Module 4.3
Anatomy of MA(q) Models

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MA (1) model

- y_t is a stationary, ergodic process
- $y_t = \mu + \epsilon_t + \theta\epsilon_{t-1}$
- μ is the mean of y_t: $\mathbf{E}(y_t) = \mu$
- θ is the MA(1) parameter
- y_t is the weighted sum of the two most recent values of the white noise sequence, ϵ_t
- $\mathbf{E}(\epsilon_t) = 0, \mathbf{Var}(\epsilon_t) = \sigma^2, \mathbf{E}(\epsilon_t\epsilon_{t-k}) = 0$ for all $k \geq 1$

Second-Order Properties of MA(1): Mean and Variance

- Mean

$$\mathbf{E}(y_t) = \mu$$

- Variance

$$\begin{aligned}
\gamma_0 = \mathbf{Var}(y_t) &= \mathbf{E}[y_t - \mathbf{E}(y_t)]^2 \\
&= \mathbf{E}[y_t - \mu]^2 \\
&= \mathbf{E}[\epsilon_t + \theta\epsilon_{t-1}]^2 \\
&= \mathbf{E}[\epsilon_t + 2\theta\epsilon_t\epsilon_{t-1} + \theta^2\epsilon_{t-1}^2] \\
&= \sigma^2 + 0 + \theta^2\sigma^2 = (1+\theta^2)\sigma^2
\end{aligned}$$

Second-order properties - autocovariance

- $$\begin{aligned}\gamma_j &= \mathbf{E}[(y_t - \mu)(y_{t-j} - \mu)] \\ &= \mathbf{E}[(\epsilon_t + \theta\epsilon_{t-1})(\epsilon_{t-j} + \theta\epsilon_{t-j-1})] \\ &= \mathbf{E}[(\epsilon_t\epsilon_{t-j} + \theta\epsilon_t\epsilon_{t-j-1} + \theta\epsilon_{t-1}\epsilon_{t-j} + \theta^2\epsilon_{t-1}\epsilon_{t-j-1})] \\ &= \begin{cases} \theta\mathbf{E}[\epsilon_{t-1}^2] = \theta\sigma^2 & \text{if } j = 1 \\ 0 & \text{otherwise} \end{cases}\end{aligned}$$

- $$\sum_{j=0}^{\infty} |\gamma_j| = \gamma_0 + \gamma_1 = (1+\theta)\sigma^2 < \infty$$

- MA(1) is ergodic in all moments

Autocorrelation function (ACF) of MA(1)

- $\rho_j = \frac{\gamma_j}{\gamma_0} = \frac{\theta\sigma^2}{(1+\theta^2)\sigma^2} = \begin{cases} \frac{\theta}{(1+\theta^2)} & \text{if } j = 1 \\ 0 & \text{if } j \neq 1 \end{cases}$
- Plot of ACF vs k, recall $\gamma_0 = 1$

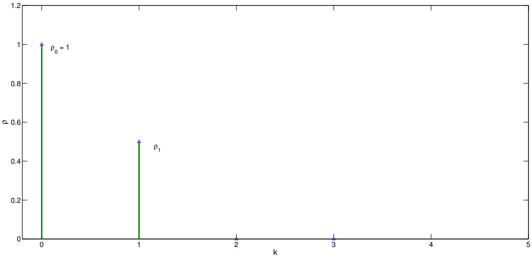


Figure: Plot of ACF vs k

- $$\rho_1(\theta) = \frac{\theta}{(1+\theta^2)}$$

- $$\rho_1(1/\theta) = \frac{1}{\theta}\frac{\theta^2}{(1+\theta^2)} = \frac{\theta}{(1+\theta^2)} = \rho_1(\theta)$$

- That is, MA(1) with θ and $1/\theta$ have the same ACF

- Plot of $\rho_1(\theta) = \frac{\theta}{(1+\theta^2)}, \rho_1(0.5) = \rho_1(2.0) = 0.4$

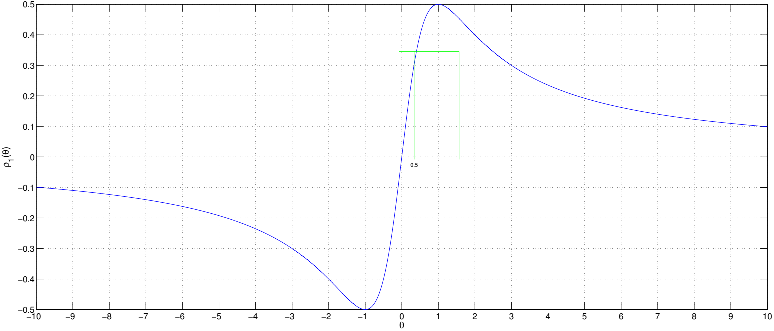


Figure: Plot of $\rho(\theta)$

Invertibility

- Rewrite:

$$y_t - \mu = \epsilon_t + \theta\epsilon_{t-1} = (1 + \theta\,\mathsf{L})\epsilon_t$$

$$\epsilon_t = (1 + \theta\,\mathsf{L})^{-1}(y_t - \mu)$$

- $$\frac{1}{1 + \theta\,\mathsf{L}} = \sum_{j=0}^{\infty}(-1)^j\theta^j\,\mathsf{L}^j, \quad \text{when} \quad |\theta| < 1$$

