Module 4.4
Anatomy of AR(p) Models

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AR (1) model

- $$\begin{cases} y_t &= c + \Phi y_{t-1} + \epsilon_t = \Phi y_{t-1} + w_t \\ w_t &= c + \epsilon_t \end{cases} \tag{1}$$

- Iterating and substituting for w_t:

$$\begin{aligned} y_t &= w_t + \Phi w_{t-1} + \Phi^2 w_{t-2} + \cdots + \Phi^k w_{t-k} + \cdots \\ &= (c + \Phi c + \Phi^2 c + \cdots + \Phi^k c + \cdots) \\ &+ (\epsilon_t + \Phi \epsilon_{t-1} + \Phi^2 \epsilon_{t-2} + \cdots \Phi^k \epsilon_{t-k} + \cdots) \end{aligned} \tag{2}$$

- y_t is called AR(1) process

AR(1) and MA(∞)

- Assuming $|\Phi| < 1 : c + \Phi c + \Phi^2 c + \cdots = \frac{c}{1-\Phi}$
- Hence

$$y_t = \frac{c}{1-\Phi} + \epsilon_t + \Phi\epsilon_{t-1} + \Phi^2\epsilon_{t-2} + \cdots \Phi^k\epsilon_{t-k} + \cdots \quad (3)$$

  which is an MA(∞) process
- When $|\Phi| < 1$, y_t is stationary and ergodic (Why?)

AR(1) Process: Mean

- From (3):

$$\mathbf{E}(y_t) = \frac{c}{1-\Phi} = \mu \tag{4}$$

- Substituting (4) in (1):

$$\begin{aligned} y_t &= \mu(1-\Phi) + \Phi y_{t-1} + \epsilon_t \\ &\text{or} \\ y_t - \mu &= \Phi(y_{t-1} - \mu) + \epsilon_t \end{aligned} \tag{5}$$

- Variance:

$$\begin{aligned}\gamma_0 = \mathbf{Var}(y_t) &= \mathbf{E}[y_t - \mu]^2 \\ &= \mathbf{E}[\epsilon_t + \Phi\epsilon_{t-1} + \Phi^2\epsilon_{t-2} + \cdots + \Phi^k\epsilon_{t-q} + \cdots]^2 \\ &= \sigma^2[1 + \Phi^2 + \Phi^4 + \Phi^6 + \cdots + \Phi^{2k} + \cdots] \\ &= \frac{\sigma^2}{1-\Phi^2}\end{aligned} \tag{6}$$

- γ_0 is finite variance $|\Phi| < 1$

Auto-Covariance and Autocorrelation Function (ACF)

- $$\begin{aligned}\gamma_j &= \mathbf{E}[(y_t - \mu)(y_{t-j} - \mu)] \\ &= \mathbf{E}[(\epsilon_t + \Phi\epsilon_{t-1} + \Phi^2\epsilon_{t-2} + \cdots + \Phi^j\epsilon_{t-j} + \Phi^{j+1}\epsilon_{t-j-1} + \cdots) \\ &\quad (\epsilon_{t-j} + \Phi\epsilon_{t-j-1} + \Phi^2\epsilon_{t-j-2} + \cdots)] \\ &= \sigma^2\left[\Phi^j + \Phi^{j+2} + \Phi^{j+4} + \cdots\right] \\ &= \sigma^2\Phi^j\left[1 + \Phi^2 + \Phi^4 + \cdots\right] = \sigma^2\frac{\Phi^j}{1-\Phi^2}\end{aligned} \tag{7}$$

- $$\rho_j = \frac{\gamma_j}{\gamma_0} = \Phi^j, \quad \text{for } \; j \geq 1 \tag{8}$$

- Since $|\Phi| < 1$, $\rho_j = \Phi^j$ is an exponentially decreasing function of j
  - when $0 < \Phi < 1$, ρ_j decreases monotonically
  - when $-1 < \Phi < 0, \rho_j$ while decreases, it also oscillates between positive and negative values

