CS5800 Algorithms

Solutions for Recitations from Sep 9, 2019 to Sep 13, 2019

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Problem 1 (Combinatorics)

You are picking 3 different numbers from 1 to 15.

1. How many ways can you do it such that the product is divisible by either 2 or 3?
2. How many ways can you do it such that the product is divisible by 4?

Solution:

1. Can count directly, but need to be careful about double counting. Can also take the complement and count products not divisble by either 2 or 3. There are five numbers to choose from (1,5,7,11,13), so we get $\binom{15}{3} - \binom{5}{3}$
2. Can count directly, but need to break down numbers into groups divisible by 4, then divisible by 2 but not 4 etc. Easier to count complement, which would be odd products or products with only one even factor (that's not divisible by 4). There are 8 odd factors, and 4 factors divisible by 2 but not 4. So either all 3 factors are odd, or 2 are odd and 1 is divisible by 2 but not by 4. So we get $\binom{15}{3} - (\binom{8}{3} + \binom{8}{2}\binom{4}{1})$

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Problem 2 (More Combinatorics)

How many ways can you rearrange the letters in the word: ALGORITHM?

1. What about the word: ALGARATHM?
2. What if no two vowels can appear together (in ALGORITHM)?
3. What if you cannot have more than two consonants together at a time (in ALGORITHM)?

Solution:

0. 9!
1. 9!/3!
2. We set the 6 consonants. There are 7 spots for the vowels to go in so that they are not next to each other. After choosing the 3 spots, we can permute the vowels. So we get $\binom{7}{3}6!3!$

3. We set the three vowels and then we have a stars and bars situation. We can think of having an equation $x_1 + x_2 + x_3 + x_4 = 6$, where each x_i represents the number of consonants at the i spot. We have the condition that $x_i < 3$. We have 6 because there are 6 total consonants. Counting this carefully gives you correct solution (if this isn't clear don't worry about this, this problem is harder than we meant).

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Problem 3 (Probability)

Suppose that Alice, Bob, and n other people line up in some random order. What is the expected number of people between Alice and Bob? (Hint: Use Linearity of Expectation)

Solution:

Let X_i denote the indicator variable of person i standing between Alice and Bob. The probability of X_i being 1 (aka person i stands between Alice and Bob) is 1/3. Let X be the random variable of the number of people standing between Alice and Bob. By Linearity of Expectation, $E[X] = E[\sum_i X_i] = \sum_i E[x_i] = n/3$

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Problem 4 (More Probability)

There are 3 white balls and 7 black balls in a bin.

1. If you pick two balls at random from the bin, what is the probability that both are white?
2. If you pick one ball, put it back and pick another ball, what is the probability that both the balls that you picked are white?
3. If you pick one ball and then pick another ball without looking at the first, what is the probability that the second ball is white?

Solution:

1. $\binom{3}{2}/\binom{10}{2}$
2. $(3/10)^2$
3. Since we don't know anything about the first ball, it's effectively like we are choosing a ball from random all 10 balls, so we get 3/10.

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Problem 5 (Logic)

The goal of this problem is to translate some assertions about binary strings into logic notation. The domain of discourse is the set of all finite-length binary strings $\lambda, 0, 1, 00, 01, 10, 11, 000, 001, \ldots$. (Here λ denotes the empty string). In your translations, you may use all the ordinary logic symbols (including $=$), variables, and the binary symbols, $0, 1$.
A string like $01x0y$ of binary symbols and variables denotes the concatenation of the symbols and the binary strings represented by the variables. For example, if the value of x is 01 and the value of y is 110 then the value of $01x0y$ is the binary string 01010110.
Here are some example of formulas and their English translations. Names for these predicates are listed in the third column so that you can reuse them in your solutions (as we do in the definition of the predicate No-1s below)

Meaning	Formula	Name
x is a prefix of y	$\exists z(xz = y)$	PREFIX(x, y)
x is a substring of y	$\exists u \exists v(uxv = y)$	SUBSTRING(x, y)
x is empty or a string of 0's	NOT(SUBSTRING(1,x))	No-1s(x)

1. x consists of three copies of some string
2. x is an even-length string of 0's
3. x does not contain both a 0 and a 1

Solution:

1. $\exists z(x = zzz)$
2. $\exists z(x = zz \wedge \text{No-1s}(z))$
3. No-1s$(x) \vee$ NOT(SUBSTRING)$(0, x)$

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Problem 6 (Pigeonhole)

Assume that friendship is a symmetric property (if Alice is friends with Bob, then Bob is friends with Alice). Show that in a group of n people, there will always be at least two people with the same number of friends.

Solution:

So a person can have at most $n-1$ friends. First case: suppose everyone in the group has at least one friend. Then there are n people, and the range of friends is between 1 and $n-1$, so there must be at least 2 people with same number of friends. Second case: suppose there is one person with no friends.

Then the other $n-1$ people all have at least one friend, and they can all have at most $n-2$ friends. So $n-1$, and range of $n-2$ values, so at least 2 people have the same number of friends.

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Problem 7 (Sequences)

Determine whether or not the expression converges to a value. If it converges, determine its value.

1. $\sum_{i=1}^{\infty} \frac{1}{2^i}$
2. $\sum_{i=1}^{\infty} \frac{1}{i}$
3. $\sum_{k=0}^{n} \binom{n}{k}$

Solution:

1. Let S denote the sum. Then $2S - S = 1$, so we get $S = 1$
2. Doesn't converge. To see this, note that

$$1/3 + 1/4 > 1/2$$
$$1/5 + 1/6 + 1/7 + 1/8 > 1/2$$
$$\text{and so on...}$$

   Always is a sequence that adds to more than 1/2, so this never converges.

3. This counts all possible subsets of a set of size n, so this is 2^n

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Problem 8 (Pointers)

1. You reversed a list in the diagnostic, now you need to reverse all c sized chunks in the list. For example, if you have the numbers 1, 2, 3, 4, 5, 6 in the list and $c = 2$, then the result would be 2, 1, 4, 3, 6, 5. Each chunk of size 2 was reversed but the ordering among the chunks is not changed.

Solution:

```
reverse(head)
{
  current = head
  preve next = null
  count = 0
  while (count<c && current! = null) {
    next = current.next
    current.next = prev
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    prev = current
    current = next
    count ++
  }
  if (next! = null)
    head.next = reverse(next)
  return prev
}
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Problem 9 (Extras - Optional)

1. Prove that $\binom{n}{k}\binom{k}{j} = \binom{n}{j}\binom{n-j}{k-j}$ by a counting a set in two different ways.
2. Suppose we randomly pick 5 cards from a standard 52 card deck.
   (a) What is the probability our hand makes a full house (three cards of the same rank, and two cards of a different rank. For example, if you picked 3 Queens and 2 Sevens, you would have a full house).
   (b) What is the probability our hand makes two pair (two cards of the same rank, two cards of a different rank, and one card of a different rank. For example, if you picked 2 Queens, 2 Sevens, and one Four, you would have two pair). Make sure you're not counting full houses and four-of-a-kinds!
3. Your class has a textbook and a final exam. Let P, Q, R be the following propositions:

$$\begin{aligned} P &:= \text{You get an A in the class} \\ Q &:= \text{You do every exercise in the book} \\ R &:= \text{You get an A in the final} \end{aligned}$$

   Translate the following statements into propositional formulas using P, Q, R and the propositional connectives $AND, NOT, IMPLIES$.

   (a) You get an A in the class, but you do not do every exercise in the book.
   (b) You get an A on the final, you do every exercise in the book, and you get an A in the class.
   (c) To get an A in the class, it is necessary for you to get an A on the final.
   (d) You get an A on the final, but you don't do every exercise in this book; nevertheless you get an A in this class.
4. In the sequence $1, 1, 2, 3, 5, 8, 3, 1, 4, \ldots$ each term starting with the third is the sum of the two preceding terms. But addition is done mod 10. Prove that the sequence is purely periodic. What is the maximum possible length of the period?
5. Let $a_1, a_2, \ldots, a_n$ be n not necessarily distinct integers. Then there always exists a subset of these numbers whose sum is divisible by n.

6. Consider a sequence of $n^2 + 1$ distinct numbers. A subsequence is monotone if the numbers are increasing or decreasing. Show that every sequence of length $n^2 + 1$ has a monotone subsequence of length at least $n + 1$ (Paul Erdos proved this in 1935).