1. (a). Given probability density function (PDF):

$$f(x,\lambda) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases} \qquad \lambda > 0.$$

$$E[X] = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \int_{-\infty}^{0} x \cdot 0\,dx + \int_{0}^{\infty} x \cdot \lambda e^{-x\lambda}\,dx.$$

$$= \int_{0}^{\infty} \lambda x e^{-\lambda x}\,dx.$$

$$= \lambda\left[x\int e^{-\lambda x}dx - \int\left(\int e^{-\lambda x}dx\right)dx\right]_0^{\infty}$$

$$= \lambda\left[\frac{xe^{-\lambda x}}{-\lambda} - \frac{e^{-\lambda x}}{\lambda^2}\right]_0^{\infty}$$

$$= \lambda\left[+\frac{1}{\lambda^2}\right]$$

$$E[X] = +\frac{1}{\lambda}. \text{ Ans.}$$

$$Var(X) = E[X^2] - (E[X])^2$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^{\infty} \lambda x^2 e^{-\lambda x}\,dx.$$

$$= \lambda\left[x^2\int e^{-\lambda x}dx - 2\int\left(x\int e^{-\lambda x}dx\right)dx\right]_0^{\infty}$$

$$= \lambda\left[\frac{x^2e^{-\lambda x}}{-\lambda} - 2\int\frac{xe^{-\lambda x}}{-\lambda}dx\right]_0^{\infty}$$

$$= \left[2\int xe^{-\lambda x}dx - x^2e^{-\lambda x}\right]_0^{\infty}$$

$$= \left[2\left(x\int e^{-\lambda x} - \int\left(\int e^{-\lambda x}dx\right)dx\right) - x^2e^{-\lambda x}\right]_0^{\infty}$$

$$= \left[2\left(\frac{xe^{-\lambda x}}{-\lambda} - \int\frac{e^{-\lambda x}}{-\lambda}dx\right) - x^2e^{-\lambda x}\right]_0^{\infty}$$

$$= \left[-\frac{2}{\lambda}xe^{-\lambda x} - \frac{2}{\lambda^2}e^{-\lambda x} - x^2e^{-\lambda x}\right]_0^{\infty}$$

$$= \frac{2}{\lambda^2} - (0) = \frac{2}{\lambda^2}$$

$$Var(X) = E[X^2] - (E[X])^2 = \frac{2}{\lambda^2} - \left(+\frac{1}{\lambda}\right)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}. \text{ Ans.}$$

1. (b) $f(x;\theta) = \begin{cases} 1/\theta & 0 \leqslant x \leqslant \theta \\ 0 & \text{otherwise.} \end{cases} \qquad \theta > 0.$

$$E(x) = \int_{-\infty}^{\infty} x.f(x)\,dx = \int_{0}^{\infty\,\theta} \frac{1}{\theta} x\,dx$$

$$E[x] = \frac{1}{\theta}\int_0^{\theta} x\,dx.$$

$$E[x] = \frac{1}{\theta}\left[\frac{\theta^2}{2}\right]$$

$$E[x] = \frac{\theta}{2}. \quad \text{Ans.}$$

$$Var[x] = E[x^2] - (E[x])^2$$

$$E[x^2] = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^{\theta} x^2 \frac{1}{\theta}\,dx$$

$$= \frac{1}{\theta}\int_0^{\theta} x^2\,dx$$

$$= \frac{1}{\theta}\left[\frac{x^3}{3}\right]_0^{\theta}$$

$$E[x^2] = \frac{\theta^2}{3}.$$

$$Var[x] = E[x^2] - (E[x])^2$$

$$= \frac{\theta^2}{3} - \frac{\theta^2}{4} = \frac{\theta^2}{12} \qquad Var[x] = \frac{\theta^2}{12}. \text{ Ans.}$$

1. (c). $E[x] = \sum_{\text{all}} x_i\, p(x_i)$

$$E[x] = (0 \times 0{\cdot}1) + (1 \times 0{\cdot}2) + (2 \times 0{\cdot}4) + 3(\times 0{\cdot}2) + (4 \times 0{\cdot}1)$$

$$E[x] = 2{\cdot}1 \text{ Ans.}$$

$$Var[x] = E[x^2] - (E[x])^2$$

$$E[x^2] = (0{\cdot}1 \times 0^2) + (0{\cdot}2 \times 1^2) + (0{\cdot}4 \times 2^2) + (0{\cdot}2 \times 3^2) + (0{\cdot}1 \times 4^2)$$

