Q1) Throwing an unbiased coin n times independently

a) Event that every 10^{th} throw lands on Head, Assume n is divisible by 10 also n is not divisible by 10

$n = 10k \qquad (k > 0)$

first 10 throw

$$\underbrace{\underset{2}{\_}\ \underset{2}{\_}\ \underset{2}{\_}\ \cdots\cdots\ \underset{2}{\_}}_{\text{possibilities}}\ \ H \rightarrow \text{fixed}$$

$\text{Prob} = \dfrac{2^9}{2^{10}} = \dfrac{1}{2}$

Sec 10 throws : $\text{prob} = 2^9/2^{10} = 1/2$

$\vdots$

k^{th} 10 throws : $\text{prob} = 2^9/2^{10} = 1/2$

$\therefore$ Total probability after $10k$ throws $= \left(\dfrac{1}{2}\right)^k$

Suppose $n = 10k + \delta \quad (k > 0,\ \delta = \{1,2,3,4,5,6,7,8,9\})$

First 10 throws : $\text{prob} = 1/2$

Second 10 throws : $\text{prob} = 1/2$

$\vdots$

k^{th} 10 throws : $\text{prob} = 1/2$

δ throws: $\text{prob} = \dfrac{2^\delta}{2^\delta} = 1$

$\therefore$ Total prob after $n = 10k + \delta$ throws $= \left(1/2\right)^k \cdot \dfrac{2^\delta}{2^\delta} = \dfrac{1}{2^k}$

b) Find the probability that atleast 1 consecutive sequence of $10 \log n$ throws exist without any heads in it.

↓

Without any heads ⇒ only tails

We must find the probability that atleast 1 consecutive sequences of $10 \log n$ throws of only tails.

Let X be a r.v such that it represents a sequence of $10 \log n$ throws of only tails.

We require to find $Pr(X \geq 1)$,

$Pr(X \geq 1) = 1 - Pr(X = 0)$

$Pr(X=0)$ ⇒ probability that there are no consecutive sequences of $10 \log n$ tails

In the total of n trials, suppose last coin flip is a Head, then

No of sequences without $10 \log n$ Consecutive tails

$f(n) = f(n-1)$ (Last is head)

If the last flip is a tail, then the sec last flip is a H, then $f(n) = f(n-2)$

$\therefore f(n) = f(n-1) + f(n-2) + \dots f(n - 10\log n)$

$Prob(X=0) = \dfrac{f(n)}{2^n}$ $\therefore P(X \geq 1) = 1 - \dfrac{f(n)}{2^n}$

Q2) Show that the way conditional probabilities as defined satisfies the axioms of probability.

Ans) Axioms of probability:

a) Axiom 1:

The prob of an event is a real num ≥ 0

b) Axiom 2:

The probability that atleast one of all the possible outcomes of a process will occur is 1

c) Axiom 3:

If 2 events A & B are mutually exclusive, the the prob of either A or B occurring is the probability of A occurring + prop of B occuring.

→ Condition probabilities satisfy usual probability axioms.

a) Axiom 1:

$P(A|B) \geq 0$ → Since normal probabilities are non-negetive, the ratio used in conditional probability is also non-negetive as long as it is well defined $(P(B) > 0)$

$$\frac{P(A \cap B)}{P(B)}$$

$P(A \cap B)$ → non-neg

$P(B)$ → positive

b) Axiom 2:

$$P(\Omega|B) = \frac{P(\Omega \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1 \quad (\Omega \cap B = B \text{ as B is asubset})$$

Ω → prob space

→ possible outcomes that dont belong to B are considered to be impossible (Assume B to be sample space)

$$P(B|B) = \frac{P(B\cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

Axiom 3:

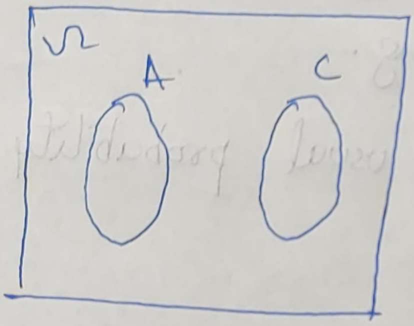
$P(A\cup C) = P(A) + P(C)$ for set A & C which are disjoint

