Limits and Colimits: Reasoning with Diagrams for AGI

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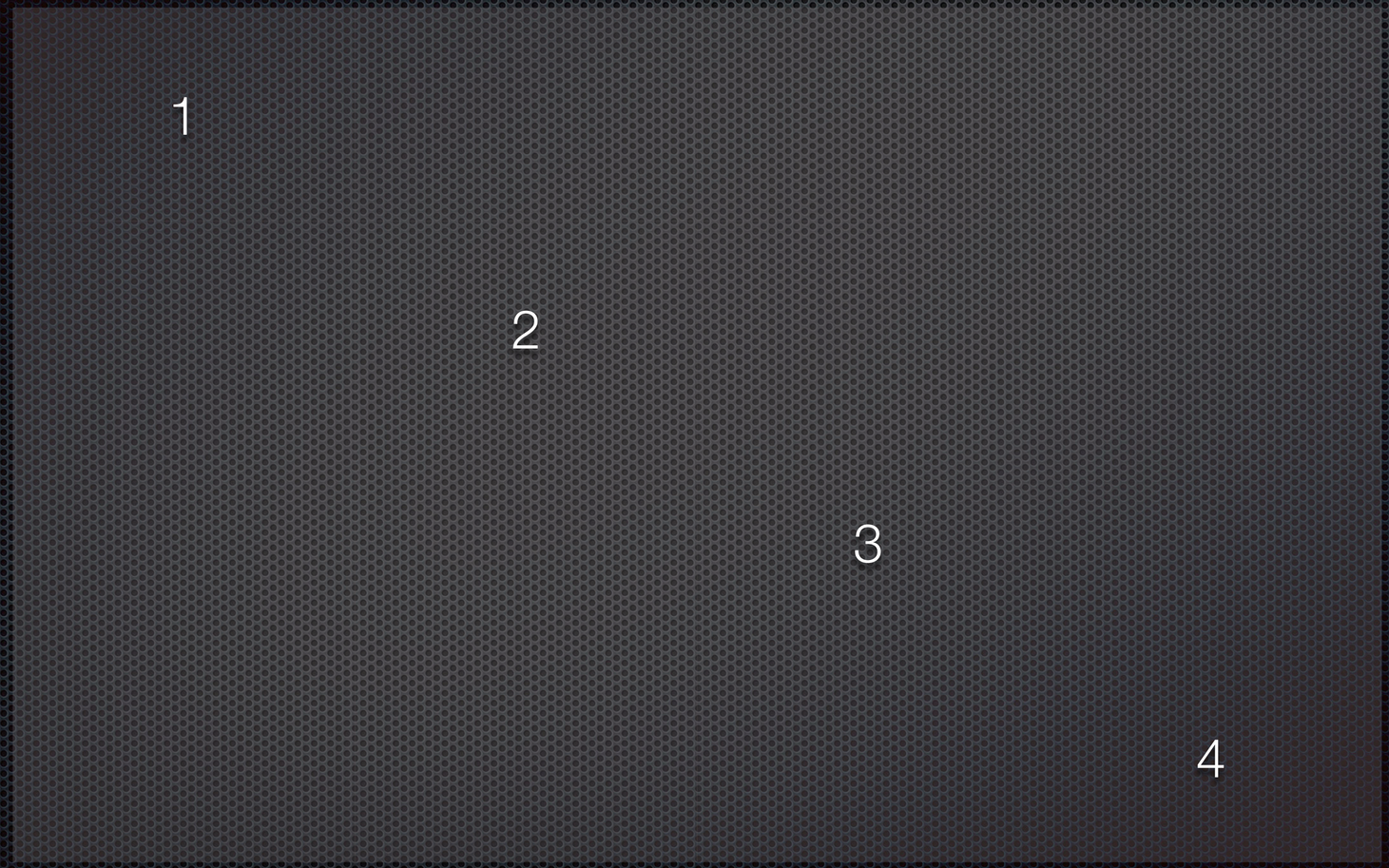
Solving Sudoku



Solving Sudoku

1	4	2	3
3	2	4	1
4	1	3	2
2	3	1	4

Diagrams and AGI

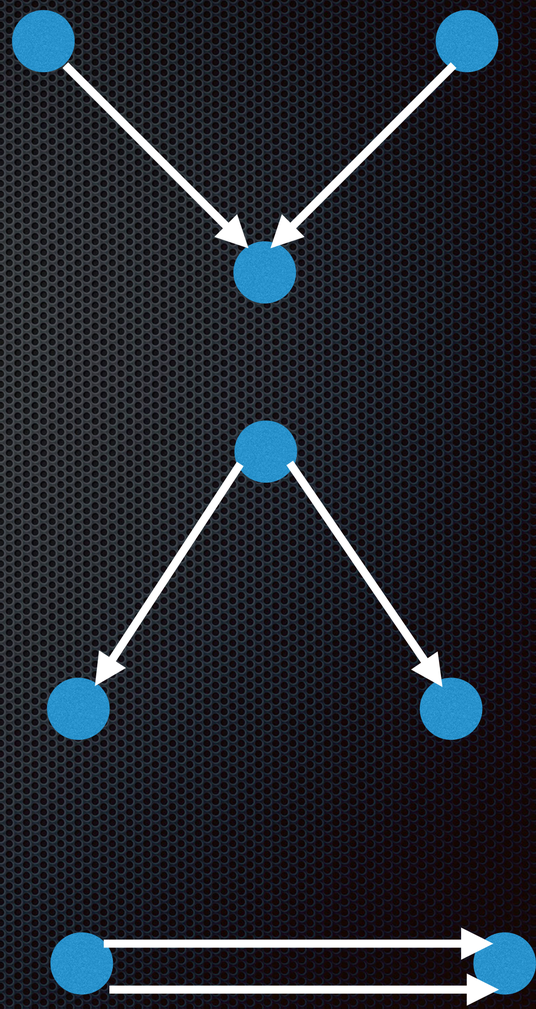
- Reasoning with diagrams is a core part of categorical AGI
- It is central to Diagrammatic Backpropagation and Geometric Transformers
- It is also essential to Topos Causal Models
- Diagrams capture constraints that are universal (Yoneda Lemma)

