SRIKAR KALE
2020701018

1) $(y^2+u^2)u_x - xy\,u_y + ux = 0$

A) $(y^2+u^2)u_x - xy\,u_y = -ux$

$$\frac{dx}{y^2+u^2} = \frac{dy}{-xy} = \frac{du}{-ux}$$

$$\frac{dy}{y} = \frac{du}{u} \Rightarrow \log y = \log u + c_1$$

$$c_1 = \log\left(y/u\right) \Rightarrow y/u = c_1'$$

$$\frac{dx}{c_1'^2u^2+u^2} = \frac{du}{-ux} \Rightarrow x\,dx = -u\left(c_1'^2+1\right)du$$

$$\Rightarrow \frac{x^2}{2} = \frac{-u^2}{2}\left(c_1'^2+1\right) + c_2$$

$$\Rightarrow \frac{x^2+y^2+u^2}{2} = c_2$$

$$\therefore F\left(y/u, \frac{x^2+y^2+u^2}{2}\right) = 0$$

2) $(x^2-y^2-uy)u_x + (x^2-y^2-ux)u_y = (x-y)u$

A) $$\frac{dx}{x^2-y^2-uy} = \frac{dy}{x^2-y^2-ux} = \frac{du}{(x-y)u}$$

$dx - dy - du = 0$ Coeff $= 1, -1, -1$

$\therefore x - y - u = c_1$

$$\frac{x\,dx}{x^2-y^2} - \frac{y\,dy}{x^2-y^2} - \frac{1}{u}\,du = 0$$

$$\Rightarrow \frac{x\,dx - y\,dy}{x^2-y^2} = \frac{1}{u}\,du$$

$$\Rightarrow \frac{1}{2}\log(x^2-y^2) = \log u + C_2$$

$$\Rightarrow \log\left(\frac{x^2-y^2}{u^2}\right) = 2C_2$$

$$\Rightarrow \frac{x^2-y^2}{u^2} = C_2'$$

$$\therefore F\left(x-y-u,\ \frac{x^2-y^2}{u^2}\right) = 0.$$

3) $L = 10$ cm

$\text{Initial}_A = 20$, $\text{Initial}_B = 40$ } steady state; $\text{final}_A = 50$, $\text{final}_B = 10^\circ$

Ans) Heat Eq: $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$

Steady state $\Rightarrow \frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow u(x) = ax + b$

$u(0) = b = 20$

$u(10) = 10a + 20 = 40$

$\Rightarrow a = 2$

$u(x,0) = 2x + 20$

$u(0,t)_F = 50$, $u(10,t)_f = 10$ } again $\frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow u_s(x) = a'x + b'$

$$u(x,t)_f = u_s(x) + u_{tr}(x,t)$$

$$u_s(0) = b' = 50$$

$$u_s(10) = 10a' + 50 = 10 \qquad \therefore \; u_s(x) = -4x + 50$$
$$\Rightarrow a' = -4$$

$$u_{tr}(x,t) = u(x,t) - u_s(x)$$

$$\left.\begin{aligned} u_{tr}(0,t) &= u(0,t) - u_s(0) = 50 - 50 = 0 \\ u_{tr}(10,t) &= u(10,t) - u_s(10) = 10 - 10 = 0 \end{aligned}\right\} \Rightarrow u_{tr} \text{ is Homogeneous B.C}$$

$$\begin{aligned} u(x,0) = f(x) &= u(x,0) - u_s(x) \\ &= 2x + 20 - (-4x + 50) \\ &= 6x - 30 \end{aligned}$$

$$u_{tr}(x,t) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{L} x \; e^{-c^2 \pi^2 n^2 t / L^2}$$

$$\text{Where } A_n = \frac{2}{L} \int_0^L (6x - 30) \sin \frac{n\pi x}{L} \, dx$$

$$\int (6x-30) \sin \frac{n\pi x}{L} = -\frac{(6x-30)L}{n\pi} \cos\left(\frac{n\pi x}{L}\right) + \frac{6L}{n\pi} \int \cos \frac{n\pi}{L} x \, dx$$

$$= \left[-\frac{(6x-30)L}{n\pi} \cos\left(\frac{n\pi x}{L}\right) + \frac{6L^2}{n^2\pi^2} \sin \frac{n\pi x}{L} \right]_0^L$$

$$= \left[-\frac{30L}{n\pi} (-1)^n + 0 \right]_L - \left[\frac{30L}{n\pi} \right]_0$$

