

M2A2 Problem Sheet 2

Lagrangian Mechanics

Solutions

1. *Particle in a central potential.*

A particle of mass m moves in R^3 under a central force

$$F(r) = -\frac{dV}{dr},$$

in spherical coordinates, so

$$(x, y, z) = (r \cos(\phi) \sin(\theta), r \sin(\phi) \sin(\theta), r \cos(\theta)).$$

Find the Lagrangian from first principles, in terms of (r, θ, ϕ) and their time derivatives.

Hence

- (a) show that h , defined by $h = mr^2\dot{\phi}\sin^2(\theta)$ is a constant of the motion.
- (b) derive the other two equations of motion.

Solution: *The kinetic energy is $T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$. We substitute*

$$x = r \sin(\theta) \cos(\phi),$$

$$y = r \sin(\theta) \sin(\phi),$$

$$z = r \cos(\theta).$$

Differentiating these, substituting into T , and simplifying, we find

$$T = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2(\theta)\dot{\phi}^2).$$

The potential energy is $V(r)$, so our Lagrangian is:

$$L = T - V = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2(\theta)\dot{\phi}^2) - V(r).$$

This Lagrangian is independent of ϕ ; so the corresponding EL equation is

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}} = \frac{\partial L}{\partial \phi} = 0.$$

Hence $h = \frac{\partial L}{\partial \dot{\phi}} = mr^2 \dot{\phi} \sin^2(\theta)$ is a constant of motion. The radial equation is:

$$\frac{d}{dt}(mr\dot{r}) = mr(\dot{\theta}^2 + \sin^2(\theta)\dot{\phi}^2) - V'(r),$$

and the θ equation is:

$$\frac{d}{dt}(mr^2\dot{\theta}) = mr^2 \sin(\theta) \cos(\theta) \dot{\phi}^2.$$

2. The spherical pendulum.

An inextensible string of length l is fixed at one end, and has a bob of mass m attached at the other. The bob swings freely in R^3 under gravity, and the string remains taut, so the system is a spherical pendulum. Find the Lagrangian in an appropriate coordinate system, and identify a conserved quantity. Write down both equations of motion.

Solution: The unconstrained Lagrangian, as in question 1, is

$$L_{\text{unconstrained}} = T - V = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2(\theta)\dot{\phi}^2) + mgr \cos(\theta).$$

Here we have taken the lowest point of the sphere as the origin of θ . Now the constraint is $r - l = 0$. Hence the extended Lagrangian is

$$L_{\text{ext}} = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2(\theta)\dot{\phi}^2) + mgr \cos(\theta) - \lambda(r - l).$$

The r equation determines λ ; the other equations are the same as we would find from the simpler Lagrangian:

$$L = \frac{ml^2}{2}(\dot{\theta}^2 + \sin^2(\theta)\dot{\phi}^2) + mgl \cos(\theta).$$

As before, ϕ is ignorable, so

$$h = \frac{\partial L}{\partial \dot{\phi}} = ml^2 \dot{\phi} \sin^2(\theta)$$

is a constant of the motion:

$$\frac{d}{dt}(ml^2\dot{\phi}\sin^2(\theta)) = 0.$$

The θ equation is:

$$\frac{d}{dt}(ml^2\dot{\theta}) = ml^2 \sin(\theta) \cos(\theta) \dot{\phi}^2 - mgl \sin(\theta).$$

3. Horizontal Atwood machine.

An inextensible taut string of length l has a mass m at each end. It passes through a hole in a smooth horizontal plane, and the lower mass hangs vertically, while the upper is free to move in the plane. Write down the Lagrangian, in terms of the two coordinates of the upper particle, and find the equations of motion. Identify two conserved quantities, and hence reduce the equations of motion to a single first-order equation.

Alternatively, treat the particles as though they moved independently, but subject to the constraint that the string is of constant length. Construct the appropriate Lagrangian, with a Lagrange multiplier λ , multiplying the length of the string. Calculate this λ , which is the tension in the string.

Solution: We use polar coordinates (r, θ) for the particle in the horizontal plane, and measure the vertical coordinate z of the other particle downwards. Then the extended constrained Lagrangian is:

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) + mgz - \lambda(r + z - l).$$

Here the r and z equations both involve λ linearly - eliminating z with the constraint $r + z - l = 0$, the Lagrangian simplifies to:

$$L = \frac{m}{2}(2\dot{r}^2 + r^2\dot{\theta}^2) + mg(l - r).$$

We see

$$h = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

is a conserved quantity, as L does not depend explicitly on θ . Since L is also independent of t , the energy is constant:

$$E = \frac{m}{2}(2\dot{r}^2 + r^2\dot{\theta}^2) + mg(r - l).$$

Returning to the extended Lagrangian, the z equation is: $m\ddot{z} = mg - \lambda$, so substituting the constraint we find:

$$\lambda = mg + m\ddot{r}.$$

4. The guitar string.

Suppose a string is tied between two fixed end points $x = 0, y = 0$ and $x = l, y = 0$. Let $y(x, t)$ be the small transverse displacement of the string from its equilibrium at position $x \in (0, l)$ and $t > 0$. The string has mass μ per unit length, and constant tension F .

