Fall 2022: Numerical Analysis
Assignment 1 (due 9/21/22, 2022 at 11:59pm ET)

Homework submission. Homework assignments must be submitted through Gradescope. Please hand in cleanly handwritten or typed homeworks. If you are required to hand in code or code listings, this will explicitly be stated on that homework assignment (if nothing is stated, you are not required to hand in code listings).

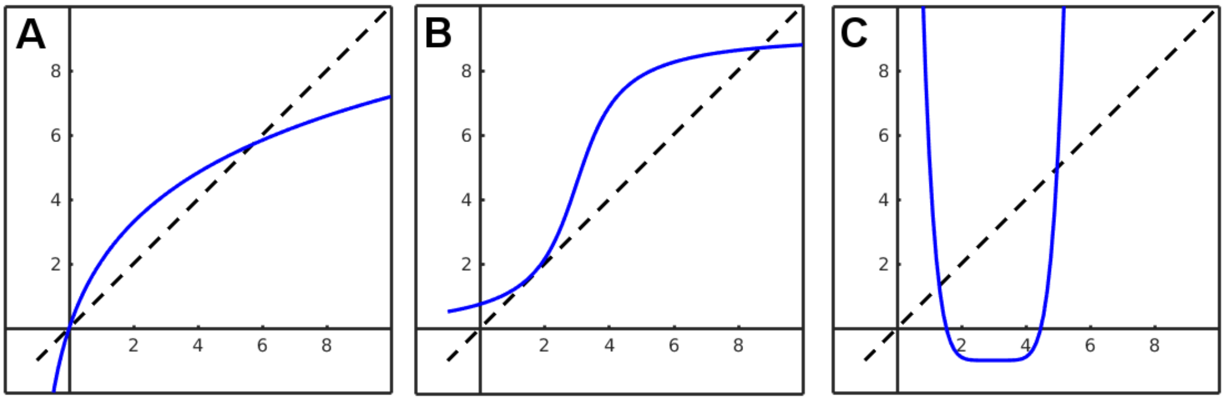
Collaboration. NYU's integrity policies will be enforced. You are encouraged to discuss the problems with other students in person or on our course discusson platform. However, you must write (i.e., type) every line of code yourself and also write up your solutions independently. Copying of any portion of someone else's solution/code or allowing others to copy your solution/code is considered cheating.

Plotting and formatting. Plot figures carefully and think about what you want to illustrate with a plot. Choose proper ranges and scales (`semilogx, semilogy, loglog`), always label axes, and give meaningful titles. Sometimes, using a table can be useful, but never submit pages filled with numbers. Discuss what we can observe in and learn from a plot. Use `format compact` and other format commands to control outputs. When you create figures in MATLAB, Python or Julia, please export them in a vector graphics format (.eps, .pdf, .dxf) rather than raster graphics or bitmaps (.jpg, .png, .gif, .tif). Vector graphics-based plots avoid pixelation and thus look much cleaner.

Programming. This is an essential part of this class. We will use Python primarily, but you are free to use other languages (MATLAB, Julia). Basic programming skills are crucial for many jobs, so this is also a good chance to get more comfortable with it, if you aren't already. In your programs, please use meaningful variable names and try to write clean, concise and easy-to-read code. Comments for explanation are a crucial part of every program.

1. **[Fixed points and contractions, 1+1+1+1pt]** Let $f(x) = e^x - x^2 - 6x - 9$ and $g(x) = 2\ln(x+3)$, where $x \in (-1, \infty)$.
   (a) Verify that the roots of $f(x)$ are the same as the fixed points of $g(x)$.
   (b) Sketch $y = g(x)$, $y = x$ and indicate all fixed points (sketch by hand or plot with a computer). You don't need to calculate them. (Hint for the sketch: Note that $g'(0) > 1$).
   (c) Use Brouwer's fixed point theorem to argue the existence of a fixed point ξ in the interval $[a, b] = [e - 3, e^2 - 3]$.
   (d) Use the contraction mapping theorem to show that ξ is the only fixed point in the interval $[e - 3, e^2 - 3]$.

2. **[Stability and speed of convergence, 2+2+1+1pt]**
   (a) For each of the three functions (solid lines) depicted below,
      (i) Write down the approximate values of the fixed points (as estimated by eye).
      (ii) State for each fixed point, whether it is stable, unstable or neither of the two.
   (b) You are given the first ten iterates of two sequences, x_k and y_k, both of which converge to zero:



k	x_k	y_k
0	1.0000000000000	1.0000000000000
1	0.3000000000000	0.6648383611734
2	0.0900000000000	0.4404850619261
3	0.0270000000000	0.2895527955097
4	0.0081000000000	0.1869046766665
5	0.0024300000000	0.1155100169867
6	0.0007290000000	0.0638472856062
7	0.0002187000000	0.0254178900244
8	0.0000656100000	0.0032236709627
9	0.0000196830000	0.0000080907744
10	0.0000059049000	0.0000000000001

(i) What is (most likely) the order of convergence of x_k? Explain your answer.

