Conditioning and Norms (Cont.)

Note: For $\underline{\underline{A}} \in \mathbb{R}^{n\times n}$ symmetric, $\underline{\underline{A}}^T \underline{\underline{A}} = \underline{\underline{A}}^2$, so $\lambda_{max}(\underline{\underline{A}}^T\underline{\underline{A}})^{1/2} = \max_{i=1,\dots,n} |\lambda_i(\underline{\underline{A}})|$

Thm: For $\|\cdot\| : \mathbb{R}^{n\times n} \to \mathbb{R}_{\geq 0}$ an induced matrix norm,

$$\|\underline{\underline{A}}\,\underline{\underline{B}}\| \leq \|\underline{\underline{A}}\| \, \|\underline{\underline{B}}\|$$

Pf:

$$\|\underline{\underline{A}}\,\underline{\underline{B}}\| = \max_{\underline{v}} \|\underline{\underline{A}}\,\underline{\underline{B}}\,\underline{v}\| / \|\underline{v}\|$$

$$\leq \|\underline{\underline{A}}\| \max_{\underline{v}} \|\underline{\underline{B}}\,\underline{v}\| / \|\underline{v}\|$$

$$= \|\underline{\underline{A}}\| \, \|\underline{\underline{B}}\|. \qquad \square$$

We now define the Condition #.

Q: How do small changes in the input to a mapping affect its output?

Def: Let $D \subset V$ where $(V, \|\cdot\|_V)$ is a normed linear space. Consider $f: D \to W$ where $(W, \|\cdot\|_W)$ is another normed linear space. We define the absolute condition #

$$\text{Cond}(f) = \sup_{\substack{x,y \in D \subset V \\ x \neq y}} \frac{\|f(y) - f(x)\|_W}{\|x - y\|_V}$$

If $\text{Cond}(f) \gg 1$, we say f is ill-conditioned.

Ex: $f(x) = x^{1/2}$, $D = [1, 2]$.

$$\text{Cond}(f) = \max_{x \neq y} \frac{|x^{1/2} - y^{1/2}|}{|x - y|} = 1/2 \qquad (\text{take } y = 1+\varepsilon,\ \varepsilon \to 0) \quad \triangle$$

Ex: $f(x) = x^{1/2}$, $D = [0,1]$ $\quad Cond(f) = \infty$ Δ

Def: The absolute local Condition # at $x \in D \subset V$ is given by

$$Cond_x(f) = \sup_{\substack{\delta x \in V \setminus \{0\} \\ x+\delta x \in D}} \frac{\| f(x+\delta x) - f(x) \|_W}{\| \delta x \|_V}$$

Ex: $f(x) = x^{1/2}$, $\quad Cond_x(f) = |f'(x)| = \frac{1}{2} \cdot x^{-1/2}$. Δ

Note: These absolute definitions depend on the magnitudes of $f(x)$ and x, which can be undesirable practically.

To avoid issues with "units", we can rescale:

Def: The relative local Condition # of f:

$$Cond_x(f) = \sup_{\substack{\delta x \in V \setminus \{0\} \\ x+\delta x \in D}} \frac{\| f(x+\delta x) - f(x) \|_W \,/\, \| f(x) \|_W}{\| \delta x \|_V \,/\, \| x \|_V}$$

Ex: $f(x) = x^{1/2}$

from earlier, $Cond(f) = \frac{1}{2} x^{-1/2} \to \infty$ as $x \to 0$.

But $Cond(f) = \left| f'(x) \cdot \frac{x}{f(x)} \right| = \frac{1}{2} \cdot x^{-1/2} \cdot x^{1/2} = \frac{1}{2}$ globally Δ

Ex: $f(x) = \sin(x)$

$Cond_x(f) = |\cos(x)|$ $\qquad Cond_x(f) = \left| \frac{\cos(x) \cdot x}{\sin(x)} \right|$

$Cond_0(f) = Cond_\pi(f) = 1$

$Cond_\varepsilon(f) = \left| \frac{\cos(\varepsilon) \cdot \varepsilon}{\sin(\varepsilon)} \right| \Rightarrow \lim_{\varepsilon \to 0} Cond_\varepsilon(f) = 1.$

$$\mathrm{Cond}_{\pi-\varepsilon}(f) = \left| \frac{\cos(\pi-\varepsilon)\cdot(\pi-\varepsilon)}{\sin(\pi-\varepsilon)} \right| \Rightarrow \lim_{\varepsilon\to 0} \mathrm{Cond}_{\pi-\varepsilon}(f) = \infty \quad \triangle$$

We want to solve $\underline{\underline{A}}\,\underline{x} = \underline{b}$. So we are interested in the Condition # of the map $f(\underline{x}) = \underline{\underline{A}}^{-1}\underline{x}$.

