Lagrange Multipliers

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In This Presentation..

- We will give a definition
- Discuss some of the lagrange multipliers
- Learn how to use it
- Do example problems

Definition

Lagrange method is used for maximizing or minimizing a general function $f(x,y,z)$ subject to a constraint (or side condition) of the form $g(x,y,z) = k$.

Assumptions made: the extreme values exist

$$\nabla g \neq 0$$

Then there is a number λ such that

$$\nabla f(x_0,y_0,z_0) = \lambda \nabla g(x_0,y_0,z_0)$$

and λ is called the Lagrange multiplier.

- Finding all values of x,y,z and λ such that

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$$

and $$g(x,y,z) = k$$

And then evaluating f at all the points, the values obtained are studied. The largest of these values is the maximum value of f; the smallest is the minimum value of f.

- Writing the vector equation $\nabla f = \lambda \nabla g$ in terms of its components, give

$$\nabla f_x = \lambda \nabla g_x \qquad \nabla f_y = \lambda \nabla g_y \qquad \nabla f_z = \lambda \nabla g_z \qquad g(x,y,z) = k$$

- It is a system of four equations in the four unknowns, however it is not necessary to find explicit values for λ.

- A similar analysis is used for functions of two variables.

Examples

- Example 1:

A rectangular box without a lid is to be made from 12 m^2 of cardboard. Find the maximum volume of such a box.

- Solution:

let x,y and z are the length, width and height, respectively, of the box in meters.

and $V = xyz$

Constraint: $g(x, y, z) = 2xz + 2yz + xy = 12$

Using Lagrange multipliers,

$V_x = \lambda g_x \qquad V_y = \lambda g_y \qquad V_z = \lambda g_z \qquad 2xz + 2yz + xy = 12$

which become

Continued..

- $yz = \lambda(2z+y)$ (1)
- $xz = \lambda(2z+x)$ (2)
- $xy = \lambda(2x+2y)$ (3)
- $2xz + 2yz + xy = 12$ (4)
- Solving these equations;
- Let's multiply (2) by x, (3) by y and (4) by z, making the left hand sides identical.
- Therefore,
- $x\,yz = \lambda(2xz+xy)$ (6)
- $x\,yz = \lambda(2yz+xy)$ (7)
- $x\,yz = \lambda(2xz+2yz)$ (8)

continued

- It is observed that $\lambda \neq 0$ therefore from (6) and (7)

$$2xz+xy=2yz+xy$$

which gives $xz = yz$. But $z \neq 0$, so $x = y$. From (7) and (8) we have

$$2yz+xy=2xz+2yz$$

which gives $2xz = xy$ and so (since $x \neq 0$) $y=2z$. If we now put $x=y=2z$ in (5), we get

$$4z^2+4z^2+4z^2=12$$

Since x, y, and z are all positive, we therefore have $z=1$ and so $x=2$ and $y = 2$.

More Examples

- Example 2:

Find the extreme values of the function $f(x,y)=x^2+2y^2$ on the circle $x^2+y^2=1$.

- Solution:

Solve equations $\nabla f= \lambda \nabla g$ and $g(x,y)=1$ using Lagrange multipliers

Constraint: $g(x, y)= x^2+y^2=1$

Using Lagrange multipliers,

$f_x= \lambda g_x$ $f_y= \lambda g_y$ $g(x,y) = 1$

which become

Continued…

- $2x = 2x\lambda$ (9)
- $4y = 2y\lambda$ (10)
- $x^2+y^2 = 1$ (11)
- From (9) we have $x=0$ or $\lambda=1$. If $x=0$, then (11) gives $y=\pm1$. If $\lambda=1$, then $y=0$ from (10), so then (11) gives $x=\pm1$. Therefore f has possible extreme values at the points (0,1), (0,-1), (1,0), (1,0). Evaluating f at these four points, we find that
- $f(0,1)=2$
- $f(0,-1)=2$
- $f(1,0)=1$
- $f(-1,0)=1$
- Therefore the maximum value of f on the circle

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- $x^2+y^2=1$ is $f(0,\pm1)=2$ and the minimum value is $f(\pm1,0)=1$.

More Examples.

- Example 3

Find the extreme values of $f(x,y)=x^2+2y^2$ on the disk $x^2+y^2\leq 1$.

- Solution:

Compare the values of f at the critical points with values at the points on the boundary. Since $f_x=2x$ and $f_y=4y$, the only critical point is (0,0). We compare the value of f at that point with the extreme values on the boundary from Example 2:

- $f(0,0)=0$
- $f(\pm 1,0)=1$
- $f(0,\pm 1)=2$
- Therefore the maximum value of f on the disk $x^2+y^2\leq 1$ is $f(0,\pm 1)=2$ and the minimum value is $f(0,0)=0$.

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- Example 4
- Find the points on the sphere $x^2+y^2+z^2=4$ that are closest to and farthest from the point (3,1,-1).
- Solution:

The distance from a point (x,y,z) to the point (3,1,-1) is

$$d=\sqrt{(x-3)2+(y-1)2+(z+1)2}$$

But the algebra is simple if we instead maximize and minimize the square of the distance:

$$d^2=f(x,y,z)=(x-3)^2+(y-1)^2+(z+1)^2$$

Constraint: $g(x,y,z)= x^2+y^2+z^2=4$

Using Lagrange multipliers, solve $\nabla f= \lambda \nabla g$ and $g=4$

This gives

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- $2(x-3)=2x\lambda$ (12)
- $2(y-1)=2y\lambda$ (13)
- $2(z+1)=2z\lambda$ (14)
- $x^2+y^2+z^2=4$ (15)
- The simplest way to solve these equations is to solve for x, y, and z in terms of λ from (12), (13), and (14), and then substitute these values into (15). From12 we have
- $x-3=x\lambda$ or
- $x(1-\lambda)=3$ or
- $x = \frac{3}{1-\lambda}$

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- Similarly (13) and (14) give
- $y = \frac{1}{1-\lambda}$
- $z = -\frac{1}{1-\lambda}$
- Therefore, from (15), we have
- $\frac{3^2}{(1-\lambda)^2} + \frac{1^2}{(1-\lambda)^2} + \frac{(-1)^2}{(1-\lambda)^2} = 4$
- Which gives $(1-\lambda)^2 = \frac{11}{4}$, $1-\lambda = \pm\frac{\sqrt{11}}{2}$, so
- $\lambda = 1 \pm \frac{\sqrt{11}}{2}$
- These values of λ then give the corresponding points (x,y,z):
- $\left(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, -\frac{2}{\sqrt{11}}\right)$ and $\left(-\frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)$

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- f has a smaller value at the first of these points, so the closest point is $(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, -\frac{2}{\sqrt{11}})$ and the farthest is $\left(-\frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)$.

Two constraints

- Say there is a new constraint, $h(x,y,z)=c$.

So there are numbers λ and μ (called Lagrange multipliers) such that

$$\nabla f(x_0,y_0,z_0) = \lambda \nabla g(x_0,y_0,z_0) + \mu \nabla h(x_0,y_0,z_0)$$

The extreme values are obtained by solving for the five unknowns x, y, z, λ and μ. This is done by writing the above equation in terms of the components and using the constraint equations:

$$f_x = \lambda g_x + \mu h_x \qquad f_y = \lambda g_y + \mu h_y \qquad f_z = \lambda g_z + \mu h_z$$

$$g(x,y,z) = k \qquad h(x,y,z) = c$$

THE ARC.

Reference

- Calculus – Stewart 6th Edition
  - Section 15.8 “Lagrange Multipliers”

Thank you!

Enjoy those lagrange multipliers…!