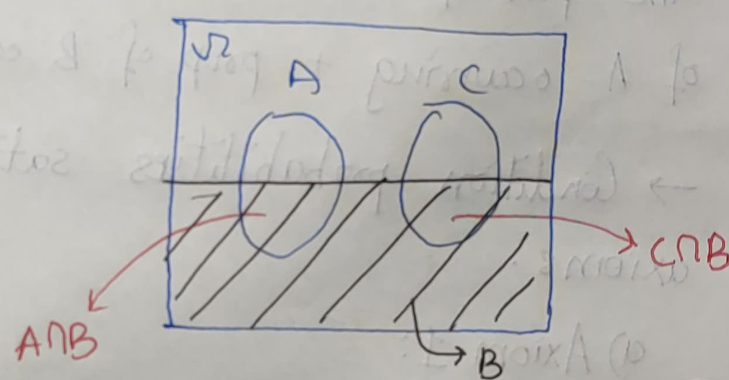
If $A\cap C = \phi$, $P(A\cup C/B) = P(A/B) + P(C/B)$

$$P(A\cup C/B) = \frac{P((A\cup C)\cap B)}{P(B)} = \frac{P(A\cap B)\cup P(B\cap C)}{P(B)}$$

disjoint property.

$$= \frac{P(A\cap B)}{P(B)} + \frac{P(B\cap C)}{P(B)} = P(A|B) + P(C|B)$$





→ We can repeat the same derivation for finite & countably many such disjoint sets

→ Any theorems derived from normal probabilities will be applicable for conditional as well.

Q3) m- Tshirts
n- participants
Each participant $\longrightarrow$ any no of Tshirts including 0

Ans a) Expected No of participants who do not get any t-shirts

Let X_j denote the number of Tshirts with j^{th} participant.

Define another r.v X_{ij} (Bernoulli R.V) such that

$$X_{ij} = \begin{cases} 1 & \text{T shirt } i \text{ is with } j^{th} \text{ participant} \\ 0 & \text{otherwise} \end{cases}$$

So, for $i = 1, 2, 3 \ldots m$ & $j = 1, 2, \ldots n$

$$Pr[X_{ij} = 1] = E[X_{ij}] = 1/n$$

Tshirt 'i' is with person 'j' uniformly at random

$$E[X_{ij}] = 1 \cdot Pr[X_{ij} = 1] + 0 \, Pr[X_{ij} = 0]$$
$$= Pr[X_{ij} = 1] = 1/n$$

$$X_j = \sum_{i=1}^{m} X_{ij} \longrightarrow$$ (summing all tshirts of person j)

$$E[X_j] = E\left[\sum_{i=1}^{m} X_{ij}\right] = \sum_{i=1}^{m} E[X_{ij}] = \underbrace{m/n}_{\text{Obvious value as } m \text{ shirts to } n \text{ people}} \rightarrow \text{Mean '}\mu\text{'}$$

linearity of Expectations

Denote another random variable Z which denotes the no of candidates with 0 t-shirts

Define a bernoulli r.v such that

$$Z_{ij} = \begin{cases} 1 & \text{Tshirt } i \text{ is not with person } j \\ 0 & \text{otherwise} \end{cases}$$

for $i = 1, 2, \ldots m$, & $j = 1, 2, \ldots n$

$$\Pr[Z_{ij} = 1] = 1 - 1/n$$

person j doesnt have tshirt i

$1/n \Rightarrow$ person j gets tshirt i

$\therefore 1 - 1/n \Rightarrow$ person j doesnt get tshirt i

$$E[Z_{ij}] = 1 \cdot \Pr[Z_{ij} = 1] + 0 \cdot \Pr[Z_{ij} = 0]$$
$$= \Pr[Z_{ij} = 1] = 1 - 1/n$$

