We now give sufficient conditions for convergence of the simple iter.

Def: Let $g \in C^0([a,b],[a,b])$. ← $g(x) \in [a,b]$ for $x \in [a,b]$

We say g is a contraction if $\exists\ 0 < L < 1$ with

$$|g(x) - g(y)| \leq L\,|x - y| \quad \forall\, x, y \in [a,b]$$

Thm (Contraction Mapping):

Let $g \in C^0([a,b],[a,b])$ and let g be a contraction.

Then g has a unique fixed point $\xi \in [a,b]$.

Moreover, (I) converges to ξ for all $x_0 \in [a,b]$.

Pf: By the previous theorem, a fixed point exists.

To see uniqueness, suppose there exists η with $g(\eta) = \eta$.

Then, $|\eta - \xi| = |g(\eta) - g(\xi)| \leq L\,|\xi - \eta|$

$$\Rightarrow (1 - L)\,|\eta - \xi| \leq 0$$

But $L \neq 1$, so $\eta = \xi$.

Now, let $x_0 \in [a,b]$. Then,

$$\begin{aligned} |x_k - \xi| &= |g(x_{k-1}) - g(\xi)| \\ &\leq L\,|x_{k-1} - \xi| \\ &\leq L^k\,|x_0 - \xi| \to 0 \text{ as } k \to \infty \quad (L \in (0,1)) \end{aligned}$$

So $|x_k - \xi| \to 0$ and $x_k \to \xi$. ☐

Accuracy

Q: How many iterations do we need to run to have the correct answer up to k digits of accuracy?

A: The following theorem.

Thm: Let $g \in C^0([a,b],[a,b])$ be a contraction mapping.

Let $\varepsilon > 0$ denote a desired accuracy.

Let $k_0(\varepsilon)$ denote the first k such that $|x_k - \xi| \le \varepsilon$.

Then,

$$k_0(\varepsilon) \le \left[\frac{\log|x_1 - x_0| - \log(\varepsilon(1-L))}{\log(1/L)} \right] + 1$$

<u>Pf</u>: From earlier, we know that

$$(*) \quad |x_k - \xi| \le L^k |x_0 - \xi|, \quad k \ge 1.$$

For $k = 1$,

$$|x_0 - \xi| = |x_0 \underbrace{- x_1 + x_1}_{=0} - \xi|$$

triangle inequality

$$\le |x_0 - x_1| + |x_1 - \xi|$$

$$\le |x_0 - x_1| + L|x_0 - \xi|$$

$$\Rightarrow \quad |x_0 - \xi| \le |x_0 - x_1| / (1-L)$$

Plugging into $(*)$,

$$|x_k - \xi| \le L^k |x_0 - x_1| / (1-L)$$

So $|x_k - \xi| \le \varepsilon$ if we have

$$\frac{L^k |x_0 - x_1|}{(1-L)} \le \varepsilon$$

Solving for k, we find

$$k \ge \frac{\log|x_1 - x_0| - \log(\varepsilon(1-L))}{\log(1/L)}$$

So $k_0(\varepsilon)$ (the first such k) cannot exceed this lower bound. $\square$

Note: As $L \to 1$, this bound diverges!

As $\varepsilon \to 0$, the bound diverges!

Why?

EX: $f(x) = e^x - 2x - 1$, $x \in [1,2]$

Roots: $f(1) = e - 3 < 0$, $f(2) = e^2 - 7 > 0$

Hence, $\exists\, \xi \in [1,2]$, $f(\xi) = 0$

FP: $f(\xi) = 0 \Rightarrow \xi = \underbrace{\log(2\xi + 1)}_{\triangleq\, g(\xi)}$

Note: $g(1) \in [1,2]$, $g(2) \in [1,2]$

also, g monotonic increasing $\Rightarrow g(x) \in [1,2]\ \forall\, x \in [1,2]$

$\therefore \exists\, \xi$, $\xi \in [1,2]$, $g(\xi) = \xi$.

Contraction: g continuous on $[1,2]$, differentiable on $(1,2)$.

Mean Value Thm: $\forall\, x, y$, $\exists\, \eta \in [x,y]$ with

$$|g(x) - g(y)| = |g'(\eta)(x-y)| = |g'(\eta)|\,|x-y|$$

$$g'(x) = \frac{2}{2x+1}, \quad g''(x) = \frac{-4}{(2x+1)^2}$$

Note $g''(x) \le 0$ for all $x \in [1,2]$, so $g'(x)$ decreasing

Then $g'(1) \geq g'(\eta) \geq g'(2)$ for all $\eta \in [1,2]$

And hence $|g'(x)| \in [2/5, 2/3]$ for $x \in [1,2]$

Then,

$$|g(x) - g(y)| \le \underbrace{2/3}_{= L} \cdot |x - y| \qquad \text{(Contraction!)}$$

Simple Iteration: $x_{k+1} = \log(2x_k + 1)$, $k = 0, 1, 2, \ldots$

Converges to ξ for any initial condition.

Q: How many iterations for 6 digits of accuracy?

A: set $\varepsilon = 0.5 \times 10^{-6}$. $L = 2/3 \Rightarrow K_0(\varepsilon) \leq 33$ Δ