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Q1) Using the probabilistic method, show that the following bipartite graph $G = (L \cup R, E)$ exists on $2n$ vertices

a) $|L| = |R| = n$

b) Every vertex v in L has a degree $n^{3/4}$ and every vertex u in R has a degree atmost $3n^{3/4}$

c) Every subset of $n^{3/4}$ vertices in L has at least $n - n^{3/4}$ neighbors in R

Ans) $|L| = |R| = n$ $\longrightarrow$ given

Degree of a vertex in $L = n^{3/4}$, total 'n' vertices

$\therefore$ No of choices $= n \cdot n^{3/4}$

Let X_i be the r.v such that i^{th} choice is V in R

$$E[X_i] = Pr(X_i = 1) = 1/n$$

$$E[X] = E\left[\sum X_i\right] = \sum E[X_i] = \text{Total choices} \times \frac{1}{n}$$

$$= n \cdot n^{3/4} \cdot 1/n = n^{3/4}$$

$\therefore \mu = n^{3/4}$

$$Pr(X \geq 3n^{3/4}) = Pr\left(X \geq \underbrace{(1+2)}_{\delta = 2 > 1} n^{3/4}\right)$$

$$= e^{-\mu\delta \ln \delta} = e^{\ln \delta^{-\mu\delta}} = \delta^{-\mu\delta}$$

$$= 2^{-n^{3/4} 2}$$

$$= \frac{1}{4^{n^{3/4}}}$$

$\longrightarrow$ very small

Let S be any subset of size $n^{3/4}$ from L
Let T be any subset of size $n - n^{3/4}$ from R

$\therefore$ No of choices in $S = n^{3/4} \cdot n^{3/4} = n^{3/2}$

Event : all neighbors of S are in T

1 vertex, 1st choice $= \dfrac{n - n^{3/4}}{n}$

$\vdots$

all $n^{3/2}$ choices of S $= \left(\dfrac{n - n^{3/4}}{n}\right)^{n^{3/2}} = \left(1 - \dfrac{1}{n^{1/4}}\right)^{n^{3/2}}$

$\therefore$ For all possibilities of S & T, the probability is upper bounded by

$$Pr(E) \leq {}^nC_{n^{3/4}} \cdot {}^nC_{n - n^{3/4}} \cdot \left(1 - \frac{1}{n^{1/4}}\right)^{n^{3/2}}$$

$${}^nC_{n^{3/4}} = {}^nC_{n - n^{3/4}}$$

$$Pr(E) = \left({}^nC_{n^{3/4}}\right)^2 \cdot \left(1 - \frac{1}{n^{1/4}}\right)^{n^{3/2}} \qquad \left({}^nC_k = \left(\frac{en}{k}\right)^k\right)$$

$$Pr(E) \leq \left(\frac{en}{n^{3/4}}\right)^{n^{3/4}} \cdot \left(\frac{en}{n^{3/4}}\right)^{n^{3/4}} \left(1 - \frac{1}{n^{1/4}}\right)^{n^{3/2}}$$

$$1 - \frac{1}{n^{1/4}} \leq e^{-1/n^{1/4}}$$

$$\therefore Pr(E) = e^{2n^{3/4}} \cdot n^{\frac{1}{2}n^{3/4}} \cdot e^{-n^{5/4}}$$

$$= e^{2n^{3/4} - n^{5/4}} \cdot n^{\frac{1}{2}n^{3/4}}$$

...ow that the function $f(x) = 1 - \left(1 - \frac{x}{k}\right)^k$ is concave
... any $k > 0$ and $x \in [0,1]$ (Note: x is a real num
... and not just an integer, k is an integer)

Ans) $f(x)$ is concave if $f''(x)$ is non positive

$$f'(x) = 0 - k\left(1 - \frac{x}{k}\right)^{k-1} \times -\frac{1}{k} = \left(1 - \frac{x}{k}\right)^{k-1}$$

$$f''(x) = -\left(\frac{k-1}{k}\right)\left(1 - \frac{x}{k}\right)^{k-2}$$

$k > 0 \Rightarrow \frac{k-1}{k}$ will be +ve and b/w 0 & 1

when $x = 1$ (RHS limit)

$1 - \frac{x}{k} \geq 0$ (0 when $x = k = 1$)

when $x = 0$ (LHS limit)