For any $j = 1, 2, \ldots m$

$$E[Z_j] = E\left[\prod_{i=1}^{m} Z_{ij}\right] = \prod_{i=1}^{m} E[Z_{ij}] = \left(1 - 1/n\right)^m$$

Z_{ij}'s are mutually independent

$$Z_j = \prod_{i=1}^{m} Z_{ij}$$ (all shirts have missed j)

$$E[Z] = E\left[\sum_{j=1}^{n} Z_j\right] = \sum_{j=1}^{n} E[Z_j] = \sum_{j=1}^{n} \left(1 - 1/n\right)^m$$
$$= n\left(1 - 1/n\right)^m$$
$$\simeq n \cdot e^{-m/n}$$

$Z = \sum_{j=1}^{n} Z_j$ (sum of all 0 shirt participants)

b) The expected no of people who get exactly 1 T-shirt.

Let Z be the r.v which is the no of participants with exactly 1 tshirt. Let Z_i be the bernoulli r.v such that

$$Z_i = \begin{cases} 1 & \text{if } i^{th} \text{ participant has 1 tshirst} \\ 0 & \text{otherwise} \end{cases}$$

$$Z = \sum_{i=1}^{n} Z_i \quad \& \quad E[Z] = E\left[\sum_{i=1}^{n} Z_i\right]$$

$$= \sum_{i=1}^{n} E[Z_i]$$

Linearity of expectations

→ Since all participants are ideal, we need to just consider one case

$$E[Z_i] = \Pr[Z_i = 1] = \frac{n \cdot (n-1)^{m-1}}{n^m}$$

$n(n-1)^{m-1}$ ⇒ n choices for a tshirt to i^{th} canditate and $(n-1)^{m-1}$ choices for other tshirts to go to $m-1$ candidates

n^m → No of ways to put n tshirts into m participants

$$\therefore E[Z] = \sum_{i=1}^{n} E[Z_i] = \frac{n \cdot n \cdot (n-1)^{m-1}}{n^m} = n\left(1 - 1/n\right)^{m-1}$$

c) Probability that some participant gets more than $10 \frac{m \log n}{n}$ shirts

We know that the mean we derived is m/n

Using markors inequalities,

$$Pr[X \geq c\mu] \leq 1/c \quad \text{for } X \text{ is a non-neg r.v}$$

We can apply it here as our R.V are also non negetive

$$Pr\left[X \geq 10 \frac{m \log n}{n}\right] = Pr\left[X \geq 10 \log n \cdot \frac{m}{n}\right]$$

$$= Pr\left[X \geq \underbrace{(10 \log n)}_{c} \cdot \mu\right]$$

proved before

$$\leq \frac{1}{c} \leq \frac{1}{10 \log n}$$

Q4) Random var are said to be identical if they have the same distribution

a) X & Y have same expectation. Are X & Y Identical?

$$E[x] = \sum_x x \Pr(X=x)$$

$$E[Y] = \sum_y y \cdot \Pr(Y=y)$$

suppose $E[x] = E[y]$, for X & Y to be identical

$$CDF(x) = CDF(Y)$$

$$\Pr(X \le x) = P(Y \le y)$$

$$\sum_x x \cdot \Pr(X=x) = \sum_y y \cdot \Pr(Y=y)$$

→ from this we cannot tell if X & Y are identical.
↳ Just 'mean'

b) X & Y have also have same variance. Are X & Y Identical?

$$Var(X) = E[x^2] - (E[x])^2$$

$$Var(y) = E[Y^2] - (E[Y])^2$$

$$\Rightarrow E[x^2] - (E[x])^2 = E[Y^2] - (E[Y])^2$$

$$\Rightarrow E[x^2] - E[Y^2] = (E[x])^2 - (E[Y])^2$$

$$= 0$$

$$\Rightarrow E[x^2] = E[Y^2]$$

If X & Y are independent & identical (iid) they must have a common mean & variance. (same distribution)

If X & Y are not independent, it is not possible to say about them being identical.