HW3 solutions

Q1

(a) left to the students

(b) $\|A\|_\infty = \max_{i=1}^{n} \underbrace{\sum_{j=1}^{n} |a_{ij}|}$, and similarly for $\|A\|_1$

$n \times n-1$ additions

$\sim n^2$

(c) see the code "HW3_Q1c.m"

(d) left to the students

Q2 $A, B \in \mathbb{R}^{n \times n}$

(a) $\|AB\| \leq \|A\| \|B\|$ (see theorem 2.10)

(b) $\|I\| = \max\limits_{v \in \mathbb{R}^n_*} \frac{\|I\vec{v}\|}{\|\vec{v}\|} = 1$ regardless of the type of norm
vector-induced

(c) $\|I\| = \|AA^{-1}\| \leq \|A\| \|A^{-1}\| \Rightarrow 1 \leq \kappa(A)$

(d) $\|I\| = 1$ in all induced norms (see (b)). Is it true for the Frobenius norm?

Q3 $A \in \mathbb{R}^{n\times n}$ and invertible

(a) $K_2(A) = \|A\| \|A^{-1}\| = \sqrt{\lambda_{max}(A^TA)}\ \underbrace{\sqrt{\lambda_{max}(A^{-1^T}A^{-1})}}_{⊛}$

⊛ Note: $(A^{-1})^T = (A^T)^{-1}$ because $A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$

$\Rightarrow ⊛ = \lambda_{max}(A^{T^{-1}}A^{-1}) = \lambda_{max}(AA^T)^{-1}$

eigenvalues of B^{-1} are reciprocal of eigenvalues of B: $B\vec{x} = \lambda\vec{x} \Rightarrow \vec{x} = B^{-1}\lambda\vec{x}$

$\Rightarrow B^{-1}\vec{x} = \frac{1}{\lambda}\vec{x}$

$\Rightarrow \lambda_{max}(AA^T)^{-1} = \dfrac{1}{\lambda_{min}(AA^T)}$

$$\Rightarrow K_2(A) = \sqrt{\frac{\lambda_{max}}{\lambda_{min}}}$$

(b) $A\vec{x} = \vec{b}$, $A\Delta\vec{x} = \Delta\vec{b}$

the eigenvalue eqn is

$$\underbrace{AA^T\vec{y} = \lambda\vec{y}}_{⊛}, \quad \vec{y} \neq \vec{0}, \quad \lambda \in \mathbb{R}$$

⊛ can be interpretted as $A\underbrace{\vec{x}}_{A^T\vec{y}} = \underbrace{\vec{b}}_{\lambda\vec{y}}$ or $A\Delta\vec{x} = \Delta\vec{b}$

- take largest eigenvalue λ_{max} and the corresponding eigenvector $\vec{y}_{max}$: $\vec{x} = A^T\vec{y}_{max}$, $\vec{b} = \lambda_{max}\vec{y}_{max}$

$\|\vec{x}\|_2^2 = \|A^T\vec{y}_{max}\|_2^2 = \vec{y}_{max}^{\,T}AA^T\vec{y}_{max} = \lambda_{max}\|\vec{y}_{max}\|_2^2 = \frac{1}{\lambda_{max}}\|\vec{b}\|^2$ ①

- take smallest eigenvalue λ_{min} and the corresponding eigenvector $\vec{y}_{min}$: $\Delta\vec{x} = A^T\vec{y}_{min}$, $\Delta\vec{b} = \lambda_{min}\vec{y}_{min}$

$\|\Delta\vec{x}\|_2^2 = \frac{1}{\lambda_{min}}\|\Delta\vec{b}\|_2^2$ ②

① & ② $\Rightarrow \dfrac{\|\Delta\vec{x}\|_2}{\|\vec{x}\|_2} = \sqrt{\dfrac{\lambda_{max}}{\lambda_{min}}}\ \dfrac{\|\Delta\vec{b}\|_2}{\|\vec{b}\|_2}$ ✓

Q4

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 1 \end{pmatrix}, \quad A^T = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} n & 1 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$n\cancel{x_1}^{0} + \underbrace{x_2 + \cdots + x_n}_{0} = \lambda \cancel{x_1}^{0}$$

$$x_1 + x_2 = \lambda x_1$$

$$\vdots$$

$$x_1 + x_n = \cancel{\lambda} x_n \rightarrow \underline{\lambda = 1}$$

Also,

$$n x_1 + (n-1)\underbrace{a}_{= x_2 = x_3 = \cdots = x_n} = \lambda x_1$$

$$x_1 + a = \lambda a \Rightarrow x_1 = a(\lambda - 1) \quad (*)$$

$(*)$

$$n\cancel{a}(\lambda-1) + (n-1)\cancel{a} = \lambda \cancel{a}(\lambda - 1) \Rightarrow \lambda^2 - (n+1)\lambda + 1 = 0$$

$$\Rightarrow \lambda = \frac{n+1 \pm \sqrt{(n+1)^2 - 4}}{2}$$

$$\rightsquigarrow \lambda_{max} = \frac{1}{2}\left(n+1+\sqrt{(n+1)^2-4}\right)$$

$$\lambda_{min} = \frac{1}{2}\left(n+1-\sqrt{(n+1)^2-4}\right)$$

Note: $\lambda_{max}\lambda_{min} = 1 \Rightarrow \lambda_{max} = \frac{1}{\lambda_{min}} \Rightarrow K_2(A) = \sqrt{\frac{\lambda_{max}}{\lambda_{min}}} = \lambda_{max}$

Condition number grows only linearly with n, and unless "n" is too large ($> 10^{10}$), we are good.

Q5 see code "HW3_Q5.m"

Q6 see code "HW3_Q6.m"