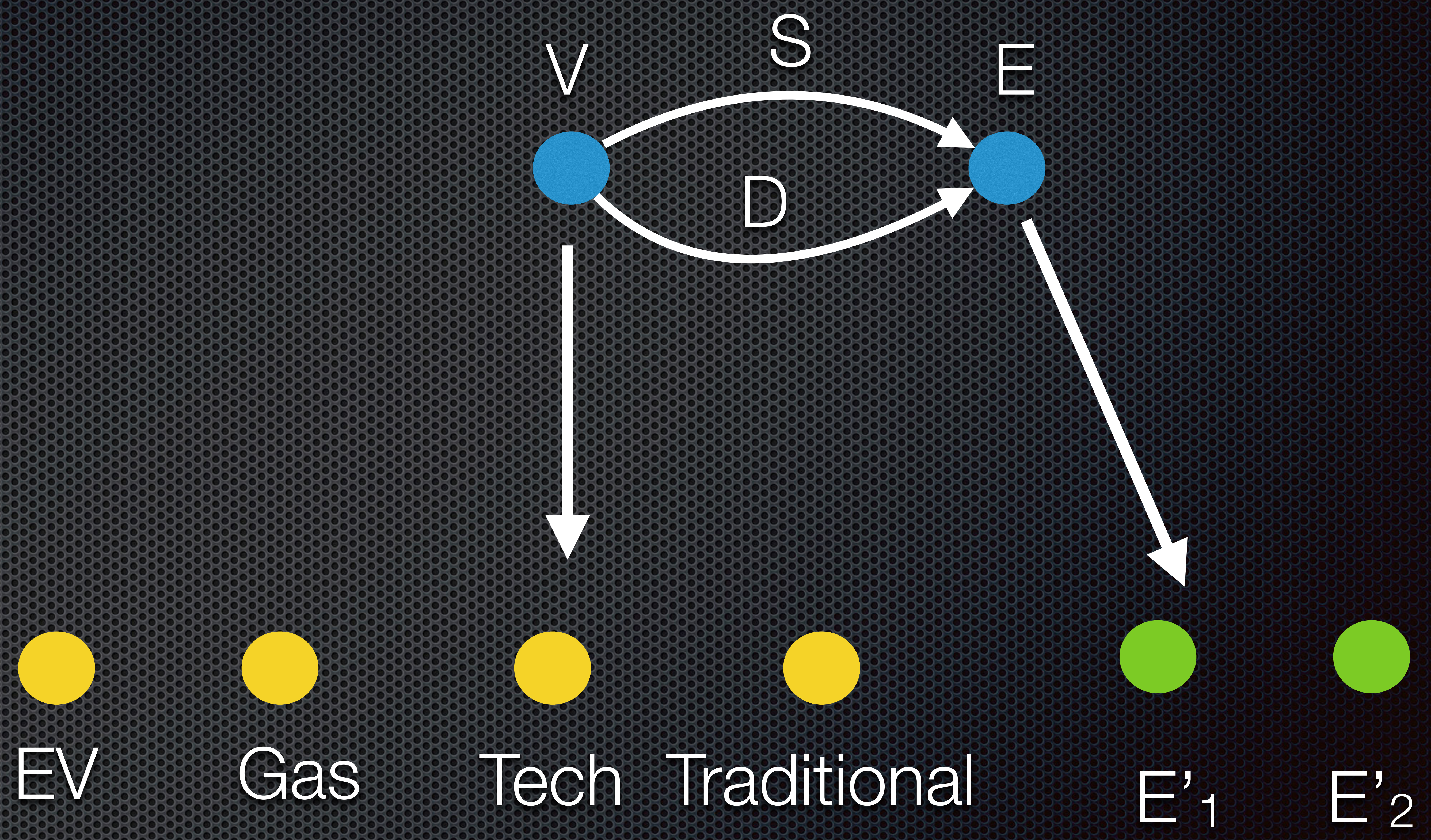
Functors map arrows to arrows

Since our graph objects are contravariant functors, they map arrows V->E into set-valued functions from E-> V



