Solution of linear systems

Given $\underline{\underline{A}} = \underline{\underline{L}}\,\underline{\underline{U}}$, we may now solve

$$\underline{\underline{A}}\,\underline{x} = \underline{b} \iff \underline{\underline{L}}\,\underline{\underline{U}}\,\underline{x} = \underline{b}$$

Define $\underline{y} = \underline{\underline{U}}\,\underline{x}$, so that

$$\underline{\underline{L}}\,\underline{y} = \underline{b}, \qquad \underline{\underline{U}}\,\underline{x} = \underline{y}$$

This leads to a simple approach: first solve for $\underline{y}$ via forward substition, then given $\underline{y}$, solve for $\underline{x}$ via back substitution.

Computational work

$\underline{Q}$: How much effort (floating point operations) to compute $\underline{\underline{A}} = \underline{\underline{L}}\,\underline{\underline{U}}$? How does it scale w/ n for $\underline{\underline{A}} \in \mathbb{R}^{n\times n}$?

Recall:

$$\begin{cases} l_{ij} = 1/u_{jj}\left[a_{ij} - \sum_{k=1}^{j-1} l_{ik}u_{kj}\right], & i = 2,\dots,n \quad j = 1,\dots,i-1 \\ u_{ij} = a_{ij} - \sum_{k=1}^{i-1} l_{ik}u_{kj}, & i = 1,\dots,n, \quad j = i,\dots,n \end{cases}$$

l_{ij}: $j-1$ $\times$ 1 $-$; $j-2$ $+$ 1 $\%$

u_{ij}: $i-1$ $\times$ 1 $-$; $i-2$ $+$ 0 $\%$

Recall:

$$\sum_{l=1}^{k} l = \frac{k\cdot(k+1)}{2}, \qquad \sum_{l=1}^{k} l^2 = \frac{k(k+1)(2k+1)}{6}$$

So in total, work is:

$$\text{work} = \underbrace{\sum_{i=2}^{n} \sum_{j=1}^{i-1} \underbrace{(2j-1)}_{l_{ij}}}_{\underline{\underline{L}}} + \underbrace{\sum_{i=1}^{n} \sum_{j=i}^{n} \underbrace{2\cdot(i-1)}_{u_{ij}}}_{\underline{\underline{U}}}$$

$$= \sum_{i=1}^{n} \left[\sum_{j=1}^{i-1} (2j-1) + \sum_{j=i}^{n} 2\cdot(i-1) \right]$$

$$= \sum_{i=1}^{n} \left[2\cdot \tfrac{1}{2}\cdot i\cdot(i-1) - (i-1) + 2\cdot(i-1)\cdot(n-i+1) \right]$$

$$= \sum_{i=1}^{n} \left[(i-1)^2 + 2(i-1)\cdot(n-(i-1)) \right]$$

$$= \sum_{i=1}^{n} \left[(i-1)^2 - 2(i-1)^2 + 2\cdot(i-1)\cdot n \right]$$

$$= \sum_{l=0}^{n-1} (2ln - l^2)$$

$$= \cancel{2}\cdot \cancel{\tfrac{1}{2}}\cdot n\cdot n\cdot(n-1) - \tfrac{1}{6}(n-1)\cdot n\cdot(2(n-1)+1)$$

$$= n\cdot(n-1)\left(n - \tfrac{1}{6}[2n-1]\right)$$

$$= \tfrac{1}{6}\, n(n-1)(6n-2n+1) = \tfrac{1}{6}\, n(n-1)(4n+1)$$

$$\sim \tfrac{2}{3}n^3 - \tfrac{1}{2}n^2 \text{ for } n \text{ large.}$$

Q: How much work to solve for $\underline{x}$?

$$\underline{\underline{L}}\,\underline{y} = \underline{b} \Rightarrow y_1 = b_1, \quad y_i = b_i - \sum_{j=1}^{i-1} l_{ij} y_j, \quad i \geq 2.$$

$$\underbrace{\qquad\qquad}_{\substack{i-1 \ \times \\ i-1 \ + \\ 1 \ -}} \Rightarrow 2i-2 \text{ ops.}$$

$$\underline{\underline{U}}\,\underline{x} = \underline{y} \Rightarrow x_n = y_n, \quad x_i = \frac{1}{u_{ii}}\Big(\underbrace{y_i - \sum_{j=i+1}^{n} u_{ij} x_j}_{\substack{n-i \ \times \quad 1 \ - \\ n-i-1 \ + \quad 1 \ \%}}\Big), \quad i \le n-1$$

$\Rightarrow 2(n-i)+1$ ops

So, the total work is:

$$\sum_{i=2}^{n}(2i-2) + \sum_{i=n-1}^{1}\big(2(n-i)+1\big) = \sum_{i=1}^{n}(2i-2) + \sum_{i=n}^{1} 2(n-i) + (n-1)$$

$$= (n-1) + \sum_{i=1}^{n} 2(i-1) + \sum_{i=1}^{n} 2(i-1)$$

$$= (n-1) + 4\sum_{i=1}^{n}(i-1) = (n-1) + 4\frac{n(n+1)}{2} - 4n$$

$$= 2n^2 + 2n + (n-1) - 4n = 2n^2 - n - 1$$

Overall Cost: $\approx 2/3\, n^3 + 3/2\, n^2$

Note: Multiple systems w/ same $\underline{\underline{A}}$?