Diagrams

Diagrams are functors F: J->C

Here, J is a finite index category

We want to define how to “solve” a diagram



Database Category

We saw that a database is a functor D: I -> S

I is a category of instances

S is a database schema

[Spivak, Simplicial Databases, Arxiv]

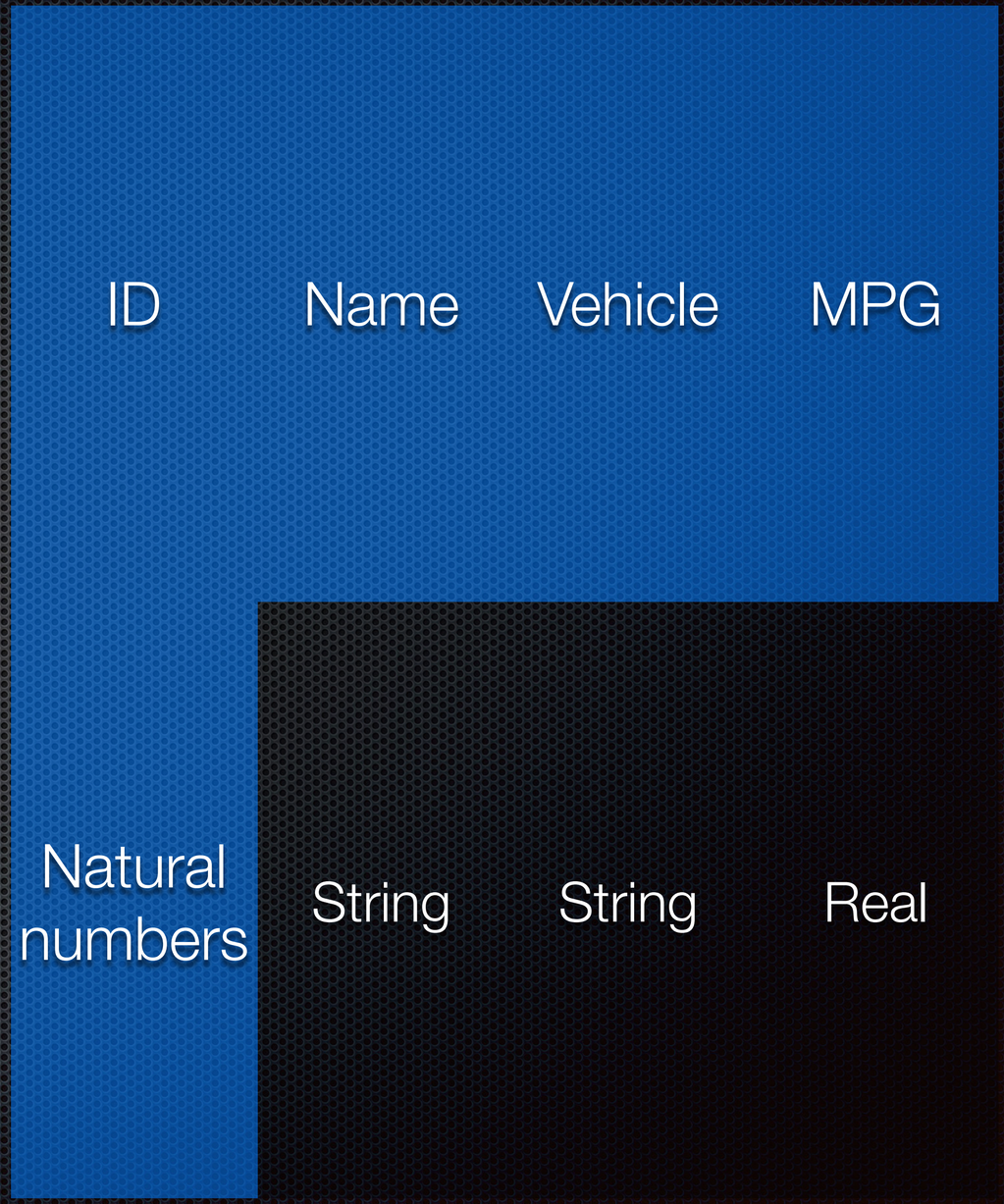
ID	Name	Vehicle	Type
1	Sridhar	Tesla	EV Sedan
2	Steve	Toyota	Gas Sedan
3	Marilyn	Honda	Gas SUV
4	Joe	Ford	Gas Pickup

