Notes on Reed-Muller codes

Yanling Chen
Q2S, Centre of Excellence
Norwegian University of Science and Technology
Trondheim, Norway

Han Vinck
Institute for Experimental Mathematics
University of Duisburg-Essen
Essen 45326, Germany
Email: {julia, vinck}@iem.uni-due.de

Abstract

In this paper, we consider the Reed-Muller (RM) codes. For the first order RM code, we prove that it is unique in the sense that any linear code with the same length, dimension and minimum distance must be the first order RM code; For the second order RM code, we give a constructive linear sub-code family for the case when m is even. This is an extension of Corollary 17 of Ch. 15 in the coding book by MacWilliams and Sloane. Furthermore, we show that the specified sub-codes of length ≤ 256 have minimum distance equal to the upper bound or the best known lower bound for all linear codes of the same length and dimension. As another interesting result, we derive an additive commutative group of the symplectic matrices with full rank.

1 Introduction

Let C be an $[n, k, d_{min}]$ binary linear code of length n, dimension k and minimum distance d_{min} . Let $V = \{0, 1\}$ and let $v = (v_1, \dots, v_m)$ range over V^m , the set of all binary m-tuples. Any function $f(v) = f(v_1, \dots, v_m)$ which takes on the values 0 and 1 is called a *Boolean function*. Reed-Muller (or RM) codes can be defined very simply in terms of Boolean functions. As stated in [2], the rth order binary RM code $\mathcal{R}(r, m)$ of length $n = 2^m$, for $0 \le r \le m$, is the set of all vectors f, where $f(v_1, \dots, v_m)$ is a Boolean function of degree at most r. In this paper, we consider the RM codes $\mathcal{R}(r, m)$

for r=1,2. The standard references are Ch. 14 and Ch. 15 in [2]. Necessary preliminaries are introduced at the beginning of Section 2 and Section 3, respectively.

The first order RM code is a linear $[2^m, 1+m, 2^{m-1}]$ code. It is optimal in the sense that it reaches the Plotkin bound (refer to Theorem 8 of Ch. 2 in [2]). Besides, it has a simple weight distribution with one all-zero codeword, one all-one codeword and $2(2^m-1)$ codewords of weight 2^{m-1} . In Section 2, we show that it has another interesting property: uniqueness, in the sense that, any linear code with the same length, dimension and minimum distance must be the first order RM code.

For the second order RM code, we recall that there is a linear sub-code family for odd m as given in Theorem 1.1 (also Corollary 17 of Ch. 15 in [2]). Note that the sub-code $\mathcal{R}_{1,2t+1}^t$ (or $\mathcal{R}_{2,2t+1}^t$) is optimal in the sense that no codes exist with the same length and weight set, but with larger dimension (refer to Proposition 14 in [3]). Correspondingly for even m, a nonlinear sub-code family (the generalized Kerdock code $\mathcal{DG}(m,d)$) is introduced in Theorem 19. of Ch. 15 in [2]. In particular, when d=m/2, $\mathcal{DG}(m,d)$ is one of the best-known nonlinear codes: the Kerdock code $\mathcal{K}(m)$.

As shown in [2], these nonlinear sub-codes turn out to have good parameters. However, a linear sub-code family for even m is still of our interest due to the advantages of linear codes in the straightforward decoding and practical implementations. In Section 3, we reconsider the case for even m and build up a linear sub-code family as given in Theorem 3.5. At the end, we conclude in Section 4.

Theorem 1.1. See Corollary 17 of Ch. 15 in [2]. Let m = 2t + 1 be odd, and let d be any number in the range $1 \le d \le t$. Then there exist two

$$[2^m, m(t-d+2)+1, 2^{m-1}-2^{m-d-1}]$$

sub-codes $\mathcal{R}^d_{1,2t+1}$ and $\mathcal{R}^d_{2,2t+1}$ of $\mathcal{R}(2,m)$. These are obtained respectively, by extending the cyclic sub-codes of $\mathcal{R}(2,m)^*$ having idempotents

$$\theta_0 + \theta_1^* + \sum_{j=d}^t \theta_{l_j}^*$$
 and $\theta_0 + \theta_1^* + \sum_{j=1}^{t-d+1} \theta_{l_j}^*$, $l_j = 1 + 2^j$.

These codes have weights 2^{m-1} and $2^{m-1} \pm 2^{m-h-1}$ for all h in the range $d \le h \le t$.

2 The first order Reed-Muller code

The first order Reed-Muller code $\mathcal{R}(1,m)$ consists of all vectors $u_0 \mathbf{1} + \sum_{i=1}^m u_i \mathbf{v_i}$, $u_i = 0$ or 1, corresponding to the linear Boolean functions. Define the *orthogonal code* \mathcal{O}_m to be the $[2^m, m, 2^{m-1}]$ code consisting of the vectors $\sum_{i=1}^m u_i \mathbf{v_i}$. Then

$$\mathcal{R}(1,m) = \mathcal{O}_m \cup (\mathbf{1} + \mathcal{O}_m),$$

is a $[2^m, 1+m, 2^{m-1}]$ code.

