LU Factorization

We saw G.E. gives $\underline{\underline{A}} = \underline{\underline{L}}\,\underline{\underline{U}}$ for $\underline{\underline{A}} \in \mathbb{R}^{n\times n}$, $\underline{\underline{L}} \in \mathbb{R}^{n\times n}$, $\underline{\underline{U}} \in \mathbb{R}^{n\times n}$,

$\underline{\underline{L}}$ unit lower triangular and $\underline{\underline{U}}$ upper triangular.

Q: Can we compute l_{ij}, u_{ij} directly?

Equate: $\underline{\underline{A}} = \underline{\underline{L}}\,\underline{\underline{U}} \Rightarrow a_{ij} = \sum_{k=1}^{n} l_{ik} u_{kj}$, $\quad i, j \in \{1, \dots, n\}$

Recall:

$$L = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ * & & 1 & \\ & & & \ddots \\ & & & & 1 \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & & & * \\ & 1 & & \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix}$$

zeros imply:

$$a_{ij} = \sum_{k=1}^{j} l_{ik} u_{kj}, \quad 1 \le j < i \le n \quad \text{(lower triangle)}$$

$$a_{ij} = \sum_{k=1}^{i} l_{ik} u_{kj}, \quad 1 \le i \le j \le n \quad \text{(upper triangle)}$$

Say $i > j$, i.e., $i = 2, \dots, n$, $j = 1, \dots, i-1$. Then,

$$a_{ij} = l_{ij} u_{jj} + \sum_{k=1}^{j-1} l_{ik} u_{kj}$$

$$l_{ij} = 1/u_{jj} \left[a_{ij} - \sum_{k=1}^{j-1} l_{ik} u_{kj} \right], \quad \begin{matrix} i = 2, \dots, n \\ j = 1, \dots, i-1 \end{matrix}$$

Similarly, for $i \le j$,

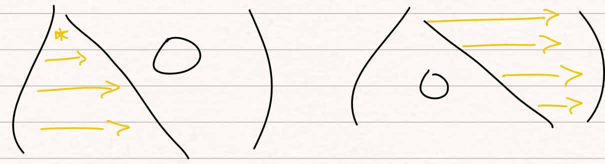
$$u_{ij} = a_{ij} - \sum_{k=1}^{i-1} l_{ik} u_{kj}, \quad \begin{matrix} i = 1, \dots, n, \\ j = i, \dots, n \end{matrix}$$

Note: Sum over empty index set $= 0$ by convention!

(e.g. $u_{1j} = a_{1j}$ for all j,
$l_{11} = 1$,
$l_{i1} = a_{i1} / u_{11}$)

For each i, we can compute l_{ij} for $i \le j$ first (in order), and u_{ij} (again in order). We can do so because for each l_{ij} we require u_{kj} for $k \le i$ (same col., previous row) and l_{ik} for $k \le i$ (same row, prev col.). At each stage we have all we need!

arrows depict flow of computation



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<u>Note</u>: Require that $u_{ii} \neq 0$ for each i!

<u>Def</u>: Let $\underline{\underline{A}} \in \mathbb{R}^{n \times n}$, $n \geq 2$, $k \leq n$. We define the matrix $\underline{\underline{A}}^{(k)} \in \mathbb{R}^{k \times k}$ by $a^{(k)}_{ij} = a_{ij}$, $i, j \leq k$. We call $\underline{\underline{A}}^{(k)}$ the leading principal submatrix of $\underline{\underline{A}}$.

$$\underbrace{\begin{pmatrix} (\underline{\underline{A}}^{(k)}) & \\ & \end{pmatrix}}_{\underline{\underline{A}}}$$

<u>Thm</u>: Let $n \geq 2$, and suppose $\underline{\underline{A}} \in \mathbb{R}^{n \times n}$ satisfies that every leading principal submatrix $\underline{\underline{A}}^{(k)} \in \mathbb{R}^{k \times k}$ is nonsingular. Then the factorization $\underline{\underline{A}} = \underline{\underline{L}}\,\underline{\underline{U}}$ with $\underline{\underline{L}} \in \mathbb{R}^{n \times n}$, $\underline{\underline{L}}$ U.L.T. and $\underline{\underline{U}} \in \mathbb{R}^{n \times n}$ U.T. exists.

<u>Pf</u>: We induct on the size of the matrix n.

$n = 2$

$$\underline{\underline{A}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a \neq 0$$

want $\underline{\underline{A}} = \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix} \begin{pmatrix} u & v \\ 0 & \eta \end{pmatrix}$ for $m, u, v, \eta \in \mathbb{R}$.

$$= \begin{pmatrix} u & v \\ mu & mv + \eta \end{pmatrix}$$

$$\Rightarrow u = a \quad v = b. \quad mu = c. \quad mv + \eta = d$$

$$\Rightarrow \; m = c/a, \quad \eta = d - bc/a.$$

$n \geq 2$: we now partition the matrix $\underline{\underline{A}} \in \mathbb{R}^{(k+1)\times(k+1)}$

$$\underline{\underline{A}} = \begin{pmatrix} \underline{\underline{A}}^{(k)} & \underline{b} \\ \underline{c}^T & d \end{pmatrix}, \qquad \underline{\underline{A}}^{(k)} \in \mathbb{R}^{k\times k} \text{ nonsingular}, \quad \underline{b}, \underline{c} \in \mathbb{R}^k, \quad d \in \mathbb{R}.$$

We know $\underline{\underline{A}}^{(k)} = \underline{\underline{L}}^{(k)} \underline{\underline{U}}^{(k)}$ by the inductive hypothesis.