Data Types

A datatype T: U -> DT

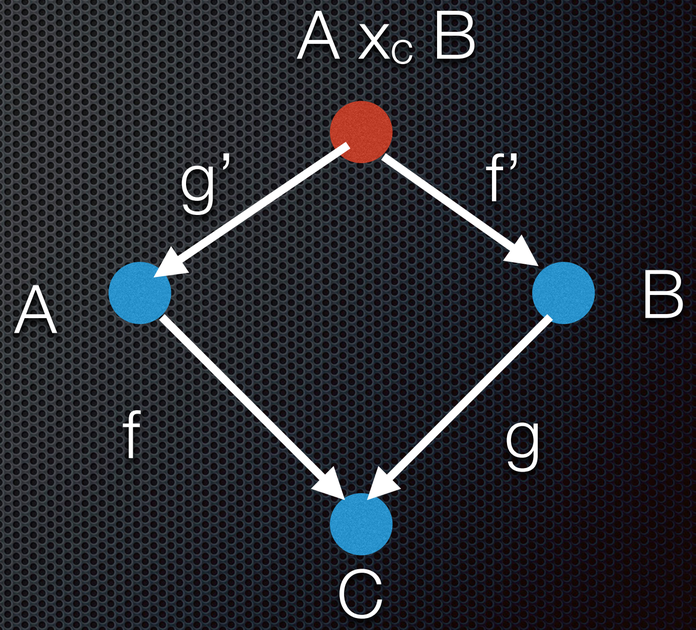
DT: Set of datatypes

U: Domains



[Spivak, Simplicial Databases, Arxiv]

Pullback



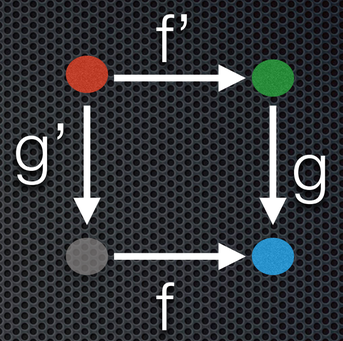
A pullback is a type of limit

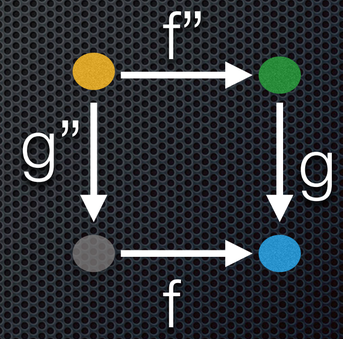
It abstracts many concrete operations: maximum, intersection, product, etc.

Pullback

In the category of sets,
pullbacks always exist

The pullback of f and g is simply
the set of ordered pairs (x,y)
such that f(x) = g(y)





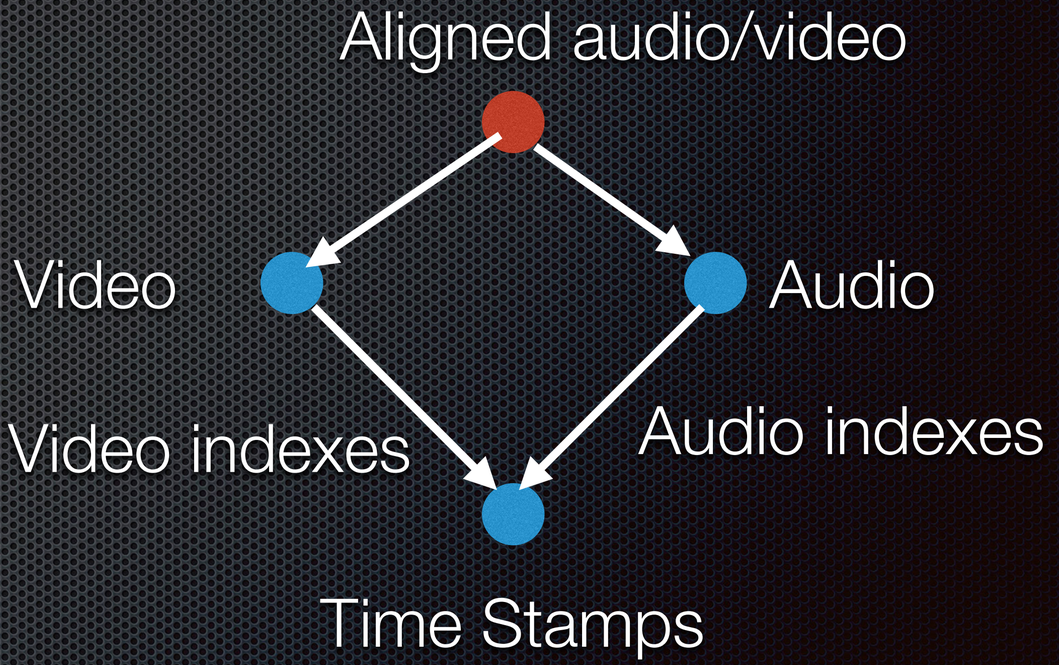
f” = f’ h

g” = g’ h

Pullback

A pullback is a type of limit

It abstracts many concrete operations: maximum, intersection, product, etc.



