Estimation theory **2021**

Homework 1: April 2021

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Instructions: *You are encouraged to discuss and collaborate with your classmates. However, you* must explicitly *mention at the top of your submission who you collaborated with. Copying is NOT permitted, and solutions must be written independently and in your own words.*

Homeworks must be submitted on Google classroom. Please scan a copy of your handwritten assignment and upload as pdf with filename <your ID>_HW<homework no>.pdf. Example: EEB19BTECH00000_HW1.pdf.

For programming questions, submit as separate files. Please use the naming convention <your ID>_HW<homework no>_problem<problem no>.. Example: EEB19BTECH00000_HW1_problem1.c*

Exercise 1.1 (35 points)**.** Suppose that we are given n iid samples $\underline{X} = X_1, X_2, \ldots, X_n$ from an unknown distribution p_X over $\{1, 2, \ldots, m\}$. We know m, but have to estimate p_X.

- Derive the maximum likelihood estimator for p_X. Is it unbiased?
- Derive the moment matching estimator for p_X. Is it unbiased? Can you comment on the relative performance of this and the ML estimator?
- Consider a third estimator (let us call this the add-1 estimator) for p_X: For $a = 1, 2, \ldots, m$, and an observation $\underline{x}$ set
$$\hat{p}_X(a) = \frac{n_a(\underline{x}) + 1}{n + m},$$
where $n_a(\underline{x})$ is the number of occurrences of symbol a in $\underline{x}$. Is this unbiased? Is it asymptotically unbiased?

Exercise 1.2 (30 points)**.** You are given a file containing symbols from $0, 1, \ldots, 9$. Compute the ML and the add-1 estimates for the pmf. Also for each estimate $\hat{p}_X$ compute the mean squared error

$$MSE = \sum_{x \in \mathcal{X}} (p_X(x) - \hat{p}_X(x))^2$$

and the total variation distance

$$TV(p_X, \hat{p}_X) = 0.5 \times \sum_{x \in \mathcal{X}} |p_X(x) - \hat{p}_X(x)|$$

Use these to estimate the entropy: find $H(\hat{p}_X)$ for each of the estimators, and compare $|H(\hat{p}_X) - H(p_X)|$.

Exercise 1.3 (35 points)**.** A major component of any communication system is estimating the channel. Consider the following setup where if we transmit a sequence of symbols $x_0, x_1, \ldots, x_{n-1}$, the receiver obtains

$$y_i = \begin{cases} h_0 x_0 + h_1 x_{n-1} + z_0 & \text{if } i = 0 \\ h_0 x_i + h_1 x_{i-1} + z_i & \text{if } i = 2, 3, \ldots, n-1 \end{cases}$$

Here, the receiver knows $x_0, \ldots, x_{n-1}$ (these are *pilot symbols* that are decided beforehand), and wants to estimate (h_0, h_1) from $y_0, y_1, \ldots, y_{n-1}$. Also, $z_0, z_1, \ldots, z_{n-1}$ are iid Gaussian random variables with mean zero and variance σ^2. We assume that σ^2 is also known.

Derive the maximum likelihood estimator for h_0, h_1.

Hint: It might help to write the above as a matrix equation.

You are also given a template for a python program `template.py` to simulate channel estimation (if you want to use a different programming language, please convert it line-by-line to the language of your choice). Use this to simulate and plot the MSE for different σ^2. You may only make changes to the function `estimator(x,y)`. Leave the rest as it is.