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Q1) Consider a Hilbert space $H$ with scalar product $( , )$. The scalar product implies a norm via $\|f\|^2 = (f, f)$, where $f \in H$.

i) Show that $\|f+g\|^2 + \|f-g\|^2 = 2(\|f\|^2 + \|g\|^2)$

ii) Assume that $(f, g) = 0$. Now show that

$$\|f+g\|^2 = \|f\|^2 + \|g\|^2$$

iii) For real and complex Hilbert space, show the following respectively

$$(f, g) = \frac{1}{4}\|f+g\|^2 - \frac{1}{4}\|f-g\|^2$$

$$(f, g) = \frac{1}{4}\|f+g\|^2 - \frac{1}{4}\|f-g\|^2 + \frac{i}{4}\|f+ig\|^2 - \frac{i}{4}\|f-ig\|^2$$

Ans) $\|f+g\|^2 = \langle f+g, f+g\rangle$

$= \langle f, f\rangle + \langle f, g\rangle + \langle g, f\rangle + \langle g, g\rangle$

$= \|f\|^2 + \|g\|^2 + \langle f, g\rangle + \langle g, f\rangle$

$= \|f\|^2 + \|g\|^2 + \langle f, g\rangle + \overline{\langle f, g\rangle}$

$\|f-g\|^2 = \langle f-g, f-g\rangle$

$= \langle f, f\rangle - \langle f, g\rangle - \langle g, f\rangle + \langle g, g\rangle$

$= \|f\|^2 + \|g\|^2 - \langle f, g\rangle - \overline{\langle f, g\rangle}$

$\|f+ig\|^2 = \langle f+ig, f+ig\rangle = \langle f, f\rangle + \langle f, ig\rangle + \langle ig, f\rangle + \langle ig, ig\rangle$

$= \langle f, f\rangle + -i\langle f, g\rangle + i\langle g, f\rangle + i \times -i\langle g, g\rangle$

$= \|f\|^2 + \|g\|^2 - i\langle f, g\rangle + i\langle g, f\rangle$

$= \|f\|^2 + \|g\|^2 - i\langle f, g\rangle + i\overline{\langle f, g\rangle}$

$\|f-ig\|^2 = \langle f-ig, f-ig\rangle = \langle f, f\rangle - \langle f, ig\rangle - \langle ig, f\rangle + \langle ig, ig\rangle$

$= \|f\|^2 + i\langle f, g\rangle - i\langle f, g\rangle + i \times -i\langle g, g\rangle$

$= \|f\|^2 + \|g\|^2 + i\langle f, g\rangle - i\overline{\langle f, g\rangle}$

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i) $\|f+g\|^2 + \|f-g\|^2$:

$$\|f\|^2 + \|g\|^2 + \langle f,g\rangle + \overline{\langle f,g\rangle} + \|f\|^2 + \|g\|^2 - \langle f,g\rangle - \overline{\langle f,g\rangle}$$

$$= 2\left(\|f\|^2 + \|g\|^2\right)$$

ii) $\|f+g\|^2 = \|f\|^2 + \|g\|^2$:

$$\|f+g\|^2 = \|f\|^2 + \|g\|^2 + \langle f,g\rangle + \overline{\langle f,g\rangle}$$
$$= \|f\|^2 + \|g\|^2 + 0 + \overline{0}$$
$$= \|f\|^2 + \|g\|^2$$

iii) $\langle f,g\rangle = \frac{1}{4}\left(\|f+g\|^2 - \|f-g\|^2\right)$ $\quad$ f, g are real Hilbert space

$$\frac{1}{4}\left(\cancel{\|f\|^2} + \cancel{\|g\|^2} + \langle f,g\rangle + \overline{\langle f,g\rangle} - \cancel{\|f\|^2} - \cancel{\|g\|^2} + \langle f,g\rangle + \overline{\langle f,g\rangle}\right)$$

$$= \frac{1}{4}\left(\langle f,g\rangle + \overline{\langle f,g\rangle} + \langle f,g\rangle + \overline{\langle f,g\rangle}\right)$$