See Deep Learning demo on pullback of AV on GitHub repo

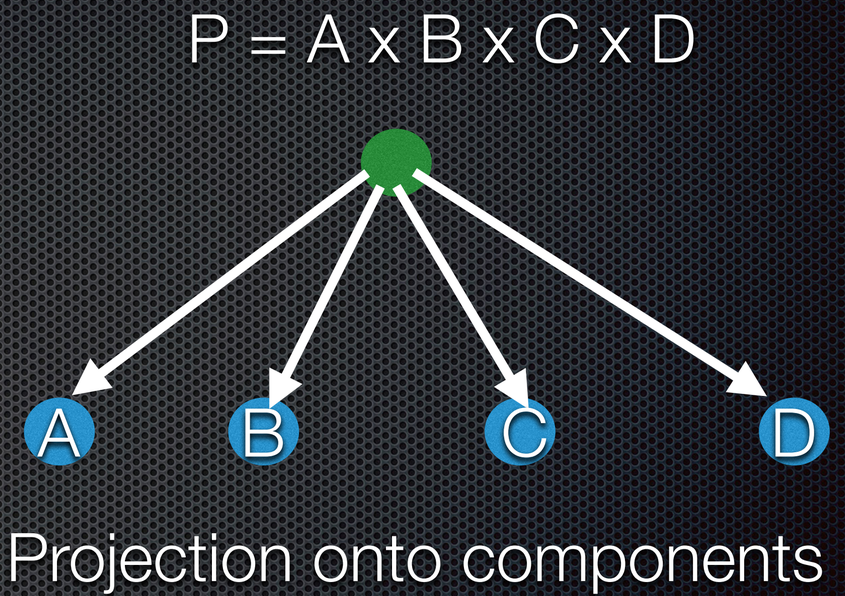
https://colab.research.google.com/github/sridharmahadevan/Category-Theory-for-AGI-UMass-CMPSCI-692CT/blob/main/notebooks/week3_synthetic_av_pullback-2.ipynb

Cartesian Product as a Limit

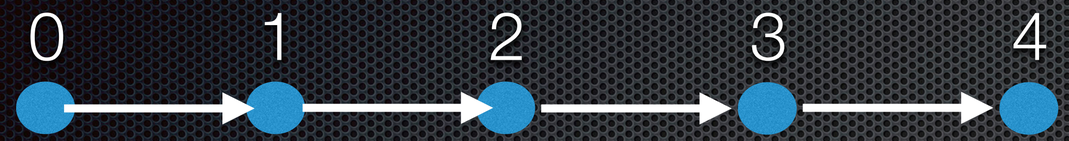
A Cartesian product of sets is defined universally in terms of a limit

It is the unique object that has projections onto component sets

Every function onto component sets must factor uniquely through the Cartesian product

