Homework 5

1

$$A = \begin{bmatrix} -6 & 2 & 0.3 & 0 & -0.7 \\ 2 & -4 & 0.1 & 0.05 & 0 \\ 0.3 & 0.1 & 2 & 0.1 & 0.1 \\ 0 & 0.05 & 0.1 & 4 & 0 \\ -0.7 & 0 & 0.1 & 0 & 6 \end{bmatrix}$$

a) symmetric (see problem 6 in HW4)

b) $R_1 = 3$, $R_2 = 2.15$, $R_3 = 0.6$, $R_4 = 0.15$, $R_5 = 0.8$

$a_{11} = -6$ $\quad a_{22} = -4$ $\quad a_{33} = 2$ $\quad a_{44} = 4$, $\quad a_{55} = 6$

$$D = \{ [-9, -1.85], [1.4, 2.6], [3.85, 4.15], [5.2, 6.8] \}$$

c) yes.

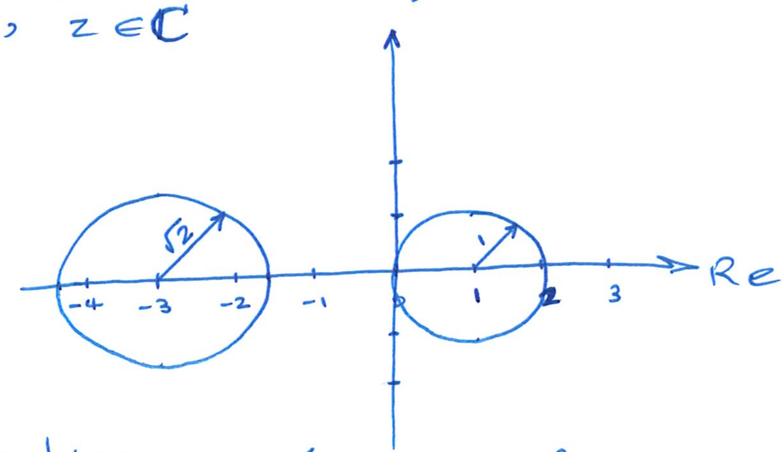
d) $\lambda \neq 0$

②

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1-i & -3 & 0 \\ 0 & 2i & z \end{bmatrix} \in \mathbb{C}^{3\times 3}, \quad z \in \mathbb{C}$$

(a) $a_{11} = 1$, $R_1 = 1$

$a_{22} = -3$, $R_2 = |-1-i| = \sqrt{2}$



(b) disk of z has a radius equal to **2**. If this disk does not overlap with any of the two disks at $a_{11} = 1$ ($R_1 = 1$) and $a_{22} = -3$ ($R_2 = \sqrt{2}$), then it is not possible to have at least two equal eigenvalues.

largest subset of $\mathbb{C}$ that z cannot be in:

$$D = \Big\{ z \in \mathbb{C} : \underbrace{|z-1|}_{\substack{\text{distance}\\ \text{between } z\\ \text{and } a_{11}}} > \underbrace{3}_{R_{11}+2} \text{ and } \underbrace{|z+3|}_{\substack{\text{distance}\\ \text{between } z\\ \text{and } a_{22}}} > \underbrace{2+\sqrt{2}}_{R_{22}+2} \Big\}$$

$\rightsquigarrow$ we require $\underline{z \in \mathbb{C} \setminus D}$

(c) disk of z has to overlap with both **disks**:

$$z \in \mathbb{C} : |z-1| < 3 \text{ and } |z+3| < 2+\sqrt{2}$$

3

(a) a code is already provided for this. See "HW5_Q3.m"

(b) $\vec{x}_0 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \rightsquigarrow \lambda = -6$

$\vec{v} = \begin{bmatrix} 0.707\ldots \\ 0 \\ -0.707\ldots \end{bmatrix}$

$\vec{x}_0 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \rightsquigarrow \lambda = 3$

$\vec{v} = \begin{bmatrix} 0.577\ldots \\ 0.577\ldots \\ 0.577\ldots \end{bmatrix}$

eig(A) by MATLAB:

$\lambda_1 = -6 \qquad \lambda_2 = 0 \qquad \lambda_3 = 3$

$\vec{v}_1 = \begin{bmatrix} 0.707\ldots \\ 0 \\ -0.707\ldots \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 0.4082\ldots \\ -0.8165\ldots \\ 0.4082\ldots \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} -0.577\ldots \\ -0.577\ldots \\ -0.577\ldots \end{bmatrix}$

this initial guess **is** perpendicular to $\vec{v}_1$, so the power method approaches to $\vec{v}_3$ (eigenvector associated with the next largest eigenvalue).

(c) See "HW5_Q3.m"

(d) trivial, knowing the eigenvalues!

