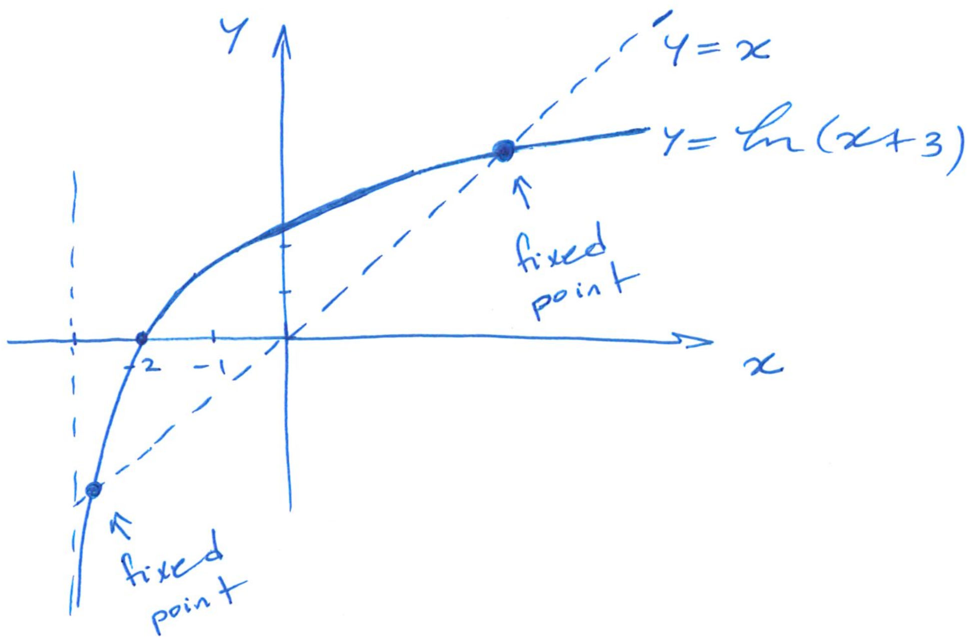
Homework 1 solutions

Problem 1

ⓐ start with $g(x) = x$, and show that $f(x) = 0$.

ⓑ



ⓒ $[a, b] = [e - 3, e^2 - 3]$

find $g(a)$, and $g(b)$; note that $g(x)$ is monotonic increasing $\Rightarrow g(x) \in [a, b] \;\; \forall \; x \in [a, b]$

$g(x)$ is continuous $\rightsquigarrow$ ...

ⓓ $g(x)$ is continuous and differentiable:

Find the Lipschitz constant through calculation of $g'(x)$ and show that it is $0 < L < 1$. Together with the results from part (c) $\rightarrow$ ...

Problem 2

(a) (i) trivial

(ii) A: unstable and stable, respectively

B: neither stable nor unstable and stable, respectively

C: unstable and unstable, respectively

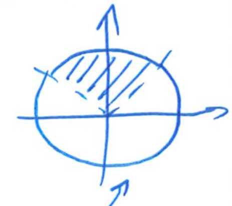
(b) Use power-law fitting by $y = [\varepsilon_{k-1}, \varepsilon_k]$, $x = [\varepsilon_{k-2}, \varepsilon_{k-1}]$ for $k \geq 2$ to find the order q for each k. What is this order as $k \to \infty$?

(c), (d) Find $\lim\limits_{k \to \infty} \dfrac{|\varepsilon_{k+1}|}{|\varepsilon_k|} = \mu$; depending on the value of μ, the speed will be linear, sublinear, or super linear.

use methods of calculus!

Problem 3

(a) Simply find max $|g'(x)|$ on the interval. Note that $g'(x) = \frac{3}{4} - \frac{1}{2}\sin x$ and $x \in [\frac{\pi}{4}, 3\frac{\pi}{4}]$



unit circle

(b) We know $|x_k - \xi| \leq L^k |x_0 - \xi| \rightsquigarrow \ldots$

(c) $x_0 = \frac{\pi}{4}$, $x_1 = g(\frac{\pi}{4})$, L from (a) : use theorem 1.9

Problem 4

ⓐ Does $g(x) = x$ yields $f(x) = 0$? g_1 ? g_2 ?

ⓑ Find the Lipschitz constant on $[1,3]$.