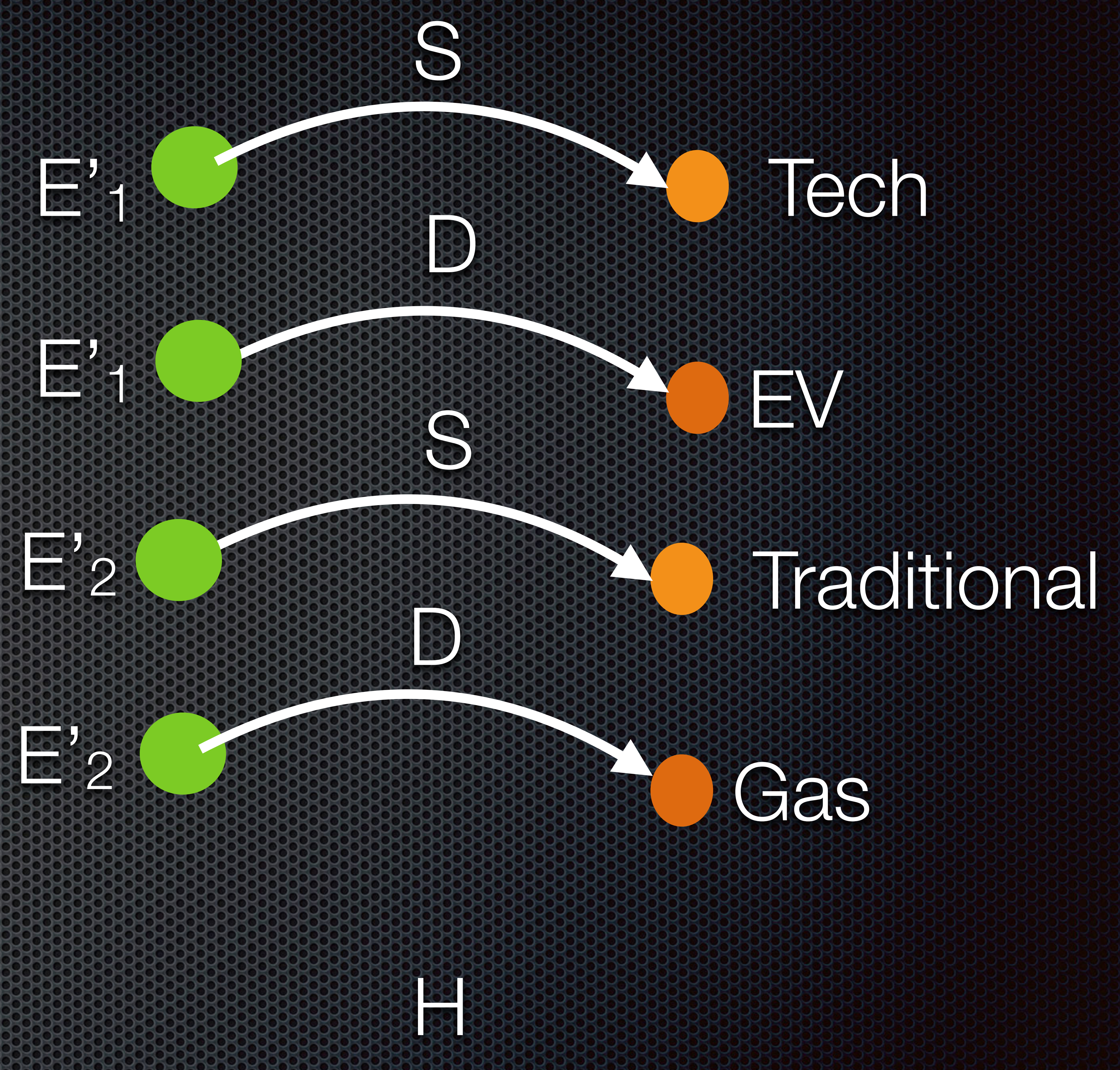
Step 1: Make Graphs into Functors

Graph H is a functor G: C^{op} —> Set



Define a graph as a functor from a category with two objects V and E to Sets

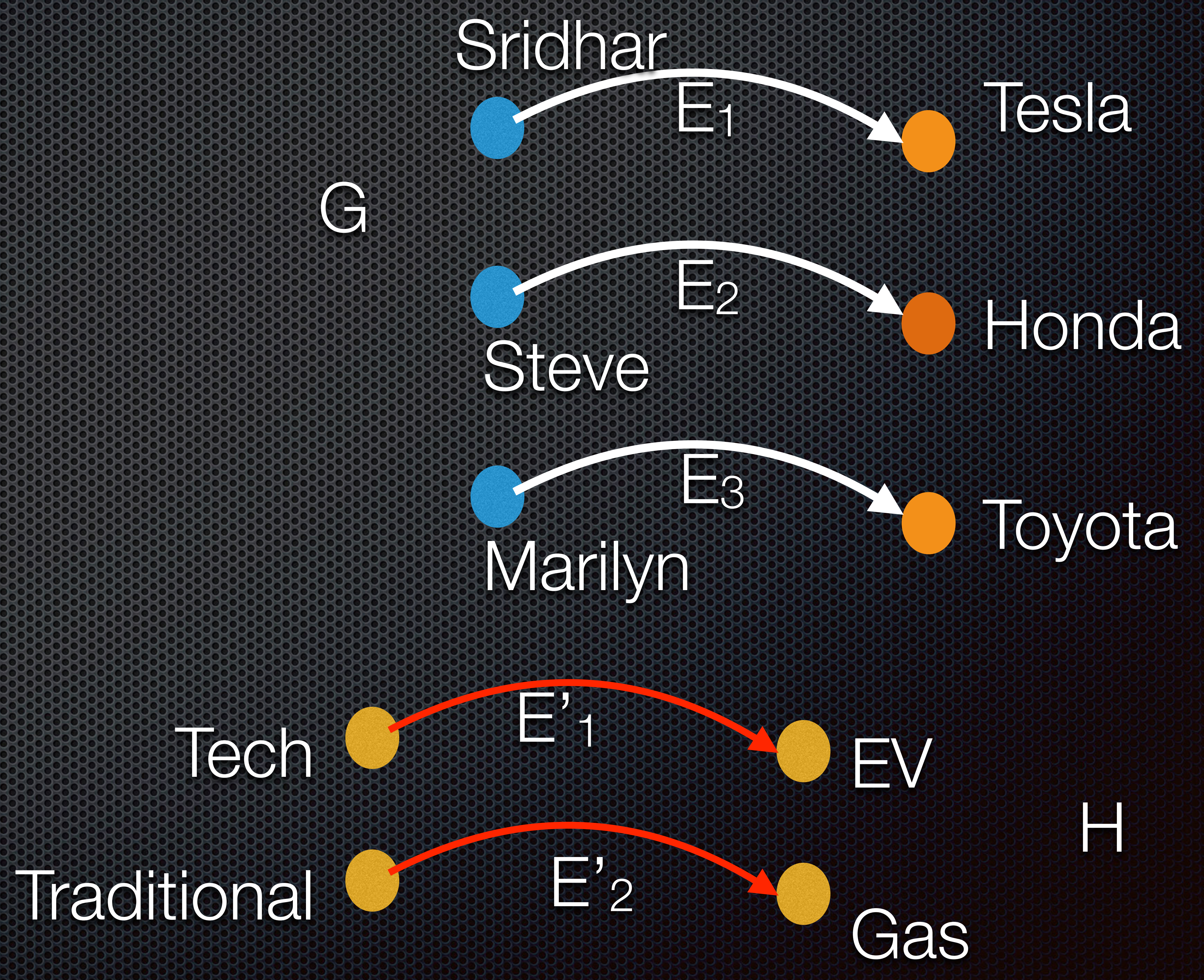
Functors map arrows to arrows