$\langle f,g\rangle = \overline{\langle f,g\rangle}$ $\quad$ (f, g real Hilbert spaces)

$$\therefore \frac{1}{4}\left(4\langle f,g\rangle\right) = \langle f,g\rangle$$

$$\langle f,g\rangle = \frac{1}{4}\left(\|f+g\|^2 + i\|f+ig\|^2 - \|f-g\|^2 - i\|f-ig\|^2\right)$$

$$= \frac{1}{4}\Big(\cancel{\|f\|^2} + \cancel{\|g\|^2} + \langle f,g\rangle + \cancel{\overline{\langle f,g\rangle}} - \cancel{\|f\|^2} - \cancel{\|g\|^2} + \langle f,g\rangle$$
$$+ \cancel{\overline{\langle f,g\rangle}} + i\cancel{\|f\|^2} + i\cancel{\|g\|^2} + \langle f,g\rangle - \cancel{\overline{\langle f,g\rangle}} - i\cancel{\|f\|^2}$$
$$- i\cancel{\|g\|^2} + \langle f,g\rangle - \cancel{\overline{\langle f,g\rangle}}\Big)$$

$$= \frac{4}{4}\langle f,g\rangle$$
$$= \langle f,g\rangle$$

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Q2) Consider the four Bell states

$$|\psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$$

$$|\phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|1\rangle \pm |10\rangle)$$

Show that these 4 vectors form a complete orthonormal basis

Ans) $|00\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$, $|01\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$, $|10\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$, $|11\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

$|\psi^{+}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$, $|\psi^{-}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}$

$|\phi^{+}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}$, $|\phi^{-}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\+1\\-1\\0\end{pmatrix}$

These are the 4 maximally entangled two-qubit Bell states, and they form a maximally entangled basis

A subset of vectors $\{v_1, v_2, \ldots v_k\}$ of a vector space $V$ with inner product $\langle , \rangle$ is called orthonormal if $\langle v_i, v_j\rangle = 0$ when $i \neq j$.
That is the vectors are mutually perpendicular. Moreover they are required to have a length = 1.

An orthonormal set must be linearly independent for it to be a basis and span a vector space.

$$\langle\psi^{+}|\psi^{+}\rangle = \frac{1}{\sqrt{2}}(1\,0\,0\,1) \times \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix} = \frac{1}{2}(1+0+0+1)$$

$$= 1$$

$$\langle\psi^{-}|\psi^{-}\rangle = \frac{1}{\sqrt{2}}(1\,0\,0\,-1) \times \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix} = \frac{1}{2}(2)$$

$$= 1$$

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$\langle \phi^+ | \psi^- \rangle = \frac{1}{\sqrt{2}} (0\,1\,1\,0) \times \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} = \frac{1}{2} (0)$

$= 0$

$\langle \psi^+ | \phi^- \rangle = \frac{1}{\sqrt{2}} (1\,0\,0\,1) \times \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} = \frac{1}{2} (0)$

$= 0$

If we check for all,

$\langle \psi^\pm | \phi^\pm \rangle = 0.$

$\langle \psi^- | \psi^+ \rangle = \frac{1}{\sqrt{2}} (1\,0\,0\,-1) \times \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$= \frac{1}{2} (1+0+0+-1) = \underline{\underline{0}}$

$\langle \psi^\pm | \psi^\mp \rangle = 0$

Similarly, $\langle \phi^\pm | \phi^\mp \rangle = 0.$

So these 4 vectors are mutually $\perp$ler; they are also linearly independent,

$c_1 |\psi^+\rangle + c_2 |\psi^-\rangle + c_3 |\phi^+\rangle + c_4 |\phi^-\rangle = 0$

$\Rightarrow \begin{pmatrix} c_1 \\ 0 \\ 0 \\ c_1 \end{pmatrix} + \begin{pmatrix} c_2 \\ 0 \\ 0 \\ -c_2 \end{pmatrix} + \begin{pmatrix} 0 \\ c_3 \\ c_3 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ c_4 \\ -c_4 \\ 0 \end{pmatrix} = 0$

$\Rightarrow \begin{pmatrix} c_1 + c_2 \\ c_3 + c_4 \\ c_3 - c_4 \\ c_1 - c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow c_1, c_2, c_3, c_4 = 0$ $\boxed{\text{only solution possible}}$

$\Rightarrow$ Linearly independent

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$\Rightarrow$ 4 Bell states are an orthonormal basis

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Q3) Consider the Linbald master Equation

$$\frac{\partial \rho}{\partial t} = \frac{i}{\hbar}[\rho, H] + \sum_i \gamma_i \left[ A_i \rho A_i^{\dagger} - \frac{1}{2} A_i^{\dagger} A_i \rho - \frac{1}{2} \rho A_i A_i^{\dagger} \right]$$

Show that the total probability remains unchanged under the action of this master equation.

Ans) $\rho$ is the density matrix, state of the system. $Tr[\rho]$ gives the total probability.

$\frac{\partial \rho}{\partial t}$ in the master Linbald form gives a infinitesimal dynamic map.