- Then

$$\begin{aligned}\epsilon_t &= (1 - \theta\,\mathsf{L} + \theta^2\,\mathsf{L}^2 - \theta^3\,\mathsf{L}^3 \cdots)(y_t - \mu)\\ &= (y_t - \mu) - \theta(y_{t-1} - \mu) + \theta^2(y_{t-2} - \mu) - \cdots\end{aligned}$$

- This is an infinite AR representation

Moral of the story

- If θ in MA(1) is such that $|\theta| < 1$, then do the analysis
- If θ in MA(1) is such that $|\theta| > 1$, then replace θ by $\overline{\theta} = 1/\theta$ and do the analysis with $\overline{\theta}$
- If $\theta = 1$, then there is no invertible version for MA(1)

MA(q) Model: Mean and Variance

- $y_t = \mu + \sum_{j=0}^{q} \theta_j \epsilon_{t-j}, q \geq 1$ and $\theta_0 = 1$
- $(\theta_1, \theta_2, \theta_3, \ldots, \theta_q)$ are the MA(q) parameters
- Mean

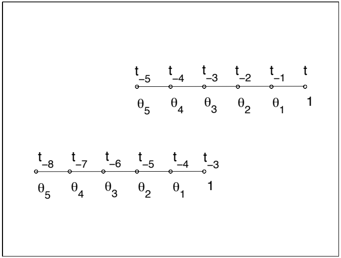
$$\mathbf{E}(y_t) = \mu$$

- Variance

$$\begin{aligned} \gamma_0 = \mathbf{Var}(y_t - \mu) &= \mathbf{E}[y_t - \mu]^2 \\ &= \mathbf{E}[\epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \cdots + \theta_q \epsilon_{t-q}]^2 \\ &= [1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2]\sigma^2 \end{aligned}$$

$$\gamma_j = \mathbf{E}[(y_t - \mu)(y_{t-j} - \mu)]$$
$$= \mathbf{E}[(\epsilon_t + \theta_1\epsilon_{t-1} + \cdots + \theta_q\epsilon_{t-q})(\epsilon_{t-j} + \theta_1\epsilon_{t-1-j} + \cdots + \theta_q\epsilon_{t-q-j})]$$

Example: $q = 5$ and $j = 3$



$$\therefore \gamma_3 = \mathbf{E}[(\theta_3\epsilon_{t-3}^2 + \theta_1\theta_4\epsilon_{t-4}^2 + \theta_2\theta_5\epsilon_{t-5}^2)] = (\theta_3 + \theta_1\theta_4 + \theta_2\theta_5)\sigma^2$$

MA(9) Model: Auto-covariance

- Generalizing:

$$\gamma_j = \begin{cases} (\theta_j + \theta_{j+1}\theta_1 + \theta_{j+2}\theta_2 + \cdots + \theta_q\theta_{q-j})\sigma^2 & \text{for } 1 \leq j \leq q \\ 0 \quad \text{for } j > q \end{cases}$$

Examples

MA(2)	MA(3)
$\gamma_0 = (1+\theta_1^2+\theta_2^2)\sigma^2$	$\gamma_0 = (1+\theta_1^2+\theta_2^2+\theta_3^2)\sigma^2$
$\gamma_1 = (\theta_1+\theta_2\theta_1)\sigma^2$	$\gamma_1 = (\theta_1+\theta_2\theta_1+\theta_3\theta_2)\sigma^2$
$\gamma_2 = \theta_2\sigma^2$	$\gamma_2 = (\theta_2+\theta_3\theta_1)\sigma^2$
$\gamma_j = 0,\ j \geq 3$	$\gamma_3 = \theta_3\sigma^2,\ \gamma_j = 0,\ j \geq 4$
$\rho_j = \frac{\gamma_j}{\gamma_0}$	$\rho_j = \frac{\gamma_j}{\gamma_0}$

MA(∞) Process: Mean and Variance

- $\{\psi_j\}, j \geq 0$ - MA(∞) parameters

$$
\begin{aligned}
y_t &= \mu + \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j} \\
&= \mu + \psi_0 \epsilon_t + \psi_1 \epsilon_{t-1} + \psi_2 \epsilon_{t-2} + \cdots
\end{aligned}
$$

- $\mathbf{E}(y_t) = \mu$
- $\mathbf{Var}(y_t) = \mathbf{E}\left[\sum_{j=0}^{\infty} \psi_j \epsilon_{t-j}^2\right] = \sigma^2 \sum_{j=0}^{\infty} \psi_j^2$
- If $\sum_{j=0}^{\infty} \psi_j^2 < \infty \implies \psi_t$ is square summable (SS), then y_t is weakly stationary
- If $\{\psi_j\}$ is SS, then MA(∞) process is ergodic