$\Rightarrow A_n = \frac{-300}{n\pi}\left(1+(-1)^n\right) \times \frac{2}{\cancel{b}\,10} = \frac{-60}{n\pi}\left(1+(-1)^n\right)$

If $n =$ odd, $A_n = 0$.

$\Rightarrow u_{tr} = 0$ for $n =$ odd

If $n =$ even, $A_n = \frac{-120}{n\pi}$

$$u_{tr}(x,t) = \sum_{n=2,4,6..}^{\infty} \frac{-120}{n\pi} \sin \frac{n\pi x}{L}\, e^{-\frac{c^2\pi^2 n^2 t}{100}} \quad (L=10)$$

$$\therefore u(x,t)_F = u_s(x) + u_{tr}(x,t)$$

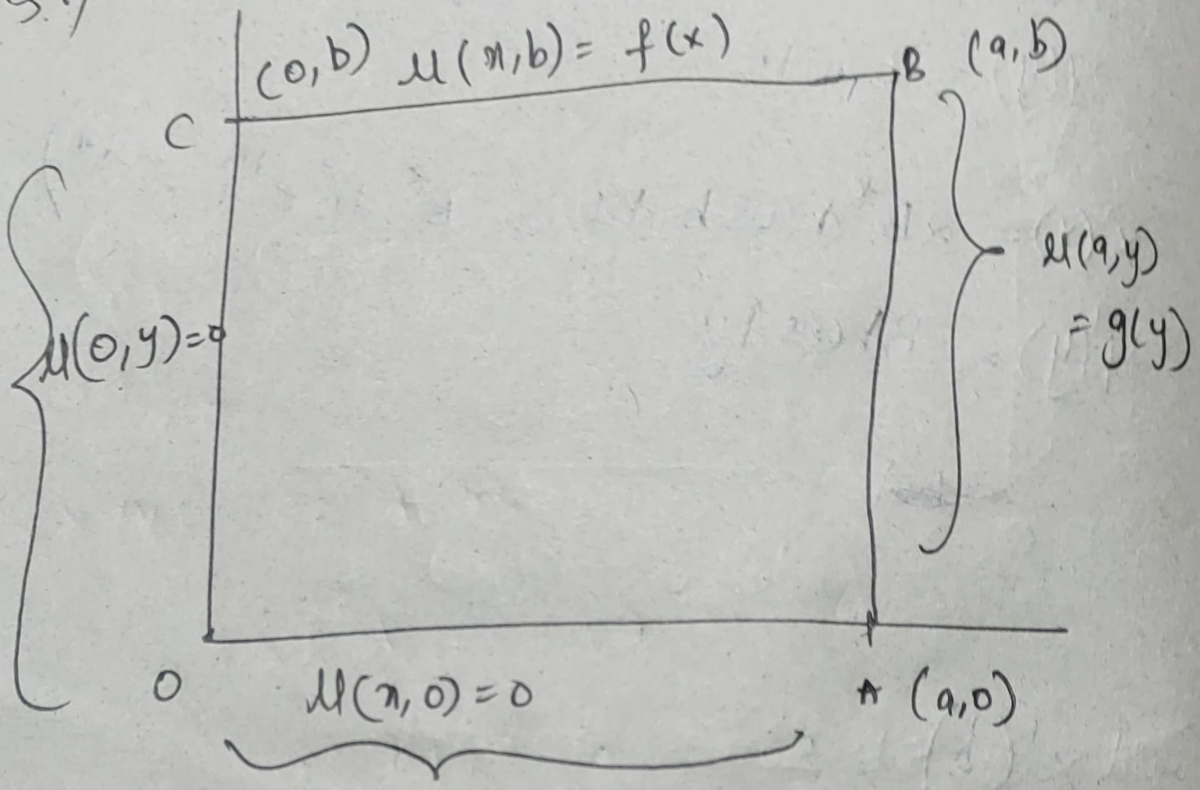
$$= -4x+50 + \sum_{n=2,4..}^{\infty} \frac{-120}{n\pi} \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{c^2\pi^2 n^2 t}{100}}$$

