# Homework 2

hw2

Q1)

$$f(x,y) = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix} = \begin{bmatrix} 2x^2+8y^2-8 \\ y - \frac{\sqrt{3}}{2}x^2 \end{bmatrix}$$

Find the roots of $f$, i.e., $(x,y) \in \mathbb{R}^2$ such that $f(x,y) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(a) $f_1: \quad y = \pm\sqrt{1 - \frac{1}{4}x^2}$

$f_2: \quad y = \frac{\sqrt{3}}{2}x^2$

(b) $\frac{\sqrt{3}}{2}x^2 = \sqrt{1 - \frac{1}{4}x^2}$

$\longrightarrow \ldots$

(c)
$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \ldots$$

(d) See the code

Q2) (a) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & 2 & 3 \end{bmatrix}$ gives $u_{11} = u_{12} = u_{13} = 1$

$l_{21} = 1$

$u_{22} = 0$ ← problem

$\det(A) \neq 0$ ✓

(b) $LU = L_1 U_1$

- multiply by $L_1^{-1}$ from the left to get $L_1^{-1} L U = U_1$
- multiply by $U^{-1}$ from the right to get $\underbrace{L_1^{-1} L}_{\text{unit lower}} = \underbrace{U_1 U^{-1}}_{\text{upper}} = I$ ← only possiblity

Note: we need $L_1, L$ and $U_1, U$ to be invertible, which they are based on the question.

Q3)

(a) Follow the proof of theorem 2.2

$n=2$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \underset{L}{\begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}} \underset{D}{\begin{pmatrix} u & 0 \\ 0 & \eta \end{pmatrix}} \underset{U}{\begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}}$$

$a \neq 0$? $\Rightarrow \checkmark$ one can find the RHS easily.

$A \in \mathbb{R}^{(k+1)\times(k+1)}$

$$\left(\begin{array}{c|c} A^{(k)} & \vec{b} \\ \hline \vec{c}^{\,T} & d \end{array}\right) = \left(\begin{array}{c|c} L^{(k)} & \vec{0} \\ \hline \vec{m}^{\,T} & 1 \end{array}\right) \left(\begin{array}{c|c} D^{(k)} & \vec{0} \\ \hline \vec{0}^{\,T} & \eta \end{array}\right) \left(\begin{array}{c|c} U^{(k)} & \vec{v} \\ \hline \vec{0}^{\,T} & 1 \end{array}\right)$$

$\Rightarrow A^{(k)} = L^{(k)} D^{(k)} U^{(k)}$

$L^{(k)} D^{(k)} \vec{v} = \vec{b}$

$\vec{m}^{\,T} D^{(k)} U^{(k)} = \vec{c}^{\,T}$

$\vec{m}^{\,T} D^{(k)} \vec{v} + \eta = d$

$\det(A^{(k)}) \neq 0 \Rightarrow \dots$

unit lower and upper triangular

(b) $A = LDU$

$\Rightarrow A^T = U^T \underbrace{D^T L^T}$

unit lower triangular

upper triangular

$\Rightarrow A^T = L' U'$

Q4) (a)

$$\begin{bmatrix} a_1 & c_1 & 0 & \cdots & & \\ b_1 & a_2 & c_2 & & & \\ & b_2 & a_3 & c_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & b_{n-2} & a_{n-1} & c_{n-1} \\ & & & & b_{n-1} & a_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & & \\ d_1 & 1 & 0 & & \\ & \ddots & 1 & & \\ & & \ddots & \ddots & \\ & & & d_{n-1} & 1 \end{bmatrix} \begin{bmatrix} e_1 & f_1 & 0 & & \\ & e_2 & f_2 & 0 & \\ & & \ddots & \ddots & \\ & & & e_{n-1} & f_{n-1} \\ & & & & e_n \end{bmatrix}$$

It is easy to show that

$$f_k = c_k \qquad k = 1, \dots, n-1$$

see for example:

$$c_1 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = f_1 \qquad \cdots$$

Also, $b_1 = \begin{bmatrix} d_1 & 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = d_1 e_1$

$$b_2 = \begin{bmatrix} 0 & d_2 & 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ e_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = d_2 e_2$$

$$b_3 = \begin{bmatrix} 0 & 0 & d_3 & 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} 0 \\ f_2 \\ e_3 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = d_3 e_3$$

$$\Rightarrow b_k = d_k e_k \qquad k = 1, \dots, n-1$$

$$\rightarrow d_k = \frac{b_k}{e_k} \qquad (e_1 = a_1)$$

And, $a_k = d_{k-1} c_{k-1} + e_k \rightarrow e_k = a_k - d_{k-1} c_{k-1} \quad k = 2, \dots, n$

$$f_k = c_k \qquad k = 1, \dots, n-1$$

$$e_1 = a_1$$

$$\begin{cases} d_k = \dfrac{b_k}{e_k} \\ e_{k+1} = a_{k+1} - d_k c_k \end{cases} \qquad k = 1, \dots, n-1$$

