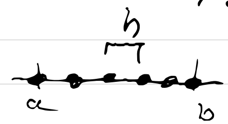
Convergence of Lagrange Interpolation

Q: Given a sequence $\{p_n\}_{n=1}^{\infty}$ of interpolation polynomials for $f:[a,b]\to\mathbb{R}$, does $p_n \to f$ as $n\to\infty$?

Note: $\{p_n\}$ depends on how the interpolation points $\{x_j\}_{j=0}^{n}$ are defined.

Uniform grid: $x_j = a + j\cdot(b-a)/n$



$h = (b-a)/n$

Recall:

$$|f(x)-p_n(x)| \le \frac{M_{n+1}}{(n+1)!}|\pi_{n+1}(x)|$$

$$M_{n+1} \triangleq \max_{\xi\in[a,b]} |f^{(n+1)}(\xi)|$$

$$\pi_{n+1}(x) = \prod_{j=0}^{n}(x-x_j)$$

So that the question becomes, for x_j given as above, does

$$\lim_{n\to\infty} \frac{M_{n+1}}{(n+1)!}\max_{x\in[a,b]}|\pi_{n+1}(x)| = 0?$$

If so, we have

$$\lim_{n\to\infty} \underbrace{\max_{x\in[a,b]} \left| f(x) - p_n(x) \right|}_{\overset{\Delta}{=}\ \|f-p_n\|_{\infty,[a,b]}} = 0$$

"infinity norm on functions"

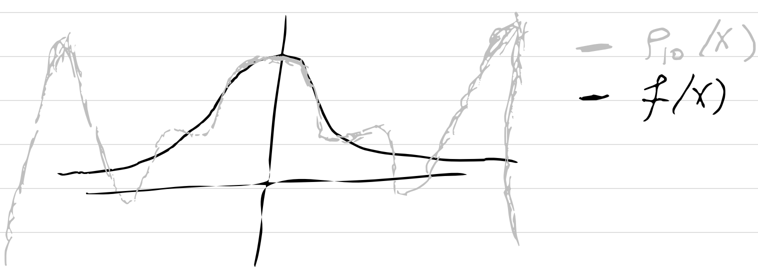
and we say that the sequence $\{p_n\}$ converges <u>uniformly</u> to f on $[a,b]$.

<u>Note:</u>

$M_{n+1} \max_{x\in[a,b]} |\pi_{n+1}(x)|$ could converge to ∞ faster than $(n+1)!$, so that the error bound diverges with n.

<u>Ex:</u> "Runge's Phenomenon"

$f(x) = (1+x^2)^{-1}, \quad x\in[-5,5]$



degree n	Max Error
2	0.65
6	0.61
10	1.92
14	7.15
18	28.74
24	252.78

Δ

Note: f is "well-behaved" in the sense that it is continuous, bounded, and so are all of its derivatives on $[-5,5]$. Failure here is related to poles in the complex plane at $\pm i$.

Hermite Interpolation

(1) Given $\{(x_i, y_i, z_i)\}_{i=0}^{n}$, find $P_{2n+1} \in \mathcal{P}_{2n+1}$ such that

$$P_{2n+1}(x_i) = y_i, \quad P_{2n+1}'(x_i) = z_i, \quad i = 0, \dots, n$$

We show that this can be done by explicit construction, similar to Lagrange interpolation.

Thm (Hermite Interpolation):

Let $n \geq 0$ and suppose that x_i, $i = 0, \dots, n$ are distinct real numbers. Given two sets $\{y_i\}_{i=0}^{n}$, $\{z_i\}_{i=0}^{n}$, there exists a unique $P_{2n+1} \in \mathcal{P}_{2n+1}$ such that