$$u(L/2,t) = -4\times5+50 + \sum_{n=2,4,6..}^{\infty} \frac{-120}{n\pi} \sin\left(\frac{n\pi}{2}\right) e^{-\frac{c^2 n^2\pi^2 t}{100}}$$

$$= 30 + 0 \quad \left(\sin\frac{n\pi}{2} \text{ as } n = \text{even} = \underset{\downarrow\, 0}{\sin k\cdot\pi}\right)$$

$= 30 \longrightarrow$ constant.

Q4)



B.C :

$$u(x,0) = 0$$
$$u(0,y) = 0$$
$$u(a,y) = g(y)$$
$$u(x,b) = f(x)$$

steady state $\Rightarrow$ $\nabla^2 u = 0$

$\Rightarrow$ $X''Y + XY'' = 0$

$\Rightarrow$ $-\dfrac{X''}{X} = \dfrac{Y''}{Y} = k$

$\Rightarrow k = -\lambda^2$, $X'' - \lambda^2 X = 0$, $Y'' + \lambda^2 Y = 0$

$X(x) = A^* e^{-\lambda x} + B^* e^{+\lambda x}$

$Y(y) = A \cos \lambda y + B \sin \lambda y$

$u(x,0) = X(x) \cdot Y(0) = 0 \Rightarrow Y(0) = 0$

$\Rightarrow Y(0) = A = 0 \Rightarrow Y(y) = B \sin \lambda y$

$u(0,y) = X(0) \cdot Y(y) = 0 \Rightarrow X(0) = 0$

$\Rightarrow X(0) = A^* + B^* = 0 \Rightarrow B^* = -A^*$

$X(x) = B^* \left(e^{\lambda x} - e^{-\lambda x} \right)$

$= 2B^* \sinh \lambda x$

Also,

2 sides, surfaces insulated

$\Rightarrow$ Any 2 of $\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0$, $\left.\frac{\partial u}{\partial x}\right|_{x=a} = 0$, $\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0$,

$\left.\frac{\partial u}{\partial y}\right|_{y=b} = 0$ can be true

$u_x = X'(x) \cdot Y(y)$

$u_x(0,y) \Rightarrow X'(0) = 0$

$X'(x) = 2\lambda B^* \cosh \lambda x$

$x'(0) = 2\lambda B^* = 0$

$\Rightarrow \lambda = 0$ or $B^* = 0$ $\longrightarrow$ Both are invalid.

$\left.\frac{\partial u}{\partial x}\right|_{x=a} = 0$

$x'(x) = 2B^* \lambda \cos h \lambda x = 0$

$= 2B^* \lambda \cos h \lambda a = 0$

$\Rightarrow \cos h \lambda a = 0$

$\frac{e^x + e^{-x}}{2} = \cos h x = 0$

$\Rightarrow e^x + e^{-x} = 0$ $\longrightarrow$ Not possible.

$\left.\frac{\partial u}{\partial y}\right|_{y=0}$

$y'(y) = B\lambda \cos \lambda y$

$y'(0) = 0 \Rightarrow B\lambda = 0$

$\Rightarrow B = 0$ or $\lambda = 0$ $\longrightarrow$ Invalid

$$\left.\frac{\partial u}{\partial y}\right|_{y=b} = Y'(b) = B\lambda \cos \lambda b = 0$$

$$\lambda b = \frac{\pi}{2}$$

$$\lambda = \frac{\pi}{2b}$$

out of 4 conditions, 3 are invalid because of the other B.C

$\therefore$ We cannot have 2 surfaces insulated

→ Question is ambigous