$$= 5{\cdot}2$$

$$Var[x] = E[x^2] - (E[x])^2$$

$$= 5{\cdot}2 - 2{\cdot}1^2$$

$$= 0{\cdot}79. \text{ Ans.}$$

2. $P(x|c_1) = \frac{3}{1000} x^2$

$P(x|c_2) = -\frac{1}{50}(x-10)$ $\quad 0 \leq x \leq 10$

(a) Given $x = 5$,

$$P(c_1|x) = \frac{P(x|c_1)\,P(c_1)}{P(x)}.$$

$$P(x|c_1)\Big|_{x=5} = \frac{3}{1000}(5)^2 \qquad P(x|c_2)\Big|_{x=5} = -\frac{1}{50}(x-10) = \frac{5}{50} = \frac{1}{10}$$

$$P(x) = P(x|c_1)\,P(c_1) + P(x|c_2)\,P(c_2)$$

$$= \frac{3}{1000} \times 25 \times P(c_1) + \frac{1}{10} \times P(c_2)$$

$$= \frac{3}{40} P(c_1) + \frac{1}{10}(1 - P(c_1)) \quad [\because P(c_1) + P(c_2) = 1].$$

$$= \frac{1}{10} - \frac{1}{40} P(c_1)$$

$$= \frac{4 - P(c_1)}{40}$$

Therefore, $P(c_1|x) = \dfrac{\frac{3}{40} P(c_1)}{\frac{(4 - P(c_1))}{40}} = \dfrac{3 P(c_1)}{4 - P(c_1)}$. Ans.

$$P(c_2|x) = \frac{P(x|c_2)\,P(c_2)}{P(x)}$$

$$= \frac{\frac{1}{10} P(c_2)}{\frac{(4 - P(c_1))}{40}} = \frac{4 P(c_2)}{4 - (1 - P(c_2))} = \frac{4 P(c_2)}{3 + P(c_2)}$$ Ans

$$\left.\begin{aligned} P(c_1|x) &= \frac{3 P(c_1)}{4 - P(c_1)} \\ P(c_2|x) &= \frac{4 P(c_2)}{3 + P(c_2)} \end{aligned}\right\} \text{Ans.}$$

2. (b). $P(C_1 | x) = P(C_2 | x)$

$$\Rightarrow \frac{P(x|C_1)\,P(C_1)}{P(x)} = \frac{P(x|C_2)\,P(C_2)}{P(x)}$$

$$\Rightarrow P(x|C_1)\,P(C_1) = P(x|C_2)\,P(C_2)$$

$$\Rightarrow P(x|C_1)\,P(C_1) = P(x|C_2)\,(1 - P(C_1))$$

$$\Rightarrow P(x|C_1)\,P(C_1) = P(x|C_2) - P(x|C_2)\,P(C_1)$$

$$\Rightarrow P(C_1)\left[P(x|C_1) + P(x|C_2)\right] = P(x|C_2)$$

$$\Rightarrow P(C_1) = \frac{P(x|C_2)}{P(x|C_1) + P(x|C_2)}$$

$$P(x|C_1)\Big|_{x=5} = \frac{3}{1000} \times 25 = \frac{3}{40} \qquad P(x|C_2)\Big|_{x=5} = \frac{1}{10}$$

therefore, $$P(C_1) = \frac{1/10}{3/40 + 1/10}$$

$$= \frac{4}{3+4} = \frac{4}{7}.$$ Ans.

2.(c)

Given that, $P(C_1) = P(C_2)$.

and ~~...~~, $P(C_1/x) > P(C_2/x)$ —— (1)

$$\frac{P(x|C_1)\,P(C_1)}{P(x)} > \frac{P(x|C_2)\,P(C_2)}{P(x)}.$$

$$P(x|C_1) > P(x|C_2) \quad [\because P(C_1) = P(C_2)].$$

$$\frac{3}{1000}x^2 > -\frac{1}{50}(x-10)$$

$$3x^2 > -20x + 200.$$

$$3x^2 + 20x - 200 > 0$$

$$(x - 5.485)(x + 12.15) > 0$$

$$\frac{-20 \pm \sqrt{400 + 2400}}{6} = \frac{-20 \pm 52.915}{6} = 5.485, -12.15$$

Therefore either $x > 5.485$ or $x < -12.15$.

but given that range of $x : 0 \leq x \leq 10$

$\therefore$ The required range of x is : $\boxed{5.485 < x \leq 10}$ Ans.