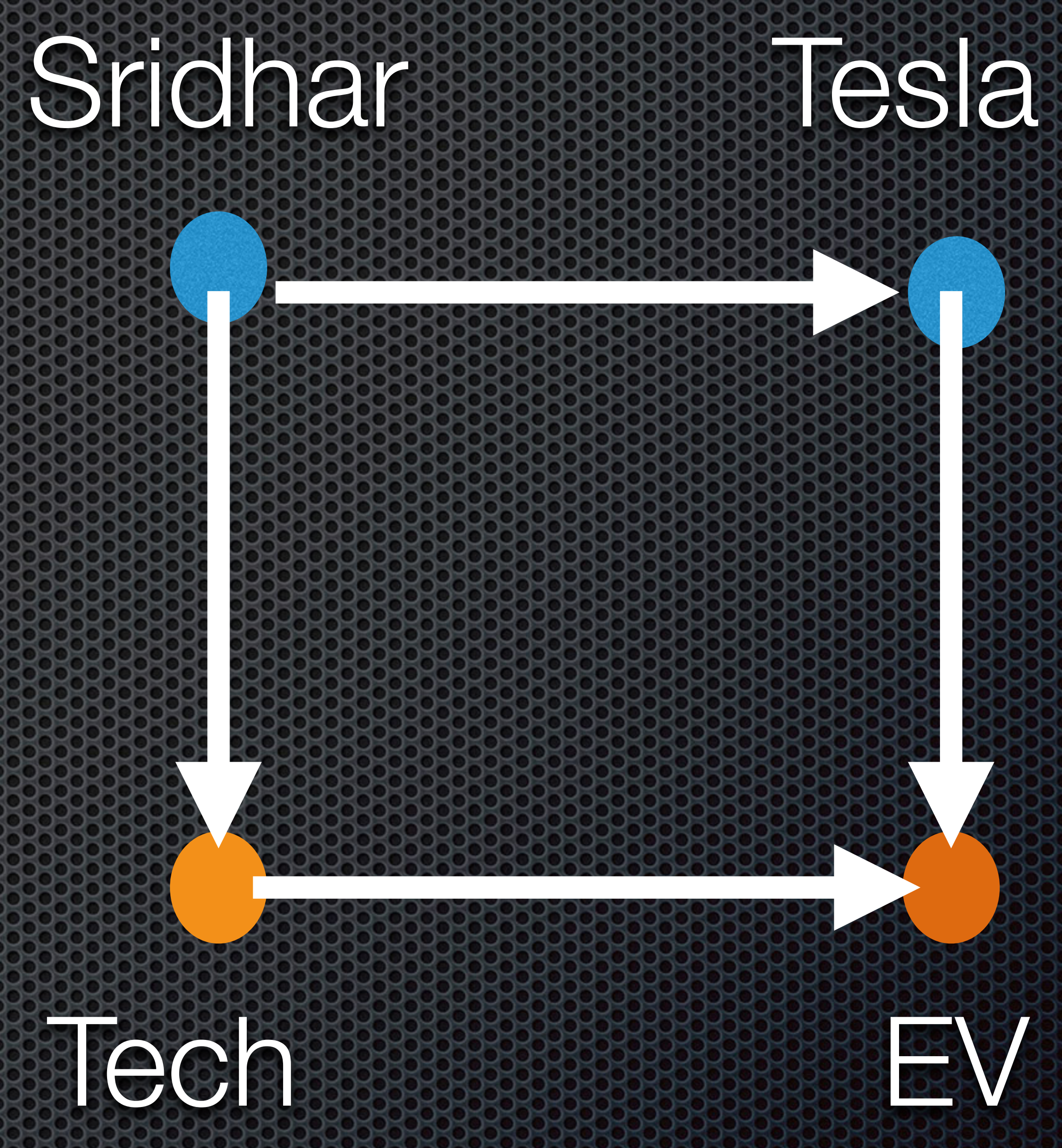
Since our graph objects are contravariant functors, they map edges V->E into set-valued functions from E-> V

Graph Homomorphism

Given two graphs G, H, a graph homorphism is a mapping of vertices from G to H that preserves adjacency