- Plot of ρ_j vs j is called the correlogram

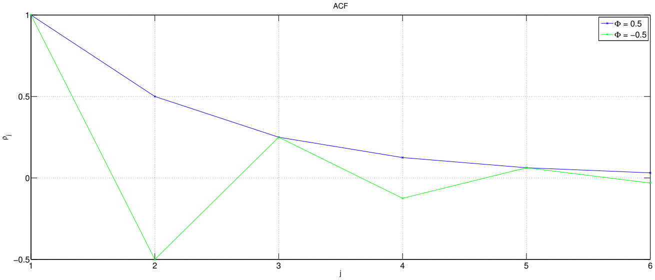

Figure : Plot of ρ_j vs j

- $$y_t = c + \Phi y_{t-1} + \epsilon_t \tag{9}$$
- Assuming weak stationarity, that is $|\Phi| < 1$:
$$\mathbf{E}(y_t) = \mu \text{ for all } t \tag{10}$$
- Taking expectations on both sides of (9) and using (10):
$$\mu = \mathbf{E}(y_t) = c + \Phi\mathbf{E}(y_{t-1}) = c + \Phi\mu \tag{11}$$
- Hence,
$$\mu = \frac{c}{1-\Phi} \tag{12}$$
same as in (4)

Variance of y_t

- Substituting (12) in (9):

$$y_t - \mu = \Phi(y_{t-1} - \mu) + \epsilon_t \tag{13}$$

- $$\begin{aligned} \mathbf{E}[y_t - \mu]^2 &= \mathbf{E}[\Phi(y_{t-1} - \mu) + \epsilon_t]^2 \\ &= \Phi^2 \mathbf{E}[y_{t-1} - \mu]^2 + \mathbf{E}(\epsilon_t^2) \end{aligned} \tag{14}$$

- The cross term vanishes since ϵ_t is uncorrelated with $(y_{t-1} - \mu)$

Variance of y_t

- Weak stationarity $\implies \mathbf{E}[y_t - \mu]^2 = \gamma_0$ for all t
- Hence, (14) becomes

$$\begin{aligned} &\gamma_0 = \Phi^2\gamma_0 + \sigma^2 \\ &\text{or} \\ &\gamma_0 = \frac{\sigma^2}{1-\Phi^2}, \text{ same as (6)} \end{aligned} \tag{15}$$

- Using (13)

$$\begin{aligned} \gamma_j &= \mathbf{E}[(y_t - \mu)(y_{t-j} - \mu)] \quad \{\text{use } (13)\} \\ &= \mathbf{E}[\Phi(y_{t-1} - \mu)(y_{t-j} - \mu) + \epsilon_t(y_{t-j} - \mu)] \\ &= \Phi\mathbf{E}[(y_{t-1} - \mu)(y_{t-j} - \mu)] \\ &= \Phi\gamma_{j-1} \end{aligned} \tag{16}$$

  since $(y_{t-j} - \mu)$ and ϵ_t are not correlated

- Iterating:

$$\gamma_j = \Phi^k \gamma_0 \tag{17}$$

- Combining (15) and (17) :

$$\gamma_j = \Phi^j \gamma_0 = \frac{\sigma^2 \Phi^j}{1 - \Phi^2} \quad \{\text{same as (8)}\} \tag{18}$$

- The behavior of ρ_j is illustrated in Figure (1)

$$\rho_j = \frac{\gamma_j}{\gamma_0} \tag{19}$$

Use of Lag operator

- $$\begin{aligned} y_t &= c + \Phi y_{t-1} + \epsilon_t \\ &= c + \Phi\,\mathrm{L}\, y_t + \epsilon_t \end{aligned} \tag{20}$$

$$y_t - \Phi\,\mathrm{L}\, y_t = (1 - \Phi\,\mathrm{L}) y_t = c + \epsilon_t \tag{21}$$

- Hence $y_t = (1 - \Phi\,\mathrm{L})^{-1} c + (1 - \Phi\,\mathrm{L})^{-1} \epsilon_t$

Relation between AR(1) and MA(∞) - Another look

- $$\begin{aligned}(1-\Phi\,\mathrm{L})^{-1}c &= c+\Phi c+\Phi^2 c+\cdots \; (\because LC = C)\\ &= \frac{c}{1-\Phi} \quad \text{if } |\Phi|<1\end{aligned}$$

- $$y_t = \frac{c}{1-\Phi} + \epsilon_t + \Phi\epsilon_{t-1} + \Phi^2\epsilon_{t-2} + \cdots \tag{22}$$

  which is MA(∞) representation of y_t

AR(2) Process

- Let
$$y_t = c + \Phi_1 y_{t-1} + \Phi_2 y_{t-2} + \epsilon \tag{23}$$
- From Module 4.2 on "Difference equations":
- Solution y_t of (23) is stable/stationary if the roots of the characteristic equation lie within the unit circle in the complex plane

Stationarity condition

- The characteristic equation for (23) is

$$\lambda^2 - \Phi_1 \lambda - \Phi_2 = 0 \tag{24}$$

$$\lambda = \frac{\Phi \pm \sqrt{\Phi_1^2 + 4\Phi_2}}{2} \tag{25}$$

- For stationarity:

$$|\lambda| < 1 \tag{26}$$

- In the following, we assume that the AR(2) parameters Φ_1 and Φ_2 are such that the condition (26) holds

Example

- Let
$$y_t = 0.6y_{t-1} + 0.3y_{t-2} + \epsilon_t \tag{27}$$
- Then $c = 0$, $\Phi_1 = 0.6$, $\Phi_2 = 0.3$
$$\lambda = \frac{\Phi \pm \sqrt{\Phi_1^2 + 4\Phi_2}}{2}$$
- Substituting: $\lambda_1 = 0.9245$, $\lambda_2 = -0.649$
- Hence, $|\lambda| < 1$ and y_t in (27) is stationary

AR(2) Process: Mean

- Let
$$y_t = c + \Phi_1 y_{t-1} + \Phi_2 y_{t-2} + \epsilon \tag{28}$$
- Since y_t is assumed yo be weakly stationary,
$$\mathbf{E}(y_t) = c + \Phi_1 \mathbf{E}(y_{t-1}) + \Phi_2 \mathbf{E}(y_{t-2})$$
- That is
$$\mu = c + \Phi_1 \mu + \Phi_2 \mu$$
or
$$\mu = \frac{c}{1 - \Phi_1 - \Phi_2} \tag{29}$$

AR(2) Process: A centered version

- Substituting (29) in (28) and simplifying:

$$(y_t - \mu) = \Phi_1(y_{t-1} - \mu) + \Phi_2(y_{t-2} - \mu) + \epsilon_t \qquad (30)$$

- $$\begin{aligned}\gamma_0 &= \mathbf{E}[(y_t-\mu)(y_t-\mu)] \quad \{\text{using (30)}\} \\ &= \mathbf{E}[\Phi_1(y_t-\mu)(y_{t-1}-\mu)] \\ &+ \mathbf{E}[\Phi_2(y_t-\mu)(y_{t-2}-\mu)] \\ &+ \mathbf{E}[(y_t-\mu)\epsilon_t]\end{aligned} \tag{31}$$

- But
$$\left.\begin{aligned}\mathbf{E}[\Phi_1(y_t-\mu)(y_{t-1}-\mu)] &= \Phi_1\gamma_1 \\ \mathbf{E}[\Phi_2(y_t-\mu)(y_{t-2}-\mu)] &= \Phi_2\gamma_2\end{aligned}\right\} \tag{32}$$

