# 39th Indian National Mathematical Olympiad – 2025

January 19, 2025

**Time:** 4.5 hours
**Instructions:**

- Calculators (in any form) and protractors are not allowed. Rulers and compasses are allowed.
- All questions carry equal marks. Maximum marks: 102.
- No marks will be awarded for stating an answer without justification.
- Answer to each question should start on a new page. Clearly indicate the question number.

## Questions

1. Consider the sequence defined by $a_1 = 2$, $a_2 = 3$, and
$$a_{2k+2} = 2 + a_k + a_{k+1}, \quad a_{2k+1} = 2 + 2a_k$$
for all integers $k \geq 1$. Determine all positive integers $n$ such that $a_n$ is an integer.

2. Let $n \geq 2$ be a positive integer. The integers $1, 2, \ldots, n$ are written on a board. In a move, Alice can pick two integers written on the board $a \neq b$ such that $a + b$ is an even number, erase both $a$ and $b$ from the board, and write the number $\frac{a+b}{2}$ on the board instead. Find all $n$ for which Alice can make a sequence of moves so that she ends up with only one number remaining on the board.
*Note:* When $n = 3$, Alice changes $(1, 2, 3)$ to $(2)$ and cannot make any further moves.

*Proof.* (Rohan Goyal) For showing that everything with the possible exception of $2, 3, 4$, and $6$ are winning for Alice:

For any $n \geq 2$, we have
$$[n-1] \cup \{n+1\} \mapsto [n-2] \cup \{n\},$$
thus, by repeating the above, for any $n \geq 3$, we have
$$[n-2] \cup \{n\} \longmapsto \{2\}.$$

Thus, for any $n \geq 5$, we have

$$[n] = ([n-3] \cup \{n-1\}) \cup \{n-2, n\} \longmapsto \{2, n-2, n\}$$

Now, if $n \neq 6$, then these 3 numbers are not in an AP. Now, if

(a) **n** is odd: $\{2, n-2, n\} \mapsto \{2, n-1\}$ and both the numbers are even and Alice ends up with exactly one number for all odd $n \geq 5$.

(b) **n** is even: Atleast one of $n$ and $n+2$ is divisible by 4. Thus, Alice can make a move such that both the numbers on the board are even and can end up with one final number as long as $n - 2 \neq \frac{n+2}{2}$. This only happens when $n = 6$.

Thus, Alice can have exactly one number for even $n \geq 8$.

**Remark**. I do think this problem is significantly more challenging and work than this solution suggests as it is very natural to go down much more convoluted paths as I did... □

3. Euclid has a tool called splitter which can only do the following two types of operations:

   (a) Given three non-collinear marked points $X, Y, Z$, it can draw the line which forms the interior angle bisector of $\angle XYZ$.

   (b) It can mark the intersection point of two previously drawn non-parallel lines.

Suppose Euclid is only given three non-collinear marked points $A, B, C$ in the plane. Prove that Euclid can use the splitter several times to draw the circle's centre passing through $A, B, C$.

*Proof.* (User HoRIDAgre8)

(a) First draw the incentre $I$ of $\triangle BIC$

(b) Mark the lines $BI$ and $CI$.Let $\ell_{XYZ}$ denote the angle bisector of $\angle XYZ$.

(c) First draw $\ell_{BAI}$ and mark $CI \cap \ell_{BAI} = E$ .Clearly $E$ lies on the opposite side of $I$ w.r.t $C$

(d) Now draw $\ell_{EIB}$ and mark $\ell_{EIB} \cap \ell_{ICB} = K_C$.Note that $K_C$ is the $C$-excentre of $\triangle BIC$.Similarly, construct $K_B$, which is the $B$ excentres of $\triangle BIC$.

(e) Now note that $K_BCBK_C$ is cyclic.So by fact 5 we have that $\ell_{K_BBK_C} \cap \ell_{K_BCK_C} = M$ lie on the circle $(K_BCBK_C)$ and $M$ is also the midpoint of arc $K_BK_C$ (on which $B$,$C$ doesn't lie).