$$P_{2n+1}(x_i) = y_i, \quad P'_{2n+1}(x_i) = z_i, \quad i = 0, \dots, n$$

Pf: For $n = 0$, consider

$$P_1(x) = y_0 + (x - x_0) z_0 \quad \checkmark$$

For $n \geq 1$, define

$$H_k(x) = [L_k(x)]^2 (1 - 2L'_k(x_k) \cdot (x - x_k))$$

$$K_k(x) = [L_k(x)]^2 \cdot (x - x_k)$$

with $L_k(x)$ the Lagrange interpolant

$$L_k(x) = \prod_{i \neq k}^{n} \left(\frac{x - x_i}{x_k - x_i} \right)$$

Now,

$$H_k(x_k) = \underbrace{L_k(x_k)^2}_{=1} [1 - \underbrace{2L'_k(x_k) \cdot (x_k - x_k)}_{=0}]$$

$$= 1$$

$$H_k(x_i) = \underbrace{L_k(x_i)}_{=0}^2 \left(1 - 2L'(x_k)(x_i - x_k)\right)$$

$$= 0 \quad (i \neq k)$$

$$K_k(x_i) = L_k(x_i)^2 \cdot (x_i - x_k) = 0$$
$$K_k(x_k) = L_k(x_k)^2 \cdot (x_k - x_k) = 0$$

Moreover,

$$H_k'(x) = 2L_k(x)L_k'(x) \cdot \left[1 - 2L_k'(x_k)(x - x_k)\right] - 2L_k(x)^2 L_k'(x_k)$$

$$K_k'(x) = L_k(x)^2 + 2L_k(x)L_k'(x) \cdot (x - x_k)$$

So that:

$$H_k'(x_k) = 2\underbrace{L_k(x_k)}_{=1} L_k'(x_k) \cdot \left[1 - \underbrace{2L_k'(x_k)(x_k - x_k)}_{=0}\right] - 2\underbrace{L_k(x_k)}_{=1}^2 L_k'(x_k)$$

$$= 2L_k'(x_k) - 2L_k'(x_k) = 0$$

$$H_k'(x_i) = 2L_k(x_i)L_k'(x_i)\left[1 - 2L_k'(x_k)(x_i - x_k)\right] - 2L_k(x_i)^2 L_k'(x_k) = 0$$

All together, then:

$$H_k(x_i) = \delta_{ik}, \quad H_k'(x_i) = 0$$

$$K_k(x_i) = 0, \quad K_k'(x_i) = \delta_{ik}$$

so that

$$P_{2n+1}(x) = \sum_{k=0}^{n} \left(H_k(x) y_k + K_k(x) z_k \right)$$

satisfies the requirements. For uniqueness, say that we had some other $q_{2n+1} \in \mathcal{P}_{2n+1}$ with

$$q_{2n+1}(x_i) = y_i, \quad q_{2n+1}'(x_i) = z_i$$

Note that $q_{2n+1} - p_{2n+1}$ has $n+1$ distinct 0's. Rolle's Theorem says that $p_{2n+1}' - q_{2n+1}'$ has another n zeros between the x_i.

But $p_{2n+1}'(x_i) = z_i = q_{2n+1}'(x_i)$, so that $p_{2n+1}' - q_{2n+1}'$ has $2n+1$ zeros and hence $p_{2n+1}' - q_{2n+1}' = 0$.

Hence $p_{2n+1} - q_{2n+1}$ is a constant function. But $(p_{2n+1} - q_{2n+1})(x_i) = 0$, so that $p_{2n+1} - q_{2n+1} = 0$. $\square$

This result motivates a definition.

Def: Let $n \geq 0$, and assume that $\{(x_i, y_i, z_i)\}_{i=0}^{n}$ are given real numbers. The polynomial

$$P_{2n+1}(x) = \sum_{k=0}^{n} \left(H_k(x) y_k + K_k(x) z_k \right)$$

is called the Hermite interpolation polynomial of degree $2n+1$.

