Homework 4

1 $n \geq 2$, $\vec{v}, \vec{v}_1 \in \mathbb{R}^n_*$ $\longrightarrow$ $H(\vec{v})$, $H_1(\vec{v}_1)$

(a)

$$H\vec{x} = \vec{x} - 2\frac{\vec{v}^T\vec{x}}{\vec{v}^T\vec{v}}\vec{v}$$

Let $\vec{x} = \vec{v}$ $\Rightarrow$ $H\vec{v} = \vec{v} - 2\vec{v} = -\vec{v}$ $\Rightarrow$ $\boxed{\lambda = -1}$

Let $\vec{x} = \vec{u}$ where $\vec{u}$ is an eigenvector perpendicular to $\vec{v}$, i.e., $\vec{u}^T\vec{v} = 0$

$$\Rightarrow H\vec{u} = \vec{u} - 2\frac{\vec{v}^T\vec{u}}{\vec{v}^T\vec{v}}\vec{v} = \vec{u} \Rightarrow \boxed{\lambda = 1}$$

this one has multiplicity of $n-1$, since there are $n-1$ independent vectors orthogonal to $\vec{v}$

(b) $\det(H) = \lambda_1 \lambda_2 \cdots \lambda_n = -1$

(c) • $(HH_1)^T HH_1 = H_1^T H^T H H_1 = H_1^T H_1 = I$

• $\det(HH_1) = \det(H)\det(H_1) = 1$

Alternative: $H_1 = I - \dfrac{2}{\vec{v}_1^T\vec{v}_1}\vec{v}_1\vec{v}_1^T$

$H = I - \dfrac{2}{\vec{v}^T\vec{v}}\vec{v}\vec{v}^T$

$$\Rightarrow HH_1 = \left(I - \frac{2}{\vec{v}^T\vec{v}}\vec{v}\vec{v}^T\right)\left(I - \frac{2}{\vec{v}_1^T\vec{v}_1}\vec{v}_1\vec{v}_1^T\right) = \ldots$$

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(a) $$A = \begin{bmatrix} \boxed{9} & -6 \\ \boxed{12} & -8 \\ \boxed{0} & 20 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} 9 \\ 12 \\ 0 \end{bmatrix} \qquad \beta = 9, \quad c = \sqrt{\vec{x}^T \vec{x}} = 15$$

$$\Rightarrow \vec{v} = \vec{x} + c\vec{e}_1 = \begin{bmatrix} 9 \\ 12 \\ 0 \end{bmatrix} + 15 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 24 \\ 12 \\ 0 \end{bmatrix} = 12 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$H_1 = I - \frac{2}{\vec{v}^T \vec{v}} \vec{v}\vec{v}^T = \begin{bmatrix} -0.6 & -0.8 & 0 \\ -0.8 & 0.6 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$H_1 A = \begin{bmatrix} -15 & 10 \\ 0 & \boxed{0} \\ 0 & \boxed{20} \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} 0 \\ 20 \end{bmatrix} \qquad \beta = 0, \quad c = \sqrt{\vec{x}^T \vec{x}} = 20$$

augment with zeros

$$\Rightarrow \vec{v} = \vec{x} + c\vec{e}_1 = \begin{bmatrix} 0 \\ 20 \end{bmatrix} + \begin{bmatrix} 20 \\ 0 \end{bmatrix} = 20 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \vec{v} = 20 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow H_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$H_2 H_1 A = \begin{bmatrix} -15 & 10 \\ 0 & -20 \\ 0 & 0 \end{bmatrix} \Rightarrow A = H_1 H_2 R = QR$$

$$Q = \begin{bmatrix} -0.6 & 0 & 0.8 \\ -0.8 & 0 & -0.6 \\ 0 & -1 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} -15 & 10 \\ 0 & -20 \\ 0 & 0 \end{bmatrix}$$

$$\hat{Q} = \begin{bmatrix} -0.6 & 0 \\ -0.8 & 0 \\ 0 & -1 \end{bmatrix}, \quad \hat{R} = \begin{bmatrix} -15 & 10 \\ 0 & -20 \end{bmatrix}$$