$1 \geq 0$

$$f''(x) = -\left(\frac{k-1}{k}\right)\left(1 - \frac{x}{k}\right)^{k-2}$$

$\therefore \underbrace{\frac{k-1}{k}, \frac{1-x}{k} \text{ is +ve}}$, $k \geq 2$ (Double differentiation)

$\hookrightarrow f''(x) =$ non positive $\Rightarrow$ Concave

for $k = 1$, $f(x) = 1 - (1 - x) = x$

Let x' & $x'' \in [0,1]$, then for $\alpha \in [0,1]$

$$f((1-\alpha)x' + \alpha x'') = (1-\alpha)x' + \alpha x''$$
$$= (1-\alpha) f(x') + \alpha f(x'')$$

$\therefore$ for $k = 1$, $f(x) = x$ is concave

It is not strictly concave.

Q3) Show a complete example of the 2 level hash scheme with your choice of p, m, n

Ans) Take $p = 23$, $m = 13$, $a = 11$, $b = 8$, $n = 8$

$h_{11,8}(k) = ((11k + 8) \bmod 23) \bmod 13$

$S = \{13, 20, 25, 36, 47, 18, 50, 99\}$

$h_{11,8}(13) = 0$ ($(151 \bmod 23) \bmod 13$) $\quad h_{11,8}(36) = 0$ ($(404 \bmod 23) \bmod 13$)

$h_{11,8}(20) = 8$ ($(228 \bmod 23) \bmod 13$) $\quad h_{11,8}(47) = 6$ ($(525 \bmod 23) \bmod 13$)

$h_{11,8}(25) = 7$ ($(283 \bmod 23) \bmod 13$) $\quad h_{11,8}(18) = 9$ ($(206 \bmod 23) \bmod 13$)

$h_{11,8}(50) = 6$ ($(558 \bmod 23) \bmod 13$) $\quad h_{11,8}(99) = 3$ ($(1097 \bmod 23) \bmod 13$)

→ Collision at 13, 36 as $h_{11,8}(13) = h_{11,8}(36) = 0$

Second hashing, $m = n^2 \Rightarrow m = 4$

Let $p = 11$, $a = 5$, $b = 2$

$h_{5,2}(k) = ((5k + 2) \bmod 11) \bmod 4$

$h_{5,2}(13) = 1 \qquad h_{5,2}(36) = 2$

→ Collision at 50, 47 as $h_{11,8}(50) = h_{11,8}(47) = 6$

Second hashing, $m = n^2 \Rightarrow m = 4$

Let $p = 11$, $a = 4$, $b = 2$, $h_{4,2} = ((4k+2) \bmod 11) \bmod 4$

$h_{4,2}(50) = 202 \bmod 4 = 0$

$h_{4,2}(47) = 8 \bmod 4 = 0$

a collision, so we need to change the values the family of hash fns.

$m = 4$, $p = 17$, $a = 1$, $b = 6$

$h_{1,6}(k) = ((1k + 6) \bmod 17) \bmod 4$

$h_{1,6}(50) = (56 \bmod 17) \bmod 4$
$= 5 \bmod 4 = 1$

$h_{1,6}(47) = (53 \bmod 17) \bmod 4$
$= 2 \bmod 4 = 2$

$\therefore$ 2nd change in hash fn got us a perfect hashing.

Q4) Write a Boolean formula on 'n' variables such that the maximum number of satisfiable clauses is exactly $\frac{1}{2}$. Repeat the case when max no of satisfiable clauses is exactly 3/4 of the total no of clauses.

Ans) 'n' variables $x_1, x_2, \ldots, x_n$

$C_1 = x_1$ $\qquad C_{n+1} = \bar{x}_1$

$C_2 = x_2$ $\qquad C_{n+2} = \bar{x}_2$

$\vdots$ $\qquad \vdots$

$C_n = x_n$ $\qquad C_{2n} = \bar{x}_n$

$\therefore f = C_1 \wedge C_2 \ldots \ldots \wedge C_{2n}$

This f will have exactly ½ of its clauses satisfied

Let $n = 3$, x_1, x_2, x_3

$C_1 = x_1 \vee x_2 \vee x_3$

$C_2 = \bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3$

$C_3 = \bar{x}_1 \vee \bar{x}_2 \vee x_3$

$C_4 = \bar{x}_1 \vee x_2 \vee \bar{x}_3$

$\therefore f = C_1 \wedge C_2 \wedge C_3 \wedge C_4$

This f will have exactly ¾ of its clauses satisfied.