Orthogonal Polynomials

Q: Is there another basis where finding the best $\mathcal{L}_2$ approximating polynomial is better conditioned than the monomials?

Goal: Find basis where the $\underline{\underline{M}}$ matrix from the previous lecture becomes diagonal.

Let $\{\varphi_j\}_{j=0}^{n}$ form a basis for $\mathbb{P}_n$. Consider

$$P_n(x) = \gamma_0 \varphi_0(x) + \dots + \gamma_n \varphi_n(x), \quad \gamma_i \in \mathbb{R}.$$

From earlier,

$$\underline{\underline{M}}\,\underline{\gamma} = \underline{\beta}, \quad M_{ij} = \langle \varphi_i, \varphi_j \rangle, \quad \beta_i = \langle f, \varphi_j \rangle.$$

$$= \int_a^b \varphi_i(x)\varphi_j(x)\underbrace{\omega(x)}_{\substack{\text{some weight fn.} \\ \omega(x) \geq 0}}dx$$

So $\underline{\underline{M}}$ will be diagonal if the $\{\varphi_i\}$ are orthogonal, i.e.

$$\langle \varphi_i, \varphi_j \rangle = D_i \cdot \delta_{ij}.$$

Def: Given $w \in C([a,b])$, $w(x) \geq 0$, we say that the sequence φ_i, $i = 0, 1, \ldots$ is a <u>system of orthogonal polynomials</u> on the interval (a,b) with respect to w if each φ_i is of exact degree i and if

$$\int_a^b \varphi_j(x)\,\varphi_i(x)\,w(x)\,dx \begin{cases} = 0 & j \neq i \\ \neq 0 & j = i \end{cases}$$

We now show that such systems exist, by G.S. orthogonalization.

Let $\varphi_0(x) = 1$.

Suppose $\varphi_i(x)$ given for $i = 0, \ldots, n$.

Define
$$\begin{cases} q(x) = x^{n+1} - a_0\varphi_0(x) - \ldots - a_n\varphi_n(x), \\ a_i = \dfrac{\langle x^{n+1}, \varphi_i \rangle}{\langle \varphi_i, \varphi_i \rangle} \end{cases}$$

Then,

$$\langle q, \varphi_i \rangle = \langle x^{n+1}, \varphi_i \rangle - a_i \langle \varphi_i, \varphi_i \rangle$$
$$= \langle x^{n+1}, \varphi_i \rangle - \langle x^{n+1}, \varphi_i \rangle = 0.$$

$\underline{EX}$: $(a,b) = (0,1)$, $\omega(x) = 1$.

$\varphi_0(x) = 1$

$\varphi_1(x) = x - a_0\varphi_0(x)$

$$\left.\begin{aligned} \langle \varphi_0, \varphi_0 \rangle &= \int_0^1 1^2 dx = 1 \\ \langle \varphi_0, x \rangle &= \int_0^1 x\, dx = 1/2 \end{aligned}\right\} \begin{aligned} &\Rightarrow a_0 = 1/2 \\ &\Rightarrow \varphi_1(x) = x - 1/2 \end{aligned}$$

$\varphi_2(x) = x^2 - d_1\varphi_1(x) - d_0\varphi_0(x)$

$$\langle \varphi_1, \varphi_1 \rangle = \int_0^1 (x - 1/2)^2 dx = \int_{-1/2}^{1/2} x^2 dx = 1/3\, x^3 \Big|_{-1/2}^{1/2} = \frac{1}{12}$$

$$\langle x^2, \varphi_1 \rangle = \int_0^1 (x^3 - 1/2\, x^2)\, dx = (1/4\, x^4 - 1/6\, x^3)\Big|_0^1$$

$$= 1/4 - 1/6 = 1/12$$

$$\Rightarrow d_1 = \frac{\langle x^2, \varphi_1 \rangle}{\langle \varphi_1, \varphi_1 \rangle} = 1$$

$$\langle x^2, \varphi_0 \rangle = \int_0^1 x^2 \cdot 1\, dx = 1/3 \Rightarrow d_0 = 1/3$$

$$\Rightarrow \varphi_2(x) = x^2 - x + 1/6.$$

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Note: $x' = (b-a)x + a$ can be used to map an orthogonal system on $(0,1)$ to any (a,b).

