Newton-Cotes Integration

(I) Given $f:[a,b]\to\mathbb{R}$, compute the definite integral $\int_a^b f(x)\,dx$.

Idea: Approximate f by a Lagrange interpolant P_n at $n+1$ evenly-spaced points, and integrate that exactly.

$$x_i = a + i\cdot h, \quad i = 0,1,\ldots,n, \quad h = (b-a)/n$$

Then,

$$P_n(x) = \sum_{k=0}^{n} L_k(x) f(x_k), \quad L_k(x) = \prod_{i\neq k} \frac{x-x_i}{x_k-x_i}$$

Hence,

$$\int_a^b f(x)\,dx \approx \int_a^b P_n(x)\,dx$$

$$= \sum_{k=0}^{n} \underbrace{\left(\int_a^b L_k(x)\,dx\right)}_{\hat{=}\, w_k} f(x_k)$$

$$= \sum_{k=0}^{n} w_k f(x_k).$$

Def: In an expression

$$(*) \quad \int_a^b f(x)\,dx \approx \sum_{k=0}^{n} \omega_k f(x_k),$$

the $\{x_k\}_{k=0}^{n}$ are called the quadrature nodes. The $\{\omega_k\}_{k=0}^{n}$ are called the quadrature weights.

If the nodes are equally spaced, we call $(*)$ a Newton-Cotes quadrature rule of order n.

Ex: (Trapezoid Rule) $n=1$

$$P_1(x) = L_0(x)\cdot f(a) + L_1(x)\cdot f(b)$$

$$L_0(x) = (x-b)/(a-b)$$

$$L_1(x) = (x-a)/(b-a)$$

$$\int_a^b (x-b)\,dx = \int_{a-b}^{0} x'\,dx' = -1/2\,(b-a)^2$$

$$\int_a^b (x-a)\,dx = \int_0^{b-a} x'\,dx' = 1/2\,(b-a)^2$$

$$\Rightarrow \int_a^b f(x)\,dx \approx \left(\frac{b-a}{2}\right)\left(f(a)+f(b)\right) \quad \triangle$$

Ex: (Simpson's Rule) $n=2$

$$w_0 = \int_a^b L_0(x)\,dx = \int_a^b \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}\,dx$$

write $x = \left(\frac{b-a}{2}\right)\cdot t + \left(\frac{b+a}{2}\right)$

Now, write that $x_0 = a$, $x_1 = b+a/2$, $x_2 = b$

$$\Rightarrow x - x_1 = \left(\frac{b-a}{2}\right)\cdot t$$

$$x - x_2 = \left(\frac{b-a}{2}\right)\cdot(t-1)$$

$$(x_0 - x_1) = (a-b)/2$$

$$(x_0 - x_2) = (a-b)$$

$$dx = (b-a)/2\,dt$$

$$\Rightarrow w_0 = \int_{-1}^{1} t\cdot(t-1)\cdot\left(\frac{b-a}{2}\right)^2\cdot(b-a)^{-2}\cdot 2\cdot\left(\frac{b-a}{2}\right)dt$$

$$= \left(\int_{-1}^{1} \frac{t\cdot(t-1)}{4}\,dt\right)\cdot(b-a)$$

$$= \left(1/3\,t^3 - 1/4\,t^2\right)\Big|_{-1}^{1}\cdot(b-a)/4$$

$$= 1/6\,(b-a)$$

A similar calculation (try it!) gives

$$w_1 = (2/3)(b-a), \quad w_2 = w_0.$$

Hence,

$$\int_a^b f(x) \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right].$$

<u>Note</u>: weights $\{w_k\}$ do not depend on f!

<u>Error Estimates</u>

We want to study the error

$$E_n(f) \triangleq \int_a^b f(x)\,dx - \sum_{k=0}^{n} w_k f(x_k).$$

<u>Thm</u>: Let $n \geq 1$, and suppose that $f \in C^{(n+1)}([a,b])$. Then,

$$|E_n(f)| \leq \frac{M_{n+1}}{(n+1)!} \int_a^b |\pi_{n+1}(x)|\,dx$$

where we have defined

$$M_{n+1} \triangleq \max_{\xi} |f^{(n+1)}(\xi)|, \quad \pi_{n+1}(x) = \prod_{k=0}^{n}(x - x_k)$$