Theorem 2.1. (Uniqueness of the first order RM code) Any linear code with parameters $[2^m, 1+m, 2^{m-1}]$ is equivalent to the first order Reed-Muller code. They have a unique weight distribution $1 + 2(2^m - 1)t^{2^{m-1}} + t^{2^m}$.

Proof. Let C be any linear $[2^m, 1+m, 2^{m-1}]$ code. Let G be its generator matrix of systematic form. We show in the following 4 steps that C must be $\mathcal{R}(1, m)$.

1st step: Construct code C_1 from C.

Deleting the first row and the first column of G, we get a new generator matrix G_1 . Let C_1 be the code generated by G_1 . Then C_1 is a linear $[2^m - 1, m, d_1]$ code, where $d_1 \geq 2^{m-1}$.

Let $n_1 = 2^m - 1$, $k_1 = m$ and assume that $d_1 \ge 2^{m-1} + 1$. According to the Plotkin bound, the number of the codewords of C_1 must satisfy the following inequality

$$|C_1| \le 2\left[\frac{d_1}{2d_1 - n_1}\right] \le 2\left[\frac{d}{3}\right] < d \le n_1 = 2^m - 1 < 2^m.$$

This contradicts the fact $|C_1| = 2^{k_1} = 2^m$. So $d_1 = 2^{m-1}$.

2nd step: C_1 is the Simplex code.

We consider the dual code of C_1 , C_1^{\perp} , a linear $[2^m - 1, 2^m - m - 1, d_1^{\perp}]$ code, where $d_1^{\perp} \geq 1$.

Suppose $d_1^{\perp} = 1$. Then there is a codeword **a** of weight 1 in C_1^{\perp} with only one 1-entry, whose coordinate is assumed to be *i*. Since $\mathbf{a} \in C_1^{\perp}$, correspondingly the *i*-th position of all codewords in C_1 must be 0. Deleting the *i*-th coordinate of code C_1 , we get a linear $[2^m - 2, m, 2^{m-1}]$ code, which is impossible due to the Plotkin bound.

Suppose $d_1^{\perp}=2$. Then there is a codeword **b** of weight 2 with only two 1-entries, whose coordinates are assumed to be i and j. Since $\mathbf{b} \in C_1^{\perp}$, correspondingly the i-th position and the j-th position of all codewords in C_1 must have the same entries. Note that it is impossible that the i-th

and j-th positions of all codewords have only 1-entry. Suppose that they are all 0-entry. Then by deleting the i-th and j-th coordinates of code C_1 , we can get a linear $[2^m - 3, m, 2^{m-1}]$ code. However, such a linear code is not existent due to the Plotkin bound. Now we consider the case when the i-th and j-th positions of all codewords in C_1 have not only 0-entry but also 1-entry. We take the codewords with 0-entry at positions i and j and then delete both coordinates. The derived codewords build up a linear $[2^m - 3, m - 1, 2^{m-1}]$ code, which is also not existent due to the Plotkin bound.

As a conclusion of above discussion, we have $d_1^{\perp} \geq 3$. It is easy to verify that when $e = \lfloor \frac{d_1^{\perp} - 1}{2} \rfloor = 1$, C_1^{\perp} reaches the Hamming bound (refer to Theorem 6 of Ch. 1 in [2]) and thus C_1^{\perp} is a perfect single-error-correcting code. Since the binary linear perfect single-error-correcting code with the same length and dimension is unique [4], so $d_1^{\perp} = 3$ and C_1^{\perp} must be the binary Hamming code. Therefore, C_1 is the Simplex code and has a unique weight enumerator $1 + (2^m - 1)t^{2^{m-1}}$. So far we have proved that any binary linear with parameters $[2^m - 1, m, 2^{m-1}]$ must be the Simplex code.

3rd step: Construct code C back from code C_1 .

Recall that G is a generator matrix of C in a systematic form. We get G_1 , the generator matric of C_1 , by deleting the first row and the first column of G. Now in order to get back C from C_1 , first we add an all-zero column to G_1 and denote the extended matrix as G_2 . Then we add a row to G_2 so as to get a matrix equivalent to G and thus code C.

Since C_1 is the Simplex code, so the columns of G_1 are the binary representations of the numbers from 1 to 2^m-1 . Note that G_2 is constructed by adding an all-0 column into G_1 . So the columns of G_2 go through all binary representations of the numbers from 0 to 2^m-1 . Clearly C_2 is equivalent to the orthogonal code \mathcal{O}_m . Without loss of generality, we use the Boolean functions v_1, \dots, v_m to denote the basis rows of G_2 . Then the basis codeword \mathbf{c} added into G_2 can be specified as a Boolean function $f(v) = f(v_1, \dots, v_m)$. Clearly $C = C_2 \cup (\mathbf{c} + C_2) = \mathcal{O}_m \cup (f + \mathcal{O}_m)$.

4th step: Code C is the first order Reed-Muller code.

