Norms and Condition #'s

Q: How do small errors in the linear system (rounding, measurement, etc.) propagate into the solution $\underline{x}$?

Q: How do we quantify this?

Def: Let $\mathcal{V}(\mathbb{R})$ denote a linear space (vector space) over the real numbers. We say that $\|\cdot\| : \mathcal{V} \to \mathbb{R}_{\geq 0}$ is a norm on $\mathcal{V}$ if:

(i) $\|v\| = 0$ if and only if $v = 0$ in $\mathcal{V}$

(ii) $\|\lambda v\| = |\lambda| \, \|v\|$ for all $\lambda \in \mathbb{R}$ for all $v \in \mathcal{V}$

(iii) $\|u + v\| \leq \|u\| + \|v\|$ for all $u, v \in \mathcal{V}$

We say that $(\mathcal{V}, \|\cdot\|)$ is a normed linear space.

If $\mathcal{V} = \mathbb{R}^n$, we say that $\|\cdot\|$ is a vector norm.

Def: The 1-norm of a vector $\underline{v} \in \mathbb{R}^n$

$$\|\underline{v}\|_1 = \sum_{i=1}^{n} |v_i|,$$

the 2-norm

$$\|\underline{v}\|_2 = \left(\sum_{i=1}^{n} v_i^2\right)^{1/2}$$

the ∞-norm

$$\|\underline{v}\|_\infty = \max_{i=1,\dots,n} |v_i|$$

and the p-norm $(p \geq 1)$

$$\|\underline{v}\|_p = \left(\sum_{i=1}^{n} |v_i|^p\right)^{1/p}$$

All of these are norms. Showing $\|\cdot\|_2$ is a norm requires the following (very important) inequality:

Lemma: (Cauchy-Schwarz)

For all $\underline{u}, \underline{v} \in \mathbb{R}^n$, $|\underline{u}^T \underline{v}| \leq \|\underline{u}\|_2 \|\underline{v}\|_2$.

Pf: Note that for all $\lambda \in \mathbb{R}$, for all $\underline{u}, \underline{v} \in \mathbb{R}^n$,

$$0 \leq \|\lambda \cdot \underline{u} + \underline{v}\|_2^2 = \sum_{i=1}^n (\lambda \cdot u_i + v_i)^2$$

$$= \sum_{i=1}^n (\lambda^2 u_i^2 + 2\lambda u_i v_i + v_i^2)$$

$$= \lambda^2 \underbrace{\left(\sum_{i=1}^n u_i^2\right)}_{a} + \lambda \cdot \underbrace{\left(\sum_{i=1}^n 2 \cdot u_i v_i\right)}_{b} + \underbrace{\sum_{i=1}^n v_i^2}_{c}$$

Note: non-negative polynomial in λ.

$$\Rightarrow \quad b^2 - 4ac \leq 0$$

$$\Rightarrow \quad \cancel{4} \cdot \left(\sum_{i=1}^n u_i v_i\right)^2 - 4\left(\sum_{i=1}^n u_i^2\right)\left(\sum_{i=1}^n v_i^2\right) \leq 0$$

$$\Rightarrow \quad \underline{u}^T \underline{v} \leq \|\underline{u}\| \|\underline{v}\| \qquad \square$$

Let us now prove the triangle inequality for $\|\cdot\|_2$.

Lemma: For all $\underline{u}, \underline{v} \in \mathbb{R}^n$, $\|\underline{u} + \underline{v}\| \leq \|\underline{u}\| + \|\underline{v}\|$.

Pf: $\|\underline{u} + \underline{v}\|_2^2 = \|\underline{u}\|_2^2 + 2\underline{u}^T \underline{v} + \|\underline{v}\|_2^2$

Cauchy - Schwarz

$$\leq \|\underline{u}\|_2^2 + 2\|\underline{u}\| \|\underline{v}\| + \|\underline{v}\|_2^2$$

$$= \left(\|\underline{u}\|_2 + \|\underline{v}\|_2 \right) \qquad \square$$

Claim: $\lim_{p \to \infty} \|\underline{u}\|_p = \|\underline{u}\|_\infty$ for all $\underline{u} \in \mathbb{R}^n$.

Pf: Let $\underline{\tilde{u}} = \left(\|\underline{u}\|_\infty \right)^{-1} \underline{u}$. Then $|\tilde{u}_i| \le 1$ for all i, so that:

$$1 \le \|\underline{\tilde{u}}\|_p \le n^{1/p}$$

$$\Rightarrow \lim_{p \to \infty} \|\underline{\tilde{u}}\|_p = 1$$

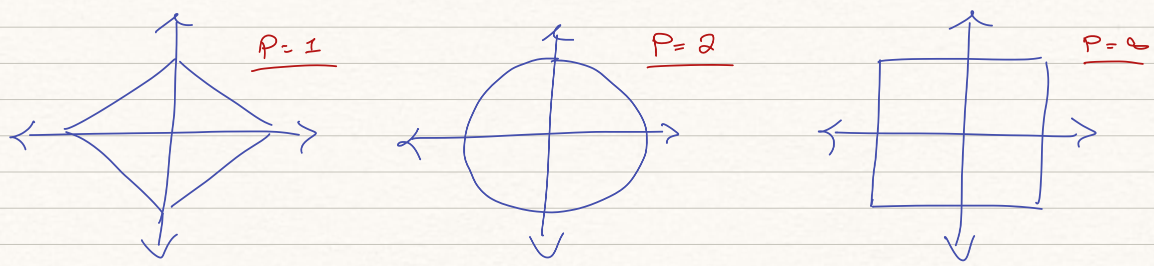
$$= \frac{\lim_{p \to \infty} \|\underline{u}\|_p}{\|\underline{u}\|_\infty} = 1$$

$$\Rightarrow \lim_{p \to \infty} \|\underline{u}\|_p = \|\underline{u}\|_\infty \qquad \square$$

Ex: Consider the "unit ball" in a given norm:

$$B_1^p \triangleq \{ \underline{x} \in \mathbb{R}^n \mid \|\underline{x}\|_p \le 1 \}$$

$n=2$



Δ

Note: We can also consider norms on matrices. Particularly important are the "induced vector norms".