Write

$$\underline{\underline{L}} = \begin{pmatrix} \underline{\underline{L}}^{(k)} & \underline{0} \\ \underline{m}^T & 1 \end{pmatrix}, \qquad \underline{\underline{U}} = \begin{pmatrix} \underline{\underline{U}}^{(k)} & \underline{v} \\ \underline{0}^T & \eta \end{pmatrix} \qquad \underline{m}, \underline{v} \in \mathbb{R}^k, \quad \eta \in \mathbb{R}$$

$$(\Rightarrow) \quad \underline{\underline{L}}\,\underline{\underline{U}} = \begin{pmatrix} \underline{\underline{L}}^{(k)} \underline{\underline{U}}^{(k)} & \underline{\underline{L}}^{(k)} \underline{v} \\ \underline{m}^T \underline{\underline{U}}^{(k)} & \underline{m}^T \underline{v} + \eta \end{pmatrix}$$

$$(\Rightarrow) \quad \begin{array}{ll} \underline{\underline{L}}^{(k)} \underline{\underline{U}}^{(k)} = \underline{\underline{A}}^{(k)} & \underline{m}^T \underline{\underline{U}}^{(k)} = \underline{c}^T \\ \underline{\underline{L}}^{(k)} \underline{v} = \underline{b} & \underline{m}^T \underline{v} + \eta = d \end{array}$$

$\underline{\underline{L}}^{(k)}$ unit lower triangular $\Rightarrow$ invertible $\Rightarrow$ $\underline{v} = (\underline{\underline{L}}^{(k)})^{-1} \underline{b}$

Moreover, $\det(\underline{\underline{A}}^{(k)}) \neq 0 = \det(\underline{\underline{L}}^{(k)} \underline{\underline{U}}^{(k)})$

$$= \det(\underline{\underline{L}}^{(k)}) \det(\underline{\underline{U}}^{(k)})$$

$$\Rightarrow \det(\underline{\underline{U}}^{(k)}) \neq 0$$

$$\Rightarrow \underline{\underline{U}}^{(k)} \text{ invertible}$$

$$\Rightarrow \underline{m} = (\underline{\underline{U}}^{k})^{-1} \underline{c}$$

Then $\eta = d - \underline{m}^T \underline{v}$. $\square$

Computational Work

$\underline{Q}$: How much effort (Floating point operations) to compute $\underset{=}{A} = \underset{=}{L}\,\underset{=}{U}$? How does it scale w/ n for $\underset{=}{A} \in \mathbb{R}^{n\times n}$?

Recall:

$$
\begin{cases}
l_{ij} = 1/u_{jj}\left[a_{ij} - \sum_{k=1}^{j-1} l_{ik} u_{kj}\right], & i = 2,\dots,n,\quad j = 1,\dots,i-1 \\
u_{ij} = a_{ij} - \sum_{k=1}^{i-1} l_{ik} u_{kj}, & i = 1,\dots,n,\quad j = i,\dots,n
\end{cases}
$$

l_{ij}:	$j-1$	$\times$	1	$-$	u_{ij}:	$i-1$ $\times$	1	$-$
	$j-2$	$+$	1	$\%$		$i-2$ $+$	0	$\%$

Recall:

$$\sum_{l=1}^{k} l = \frac{k\cdot(k+1)}{2}, \qquad \sum_{l=1}^{k} l^2 = \frac{k(k+1)(2k+1)}{6}$$

So in total, work is:

$$\text{Work} = \underbrace{\sum_{i=2}^{n}\sum_{j=1}^{i-1} \underbrace{(2j-1)}_{l_{ij}}}_{\underset{=}{L}} + \underbrace{\sum_{i=1}^{n}\sum_{j=i}^{n} \underbrace{2\cdot(i-1)}_{u_{ij}}}_{\underset{=}{U}}$$

$$= \sum_{i=1}^{n}\left[\sum_{j=1}^{i-1}(2j-1) + \sum_{j=i}^{n} 2\cdot(i-1)\right]$$

$$= \sum_{i=1}^{n}\left[2\cdot \tfrac{1}{2}\cdot i\cdot(i-1) - (i-1) + 2\cdot(i-1)\cdot(n-i+1)\right]$$

$$= \sum_{i=1}^{n} \left[(i-1)^2 + 2(i-1)\cdot(n - i-1)) \right]$$

$$= \sum_{i=1}^{n} \left[(i-1)^2 - 2(i-1)^2 + 2\cdot(i-1)\cdot n \right]$$

$$= \sum_{l=0}^{n-1} (2ln - l^2)$$

$$= \cancel{2}\cdot \cancel{1/2}\cdot n\cdot n\cdot(n-1) - 1/6(n-1)\cdot n\cdot(2(n-1)+1)$$

$$= n\cdot(n-1)\,(n - 1/6\,[2n-1])$$

$$= 1/6\, n(n-1)\,(6n-2n+1) = 1/6\, n(n-1)(4n+1)$$

$\sim 2/3\, n^3 - 1/2\, n^2$ for n large.