

Week 7 Stats 25-26

Srikanth Pai, Asst. Prof, MSE.

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1 Homework for the week

1. Do the following exercises

- (a) Let $\underline{X} \sim \mathcal{N}(\underline{\mu}, \underline{\Sigma})$. Write down the pdf.
- (b) Prove that the normal random vector is independent if the covariance matrix is diagonal. Determine the condition for coordinates to be i.i.d.
- (c) For the two-variable case, assume the means are μ_x, μ_y correlation is ρ and variances are σ_x^2, σ_y^2 . Write down the pdf explicitly as an algebraic expression.
- (d) Compute the conditional pdf $p_{X|Y}(x | y)$ from the previous exercise.
- (e) Compute the conditional expectation of X given $Y = y$ using the previous exercise. (Feel free to use: Gaussian single variable pdf marginalises to 1.)

2. Let X_1, X_2, X_3, \dots be a sequence of i.i.d. Uniform(0, 1) random variables. Define

$$Y_n = \min(X_1, X_2, \dots, X_n).$$

Prove the following convergence results *separately* (i.e., do not conclude the weaker convergence modes from the stronger ones).

- (a) $Y_n \xrightarrow{d} 0$.
- (b) $Y_n \xrightarrow{p} 0$.
- (c) $Y_n \xrightarrow{L^p} 0$, for all $p \geq 1$.
- (d) $Y_n \xrightarrow{a.s.} 0$. [You may use Borel-Cantelli I stated in Problem 1 of Section 2.]

3. The Gamma distribution with shape parameter $\alpha > 0$ and rate parameter $\lambda > 0$ has density

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \quad x > 0.$$

Recall that the Gamma function is defined for $\alpha > 0$ by the integral

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx.$$

More generally, for $\alpha > 0$ and $\beta > 0$,

$$\int_0^\infty x^{\alpha-1} e^{-\beta x} dx = \frac{\Gamma(\alpha)}{\beta^\alpha}.$$

(a) Prove that if $X \sim \text{Gamma}(\alpha, \lambda)$ then its moment generating function is

$$M_X(t) = \mathbb{E}[e^{tX}] = \left(\frac{\lambda}{\lambda - t} \right)^\alpha, \quad t < \lambda.$$

(b) Let X_1, \dots, X_n be independent and identically distributed $\text{Exp}(\lambda)$ random variables. Show that

$$S_n = X_1 + \dots + X_n \sim \text{Gamma}(n, \lambda).$$

In this case, it's called $\text{Erlang}(n, \lambda)$.

(c) Let $Z \sim N(0, 1)$. Show that $Z^2 \sim \text{Gamma}(\frac{1}{2}, \frac{1}{2})$. Deduce that if Z_1, \dots, Z_n are independent $N(0, 1)$ then

$$\sum_{i=1}^n Z_i^2 \sim \text{Gamma}\left(\frac{n}{2}, \frac{1}{2}\right).$$

This distribution is called Chi-squared with ν degrees of freedom.

(d) Let $X \in \mathbb{R}^n$ be a standard normal vector $X \sim N(0, I_n)$. Let A be a symmetric idempotent matrix with $\text{rank}(A) = k$. Show that

$$Q = X^\top A X \sim \chi_k^2.$$

[Hint: Show that if vector X is iid standard normal and R is an orthogonal matrix, then so is RX . The proof follows from computing the covariance matrix of RX .]

2 Bonus problems on understanding convergence

The bonus problems in the upcoming internals (and weekly tests) may require a thorough understanding of the ideas in this section. If you are planning to apply to top schools such as MIT, Oxford, Columbia, or LSE, they assume that your econometrics background is on the level of Greene's book.¹

The statistics used in rigorous econometrics builds directly on the results from this section. If you learn this material properly, then during the next semester break you will be able to read Greene on your own. It is one of the most rigorous and respected texts in econometrics.

The problems here are not harder in terms of calculations, but they require careful reasoning. If you are still struggling with the basics, it is better to consolidate those first. For the rest, this is a good opportunity to score bonus marks and strengthen your preparation.

As an incentive, I will include one bonus question directly from this section in a test or exam.

¹https://www.ctanujit.org/uploads/2/5/3/9/25393293/_econometric_analysis_by_greene.pdf

1. Let $(A_n)_{n \geq 1}$ be a sequence of events in a probability space. Prove the following (called *Borel-Cantelli I*): If

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty,$$

then

$$\mathbb{P}\left(\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m\right) = 0.$$

[**Hint:** The event written inside the probability function is actually simple to say in English “ A_n infinitely often”, i.e.

$$\{A_n \text{ infinitely often}\} = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m,$$

and use continuity of probability from above (Kolmogorov Axiom 3') along with the tail sums $\sum_{m=n}^{\infty} \mathbb{P}(A_m)$.]

2. For each of the following sequences of random variables, determine whether $X_n \rightarrow X$ almost surely. Use the *definition* of a.s. convergence directly.

