Hilbert Matrix

Define $\underline{H}_n \in \mathbb{R}^{n\times n}$ by $h_{ij} = 1/(i+j-1)$, $i,j = 1,2,\dots,n$

Note: $\underline{H}_n^T = \underline{H}_n$. Moreover, possible to show $\lambda_i(\underline{H}_n) \geq 0$ for all i.

Can be shown that $\lambda_{min} \to 0$ as $n \to \infty$. This causes

$$\kappa_2(\underline{H}_n) \sim \underbrace{\frac{(\sqrt{2}+1)^{4n+4}}{2^{15/4}\cdot(\pi n)^{1/2}}}_{\text{exponential growth!}} \quad \text{as } n\to\infty$$

Consider $b_i = \sum_{j=1}^{n} j/(i+j-1)$. $(\underline{b} \in \mathbb{R}^n)$ Note that:

$$\sum_{j=1}^{n} h_{ij}\cdot j = \sum_{j=1}^{n}\left(\frac{j}{i+j-1}\right) = b_i$$

So $\underline{x} \in \mathbb{R}^n$ with $x_i = i$ solves $\underline{H}_n \cdot \underline{x} = \underline{b}$.

Solve $\underline{H}_n \underline{x} = \underline{b}$ for $\underbrace{\underline{x}+\delta x}_{\text{due to rounding error}}$ via LU.

n	$\|\delta b\|_2/\|b\|_2$	$\|\delta x\|_2/\|x\|_2$
5	1.2×10^{-15}	8.5×10^{-11}
10	1.7×10^{-15}	1.3×10^{-3}
15	2.8×10^{-15}	4.1
20	6.3×10^{-15}	8.7
25	1.9×10^{-13}	5.5×10^{2}

Note: for $n \geq 15$, error is larger than the solution itself!

Least Squares Problems

Often (e.g. data fitting), want to solve:

$$(P) \quad \min_{\underline{x}} \|\underline{A}\underline{x} - \underline{b}\|_2 \qquad \underline{x}\in\mathbb{R}^n,\ \underline{b}\in\mathbb{R}^m,\ A\in\mathbb{R}^{m\times n}$$

$m > n$? $\underline{\underline{A}}\,\underline{x} = \underline{b}$ does not exist in general!

<u>Ex:</u> $\begin{pmatrix} 3 & 1 \\ 1 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ Δ

$m < n$? Infinitely many solutions!

<u>Ex:</u> $\begin{pmatrix} 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 1 \Rightarrow \underline{x} = \begin{pmatrix} \mu \\ 1-3\mu \end{pmatrix}$ for $\mu \in \mathbb{R}$. Δ

Note: Convenient to solve instead:

$$\min_{\underline{x}} \ \tfrac{1}{2} \| \underline{\underline{A}}\,\underline{x} - \underline{b} \|_2^2$$

By expansion,

$$\tfrac{1}{2} \| \underline{\underline{A}}\,\underline{x} - \underline{b} \|^2 = \tfrac{1}{2} (\underline{\underline{A}}\,\underline{x} - \underline{b})^T (\underline{\underline{A}}\,\underline{x} - \underline{b})$$

$$= \tfrac{1}{2}\, \underline{x}^T \underline{\underline{A}}^T \underline{\underline{A}}\, \underline{x} - 2\, \underline{x}^T \underline{\underline{A}}^T \underline{b} + \underline{b}^T \underline{b}$$

Minimum can be found by setting gradient wrt $\underline{x} = \underline{0}$!

This gives the normal equations

$$\underline{\underline{A}}^T \underline{\underline{A}}\, \underline{x} = \underline{\underline{A}}^T \underline{b}$$

Note: $\underline{\underline{A}}^T \underline{\underline{A}} \in \mathbb{R}^{n \times n}$

$\underline{\underline{A}}^T \underline{b} \in \mathbb{R}^n$

square system!

<u>Problem:</u> $\kappa(\underline{\underline{A}}^T \underline{\underline{A}}) = \kappa(\underline{\underline{A}})^2$! The condition # of $\underline{\underline{A}}$ gets squared, making the normal equations often quite ill-conditioned.

Ex:

$\underline{\underline{A}} = \begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix}$, $\varepsilon \in (0,1)$ $\quad \underline{\underline{A}}^{-1} = \begin{pmatrix} 1/\varepsilon & 0 \\ 0 & 1 \end{pmatrix}$

$\|\underline{\underline{A}}\| = 1$, $\|\underline{\underline{A}}^{-1}\| = 1/\varepsilon \Rightarrow \kappa_2(\underline{\underline{A}}) = 1/\varepsilon$

$\underline{\underline{A}}^T \underline{\underline{A}} = \underline{\underline{A}}^2 = \begin{pmatrix} \varepsilon^2 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \kappa_2(\underline{\underline{A}}^T\underline{\underline{A}}) = 1/\varepsilon^2$

For ε small (e.g. 10^{-2}, 10^{-3}), extra factor of $1/\varepsilon$ huge!

Q: Can we devise a method that avoids squaring $\kappa(\underline{\underline{A}})$, e.g. by working directly w/ the matrix $\underline{\underline{A}}$?