Note: $f: \mathbb{R}^n \to \mathbb{R}^n$, so we fix a norm $\|\cdot\|$ on $\mathbb{R}^n$.

$$\mathrm{Cond}_{\underline{b}}(f) = \sup_{\delta\underline{b}\in\mathbb{R}^n\setminus\{\underline{0}\}} \frac{\|\underline{\underline{A}}^{-1}(\underline{b}+\delta\underline{b}) - \underline{\underline{A}}^{-1}\underline{b}\| \,/\, \|\underline{\underline{A}}^{-1}\underline{b}\|}{\|\delta\underline{b}\| \,/\, \|\underline{b}\|}$$

$$= \sup_{\delta\underline{b}\in\mathbb{R}^n\setminus\{\underline{0}\}} \frac{\|\underline{\underline{A}}^{-1}\delta\underline{b}\|}{\|\delta\underline{b}\|} \cdot \frac{\|\underline{b}\|}{\|\underline{\underline{A}}^{-1}\underline{b}\|}$$

$$= \|\underline{\underline{A}}^{-1}\| \cdot \frac{\|\underline{b}\|}{\|\underline{\underline{A}}^{-1}\underline{b}\|}$$

$$= \|\underline{\underline{A}}^{-1}\| \, \|\underline{\underline{A}}\,\underline{\underline{A}}^{-1}\underline{b}\| \,/\, \|\underline{\underline{A}}^{-1}\underline{b}\|$$

$$\leq \|\underline{\underline{A}}^{-1}\| \, \|\underline{\underline{A}}\| \, \|\cancel{\underline{\underline{A}}^{-1}\underline{b}}\| \,/\, \|\cancel{\underline{\underline{A}}^{-1}\underline{b}}\|$$

$$= \|\underline{\underline{A}}^{-1}\| \, \|\underline{\underline{A}}\|$$

Swapping roles of $\underline{\underline{A}}$, $\underline{\underline{A}}^{-1}$, for $g(\underline{x}) = \underline{\underline{A}}\,\underline{x}$ we find

$$\forall \underline{x},\ \mathrm{Cond}_{\underline{x}}(g) \leq \|\underline{\underline{A}}^{-1}\| \, \|\underline{\underline{A}}\|$$

Motivated by these calculations,

Def: The condition number of a matrix $\underline{\underline{A}}$ is given by

$$\kappa(\underline{\underline{A}}) = \|\underline{\underline{A}}\| \, \|\underline{\underline{A}}^{-1}\|$$

Note: $\mathcal{K}(\underline{\underline{A}}) = \mathcal{K}(\underline{\underline{A}}^{-1})$. Moreover, $\mathcal{K}(\underline{\underline{A}}) \geq 1$ for all $\underline{\underline{A}}$.

Note: $\mathcal{K}(\underline{\underline{A}})$ depends on the norm chosen!

Ex: $\underline{\underline{A}} \in \mathbb{R}^{n\times n}$,

$$\underline{\underline{A}} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 \\ 1 & 0 & \cdots & 0 & 1 \end{pmatrix} \qquad \|\underline{\underline{A}}\|_1 = n, \quad \|\underline{\underline{A}}\|_\infty = 2$$

$$\underline{\underline{A}}^{-1} = \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & & 0 & \vdots \\ -1 & & 1 & & \vdots \\ \vdots & 0 & & \ddots & 0 \\ -1 & & & & 1 \end{pmatrix} \qquad \|\underline{\underline{A}}^{-1}\|_1 = n, \quad \|\underline{\underline{A}}^{-1}\|_\infty = 2$$

$$\mathcal{K}_1(A) = n^2, \quad \mathcal{K}_\infty(A) = 4 \qquad \triangle$$

Thm: Let $\underline{\underline{A}} \in \mathbb{R}^{n\times n}$, $\underline{b} \in \mathbb{R}^n$, and say $\underline{\underline{A}}\,\underline{x} = \underline{b}$, $\underline{\underline{A}}(\underline{x} + \delta\underline{x}) = \underline{b} + \delta\underline{b}$.

Then, for $\underline{\underline{A}}$ nonsingular, $\underline{b} \neq \underline{0}$,

$$\underbrace{\frac{\|\delta\underline{x}\|}{\|\underline{x}\|}}_{\text{relative change in } \underline{x}} \leq \mathcal{K}(\underline{\underline{A}}) \underbrace{\frac{\|\delta\underline{b}\|}{\|\underline{b}\|}}_{\text{relative change in } \underline{b}}$$

Pf: Solve $\underline{\underline{A}}\,\delta\underline{x} = \underline{b} - \underline{\underline{A}}\,\underline{x} + \delta\underline{b} = \delta\underline{b}$

$$\Rightarrow \delta\underline{x} = \underline{\underline{A}}^{-1}\delta\underline{b}$$

Then, $\|\underline{b}\| \leq \|\underline{\underline{A}}\|\,\|\underline{x}\|$, $\quad \|\delta\underline{x}\| \leq \|\underline{\underline{A}}^{-1}\|\,\|\delta\underline{b}\|$

So that:

$$\frac{\|\delta\underline{x}\|}{\|\underline{x}\|} \leq \frac{\|\underline{\underline{A}}^{-1}\|\,\|\delta\underline{b}\|}{\|\underline{x}\|}$$

$$\leq \frac{\|\underline{\underline{A}}^{-1}\| \, \|\underline{\underline{A}}\| \, \|\delta \underline{b}\|}{\|\underline{b}\|}$$

$$= \frac{\kappa(\underline{\underline{A}}) \, \|\delta \underline{b}\|}{\|\underline{b}\|} \qquad \square$$

Note: Due to rounding errors, we never solve $\underline{\underline{A}}\,\underline{x} = \underline{b}$ exactly.

we always obtain $\underline{x} + \delta \underline{x}$ for some $\delta \underline{x}$.

Generally, $\underline{\underline{A}}(\underline{x} + \delta \underline{x}) = \underline{b} + \delta \underline{b}$ for $\delta \underline{b}$ small.

If $\kappa(\underline{\underline{A}})$ large, $\delta \underline{x}$ may not be small!