Ex: want to construct a cubic P_3 with

$$P_3(0) = 0, \quad P_3(1) = 1, \quad P_3'(0) = 1, \quad P_3'(1) = 0$$

for $2n+1 = 3$, $n = 1$, so that

$$P_3(x) = \sum_{k=0}^{n} \left(H_k(x) y_k + K_k(x) z_k \right)$$

$$= H_0(x) \cdot 0 + K_0(x) \cdot 1$$

$$+ H_1(x) \cdot 1 + K_1(x) \cdot 0$$

$$= H_1(x) + K_0(x)$$

Now,

$$L_0(x) = \frac{(x-1)}{(0-1)} = (1-x)$$

$$L_1(x) = \frac{x}{1-0} = x$$

So that:

$$H_1(x) = [L_1(x)]^2(1 - 2L_1'(x_1)\cdot(x - x_1))$$
$$= x^2\cdot(1 - 2(x-1))$$
$$= x^2\cdot(3 - 2x)$$
$$K_0(x) = [L_0(x)]^2\cdot(x - x_0) = (1-x)^2\cdot x$$

This gives the interpolant

$$P_3(x) = (1-x)^2\cdot x + x^2\cdot(3-2x)$$
$$= (1 - 2x + x^2)x + x^2\cdot(3-2x)$$
$$= x - 2x^2 + x^3 + 3x^2 - 2x^3$$
$$= -x^3 + x^2 + x \qquad \Delta$$

<u>Def:</u> Let $f \in C^1([a,b])$. Suppose $\{x_i\}_{i=0}^{n}$ are distinct points in $[a,b]$. Then,

$$P_{2n+1}(x) = \sum_{k=0}^{n} H_k(x) f(x_k) + K_k(x) f'(x_k)$$

is the <u>Hermite interpolation polynomial</u> of degree $2n+1$ for f. It satisfies

$$P_{2n+1}(x_i) = f(x_i), \quad P_{2n+1}'(x_i) = f'(x_i)$$

Last, we provide an error bound.

Thm: Suppose $n \geq 0$ and let $f \in C^{2n+2}([a,b])$.

Let P_{2n+1} denote the Hermite interpolation polynomial of f. Then, for every $x \in [a,b]$, there exists $\xi = \xi(x)$ in (a,b) such that

$$f(x) - P_{2n+1}(x) = \frac{f^{(2n+2)}(\xi)}{(2n+2)!}\left(\pi_{n+1}(x)\right)^2$$

Moreover,

$$|f(x) - P_{2n+1}(x)| \leq \frac{M_{2n+2}}{(2n+2)!}\left(\pi_{n+1}(x)\right)^2$$

with $M_{2n+2} \triangleq \max_{\xi \in [a,b]} |f^{(2n+2)}(\xi)|$

Pf: Define $\psi : [a,b] \to \mathbb{R}$ by

$$\psi(t) = f(t) - P_{2n+1}(t) - \frac{f(x) - P_{2n+1}(x)}{(\pi_{n+1}(x))^2}\left(\pi_{n+1}(t)\right)^2$$

Note $\psi(x_i) = \psi(x) = 0$, so ψ has $n+2$ 0's.

By Rolle's theorem, ψ' has $n+1$ 0's that interlace the 0's of ψ.

Moreover,

$$\psi'(t) = f'(t) - P_{2n+1}'(t) - 2\frac{(f(x) - P_{2n+1}(x))}{(\pi_{n+1}(x))^2} \cdot \pi_{n+1}(t)\,\pi_{n+1}'(t).$$

So that $\psi'(x_i) = 0 \ \forall i$.

Hence, ψ' has $2n+2$ distinct 0's.

Applying Rolle's theorem repeatedly, we find that $\psi^{(2n+2)}$ vanishes at some $\xi \in (a,b)$.

Because $P_{2n+1}^{(2n+2)} = 0$,

$$\psi^{(2n+2)}(\xi) = 0 = f^{(2n+2)}(\xi) - \frac{f(x) - P_{2n+1}(x)}{(\pi_{n+1}(x))^2} \cdot (2n+2)!$$

$$\Rightarrow \quad f(x) - P_{2n+1}(x) = \frac{f^{(2n+2)}(\xi) \cdot \pi_{n+1}(x)^2}{(2n+2)!} \qquad \square$$