So for the total probability to remain unchanged,

$$\frac{\partial}{\partial t}[Tr\,\rho] = 0.$$

Here,

$$\frac{\partial \rho}{\partial t} = \frac{i}{\hbar}[\rho, H] + \sum_i \gamma_i \left[ A_i \rho A_i^{\dagger} - \frac{1}{2} A_i^{\dagger} A_i \rho - \frac{1}{2} \rho A_i A_i^{\dagger} \right]$$

$$\frac{\partial}{\partial t}[Tr\,\rho] = \frac{i}{\hbar} Tr\left[[\rho, H]\right] + \sum_i \gamma_i\, Tr\left[ A_i \rho A_i^{\dagger} - \frac{1}{2} A_i^{\dagger} A_i \rho - \frac{1}{2} \rho A_i A_i^{\dagger} \right]$$

Using the cyclic property of trace,

$$Tr[A_i \rho A_i^{\dagger}] = Tr[A_i^{\dagger} A_i \rho] = Tr[\rho A_i A_i^{\dagger}]$$

also, $Tr([\rho, H]) = 0$

$$\Rightarrow \frac{\partial}{\partial t}[Tr[\rho]] = \frac{i}{\hbar} \times 0 + \sum_i \gamma_i [0]$$

$$= 0$$

So the total probability, given by $tr(\rho)$ remains constant under the action of this master Equation.

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Q5) Show that, for the qubit, the transposition map is not completely positive.

Ans) Let us take a maximally entangled state in 2x2 dimension

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$$

$$|\psi\rangle\langle\psi| = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix} \times \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 0 & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 0 & 0 & 1\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\1 & 0 & 0 & 1\end{pmatrix}$$

Let us take the transposition map T.

$$\rho = \begin{pmatrix}\rho_{11} & \rho_{12}\\ \rho_{21} & \rho_{22}\end{pmatrix}, \quad T(\rho) = \begin{pmatrix}\rho_{11} & \rho_{21}\\ \rho_{12} & \rho_{22}\end{pmatrix}$$

Positivity:

$$\rho_{11}\cdot\rho_{22} \geq |\rho_{12}|^2$$

$$\rho_{11}, \rho_{22} \geq 0.$$

Checking complete positivity:

$$\mathbb{I} \otimes T(|\psi\rangle\langle\psi|) = \begin{pmatrix} T\begin{pmatrix}1/2 & 0\\0 & 0\end{pmatrix} & T\begin{pmatrix}0 & 1/2\\0 & 0\end{pmatrix}\\ T\begin{pmatrix}0 & 0\\1/2 & 0\end{pmatrix} & T\begin{pmatrix}0 & 0\\0 & 1/2\end{pmatrix}\end{pmatrix}$$

$$= \begin{pmatrix}1/2 & 0 & 0 & 0\\0 & 0 & 1/2 & 0\\0 & 1/2 & 0 & 0\\0 & 0 & 0 & 1/2\end{pmatrix}$$

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Finding the eigen values for this matrix

$$\begin{vmatrix} \frac{1}{2}-\lambda & 0 & 0 & 0 \\ 0 & -\lambda & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\lambda & 0 \\ 0 & 0 & 0 & \frac{1}{2}-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \left(\frac{1}{2}-\lambda\right)\left( \begin{vmatrix} -\lambda & \frac{1}{2} & 0 \\ \frac{1}{2} & -\lambda & 0 \\ 0 & 0 & \frac{1}{2}-\lambda \end{vmatrix} \right) = 0$$

$$\Rightarrow \left(\frac{1}{2}-\lambda\right)\left(-\lambda\left(-\lambda\left(\frac{1}{2}-\lambda\right)\right) - \frac{1}{2}\left(\frac{1}{2}\left(\frac{1}{2}-\lambda\right)\right)\right)$$

$$\Rightarrow \left(\frac{1}{2}-\lambda\right)\left(\lambda^2\left(\frac{1}{2}-\lambda\right) - \frac{1}{4}\left(\frac{1}{2}-\lambda\right)\right)$$

$$= \left(\frac{1}{2}-\lambda\right)^2\left(\lambda^2 - \frac{1}{4}\right) \Rightarrow \lambda = \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}$$

We have one neg eigen value

$\therefore$ Transposition map is not completely +ve, though it is positive.

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Q4) i) Solve the master Eq for a qubit

$$\frac{\partial \rho}{\partial t} = \gamma_x(\sigma_x \rho \sigma_x - \rho) + \gamma_y(\sigma_y \rho \sigma_y - \rho) + \gamma_z(\sigma_z \rho \sigma_z - \rho)$$

where $\sigma_i$'s are the pauli's matrices

ii) Show how the maximally mixed state changes after this operation.