④ $A \in \mathbb{R}^{n\times n}$, symmetric with eigenvalues

$$|\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_n|$$

and eigenvectors $\vec{\eta}_1, \vec{\eta}_2, \ldots, \vec{\eta}_n$.

(a) $\vec{x}_K = \sum_{i=1}^{n} c_i \lambda_i^K \vec{\eta}_i$

$$\Rightarrow \vec{x}_K^T \vec{x}_K = \left(\sum_{i=1}^{n} c_i \lambda_i^K \vec{\eta}_i^T\right)\left(\sum_{i=1}^{n} c_i \lambda_i^K \vec{\eta}_i\right) = \sum_{i=1}^{n} c_i^2 \lambda_i^{2K}$$

$$\vec{x}_K^T A \vec{x}_K = \left(\sum_{i=1}^{n} c_i \lambda_i^K \vec{\eta}_i^T\right)\left(\sum_{i=1}^{n} c_i \lambda_i^{K+1} \vec{\eta}_i\right) = \sum_{i=1}^{n} c_i^2 \lambda_i^{2K+1}$$

$$\rightsquigarrow r_K = \frac{\sum_{i=1}^{n} c_i^2 \lambda_i^{2K+1}}{\sum_{i=1}^{n} c_i^2 \lambda_i^{2K}} = \frac{\lambda_1^{2K+1} c_1^2 \left(1 + \sum_{i=2}^{n} \left(\frac{c_i}{c_1}\right)^2 \left(\frac{\lambda_i}{\lambda_1}\right)^{2K} \left(\frac{\lambda_i}{\lambda_1}\right)\right)}{\lambda_1^{2K} c_1^2 \left(1 + \sum_{i=2}^{n} \left(\frac{c_i}{c_1}\right)^2 \left(\frac{\lambda_i}{\lambda_1}\right)^{2K}\right)}$$

$$\rightsquigarrow r_K = \lambda_1 \frac{1 + \sum_{i=2}^{n} \left(\frac{c_i}{c_1}\right)^2 \left(\frac{\lambda_i}{\lambda_1}\right)^{2K} + \sum_{i=2}^{n} \left(\frac{c_i}{c_1}\right)^2 \left(\frac{\lambda_i}{\lambda_1}\right)^{2K} \left(\frac{\lambda_i}{\lambda_1} - 1\right)}{1 + \sum_{i=2}^{n} \left(\frac{c_i}{c_1}\right)^2 \left(\frac{\lambda_i}{\lambda_1}\right)^{2K}}$$

$$\rightsquigarrow r_K = \lambda_1 \left(1 + \underbrace{\frac{\sum_{i=2}^{n} \left(\frac{c_i}{c_1}\right)^2 \left(\frac{\lambda_i}{\lambda_1}\right)^{2K} \left(\frac{\lambda_i}{\lambda_1} - 1\right)}{1 + \sum_{i=2}^{n} \left(\frac{c_i}{c_1}\right)^2 \left(\frac{\lambda_i}{\lambda_1}\right)^{2K}}}_{a_K}\right)$$

$$\chi = a_K \left(\frac{\lambda_1}{\lambda_2}\right)^{2K} = \frac{\sum_{i=2}^{n} \left(\frac{c_i}{c_1}\right)^2 \left(\frac{\lambda_i}{\lambda_2}\right)^{2K} \left(\frac{\lambda_i}{\lambda_1} - 1\right)}{1 + \sum_{i=2}^{n} \left(\frac{c_i}{c_1}\right)^2 \left(\frac{\lambda_i}{\lambda_1}\right)^{2K}}$$

as $K \to \infty$, denominator goes to 1. Now suppose λ_j is the largest eigenvalue that is smaller than λ_2: the sum in the numerator goes to zero for $i = j, \ldots, n$ $\rightarrow$ $\lim_{K\to\infty} \chi = \sum_{i=2}^{j-1} \left(\frac{c_i}{c_1}\right)^2 \left(\frac{\lambda_i}{\lambda_1} - 1\right)$

Note: $|\lambda_2| = |\lambda_3| = \cdots = |\lambda_{j-1}| > |\lambda_j|$

(b) We know that there is an eigenvalue $\lambda \in [20, 22]$. Denote the corresponding eigenvector by $\vec{v}$, i.e., $A\vec{v} = \lambda \vec{v}$.