For any vector $u = (u_1, \dots, u_m) \in V^m$, f(u) denote the value of the Boolean function f at u. Let $F(u) = (-1)^{f(u)}$. The Hadamard transform of F (refer to p. 414 of Ch. 14 in [2]) is given by

$$\hat{F}(u) = \sum_{v \in V^m} (-1)^{u \cdot v} F(v), \quad u \in V^m,$$
$$= \sum_{v \in V^m} (-1)^{u \cdot v + f(v)}.$$

 $\hat{F}(u)$ is equal to the number of 0's minus the number of 1's in the binary vector $f + \sum_{i=1}^{m} u_i v_i$. Thus

$$\hat{F}(u) = 2^m - 2 \text{dist}\{f, \sum_{i=1}^m u_i v_i\}.$$

Here dist $\{\mathbf{a}, \mathbf{b}\}$ is the Hamming distance of two binary vectors \mathbf{a} and \mathbf{b} . Note that $f + \sum_{i=1}^{m} u_i v_i \in C$ and code C has minimum distance 2^{m-1} , i.e.,

$$\operatorname{dist}\{f, \sum_{i=1}^{m} u_i v_i\} \ge 2^{m-1}.$$

Therefore,

$$\hat{F}(u) < 0$$
, for any $u \in V^m$.

According to Lemma 2 of Ch. 14 in [2], for any $v \neq 0$, $v \in V^m$,

$$\sum_{u \in V^m} \hat{F}(u)\hat{F}(u+v) = 0.$$

Let W be the set such that $\hat{F}(u) = 0$ for $u \in W$. Then for $u \in V^m \setminus W$, we have $\hat{F}(u) < 0$. For any $v \neq 0$, $v \in V^m$,

$$\sum_{u \in V^m} \hat{F}(u)\hat{F}(u+v) = \sum_{u \in \{W \cup (-v+W)\}} \hat{F}(u)\hat{F}(u+v) + \sum_{u \in V^m \setminus \{W \cup (-v+W)\}} \hat{F}(u)\hat{F}(u+v)$$

$$= \sum_{u \in V^m \setminus \{W \cup (-v+W)\}} \hat{F}(u)\hat{F}(u+v)$$

$$= 0,$$

where $-v + W = \{u - v | u \in W\}$. Clearly, we have

$$V^m = W \cup (-v + W)$$
, for any $v \in V^m \setminus \{0\}$.

In the following we will prove that $|W| = 2^m - 1$. First we show that $|W| \neq 2^m$, i.e., $W \neq V^m$. Suppose that $W = V^m$, then we have

$$\sum_{u \in V^m} \hat{F}(u)^2 = \sum_{u \in W} \hat{F}(u)^2 = 0.$$

This is a contradiction to the Parseval's equation (refer to Corollary 3 of Ch. 14 in [2])

$$\sum_{u \in V^m} \hat{F}(u)^2 = 2^{2m}.$$

Now we prove that $|W| > 2^m - 2$. Suppose that $|W| \le 2^m - 2$. Then there are at least two elements $i, j \in V^m$, $i \ne j$ and $i, j \notin W$. Note that if we choose $v = j - i \ne 0$, then we have $i \notin -v + W$ since $j \notin W$. Thus $i \in V^m$ but $i \notin \{W \cup (-v + W)\}$ for $v = j - i \ne 0$. This contradicts to the fact that

$$V^m = W \cup (-v + W)$$
, for any $v \in V^m \setminus \{0\}$.

As a result, we can conclude that $|W| = 2^m - 1$. Assume that $\tilde{u} \notin W$. Recall the Parseval's equation. We have

$$\sum_{u \in V^m} \hat{F}(u)^2 = \hat{F}(\widetilde{u})^2 = 2^{2m}.$$

Since $\hat{F}(\widetilde{u}) < 0$, it is clear that $\hat{F}(\widetilde{u}) = -2^m$. Note that $\hat{F}(u) = 2^m - 2 \text{dist} \{f, \sum_{i=1}^m u_i v_i\}$. So

$$\operatorname{dist}\{f, \sum_{i=1}^{m} u_i v_i\} = \frac{1}{2} \{2^m - \hat{F}(u)\} = \begin{cases} 2^{m-1} & u \in W, \\ 2^m & u = \widetilde{u}. \end{cases}$$

In other words, the coset $\mathbf{c} + C_2$ or $f + \mathcal{O}_m$ has one codeword of weight 2^m and $2^m - 1$ codewords of weight 2^{m-1} . Thus, we have proved that $\mathbf{1} \in \mathbf{c} + C_2$. So

$$C = C_2 \cup (1 + C_2) = \mathcal{O}_m \cup (1 + \mathcal{O}_m),$$

is the first order Reed-Muller code and has the unique weight enumerator $1+(2^{m+1}-2)t^{2^{m-1}}+t^{2^m}$.

