Iterative Solution to Nonlinear Equations (1D)

Given a function $f: \mathbb{R} \to \mathbb{R}$, consider the root-finding problem

(P) Find $\xi \in \mathbb{R}$ such that $f(\xi) = 0$.

We've all seen linear equations: $ax + b = 0 \Rightarrow x = -b/a$

Quadratic equations: $ax^2 + bx + c = 0 \Rightarrow x_\pm = \dfrac{-b \pm (b^2 - 4ac)^{1/2}}{2a}$

Q: What about higher-order polynomials?

A: No! For all $n \geq 5$, there exists a polynomial with integer coefficients with solution that cannot be written in terms of radicals.

So: No general formula for ξ can exist! Instead, we construct an approximation to ξ via iteration.

Simple Iteration

"f is a continuous fn. on $[a,b]$"

Let $f: [a,b] \to \mathbb{R}$, $a < b$, $f \in C^0[a,b]$

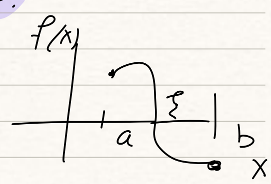
We want to solve the problem (P).

Ex: $f(x) = x^2 + 1$. $\xi = \pm i$, so $\xi \notin \mathbb{R}$!

"$f(a)$ and $f(b)$ have opp. sign"

Thm: Let $f \in C^0[a,b]$, $f: [a,b] \to \mathbb{R}$. Assume $f(a) \cdot f(b) \leq 0$.

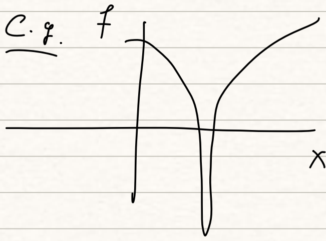
Then $\exists\, \xi \in [a,b]$ with $f(\xi) = 0$.



Pf: Assume $f(a) \cdot f(b) \neq 0$, otherwise we are done.

Then $f(a) \cdot f(b) < 0$, so $0 \in (f(a), f(b))$. By the intermediate value theorem, done. ∎

Note: Can be hard to use the above theorem because such an interval $[a,b]$ could be hard to find! (see book for example)

e.g. 

Instead, we often transform the problem to be a fixed-point problem by writing

$$x - g(x) = 0 \iff f(x) = 0$$

(for example, $g = f + x$)

Def: We say that ξ is a fixed-point of g if $g(\xi) = \xi$.

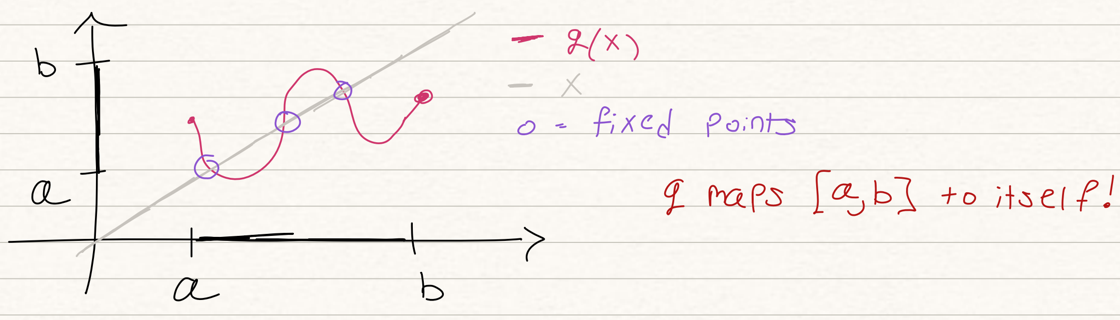
Thm (Brouwer): Let $g:[a,b] \to \mathbb{R}$, $g \in C([a,b])$. Further assume that $g(x) \in [a,b]$ for all $x \in [a,b]$. Then $\exists\, \xi \in [a,b]$ such that $\xi = g(\xi)$.

Pf: Let $f(x) = x - g(x)$. Then:

(i) $f(a) = a - g(a) \leq 0 \quad (g(x) \geq a)$

(ii) $f(b) = b - g(b) \geq 0 \quad (g(x) \leq b)$

So $f(a)f(b) \leq 0$, $f \in C([a,b])$. Hence, by the previous theorem, $\exists\, \xi$ with $f(\xi) = 0 \Rightarrow g(\xi) = \xi$. $\square$



Note: Many ways to transform root-finding problem to a fixed-point problem, only need to find one with $g(x) \in [a,b]$.

Note: So far we have verified the existence of solutions to the problem $f(x)=0$. The next def. moves towards algorithms.

Def: Let $g \in C^0([a,b])$ and assume $g(x) \in [a,b]$ for $x \in [a,b]$. Let $x_0 \in [a,b]$. We call the recursion

$$(1) \qquad x_{k+1} = g(x_k), \quad k = 0, 1, \dots,$$

a simple iteration. The $\{x_k\}$ are iterates.

Claim: If the iteration (1) converges to some ξ then ξ is a fixed point of g.

Pf: By continuity,

$$\xi = \lim_{k\to\infty} x_{k+1} = \lim_{k\to\infty} g(x_k) = g\left(\lim_{k\to\infty} x_k\right) = g(\xi) \quad \square$$

We now give sufficient conditions for convergence of (1).

$g(x) \in [a,b]$ for $x \in [a,b]$

Def: Let $g \in C^0([a,b],[a,b])$.
We say g is a contraction if $\exists\ 0 < L < 1$ with

$$|g(x) - g(y)| \le L\,|x-y| \quad \forall\, x,y \in [a,b]$$