(f) Now since $\triangle MK_BK_C$ is isosceles, we have that $\ell_{K_BMK_C}$ is perpendicular bisector of $K_BK_C$ , since $K_BK_C$ is nothing other than $\ell_{EIB}$ , we can mark $\ell_{K_BMK_C} \cap K_BK_C = L$.Now note that $L$ is the centre of $(K_BCBK_C)$ Thus we have $\ell_{BLC}$ is the perpendicular bisector of $BC$

Thuss we have drawn the perpendicular bisector of $BC$.Repeat this process with $CA$ or $AB$ and mark the intersection of the perpendicular bisectors to get the circumcentre □

4. Let $n \geq 3$ be a positive integer. Find the largest real number $t_n$ as a function of $n$ such that the inequality

$$\max(|a_1 + a_2|, |a_2 + a_3|, \ldots, |a_n - 1 + a_n|, |a_n + a_1|) \geq t_n \cdot \max(|a_1|, |a_2|, \ldots, |a_n|)$$

holds for all real numbers $a_1, a_2, \ldots, a_n$.

*Proof.* (Rohan and Rijul) For the case when $n$ is even, the sequence $a_i = (-1)^i$ implies that $t_n = 0$. Now, more interesting when $n$ is odd:

$$\sum_{i=1}^{n} |a_i + a_{i+1}| \geq |\sum_{i=1}^{n} (-1)^i (a_i + a_{i+1})| = 2|a_1|$$

The same could have been done with $a_2$ or $a_3$ or so on. Thus,

$$n \max(|a_1 + a_2|, \ldots, |a_n + a_1|) \geq \sum_{i=1}^{n} |a_i + a_{i+1}| \geq 2 \max(|a_1|, \ldots, |a_n|) \implies t_n \geq \frac{2}{n}.$$

Now, the equality case is directly motivated from the triangle inequality. Let $a_n$ be the largest. For simplicity, we just let it be $n$ so that RHS is just $2n$. Now, each of the $a_i + a_{i+1}$ must add up to 2 or $-2$ for the bound to be tight. This gives us the construction, $a_i = (-1)^i(n - 2i)$ which indeed attains the desired bound. □

5. Greedy goblin Griphook has a regular 2000-gon, whose every vertex has a single coin. In a move, he chooses a vertex, removes one coin each from the two adjacent vertices, and adds one coin to the selected vertex, keeping the remaining coin for himself. He can only make such a move if both adjacent vertices have at least one coin. Griphook stops only when he cannot make any more moves. What is the **maximum** and **minimum** number of coins that he could have collected?

*Proof.* 1. Griphook can pick 1998 coins:

**Lemma 1.** $1^n 01s \mapsto 010^n s$ *for any* $n \geq 1$ *and any string* $s$.

*Proof.* Observe that $1^n 01 \mapsto 1^{n-1}010$. Repeating this move $n$ times gives the desired result. □

Now,

$$1^{2000} \mapsto 1^{1996}0201 \mapsto 1^{1995}01010 \overset{\text{Lemma.}}{\mapsto} 010^{1996}10$$

2. Griphook can collect exactly 668 coins:

$$1^{2000} \mapsto 1(020)^{666}1 = 1020(020)^{664}0201 \mapsto 0110(020)^{664}0110$$

3. Griphook cannot collect more than 1998 coins: Observe that if $o$ is the number of coins on odd positions and $e$ on even positions then $o - e$ is invariant $\pmod 3$. Thus,

$o \equiv e \pmod 3$ always! Now, we must end with atleast one coin so atleast 2 must be remaining since $o \equiv e \pmod 3$.

The below proof is not mine and much better than mine which was extremely convoluted. It was shared by an anonymous (to me) reviewer.

4. Griphook could not have collected less than 668 coins when the game stops: We show that Griphook makes atleast 668 moves even in the weaker game where no coin is placed back and both are pocketed. This cannot increase the length of the game and thus we just bound the minimum number of moves Griphook has to make. Observe that this game is now simply on two distinct circles of length 1000 where you can pick up two adjacent coins and pocket them. Clearly each circle requires atleast 334 moves to ensure that there are no adjacent coins! Thus, atleast 668 moves are required. □

6. Let $b \geq 2$ be a positive integer. Anu has an infinite collection of notes with exactly $b-1$ copies of a note worth $b^k - 1$ rupees, for every integer $k \geq 1$. A positive integer $n$ is called payable if Anu can pay exactly $n^2 + 1$ rupees by using some collection of her notes. Prove that if there is a payable number, there are infinitely many payable numbers.