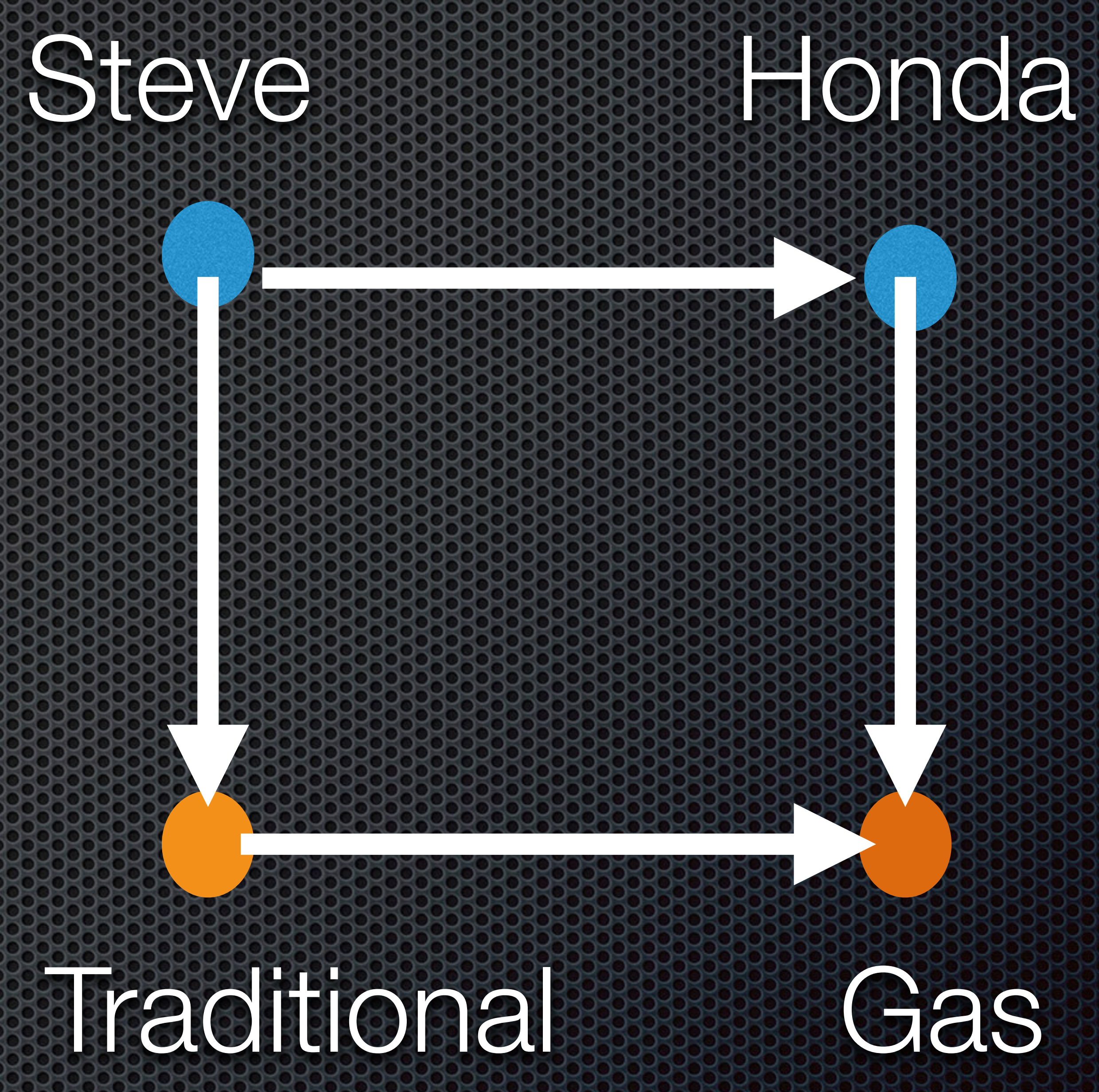
Natural Transformation



Map and Translate

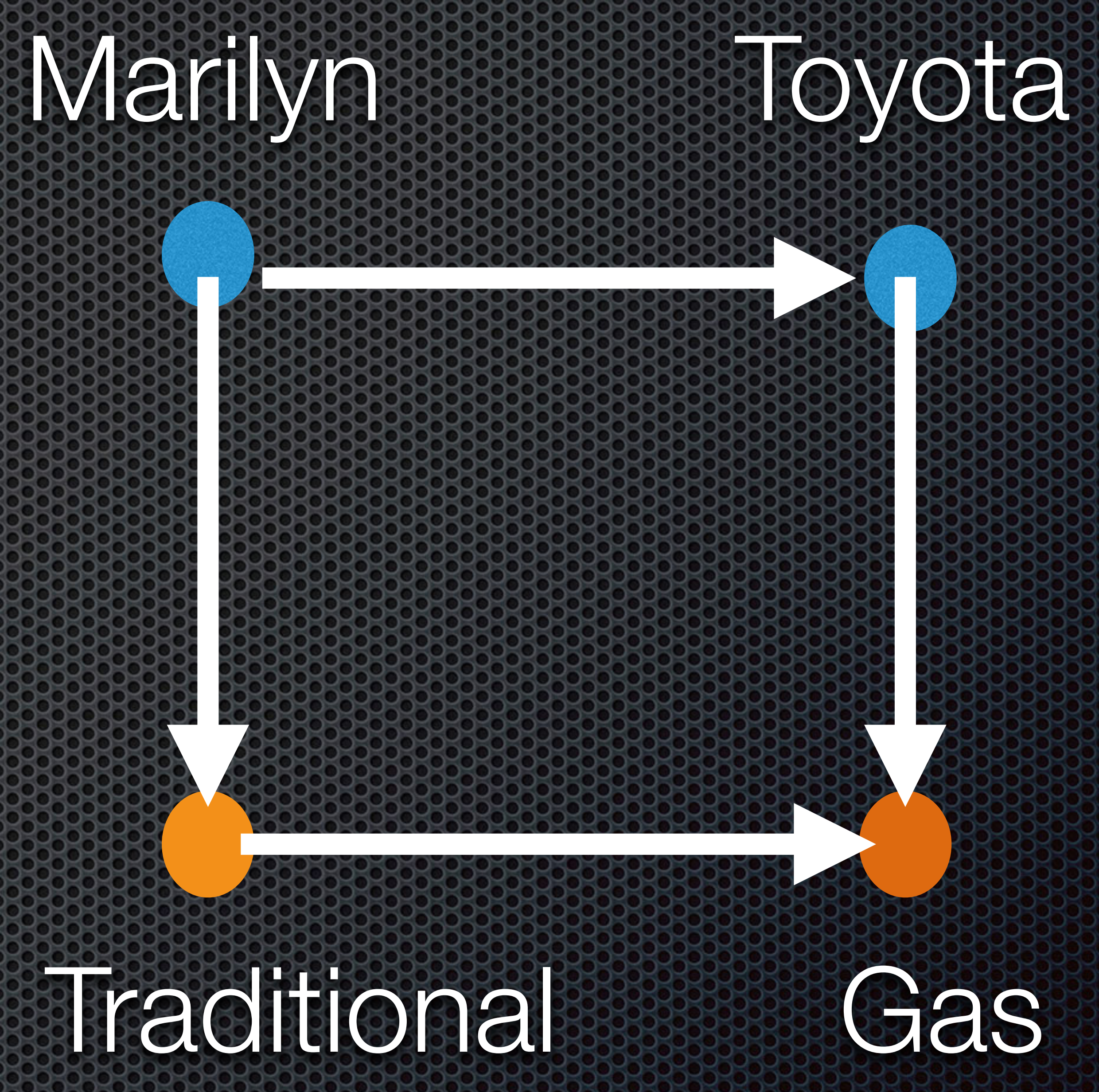
Natural Transformation

Map and Translate



Natural Transformation

Map and Translate



Statement of Yoneda lemma

- For any set-valued functor F:C -> Set, whose domain is a “locally small” category C, and any object x in C, the following bijection exists

  Nat(C(-, c), F) ~ Fc

  associating a natural transformation C(-, c) => F to each element in Fc

- This transformation is natural in both c and F

The “Holographic” Yoneda Principle

- A natural transformation $\alpha : C(- , c) \Rightarrow F$

  is a set of maps defined over each object $\alpha_x : C(x, c) \to F(x)$
- Is this even expressible as a set?
  - For each object, we get a set (not necessarily finite)

Proving the Yoneda lemma

- Understanding the proof of the Yoneda lemma is highly instructive.
- It contains within it an amazing idea about data compression
  - A “doubly infinite” structure — a natural transformation defined on every object of a category — is in fact determined by a single element u in Fc!
- The “magic” behind the proof is knowing which element u in Fc to pick.

The Magic Element to pick

$$u = \alpha_c(\mathrm{id}_c)$$

Consider a probe from any d to c

$$f : d \to c$$

$$\begin{tikzcd}
C(c,c) \arrow[r, "\alpha_c"'] \arrow[d, "f_*"] & Fc \arrow[d, "Ff"'] \\
C(d,c) \arrow[r, "\alpha_d"] & Fd
\end{tikzcd}$$

The Magic Element to pick

$$u = \alpha_c(\mathrm{id}_c)$$

Let's pullback id_c

through our probe $f : d \to c$

$$f \circ \mathrm{id}_c = f$$

$$\begin{tikzcd}
C(c,c) \arrow[r, "\alpha_c"'] \arrow[d, "f_*"] & Fc \arrow[d, "Ff"'] \\
C(d,c) \arrow[r, "\alpha_d"] & Fd
\end{tikzcd}$$

Naturality now forces us to conclude that