If $P(C_1) = 0.4$ & $P(C_2) = 0.6$

then from equation (1) : $P(C_1|x) > P(C_2|x)$

$$\frac{P(x|C_1)\,P(C_1)}{P(x)} > \frac{P(x|C_2)\,P(C_2)}{P(x)}$$

$$P(x|C_1)\,\frac{4}{10} > \frac{6}{10}\,P(C_2)$$

$$2 \times \frac{\cancel{3}}{\cancel{1000}\ 10}x^2 > -\frac{\cancel{3}}{\cancel{50}}(x-10)$$

$$x^2 > -10x + 100$$

$$x^2 + 10x - 100 > 0$$

$$(x - 6.18)(x + 16.18) > 0.$$

$$\frac{-10 \pm \sqrt{100 + 400}}{2} = \frac{-10 \pm 22.36}{2} = 6.18, -16.18$$

Therefore, either $x > 6.18$ or $x < 16.18$

$\therefore$ The required range of x is :

$\boxed{6.18 < x \leq 10}$ Ans.

3. Probability of testing positive when I have disease : $P(P/H) = 0.983$,

Probability of testing negative when I have disease : $P(N/H) = 1 - 0.983$

$= 0.017$

Probability of testing negative when I don't have disease : $P(N/DH) = 0.945$

Probability of testing positive when I don't have disease : $P(P/DH) = 1 - 0.945$

$= 0.055$

Probability of the disease striking $= P(R) = 0.0001 = P(H)$

therefore, $P(H/P) = \dfrac{P(P/H)\,P(H)}{P(P)}$

Here, $P(P) = P(P/H)\,P(H) + P(P/DH)\,P(DH)$

$= (0.983 \times 0.0001) + (0.0055 \times (1 - 0.0001))$

$= 0.0550928$

$\therefore\ P(H/P) = \dfrac{0.983 \times 0.0001}{0.0550928} = 0.00178426$

3. (b). $P(DH/P) = \dfrac{P(P/DH)\,P(DH)}{P(P)}$

$P(DH) = 1 - P(H) = (1 - 0'0001)$

$P(DH/P) = \dfrac{0'055 \times (1 - 0'0001)}{0'0550928} = 0'998216$

$P(H/P) = 0'00178426$

Cost for treatement having no disease = \$1000

Cost for not treatement even after having disease = \$1000000

Expected cost (Risk) assuming the test comes out positive and undergo treatement = $1000 \times 0'998216 = \underline{998'216}$

(c) Expected cost (Risk) assuming the test comes out positive and did not undergo treatement = $1000000 \times 0'00178426 = 1784{\cdot}26$

$(\text{Risk})_{\text{no treatement}} > (\text{Risk})_{\text{treatement}}$

$1784{\cdot}26 > 998{\cdot}216$

Therefore, the decision should be to undergo treatement.

4. $\phi : \mathbb{R}^3 \to \mathbb{R}^4$

$$\phi\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = A_\phi \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{3\times 1} = \begin{bmatrix} 3x_1 + 2x_2 + x_3 \\ x_1 + x_2 + x_3 \\ x_1 - 3x_2 \\ 2x_1 + 3x_2 + x_3 \end{bmatrix}_{4\times 1}$$

Therefore, the dimension of A_ϕ would be (4×3) which can be assumed to be

$$A_\phi = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cancel{a_{14}} \\ a_{21} & a_{22} & a_{23} & \cancel{a_{24}} \\ a_{31} & a_{32} & a_{33} & \cancel{a_{34}} \\ a_{41} & a_{42} & a_{43} & \cancel{a_{44}} \end{bmatrix}_{4\times 3}.$$

Therefore,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_1 + 2x_2 + x_3 \\ x_1 + x_2 + x_3 \\ x_1 - 3x_2 \\ 2x_1 + 3x_2 + x_3 \end{bmatrix}$$

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \\ a_{41}x_1 + a_{42}x_2 + a_{43}x_3 \end{bmatrix} = \begin{bmatrix} 3x_1 + 2x_2 + x_3 \\ x_1 + x_2 + x_3 \\ x_1 - 3x_2 \\ 2x_1 + 3x_2 + x_3 \end{bmatrix}$$

compairing the matrices,

$a_{11} = 3 \quad a_{12} = 2 \quad a_{13} = 1$

$a_{21} = 1 \quad a_{22} = 1 \quad a_{23} = 1$

$a_{31} = 1 \quad a_{32} = -3 \quad a_{33} = 0$

$a_{41} = 2 \quad a_{42} = 3 \quad a_{43} = 1$

Therefore the $(A_\phi)_{4\times 3} = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & -3 & 0 \\ 2 & 3 & 1 \end{bmatrix}_{4\times 3}$ Ans.