Assume $\vec{x}_0 \perp \vec{v} \Rightarrow v_5 = 0$ (entry number 5 of $\vec{v}$ has to be zero)

$$\rightsquigarrow \left[\begin{array}{c|c} \underbrace{\bar{A}}_{\in \mathbb{R}^{4\times 4}} & \begin{matrix} * \\ * \\ * \\ * \end{matrix} \\ \hline * \; * \; * \; * & 21 \end{array}\right] \begin{bmatrix} \bar{v} \\ \hline 0 \end{bmatrix} = \lambda \begin{bmatrix} \bar{v} \\ \hline 0 \end{bmatrix} \qquad \underbrace{\vec{v}}_{\in \mathbb{R}^5} = \begin{bmatrix} \underbrace{\bar{v}}_{\in \mathbb{R}^4} \\ 0 \end{bmatrix}$$

$\Rightarrow \bar{A}\bar{v} = \lambda \bar{v} \rightsquigarrow \lambda \in [20, 22]$ is an eigenvalue of $\bar{A}$, which is impossible due to Gerschgorin's theorem.

$\Rightarrow \vec{x}_0$ is not perpendicular to $\vec{v}$

(c) $\alpha_k \sim \left|\frac{\lambda_2}{\lambda_1}\right|^{2k} C \qquad \alpha_{k+5} \sim \left|\frac{\lambda_2}{\lambda_1}\right|^{2k+10} C$

$|\lambda_1| \approx 21$, $|\lambda_2| \approx 9 \rightsquigarrow \left|\frac{\lambda_2}{\lambda_1}\right| \approx \frac{1}{2}$

$\rightsquigarrow$

$$-\log_{10}\left(\frac{\left|\frac{\lambda_2}{\lambda_1}\right|^{2k+10} \cancel{C}\, \cancel{X_1}}{\left|\frac{\lambda_2}{\lambda_1}\right|^{2k} \cancel{C}\, \cancel{X_1}}\right) = -\log_{10}\left(\frac{1}{2}\right)^{10} \approx 3 \text{ digits}$$

5

$$p(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_{n-1} x^{n-1} + x^n$$

(a)

$$A_p - \lambda I = \begin{bmatrix} -\lambda & & & & -\alpha_0 \\ 1 & \ddots & & & -\alpha_1 \\ & \ddots & & & \vdots \\ & & & -\lambda & -\alpha_{n-2} \\ & & & 1 & -\alpha_{n-1} - \lambda \end{bmatrix}$$

$$\det(A_p - \lambda I) = -\alpha_0 + \alpha_1(-\lambda) - \alpha_2(\lambda^2) + \cdots - (\alpha_{n-1} + \lambda)(\lambda^{n-1})$$

$$\Rightarrow \det(A_p - \lambda I) = -\alpha_0 - \alpha_1 \lambda - \alpha_2 \lambda^2 \cdots - \alpha_{n-1}\lambda^{n-1} - \lambda^n \qquad \square$$

(b) $P_w(x) = (x-1)\cdot(x-2)\cdots(x-15)$

Note: MATLAB poly gives the coefficients in the following order:

$$p(1)x^n + p(2)x^{n-1} + \cdots + p(n)x + p(n+1)$$

So, to find $\alpha_0, \ldots, \alpha_{n-1}$, we do the following:

$p = \text{poly}(1:n)$ $\qquad n=15$

$\alpha = p(n+1:-1:2)/p(1) \rightsquigarrow \alpha = [\alpha_0, \alpha_1, \ldots, \alpha_{n-1}]$

See code "HW5_Q5.m"

⑥ see the code "HW5_Q6.m"

⑦ see the code "HW5_Q7.m"

Note: $|\varepsilon < 1$ magnifies the difference between λ_1 and other eigenvalues $\rightarrow$ faster convergence of the power method

⑧

$$L_k(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{x - x_i}{x_k - x_i} \implies L_k(x_i) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$$

(a)

$$\rightsquigarrow \sum_{k=0}^{n} \alpha_k L_k(x) = 0 \rightsquigarrow \text{at } x = x_i: \sum_{k=0}^{n} \alpha_k L_k(x_i) = \alpha_i = 0 \qquad \square$$

(b)

$$\sum_{k=0}^{n} \alpha_k L_k(x) = \sum_{k=0}^{n} \beta_k x^k = \beta_0 + \beta_1 x + \beta_2 x^2 + \dots + \beta_n x^n$$

$$= \begin{bmatrix} 1 & x & x^2 & \dots & x^n \end{bmatrix} \underbrace{\begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_n \end{bmatrix}}_{\vec{\beta}}$$

$x = x_0$

$$\implies \alpha_0 = \begin{bmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \end{bmatrix} \vec{\beta}$$

$x = x_1$

$$\implies \alpha_1 = \begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^n \end{bmatrix} \vec{\beta}$$

$\vdots$

$$\implies \vec{\alpha} = \underbrace{\begin{bmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^n \end{bmatrix}}_{V} \vec{\beta}$$

(c) see code "HW5_Q8.m"