$$\alpha_d(f) = F(f)(u)$$

Yoneda lemma covariant formulation

- For the covariant formulation, the statement of the Yoneda lemma is

$$\mathrm{Hom}(\mathscr{C}(c, -\ ), F) \cong Fc$$

PROOF OF THE BIJECTION. There is clearly a function $\Phi\colon \mathrm{Hom}(\mathsf{C}(c,-),F) \to Fc$ that maps a natural transformation $\alpha\colon \mathsf{C}(c,-) \Rightarrow F$ to the image of 1_c under the component function $\alpha_c\colon \mathsf{C}(c,c) \to Fc$, i.e.,

$$\Phi\colon \mathrm{Hom}(\mathsf{C}(c,-),F) \to Fc \qquad\qquad \Phi(\alpha) := \alpha_c(1_c)\,.$$

Our first aim is to define an inverse function $\Psi\colon Fc \to \mathrm{Hom}(\mathsf{C}(c,-),F)$ that constructs a natural transformation $\Psi(x)\colon \mathsf{C}(c,-) \Rightarrow F$ from any $x \in Fc$. To this end, we must define components $\Psi(x)_d\colon \mathsf{C}(c,d) \to Fd$ so that naturality squares, such as displayed for $f\colon c \to d$ in C, commute:

$$\begin{tikzcd} \mathsf{C}(c,c) \arrow[r, "\Psi(x)_c"] \arrow[d, "f_*"'] & Fc \arrow[d, "Ff"] \\ \mathsf{C}(c,d) \arrow[r, "\Psi(x)_d"'] & Fd \end{tikzcd}$$

The image of the identity element $1_c \in \mathsf{C}(c,c)$ under the left-bottom composite is $\Psi(x)_d(f) \in Fd$, the value of the component $\Psi(x)_d$ at the element $f \in \mathsf{C}(c,d)$. The image under the top-right composite is $Ff(\Psi(x)_c(1_c))$. For Ψ to define an inverse for Φ, we must define $\Psi(x)_c(1_c) = x$. Now, naturality forces us to define

$$(2.2.5) \qquad \Psi\colon Fc \to \mathrm{Hom}(\mathsf{C}(c,-),F) \qquad\qquad \Psi(x)_d(f) := Ff(x)\,.$$

This condition completely determines the components $\Psi(x)_d$ of $\Psi(x)$.

Figure source: Emily Riehl, Category Theory in Context, page 57

Why is Yoneda lemma useful?

- It is the basis of the Diagrammatic Backpropagation method (chapter 6)
  - DB works by minimizing ``diagrammatic curvature”
  - Diagrams are functors, and (co)limits of diagrams are defined using the Yoneda lemma as representable functors over diagrams
  - DB numerically approximates these (co)limits

Why is the Yoneda lemma useful?

- The Yoneda lemma is central to Universal Causality (chapter 12-15)

  Causal reproducing property: Hom(C(-,x),C(-y)) ~ C(x,y)

  - Topos causal models are based on Yoneda embeddings

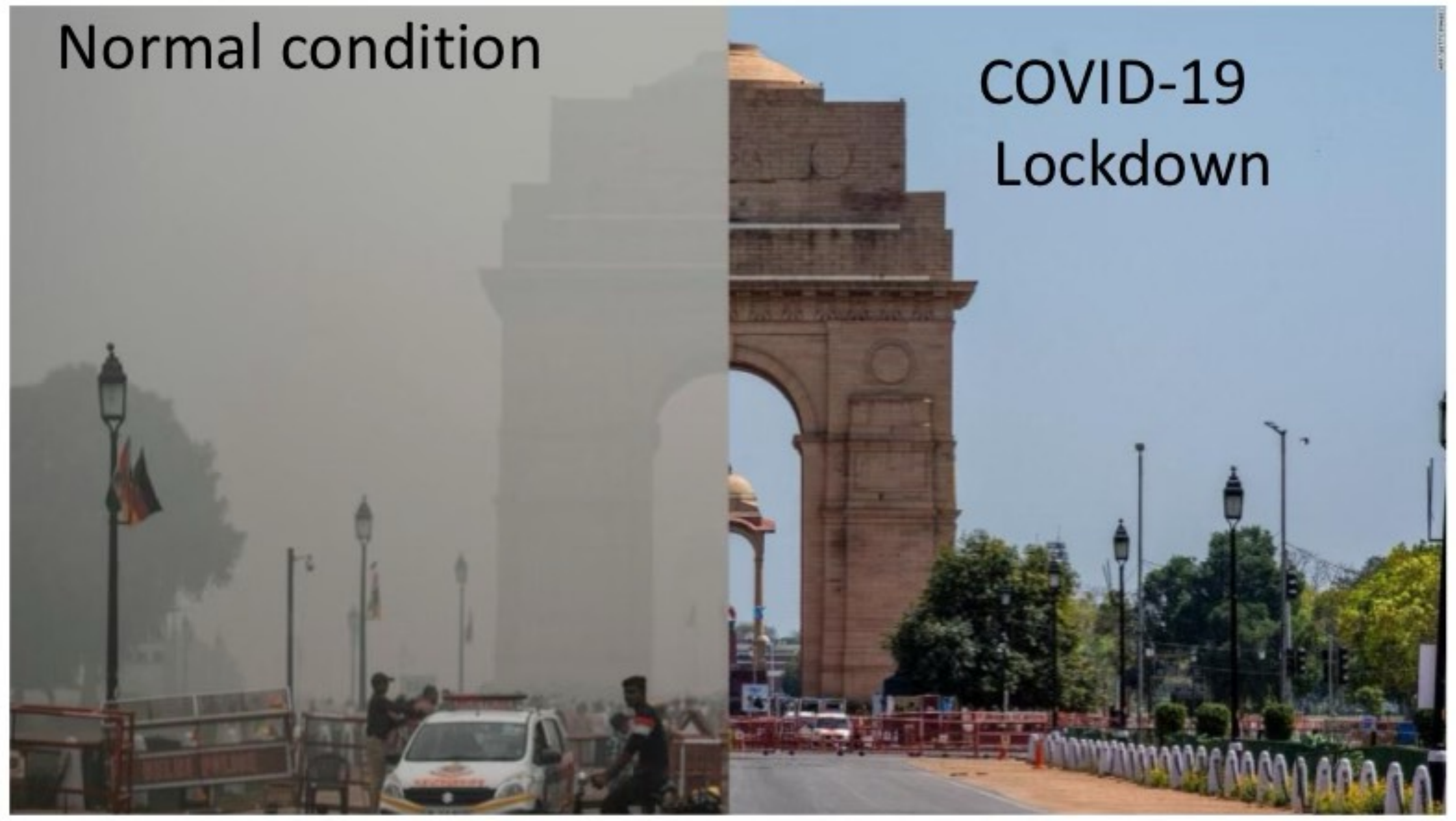
    - C(-,x) are set-valued functors called (pre)sheaves

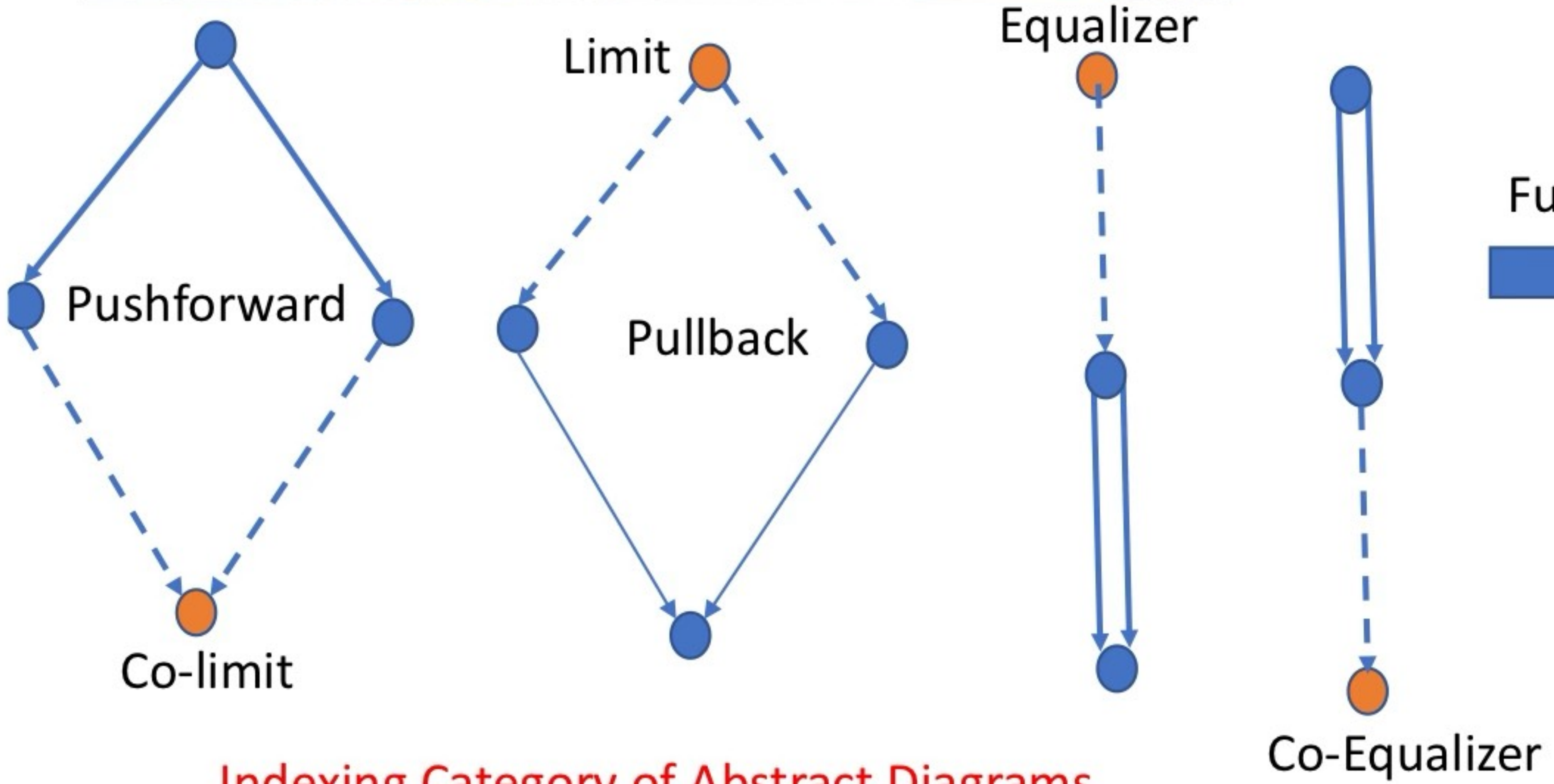
    - They form a topos category of functor objects

Yoneda lemma and Universal Causality

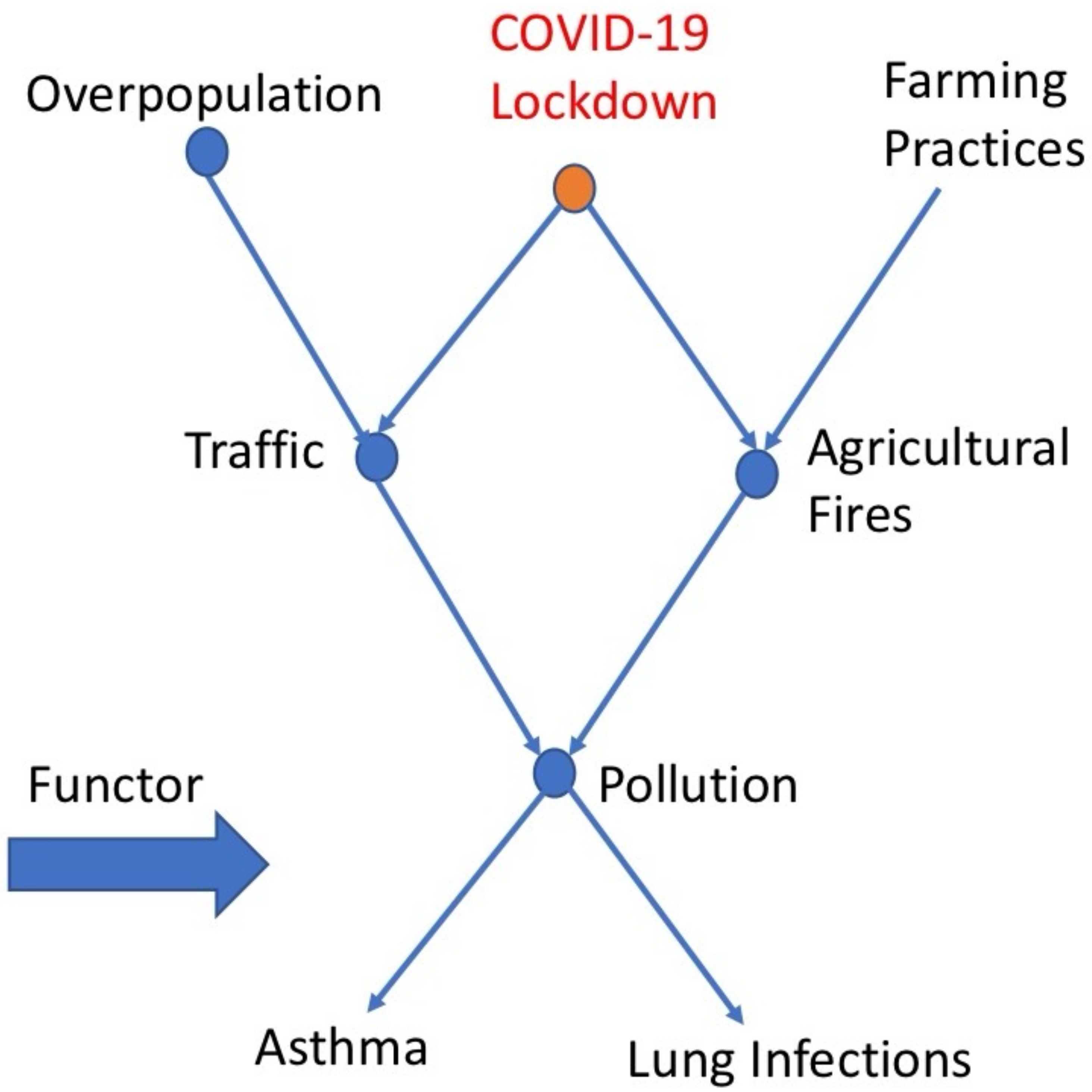
[Mahadevan, Entropy, 2023]