- The third term on the r.h.s of (31):

$$\begin{aligned} \mathbf{E}[(y_t - \mu)\epsilon_t] &= \Phi_1 \mathbf{E}[(y_{t-1} - \mu)\epsilon_t)] \\ &+ \Phi_2 \mathbf{E}[(y_{t-2} - \mu)\epsilon_t)] \\ &+ \mathbf{E}[\epsilon_t^2] \end{aligned} \tag{33}$$

- Since ϵ_t is <u>not</u> correlated with $(y_{t-1} - \mu)$ and $(y_{t-2} - \mu)$

$$\mathbf{E}[(y_t - \mu)\epsilon_t] = \mathbf{E}[\epsilon_t^2] = \sigma^2 \tag{34}$$

- Substituting (32) and (34) into (3):

$$\gamma_0 = \Phi_1\gamma_1 + \Phi_2\gamma_2 + \sigma^2 \tag{35}$$

- Notice the dependence of γ_0 on γ_1 and γ_2
- We now move on to computing γ_j for $j \geq 1$

- $$\gamma_j = \mathbf{E}[\Phi_1(y_t - \mu)(y_{t-j} - \mu)] \tag{36}$$
- Using (30)
$$\begin{aligned}\gamma_j &= \Phi_1 \mathbf{E}[(y_{t-1} - \mu)(y_{t-j} - \mu)] \\ &+ \Phi_2 \mathbf{E}[(y_{t-2} - \mu)(y_{t-j} - \mu)] \\ &+ \underbrace{\mathbf{E}[\epsilon_t(y_{t-j} - \mu)]}_{=0}\end{aligned} \tag{37}$$
- $$\gamma_j = \Phi_1 \gamma_{j-1} + \Phi_2 \gamma_{j-2} \tag{38}$$

- Combining (35) and (38)

$$\left.\begin{aligned} \gamma_0 &= \Phi_1\gamma_1 + \Phi_2\gamma_2 + \sigma^2 \\ \gamma_j &= \Phi_1\gamma_{j-1} + \Phi_2\gamma_{j-2} \ \text{ for } j \geq 1 \end{aligned}\right\} \tag{39}$$

- Notice the close relationship between AR(2) model in (30) and the dynamics of evolution auto-covariance γ_j in (38)
- The system in (38) is known as the Yule-Walker equation

ACF for AR(2)

- Dividing the second equation in (39) by γ_0:

$$\frac{\gamma_j}{\gamma_0} = \Phi_1 \frac{\gamma_{j-1}}{\gamma_0} + \Phi_2 \frac{\gamma_{j-2}}{\gamma_0},$$

or

$$\rho_j = \Phi_1 \rho_{j-1} + \Phi_2 \ \rho_{j-2}, \quad \text{for} \ \ j \geq 1 \tag{40}$$

- Setting $j = 1$, since $\rho_0 = 1$, using $\rho_k = \rho_{-k}$, from (40)

$$\rho_1 = \Phi_1 \rho_0 + \Phi_2 \rho_{-1} = \Phi_1 + \Phi_2 \rho_1$$

or

$$\rho_1 = \frac{\Phi_1}{1 - \Phi_2} \tag{41}$$

- Setting $j = 2$ in (40):

$$\rho_2 = \Phi_1 \rho_1 + \Phi_2 \rho_0 = \Phi_1 \rho_1 + \Phi_2 \tag{42}$$

Expression for $\gamma_0 = \textbf{Var}(y_t)$

- Recall:

$$\begin{aligned}\gamma_0 &= \Phi_1\gamma_1 + \Phi_2\gamma_2 + \sigma^2 \\ &= \Phi_1\rho_1\gamma_0 + \Phi_2\rho_2\gamma_0 + \sigma^2\end{aligned}$$

or

$$[1 - \Phi_1\rho_1 - \Phi_2\rho_2]\gamma_0 = \sigma^2 \tag{43}$$

- Substituting (41) and (42) in (43):

$$\gamma_0 = \frac{(1-\Phi_2)\sigma^2}{(1+\Phi_2)[(1-\Phi_2)^2 - \Phi_1^2]} \tag{44}$$