(b) $H\vec{x} = \alpha\vec{y}$, $\quad \vec{x},\vec{y}\in\mathbb{R}^n_*$

$$\vec{v} = \vec{x} + c\vec{y}$$

$$H\vec{x} = \vec{x} - \frac{2}{\vec{v}^T\vec{v}}(\vec{v}^T\vec{x})\vec{v}$$

$$\vec{v}^T\vec{x} = (\vec{x}^T + c\vec{y}^T)\vec{x} = \vec{x}^T\vec{x} + c\vec{y}^T\vec{x}$$

$$\vec{v}^T\vec{v} = (\vec{x}^T + c\vec{y}^T)(\vec{x} + c\vec{y}) = \vec{x}^T\vec{x} + 2c\vec{y}^T\vec{x} + c^2\vec{y}^T\vec{y}$$

$$\Rightarrow H\vec{x} = \frac{(\vec{x}^T\vec{x} + 2c\vec{y}^T\vec{x} + c^2\vec{y}^T\vec{y})\vec{x} - 2(\vec{x}^T\vec{x} + c\vec{y}^T\vec{x})(\vec{x} + c\vec{y})}{\vec{x}^T\vec{x} + 2c\vec{y}^T\vec{x} + c^2\vec{y}^T\vec{y}}$$

$$= \frac{\cancel{\vec{x}^T\vec{x}}\,x + \cancel{2c\vec{y}^T x}\,\vec{x} + \cancel{c^2\vec{y}^T}\vec{y}\vec{x} - \cancel{2\vec{x}^T}\vec{x}\,\vec{x} - 2\vec{x}^T\vec{x}\,c\vec{y} - \cancel{2c\vec{y}^T}\vec{x}\,\vec{x} - 2c^2\vec{y}^T\vec{x}\,\vec{y}}{\vec{x}^T\vec{x} + 2c\vec{y}^T\vec{x} + c^2\vec{y}^T\vec{y}}$$

$$\Rightarrow H\vec{x} = \frac{\overbrace{(c^2\vec{y}^T\vec{y} - \vec{x}^T\vec{x})}\vec{x} - 2c(\vec{x}^T\vec{x} + c\vec{y}^T\vec{x})\vec{y}}{\vec{x}^T\vec{x} + 2c\vec{y}^T\vec{x} + c^2\vec{y}^T\vec{y}}$$

$$= 0 \Rightarrow c^2 = \frac{\vec{x}^T\vec{x}}{\vec{y}^T\vec{y}} \rightsquigarrow c = \begin{cases} \operatorname{sign}(\vec{y}^T\vec{x})\sqrt{\dfrac{\vec{x}^T\vec{x}}{\vec{y}^T\vec{y}}} & \operatorname{sign}(\vec{y}^T\vec{x}) \neq 0 \\ \sqrt{\dfrac{\vec{x}^T\vec{x}}{\vec{y}^T\vec{y}}} & \operatorname{sign}(\vec{y}^T\vec{x}) = 0 \end{cases}$$

$$\rightsquigarrow H = I - \frac{2}{\vec{v}^T\vec{v}}\vec{v}\vec{v}^T \quad \text{where} \quad \vec{v} = \vec{x} + c\vec{y}$$

3 (a)

$$\underbrace{\kappa_2(A^TA)} = \kappa_2((QR)^T QR) = \kappa_2(R^T Q^T Q R) = \kappa_2(R^T R)$$

$$\hookrightarrow = \frac{|\lambda_{max}(A^TA)|}{|\lambda_{min}(A^TA)|}$$

(b) $QR\vec{x} = \vec{b} \Rightarrow \underbrace{R\vec{x}} = \underbrace{Q^T\vec{b}}$

$n \times (n + \underbrace{n-1}_{\text{additions}}) = 2n^2 - n$

multiplications

solve by backward substitution needs n^2 operations

$\rightarrow$ total for QR solve: $3n^2 - n$

total for LU solve: $\underbrace{2n^2 - n}$

$L\vec{y} = \vec{b}$ $\quad n^2 - n$ operations

$U\vec{x} = \vec{y}$ $\quad n^2$ "

④ (a) $\vec{b}^{(2)} = \vec{a}^{(2)} - (\vec{q}^{(1)})^T \vec{a}^{(2)} \vec{q}^{(1)}$

$$(\vec{q}^{(1)})^T \vec{b}^{(2)} = (\vec{q}^{(1)})^T \vec{a}^{(2)} - (\vec{q}^{(1)})^T \vec{a}^{(2)} \cancel{(\vec{q}^{(1)})^T \vec{q}^{(1)}}^{\,1} = 0$$

$\Rightarrow$ $b^{(2)}$, and hence $q^{(2)}$, is perpendicular to $q^{(1)}$.

