Newton's Method (Continued)

We now return to the root-finding problem:

$$(P) \quad \text{Find } \xi \in \mathbb{R} \text{ s.t. } f(\xi) = 0 \text{ for } f: \mathbb{R} \to \mathbb{R}$$

We look for more systematic approaches.

<u>Def</u>: Let $f: \mathbb{R} \to \mathbb{R}$ be continuous around $\xi \in \mathbb{R}$.

Relaxation uses the sequence

$$x_{k+1} = x_k - \lambda f(x_k), \quad k = 0, 1, \dots, \quad \lambda \neq 0$$

<u>Note</u>: This is a simple iteration w/ $g(x) = x - \lambda f(x)$

<u>Note</u>: $g'(x) = 1 - \lambda f'(x)$. This lets us "tune" the Lipschitz constant.

<u>Thm</u>: Let $f: \mathbb{R} \to \mathbb{R}$ be continuous around ξ with $f(\xi) = 0$.

Let f' be defined and continuous around ξ with $f'(\xi) \neq 0$.

Then, $\exists \lambda > 0$, $\delta > 0$ such that relaxation will converge for any $x_0 \in [\xi - \delta, \xi + \delta]$.

<u>Pf</u>: Let $f'(\xi) \triangleq \gamma$ and assume $\gamma > 0$ WLOG.

Because f' cts. around ξ, $\exists \delta > 0$ such that $f'(x) \geq \gamma/2$ for all $x \in \underbrace{[\xi - \delta, \xi + \delta]}_{= I}$. Let $M \triangleq \max\limits_{x \in I} f'(x)$.

$(\Rightarrow f'(x) \leq M \;\forall x \in I)$

Then, for all $x \in I$, for $\lambda > 0$,

$$1 - \lambda M \leq 1 - \lambda f'(x) \leq 1 - \lambda \gamma/2$$

We solve for λ so that

$$1 - \lambda M = -\vartheta, \quad 1 - \lambda \gamma/2 = \vartheta$$

$$\Rightarrow \begin{cases} \vartheta = \dfrac{2M - \gamma}{2M + \gamma} < 1, \end{cases}$$

$(\lambda = 4/(2M+\gamma))$

Then $g(x) = x - \lambda f(x)$ has $|g'(x)| < \theta$ for $x \in I$, and so is a local contraction mapping around ξ. $\square$

<u>Q</u>: What if we allow λ to depend on x?

$$x_{k+1} = x_k - \lambda(x_k) f(x_k)$$

<u>Note</u>: If $x_k \to \xi$, we have

$$\xi = \xi - \lambda(\xi) f(\xi)$$

$$\Rightarrow f(\xi) = 0 \quad \text{if} \quad \lambda(\xi) \neq 0$$

Rate of convergence given by $g'(\xi)$. But:

$$g'(\xi) = 1 - \lambda(\xi) f'(\xi) - \lambda'(\xi) \underbrace{f(\xi)}_{=0}$$

$$= 1 - \lambda(\xi) f'(\xi)$$

So we should set $\lambda(x)$ so that $g'(\xi)$ <u>small</u>.

<u>Def</u>: <u>Newton's method</u> for the solution of $f(x) = 0$ with $f: \mathbb{R} \to \mathbb{R}$ is given by

$$x_{k+1} = x_k - f(x_k)/f'(x_k), \quad k = 0, 1, 2, \ldots$$

We assume that $f'(x_k) \neq 0$ for all x_k.

<u>Geometry</u>:

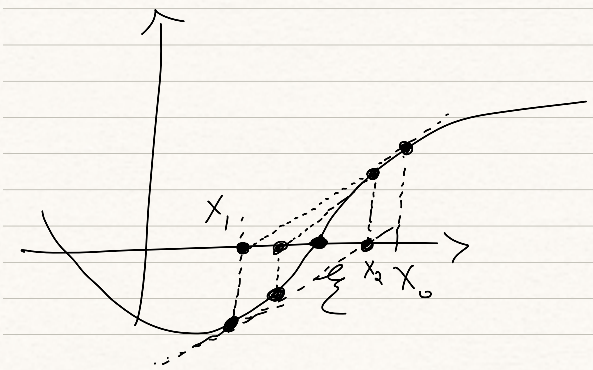
Note: By Taylor expansion,

$$0 = f(\xi) = f(x_k) + (\xi - x_k) f'(x_k) + O(|\xi - x_k|^2)$$

For $|\xi - x_k|$ "small",

$$0 \approx f(x_k) + (\xi - x_k) f'(x_k)$$

$$\Rightarrow \xi \approx x_k - f(x_k)/f'(x_k)$$



Find the next iterate by solving for where the tangent line intersects the x-axis!

Convergence

We first need a definition.

Def: Suppose $\xi = \lim_{k \to \infty} x_k$. We say $x_k \to \xi$ with at least order q if there exists a sequence $\{\varepsilon_k\}_{k=0}^{\infty}$, $\varepsilon_k \geq 0$, $\varepsilon_k \to 0$ and a $\mu > 0$ such that

$$|x_k - \xi| \leq \varepsilon_k, \ k = 0, 1, 2, \ldots \quad \text{and} \quad \lim_{k \to \infty} \varepsilon_{k+1} / \varepsilon_k^q = \mu.$$

If $\varepsilon_k = |x_k - \xi|$, we say $\{x_k\}$ converges w/ order q.
If $q = 2$, we say $\{x_k\}$ converges at least quadratically.

Note: Require $\mu > 0$, not $\mu \in (0,1)$ like for linear convergence.

Ex: Let $c > 1$, $q > 1$. Then $x_k = c^{-q^k}$ converges to 0 with order q. $\triangle$

Thm: (Convergence of Newton)

Let $f \in C^2(I_\delta, \mathbb{R})$ with $I =]\xi - \delta, \xi + \delta[$, $\delta > 0$, with $f(\xi) = 0$ and $f'(\xi) \neq 0$. Suppose $\exists A > 0$ with

$$\frac{|f''(x)|}{|f'(y)|} \leq A \quad \forall\, x, y \in I_\delta.$$

Then, for all x_0 with $|\xi - x_0| \leq h$, $h = \min(\delta, 1/A)$, the Newton sequence $\{x_k\}$ converges to ξ quadratically.

Pf: We proceed by induction.

Suppose $|\xi - x_k| \leq h$. Then $x_k \in I_\delta$.

By Taylor's Theorem, $\eta_k \in [x_k, \xi]$

$$0 = f(\xi) = f(x_k) + (\xi - x_k) f'(x_k) + \frac{(\xi - x_k)^2}{2} f''(\eta_k)$$

$$0 = \xi - \underbrace{\left(x_k - f(x_k)/f'(x_k)\right)}_{x_{k+1}} + \frac{(\xi - x_k)^2 f''(\eta_k)}{2 f'(x_k)}$$

$$\Rightarrow (\xi - x_{k+1}) = -\frac{(\xi - x_k)^2 f''(\eta_k)}{2 f'(x_k)}$$

Now, $\eta_k \in I_\delta$ (it's between x_k and ξ). Also,

$$|\xi - x_k| \leq h \leq 1/A \quad \text{by assumption.}$$

Hence,

$$|\xi - x_{k+1}| \leq \frac{|\xi - x_k| \cdot A \cdot 1/A}{2} = \frac{|\xi - x_k|}{2}.$$

Then, for $|\xi - x_0| \leq h$, $|\xi - x_k| \leq 2^{-k} \cdot h$ so $x_k \to \xi$.

Because $\eta_k \in (x_k, \xi)$, $\eta_k \to \xi$ as well. Then

$$\lim_{k \to \infty} \frac{|x_{k+1} - \xi|}{|x_k - \xi|} = \underbrace{\left| \frac{f''(\xi)}{2f'(\xi)} \right|}_{\triangleq \mu} \qquad \square$$

<u>Note</u>: Theorem requires that $f'(\xi) \neq 0$ so we can bound $|f''(x) / f'(y)|$ around ξ.

<u>Note</u>: Convergence is <u>asymptotically</u> quadratic but it can be quite slow at first!