Lagrange Interpolation

(I) Given data $(x_i, f(x_i))_{i=0}^{n}$ for a function $f : \mathbb{R} \to \mathbb{R}$, how do we find a polynomial $p : \mathbb{R} \to \mathbb{R}$ such that $p(x_i) = f(x_i)$ for $i = 1, \dots, n$?

Let $\mathcal{P}_n$ denote the set of real-valued polynomials of degree $\leq n$, i.e.

$$\mathcal{P}_n \triangleq \left\{ \sum_{i=0}^{n} a_i x^i \;\middle|\; a_i \in \mathbb{R} \right\}$$

<u>Lemma</u>: Suppose that $n \geq 1$. Then there exists $L_k \in \mathcal{P}_n$, $k = 0, 1, \dots, n$ such that

$$L_k(x_i) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$$

for all i, k. Moreover,

$$p_n(x) = \sum_{k=0}^{n} L_k(x) f(x_k)$$

solves problem (I).

<u>Pf</u>: We proceed by explicit construction.

For each k, we require $L_k(x_i) = 0$ for $i \neq k$. So we may write

$$L_k(x) = C_k \prod_{i \neq k}^{n} (x - x_i)$$

for some C_k. Setting

$$L_k(x_k) = 1 \Rightarrow C_k = \prod_{i \neq k}^{n} (x_k - x_i)^{-1}$$

gives the Lagrange Polynomial

$$L_k(x) = \frac{\prod_{i \neq k} (x - x_i)}{\prod_{i \neq k} (x_k - x_i)} \qquad \square$$

Note: For $n = 0$, we define $L_0(x) = 1$.

Note: The Lagrange Polynomials have a degree that depends on n!

We now show that the interpolating Lagrange polynomial is unique!

Thm: (Lagrange Interpolation Thm)

Let $n \geq 0$, and let $x_i \in \mathbb{R}$, $i = 0, \ldots, n$ be distinct. Let $y_i \in \mathbb{R}$, $i = 0, \dots, n$ (not necessarily distinct). Then, there exists a unique polynomial $p_n \in \mathbb{P}_n$ with $p_n(x_i) = y_i$ for $i = 0, \dots, n$.

Pf: For $n = 0$ trivial. For $n \geq 1$, we know that

$$p_n(x) = \sum_{k=0}^{n} L_k(x) y_k$$

satisfies the requirements, so we have existence. Suppose there exists some other $q_n \in \mathbb{P}_n$ with $q_n(x_i) = y_i \; \forall i$. Then $(q_n - p_n) \in \mathbb{P}_n$. Moreover, q_n and p_n agree on all the x_i, so that $q_n - p_n$ is a degree n polynomial with $n+1$ roots. Hence, $q_n - p_n = 0$ and p_n is unique. □

Motivated by the above, we define

Def: The Lagrange Interpolant on the $n+1$ points $(x_i, y_i)_{i=0}^{n}$ is given by

$$p_n(x) = \sum_{k=0}^{n} L_k(x) y_k$$

The numbers x_i, $i=0,\dots,n$ are called the interpolation points.

Def: Given $f:[a,b]\to\mathbb{R}$ and distinct interpolation points $x_i \in [a,b]$ for $i=0,\dots,n$, the polynomial

$$P_n(x) = \sum_{k=0}^{n} L_k(x) f(x_k)$$

is called the Lagrange interpolating polynomial of degree n for f.

Ex: $f(x) = \exp(x)$

$x_0 = -1,\ x_1 = 0,\ x_2 = 1 \quad (n=2)$

$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$$

$$= \frac{x(x-1)}{(-1)(-2)} = \tfrac{1}{2}\, x(x-1)$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$$

$$= -(x+1)(x-1) = 1-x^2$$

$$l_2(x) = \frac{(x+1)x}{2}$$

So that, all together,

$$P_2(x) = \frac{e^{-1}}{2} x(x-1) + (1-x^2)e^0 + \tfrac{1}{2} x(x+1)e^1 \quad \Delta$$

<u>Note</u>: $P_n(x)$ and f agree <u>on</u> the x_i, but can be very different off the x_i!

<u>Q</u>: How far off can they be?

<u>Thm</u>: Let $n \geq 0$, and let $f: [a,b] \to \mathbb{R}$. Assume $f \in C^{n+1}([a,b])$. Then, there exists $\xi(x) \in (a,b)$ such that

$$f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \pi_{n+1}(x)$$

$$\pi_{n+1}(x) \triangleq (x-x_0)(x-x_1)\cdots(x-x_n)$$

Moreover,

$$|f(x) - P_n(x)| \leq \frac{M_{n+1}}{(n+1)!} |\pi_{n+1}(x)|$$

$$M_{n+1} \triangleq \max_{\xi \in [a,b]} |f^{(n+1)}(\xi)|$$

<u>Pf</u>: Note for $x = x_i$, the statement is trivial, as both sides $= 0$.

So, consider $x \in [a,b]$, $x \neq x_i$ for any i.

Now, define $\varphi_x : [a,b] \to \mathbb{R}$

$$\varphi_x(t) = f(t) - P_n(t) - \frac{f(x) - P_n(x)}{\pi_{n+1}(x)} \pi_{n+1}(t)$$

Clearly,

$$\varphi_x(x_i) = \underbrace{f(x_i) - P_n(x_i)}_{=0} - \frac{f(x) - P_n(x)}{\pi_{n+1}(x)} \underbrace{\pi_{n+1}(x_i)}_{=0}$$

$$\varphi_x(x) = f(x) - P_n(x) - \frac{(f(x) - P_n(x))}{\cancel{\pi_{n+1}(x)}} \cancel{\pi_{n+1}(x)}$$

$= 0$!

So φ_x vanishes at the $n+2$ points $x, x_0, x_1, \ldots, x_n$.

By Rolle's Theorem, φ_x' vanishes at $n+1$ points between the points at which φ_x vanishes.

Induction:

For $n=0$, φ_x', $\exists\, \xi \in (a,b)$ with $\varphi_x'(\xi) = 0$.

$$0 = \varphi_x'(\xi) = f'(\xi) - \underbrace{P_0'(\xi)}_{=0}$$

$$- f(x) - P_n(x) / \pi_{n+1}(x)$$

$$\Rightarrow \quad 0 = f'(\xi) - \frac{f(x) - P_0(x)}{\pi_1(x)} \quad \checkmark$$

Now, Consider $n \geq 1$.

Applying Rolle's theorem again, $\varphi_x''(t)$ must vanish at n points. Applying recursively, $\varphi^{(n+1)}$ vanishes at some single point $\xi \in (a,b)$.

$$0 = \varphi_x^{(n+1)}(\xi) = f^{(n+1)}(\xi) - \underbrace{P_n^{(n+1)}(\xi)}_{=0}$$

$$- \left[\frac{f(x) - P_n(x)}{\pi_{n+1}(x)}\right](n+1)!$$

$$\Rightarrow f(x) - P_n(x) = \frac{f^{(n+1)}(\xi) \cdot \pi_{n+1}(x)}{(n+1)!} \quad \forall x$$

Maximizing over $x \in [a,b]$,

$$\max_{x \in [a,b]} |f(x) - P_n(x)| \leq \frac{|f^{(n+1)}(x)| \, M_{n+1}}{(n+1)!} \quad \square$$