3 The second order Reed-Muller code

The second order binary RM code $\mathcal{R}(2,m)$ is the set of all vectors f, where $f(v_1, \dots, v_m)$ is a Boolean function of degree ≤ 2 . $\mathcal{R}(2,m)$ is of length 2^m , of dimension $1 + m + {m \choose 2}$ and of minimum distance 2^{m-2} . A typical codeword of $\mathcal{R}(2,m)$ is given by the Boolean function

$$S(v) = \sum_{1 \le i \le j \le m} q_{ij} v_i v_j + \sum_{1 \le i \le m} u_i v_i + \epsilon$$
$$= vQv^T + Lv^T + \epsilon,$$

where $v = (v_1, \dots, v_m)$, $Q = (q_{ij})$ is an upper triangular binary matrix, $L = (u_1, \dots, u_m)$ is a binary vector and ϵ is 0 or 1. Note that, if Q is

fixed and the linear function $Lv^T + \epsilon$ varies over the first-order RM code $\mathcal{R}(1,m)$, then S(v) runs through a coset of $\mathcal{R}(1,m)$ in $\mathcal{R}(2,m)$. This coset is characterized by Q, or alternatively by the symmetric matrix $B = Q + Q^T$. B is a binary symmetric matrix with zero diagonal, which is called *symplectic matrix*. The *symplectic form* associated with B, as defined by (4) of Ch. 15 in [2], is

$$\mathcal{B}(u, v) = u(Q + Q^{\mathrm{T}})v^{\mathrm{T}}$$

= $S(u + v) + S(u) + S(v) + \epsilon$.

As stated in Lemma 3.1, the weight distribution of the coset associated with \mathcal{B} depends only on the rank of the matrix B.

Lemma 3.1. See Theorem 5 of Ch. 15 in [2]. If the matrix B has rank 2h, the weight distribution of the corresponding coset of $\mathcal{R}(1,m)$ in $\mathcal{R}(2,m)$ is as follows:

Since rank of B satisfies $2h \leq m$, the coset with largest possible minimum weight occurs when 2h = m. In this case m must be even. Furthermore, the Boolean functions associated with such cosets are quadratic bent functions, in the sense that they are furthest away from the linear Boolean functions (refer to Theorem 6 of Ch. 14 in [2]).

Note that the binary RM code is also conveniently defined as an extension of a cyclic code. In this section, we consider the punctured second order RM code $\mathcal{R}(2,m)^*$. As shown in Theorem 3.5, we obtain a sub-code family of $\mathcal{R}(2,m)$ for even m by extending a family of sub-codes of $\mathcal{R}(2,m)^*$.

Let F = GF(2) and F[x] be the set of polynomials in x with coefficients from F. Define the ring $R_n = F[x]/(x^n - 1)$, which consists of the residue classes of F[x] modulo $x^n - 1$. A cyclic code of length n is an ideal of $F[x]/(x^n - 1)$. Let $GF(2^m)$ be the splitting field of $x^n - 1$ over F(m) is the smallest positive integer such that n divides $2^m - 1$). Let $\alpha \in GF(2^m)$ be a primitive n-th root of unity. If $c(x) = \sum_{i=0}^{n-1} c_i x^i$ is a polynomial of R_n , then the Mattson-Solomon polynomial of c(x) is a polynomial in F[z] defined by

$$A(z) = \sum_{j=1}^{n} c(\alpha^{j}) z^{n-j}.$$

It has the following property.

Theorem 3.2. See Theorem 1.2 in [5]. $c_i = A(\alpha^i)$.

We partition the integers mod n into sets called *cyclotomic cosets* mod n. The cyclotomic coset containing s is

$$C_s = \{s, 2s, 2^2s, 2^3s, \cdots, 2^{m_s - 1}s\},\$$

where m_s is the smallest positive integer such that $2^{m_s} \cdot s \equiv s \pmod{n}$. Clearly, the numbers $-s, -2s, -2^2s, -2^3s, \cdots, -2^{m_s-1}s$ also form a cyclotomic coset denoted C_{-s} . Let

$$T_{m_s}(\gamma) = \gamma + \gamma^2 + \gamma^{2^2} + \dots + \gamma^{2^{m_s - 1}}.$$

If $\gamma \in GF(2^{m_s})$, then it is called the *trace* of γ from $GF(2^{m_s})$ to GF(2).

It is known that R_n is the direct sum of its minimal ideals and each minimal ideal is generated by a *primitive idempotent*. A primitive idempotent θ_s is defined by the property

$$\theta_s(\alpha^j) = \begin{cases} 1 & \text{if } j \in C_s, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\theta_s^*(x) = \theta_{s'}(x)$, where $s' \in C_{-s}$. The polynomial $\theta_s^*(x)$ is also a primitive idempotent. Its Mattson-Solomon polynomial is $A(z) = \sum_{j \in C_s} z^j$.

Let Ψ be a cyclic code of length n over F. Then there is a unique polynomial in Ψ which is both an idempotent and a generator. For the punctured second order Reed-Muller code $\mathcal{R}(2,m)^*$, its idempotent is

$$\theta_0 + \theta_1^* + \sum_{i=1}^{\left[\frac{m}{2}\right]} \theta_{l_i}^*, \quad l_j = 1 + 2^j.$$

Before proving Theorem 3.5, we introduce the following two lemmata.