Ans) Given this master Eq, we want to derive a map $\Phi$ such that

$$\rho(t) = \Phi_t[\rho(0)]$$

We will take 3 pauli's matrices & $I$ (in 2-dim) as the orthonormal basis for the Hilbert space

$$\phi(\rho) = (\vec{F} \cdot \vec{r})^T \vec{G} \longrightarrow (i)$$

For any map $\phi$

$$F_{k,l} = tr[G_k \phi(G_l)] \longrightarrow (ii)$$

$$r_l = tr[G_l \rho] \longrightarrow (iii)$$

Also, given a master eq of the form $\frac{\partial \rho}{\partial t} = d(\rho)$ can be rewriten as:

$$\frac{\partial \rho}{\partial t} = [\vec{L} \cdot r(\vec{t})]^T G \longrightarrow (iv)$$

$$L_{kl} = tr[G_k d(G_l)] \longrightarrow (v)$$

Comparing eq (i) & eq (iv)

$$\dot{\vec{F}}\vec{r}(0) = \vec{L} \cdot r(t) \longrightarrow (vi)$$

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We get $\dot{\vec{F}} = \vec{L} \cdot \vec{F}$ $\longrightarrow$ (vii)

or $\vec{L} = \dot{\vec{F}} \cdot F^{-1}$ $\longrightarrow$ (viii)

To solve this eq,

$$L_{kl} = tr\left[ G_k \, \lambda \left( G_l \right) \right]$$

$$\lambda(\sigma_x) = \gamma_x \left( \sigma_x \sigma_x \sigma_x - \sigma_x \right) + \gamma_y \left( \sigma_y \sigma_x \sigma_y - \sigma_x \right)$$
$$+ \gamma_z \left( \sigma_z \sigma_x \sigma_z - \sigma_x \right)$$

$$\sigma_x \sigma_x \sigma_x - \sigma_x = 0$$

$$\sigma_y \sigma_x \sigma_y - \sigma_x = i \sigma_y \sigma_z - \sigma_x$$
$$= -\sigma_x - \sigma_x$$
$$= -2\sigma_x$$

$$\sigma_z \sigma_x \sigma_z - \sigma_x = -i \sigma_z \sigma_y - \sigma_x$$
$$= -\sigma_x - \sigma_x$$
$$= -2\sigma_x$$

$$\therefore \lambda(\sigma_x) = -2\sigma_x \left( \gamma_z + \gamma_y \right)$$

Similarly

$$\lambda(\sigma_y) = -2\sigma_y \left( \gamma_x + \gamma_z \right)$$

$$\lambda(\sigma_z) = -2\sigma_z \left( \gamma_x + \gamma_y \right)$$

$$\lambda(I) = \gamma_x \left( \sigma_x^2 - I \right) + \gamma_y \left( \sigma_y^2 - I \right)$$
$$+ \gamma_z \left( \sigma_z^2 - I \right)$$
$$= 0.$$

Only $tr\{\sigma_i \lambda(\sigma_i)\}$ = non zero.

$tr\{\sigma_x \lambda(\sigma_x)\} = -2(\gamma_z + \gamma_y)$

$tr\{\sigma_z \lambda(\sigma_x)\} = tr[-2\,\sigma_z \sigma_x (\gamma_z + \gamma_y)]$

$= -2(\gamma_z + \gamma_y)\, \underbrace{tr[\sigma_z \sigma_x]}_{\to 0}$

$= 0$

$\therefore tr\{\sigma_y \lambda(\sigma_y)\} = -2(\gamma_x + \gamma_z)$

$tr\{\sigma_z \lambda(\sigma_z)\} = -2(\gamma_x + \gamma_y)$

$tr\{\sigma_i \lambda(\sigma_j)\} = 0$

$tr\{\sigma_i \lambda(I)\} = 0$

So L is a diagonal matrix with entries, 0, $-2(\gamma_y + \gamma_z)$, $-2(\gamma_x + \gamma_z)$, & $-2(\gamma_x + \gamma_y)$

Solving $\dot{F} = LF$, we get

$F(t) = \exp(Lt)$ as $L$ is a const in 't'

& $F(0) = I$.

$\therefore$ $F(t)$ is a diagonal matrix with entries,

$e^0 (=1)$, $e^{-2(\gamma_y + \gamma_z)t}$, $e^{-2(\gamma_x + \gamma_z)t}$, $e^{-2(\gamma_x + \gamma_y)t}$