EE5357

Estimation theory

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Assignment – 01

Question – 1.1

(i) $\underline{X} = x_1 x_2 x_3 \ldots x_n.$

ML Estimeter parameters be as follows.

$\theta_1, \theta_2, \theta_3, \theta_4 \ldots \theta_m$

$$\delta(x_i = j) = \begin{cases} 1 & \text{if } x_i = j \\ 0 & \text{else} \end{cases}$$

$$P(x_i/\theta) = \prod_{j=1}^{m} \theta_j^{I(x_j = j)}$$

$$\therefore L(\theta_1, \theta_2 \ldots \theta_m) = \underset{\Sigma\theta_i = 1\, \theta_i}{\arg\max}\; P(\underline{X}/\theta)$$

$$= \prod_{i=1}^{n} P(x_i/\theta)$$

$$= \prod_{i=1}^{m} \prod_{j=1}^{n} \theta_j^{I(x_i = j)}$$

$$= \prod_{j=1}^{m} \theta_j^{\sum_{i=1}^{n} I(x_i = j)} \quad \text{———} (1)$$

$\Rightarrow$ $n_j(\underline{X}) = \sum_{i=1}^{n} I(x_i = j)$ = Number of data points in $\underline{X}$ for which $x_i = j$

$= n_j(\underline{X})$ = Number of Occurances of j

$\therefore$ Equation (1) can be written as

$$L(\theta_1\theta_2\ldots\theta_n) = \prod_{j=1}^{m} \theta_j^{n_j(x)}.$$

taking $\frac{1}{n}\cdot\log$ on both sides

$$\frac{1}{n}\log(L(\theta_1\theta_2\theta_3\ldots\theta_n)) = \sum_{j=1}^{m} \frac{n_j(x)}{n}\log(\theta_j).$$

$$\boxed{\text{let } q_j = \frac{n_j(x)}{n}} \quad \text{--- (2)}$$

and this q_j is the Empirical pmf.

$$\Rightarrow \frac{1}{n}\log[L(\theta_1\theta_2\ldots\theta_n)] = \sum_{j=1}^{m} q_j\log\theta_j - q_j\log q_j + q_j\log q_j$$

$$= -\sum_{j=1}^{m} q_j\log\frac{q_j}{\theta_j} + \sum_{j=1}^{m} q_j\log q_j.$$

$$= -D(q\|\theta) - H(q).$$

for the Maximize log likelihood $D(q\|\theta)$ should be 0.

$$\therefore D(q\|\theta) = 0$$

$$\boxed{q = \theta} \quad \text{--- (3)}$$

$\therefore$ from (2) and (3)

$$\boxed{q_j = \theta_j = \frac{n_j(x)}{n}}$$

We can see that the ML Estimator θ Equal to Empirical pmf.

$$\Rightarrow E[\theta_j] = E\left[\frac{n_j(\underline{x})}{n}\right]$$

as $n_j(\underline{x}) = \sum_{i=1}^{n} I(x_i = j)$

$$\therefore E[I(x_i = j)] = 0 + 1 \times p[I_a(x_i = j) =]$$
$$= p[x_i = j] = P_x(j).$$

$$\Rightarrow E[\theta_j] = \frac{E(n_j(\underline{x}))}{n}$$
$$= \frac{\sum_{i=1}^{n} E[I(x_i = j)]}{n}$$
$$= \frac{\sum_{i=1}^{n} P_x(j)}{n} = P_x(j).$$

as $E(\theta_j) = P_x(j)$, then this is a Unbiased Estimator.

(ii) $m_i = \frac{1}{n}\sum_{j=1}^{n} x_j^i = E[x^i]$

$$m_i = \frac{1}{n}\sum_{j=1}^{n} x_j^i$$
$$= \frac{1}{n}\sum_{a=1}^{m} n_a(\underline{x})a^i$$

$$m_1 = \sum_{a=1}^{m} \frac{n_a(\underline{x})}{n} a, \quad m_2 = \sum_{a=1}^{m} \frac{n_a(\underline{x})}{n} a^2$$

$$\therefore m_m = \sum_{a_1}^{m} \frac{n_a(\underline{x})}{n} a^m$$

$$\therefore \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_m \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & \cdots & m \\ 1^2 & 2^2 & 3^2 & \cdots & m^2 \\ \vdots & & & & \\ 1^m & 2^m & 3^m & \cdots & m^m \end{bmatrix} \begin{bmatrix} \frac{n_1(\underline{x})}{n} \\ n_2(\frac{x}{n}) \\ \vdots \\ \frac{n_m(\underline{x})}{m} \end{bmatrix} \quad \text{— (1)}$$

