Practice Final Exam, Fall 2022

1. **Short answers and true/false**

   Write down a table with short answers to the following questions. Below the table, include a brief justification/reasoning for your answer for each question (you will not get points without an explanation!).

   (a) How many iterations does Newton's method require to solve a linear equation?

   (b) The 2-norm condition number of an $n \times n$ matrix with singular values $\sigma_1 \geq \sigma_2 \cdots \geq \sigma_n$ is ...

   (c) (3 pts) To leading order, how many FLOPs are required to solve an $[n \times n]$ linear system if we already know the SVD decomposition/factorization of the matrix. For full points, give the constant in front of the leading order term.

   (d) (3 pts) Using the initial guess $\mathbf{x}_0 = [0, 1, 0]^T$, the power method applied to the matrix

   $$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   will converge to what eigenvalue if exact arithmetic (no roundoff error) is used?

   (e) True or False: For the nodes $x_0 = -1, x_1 = 0, x_2 = 1$, the Lagrange interpolation polynomial $L_0(x)$ corresponding to node x_0 is $\frac{1}{2}x^2 - \frac{1}{2}x$.

   (f) True or false: The $n + 1$ point Lagrange polynomial

   $$p_n(x) = \sum_{k=0}^{n} \exp(x_k) L_k(x) + \sum_{k=0}^{n} L_k(x)$$

   interpolates the function $\exp(x)$.

   (g) The Legendre polynomial of degree n is orthogonal to *every* polynomial with degree $(n - 1)$ or less.

   (h) True or False: The three point (i.e., non-composite) Simpsons rule and the three point Gauss(-Legendre) quadrature formulas are *equally* accurate approximations of

$$\int_{-1}^{1} (x^4 + 1)dx$$

(i) Let $I = \int_a^b f(x)\, dx$, and let I_n be the result of the composite Simpson's rule approximation to I with $n+1$ quadrature points, and denote $e_n = |I - I_n|$. For n large, what does e_{2n}/e_n converge to?

2. How many FLOPs (additions/subtractions/multiplications/divisions) are required to:

   (a) Add two vectors of length n?

   (b) Multiply an $[m \times n]$ matrix with a vector of length n?

   (c) Solve an $[n \times n]$ linear system if we already know the LU factorization of the matrix.

   An explanation/derivation is required to get points. Provide the leading order term including the constant in front.

3. **Householder reflectors and QR factorization**. Given $\boldsymbol{v} \in \mathbb{R}^n \backslash \{0\}$, recall that the corresponding Householder matrix H is defined as

$$H = I - \frac{2}{\boldsymbol{v}^T\boldsymbol{v}} \boldsymbol{v}\boldsymbol{v}^T,$$

   where $I \in \mathbb{R}^{n \times n}$ is the identity matrix.

   (a) Given any two nonzero vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbb{R}^n$ such that $||\boldsymbol{x}||_2 = ||\boldsymbol{y}||_2$, construct a Householder matrix H, such that $H\boldsymbol{x}$ is a scalar multiple of $\boldsymbol{y}$.

   (b) For the matrix

$$A = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 2 \\ 1/2 & 2 \\ 1/2 & 0 \end{bmatrix},$$

   use Househlder reflections to compute the QR factorization of A. The entries in R should be integer.

   (c) What is the value of $\kappa_2(R)/\kappa_2(A)$? [where $\kappa_2(\cdot)$ is the condition number with respect to the 2 norm]

4. Consider the Python code:

```
A = np.array([[7, -sqrt(3)], [-sqrt(3), 5]]
x = np.array([0,1])
n = 100; # Large integer
```

```python
for k in range(n):
    x = np.dot(A, x)   # or x = A @ x
    x = x / np.linalg.norm(x);
return x
```

(a) What does the output of the Python code converge to as n becomes larger and larger? Call this limit x_0.

(b) If we print

```python
x @ (A @ x)
```

at the end of the program, what would we get as output for very large n and why?

(c) Approximately how big does n need to be for $\|x - x_0\|_2$ to be smaller than $1/2^{10} \approx 10^{-3}$? You do not need to give an exact number, in numerical analysis we only need an error *estimate*.

5. In this problem you will derive the most accurate quadrature rule possible that uses some values of the derivative of the function in addition to the values of the function.

(a) For an arbitrary/generic smooth function $f(x)$, find the best values for the weights w_{-1}, w_0, and w_1 in the quadrature rule:

$$\int_{-1}^{1} f(x)dx \approx w_{-1}f'(-1) + w_0 f(0) + w_1 f'(1)$$

Do this by requiring that polynomials up to the highest possible degree are integrated exactly.