- Auto-covariance:

$$
\begin{aligned}
\gamma_j &= \mathbf{E}[(y_t - \mu)(y_{t-j} - \mu)] \\
&= \mathbf{E}[(\psi_0\epsilon_t + \psi_1\epsilon_{t-1} + \psi_2\epsilon_{t-2} + \cdots + \psi_j\epsilon_{t-j} + \psi_{j+1}\epsilon_{t-j-1} + \cdots) \\
&\quad (\psi_0\epsilon_{t-j} + \psi_1\epsilon_{t-j-1} + \psi_2\epsilon_{t-j-2} + \cdots)] \\
&= \sigma^2\left[\psi_j\psi_0 + \psi_{j+1}\psi_1 + \psi_{j+2}\psi_2 + \cdots\right]
\end{aligned}
$$

Summary: Structure of ACF MA(q)

- $$\rho_j = \begin{cases} \frac{\gamma_j}{\gamma_0} \neq 0 & \text{for } 1 \leq j \leq q \\ 0 & \text{for } j > q \end{cases}$$
- This key signature is used in identification of MA(q) process

Absolute vs square summability

- Let $\{\psi_j\}_{j\geq 0}$ be a sequence
- This sequence is said to be:
  - absolutely summable (AS) if $\sum_{j=0}^{\infty} |\psi_j| < \infty$
  - square summable (SS) if $\sum_{j=0}^{\infty} \psi_j^2 < \infty$
- Claim:
  - $\{\text{AS}\} \implies \{\text{SS}\}$
  - $\{\text{AS}\} \not\Longleftarrow \{\text{SS}\}$

- Let $\psi_j = 1/j$ for $j \geq 1$

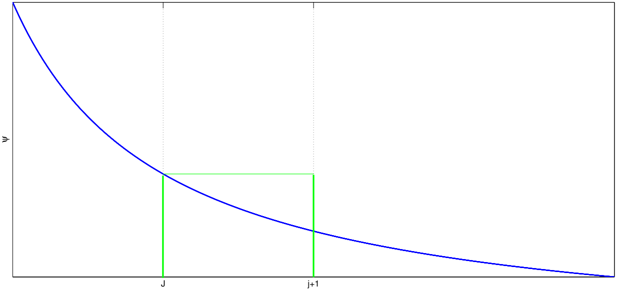


Figure: Plot of $1/j$ vs j

- From the plot of $1/j$ vs j, it is immediate that the area of the rectangle $1 \cdot 1/j$ is larger than the area under the curve $1/j$ from j to $(j+1)$

Example - AS

- That is $\frac{1}{j} = 1 \cdot \frac{1}{j} > \int_j^{j+1} \frac{\mathrm{d}\,x}{x} = \log(j+1) - \log(j)$
- Thus, $\sum_{j=1}^{N} \frac{1}{j} > \int_1^{N+1} \frac{\mathrm{d}\,x}{x} = \log(N+1)$
- That is,
  $\sum_{j=1}^{\infty} \frac{1}{j} > \lim_{N\to\infty} \sum_{j=1}^{N} \frac{1}{j} = \lim_{N\to\infty} \log(N+1) = \infty$
- Hence, $\sum_{j=1}^{\infty} \frac{1}{j}$ is divergent, that is not AS

Example - SS

- Let $\psi_j = 1/j^2$ for $j \geq 1$

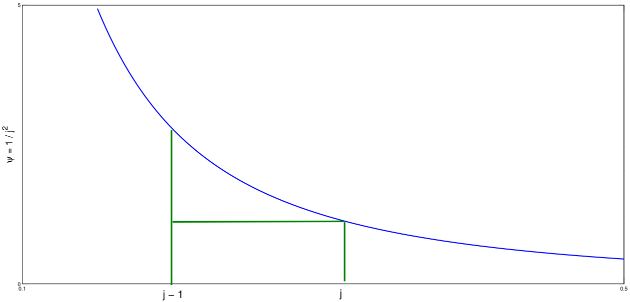


Figure: Plot of $1/j^2$ vs j

- From the plot of $1/j^2$ vs j, it is immediate that the area of the rectangle between $j-1$ and j is less than the area under the curve $1/j^2$ from $j-1$ to j

Example - SS

- That is $\frac{1}{j^2} = 1 \cdot \frac{1}{j^2} < \int_{j-1}^{j} \frac{\mathrm{d}\,x}{x^2} = -x^{-1}\big|_{j-1}^{j} = \frac{1}{j-1} - \frac{1}{j}$
- Thus,

$$
\begin{aligned}
\sum_{j=1}^{N} \frac{1}{j^2} &< 1 + \sum_{j=1}^{N} \left(\frac{1}{j-1} - \frac{1}{j} \right) \\
&= 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{N-1} - \frac{1}{N}\right) \\
&= 2 - \frac{1}{N} < 2 \ \text{ as } \ N \to \infty
\end{aligned}
$$

- Hence, $\sum_{j=1}^{\infty} \frac{1}{j^2} < \infty \implies$ SS