4. (b) $(A_\phi)_{4\times 3}$ $(A_\phi^T)_{3\times 4}$ ~~$(A_\phi A_\phi^T)_{4\times 4}$~~

Rank of A_ϕ or A_ϕ^T will be min of $(3,4)$ = ~~min~~ can be max 3.

Therefore, A_ϕ or A_ϕ^T will have of max 3.

But, $(A_\phi)_{4\times 3} (A_\phi)^T_{3\times 4} = (A_\phi A_\phi^T)_{4\times 4}$

Therefore $(A_\phi A_\phi^T)$ can have max rank of 4.

If rank $(A_\phi A_\phi^T) < 4$ then $|A_\phi A_\phi^T| = 0$.

but, rank $(A_\phi A_\phi^T) \leq$ ~~rank~~ min (rank (A_ϕ), rank (A_ϕ^T))

rank $(A_\phi A_\phi^T) \leq$ (can be max 3)

rank $(A_\phi A_\phi^T) \leq 3 < 4$.

Therefore, rank $(A_\phi A_\phi^T) < 4 \rightarrow |A_\phi A_\phi^T| = 0$. Proved

4. (c). $(A_\phi)_{4\times 3}$.

Therefore, rank (A_ϕ) = minimum of $(4,3)$ = can be maximum of 3.

therefore rank $(A_\phi) \leq 3$. Ans

5. (a). $\|y - Xw\|^2 = \langle y - Xw,\ y - Xw \rangle$

$$= y^Ty - 2w^TX^Ty + w^TX^TXw$$

To $\underset{w}{\text{minimize}}$ $\|y - Xw\|^2$ we take the derivatére wrt. w ~~x~~ and equate to 0,

Therefore, $-2X^Ty + 2X^TXw = 0$

$$X^Ty = X^TXw$$

$$\therefore \boxed{w = (X^TX)^{-1}X^Ty}.$$

5. (b) $\|Xw - y\|^2 + \lambda \|w\|^2 \longrightarrow$ objective function of Ridge Regression.

$$= (Xw - y)^T (Xw - y) + \lambda w^T w$$

$$= y^T y - w^T X^T y - w^T X^T y + w^T X^T X w + w^T w \lambda$$

$$= y^T y - 2 w^T X^T y + w^T w (X^T X + \lambda I)$$

Taking derivative wrt. w and equating to 0. (to minimise the function).

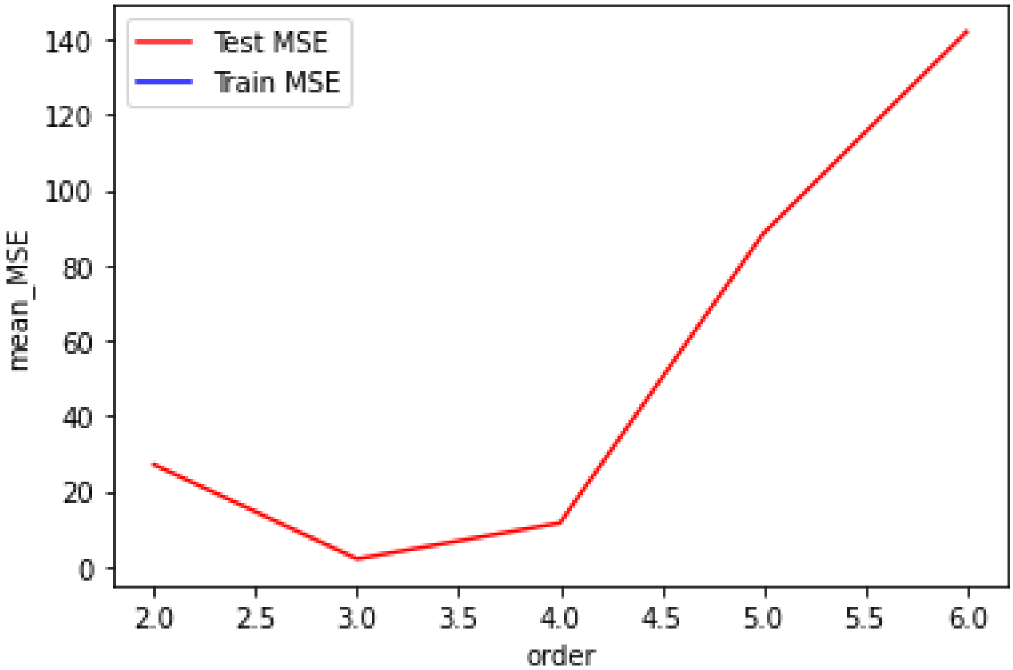
$$-2X^T y + 2w (X^T X + \lambda I) = 0$$