Lemma 3.3.

$$\gcd(2^m-1,2^i+1) = \begin{cases} 1 & \text{if } \gcd(m,2i) = \gcd(m,i), \\ 2^{\gcd(m,i)}+1 & \text{if } \gcd(m,2i) = 2\gcd(m,i). \end{cases}$$

Proof.

$$\gcd(2^{m} - 1, 2^{i} + 1) \stackrel{(a)}{=} \gcd(2^{m} - 1, 2^{i} + 1, 2^{m-i} - 2^{i})$$

$$\stackrel{(b)}{=} \gcd(2^{m} - 1, 2^{i} + 1, 2^{2i} - 1)$$

$$\stackrel{(c)}{=} \gcd(2^{\gcd(m, 2i)} - 1, 2^{i} + 1),$$

where (a) is from the fact that $2^m - 1 = (2^{m-i} - 1)(2^i + 1) - (2^{m-i} - 2^i)$; (b) is from the fact that $2^m - 1 = 2^i(2^{m-i} - 2^i) + (2^{2i} - 1)$; (c) is from the fact that $\gcd(2^x - 1, 2^y - 1) = 2^{\gcd(x,y)} - 1$.

If gcd(m, 2i) = gcd(m, i), we have

$$\begin{split} \gcd(2^m-1,2^i+1) &= \gcd(2^{\gcd(m,2i)}-1,2^i+1) \\ &= \gcd(2^{\gcd(m,i)}-1,2^i+1) \\ &\stackrel{(d)}{\leq} \gcd(2^i-1,2^i+1) \\ &\stackrel{(e)}{=} 1, \end{split}$$

where (d) is from the fact that $2^{\gcd(m,i)} - 1 \mid 2^i - 1$; (e) is from the fact that $\gcd(2^i - 1, 2^i + 1) = 1$. Clearly we have in this case $\gcd(2^m - 1, 2^i + 1) = 1$. If $\gcd(m, 2^i) = 2\gcd(m, i)$, we have

$$\gcd(2^{m} - 1, 2^{i} + 1) = \gcd(2^{\gcd(m, 2i)} - 1, 2^{i} + 1)$$

$$= \gcd(2^{2\gcd(m, i)} - 1, 2^{i} + 1)$$

$$\stackrel{(f)}{=} \gcd(2^{\gcd(m, i)} - 1, 2^{i} + 1) \cdot \gcd(2^{\gcd(m, i)} + 1, 2^{i} + 1)$$

$$= \gcd(2^{\gcd(m, i)} + 1, 2^{i} + 1),$$

$$\stackrel{(g)}{=} 2^{\gcd(m, i)} + 1,$$

where (f) is from the fact that $2^{2\gcd(m,i)}-1=(2^{\gcd(m,i)}-1)(2^{\gcd(m,i)}+1)$ and $\gcd(2^{\gcd(m,i)}-1,2^{\gcd(m,i)}+1)=1$. (g) is from the fact that $2^{\gcd(m,i)}+1\mid 2^i+1$, since $\frac{i}{\gcd(m,i)}$ is odd due to $\gcd(m,2i)=2\gcd(m,i)$.

Lemma 3.4. If m = 2t + 1 is odd, then for $l_i = 1 + 2^i$,

$$|C_{l_i}| = m, \quad 1 \le i \le t.$$

If m = 2t + 2 is even, then for $l_i = 1 + 2^i$,

$$|C_{l_i}| = \begin{cases} m & 1 \le i \le t, \\ m/2 & i = t+1. \end{cases}$$

Proof. If m=2t+1 is odd, for $1 \le i \le t$, l_i and 2^m-1 are relatively prime according to Lemma 3.3. In this case, it is clear that $|C_{l_i}|=m$.

If m = 2t + 2 is even, for $1 \le i \le t$, C_{l_i} consists of

$$\{1+2^{i},2+2^{i+1},2^2+2^{i+2},\cdots,2^t+2^{i+t},2^t+2^{i+t},2^{t+1}+2^{i+t+1}\cdots\}.$$

It is easy to see that $|C_{l_i}| \ge t + 2 = m/2 + 1$. Note that $|C_{l_i}|$ must be either m or a divisor of m. Therefore, $|C_{l_i}| = m$ for $1 \le i \le t$. Consider the case i = t + 1. $C_{l_{t+1}}$ consists of

$$\{1+2^{t+1}, 2+2^{t+2}, 2^2+2^{t+3}, \cdots, 2^t+2^{2t+1}\}.$$

It is easy to count that $|C_{l_{t+1}}| = t + 1 = m/2$.

