Note: Example above suggests the following sufficient condition.

By mean value theorem,

$$\forall x \neq y, \quad \frac{|g(x)-g(y)|}{|x-y|} = |g'(\eta)| \quad \text{for some } \eta \in [x,y]$$

So for g differentiable on (a,b), if $0 < |g'(x)| < 1$ for all $x \in (a,b)$, then g is a contraction!

We now consider a local version of the contraction mapping theorem.

Thm: Let $g \in C^0([a,b],[a,b])$.

Let $\xi = g(\xi)$, $\xi \in [a,b]$ denote a fixed point.

Let g have a continuous derivative in a neighborhood around ξ w/ $|g'(x)| < 1$.

Then the sequence $\{x_k\}_{k=0}^{\infty}$ with $x_{k+1} = g(x_k)$ converges to ξ as $k \to \infty$ as long as x_0 is sufficiently close to ξ.

Pf: (Sketch) By assumption, g is a contraction locally around ξ. Take x_k to be inside this contraction region I.

Then,
$$x_{k+1} - \xi = g(x_k) - g(\xi) = g'(\eta)\cdot(x_k - \xi)$$
with $\eta \in [x_k, \xi]$.

Because $x_k \in I$ and $\xi \in I$, $\eta \in I$. Hence $|g'(\eta)| \leq L$ with $L \in (0,1)$. Then,

$$|x_{k+1} - \xi| \leq L \cdot |x_k - \xi|$$

So $x_{k+1} \in I$ as well. Moreover,

$$|x_k - \xi| \leq L^k |x_0 - \xi|$$

Proof same as global result from here. $\square$

Q: What if contraction conditions are not satisfied?

Def: Let $g \in C^0([a,b],[a,b])$, let ξ denote a fixed point.

We say ξ is a stable fixed point if $x_k \to \xi$ whenever x_0 is sufficiently close to ξ.

We say ξ is unstable if no x_0 gives $x_k \to \xi$ except for $x_0 = \xi$.

Q: At what rate does $x_k \to \xi$ for a stable fixed point?

$$\lim_{k \to \infty} \frac{|x_{k+1} - \xi|}{|x_k - \xi|} = \lim_{k \to \infty} \left| \frac{g(x_k) - g(\xi)}{x_k - \xi} \right| = g'(\xi)$$

So we can think of $g'(\xi)$ as providing a local measure of a convergence rate (which becomes true asymptotically).

Def: We say a sequence converges linearly if

$$\exists \mu \in (0,1), \lim_{k \to \infty} \frac{|x_{k+1} - \xi|}{|x_k - \xi|} = \mu$$

if $\mu = 0$, we say superlinear convergence.

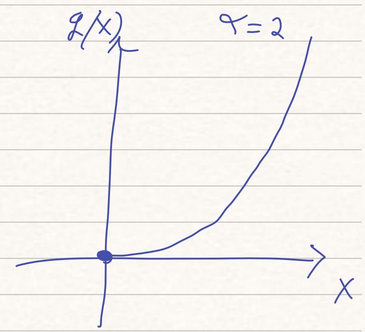
if $\mu = 1$, we say sublinear convergence.

Def: We say $x_k \to \xi$ at least linearly if $|x_k - \xi| \leq \varepsilon_k$ where $\varepsilon_k \to 0$ linearly.

Def: We call $\rho \triangleq -\log_{10}(\mu)$ the asymptotic rate of convergence of the sequence. ρ measures the # of digits of accuracy gained from each iteration.

$$(\; 2^{-(1+\log_2(1/x)^{1/q})^{\tau}} \qquad 0 < x \leq 1 \qquad q > 0$$

$\underline{\text{Ex}}$: $g(x) = \begin{cases} 0 & x = 0 \end{cases}$



Note: $\lim_{x \to 0^+} g(x) = 0$, so g is continuous.

g monotonic increasing on $[0,1]$

$g(x) \in [0, 1/2]$ for $x \in [0,1]$

$g(0) = 0$ (fixed point)

Simple iteration: $x_{k+1} = g(x_k)$ $\quad x_0 = 1 = 2^{-0^\alpha}$

$x_1 = 2^{-1^\alpha}$

$x_2 = 2^{-(1 + \log_2(2)^{1/\alpha})^\alpha} = 2^{-2^\alpha}$

$\vdots$

$x_k = 2^{-k^\alpha} \to \xi = 0$ as $k \to \infty$.