(b) trivial given the above formulas

Q5)

$$A = LDU = A^T = U^T D^T L^T$$

(a)

$$\Rightarrow LDU = U^T D^T L^T$$

$$\Rightarrow DU(L^T)^{-1} = L^{-1} U^T D^T$$

$$\rightsquigarrow U'(L^T)^{-1} = L^{-1} U'^T$$

upper triangular

lower triangular

$$\Rightarrow U'(L^T)^{-1} = D^*$$ a diagonal matrix

$$\rightsquigarrow U' = D^* L^T$$

diagonal values are the same as those of D

diagonal values are 1

$$\Rightarrow D^* = D \Rightarrow U' = DL^T = DU$$

$$\rightsquigarrow U = L^T$$

$$\Rightarrow A = LDL^T = RR^T$$

with $R = LD'$ ← $D'$ is a diagonal matrix with elements $\sqrt{d_{ii}}$

(b) see the MATLAB code.

# (Q6)

(a) the system is

$$\left[\begin{array}{c|c} L_1\in\mathbb{R}^{K\times K} & \overset{\text{zeros}}{B\in\mathbb{R}^{K\times(N-K)}} \\ \hline A\in\mathbb{R}^{(N-K)\times K} & L_2\in\mathbb{R}^{(N-K)\times(N-K)}\end{array}\right] \left[\begin{array}{c}\overset{\in\mathbb{R}^{K}}{\vec{y}_1}\\ \hline \underset{\in\mathbb{R}^{N-K}}{\vec{y}_2}\end{array}\right] = \left[\begin{array}{c}\overset{\text{zeros}\quad\in\mathbb{R}^{K}}{\vec{b}_1}\\ \hline \underset{\in\mathbb{R}^{N-K}}{\vec{b}_2}\end{array}\right]$$

$$\Rightarrow L_1\vec{y}_1=\vec{b}_1 \Rightarrow \vec{y}_1=[y_1, y_2 \dots, y_K]^T=\vec{0}^{\,T}$$

(b) It follows directly from part (a), and using the method described in Q7 to find the inverse of a matrix:

$$\overset{C=L^{-1}}{LC=I} \Rightarrow L\underset{i\text{th column of } C}{\begin{bmatrix} c_{1i}\\ c_{2i}\\ \vdots\\ c_{ni}\end{bmatrix}} = \begin{bmatrix}0\\0\\ \vdots\\ 1\\ \vdots\\ 0\end{bmatrix}$$

ith element

ith column of identity matrix

$$\Rightarrow c_{ji}=0 \quad \text{for } j=1,\dots,i-1$$

$\Rightarrow$ the matrix $C$ will be lower triangular.

(7)

$A \in \mathbb{R}^{n \times n}$

(a) $A A^{-1} = I$, with $A^{-1} = C$

$AC = I \Rightarrow LUC = I$

for each $i = 1, \ldots, n$

$\rightsquigarrow LU \underbrace{\begin{bmatrix} c_{1i} \\ c_{2i} \\ \vdots \\ c_{ni} \end{bmatrix}}_{\vec{y}} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$ ← ith column of I

ith column of C

$\Rightarrow L\vec{y} = I_i \Rightarrow$ find $\vec{y}$ by forward substitution

$U\vec{c}_i = \vec{y} \Rightarrow$ find $\vec{c}_i$ (ith column of C by backward substitution)

(b) $n \times (n(n-1) + n^2) = 2n^3 - n^2 \sim \boxed{2n^3}$

n systems — forward step — backward step

(c) for each column K, we write

$L\vec{y} = I_K \Rightarrow y_K = 1 \qquad y_i = -\underbrace{\sum_{j=K}^{i-1} l_{ij} y_j}_{2(i-K)+1 \text{ operations}} \qquad j = K+1, \ldots, n$

$\Rightarrow \sum_{i=K+1}^{n} 2(i-K)+1 = \cdots = (n-K)^2$

$$\sum_{k=1}^{n}(n-k)^2 = \sum_{k=1}^{n} n^2 + k^2 - 2nk$$

$$= n^3 + \frac{1}{6}n(n+1)(2n+1) - n^2(n+1)$$

$U\vec{c}_k = \vec{y}$ ⇝ no change in the number of operations $n \times n^2 = n^3$

⟹ Total number of operations:

$$\cancel{n^3} + \frac{1}{\cancel{3}}\cancel{n^3} + \frac{1}{6}n^2 + \frac{1}{3}n^2 + \frac{n}{6} - \cancel{n^3} - n^2 \cancel{+n^3}$$

$$= \frac{4}{3}n^3 - \frac{1}{2}n^2 + \frac{n}{6} \sim \frac{4}{3}n^3$$

If we consider the number of operations for $LU$, $\sim \frac{2}{3}n^3$ more operations will be required:

(b): $\sim \frac{8}{3}n^3$ (c): $\sim 2n^3$