ACF for y_t - Summary

- $$\left.\begin{aligned} \rho_0 &= 1 \\ \rho_1 &= \tfrac{\Phi_1}{1-\Phi_2}, \\ \rho_2 &= \Phi_1\rho_1 + \Phi_2 \end{aligned}\right\} \tag{45}$$

$$\rho_j = \Phi_1\rho_{j-1} + \Phi_2\rho_{j-2},\ j \geq 3 \tag{46}$$

- (45) gives the I.C. for the second-order recurrence in (46)
- We can recursively compute ρ_j using (45)-(46) and plot ρ_j vs j

- Let $y_t = 0.6y_{t-1} + 0.3y_{t-2} + \epsilon_t$
- $\Phi_1 = 0.6$ and $\Phi_2 = 0.3$
- $\rho_0 = 1, \rho_1 = \frac{\Phi_1}{1-\Phi_2} = \frac{0.6}{0.7} = 0.8571$
- $\rho_2 = \Phi_1\rho_1 + \Phi_2 = 0.8143$
- $\rho_3 = \Phi_1\rho_2 + \Phi_2\rho_1 = 0.7457$, etc.

AR(p) Model

- Let

$$y_t = c + \Phi_1 y_{t-1} + \Phi_2 y_{t-2} + \cdots + \Phi_p y_{t-p} + \epsilon_t \tag{47}$$

- The lag polynomial for (47) is

$$\Phi(r) = 1 - \Phi_1 r - \Phi_2 r^2 - \cdots - \Phi_p r^p = 0 \tag{48}$$

- If the p roots of $\Phi(r) = 0$ in (48) lie outside the 0 unit circle in the complex plane, then, y_t is a stable and a weakly stationary process.
- In the following , it is assumed that y_k in (47) is weakly stationary

Mean of AR(p) process

- Taking expectations on both sides of (47):

$$\mathbf{E}[y_t] = c + \Phi_1 \mathbf{E}[y_{t-1}] + \Phi_2 \mathbf{E}[y_{t-2}] + \cdots + \Phi_p \mathbf{E}[y_{t-p}]$$

- Since $\mathbf{E}[y_t] = \mu$ for all μ:

$$\mu = \frac{c}{1 - \Phi_1 - \Phi_2 - \cdots - \Phi_p} \tag{49}$$

Centered AR(p) process

- Substituting (49) into (47):

$$y_t - \mu = c + \Phi_1(y_{t-1} - \mu) + \Phi_2(y_{t-2} - \mu) + \cdots + \Phi_p(y_{t-p} - \mu) + \epsilon_t \quad (50)$$

- This form is useful to characterize the ACF

Variance of y_t

- $$\begin{aligned}\gamma_0 &= \mathbf{E}[(y_t-\mu)(y_t-\mu)] \quad \{\text{using (50)}\} \\ &= \mathbf{E}[(y_t-\mu)\{\Phi_1(y_{t-1}-\mu) \\ &\quad + \Phi_2(y_{t-2}-\mu)+\cdots+\Phi_p(y_{t-p}-\mu)+\epsilon_t\}] \\ &= \Phi_1\mathbf{E}[(y_t-\mu)\{(y_{t-1}-\mu)] \\ &\quad + \Phi_2\mathbf{E}[(y_t-\mu)(y_{t-2}-\mu)]+\cdots \\ &\quad + \Phi_p\mathbf{E}[(y_t-\mu)(y_{t-p}-\mu)]+\mathbf{E}[(y_t-\mu)\epsilon_t]\end{aligned} \tag{51}$$

Variance of y_t

- Since ϵ_t is uncorrelated with $(y_{t-j} - \mu)$ for all $j > 0$, the last term on the r.h.s of (51) reduces to

$$\mathbf{E}[(y_t - \mu)\epsilon_t] = \mathbf{E}[\epsilon_t^2] = \sigma^2 \tag{52}$$

- Using (52) and from the definition of γ_j, we obtain

$$\gamma_0 = \Phi_1\gamma_1 + \Phi_2\gamma_2 + \cdots + \Phi_p\gamma_p + \sigma^2 \tag{53}$$

- $$\begin{aligned}\gamma_j &= \mathbf{E}[(y_t - \mu)(y_{t-j} - \mu)] \\ &= \Phi_1 \mathbf{E}[(y_{t-1} - \mu)\{(y_{t-j} - \mu)] \\ &\quad + \Phi_2 \mathbf{E}[(y_{t-2} - \mu)(y_{t-j} - \mu)] + \cdots \\ &\quad + \Phi_p \mathbf{E}[(y_{t-p} - \mu)(y_{t-j} - \mu)] + \underbrace{\mathbf{E}[(y_{t-j} - \mu)\epsilon_t]}_{=0}\end{aligned} \tag{54}$$

$$\gamma_j = \Phi_1 \gamma_{j-1} + \Phi_2 \gamma_{j-2} + \cdots + \Phi_p \gamma_{j-p} \text{ for } j \geq 1 \tag{55}$$

Yule-Walker expectation

- Dividing (55) by γ_0: For $j \geq 1$

$$\rho_j = \Phi_1 \rho_{j-1} + \Phi_2 \rho_{j-2} + \cdots + \Phi_p \rho_{j-p}, \tag{56}$$

  which is a p^{th} order recurrence relation.
- We need a set of p initial conditions to solve for ρ_j for $j > p$

- Set $\rho_j = \lambda^j$ in (56)
- Substituting in (56):

$$\lambda^{j-p}\left[\lambda^p - \Phi_1\lambda^{p-1} - \Phi_2\lambda^{p-2} - \cdots - \Phi_p\right] = 0$$

- Since $\lambda \neq 0$, we get the characteristic equation:

$$\lambda^p - \Phi_1\lambda^{p-1} - \Phi_2\lambda^{p-2} - \cdots - \Phi_p = 0 \tag{57}$$

General Solution

- Let $\lambda_1, \lambda_2, \ldots, \lambda_p$ be the roots of (57), where we are guaranteed that $|\lambda_i| < 1$ for $1 \leq i \leq p$ by weak stationary condition
- Then

$$\rho_j = a_1\lambda_1^j + a_2\lambda_2^j + \cdots + a_p\lambda_p^j \tag{58}$$

  is a general solution where $a_1, a_2, \ldots, a_p$ are unknown constants
- a_i's are determined using the p-initial conditions on $\rho_j, 1 \leq j \leq p$

- When $p = 2$, the solution of the Yule-Walker equation was given earlier: From (56)

$$\left.\begin{aligned} \Phi_1\rho_0 + \Phi_2\rho_1 = \rho_1 \\ \Phi_2\rho_0 + \Phi_1\rho_1 = \rho_2 \end{aligned}\right\} \tag{59}$$

or

$$\begin{bmatrix} \Phi_1 & \Phi_2 \\ \Phi_2 & \Phi_1 \end{bmatrix} \begin{bmatrix} \rho_0 \\ \rho_1 \end{bmatrix} = \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} \tag{60}$$

An Illustration

- Let $p = 3$. Then from (56) :

$$\begin{bmatrix} \Phi_1 & \Phi_2 & \Phi_3 \\ \Phi_2 & \Phi_1 & \Phi_2 \\ \Phi_3 & \Phi_2 & \Phi_1 \end{bmatrix} \begin{bmatrix} \rho_0 \\ \rho_1 \\ \rho_2 \end{bmatrix} = \begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{bmatrix} \tag{61}$$