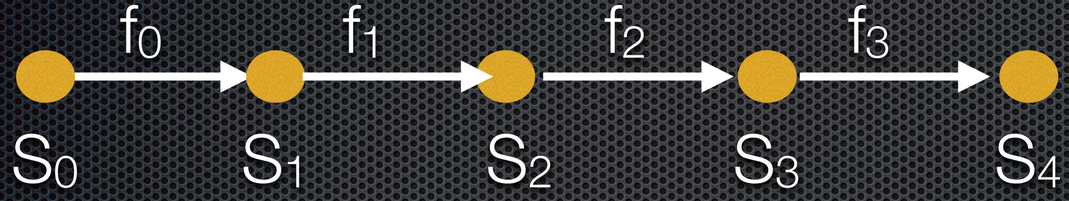


Limits over a Time Series



Indexing category

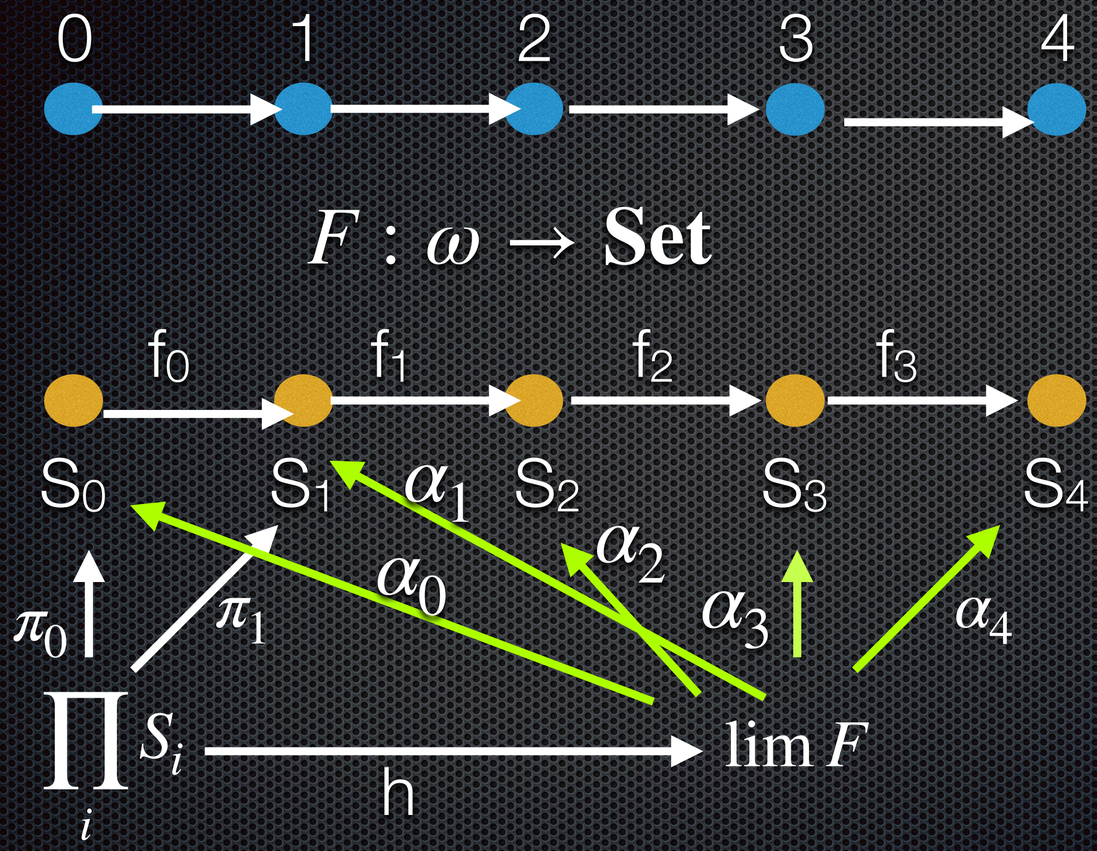
$$F : \omega \to \mathbf{Set}$$



Sets

Note that the limit here cannot be Cartesian product!

Limits over a Time Series



Indexing category

Sets

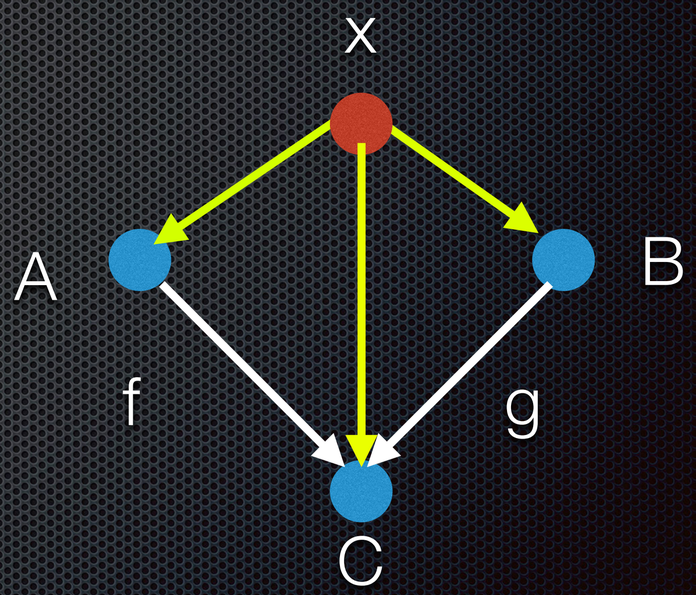
$\alpha_{n+1} = f_{n+1}\alpha_n$

Cones

A cone over diagram F: J->C with apex x is a natural transformation from the constant functor at x to F

The constant functor at x maps every object in J to x, and every arrow in J to the identity arrow at x

$$\Delta : C \to C^J$$



Cones and Cocones: Formal Definition

DEFINITION 3.1.2. A **cone over** a diagram $F\colon \mathsf{J} \to \mathsf{C}$ with **summit** or **apex** $c \in \mathsf{C}$ is a natural transformation $\lambda\colon c \Rightarrow F$ whose domain is the constant functor at c. The components $(\lambda_j\colon c \to Fj)_{j\in\mathsf{J}}$ of the natural transformation are called the **legs** of the cone. Explicitly:

- The data of a cone over $F\colon \mathsf{J} \to \mathsf{C}$ with summit c is a collection of morphisms $\lambda_j\colon c \to Fj$, indexed by the objects $j \in \mathsf{J}$.
- A family of morphisms $(\lambda_j\colon c \to F_j)_{j\in\mathsf{J}}$ defines a cone over F if and only if, for each morphism $f\colon j \to k$ in J, the following triangle commutes in C:

(3.1.3)

$$\begin{array}{ccc} & c & \\ {\scriptstyle\lambda_j}\swarrow & & \searrow{\scriptstyle\lambda_k} \\ Fj & \xrightarrow[Ff]{} & Fk \end{array}$$

Dually, a **cone under** F with **nadir** c is a natural transformation $\lambda\colon F \Rightarrow c$, whose **legs** are the components $(\lambda_j\colon F_j \to c)_{j\in\mathsf{J}}$. The naturality condition asserts that, for each morphism $f\colon j \to k$ of J, the triangle

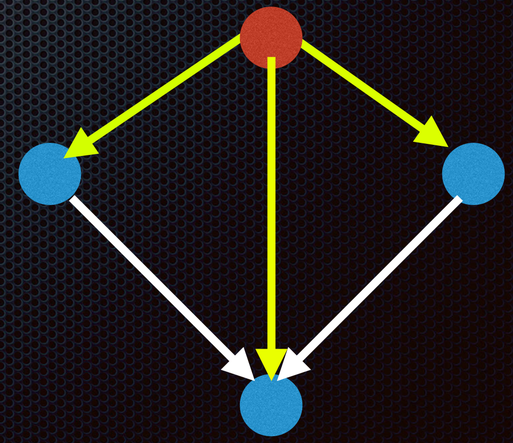
$$\begin{array}{ccc} Fj & \xrightarrow{Ff} & Fk \\ & {\scriptstyle\lambda_j}\searrow \quad \swarrow{\scriptstyle\lambda_k} & \\ & c & \end{array}$$

commutes in C.