<u>Pf</u>: Recall that

$$\sum_{k=0}^{n} w_k f(x_k) = \int_a^b p_n(x)\,dx$$

Hence,

$$E_n(f) = \int_a^b \left(f(x) - p_n(x)\right) dx$$

$$\Rightarrow |E_n(f)| \leq \int_a^b |f(x) - p_n(x)| \, dx$$

The bound then follow from our previous bound on $|f(x) - p_n(x)|$. $\square$

<u>Ex</u> (Trapezoid):

$$|E_1(f)| \leq \frac{M_2}{2} \int_a^b |(x-a)(x-b)| \, dx$$

$$= \frac{M_2}{2} \int_a^b (b-x)(x-a) \, dx$$

$$= \frac{(b-a)^3}{12} \cdot M_2$$

<u>Ex</u>: (Simpson)

Analogously,

$$E_2(f) \leq \frac{M_3}{6} \int_a^b \left|(x-a) \cdot \left(x - \frac{b+a}{2}\right)(x-b)\right|$$

$$= \frac{(b-a)^4}{196} \cdot M_3.$$

Note: Bound for Simpson pessimistic! In particular, does not show that Simpson has $E_2''(f) = 0$ for $f \in P_3$.

Thm: Let $f \in C^4([a,b])$. Then,

$$\int_a^b f(x)\,dx - \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] = -\frac{(b-a)^5}{2880} f^{(4)}(\xi)$$

for some $\xi \in (a,b)$.

Pf: See book.

This theorem yields the bound

$$E_2(f) \leq \frac{(b-a)^5}{2880} M_4$$

Note: Because of Runge's Phenomenon,

$$\int_{-5}^{5} 1/(1+x^2)\,dx$$

may not become more accurate as $n \to \infty$!

Actually, $E_n(f) \to \infty$ with n for this choice of f.

Better Idea: "Composite" integration.

Composite Quadrature Rules

Basic Idea:

$$\int_a^b f(x)\,dx = \sum_{i=1}^{m} \int_{x_{i-1}}^{x_i} f(x)\,dx$$

$$x_i = a + i\cdot h, \quad h = (b-a)/m, \quad i = 0, 1, \dots, m$$

Then apply quadrature to each of the sub-integrals.

Ex: Composite trapezoid

$$\int_{x_i}^{x_{i-1}} f(x)\,dx \approx h/2 \cdot \left[f(x_{i-1}) + f(x_i) \right]$$

$$\Rightarrow \int_a^b f(x)\,dx \approx h\cdot\left(1/2\, f(x_0) + f(x_1) + \cdots f(x_{m-1}) + 1/2\, f(x_m) \right)$$

Error:

$$\mathcal{E}_1(f) \triangleq \int_a^b f(x)\,dx - h\cdot\left(1/2\, f_0 + f_1 + \dots + f_{m-1} + 1/2\, f_m \right)$$

$$= \sum_{i=1}^{m} \left[\int_{x_{i-1}}^{x_i} f(x)\,dx - h/2\left(f_{i-1} + f_i \right) \right]$$

From our earlier error bound for the trapezoid rule, we know that

$$\left| \int_{x_{i-1}}^{x_i} f(x)\,dx - \tfrac{1}{2}h\,(f_{i-1} + f_i) \right| \leq \frac{h^3}{12} M_2$$

So that, all together,

$$|E_1(f)| \leq \frac{m \cdot h^3}{12} \cdot M_2 = \frac{(b-a)^3}{12 \cdot m^2} M_2$$

Ex: Composite Simpson

Divide into $\underline{2m}$ subintervals

$$x_i = a + i \cdot h, \quad h = b-a/2m, \quad i = 0, 1, \ldots, 2m$$

$$\int_a^b f(x)\,dx = \sum_{i=1}^{m} \int_{x_{2i-2}}^{x_{2i}} f(x)\,dx$$

$$\approx \sum_{i=1}^{m} \frac{h}{3} \left(f_{2i-2} + 4 f_{2i-1} + f_{2i} \right)$$

$$= h/3 \,(f_0 + 4f_1 + 2f_2 + 4f_3 + \cdots$$

$$+ 2 f_{2m-2} + 4 f_{2m-1} + f_{2m})$$

Splitting the integral as before, we may write that

$$|E_2(f)| \leq \sum_{i=1}^{m} \left| \int_{x_{2i-1}}^{x_{2i}} f(x)\,dx - \frac{h}{3} (f_{2i-2} + 4 f_{2i-1} + f_{2i}) \right|$$

$$\Rightarrow \quad \mathcal{E}_2(f) \leq \frac{(b-a)^5}{2880 m^4} M_4$$