$$w (X^T X + \lambda I) = X^T y$$

$$\boxed{w = (X^T X + \lambda I)^{-1} X^T y}.$$

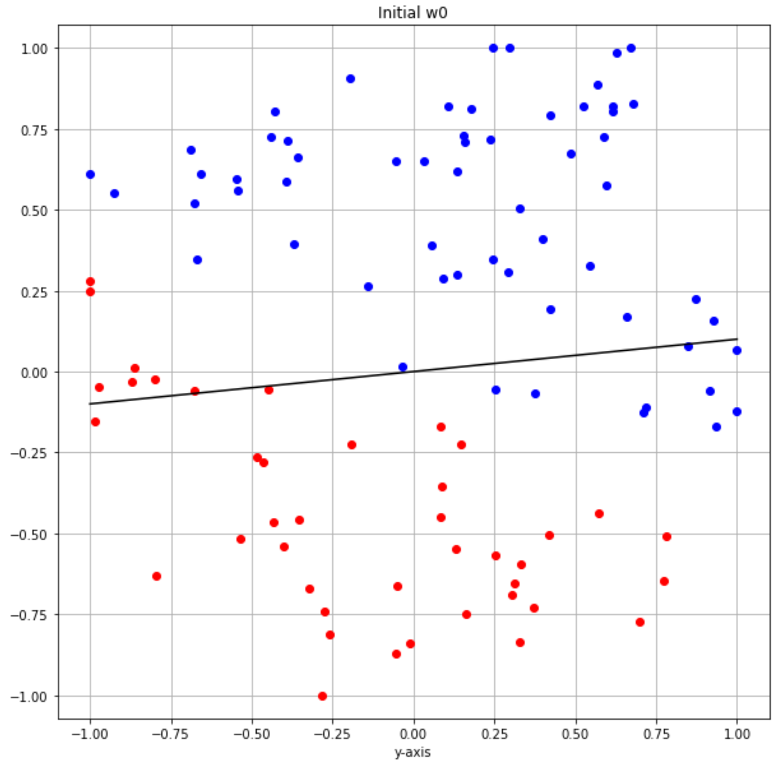
optimal w.

Q6. Error Figure -

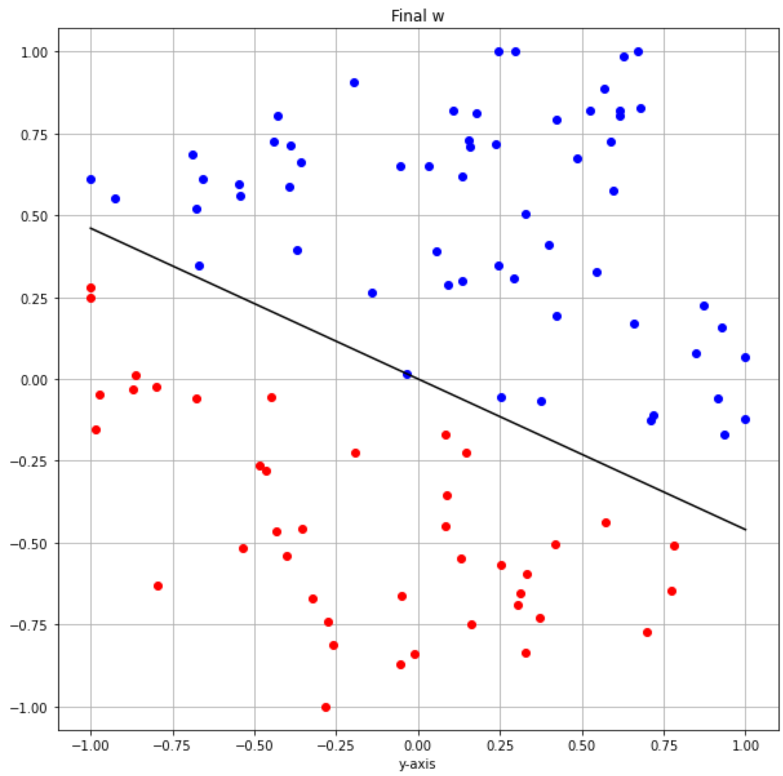


The MSE is lowest for order=3 therefore, the model is underfitting under the order < 3 and overfitting for order > 3.

Q7.



Initial_result_w0



Perceptron result