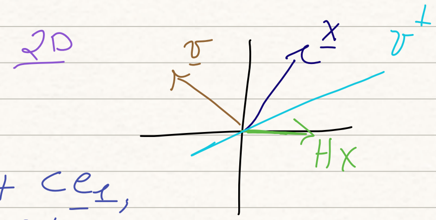
Lemma: For all $\underline{x} \in \mathbb{R}^n$, there exists a Householder reflector $\underline{\underline{H}} \in \mathbb{R}^{n \times n}_{sym}$ such that $\underline{\underline{H}}\underline{x} = \lambda \cdot \underline{e}_1$, i.e., all components but the first are nonzero.



Pf: Recall that $\underline{v}$, $\underline{x}$, $\underline{\underline{H}}\underline{x}$ are coplanar

We want $\underline{\underline{H}}\underline{x} = \lambda \cdot \underline{e}_1$, so take $\underline{v} = \underline{x} + c\underline{e}_1$, $c \neq 0$.

Then $\underline{\underline{H}} = \underline{\underline{I}} - \left(2/\|\underline{v}\|^2\right) \underline{v}\,\underline{v}^T$.

$$\underline{v}^T \underline{x} = \|\underline{x}\|^2 + c x_1$$

$$\underline{v}^T \underline{v} = \|\underline{x}\|^2 + 2 c x_1 + c^2$$

So that:

$$\underline{\underline{H}}\underline{x} = \underline{x} - \left(\frac{2\left(\|\underline{x}\|^2 + c x_1\right)}{\|x\|^2 + 2 c x_1 + c^2}\right)\left(\underline{x} + c\underline{e}_1\right)$$

$$= \left(\frac{\|x\|^2 + \cancel{2cx_1} + c^2 - 2\|x\|^2 - \cancel{2cx_1}}{\|x\|^2 + 2 c x_1 + c^2}\right) \underline{x}$$

$$- \left(\frac{2\left(\|x\|^2 + c x_1\right)}{\|x\|^2 + 2 c x_1 + c^2}\right) c\underline{e}_1$$

$$= \left(\frac{c^2 - \|x\|^2}{\|x\|^2 + 2 c x_1 + c^2}\right) \underline{x}$$

$$- \left(\frac{2\left(\|x\|^2 + c x_1\right)}{\|x\|^2 + 2 c x_1 + c^2}\right) c\underline{e}_1$$

So, will be a multiple of $\underline{e}_1$ if:

$$c = \|x\|.$$

$$\underbrace{\|x\|^2 + 2cx_1 + c^2}\neq 0$$

$$= (x_1+c)^2 + \underbrace{\|x\|^2 - x_1^2}_{\geq 0} \geq (x_1+c)^2 \geq 0$$

as long as $x_1 + c \neq 0$, i.e., $c = \begin{cases} \operatorname{sign}(x_1)\,\|x\| & x_1 \neq 0 \\ \|x\| & x_1 = 0 \end{cases}$ $\square$

Ex:

$\underline{x} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ want $\underline{v} = \underline{x} + \operatorname{sign}(x_1)\|x\| \cdot \underline{e}_1$

$\|\underline{x}\| = (1+4+4)^{1/2} = 3$, $\operatorname{sign}(x_1) = 1$

So $\underline{v} = (4\ \ -2\ \ 2)^T$

Now,

$$\underline{\underline{H}}(\underline{v}) = \underline{\underline{I}} - 2\frac{\underline{v}\underline{v}^T}{\|\underline{v}\|_2^2}$$

$$\|\underline{v}\|_2^2 = 16 + 4 + 4 = 24$$

$$(\underline{v}\underline{v}^T)_{ij} = v_i v_j \Rightarrow \underline{v}\underline{v}^T = \begin{pmatrix} 16 & -8 & 8 \\ -8 & 4 & -4 \\ 8 & -4 & 4 \end{pmatrix}$$

So:

$$\frac{-2}{\underline{v}^T\underline{v}}\underline{v}\underline{v}^T = -1/12 \begin{pmatrix} 16 & -8 & 8 \\ -8 & 4 & -4 \\ 8 & -4 & 4 \end{pmatrix}$$

$$= -\begin{pmatrix} 4/3 & -2/3 & 2/3 \\ -2/3 & 1/3 & -1/3 \\ 2/3 & -1/3 & 1/3 \end{pmatrix}$$

$$\underline{\underline{H}}(\underline{v}) = \underline{\underline{I}} - \begin{pmatrix} 4/3 & -2/3 & 2/3 \\ -2/3 & 1/3 & -1/3 \\ 2/3 & -1/3 & 1/3 \end{pmatrix}$$

$$\Rightarrow \underline{\underline{H}}(\underline{v})\underline{x} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} - \begin{pmatrix} 4/3 & -2/3 & 2/3 \\ -2/3 & 1/3 & -1/3 \\ 2/3 & -1/3 & 1/3 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} - \begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ 0 \end{pmatrix} \checkmark \qquad \Delta$$