Theorem 3.5. Let m = 2t + 2 be even, and let d be any number in the range $1 \le d \le t + 1$. Then there exists a

$$[2^m, m(t-d+2) + m/2 + 1, 2^{m-1} - 2^{m-d-1}]$$

sub-code \mathcal{R}^d_{2t+2} of $\mathcal{R}(2,m)$. It is obtained by extending the cyclic sub-code of $\mathcal{R}(2,m)^*$ having idempotent

$$\theta_0 + \theta_1^* + \sum_{j=d}^{t+1} \theta_{l_j}^*.$$

The code has codewords of weights 2^{m-1} and $2^{m-1} \pm 2^{m-h-1}$ for all h in the range $d \le h \le t+1$.

Proof. Let m = 2t + 2. The general codeword of $\mathcal{R}(2, m)^*$ is

$$b\theta_0 + a_0 x^{i_0} \theta_1^* + \sum_{j=1}^{t+1} a_j x^{i_j} \theta_{l_j}^*, \quad l_j = 1 + 2^j,$$

where $b, a_0, a_j \in GF(2), 0 \le i_k \le 2^m - 2$ for $0 \le k \le t$ and $0 \le i_{t+1} \le 2^{m/2} - 2$. Consider the case b = 0. Its Mattson-Solomon polynomial is

$$\sum_{s \in C_1} (\gamma_0 z)^s + \sum_{j=1}^{t+1} \sum_{s \in C_{l_j}} (\gamma_j z)^s,$$

where $\gamma_0, \gamma_j \in GF(2^m)$. Due to Theorem 3.2, the corresponding Boolean function is

$$S(\xi) = \sum_{j=0}^{t+1} T_{|C_{l_j}|} (\gamma_j \xi)^{l_j} \text{ for all } \xi \in GF(2^m)^*,$$

$$\stackrel{(a)}{=} \sum_{j=0}^{t} T_m (\gamma_j \xi)^{1+2^j} + T_{m/2} (\gamma_{t+1} \xi)^{1+2^{t+1}}.$$

Here (a) is due to Lemma 3.4. The corresponding symplectic form is

$$\mathcal{B}(\xi,\eta) = S(\xi+\eta) + S(\xi) + S(\eta)$$

$$\stackrel{(b)}{=} \sum_{j=1}^{t} T_m(\gamma_j^{1+2^j}(\xi\eta^{2^j} + \xi^{2^j}\eta)) + T_{m/2}(\gamma_{t+1}^{1+2^{t+1}}(\xi\eta^{2^{t+1}} + \xi^{2^{t+1}}\eta))$$

$$\stackrel{(c)}{=} T_m\{\sum_{j=1}^{t} (\gamma_j^{1+2^j}\xi\eta^{2^j} + \gamma_j^{1+2^{2t+2-j}}\xi\eta^{2^{2t+2-j}}) + \gamma_{t+1}^{1+2^{t+1}}\xi\eta^{2^{t+1}}\}$$

$$= T_m(\xi L_B(\eta)),$$

where $\xi, \eta \in GF(2^m)^*$ and

$$L_B(\eta) = \sum_{j=1}^{t} \gamma_j [(\gamma_j \eta)^{2^j} + (\gamma_j \eta)^{2^{2t+2-j}}] + \gamma_{t+1} (\gamma_{t+1} \eta)^{2^{t+1}}.$$

Note that (b) is from the fact that $T_m(\alpha + \beta) = T_m(\alpha) + T_m(\beta)$ and

$$(\xi + \eta)^{1+2^{j}} = (\xi + \eta)(\xi^{2^{j}} + \eta^{2^{j}}) = \xi^{1+2^{j}} + \xi \eta^{2^{j}} + \xi^{2^{j}} \eta + \eta^{1+2^{j}};$$

(c) is due to the fact that

$$T_m(\gamma_j^{1+2^j}\xi^{2^j}\eta) = T_m(\gamma_j^{1+2^{2t+2-j}}\xi\eta^{2^{2t+2-j}}),$$

$$T_m(\gamma_{t+1}^{1+2^{t+1}}\xi\eta^{2^{t+1}}) = T_{m/2}(\gamma_{t+1}^{1+2^{t+1}}(\xi\eta^{2^{t+1}}+\xi^{2^{t+1}}\eta)).$$

Let $1 \le d \le t+1$ and $\gamma_1 = \gamma_2 = \cdots = \gamma_{d-1} = 0$. Then

$$L_{B}(\eta) = \sum_{j=1}^{t} \gamma_{j} [(\gamma_{j}\eta)^{2^{j}} + (\gamma_{j}\eta)^{2^{2t+2-j}}] + \gamma_{t+1}(\gamma_{t+1}\eta)^{2^{t+1}}$$

$$= \gamma_{d}(\gamma_{d}\eta)^{2^{d}} + \gamma_{d+1}(\gamma_{d+1}\eta)^{2^{d+1}} + \dots + \gamma_{t}(\gamma_{t}\eta)^{2^{t}} + \gamma_{t+1}(\gamma_{t+1}\eta)^{2^{t+1}}$$

$$+ \gamma_{t}(\gamma_{t}\eta)^{2^{t+2}} + \dots + \gamma_{d+1}(\gamma_{d+1}\eta)^{2^{2t+1-d}} + \gamma_{d}(\gamma_{d}\eta)^{2^{2t+2-d}}$$

$$= L'_{B}(\eta)^{2^{d}},$$

where degree $L'_B(\eta) \leq 2^{2t+2-2d}$. Thus the dimension of the space of η for which $L_B(\eta) = 0$ is at most 2t+2-2d. So rank $B \geq 2t+2-(2t+2-2d) = 2d$ (refer to (20) of Ch. 15 in [2]). In particular, when d = t+1, rank B = 2t+2 and the symplectic matrix B is corresponding to a quadratic bent function.

Note that by setting $\gamma_i = 0$ we are removing the idempotent $\theta_{l_i}^*$ from the code. Setting the first d-1 γ_i 's equal to 0, we derive a sub-code $\mathcal{R}_{2t+2}^{d'}$,

which has a corresponding symplectic form of rank $\geq 2d$. Clearly the code $\mathcal{R}_{2t+2}^{d'}$ has idempotent

$$\theta_1^* + \sum_{j=d}^{t+1} \theta_{l_j}^*, \quad l_j = 1 + 2^j.$$

According to Lemma 3.4, the code has dimension $\sum_{j=d}^{t+1} |C_{l_j}| = (t+2-d)m+m/2$. Due to Lemma 3.1, the code has codewords of weights 2^{m-1} and $2^{m-1} \pm 2^{m-h-1}$ for all h in the range $d \leq h \leq t+1$. Adding the all-one codeword into $\mathcal{R}_{2t+2}^{d'}$, we get a sub-code $\mathcal{R}_{2t+2}^{d^*}$ of $\mathcal{R}(2,m)^*$ having idempotent

$$\theta_0 + \theta_1^* + \sum_{j=d}^{t+1} \theta_{l_j}^*, \quad l_j = 1 + 2^j,$$

of dimension (t+2-d)m+m/2+1 and minimum distance $2^{m-1}-2^{m-d-1}-1$. Adding a parity check bit, we get the extended code \mathcal{R}^d_{2t+2} .

Corollary 3.6.

$$\mathcal{R}^{t+1}_{2t+2} \subset \mathcal{R}^t_{2t+2} \subset \cdots \subset \mathcal{R}^1_{2t+2}.$$

Proof. Recall that \mathcal{R}^d_{2t+2} for $1 \leq d \leq t+1$ is by extending the sub-code of $\mathcal{R}(2,m)^*$, $\mathcal{R}^{d^*}_{2t+2}$, which has idempotent

$$\theta_0 + \theta_1^* + \sum_{j=d}^{t+1} \theta_{l_j}^*, \quad l_j = 1 + 2^j.$$

The corollary follows directly from the fact that

$$\mathcal{R}^{t+1^*}_{2t+2} \subset \mathcal{R}^{t^*}_{2t+2} \subset \cdots \subset \mathcal{R}^{1^*}_{2t+2}.$$

Clearly the sub-codes in the sub-code family for even m satisfy the nested structure. By a similar proof, the sub-codes in the sub-code family for odd m by Theorem 1.1 have the same property.

Corollary 3.7.

$$\mathcal{R}_{1,2t+1}^t \subset \mathcal{R}_{1,2t+1}^{t-1} \subset \dots \subset \mathcal{R}_{1,2t+1}^1;$$

$$\mathcal{R}_{2,2t+1}^t \subset \mathcal{R}_{2,2t+1}^{t-1} \subset \dots \subset \mathcal{R}_{2,2t+1}^1.$$

We say that a linear code is *minimum distance optimal* if it achieves the largest minimum distance for given length and dimension. If the binary code contains the all-one sequence, then we say that it is *self-complementary*.

Theorem 3.8. If a linear $[2^m, 1+3m/2, 2^{m-1}-2^{m/2-1}]$ code is self-complementary, then it is minimum distance optimal. Here m is even.

Proof. Let C be a self-complementary linear $[2^m, 1 + 3m/2, 2^{m-1} - 2^{m/2-1}]$ code. If C contains $\mathcal{R}(1, m)$, due to Theorem 6 in Ch.14 in [2], it is clear that C is minimum distance optimal. Here we prove its is already true if C contains the all-one codeword, i.e., C is self-complementary.

Suppose that the minimum distance d for given $n=2^m, k=1+3m/2$ can be larger, i.e., $d=2^{m-1}-2^{m/2-1}+\delta$, where δ is a positive integer and $0<\delta<2^{m/2-1}$. Due to the Grey-Rankin bound (refer to (46) of Ch. 17 in [2]) for the binary self-complementary code, we have

$$|C| \leq \frac{8d(n-d)}{n-(n-2d)^2}$$

$$= \frac{8(2^{m-1}-2^{m/2-1}+\delta)(2^{m-1}+2^{m/2-1}-\delta)}{2^m-(2^{m/2}-2\delta)^2}$$

$$= 2 + \frac{2^{m-1}(2^m-1)}{\delta(2^{m/2}-\delta)}$$

$$\stackrel{(a)}{\leq} 2^{3m/2-1} + 2^{m-1} + 2,$$

where (a) follows from the fact that $\delta(2^{m/2}-\delta)\geq 2^{m/2}-1$. However, $|C|=2^{1+3m/2}>2^{3m/2-1}+2^{m-1}+2$, which is impossible for even m>0. Thus the self-complementary linear $[2^m,1+3m/2,2^{m-1}-2^{m/2-1}]$ code has optimal minimum distance.