(ii) What is (most likely) the order of convergence of y_k? Explain your answer[1].

(c) At what speed does the sequence $x_n := 1 + (-1/n)^n$ converge to 1?

(d) At what speed does the sequence $x_n := (n/(n+1)^3)$ converge to 0?

3. **[Fixed point convergence, 2+1+1pt]** Let g be defined on $[\pi/3, 3\pi/4]$.

$$g(x) = \frac{3}{4}x + \frac{1}{2}\cos x.$$

(a) Determine the (smallest possible) Lipschitz constant L.

(b) How many iterations are required to increase the accuracy by a factor of 100, i.e., given some x_0 in the interval, what is k such that you can guarantee that $|x_k - \xi| \leq 10^{-2}|x_0 - \xi|$?

(c) Starting with initial guess $x_0 = \pi/3$, compute the first fixed point iterate x_1 and use it, together with the Lipschitz constant you found, to compute after how many fixed point iterations k you can be certain that $|\xi - x_k| < 10^{-10}$.

4. **[More fixed point algorithm, 1+1+1+1pt]** The equation

$$f(x) := x^2 - 5 = 0,$$

has a single root $\xi = \sqrt{5} \approx 2.2361\ldots$ in the interval $[1, 3]$. Consider the fixed point iteration $x_{k+1} = g(x_k)$, where g is defined as one of the following options:

[1]Note that the definition of convergence is asymptotic, i.e., close to the limit point.

- $g_1(x) = 5 + x - x^2$,
- $g_2(x) = 1 + x - \frac{1}{5}x^2$,
- $g_3(x) = \frac{1}{2}x + \frac{5}{2x^2}$.

(a) Identify the fixed point functions for which the fixed point is also a root of f.

(b) For the cases where computing the fixed point is equivalent with solving $f(x) = 0$, discuss whether the fixed point iteration is guaranteed to converge in some neighborhood of ξ.

(c) If the iteration in b) is guaranteed to converge, compute the value of

$$\lim_{k\to\infty} \frac{|x_{k+1} - \xi|}{|x_k - \xi|}$$

Hint: Everything is easier with the mean value theorem!

(d) Give Newton's method for solving $f(x) = 0$ and, for given starting value $x_0 = 2$ compute the first Newton iterate x_1.

5. **[Stability, 3pt]** Define the function g by $g(0) = 0$, $g(x) = -x\sin^2(1/x)$ for $0 < x \le 1$. Show that g is continuous, and that 0 is the only fixed point of g in the interval $[0, 1]$. By considering the iteration $x_{n+1} = g(x_n)$, with $x_0 = 1/(k\pi)$ and $x_0 = 2/((2k+1)\pi)$, where k is an integer, show that using the definition of stability provided in class, $\xi = 0$ is neither stable nor unstable.

6. **[Newton's method, 1+1+1pt]**

(a) Show how Newton's method can be view as a fixed point iteration, i.e., what is the function g so that the Newton iterate can be expressed as $x_{k+1} = g(x_k)$?

(b) Take $f(x) = (x+1)^2 - 1$ and assume x_0 is close enough to the root $\xi = 0$. Show that Newton's method applied to this function gives asymptotically quadratic convergence. Why is the convergence quadratic?

(c) Take $f(x) = x^2$. Show that Newton's method applied to this function gives linear convergence to the root of f (**hint**: just plug f into the formula for the Newton iterates and do some algebra). Why is the convergence *only linear*?

7. **[Steffensen's method 1+1+2pt]** Steffensen's method for solving $f(x) = 0$ for $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$x_{k+1} = x_k - \frac{f(x_k)}{g(x_k)}$$
$$g(x_k) = \frac{f(x_k + f(x_k)) - f(x_k)}{f(x_k)}$$

(a) Show that Steffensen's method can be viewed as a fixed point iteration. Assuming that f is one-to-one, show that a fixed point ξ of the iteration satisfies $f(\xi) = 0$.

(b) Recall the definition of the derivative

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h},$$

and show that $g(x_k)$ can be viewed as an approximation to $f'(x_k)$ along the relaxation

$$x_{k+1} = x_k + f(x_k).$$

(c) Show that Steffensen's method converges quadratically under suitable hypotheses.