Computing QR

Thm: Let $\underline{\underline{A}} \in \mathbb{R}^{m \times n}$. Then there exists a sequence of Householder matrices $\underline{\underline{H}}_i$, $i = 1, .., n$ $\underline{\underline{H}}_i \in \mathbb{R}^{m \times m}$ so that

$$\underline{\underline{H}}_n \underline{\underline{H}}_{n-1} \cdots \underline{\underline{H}}_1 \underline{\underline{A}} = \underline{\underline{R}} \in \mathbb{R}^{m \times n}$$

with $\underline{\underline{R}}$ upper triangular. Moreover,

$$\underline{\underline{Q}} = \underline{\underline{H}}_1 \cdots \underline{\underline{H}}_n \in \mathbb{R}^{m \times m} \quad \text{is orthogonal.}$$

Note:

$$\underline{\underline{A}} = \underline{\underline{Q}}\,\underline{\underline{R}} = \begin{pmatrix} | & | & & | \\ \underline{q}_1 & \underline{q}_2 & \cdots & \underline{q}_m \\ | & | & & | \end{pmatrix} \begin{pmatrix} \hat{\underline{\underline{R}}} \\ \hline \underline{\underline{0}} \end{pmatrix} \qquad \hat{\underline{\underline{R}}} \in \mathbb{R}^{n \times n}$$

Called the full QR factorization. Earlier, we saw

$$\underline{\underline{A}} = \hat{\underline{\underline{Q}}}\,\hat{\underline{\underline{R}}}, \quad \hat{\underline{\underline{Q}}} = \begin{pmatrix} | & | & & | \\ \underline{q}_1 & \underline{q}_2 & \cdots & \underline{q}_m \\ | & | & & | \end{pmatrix}, \quad \underline{\underline{Q}} = \begin{pmatrix} & | & & | \\ \hat{\underline{\underline{Q}}} & \underline{q}_{n+1} & \cdots & \underline{q}_m \\ & | & & | \end{pmatrix}$$

called the reduced QR factorization.

Pf: Write

$$\underline{\underline{A}} = \begin{pmatrix} | & \\ \underline{a}_1 & B \\ | & \end{pmatrix}, \qquad \underline{\underline{A}}_1 \in \mathbb{R}^{m \times (n-1)}$$

From earlier, there exists $\underline{\underline{H}}_1$ with $\underline{\underline{H}}_1 \underline{a}_1 = -\|\underline{a}_1\| \underline{e}_1$.

Then,

$$\underline{\underline{H}}_1 \underline{\underline{A}} = \begin{pmatrix} \|\underline{a}_1\| & \\ \underline{0}_{m-1} & \underline{\underline{H}}_1 \underline{\underline{B}} \end{pmatrix}$$

Now, write $\quad , \quad \quad {}^{m-1}$

$$\underline{\underline{H}}_1 \underline{\underline{B}} = \begin{pmatrix} & b & \\ \underline{b}_1 & & \underline{\underline{C}} \end{pmatrix} \qquad \begin{array}{l} \underline{b}_1 \in \mathbb{R} \\ \underline{\underline{C}} \in \mathbb{R}^{m\times(n-2)} \end{array}$$

There exists $\underline{\underline{H}}_2' \in \mathbb{R}^{(m-1)\times(m-1)}$ so that $\underline{\underline{H}}_2' \underline{b}_1 = \pm\|\underline{b}_1\|_2 \underline{e}_1$.