Table 1: Some sub-codes of $\mathcal{R}(2,m)$

m	4	6	6	8	8	8	3	5	7	7
length	16	64	64	256	256	256	8	32	128	128
dimension	7	10	16	13	21	29	7	11	15	22
d_{-}	6	28	24	120	112	96	2	12	56	48
minimum distance	6	28	24	120	112	96	2	12	56	48
d_{+}	6	28	24	122	116	111	2	12	56	52

Remark: In Table 1, we show the minimum distance of the sub-codes of $\mathcal{R}(2,m)$ constructed by Theorem 3.5 and Theorem 1.1 for $m \leq 8$. Note that d_+ is the upper bound of the minimum distance for all the linear codes of the same length and dimension; d_- is the largest minimum distance, of which a linear code with the same length and dimension has been discovered so far (refer to [1]). From Table 1, we see that although all these sub-codes contain of $\mathcal{R}(1,m)$ and have specified weight sets, they have minimum distances reaching the upper bound d_+ or achieving the largest known minimum distance d_- of which a linear code with the same length and dimension can be constructed (maybe not a sup-code of $\mathcal{R}(1,m)$) or a sub-code of $\mathcal{R}(2,m)$). To some extend, we can say that the $\mathcal{R}(2,m)$ has good sub-codes that can be constructed by Theorem 3.5 and Theorem 1.1.

Compare the linear code \mathcal{R}^{t+1}_{2t+2} to the Kerdock code $\mathcal{K}(m)$. We consider \mathcal{R}^{t+1}_{2t+2} as $\mathcal{R}(1,m)$ together with $2^{m/2}-1$ cosets of $\mathcal{R}(1,m)$, and $\mathcal{K}(m)$ as $\mathcal{R}(1,m)$ together with $2^{m-1}-1$ cosets of $\mathcal{R}(1,m)$. Note that every coset is corresponding to a quadratic bent function and therefore associated to a symplectic matrix of full rank. It is well known that the cosets of $\mathcal{K}(m)$ are corresponding to the maximal set of symplectic forms with the property that the rank of the sum of any two in the set is still full rank. Clearly $\mathcal{K}(m)$ has much more codewords. However, \mathcal{R}^{t+1}_{2t+2} enjoys a linear structure. One can correspondingly obtain a set of $2^{m/2}-1$ symplectic matrices of full rank, denoted as \mathcal{G}^* . Introducing a matrix with all zero elements into \mathcal{G}^* , we get a set \mathcal{G} . Due to the linearity of \mathcal{R}^{t+1}_{2t+2} , \mathcal{G} is a commutative group with respect to the addition operation.

Theorem 3.9. For any even number m, there exists a group \mathcal{G} of $m \times m$ symplectic matrices with respect to the addition operation. There are $2^{m/2}$ symplectic matrices in \mathcal{G} . In particular, all the matrices except the matrix with all zero elements have full rank m.

4 Conclusion

In this paper, we consider the first order and the second order Reed-Muller codes. Our main contributions are twofold. First, we prove the uniqueness of the first order Reed-Muller code. Secondly, we give a linear sub-code family of the second order Reed-Muller code $\mathcal{R}(2,m)$ for even m, which is an extension of Corollary 17 of Ch. 15 in [2]. We also show that for $m \leq 8$, these specified sub-codes have good minimum distance equal to the upper bound or the largest constructive minimum distance for linear codes of the same length and dimension. As an additional result, we obtain an additive

commutative group of $m \times m$ symplectic matrices of full rank with respect to the addition operation, which is new to our knowledge.

References

- [1] M. Grassl, "Linear Block Codes," http://www. codetables.de.
- [2] F. J. MacWilliams and N. J. A. Sloane, *The Theory of Error-Correcting Codes*, 5th printing, Elsevier Science Publishers, The Netherlands, 1986.
- [3] Johannes Maks and Juriaan Simonis, "Optimal Subcodes of Second Order Reed-Muller Codes and Maximal Linear Spaces of Bivectors of Maximal Rank," *Designs, Codes and Cryptography*, vol. 21, pp. 165-180, 2000.
- [4] Aimo Tietäväinen, "On the Nonexistence of Perfect Codes over Finite Fields," SIAM J. Appl. Math., vol. 24, no. 1, pp. 88-96, Jan. 1973.
- [5] Anthony M. Kerdock, F. Jessie MacWilliams and Andrew M. Odlyzko, "A New Theorem about the Mattson-Solomon Polynomial and Some Applications," *IEEE Transactions on Information Theory*, vol. 20, pp. 85-89, Jan. 1974.