Factorize $\underline{\underline{A}} = \underline{\underline{L}}\,\underline{\underline{U}}$ once

Pivoting

Ex: $\underline{\underline{A}} = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & 5 & -4 \end{pmatrix}$ will fail, because $A_{11} = 0$!

permute row 1 w/ row 2: $\hat{\underline{\underline{A}}} = \begin{pmatrix} 2 & 4 & 2 \\ 0 & 1 & 1 \\ -1 & 5 & -4 \end{pmatrix}$

$\tilde{A}^{(1)} = 2$ ✓, $\tilde{A}^{(2)} = \begin{pmatrix} 2 & 4 \\ 0 & 1 \end{pmatrix}$ ✓ fine! Δ

The above interchange of rows is called a Pivot.

In addition, even if $\det \underline{A}^{(k)} \neq 0$ for all k, rounding errors stemming from multiplication by large or division by small #'s can be problematic. Pivoting can help w/ this.

Def: Let $n \geq 2$, $\underline{P} \in \mathbb{R}^{n \times n}$. We call $\underline{P}$ a permutation matrix if $P_{ij} \in \{0, 1\}$ for all i, j and each row and column contain exactly one 1.

Ex: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ Δ

Permutation matrices have some useful properties:

Lemma: Let $n \geq 2$ and let $\underline{P}$ be a permutation matrix. Then,

(i) $\underline{P}$ can be obtained from $\underline{I}$ by permuting rows.

(ii) If $\underline{Q} \in \mathbb{R}^{n \times n}$ is another permutation, so are $\underline{\underline{QP}}$ and $\underline{\underline{PQ}}$.

(iii) Let $\underline{P}^{(rs)}$ be the interchange matrix obtained from $\underline{\underline{I}}$ by swapping rows r and s. Then $\underline{P}^{(rs)}$ is a permutation and every permutation can be obtained a product of interchange matrices.

(iv) $\det(\underline{P}) = \pm 1$

We now state a positive result about pivoting.

Thm: Let $n \geq 2$, $A \in \mathbb{R}^{n \times n}$. There exists a permutation matrix $P \in \mathbb{R}^{n \times n}$ a unit lower triangular matrix $L \in \mathbb{R}^{n \times n}$ and an upper triangular matrix $U \in \mathbb{R}^{n \times n}$ so that

$$PA = LU$$

Pf: $n=2$ Induction!

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad a \neq 0 \Rightarrow P = I$ from LU fact.

$a = 0,\ c \neq 0$: $\quad P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow PA = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} c & d \\ 0 & b \end{pmatrix}}_{U}$

$a = 0,\ c = 0$: $\quad A = \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}}_{U}$ $\quad (P = I)$

$n = k+1$

Say that $\max_i |a_{i1}|$ occurs in row k. Call this element γ, and permute row 1 w/ row k.

$$P^{(1k)} A = \begin{pmatrix} \gamma & w^T \\ p & B \end{pmatrix} = \begin{pmatrix} 1 & 0^T \\ m & I \end{pmatrix} \begin{pmatrix} \gamma & v^T \\ 0 & C \end{pmatrix} \quad B, C \in \mathbb{R}^{k \times k},\ p, m, v, w \in \mathbb{R}^k$$

By induction, $P^* C = L^* U^*$ for some $P^*, L^*, U^* \in \mathbb{R}^{k \times k}$.

Define $P = \begin{pmatrix} 1 & 0^T \\ 0 & (P^*)^{-1} \end{pmatrix} P^{(1k)}$. Then,

$$PA = \begin{pmatrix} 1 & 0^T \\ 0 & (P^*)^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0^T \\ m & I \end{pmatrix} \begin{pmatrix} \gamma & v^T \\ 0 & C \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \underline{0}^T \\ (\underline{\underline{P}}^*)^{-1}\underline{m} & (\underline{\underline{P}}^*)^{-1} \end{pmatrix} \begin{pmatrix} \alpha & \underline{v}^T \\ \underline{0} & \underline{\underline{C}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \underline{0}^T \\ (\underline{\underline{P}}^*)^{-1}\underline{m} & (\underline{\underline{P}}^*)^{-1} \end{pmatrix} \begin{pmatrix} \alpha & \underline{v}^T \\ \underline{0} & \underline{\underline{P}}^*\underline{\underline{L}}^*\underline{\underline{U}}^* \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 1 & \underline{0}^T \\ (\underline{\underline{P}}^*)^{-1}\underline{m} & \underline{\underline{L}}^* \end{pmatrix}}_{\underline{\underline{L}}} \underbrace{\begin{pmatrix} \alpha & \underline{v}^T \\ \underline{0} & \underline{\underline{U}}^* \end{pmatrix}}_{\underline{\underline{U}}} \qquad \square$$