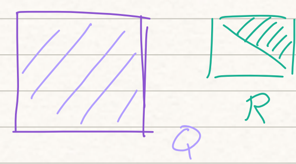
QR Factorization

We proceed to solve $\min_{x} \|Ax - b\|_2^2$ by factorization.

Thm: Let $A \in \mathbb{R}^{m\times n}$ where $m \geq n$. Then, there exists an upper triangular matrix $R \in \mathbb{R}^{n\times n}$ and an orthogonal matrix $Q \in \mathbb{R}^{m\times n}$ ($Q^T Q = I$) such that

$$A = QR.$$



Moreover, if $\operatorname{rank}(A) = n$, then R is nonsingular.

Pf: By induction on n.

$n=1$: Then $A = c \in \mathbb{R}^m$. Write $Q = c/\|c\|_2$, $R = \|c\|_2$.

$n = k+1$ (assume true for $n \leq k$, and take $k < m$)

Write $A = (A_k, a)$ for $A \in \mathbb{R}^{m\times(k+1)}$, $A_k \in \mathbb{R}^{m\times k}$, $a \in \mathbb{R}^m$

We want $Q = (Q_k, q)$ and

$$R = \begin{pmatrix} R_k & r \\ 0 & \tau \end{pmatrix} \qquad R_k \in \mathbb{R}^{k\times k} \text{ U.T.}, \quad r \in \mathbb{R}^k, \ \tau \in \mathbb{R}$$

and we need $(A_k \ a) = (Q_k \ q)\begin{pmatrix} R_k & r \\ 0 & \tau \end{pmatrix}$

$$\Rightarrow \left.\begin{aligned} A_k &= Q_k R_k \\ a &= Q_k r + \tau q \end{aligned}\right\} A = QR$$

$Q_k^T Q_k = I_k$

$$\left.\begin{array}{l} \underline{q}^T \underline{\underline{Q}}_k = \underline{0} \\ \underline{q}^T \underline{q} = 1 \end{array}\right\} \quad \underline{\underline{Q}}^T \underline{\underline{Q}} = \underline{\underline{I}}$$

By induction, $\underline{\underline{A}}_k = \underline{\underline{Q}}_k \underline{\underline{R}}_k$ exists.

Solving gives $\underline{r} = \underline{\underline{Q}}_k^T \underline{a}$ (using $\underline{q}^T \underline{\underline{Q}}_k = (\underline{\underline{Q}}_k^T \underline{q})^T = 0$)

$$\underline{q} = 1/r(\underline{a} - \underline{\underline{Q}}_k \underline{\underline{Q}}_k^T \underline{a})$$

$$r = \|\underline{a} - \underline{\underline{Q}}_k \underline{\underline{Q}}_k^T \underline{a}\|_2 \quad \text{so} \quad \|\underline{q}\| = 1.$$

Note: fails for $\underline{a} - \underline{\underline{Q}}_k \underline{\underline{Q}}_k^T \underline{a} = \underline{0}$, because $r = 0$.

Then, choose $\underline{q} \in \mathbb{R}^m$ <u>arbitrarily</u> with $\underline{\underline{Q}}_k^T \underline{q} = 0$.

<u>Note</u>: Require $k < m$ so $\underline{\underline{Q}}_k$ not square, otherwise $\underline{\underline{Q}}_k^T \underline{q} = \underline{0}$ not possible!

Last, assume $\operatorname{rank}(\underline{\underline{A}}) = n$. If $\underline{\underline{R}}$ were singular, $\exists\, \underline{p} \in \mathbb{R}^n$ so that $\underline{\underline{Q}}\,\underline{\underline{R}}\,\underline{p} = \underline{0}$. But then $\underline{\underline{A}}\underline{p} = 0$ ($\underline{\underline{A}} = \underline{\underline{Q}}\,\underline{\underline{R}}$), so that $\underline{\underline{A}}$ couldn't have rank n. $\square$

Let's now apply this to least squares problems.

<u>Thm</u>: Let $\underline{\underline{A}} \in \mathbb{R}^{m \times n}$, $\operatorname{rank}(\underline{\underline{A}}) = n$, $m \geq n$. Let $\underline{b} \in \mathbb{R}^m$.

Then, there exists a unique least-squares solution of the equations $\underline{\underline{A}}\,\underline{x} = \underline{b}$ given by the solution of

$$\underline{\underline{R}}\,\underline{x} = \underline{\underline{Q}}^T \underline{b}$$

where $\underline{\underline{A}} = \underline{\underline{Q}}\,\underline{\underline{R}}$.

<u>Pf</u>: Say $m = n$. Then $\underline{x} = \underline{\underline{A}}^{-1} \underline{b} = (\underline{\underline{Q}}\,\underline{\underline{R}})^{-1} \underline{b} = \underline{\underline{R}}^{-1} \underline{\underline{Q}}^T \underline{b}$

Now say $m > n$. Write

$b = b_q + b_r$ for $b_q \in \operatorname{Range}(Q)$, orthogonal comp…

$$b_r \in \text{Range}(Q)^{\perp}$$ rlement

i.e., $b_q = Qc$ for some $c \in \mathbb{R}^n$,

$Q^T b_r = 0$.

Now, say that $Rx = Q^T b$, and let $y \in \mathbb{R}^n$ be arbitrary.

$$
\begin{aligned}
Ay - b &= QRy - b \\
&= QR(y-x) + QRx - b \\
&= QR(y-x) + Q(Q^T b) - b \\
&= QR(y-x) + \underbrace{Q(Q^T b_q)}_{= QQ^T Qc = Qc = b_q} - b \\
&= QR(y-x) + b_q - b \\
&= QR(y-x) - b_r
\end{aligned}
$$

So that:

$$
\begin{aligned}
\|Ay - b\|_2^2 &= \underbrace{\|R(y-x)\|_2^2}_{\geq 0} + \|b_r\|^2 - \underbrace{2 b_r^T QR(y-x)}_{=0} \\
&\geq \|b_r\|_2^2
\end{aligned}
$$

So $\|Ay-b\|_2^2$ smallest when $R(y-x) = 0 \iff y = x$. $\square$