Pollution in New Delhi, India





Indexing Category of Abstract Diagrams



Actual Causal Model

What is meaning?

How do you understand any object?

Take the Dow Jones Industrial Average (DJIA)

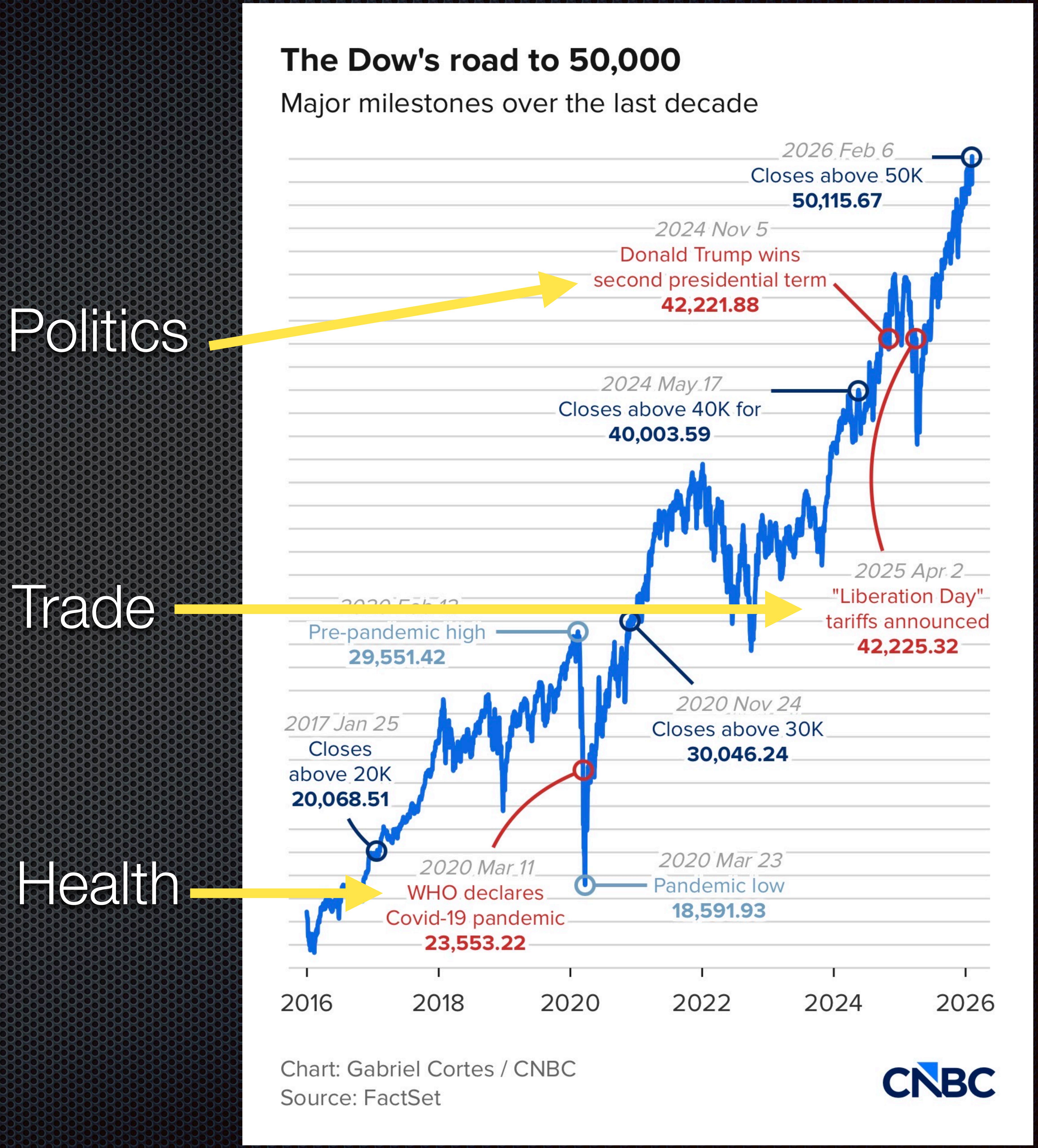
Classical View

The Dow Jones Industrial Average (DJIA) is a price-weighted stock market index that tracks the performance of 30 large, blue-chip companies listed on the New York Stock Exchange (NYSE) and the Nasdaq.

Unlike indices like the S&P 500, which are weighted by market capitalization, the Dow is defined by the actual share prices of its member companies.

Yoneda Definition

View DJIA by the cumulative influences on it from other objects



Yoneda Embedding

Turn an object into a (set-valued) functor!

Y: DJIA —> C(-, DJIA)