Define the extension:

$$\underline{\underline{H}}_2 = \begin{pmatrix} I & \underline{0}^T \\ \underline{0} & \underline{\underline{H}}_2' \end{pmatrix} \in \mathbb{R}^{m\times m}$$

Then,

$$\underline{\underline{H}}_2 \underline{\underline{H}}_1 \underline{\underline{A}} = \begin{pmatrix} I & \underline{0}^T \\ \underline{0} & \underline{\underline{H}}_2' \end{pmatrix} \begin{pmatrix} \pm\|a\| & b & \\ \underline{0}_{m-1} & \underline{b}_1 & \underline{\underline{C}} \end{pmatrix}$$

$$= \begin{pmatrix} \pm\|a\| & b & \\ 0 & \pm\|b\| & \underline{\underline{D}} \\ \underline{0}_{m-2} & \underline{0}_{m-2} & \end{pmatrix} \qquad \begin{array}{l} \underline{\underline{D}} \in \mathbb{R}^{m\times n-2} \\ \underline{\underline{D}} = \underline{\underline{H}}_2 \underline{\underline{C}} \end{array}$$

Proceeding recursively in this way, we have

$$\underline{\underline{H}}_n \cdots \underline{\underline{H}}_1 \underline{\underline{A}} = \underline{\underline{R}}$$

Now, then $\underline{\underline{A}} = \underbrace{\underline{\underline{H}}_1 \cdots \underline{\underline{H}}_n}_{\underline{\underline{Q}}} \underline{\underline{R}}$

$\underline{\underline{Q}}^T = \underline{\underline{H}}_n^T \cdots \underline{\underline{H}}_1^T = \underline{\underline{H}}_n \cdots \underline{\underline{H}}_1 = \underline{\underline{Q}}^{-1}$ because $\underline{\underline{H}}_i^2 = \underline{\underline{I}} \; \forall i$. $\square$

Ex: $\underline{\underline{A}} = \begin{pmatrix} 1 & -3 \\ -2 & 2 \\ 2 & -1 \end{pmatrix}$ we computed $\underline{\underline{H}}_1$ earlier.

$$\underline{\underline{H}}_1 = \underline{\underline{I}} - \begin{pmatrix} 4/3 & -2/3 & 2/3 \\ -2/3 & 1/3 & -1/3 \\ 2/3 & -1/3 & 1/3 \end{pmatrix} = \begin{pmatrix} -1/3 & 2/3 & -2/3 \\ 2/3 & 2/3 & 1/3 \\ -2/3 & 1/3 & 2/3 \end{pmatrix}$$

$$\underline{\underline{H}}_1 \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix} = 1/3 \begin{pmatrix} -1 & 2 & -2 \\ 2 & 2 & 1 \\ -2 & 1 & 2 \end{pmatrix} \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix}$$

$$= 1/3 \begin{pmatrix} 9 \\ -3 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

Now, want $\underline{\underline{H}}_2' = \underline{\underline{I}} - \dfrac{2 v' v'^T}{\|v'\|^2}$, $\underline{\underline{H}}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & \multicolumn{2}{c}{\underline{\underline{H}}_2'} \end{pmatrix}$

w/ $\underline{\underline{H}}_2' \underbrace{\begin{pmatrix} -1 \\ 2 \end{pmatrix}}_{x'} = \begin{pmatrix} \|x'\| \\ 0 \end{pmatrix}$

$\|x'\| = 5^{1/2}$, $\text{sign}(x_1) = -1$

$\underline{v}' = \underline{x}' - \sqrt{5}\,\underline{e}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} - \begin{pmatrix} \sqrt{5} \\ 0 \end{pmatrix} = \begin{pmatrix} -(1+5^{1/2}) \\ 2 \end{pmatrix}$

$\|\underline{v}'\|^2 = 4 + (1+5^{1/2})^2$

$$\underline{v}'\underline{v}'^T = \begin{pmatrix} (1+5^{1/2})^2 & -2\cdot(1+5^{1/2}) \\ -2\cdot(1+5^{1/2}) & 4 \end{pmatrix}$$

$$\underline{\underline{H}}_2' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{2}{4+(1+5^{1/2})^2} \begin{pmatrix} (1+5^{1/2})^2 & -2\cdot(1+5^{1/2}) \\ -2\cdot(1+5^{1/2}) & 4 \end{pmatrix}$$

Then $\underline{\underline{H}}_2' \underline{x}' = \begin{pmatrix} 5^{1/2} \\ 0 \end{pmatrix}$, $\underline{\underline{H}}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & \multicolumn{2}{c}{\underline{\underline{H}}_2'} \end{pmatrix}$

$$\underline{\underline{H}}_2 \underline{\underline{H}}_1 \underline{\underline{A}} = \underbrace{\begin{pmatrix} -3 & 3 \\ 0 & 5^{1/2} \\ 0 & 0 \end{pmatrix}}_{\underline{\underline{R}}_1}$$

$\underline{\underline{Q}} = \underline{\underline{H}}_1 \underline{\underline{H}}_2$