A COLLECTION OF RESULTS ON M.C.

- Let P be the transition matrix of a M.C.
- This M.C. is said to have a regular transition matrix if there is no absorbing state and there is a non-zero probability of going from any state to any other state. That is, the graph is strongly connected.

<u>Theorem 1</u>: Let $P = [p_{ij}]$ and $p_{ij} > 0$. Let ε be the smallest entry in P and $\varepsilon > 0$. Let $x \in R^n$ and $M_0 = \max_i \{x_i\}$, $m_0 = \min_i \{x_i\}$ and let M_1 and m_1 be the corresponding quantities for Px. Then

$$M_1 \le M_0 \quad \text{and} \quad m_1 \geq m_0$$

$$(M_1 - m_1) \le (1-2\varepsilon)(M_0 - m_0)$$

<u>Proof</u>: Let $\bar{x}$ be obtained from x by replacing all of its elements except m_0 by M_0. Then $x \le \bar{x}$ (element wise comparison)

- Let $y = P\bar{x}$ be a vector. Then, each y_i is the average of the elements of $\bar{x}$

$$\text{(ie)} \quad y_i = \sum_{j=1}^{n} p_{ij} \bar{x} \qquad 1 \le i \le n.$$

- Hence, each component of y is of the form (with $a \geq \varepsilon$)

$$a\, m_0 + (1-a) M_0 = M_0 - a(M_0 - m_0).$$

$\therefore$ Each element y_i of $y = P\bar{x}$ is such that

$$[1 \leq i \leq n) \quad y_i \leq M_0 - \varepsilon(M_0 - m_0) \qquad (\varepsilon < a)$$

~~Since $x \leq \bar{x} \Rightarrow Px \leq P\bar{x}$~~

$$\therefore \max_i \{y_i\} \leq M_0 - \varepsilon(M_0 - m_0)$$

- Recall $x \leq \bar{x} \Rightarrow Px \leq P\bar{x} = y$

$$\therefore \left\{\begin{matrix}\text{Max. element} \\ \text{of } Px = M_1\end{matrix}\right\} \leq \max_i \{y_i\}$$

$$M_1 \leq M_0 - \varepsilon(M_0 - m_0) \rightarrow (1)$$

- Now apply the same argument to $-x$, $\Rightarrow$

$$-m_1 \leq -m_0 - \varepsilon[-m_0 + M_0] \longrightarrow (2)$$

- Add (1) & (2):

$$M_1 - m_1 \leq M_0 - m_0 - 2\varepsilon(M_0 - m_0)$$

$$= (1 - 2\varepsilon)(M_0 - m_0).$$

<u>Theorem 2</u>: Let $P = [p_{ij}]$, $p_{ij} > 0$. Then

a) $P^n \to A$

b) Each row of A is the same: $\alpha = (\alpha_1, \alpha_2 \cdots \alpha_n)$

(ii) $A = u\alpha$ where $u = (1, 1, \cdots 1)^T$

c) $\alpha_i > 0$ for all i

<u>Proof</u>: Let $P = [p_{ij}]$ and let $\varepsilon > 0$ be the min. element of P. Let e_j be the j^{th} unit vector.

. Let M_k, m_k be the max and min elements of $P^k e_j$.

. From $P^k e_j = P(P^{k-1} e_j)$ and from Theorem (1) above, it follows that

$$M_1 \geq M_2 \geq M_3 \geq \cdots$$

$$m_0 \leq m_1 \leq m_3 \leq \cdots$$

and
$$\underbrace{M_k - m_k}_{d_k} \leq (1-2\varepsilon)\underbrace{(M_{k-1} - m_{k-1})}_{d_{k-1}}$$

(ii) $$d_k \leq (1-2\varepsilon)\, d_{k-1}$$

. Iterating $d_k \leq (1-2\varepsilon)^k d_0 \to 0$ as $k \to \infty$

Thus, $P^k e_j \to$ <u>to a vector with the same element</u>.

. Let α_j be the common value. Then

$0 < m_k \le \alpha_j \le M_k$ for all k.

(ii) $P^k e_j \to \alpha_j > 0$ as $h \to \infty$

· Since the row sum of $P^k = 1$ for each k,

$$\sum_{j=1}^{n} \alpha_j = 1$$

(ii) $P^k \to \begin{bmatrix} \alpha_j \\ \alpha_j \\ \vdots \\ \alpha_j \end{bmatrix}$ as $h \to \infty$ $\begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \vdots & \vdots & & \vdots \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{bmatrix}$

<u>Theorem 3:</u> Let P, A, α be as in Theorem 2.

(a) If π is a prob. vector, then $\pi^T P^k \to \alpha$ as $k \to \infty$ (π is column)

(b) α is the unique vector: $\alpha P = \alpha$ (α-row)

(c) $PA = AP = A$.

<u>Proof:</u> · If π is a prob vector $\Rightarrow \sum_{i=1}^{n} \pi_i = 1$.

Hence $\pi^T P^k \to \pi^T A = \pi^T \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \vdots & \vdots & & \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{bmatrix}$

$= (\alpha_1, \alpha_2, \ldots \alpha_n) = \alpha$

· Consider $P^{k+1} = P \cdot P^k = P^k \cdot P$

let $k \to \infty$ $\underline{A = PA = A \cdot P}$ $\Rightarrow$ (c) above

· To show α is unique: $\alpha = \alpha P$.

Let β be another vector: $\beta = \beta P$ $\left(\sum_{j=1}^{n} \beta_j = 1\right)$

By (a): $\beta P^n \to \beta A = \alpha. \longrightarrow (*)$

But $\beta P = \beta$

$\therefore \beta P^2 = (\beta P)P = \beta P = \beta$

$\therefore \beta P^n = \beta \longrightarrow (**)$

Combining $(*)$ and $(**)$ $\Rightarrow$ $\alpha = \beta$ (ie) unique.

<u>Summary</u>: Let $P = [p_{ij}]$ $\quad p_{ij} > 0$

1) $P^n \longrightarrow A = \begin{bmatrix} \alpha \\ \alpha \\ \vdots \\ \alpha \end{bmatrix}$ $\qquad \alpha = (\alpha_1, \alpha_2 \ldots \alpha_n)$

$\alpha_i > 0, \quad \sum_{i=1}^{n} \alpha_i = 1$

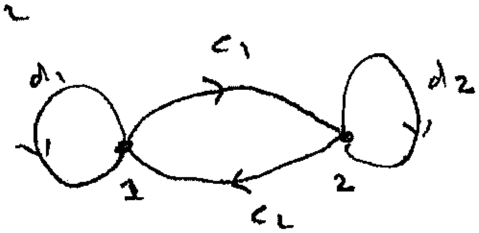
2) $\alpha P = \alpha$ (ie) α is the left eigenvector corresponding to the e.v $\lambda = 1$.

Thus, given P, we can easily find P^* the limit of P^k as $k \to \infty$, by simply solving for

$$\alpha P = \alpha.$$

3) α is called the stationary distribution.

<u>Example</u>: $n=2$



$$P = \begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} d_1 & c_1 \\ c_2 & d_2 \end{bmatrix} \end{array}$$

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• Find $\alpha = (\alpha_1, \alpha_2)$: $\alpha = \alpha P$

$$(\alpha_1, \alpha_2) = (\alpha_1, \alpha_2) \begin{bmatrix} d_1 & c_1 \\ c_2 & d_2 \end{bmatrix}$$

$$\therefore \boxed{\alpha_1 = \alpha_1 d_1 + \alpha_2 c_2} , \quad \boxed{\alpha_2 = \alpha_1 c_1 + \alpha_2 d_2}$$

$$\boxed{\alpha_1 + \alpha_2 = 1}$$

$$\therefore \alpha_1 = 1 - \alpha_2$$

$$(d_i = 1 - c_i)$$

$$\therefore \cancel{\alpha_2} = (1 - \alpha_2) c_1 + \alpha_2 d_2$$

$$= c_1 - \alpha_2 c_1 + \alpha_2 d_2$$

$$= c_1 - \alpha_2 (c_1 - d_2) = c_1 - \alpha_2 [c_1 - (1 - c_2)]$$

$$= c_1 - \alpha_2 c_1 - \alpha_2 c_2 + \cancel{\alpha_2}$$

$$\therefore c_1 = \alpha_2 (c_1 + c_2) \Rightarrow \alpha_2 = \frac{c_1}{c_1 + c_2}$$

$$\therefore \alpha_1 = \frac{c_2}{c_1 + c_2}$$

$$\therefore \alpha = \left(\frac{c_2}{c_1 + c_2}, \frac{c_1}{c_1 + c_2} \right)$$

$$\therefore \boxed{\left.\begin{matrix} c_2 > c_1 \\ < c_1 \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} \alpha_1 > \alpha_2 \\ < \alpha_2 \end{matrix}\right.}$$