Practice Midterm Exam

This exam is scheduled for 70 minutes. You can use a one page cheat sheet, but no other sources including calculators, lecture notes, or textbooks.

Name: netID:

1. **Short answers and true/false [40 points]**

Please answer the following questions either with True/False or provide a short answer. You do not need to give details/explanations for these answers.		
1	True of False? Any square matrix can be LU factorized with pivoting.	
2	When using the bisection method to find a root of $f(x) = x^3 - x^2 - 1$ with starting interval bounds $(a_0, b_0) = (-2, 2)$, what are the interval bounds (a_1, b_1) after one step?	
3	True or False? The secant method will always converge to the solution with any given pairs of initial value.	
4	At which convergence order (sublinear, linear, superlinear, quadratic) does the sequence $x_k := 1 + (-1)^k \exp(-k)$ for $k = 1, 2, \ldots$, converge to 1?	
5	Computing the product of a matrix $A \in \mathbb{R}^{n\times n}$ with a vector $\boldsymbol{v} \in \mathbb{R}^n$ takes 0.01 seconds for $n = 10,000$. Based on the flop count, how long will it take to multiply a matrix with a vector when $n = 100,000$?	
6	Give a 2×2 matrix whose induced 1-norm and induced ∞-norm coincide.	
7	True or False? For $\boldsymbol{v} \in \mathbb{R}^n$, the function $N(v) := 10\|v\|_2$ is a norm.	
8	Give the 3×3 permutation matrix P that permutes the first and the second row of a matrix when multiplied from the left.	
9	True of False? The Frobenius matrix norm is not induced by any vector norm.	
10	Consider the pseudo code snippet below this table. To leading order in n, including the constant (prefactor), give the operation count.	

```
z = 0;
for i in {1, ..., n}
    for j in {1, ..., n}
        z = z + i*j;
    end
end
```

2. **Fixed points [20 pts]** The equation
$$f(x) := x^2 - 5 = 0,$$
has a single root $\xi = \sqrt{5} \approx 2.2361\ldots$ in the interval $[1, 3]$. Consider the fixed point iteration $x_{k+1} = g(x_k)$, where g is defined as one of the following options:
   - $g_1(x) = 5 + x - x^2$,
   - $g_2(x) = 1 + x - \frac{1}{5}x^2$,
   - $g_3(x) = \frac{1}{2}x + \frac{5}{2x^2}$.

   (a) Identify the fixed point functions for which the fixed point is also a root of f.

   (b) For the cases where computing the fixed point is equivalent with solving $f(x) = 0$, discuss whether the fixed point iteration is guaranteed to converge in some neighborhood of ξ.

   (c) If the iteration in b) is guaranteed to converge, compute the value of
$$\lim_{k\to\infty} \frac{|x_{k+1} - \xi|}{|x_k - \xi|}$$
   Hint: Everything is easier with the mean value theorem!

3. **Convergence [15 points]** Recall that a sequence $\{x_k\}$ converges to ξ with order q if
$$\lim_{k\to\infty} \frac{|x_{k+1} - \xi|}{|x_k - \xi|^q} = \mu > 0.$$

   (a) Show that the sequence $\{a_k\}$ with
$$a_k = \left(\frac{1}{3}\right)^{2^k}$$
   converges quadratically to 0.

   (b) Show that the sequence $\{b_k\}$ with
$$b_k = 1 - \frac{1}{100^k}$$
   converges linearly to 1. What is the rate of convergence?

4. **LU factorization [20 points]** Consider the matrix
$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

   (a) Since A is tridiagonal, its LU factorization can be written
$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ l_2 & 1 & 0 \\ 0 & l_3 & 1 \end{pmatrix} \begin{pmatrix} u_1 & v_1 & 0 \\ 0 & u_2 & v_2 \\ 0 & 0 & u_3 \end{pmatrix}$$
   Compute the value of l_2, l_3, v_1, v_2, u_1, u_2, and u_3.

(b) Show that $\det(A) = \det(U)$.

(c) Provide a matrix $B \in \mathbb{R}^{3\times 3}$, where $b_{11} \neq 0$, for which the LU factorization algorithm without pivoting fails.

5. **Norms and Condition numbers [20 points]**

(a) Find $||A||_1$ for the matrix

$$A = \begin{bmatrix} 1 & \epsilon \\ \epsilon & 1 \end{bmatrix},$$

where $\epsilon \in (0, 1)$.

(b) Suppose that you have two systems

$$\begin{aligned} x_1 + \epsilon x_2 &= b_1 \\ \epsilon x_1 + x_2 &= b_2 \end{aligned} \qquad \text{and} \qquad \begin{aligned} x_1' + \epsilon x_2' &= b_1' \\ \epsilon x_1' + x_2' &= b_2' \end{aligned}$$

where $\mathbf{b}' = (b_1', b_2')^T$ is approximately equal to $\mathbf{b} = (b_1, b_2)^T$, with a 5% relative error, that is $\frac{||\mathbf{b}'-\mathbf{b}||_1}{||\mathbf{b}||_1} \leq 0.05$. Using part (a), find an upper bound for the relative error $\frac{||\mathbf{x}'-\mathbf{x}||_1}{||\mathbf{x}||_1}$ where $\mathbf{x}' = (x_1', x_2')^T$ and $\mathbf{x} = (x_1, x_2)^T$. This upper bound will depend on ϵ.

(c) For what values of ϵ is A ill-conditioned?

6. **Least Squares Problem [XX points]**

(a) Given is a matrx $A \in \mathbb{R}^{m\times n}$ with $m > n$ and a vector $\boldsymbol{b} \in \mathbb{R}^m$

(i) Write down the least squares problem for the overdetermined system $A\mathbf{x} = \boldsymbol{b}$ with $\mathbf{x} \in \mathbb{R}^n$.

(ii) State the associated normal equations that define the solution $\mathbf{x}$ to the least squares problem in (a).

(iii) Why is solving the normal equations numerically generally not the preferred method?

(b) You are given four pairs of datapoints (t_i, r_i) $i = 1, 2, 3, 4$ shown in the table below.

	$i=1$	$i=2$	$i=3$	$i=4$
t	-1	0	2	4
r	3	2	-1	0

You expect that these can be approximated by a function of the form

$$r(t) = x_1 e^t + x_2 t$$

with appropriate coefficients x_1 and x_2. Write down the associated least squares system, i.e. define the matrix A and the vector $\boldsymbol{b}$ (you don't have to solve the system).