EX: [Legendre polynomials on $(-1,1)$]

Replace x by $\frac{x-a}{b-a} = \frac{x+1}{2}$.

$$\varphi_0'(x) = 1, \quad \varphi_1'(x) = 1/2\, x, \quad \varphi_2'(x) = \left(\frac{x+1}{2}\right)^2 - \frac{x}{2} - 1/2 + 1/6$$

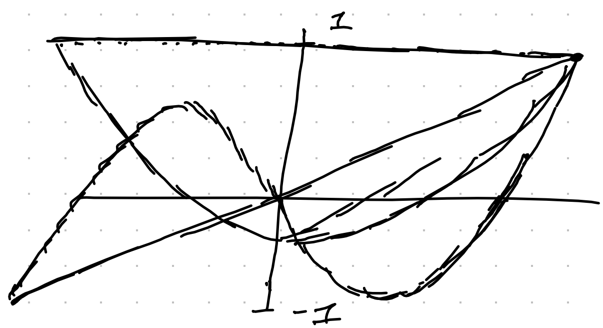
$$= \frac{x^2}{4} + \cancel{x/2} + 1/4 - \cancel{x/2} - 1/2 + 1/6$$

$$= 1/4\, x^2 - 1/12$$

Normalize so $\varphi_i(1) = 1$:

$$\varphi_0(x) = 1, \quad \varphi_1(x) = x, \quad \varphi_2(x) = 3/2\, x^2 - 1/2$$

Can show that $\varphi_3(x) = 5/2\, x^3 - 3/2\, x$.



<u>Ex:</u> (Chebyshev polynomials)

$T_n(x) = \cos(n \cdot \cos^{-1} x)$ forms an orthogonal system on $(-1, 1)$
with respect to the weight $w(x) = (1-x^2)^{-1/2}$.

We use the change of variables $x \in (0, \pi) \mapsto \cos(x) \in (-1, 1)$

$$\langle T_m, T_n \rangle = \int_{-1}^{1} 1/(1-x^2)^{1/2} \cos(m\cos^{-1}x) \cdot \cos(n\cos^{-1}x)\, dx$$

$$t = \cos^{-1}(x) \Rightarrow dt = -\frac{1}{(1-x^2)^{1/2}} dx$$

$$= \int_0^{\pi} \cos(mt)\cos(nt)\, dt = \tfrac{1}{2}\int_0^{\pi} (\cos((m+n)t) + \cos((m-n)t))\, dt$$

$$= \begin{cases} 0 & m \neq n \\ \pi/2 & m = n \end{cases}$$

Thm: Given $f \in L^2_w(a,b)$, there exists a unique polynomial $P_n \in \mathbb{P}_n$ such that $\|f - P_n\|_2 = \min_{q \in \mathbb{P}_n} \|f - q\|_2$.

Pf: Let $\{\varphi_j\}_{j=0}^{n}$ be a system of orthonormal polynomials for $\mathbb{P}_n$, i.e., $\langle \varphi_i, \varphi_j \rangle = \delta_{ij}$.

We may write

$$q_n^{(x)} = \sum_{j=0}^{n} \beta_j \varphi_j(x).$$

Define $E(\underline{\beta}) = \|f - q_n\|_2^2$, i.e.,

$$E(\beta) = \langle f - q_n, f - q_n \rangle$$

$$= \|f\|_2^2 - 2\langle f, q_n \rangle + \|q_n\|_2^2$$

Observe that: $\|q_n\|_2^2 = \sum_{i,j=0}^{n} \beta_i \beta_j \langle \varphi_i, \varphi_j \rangle = \sum_{j=0}^{n} \beta_j^2$.

$$\langle f, q_n \rangle = \sum_{j=0}^{n} \beta_j \langle f, q_n \rangle,$$

Hence,

$$E(\underline{\beta}) = \|f\|_2^2 - 2\sum_{j=0}^{n} \beta_j \langle f, \psi_j \rangle + \sum_{j=0}^{n} \beta_j^2$$

$$= \sum_{j=0}^{n} \left(\beta_j - \langle f, \psi_j \rangle\right)^2 + \|f\|_2^2 - \sum_{j=0}^{n} |\langle f, \psi_j \rangle|^2$$

Clearly minimized for $\overline{\beta_j = \langle f, \psi_j \rangle}$.

So that $P_n(x) = \sum_{j=0}^{n} \langle f, \psi_j \rangle \psi_j$. $\square$