Induction: suppose that $q^{(1)}, q^{(2)}, \ldots, q^{(i-1)}$ are orthogonal to each other. Then $q^{(i)}$ is orthognal to them:

$$\vec{b}^{(i)} = \vec{a}^{(i)} - \sum_{k=1}^{i-1} (\vec{q}^{(k)})^T \vec{a}^{(i)} \vec{q}^{(k)} \quad (*)$$

$$\Rightarrow (\vec{q}^{(j)})^T \vec{a}^{(i)} - \underbrace{\sum_{k=1}^{i-1} (\vec{q}^{(k)})^T \vec{a}^{(i)} (\vec{q}^{(j)})^T \vec{q}^{(k)}}_{(\vec{q}^{(j)})^T \vec{a}^{(i)}} = 0$$

all terms are zero except when $j = k$

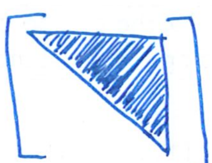
(b)

$(*)$: $\vec{a}^{(i)} = \vec{b}^{(i)} + \sum_{k=1}^{i-1} (\vec{q}^{(k)})^T \vec{a}^{(i)} \vec{q}^{(k)}$

$$\Rightarrow \vec{a}^{(i)} = \|\vec{b}^{(i)}\|_2 \vec{q}^{(i)} + \sum_{k=1}^{i-1} (\vec{q}^{(k)})^T \vec{a}^{(i)} \vec{q}^{(k)}$$

$$\Rightarrow \vec{a}^{(i+1)} = \|\vec{b}^{(i+1)}\|_2 \vec{q}^{(i+1)} + \sum_{k=1}^{i} \left((\vec{q}^{(k)})^T \vec{a}^{(i+1)}\right) \vec{q}^{(k)} \quad (**)$$

(c)

$$A = \hat{Q}\hat{R} = [\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_n]$$ 

$\hat{R}$

$$\Rightarrow \vec{a}_{i+1} = \vec{q}_1 \hat{r}_{1,i+1} + \vec{q}_2 \hat{r}_{2,i+1} + \cdots + \vec{q}_{i+1} \hat{r}_{i+1,i+1}$$

compare with $(**)$ $\Rightarrow$ $\hat{r}_{k,i+1} = (\vec{q}^{(k)})^T \vec{a}^{(i+1)}$ $\quad k = 1, \ldots, i$

$\hat{r}_{i+1,i+1} = \|\vec{b}^{(i+1)}\|_2$

$\hat{r}_{1,1} = \|\vec{a}_1\|_2$

see "Gramschmidt.m"

(d)

$$A = \begin{bmatrix} 1+\varepsilon & 1 & 1 \\ 1 & 1+\varepsilon & 1 \\ 1 & 1 & 1+\varepsilon \end{bmatrix} \qquad 0 < \varepsilon \ll 1$$

$$\vec{q}^{\,(1)} = \frac{1}{\sqrt{(1+\varepsilon)^2+1+1}} \begin{bmatrix} 1+\varepsilon \\ 1 \\ 1 \end{bmatrix} \approx \frac{1}{\sqrt{3+2\varepsilon}} \begin{bmatrix} 1+\varepsilon \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{b}^{\,(2)} = \begin{bmatrix} 1 \\ 1+\varepsilon \\ 1 \end{bmatrix} - \underbrace{(\vec{q}^{\,(1)})^T \begin{bmatrix} 1 \\ 1+\varepsilon \\ 1 \end{bmatrix}}_{\sqrt{3+2\varepsilon}} \vec{q}^{\,(1)} = \begin{bmatrix} -\varepsilon \\ \varepsilon \\ 0 \end{bmatrix} \Rightarrow \vec{q}^{\,(2)} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

$$\vec{b}^{\,(3)} = \begin{bmatrix} 1 \\ 1 \\ 1+\varepsilon \end{bmatrix} - \underbrace{(\vec{q}^{\,(1)})^T \begin{bmatrix} 1 \\ 1 \\ 1+\varepsilon \end{bmatrix}}_{} \vec{q}^{\,(1)} - \underbrace{(\vec{q}^{\,(2)})^T \begin{bmatrix} 1 \\ 1 \\ 1+\varepsilon \end{bmatrix}}_{0} \vec{q}^{\,(2)} = \begin{bmatrix} -\varepsilon \\ 0 \\ \varepsilon \end{bmatrix}$$

$$= \begin{bmatrix} 1+\varepsilon \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \vec{q}^{\,(3)} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\Rightarrow Q = \begin{bmatrix} \frac{1+\varepsilon}{\sqrt{3+2\varepsilon}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3+2\varepsilon}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3+2\varepsilon}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\cos\theta_{12} = \frac{(\vec{q}^{\,(1)})^T \vec{q}^{\,(2)}}{\|\vec{q}^{\,(1)}\|_2 \|\vec{q}^{\,(2)}\|_2} = 0 \Rightarrow \theta_{12} = \frac{\pi}{2}$$