Def: Given a norm on vectors $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$, the induced matrix norm is defined by

$$\|\underline{\underline{A}}\| = \max_{\underline{v}\in\mathbb{R}^n} \frac{\|\underline{\underline{A}}\,\underline{v}\|}{\|\underline{v}\|} \qquad (\underline{v}\neq 0)$$

Note: By definition, for any $\underline{v}\in\mathbb{R}^n$, $\underbrace{\|\underline{\underline{A}}\,\underline{v}\|}_{\text{v. norm}} \le \underbrace{\|\underline{\underline{A}}\|}_{\text{m. norm}}\,\underbrace{\|\underline{v}\|}_{\text{v. norm}}$

Note: For every vector norm, $\|\underline{\underline{I}}\| = 1$.

Note: In practice, computing $\|\underline{\underline{A}}\|$ for a matrix $\underline{\underline{A}}$ via the def. is challenging. The following results fix this.

Thm: The following equalities hold for $\underline{\underline{A}}\in\mathbb{R}^{n\times n}$,

(a) $$\|\underline{\underline{A}}\|_\infty = \max_{i=1,\dots,n} \sum_{j=1}^{n} |a_{ij}|,$$

(b) $$\|\underline{\underline{A}}\|_1 = \max_{j=1,\dots,n} \sum_{i=1}^{n} |a_{ij}|,$$

(c) $$\|\underline{\underline{A}}\|_2 = \lambda_{max}(\underline{\underline{A}}^T\underline{\underline{A}})^{1/2}$$

Pf:

(a) $$|(\underline{\underline{A}}\,\underline{v})_i| = \left|\sum_{j=1}^{n} a_{ij} v_j\right| \le \sum_{j=1}^{n} |a_{ij}|\,|v_j|$$

$$\le \|\underline{v}\|_\infty \sum_{j=1}^{n} |a_{ij}|$$

$$\Rightarrow \frac{|(A v)_i|}{\|\underline{v}\|_\infty} \le \sum_{j=1}^{n} |a_{ij}|$$

$$\Rightarrow \frac{\|\underline{\underline{A}}\,\underline{v}\|_\infty}{\|v\|} \le \max_{i=1,\dots,n} \sum_{j=1}^{n} |a_{ij}|$$

$\|v\|_\infty$ ...

Now, let i^* be the maximizer. Let $v_j^* = \text{sign}(a_{i^*j})$.

Then,

because of max

$$\|A\|_\infty = \max_{v} \frac{\|Av\|_\infty}{\|v\|_\infty} \geq \frac{\|Av^*\|_\infty}{\underbrace{\|v^*\|_\infty}_{=1}}$$

$$= \max_{i=1,\dots,n} |(Av^*)_i|$$

$$= \max_{i=1,\dots,n} \left| \sum_{j=1}^{n} a_{ij} \,\text{sign}(a_{i^*j}) \right|$$

because of max

$$\geq \left| \sum_{j=1}^{n} a_{i^*j} \,\text{sign}(a_{i^*j}) \right|$$

$$= \sum_{j=1}^{n} |a_{i^*j}|$$

$$= \max_{i=1,\dots,n} \sum_{j=1}^{n} |a_{i^*j}|$$

(b) Similar.

(c) Note: $(A^TA)^T = A^TA$ (symmetric matrix)

Recall: symmetric matrices have real eigenvalues and an orthonormal basis of eigenvectors.

Note: $A^TAv = \lambda v$

$$\Rightarrow \lambda = (v^TA^TAv) / (v^Tv)$$

$$= \|\underline{\underline{A}}\,\underline{v}\|_2 \,/\, \|\underline{v}\|_2 \geq 0$$

Let $\{\underline{w}_i\}_{i=1}^n$ be an orthonormal basis of eigenvectors.

Then for any $\underline{u} \in \mathbb{R}^n$, $\underline{u} = \sum_{i=1}^n c_i \underline{w}_i$ for some $\{c_i\}$.

Then $\underline{\underline{A}}^T \underline{\underline{A}}\,\underline{u} = \sum_{i=1}^n c_i \lambda_i \underline{w}_i$.

$$\Rightarrow \underline{u}^T \underline{\underline{A}}^T \underline{\underline{A}}\,\underline{u} = \|\underline{\underline{A}}\,\underline{u}\|^2$$

orthonormality

$$= \sum_{i=1}^n c_i^2 \lambda_i$$

$$\leq \lambda_{max}(\underline{\underline{A}}^T \underline{\underline{A}}) \underbrace{\sum_{i=1}^n c_i^2}_{= \|\underline{u}\|_2^2}$$

$$\Rightarrow \frac{\|\underline{\underline{A}}\,\underline{u}\|_2^2}{\|\underline{u}\|_2^2} \leq \lambda_{max}(\underline{\underline{A}}^T \underline{\underline{A}})$$

For equality, set $\underline{u} = \underline{w}_{i^*}$ with $\lambda_{i^*} = \lambda_{max}$. $\square$