Check if $g(x) \in [1,3]$ $\forall$ $x \in [1,3]$

or rather $g'(\xi)$

g_2 ?

ⓒ Consider two points on the $g'(x)$ curve:

(ξ, ξ) and $(x_k, g(x_k))$

$\Rightarrow$ By mean value theorem:

$$\frac{|g(x_k) - \xi|}{|x_k - \xi|} = g'(\eta)$$ for some η between x_k and ξ

⋮

ⓓ $x_0 = 2$: $f(x_0) = 2^2 - 5 = -1$

$f'(x_0) = 2 \times 2 = 4$

$$\Rightarrow x_1 = 2 - \frac{f(x_0)}{f'(x_0)} = \dots$$

Problem 5

$g(0) = 0, \quad g(x) = -x \sin^2\left(\frac{1}{x}\right) \quad$ for $\quad 0 < x \leq 1$

g continuous: $\quad \lim_{x \to 0^+} g(x) = ?$

$\xi = 0$ is the only fixed point: what is the sign of $y = g(x)$? Can it ever intersect with $y = x$?

$$x_{n+1} = g(x_n) = -x_n \sin^2\left(\frac{1}{x_n}\right)$$

$$x_0 = \frac{1}{k\pi} \Rightarrow x_1 = -\frac{1}{k\pi} \sin^2 k\pi = 0$$

$$x_0 = \frac{2}{(2k+1)\pi} \Rightarrow x_1 = -\frac{2}{(2k+1)\pi} \sin^2 \frac{(2k+1)\pi}{2} = -\frac{2}{(2k+1)\pi} \sin^2\left(k\pi + \frac{\pi}{2}\right)$$

± 1 depending on k

Depending on x_0, the fixed point iteration is either going to converge to ξ or it is not.

$\rightarrow$ niether stable nor unstable

Problem 6

(a) $$x_{k+1} = \underbrace{x_k - \frac{f(x_k)}{f'(x_k)}}_{\text{RHS is a function of } x_k \text{ only}}$$

(b) $f(x) = (x+1)^2 - 1$, x_0 close to ξ

$$x_{k+1} = x_k - \frac{(x_k+1)^2 - 1}{2(x_k+1)} = \frac{1}{2}\frac{x_k^2}{x_k+1}$$

$$\lim_{k\to\infty} \frac{\varepsilon_{k+1}}{\varepsilon_k^2} = \lim_{k\to\infty} \frac{\frac{1}{2}\left|\frac{x_k^2}{x_k+1}\right|}{x_k^2} = \lim_{k\to\infty} \frac{\frac{1}{2}}{|x_k+1|} = \frac{1}{2}$$

(c) $$x_{k+1} = x_k - \frac{x_k^2}{2x_k} = \frac{1}{2}x_k$$

$$\Rightarrow \lim_{k\to\infty} \frac{\varepsilon_{k+1}}{\varepsilon_k} = \frac{1}{2}$$

Problem 7

$$x_{k+1} = x_k - \frac{f(x_k)}{g(x_k)}$$

$$g(x_k) = \frac{f(x_k + f(x_k)) - f(x_k)}{f(x_k)}$$

ⓐ $$x_{k+1} = \underbrace{x_k - \frac{f(x_k)}{\frac{f(x_k+f(x_k)) - f(x_k)}{f(x_k)}}}_{\text{RHS is a function of } x_k \checkmark} \qquad F(x_k)$$

ⓑ as $k \to \infty$, $f(x_k) \to 0$, take it as h ... (similar to the secant method)

$x_{k+1} - x_k$

ⓒ $$\xi - x_{k+1} = \xi - F(x_k) \qquad F(\xi) = \xi$$

$$= \xi - \left(F(\xi) + (x_k - \xi) F'(\xi) + \frac{(x_k-\xi)^2}{2} F'(\eta) \right)$$

$$\Rightarrow \xi - x_{k+1} = -(x_k - \xi)\cancel{F'(\xi)} - \frac{(x_k-\xi)^2}{2} F'(\eta)$$

$\to 0$ (can be shown)

$\eta \to \xi$ as $k \to \infty$

$$\Rightarrow \lim_{k\to\infty} \frac{|\xi - x_{k+1}|}{|\xi - x_k|} = \frac{|F'(\xi)|}{2}$$