Figure source: Riehl, Category Theory in Context

Cones as set-valued functors

- Given a diagram F: J->C, we can define a set-valued functor
  - Cone(-, F): the set of cones over F with summit c
  - Cone(F, -): the set of cones under F with nadir c



Limits and Colimits: Representable functors using the Yoneda Lemma

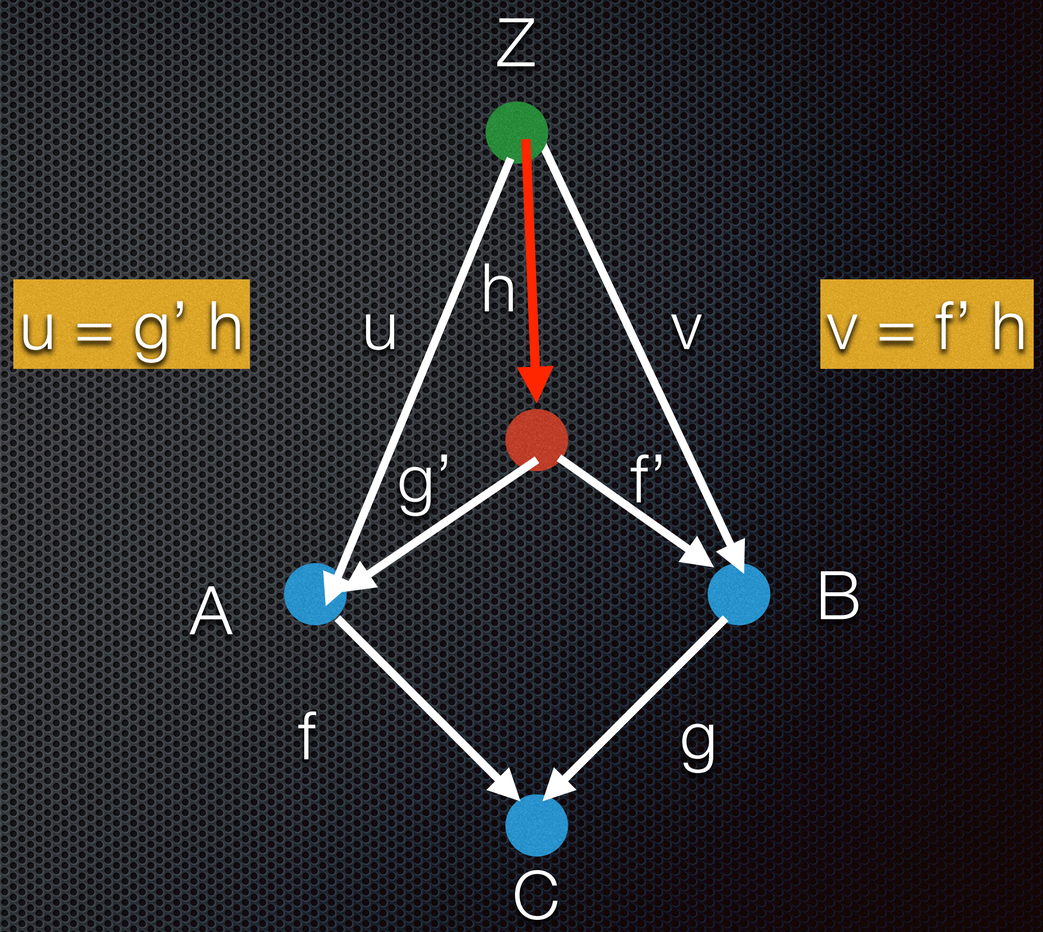
- A limit of diagram F is a representation for Cone(-, F)
  - C(-, lim F) ~ Cone(-, F)
- A colimit of diagram F is a representation of Cone(F, -)
  - C(colim F, -) ~ Cone(F, -)

Limits using Category of Elements

- We can model limits in terms of a category of elements
  - $\int Cone(-, F)$
- Each object in this category is an actual cone Cone(x, F)
- Arrows are mappings from one cone to another
- A limit is the terminal object in the category of elements $\int Cone(-, F)$

