More Orthogonal Polynomials

Ex: Find best (in L_2) degree 2 polynomial of $f(x) = e^x$, $w(x) = 1$.

We already computed $\varphi_0, \varphi_1, \varphi_2$.

Write $p_2(x) = \gamma_0 \varphi_0(x) + \gamma_1 \varphi_1(x) + \gamma_2 \varphi_2(x)$

$$\gamma_i = \langle \varphi_i, f \rangle / \langle \varphi_i, \varphi_i \rangle$$

$\varphi_0 = 1$, $\varphi_1 = x - 1/2$, $\varphi_2 = x^2 - x + 1/6$

$\langle \varphi_i, f \rangle = \int \varphi_i(x) f(x) dx$

This gives: $\gamma_0 = e - 1$

$\gamma_1 = 18 - 6e$

$\gamma_2 = 210e - 570$

Thm: Suppose q_i, $i=0,1,\ldots$ is a system of orthogonal polynomials on the interval (a,b) w.r.t. a positive, continuous weight ω. For $i \geq 1$, the zeros of q_i are real and distinct, and lie in (a,b).

Pf: Let ξ_i, $i=1,\ldots,k$ denote the points at which $q_i(x)$ changes sign. Note $k \geq 1$, because

$$\int_a^b \omega(x)\, q_i(x)\, q_0(x)\, dx = \int_a^b \underbrace{\omega(x)}_{\geq 0} q_i(x)\, dx = 0$$

Define $\pi_k(x) = (x-\xi_1)(x-\xi_2)\cdots(x-\xi_k)$

Note: $q_i(x)\,\pi_k(x)$ does not change sign, because q_i and π_k change sign at the same locations.

Hence,

$$\int_a^b q_i(x)\, \pi_k(x)\, \omega(x)\, dx \neq 0.$$

But φ_i is orthogonal to every polynomial of degree $\neq i$, because each such p may be written

$$p(x) = \sum_{k \neq i} \beta_k \varphi_k(x) \quad \text{for some } \{\beta_k\}$$

$$\Rightarrow \langle p, \varphi_j \rangle = \sum_{k \neq i} \beta_k \underbrace{\langle \varphi_k, \varphi_i \rangle}_{= 0 \text{ by orthogonality}} = 0$$

So $\tilde{\pi}_k$ must have degree exactly i.
Hence, the points $\xi_i \in (a, b)$ must be all the 0's of φ_j. $\square$

Gaussian Quadrature

(P) We want to compute the weighted integral

$$\int_a^b \omega(x) f(x)\, dx$$

for $f \in C^0([a,b])$, $\omega \in C^0([a,b])$, $\omega(x) \geq 0$.

Let $\{x_i\}_{i=0}^{n}$ denote $(n+1)$ nodes (not equally spaced).

Recall: the Hermite Interpolant:

$$P_{2n+1}(x) = \sum_{k=0}^{n} H_k(x) f(x_k) + K_k(x) f'(x_k)$$

$$H_k(x) = L_k(x)^2 (1 - 2L_k'(x_k)(x - x_k))$$

$$K_k(x) = L_k(x)^2 (x - x_k)$$

$$L_k(x) = \prod_{j \neq k} \left(\frac{x - x_j}{x_k - x_j} \right), \quad (L_0(x) = 1 \text{ w/ } n = 0)$$

Hence, using this interpolant,

$$\int w(x) f(x) dx \simeq \int_a^b w(x) P_{2n+1}(x) dx$$

$$= \sum_{k=0}^{n} [W_k(x) f(x_k) + V_k f'(x_k)]$$

$$W_k \triangleq \int_a^b w(x) H_k(x) dx, \quad V_k = \int_a^b w(x) K_k(x) dx$$

Note: If we can choose the $\{x_k\}$ so that the V_k vanish, we will not require the derivative values f'!

$$V_k = \int_a^b w(x) L_k(x)^2 \cdot (x - x_k)\, dx$$

$$= C_n \int_a^b w(x)\, \tilde{\pi}_{n+1}(x) L_k(x)\, dx,$$

$$C_n \triangleq \begin{cases} 1 & n=0 \\ \prod_{j \neq k} (x_k - x_j)^{-1} & \text{else}, \end{cases} \quad \tilde{\pi}_{n+1}(x) \triangleq \prod_{j=0}^{n} (x - x_j)$$

Note: $\tilde{\pi}_{n+1}$ is of degree $n+1$. L_k is of degree n.

so $V_k = 0$ if $\tilde{\pi}_{n+1}$ is orthogonal to every polynomial of lower degree! (w.r.t. weight w)

Hence, we want $\tilde{\pi}_{n+1} = \varphi_{n+1}$ in a system of orthogonal polynomials w.r.t. w. i.e., we should choose the $\{x_k\}$ to be the roots of φ_{n+1}!

Now, consider W_k,

$$W_k = \int_a^b \omega(x) H_k(x)\,dx$$

$$= \int_a^b \omega(x) L_k(x)^2 \left(1 - 2L_k'(x_k)(x - x_k)\right) dx$$

$$= \int_a^b \omega(x) L_k(x)^2 - 2L_k'(x_k) \underbrace{V_k}_{=0}.$$

This gives the <u>Gauss Quadrature rule</u>

$$\int_a^b f(x)\,\omega(x)\,dx \approx \sum_{k=0}^{n} W_k f(x_k),$$

$$W_k \triangleq \int_a^b \omega(x) L_k(x)^2\,dx,$$

$$\{x_k\}_{k=0}^{n} \text{ roots of } \varphi_{n+1}!$$