(a) Let

$$X_n = \begin{cases} -\frac{1}{n}, & \text{with probability } \frac{1}{2}, \\ +\frac{1}{n}, & \text{with probability } \frac{1}{2}. \end{cases}$$

Show that $X_n \rightarrow 0$ almost surely.

(b) Consider a random variable s that is uniformly distributed on $[0, 1]$. Define

$$X_n(s) = \begin{cases} 1, & 0 \leq s < \frac{n+1}{2n}, \\ 0, & \text{otherwise,} \end{cases} \quad X(s) = \begin{cases} 1, & 0 \leq s < \frac{1}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

Show that $X_n \rightarrow X$ almost surely.

3. This exercise shows that ‘almost sure convergence’ implies ‘convergence in probability’:

(a) State the definition of convergence in probability.

(b) Prove that if $X_n \rightarrow X$ almost surely, then $X_n \rightarrow X$ in probability. (Hint: Use Borel–Cantelli I on the sets $A_n = \{|X_n - X| > \varepsilon\}$ for fixed $\varepsilon > 0$.)

4. Consider a random variable ω that is uniformly distributed on $[0, 1]$. Define

$$X_n(\omega) = \begin{cases} 1, & \text{if } \omega \in \left[\frac{k}{n}, \frac{k+1}{n}\right) \text{ with } k \text{ odd,} \\ 0, & \text{otherwise.} \end{cases}$$

(a) Show that for almost every ω , the sequence $(X_n(\omega))$ oscillates between 0 and 1 infinitely often, and hence does not converge almost surely.

- (b) Show instead that $X_n \rightarrow \frac{1}{2}$ in probability.
5. In this problem, we show that convergence in probability implies convergence in distribution.

Definition. We say that a sequence of random variables X_n *converges in distribution* to a random variable X , written

$$X_n \xrightarrow{d} X,$$

if for every real number x at which the cumulative distribution function $F_X(t) = \mathbb{P}(X \leq t)$ is continuous, we have

$$\lim_{n \rightarrow \infty} \mathbb{P}(X_n \leq x) = F_X(x).$$

Show that if $X_n \xrightarrow{p} X$ (convergence in probability), then $X_n \xrightarrow{d} X$. [**Hint.** Fix x at which F_X is continuous, and fix $\varepsilon > 0$. Use the inclusions

$$\{X \leq x - \varepsilon\} \subseteq \{X_n \leq x\} \cup \{|X_n - X| > \varepsilon\},$$

and

$$\{X_n \leq x\} \subseteq \{X \leq x + \varepsilon\} \cup \{|X_n - X| > \varepsilon\}.$$

Take probabilities, then use the fact that $\mathbb{P}(|X_n - X| > \varepsilon) \rightarrow 0$ as $n \rightarrow \infty$, and finally let $\varepsilon \downarrow 0$.]

3 Optional question on T distribution

The following material won't appear for your exam, for your knowledge only. We will use the density in stats but no derivation needs to be memorised.

1. Let $X \sim \text{Gamma}(\frac{\nu}{2}, \frac{1}{2})$ and let $Y \sim N(0, 1)$ be independent of X . Define

$$T = \frac{Y}{\sqrt{X/\nu}}.$$

ChatGPT-5 says it can compute the distribution of T . The pdf given below is right. Is the proof given below correct?

Problem: Show that the density of T is

$$f_T(t) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi} \Gamma(\frac{\nu}{2})} \left(1 + \frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}},$$

i.e. T has the Student- t distribution with ν degrees of freedom.

Proof. The joint density of (Y, V) is

$$f_{Y,V}(y, w) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \cdot \frac{1}{2^{\nu/2} \Gamma(\nu/2)} w^{\nu/2-1} e^{-w/2}, \quad w > 0.$$

Use the transformation $y = t\sqrt{w/\nu}$ (together with w unchanged). The Jacobian is $\sqrt{w/\nu}$. Hence

$$f_T(t) = \int_0^\infty f_{Y,V}(t\sqrt{w/\nu}, w) \sqrt{\frac{w}{\nu}} dw.$$

Substitute and simplify:

$$f_T(t) = \frac{1}{\sqrt{2\pi} 2^{\nu/2} \Gamma(\nu/2) \sqrt{\nu}} \int_0^\infty w^{(\nu-1)/2} \exp\left(-\frac{w}{2}\left(1 + \frac{t^2}{\nu}\right)\right) dw.$$

The integral is a Gamma integral with parameter $a = (\nu+1)/2$ and rate $b = \frac{1}{2}\left(1 + \frac{t^2}{\nu}\right)$, so it equals

$$\Gamma\left(\frac{\nu+1}{2}\right) \left(\frac{1}{2}\left(1 + \frac{t^2}{\nu}\right)\right)^{-(\nu+1)/2}.$$

Putting constants together and simplifying yields

$$f_T(t) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi} \Gamma\left(\frac{\nu}{2}\right)} \left(1 + \frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}},$$

as required.