Universal Property

Universal property of a pullback defines it as a type of limit



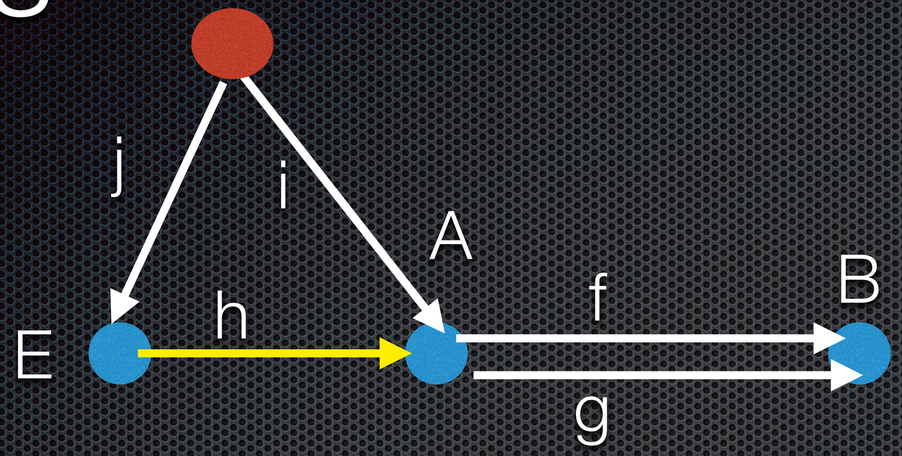
Colimits using Category of Elements

- We can model colimits in terms of a category of elements
  - $\int Cone(F, -)$
- Each object in this category is an actual cone Cone(x, F)
- Arrows are mappings from one cone to another
- A colimit is the initial object in the category of elements $\int Cone(F, -)$

Cartesian Products as Limits

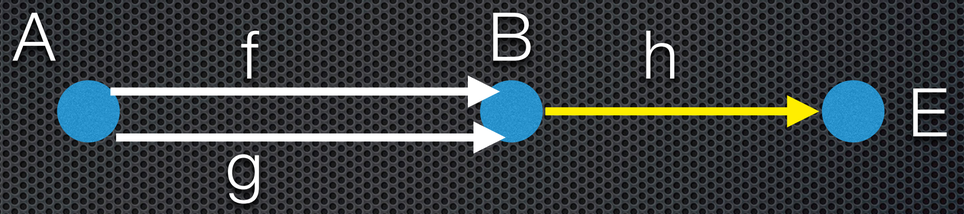
- A Cartesian product is a limit of a diagram indexed by a discrete category with only identity arrows
  - Limit is $\prod_j F_j$
  - The legs of the limit cone are projections $\pi_k : \prod_j F_j \to F_k$

Equalizers



- An equalizer is a limit of a diagram with two parallel arrows
- Note that equalizers are “right cancellable”:
  - f h = g h implies that f = g

Coequalizers



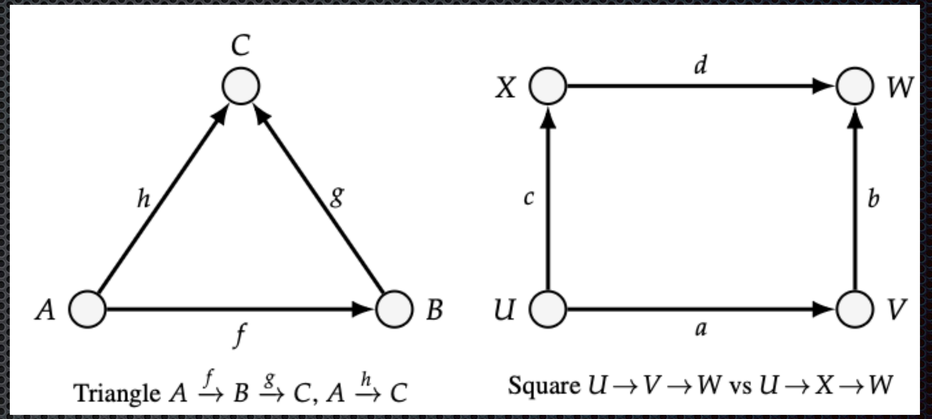
- A coequalizer is a colimit of a diagram with two parallel arrows
- Note that equalizers are “left cancellable”:
  - h f = h g implies that f = g

(Co)complete Categories

- A digram is small if its indexing category is small
- A category is complete if it admits limits of all small diagrams
- A category is cocomplete if it admits colimits of all small diagrams
- The category of Sets is complete and cocomplete

Diagrammatic Backpropagation

DB extends vanilla backpropagation to include diagrammatic reasoning



Mahadevan, Categories for AGI, 2026

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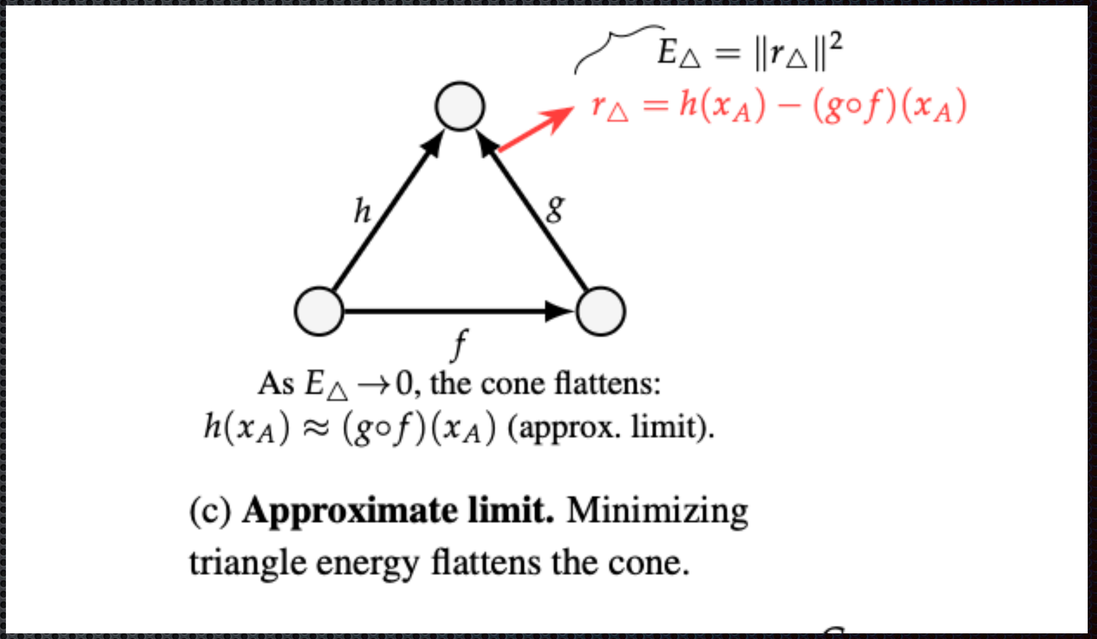