Now, $\lim_{k \to \infty} \dfrac{|x_{k+1} - \xi|}{|x_k - \xi|} = \dfrac{|x_{k+1}|}{|x_k|} = \dfrac{2^{-(k+1)^\alpha}}{2^{-k^\alpha}}$

$= 2^{-((k+1)^\alpha - k^\alpha)}$

$= \begin{cases} 1 & 0 < \alpha < 1 & \text{Sublinear} \\ 1/2 & \alpha = 1 & \text{linear} \\ 0 & \alpha > 1 & \text{Superlinear} \end{cases}$ $\qquad \triangle$

$\underline{\text{Q}}$: what about unstable fixed points?

$\underline{\text{Thm}}$: Let $g \in C^1([a,b], [a,b])$, say $\xi = g(\xi)$.
Let $|g'(\xi)| > 1$. Then $x_{k+1} = g(x_k)$ does not converge to ξ for $\underline{\text{any}}$ $x_0 \neq \xi$.

$\underline{\text{Pf}}$: say $x_0 \neq \xi$. By continuity, $\exists$ an interval I around ξ with $|g'(x)| > 1$

for all $x \in \xi$. Let $\min_{x \in I} |g'(x)| \triangleq L \geq 1$

For $x_k \in I$, $|x_{k+1} - \xi| = |g(x_k) - g(\xi)|$

$$= |(x_k - \xi) g'(\eta_k)| \quad (\eta_k \in [x_k, \xi])$$
$$\geq L |x_k - \xi|.$$

Similarly, if $x_{k+1} \in I$ still,

$$|x_{k+2} - \xi| \geq L^2 |x_k - \xi|$$

After some finite # of steps, the sequence must leave I because $L > 1$. Hence, sequence cannot converge. $\square$

Note: See book for examples of unstable fixed points, asymptotic conv.

(Ex. 1.5, Ex. 1.6, Sec. 1.3)

Newton's Method

We now return to the root-finding problem:

(P) Find $\xi \in \mathbb{R}$ s.t. $f(\xi) = 0$ for $f : \mathbb{R} \to \mathbb{R}$

We look for more systematic approaches.

Def: Let $f : \mathbb{R} \to \mathbb{R}$ be continuous around $\xi \in \mathbb{R}$.

Relaxation uses the sequence

$$x_{k+1} = x_k - \lambda f(x_k), \quad k = 0, 1, \ldots, \quad \lambda \neq 0$$

Note: This is a simple iteration w/ $g(\cdot) = \cdot - \lambda f(\cdot)$

Note: $g'(x) = 1 - \lambda f'(x)$. This let's us "tune" the Lipschitz constant.

Thm: Let $f : \mathbb{R} \to \mathbb{R}$ be continuous around ξ with $f(\xi) = 0$.

Let f' be defined and continuous around ξ with $f'(\xi) \neq 0$.

Then, $\exists \lambda > 0, \delta > 0$ such that relaxation will converge for any $x_0 \in [\xi - \delta, \xi + \delta]$.

Pf: Let $f'(\xi) \triangleq \gamma$ and assume $\gamma > 0$ WLOG.

Because f' cts. around ξ, $\exists \delta$ such that $f'(x) \geq \gamma/2$ for all $x \in \underbrace{[\xi - \delta, \xi + \delta]}_{= I}$. Let $M \triangleq \max_{x \in I} f'(x)$.

Then, for all $x \in I$, for $\lambda > 0$,

$$1 - \lambda M \leq 1 - \lambda f'(x) \leq 1 - \lambda \gamma/2$$

We solve for λ so that

$$1 - \lambda M = -\vartheta, \quad 1 - \lambda \gamma/2 = \vartheta$$

$$\Rightarrow \begin{cases} \vartheta = \dfrac{2M - \gamma}{2M + \gamma} < 1, \\ \lambda = 4/(2M + \gamma) \end{cases}$$

Then, $g(x) = x - \lambda f(x)$ has $|g'(x)| \leq \vartheta$ for $x \in I$, and so is a local contraction mapping around ξ. $\square$

Q: What if we allow λ to depend on x?

$$x_{k+1} = x_k - \lambda(x_k) f(x_k)$$

Note: If $x_k \to \xi$, we have

$$\xi = \xi - \lambda(\xi) f(\xi)$$

$$\Rightarrow f(\xi) = 0 \quad \text{if } \lambda(\xi) \neq 0.$$

Rate of convergence given by $g'(\xi)$. But:

$$g'(\xi) = 1 - \lambda(\xi) f'(\xi) - \lambda'(\xi) \underbrace{f(\xi)}_{= 0}$$

$$= 1 - \lambda(\xi) f'(\xi)$$

So we should set $\lambda(x)$ so that $g'(\xi)$ small.

Def: Newton's method for the solution of $f(x)=0$ with $f:\mathbb{R}\to\mathbb{R}$ is given by

$$x_{k+1} = x_k - f(x_k)/f'(x_k), \quad k=0,1,2,\dots$$

we assume that $f'(x_k) \neq 0$ for all x_k.

Geometry: