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Author(s): Haim Reisman
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A GENERAL APPROACH TO THE ARBITRAGE PRICING THEORY (APT)

BY HAIM REISMAN

1. INTRODUCTION

THE ARBITRAGE PRICING THEORY (APT) first studied by Ross (1976) is one of the central models of the theory of capital markets. One of the reasons for the attractiveness of this model lies in the fact that relatively strong results are obtained under very weak assumptions. A description of the version of the model studied by Chamberlain and Rothschild (1983) is given below.

Given are a sequence of assets with payoffs in some L^2 space and a K dimensional subspace called the factor space. The model studies in what sense the price of an asset can be approximately determined by the projection of its payoff on the factor space. The model provides a bound on the sum of the squares of the pricing mistakes. This bound depends on the norm of the price functional and on an independence module which measures the correlation between the projections of the asset payoffs on the orthogonal complement of the factor space.

In the present paper, the APT is studied in economies in which the choice space is an arbitrary normed vector space. No notion of positivity of the price functional is needed in formulating the theory. The definition of an approximate factor structure in the style of Chamberlain and Rothschild (1983) is generalized to general normed spaces, and the no arbitrage assumption is replaced by the assumption that the price functional is continuous. The proof of the main theorem is simpler than the proofs in the existing less general versions, e.g., Huberman (1982), Ingersoll (1984), and Chamberlain and Rothschild (1983).

The plan of the paper is as follows. Formulation of the problem in general choice spaces is given in Section 2. A special case of the theory is considered in Section 3. The special case considered in Section 3 is the L^2 theory as was studied by Ross (1976) and by Chamberlain and Rothschild (1983). The proof of the main theorem is given in Section 4. The applications of the theory are discussed in Section 5.

2. FORMULATION

The formulation given below is a generalization of the version of the APT studied by Chamberlain and Rothschild (1983). Let B be a normed vector space, $M \subset B$ a subspace of B and $\Pi \in M^*$ (M^* is the dual of M). Interpret B as a part of the choice space in our economy, M as a part of the market, and Π as the restriction of the price functional to M.

Let $F \subset B$ be a subspace and assume a projection $P: B \to F$ exists. F is called the factor space and every $x \in M$ has the decomposition

$$x = Px + \varepsilon. \tag{1}$$

P_x is called the factor part of x and ε is called the noisy part of x.

For $1 \leqslant p \leqslant \infty$ let ℓ_p be the Banach space of all real sequences $\alpha = (\alpha_1, \alpha_2, \dots)$ with the norm

$$|\alpha|_p = \left(\sum_{i=1}^{\infty} |\alpha_i|^p \right)^{1/p} \quad \text{if} \quad 1 \leqslant p < \infty,$$

and

$$|\alpha|_\infty = \sup_{i \geqslant 1} |\alpha_i|.$$

Define ℓ_p^0 to be the set of all $\alpha \in \ell_p$ such that only a finite number of the α_i's are not zero.

Under consideration is the set $\{x_i \in M;\ i = 1, 2, \dots\}$. For each i define

$$\varepsilon_i = x_i - Px_i. \tag{2}$$

Let $1 \leqslant p < \infty$. The independence module of type p of the ε_i's is defined as

$$\Lambda_p = \sup_{\alpha \in \ell_p^0,\, |\alpha|_p = 1} \left\| \sum_{i=1}^{\infty} \alpha_i \varepsilon_i \right\|_B. \tag{3}$$

If $\Lambda_p < \infty$ then we say that $\{x_i\}_{i=1}^{\infty}$ satisfies an approximate factor structure of type p. The relation of the above to the classical theory is demonstrated in Section 3. The main theorem of the APT is proved in Section 4.

3. A SPECIAL CASE

A special case of the economic setting described in Section 2 is the version of the APT studied by Chamberlain and Rothschild (1983). Let $B = L^2(\Omega, H, Q)$ for some probability space (Ω, H, Q). Interpret $x_i \in B$ as an asset with a payoff of $x_i(\omega)$ if the state is $\omega \in \Omega$. Let $\delta_1, \dots, \delta_K \in B$ and let $F = \operatorname{span}\{1, \delta_1, \dots, \delta_K\}$. Let P be the orthogonal projection of F. For each i, Px_i is given by

$$Px_i = \sum_{k=1}^{K} \beta_{ki}\delta_k + \beta_{0i} \qquad (i = 1, 2, \dots), \tag{4}$$

where $\beta_{ki} \in R$. Each x_i has the decomposition given in (2). Obviously

$$E\varepsilon_i = E\varepsilon_i\delta_k = 0 \quad \forall i = 1, 2, \dots \quad \text{and} \quad 1 \leqslant k \leqslant K.$$

If $E\varepsilon_i\varepsilon_j = 0$ for every $i \neq j$, then

$$\Lambda_2^2 = \sup_{1 \leqslant i} E\varepsilon_i^2. \tag{5}$$

If $\Lambda_2 < \infty$ then $\{x_i\}_{i=1}^{\infty}$ has a strict linear factor structure as defined in Ross (1976). Intuitively, the payoff of each asset depends on a small number of factors common to all assets in the economy and a noise term specific to the asset and uncorrelated with noise terms of other assets.

If the ε_i's are not orthogonal to each other and $\Lambda_2 < \infty$ then we have an approximate factor structure in the style of Chamberlain and Rothschild (1983). Let Ω_N be an $N \times N$ matrix defined by

$$\Omega_N = \left(E\varepsilon_i\varepsilon_j\right)_{1, j = 1, \dots, N}. \tag{6}$$

Observe that

$$\Lambda_2^2 = \sup_{1 \leqslant N} \sup_{\substack{\alpha \in R^N \\ |\alpha|_2 = 1}} \alpha' \Omega_N \alpha \tag{7}$$

and notice that boundedness of the right-hand side of (7) is defined as the existence of an approximate factor structure in Chamberlain and Rothschild (1983).

4. THE MAIN THEOREM

The following theorem is a generalization of the classical APT with an approximate factor structure to choice spaces which are normed vector spaces. The proof given here is simpler than the proofs given in the existing less general versions of the APT and reduces the argument down to the barest essentials.

THEOREM 1: *Let* $1 \leqslant p < \infty$ *and assume that* $\Lambda_p < \infty$. *Then there exists a* $\psi \in F^*$ *such that*

$$\Pi(x_i) = \psi(Px_i) + \rho_i \tag{8}$$

and $\rho = (\rho_1, \rho_2, \dots)$ *satisfies*

$$|\rho|_q \leqslant \|\Pi\|_{M^*} \Lambda_p, \tag{9}$$

where $q = p/(p-1)$ *if* $p > 1$ *and* $q = \infty$ *if* $p = 1$.

PROOF: By the Hahn Banach theorem there exists $\Pi^* \in B^*$ such that $\Pi^*(x) = \Pi(x)$ for every $x \in M$ and $\|\Pi^*\|_{B^*} = \|\Pi\|_{M^*}$. Define $\psi \in F^*$ by $\psi(x) = \Pi^*(x)$ for every $x \in F$. This implies that $\rho_i = \Pi^*(\varepsilon_i)$. For $\alpha \in \ell_p^0$, $|\alpha| = 1$,

$$\left| \Pi^* \left(\sum_{i=1}^{\infty} \alpha_i \varepsilon_i \right) \right| \leqslant \|\Pi^*\|_{B^*} \left\| \sum_{i=1}^{\infty} \alpha_i \varepsilon_i \right\|_B. \tag{10}$$

This implies that for every $\alpha \in \ell_p$, $|\alpha|_p = 1$ and for every $N \geqslant 1$,

$$\left| \sum_{i=1}^{N} \alpha_i \rho_i \right| \leqslant \|\Pi\|_{M^*} \Lambda_p. \tag{11}$$

The duality of ℓ_p and ℓ_q implies that

$$|\rho^N|_q \leqslant \|\Pi\|_{M^*} \Lambda_p, \tag{12}$$

where $\rho^N = (\rho_1, \dots, \rho_N, 0, \dots)$ and the theorem follows. *Q.E.D.*

REMARK: ψP is a linear functional on B and it is continuous if P is continuous. The fact the P is a linear projection was not used in the proof and P can be replaced by any mapping from B to F.

5. EXAMPLE

Let (Ω, H, Q) be a complete probability space, let $B = L^2(\Omega, H, Q)$, let $F = L^2(\Omega, H_f, Q)$, where H_f is a sub σ-algebra of H which includes all the null sets of H, let $M \subset B$, and let

$$Px = E\left[x \mid H_f\right] \quad \forall x \in B, \tag{13}$$

and interpret $x \in B$ as a claim for $x(\omega)$ if the state is $\omega \in \Omega$. Theorem 1 implies that if $\Lambda_2 < \infty$ then there exists $\psi \in F^*$ such that

$$\Pi(x_i) = \psi\left(E\left[x_i \mid H_f\right]\right) + \rho_i \tag{14}$$

and the ρ_i's satisfy inequality (9).

Of special interest is the case where H_f is the σ algebra generated by some continuously traded securities and the prices of all $x \in F = L^2(\Omega, H_f, Q)$ are determined by arbitrage by those securities. Extensions of this method of approximate valuation in dynamic markets were studied by Reisman (1985, 1986) and by Chamberlain (1985).

6. CONCLUSION

The contribution of the paper is mainly in demonstrating that the APT is a simple consequence of the Hahn Banach Theorem when the asset payoffs have an approximate factor structure.

In addition the general formulation of the APT studied in this paper, which allows an infinite dimensional factor space, may be used in the extension of the theory to the dynamic case.

Department of Finance, School of Management, University of Minnesota, Minneapolis, MN 55455, U.S.A.

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REFERENCES

CHAMBERLAIN, G. (1985): "Asset Pricing in Multiperiod Securities Markets," Working Paper, University of Wisconsin.

CHAMBERLAIN, G., AND M. ROTHSCHILD (1983): "Arbitrage Factor Structure and Mean-Variance Analysis on Large Asset Markets," *Econometrica*, 51, 1281–1304.

HUBERMAN, G. (1982): "A Simple Approach to Arbitrage Pricing Theory," *Journal of Economic Theory*, 28, 183–191.

INGERSOLL, J. (1984): "Some Results on the APT," *Journal of Finance*, 34, 1021–1039.

REISMAN, H. (1985): "Valuation of Noisy Streams," Working Paper, University of Minnesota.

——— (1986): "The Intertemporal APT," Working Paper, University of Minnesota.

ROSS, S. (1976): "The Arbitrage Theory of Capital Asset Pricing," *Journal of Economic Theory*, 13, 341–360.