(c) **Approximate limit.** Minimizing triangle energy flattens the cone.

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Diagrammatic Backpropagation

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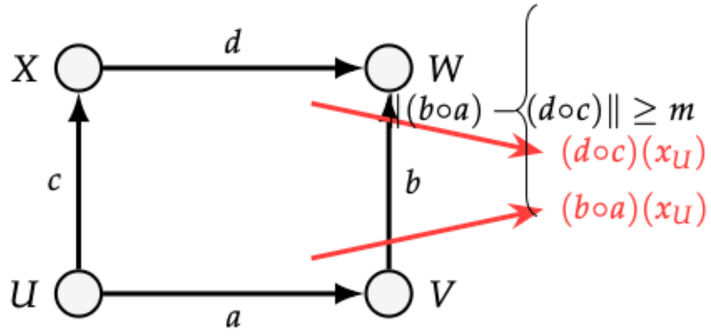

Contrastive margin m prevents collapse of distinct branches (approx. colimit separation).

(d) **Approximate colimit.** Margin keeps branches distinct.

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Diagrammatic Backpropagation

DB extends vanilla backpropagation to include diagrammatic reasoning

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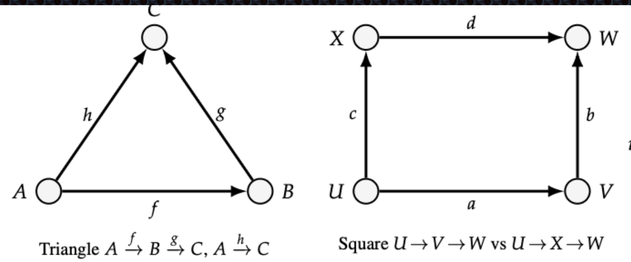
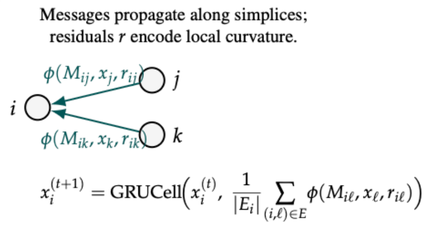
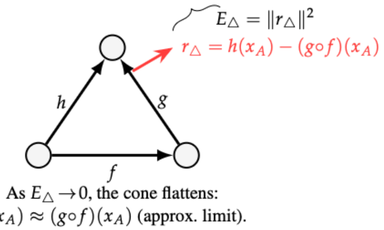


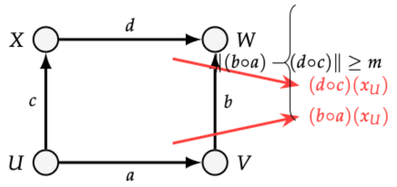
(a) **Simplicial diagrams.** Edges carry morphisms $M_f, M_g, \ldots$.



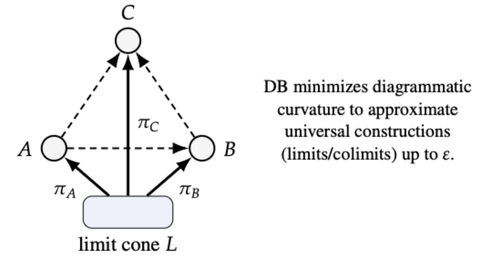
(b) **Simplicial message passing.** Curvature diffuses via messages.



(c) **Approximate limit.** Minimizing triangle energy flattens the cone.



(d) **Approximate colimit.** Margin keeps branches distinct.



(e) **Universal view.** Curvature $\to$ 0 $\Rightarrow$ numeric limit; margin $\Rightarrow$ numeric colimit.

Figure 12: **Simplicial message passing and limit/colimit approximation in DB.** (A) Triangle and square simplices encode compositional constraints. (B) DB aggregates messages along simplices; residuals carry local curvature. (C) Minimizing triangle energy $\|h(x_A) - (g \circ f)(x_A)\|^2$ flattens the cone (approx. limit). (D) A margin on square paths prevents collapse (approx. colimit).

Background Reading

- Chapter 3, Emily Riehl, Category Theory in Context
- Chapter 3, Sridhar Mahadevan, Categories for AGI
- GitHub code:
  - Synthetic AV demo of pullback using deep learning
  - Python Suduko GT solver uses diagrammatic constraints