Cauchy-Schwarz inequality:

Proof: Let u, v be non-zero vectors in Hilbert space H, such that they cannot be represented by a linear multiple of each other.

Then for an arbitrary complex number λ,

$$\|u - \lambda v\|^2 \geq 0$$

$$\Rightarrow (u,u) - \lambda(u,v) - \bar{\lambda}(v,u) + \lambda\bar{\lambda}(v,v) \geq 0$$

$$\Rightarrow \|u\|^2 + \|v\|^2 \cdot \lambda\bar{\lambda} - \lambda(u,v) - \bar{\lambda}(v,u) \geq 0$$

Consider $\lambda = \dfrac{(v,u)}{\|v\|^2}$

Then we have

$$\Rightarrow \|u\|^2 + \frac{\|(u,v)\|^2}{\|v\|^4}\|v\|^2 - 2\frac{\|(u,v)\|^2}{\|v\|^2} \geq 0$$

$$\Rightarrow \|u\|^2 \geq \frac{\|(u,v)\|^2}{\|v\|^2}$$

$$\Rightarrow \|u\|^2\|v\|^2 \geq \|(u,v)\|^2$$

$$\Rightarrow \boxed{\|(u,v)\|^2 \leq (u,u)(v,v)}$$