$$E[x_i] = \sum_{a=1}^{m} a^i P_x(a)$$

$$\therefore \quad E(x_1) = \sum_{a=1}^{m} a P_x(a), \quad E[x_2] = \sum_{a=1}^{m} a^2 P_x(a)$$

$$E(x_m) = \sum_{a=1}^{m} a^m P_x(a).$$

$$\begin{bmatrix} E(x_1) \\ E(x_2) \\ \vdots \\ E(x_m) \end{bmatrix} = \begin{bmatrix} 1 & 2 & \cdots & m \\ 1^2 & 2^2 & \cdots & m^2 \\ \vdots & & & \\ 1^m & 2^m & \cdots & m^m \end{bmatrix} \begin{bmatrix} P_x(1) \\ P_x(2) \\ \vdots \\ P_x(m) \end{bmatrix} \quad \text{——— (2)}$$

As
$$\begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_m \end{bmatrix} = \begin{bmatrix} E(x_1) \\ E(x_2) \\ \vdots \\ E(x_m) \end{bmatrix}$$

$$\therefore \begin{bmatrix} \frac{n_1(x)}{n} \\ \frac{n_2(x)}{n} \\ \vdots \\ \frac{n_m(x)}{n} \end{bmatrix} = \begin{bmatrix} P_x(1) \\ P_x(2) \\ \vdots \\ P_x(m) \end{bmatrix}.$$

$$\therefore \quad P_x(a) = \frac{n_a(x)}{n}.$$

~~the~~ Here the Empirical pmf's is equal to Estimated pmf using moment method

~~moment~~ Moment matching and ML Estimator gives us the same estimator. Both are equal.

(iii)

$$P\hat{x}(a) = \frac{n_a(x)+1}{n+m}$$

$$E[\hat{P}_x(a)] = \frac{E[n_a(x)+1]}{n+m}$$

$$= \frac{E[n_a(x)]+1}{n+m}$$

$$= \frac{nP_x(a) + 1/n}{1+m/n}$$

$\therefore$ $E[\hat{P}_x(a)] \neq P_x(a)$.

it is Biased

for $\lim_{n\to\infty} E[\hat{P}_x(a)] = \frac{P_x(a)+0}{1+0} = P_x(a)$

so, Asymptotically add-1 Estimator is a unbiased Estimator.

Question 1.3

$$y_i = \begin{cases} h_0 x_0 + h_1 x_{n-1} + z_0 & \text{if } i=0 \\ h_0 x_i + h_1 x_{i-1} + z_i & \text{if } i=1,2,3\ldots n-1 \end{cases}$$

where $z_i \sim N(0,\sigma^2)$

$$\therefore L(h_0 h_1) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y_i - h_0 x_i - h_1 x_{i-1})^2}{2\sigma^2}}$$

$$= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^N e^{-\sum_{i=1}^{n}\frac{(y_i - h_0 x_i - h_1 x_{i-1})^2}{2\sigma^2}}$$

$$\log L(h_0,h_1) = -\frac{N}{2}\log(2\pi\sigma^2) - \sum_{i=1}^{n}\frac{(y_i - h_0 x_i - h x_{i-1})^2}{2\sigma^2}$$

as $\frac{d}{dh_0} \log(L(h_0, h_1)) = 0$.

$$\therefore \boxed{\sum x_i y_i - h_0 \sum x_i^2 - h_1 \sum x_i x_{i-1} = 0} \quad \text{— (1)}$$

$$\frac{d}{dh_1} \log\left[L(h_0, h_1)\right] = 0$$

$$\therefore \boxed{\sum x_{i-1} y_i - h_0 \sum x_i x_{i-1} - h_1 \sum x_{i-1}^2 = 0} \quad \text{— (2)}$$

from (1) and (2)

$$\sum x_i^2 = \sum x_{i-1}^2$$

$$\Rightarrow \begin{bmatrix} \sum x_i y_i \\ \sum x_{i-1} y_i \end{bmatrix} = \begin{bmatrix} \sum x_i^2 & \sum x_i x_{i-1} \\ \sum x_i x_{i-1} & \sum x_i^2 \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \end{bmatrix}.$$

$$\therefore \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} = \begin{bmatrix} \sum x_i^2 & \sum x_i x_{i-1} \\ \sum x_i x_{i-1} & \sum x_i^2 \end{bmatrix}^{-1} \begin{bmatrix} \sum x_i y_i \\ \sum x_{i-1} y_i \end{bmatrix}$$