Show that the kinetic and potential energies are given by:

$$\begin{aligned} T &= \frac{\mu}{2} \int_0^l y_t^2 dx, \\ V &= F \int_0^l (\sqrt{1 + y_x^2} - 1) dx. \end{aligned}$$

Here subscripts denote partial derivatives. You may neglect the effect of gravity. If the displacement y is small, so that $|y_x| \ll 1$, show that the Lagrangian can be approximated by an expression quadratic in y , and find the Euler-Lagrange equation for the approximate Lagrangian.

Solution: The change in length of the displaced string is:

$$\Delta l = \int_0^l (\sqrt{1 + y_x^2} - 1) dx,$$

so the potential energy (the work done in displacing the string) is $F\Delta l$ + times this:

$$V = F \int_0^l (\sqrt{1 + y_x^2} - 1) dx,$$

as required. The string is supposed to move only in the y -direction, so its kinetic energy is, as usual, the integral of the mass density times half the square of the speed:

$$T = \frac{\mu}{2} \int_0^l y_t^2 dx.$$

Hence

$$L = \int_0^l \frac{\mu}{2} y_t^2 - F(\sqrt{1 + y_x^2} - 1) dx.$$

Now let y_x be small; the leading approximation to L is then:

$$L^{(2)} = \int_0^l \frac{\mu}{2} y_t^2 - \frac{F}{2} y_x^2 dx,$$

where terms quartic in y_x have been neglected. The EL equation for this approximate Lagrangian is:

$$\mu y_{tt} = F y_{xx},$$

which is the familiar wave equation; the speed c of these waves is given by $c^2 = F/\mu$.

5. The diatomic molecule.

Two atoms of masses m_1, m_2 move freely in the plane, with the constraint that the distance between them $|\mathbf{x}_1 - \mathbf{x}_2| - l = 0$, where l is a constant.

- (a) Write down the kinetic energy and the constrained Lagrangian in Cartesian coordinates, and find the the Lagrange multiplier of the constraint, which is the force in the bond between the two atoms.
- (b) Rewrite the Lagrangian in new coordinates (\mathbf{X}, \mathbf{r}) , where \mathbf{X} is the centre of mass, and $(\mathbf{x}_1 - \mathbf{x}_2) = \mathbf{r}$.

Identify six symmetries of the system and write down the corresponding Noether integrals.

- If the particles are subjected to a gravitational potential $\sum_{i=1}^2 m_i \mathbf{x}_i \cdot \mathbf{j} g$, where \mathbf{j} is the unit vector $(0, 1)$, write down the modified Lagrangian.

- If the particles are subjected instead to a harmonic potential $\sum_{i=1}^2 m_i \omega^2 |\mathbf{x}_i|^2$, write down the modified Lagrangian.
- If the particles are no longer subjected to the constraint, but instead there is a force between them due to a potential $V(|\mathbf{x}_1 - \mathbf{x}_2|)$, write down the Lagrangian.

State which of the symmetries and conservation laws survive in each of these cases.

Solution:

(a) *Let the particles have Cartesian coordinates (x_1, y_1) , (x_2, y_2) . The extended Lagrangian is:*

$$L = \frac{m_1}{2}(\dot{x}_1^2 + \dot{y}_1^2) + \frac{m_2}{2}(\dot{x}_2^2 + \dot{y}_2^2) - \lambda(|\mathbf{x}_1 - \mathbf{x}_2| - l).$$

The EL equations are

$$m_1 \ddot{\mathbf{x}}_1 = \lambda(\mathbf{x}_1 - \mathbf{x}_2)/l,$$

$$m_2 \ddot{\mathbf{x}}_2 = \lambda(\mathbf{x}_2 - \mathbf{x}_1)/l,$$

where we have substituted in the constraint. If we dot each equation with $(\mathbf{x}_1 - \mathbf{x}_2)$, divide by the masses, and subtract, we find

$$(\ddot{\mathbf{x}}_1 - \ddot{\mathbf{x}}_2) \cdot (\mathbf{x}_1 - \mathbf{x}_2) = \lambda(1/m_1 + 1/m_2).$$

Now the second derivative of the constraint gives:

$$(\ddot{\mathbf{x}}_1 - \ddot{\mathbf{x}}_2) \cdot (\mathbf{x}_1 - \mathbf{x}_2) + (\dot{\mathbf{x}}_1 - \dot{\mathbf{x}}_2) \cdot (\dot{\mathbf{x}}_1 - \dot{\mathbf{x}}_2) = 0.$$

Thus

$$\lambda = -\frac{m_1 m_2}{l(m_1 + m_2)} |\dot{\mathbf{x}}_1 - \dot{\mathbf{x}}_2|^2.$$