$$\cos\theta_{13} = \frac{(\vec{q}^{\,(1)})^T \vec{q}^{\,(3)}}{\|\vec{q}^{\,(1)}\|_2 \|\vec{q}^{\,(3)}\|_2} = 0 \Rightarrow \theta_{13} = \frac{\pi}{2}$$

$$\cos\theta_{23} = \frac{(\vec{q}^{\,(2)})^T \vec{q}^{\,(3)}}{\|\vec{q}^{\,(2)}\|_2 \|\vec{q}^{\,(3)}\|_2} = \frac{\frac{1}{2}}{1} = \frac{1}{2} \Rightarrow \underline{\theta_{23} = \frac{\pi}{3}}$$

5 (a) $a = (1-1) + 10^{-16} = 0 + 10^{-16} = 10^{-16}$

$b = 1 - (1+10^{-16}) = 1 - 1 = 0$

(b) see the code "HW4_Q5b.m"

(c) see the code "HW4_Q5c.m"

6

(a) $(A+\alpha I)\vec{x} = (\lambda+\alpha)\vec{x}$

$\Rightarrow \cancel{A\vec{x}} + \alpha\vec{x} = \cancel{\lambda\vec{x}} + \alpha\vec{x}$ ✓

(b) $A\vec{x} = \lambda\vec{x} \Rightarrow \underbrace{\alpha A}\vec{x} = \underbrace{\alpha\lambda}\vec{x}$ ✓

(c) $A\vec{x} = \lambda\vec{x} \Rightarrow \vec{x} = \lambda A^{-1}\vec{x} \Rightarrow A^{-1}\vec{x} = \frac{1}{\lambda}\vec{x}$ ✓

(d) $A^2\vec{x} = AA\vec{x} = A\lambda\vec{x} = \lambda A\vec{x} = \lambda^2\vec{x}$

$A^3\vec{x} = AA^2\vec{x} = A\lambda^2\vec{x} = \lambda^2 A\vec{x} = \lambda^3\vec{x}$

$\vdots$

$A^k\vec{x} = \quad \cdots \quad = \lambda^k\vec{x}$ ✓

(e) $A\vec{x} = \lambda\vec{x} \Rightarrow AS^{-1}S\vec{x} = \lambda\vec{x} \Rightarrow \underbrace{SAS^{-1}}_{B}S\vec{x} = \lambda S\vec{x}$

$\Rightarrow BS\vec{x} = \lambda S\vec{x}$

- B has the same eigenvalues λ. ✓
- Eigenvectors of B are $S\vec{x}$.

(f) not true X

counterexample: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \lambda_1 = \lambda_2 = 1$

(g) not true X

counterexample: $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \Rightarrow \lambda = \pm i$

(h) $\det(A) = \lambda_1\lambda_2 \cdots \lambda_n = 0 \Rightarrow$ at least one of the eigenvalues has to be zero. ✓

(i) not true X

counterexample: $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ any strictly triangular matrix works as a counterexample.

(j) $A^T = A$

$A\vec{x} = \lambda\vec{x}$ ① $\Rightarrow A\vec{x}^* = \lambda^*\vec{x}^*$ ② (* denotes complex conjugate)

①: $\vec{x}^{*T} A\vec{x} = \lambda\vec{x}^{*T}\vec{x}$ ①'

②: $\underbrace{\vec{x}^T A\vec{x}^*}_{} = \lambda^*\vec{x}^T\vec{x}^*$ ②'

$\hookrightarrow = (A\vec{x}^*)^T\vec{x} = \vec{x}^{*T}A^T\vec{x} = \vec{x}^{*T}A\vec{x}$

$\Rightarrow$ ①' − ②': $0 = (\lambda - \lambda^*)\vec{x}^T\vec{x}^* \Rightarrow \lambda = \lambda^*$ (λ is real) ✓

(k) $A\vec{x} = \lambda_1\vec{x} \Rightarrow \vec{y}^T A\vec{x} = \lambda_1\vec{y}^T\vec{x}$ ①

$A\vec{y} = \lambda_2\vec{y} \Rightarrow \underbrace{\vec{x}^T A\vec{y}}_{} = \lambda_2\vec{x}^T\vec{y}$ ②

$\hookrightarrow = (A\vec{y})^T\vec{x} = \vec{y}^T A^T\vec{x} = \vec{y}^T A\vec{x}$

① − ②: $0 = (\lambda_1 - \lambda_2)\vec{x}^T\vec{y} \xRightarrow{\lambda_1 \neq \lambda_2} \vec{x}^T\vec{y} = 0$ ✓