(b) Use this rule to estimate $\int_{-1}^{1} \cos(\pi x/2)\, dx$ and compare to the answer from (the non-composite) Simpson's rule. Which rule is more accurate for this specific problem?

6. An inner product (f, g) for two real-valued functions $f(x)$ and $g(x)$ on an interval $[a, b]$ defines a (non-unique) set of orthogonal polynomials with respect to (w.r.t.) that inner product. Consider the interval $[-1, 1]$ and the inner product given by the n-point Gaussian quadrature rule for approximating the standard L_2 inner product $(f, g)_2 = \int_a^b f(x)g(x)\,dx$, i.e.,

$$(f, g) = \sum_{k=1}^{n} w_k f(x_k) g(x_k) \tag{1}$$

where w_k are the Gaussian quadrature weights and x_k are the corresponding nodes. This inner product defines a new weighted l_2 norm

$$\|f\|_{l_2} = \sum_{k=1}^{n} w_k |f(x_k)|^2. \tag{2}$$

Take $n = 3$ for this problem. Note that to answer the questions below you do *not* need to know the values of w_k and x_k, but if you want them, they are $x_1 = -\sqrt{3/5}$, $x_2 = 0$, $x_3 = \sqrt{3/5}$, $w_1 = w_3 = 5/9$ and $w_2 = 8/9$.

(a) Write down a basis for $\mathcal{P}_2$, the space of polynomials of degree at most two, that are orthogonal w.r.t. (1), and explain how you got it and why you know they are orthogonal. You will get 1/2 of the points for any answer that is an orthogonal basis. You will get full points only if you write down a basis where the degree of the polynomial increases, i.e., the first polynomial is of degree zero, the second of degree one, etc. Since orthogonal polynomials are only defined up to a constant, normalize your answers so that the coefficient in front of the highest power of x is unity.

(b) Given the three function values $y_k = f(x_k)$, $k = 1, \ldots, 3$, write down an explicit formula for the optimal polynomial approximation of $f(x)$ in $\mathcal{P}_2$ w.r.t. the norm (2), i.e., find

$$p_2(x) = \arg\min_{p_2 \in \mathcal{P}_2} \|f - p\|_{l_2}.$$

Note: You do not need to write one complete (long) formula, you can break it into pieces and define intermediate constants as appropriate, but make sure everything is defined before it is used, and the only input is y_1, y_2 and y_3.

Hint: Both parts of this question can also be answered with minimal to no algebraic calculations.

7. **[QR Algorithm for finding eigenvalues]**

(a) What is the advantage of the QR-algorithm for finding eigenvalue over the power method?

(b) Let $A^{(k)}$ be the sequence of matrices produced by the QR-algorithm. Show that the eigenvalues of $A^{(k+1)}$ are the same as those of $A^{(k)}$.

(c) For the symmetric tridiagonal matrix

$$T = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & 2 \\ 0 & 2 & -1 \end{bmatrix},$$

the QR iteration will result in a diagonal matrix $\Lambda \in \mathbb{R}^{3\times 3}$. What is the determinant of Λ, i.e., the product of its eigenvalues? *Hint:* You might not have to compute Λ to answer this question.

8. Consider the inner product

$$\langle f, g \rangle = \int_{[0,2]} x f(x) g(x) dx \tag{3}$$

(a) Calculate orthogonal polynomials ϕ_0, ϕ_1, and ϕ_2 in this inner product.

(b) Calculate the best quadratic approximation to $f(x) = x^3$ in the norm induced by this inner-product.

(c) Consider the function $f(x) = \exp(\frac{x^2}{4})$. Prove or disprove that

$$p_2(x) = 1 - \frac{1}{2}x + \frac{\exp(1)}{4}x^2 \tag{4}$$

is the best quadratic approximation to f in the inner-product defined above. *Hint: recall that the residual $p_2 - f$ of the best approximation should be orthogonal to all polynomials of degree ≤ 2.*

9. Let $f(x)$ be a function that satisfies $f(0) = 1$, $f'(1) = 0$, and $f(2) = -1$. Find a quadratic approximation to $f(x)$ on $[0, 2]$ that interpolates the given values of the function and its derivative. *Hint:* This is not Lagrange nor Hermite interpolation, so you have to find that polynomial by means other than we discussed in class.