(b) *In the relative coordinates,*

$$\mathbf{X} = (m_1 \mathbf{x}_1 + m_2 \mathbf{x}_2)/(m_1 + m_2),$$

$$\mathbf{r} = (\mathbf{x}_1 - \mathbf{x}_2),$$

we get

$$\mathbf{x}_1 = \mathbf{X} + \frac{m_2}{m_1 + m_2} \mathbf{r},$$

$$\mathbf{x}_2 = \mathbf{X} - \frac{m_1}{m_1 + m_2} \mathbf{r}.$$

Substituting these expressions and their derivatives into L we get, after some cancellation:

$$L = \frac{m_1 + m_2}{2} \dot{\mathbf{X}}^2 + \frac{m_1 m_2}{2(m_1 + m_2)} \dot{\mathbf{r}}^2 - \lambda(|\mathbf{r}| - l).$$

The centre of mass motion and the relative motion are now decoupled.

The system has several symmetries -

- L is independent of t , so the energy

$$E = \frac{m_1 + m_2}{2} \dot{\mathbf{X}}^2 + \frac{m_1 m_2}{2(m_1 + m_2)} \dot{\mathbf{r}}^2$$

is conserved.

- L is independent of \mathbf{X} , so both components of the total momentum

$$\mathbf{P} = \frac{\partial L}{\partial \dot{\mathbf{X}}} = (m_1 + m_2) \dot{\mathbf{X}},$$

are conserved.

- L is unchanged under rotations of the vectors \mathbf{X} or of \mathbf{r} , separately. Thus the two quantities:

$$J = (m_1 + m_2) \mathbf{X} \wedge \dot{\mathbf{X}},$$

and

$$K = \frac{m_1 m_2}{m_1 + m_2} \mathbf{r} \wedge \dot{\mathbf{r}},$$

are each conserved.

- The system is invariant under Galilean transformations -

$$\mathbf{X} \rightarrow \mathbf{X} + \epsilon \mathbf{V} t,$$

$$\dot{\mathbf{X}} \rightarrow \dot{\mathbf{X}} + \epsilon \mathbf{V},$$

for any constant vector \mathbf{V} . L is not unchanged under this transformation; rather

$$L \rightarrow L + \epsilon(m_1 + m_2) \dot{\mathbf{X}} \cdot \mathbf{V}$$

$$= L + \epsilon \frac{d}{dt} (m_1 + m_2) \mathbf{X} \cdot \mathbf{V}.$$

Hence the Noether integral is here

$$\mathbf{I} \cdot \mathbf{V} = (m_1 + m_2) \mathbf{X} \cdot \mathbf{V} - \mathbf{P} \cdot \mathbf{V} t,$$

so the centre of mass moves at constant speed.

Note that the unconstrained system has 4 degrees of freedom, so it consists of 4 second order equations. With these 7 conserved quantities, the motion is reduced to a single first order system. In fact, including the constraint means the equation for the time dependence of $|\mathbf{r}|$ is trivial - the motion is determined completely.

- If the particles are subjected to a gravitational potential $\sum_{i=1}^2 m_i \mathbf{x}_i \cdot \mathbf{j} g$, where \mathbf{j} is the unit vector $(0, 1)$, the modified Lagrangian is:

$$L = \frac{m_1}{2} (\dot{x}_1^2 + \dot{y}_1^2) + \frac{m_2}{2} (\dot{x}_2^2 + \dot{y}_2^2) - \lambda (|\mathbf{x}_1 - \mathbf{x}_2| - l) - \sum_{i=1}^2 m_i \mathbf{x}_i \cdot \mathbf{j} g.$$

The new potential term is not invariant under translations in the \mathbf{j} direction, or under rotations. The other symmetries survive.

- If the particles are subjected instead to a harmonic potential $\sum_{i=1}^2 m_i \omega^2 |\mathbf{x}_i|^2$, the modified Lagrangian is:

$$L = \frac{m_1}{2} (\dot{x}_1^2 + \dot{y}_1^2) + \frac{m_2}{2} (\dot{x}_2^2 + \dot{y}_2^2) - \lambda (|\mathbf{x}_1 - \mathbf{x}_2| - l) - \sum_{i=1}^2 m_i \omega^2 |\mathbf{x}_i|^2.$$

Both rotational symmetries still survive under this perturbation, so J and K , defined as before, are still conserved separately. The system is no longer Galilean invariant, so \mathbf{I} is not conserved for this system.

- If the particles are no longer subjected to the constraint, but instead there is a force between them due to a potential $V(|\mathbf{x}_1 - \mathbf{x}_2|)$, the Lagrangian is:

$$L = \frac{m_1}{2} (\dot{x}_1^2 + \dot{y}_1^2) + \frac{m_2}{2} (\dot{x}_2^2 + \dot{y}_2^2) - V(|\mathbf{x}_1 - \mathbf{x}_2|).$$

All 7 symmetries survive in this model.