Householder's Method

(P) Given $\underline{\underline{A}} \in \mathbb{R}^{n \times n}$, find $\underline{v} \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$ with

$$\underline{\underline{A}}\,\underline{v} = \lambda \underline{v}$$

We call $(\underline{v}, \lambda)$ an eigenvector-eigenvalue pair.

We will find $(\underline{v}, \lambda)$ via QR factorization!

Q: How do we compute $\underline{\underline{A}} = \underline{\underline{Q}}\,\underline{\underline{R}}$?

Def: Let $\underline{v} \in \mathbb{R}^n$, $\underline{v} \neq \underline{0}$. The Householder matrix of order n is

$$\underline{\underline{H}}(\underline{v}) = \underline{\underline{I}} - \left(\frac{2}{\underline{v}^T \underline{v}}\right) \underline{v}\,\underline{v}^T$$

Note: For all $\underline{x} \in \mathbb{R}^n$, $\underline{\underline{H}}(\underline{v})\,\underline{x} = \underline{x} - 2\left(\frac{\underline{x}^T \underline{v}}{\underline{v}^T \underline{v}}\right) \underline{v}$

So $\underline{\underline{H}}(\underline{v})\underline{x}$, $\underline{x}$, and $\underline{v}$ are coplanar!

Note: Define $\underline{v}^{\perp} = \{ \underline{x} \mid \underline{v}^T \underline{x} = 0 \}$

Then $\underline{\underline{H}}(\underline{v})\, \underline{v}^{\perp} = \{ \underline{\underline{H}}(\underline{v})\,\underline{x} \mid \underline{v}^T \underline{x} = 0 \} = \underline{v}^{\perp}$!

(i.e. vectors orthogonal to $\underline{v}$ left invariant by $H(\underline{v})$)

Note: $\forall\, \underline{x} \in \mathbb{R}^n$, $\underline{v}^T \underline{\underline{H}}\,\underline{x} = -\underline{v}^T \underline{x}$

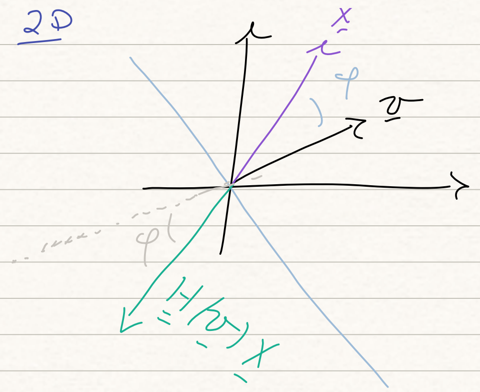
So if $\angle(\underline{x}, \underline{v}) = \varphi$, $\angle(\underline{\underline{H}}(\underline{v})\,\underline{x}, \underline{v}) = \pi + \varphi$

angle between

Geometrically, this is a reflection over $\underline{v}^{\perp}$, the hyperplane orthogonal to $\underline{v}$!

We call $H(\underline{v})$ a Householder Reflector.

2D



Lemma: Every Householder Reflector is symmetric ($H^T = H$) and orthogonal ($H^T H = H H^T = I$)

Pf:

$$H(v)^T = \left(\underline{\underline{I}} - \frac{2}{\underline{v}^T \underline{v}} \underline{v}\underline{v}^T\right)^T = \underline{\underline{I}}^T - \left(\frac{2}{\underline{v}^T \underline{v}}\right)\left(\underline{v}\underline{v}^T\right)^T$$

$$= \underline{\underline{I}} - \left(2 / \underline{v}^T \underline{v}\right) \underline{v}\,\underline{v}^T = H(v) \quad \checkmark$$

$$H(v)^2 = \left(I - \frac{2}{\|v\|_2^2} vv^T\right)\left(I - \frac{2}{\|v\|_2^2} vv^T\right)$$

$$= \underline{\underline{I}} - \frac{4}{\|\underline{v}\|_2^2} \underline{v}\underline{v}^T + \frac{4}{\|v\|_2^4} \underbrace{\left(vv^T\right)\left(vv^T\right)}_{\substack{= v(v^T v) v^T \\ = \|v\|_2^2 \underline{v}\underline{v}^T}}$$

$$= \underline{\underline{I}} - \frac{4}{\cancel{\|v\|^2}} vv^T + \frac{4}{\cancel{\|v\|^2}} vv^T$$

$$= \underline{\underline{I}} \quad \checkmark \qquad \square$$

Lemma: Let $1 \leq k < n$ and let $\underline{\underline{H}}_k \in \mathbb{R}^{k \times k}$ be a Householder Reflector. Then the matrix

$$\underline{\underline{H}} = \begin{pmatrix} \underline{\underline{I}}_{n-k} & \underline{\underline{0}} \\ \underline{\underline{0}}^T & \underline{\underline{H}}_k \end{pmatrix} \in \mathbb{R}^{n \times n}, \qquad \underline{\underline{I}} \in \mathbb{R}^{(n-k) \times (n-k)}, \quad \underline{\underline{0}} \in \mathbb{R}^{(n-k) \times k}$$

is a Householder Reflector.

<u>Note</u>: $\underline{\underline{H}}$ keeps an $(n-k)$-dim subspace fixed and reflects in the complement of dimension k.

<u>Pf</u>: $\underline{\underline{H}}_k = \underline{\underline{I}}_k - 2/\|\underline{v}_k\|^2 \, \underline{v}_k \underline{v}_k^T$, $\underline{v}_k \in \mathbb{R}^k$

Define $\underline{v} = (\underline{0}_{n-k}^T, \underline{v}_k^T)^T$.

Then $\underline{\underline{H}} = \underline{\underline{I}}_n - 2/\|\underline{v}\|^2 \, \underline{v}\underline{v}^T$ $\square$