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Assignment II FUNCTIONAL ANALYSIS

Q) Describe the span of $M = \{(1,1,1), (0,0,2)\}$ in $\mathbb{R}^3$

Ans) $V_1 = (1,1,1)$
$V_2 = (0,0,2)$

The span of M is the linear combinations of the vectors of M.

$\therefore$ span of $M = \alpha V_1 + \beta V_2$ where α, β are arbitrary.

$$= \alpha(1,1,1) + \beta(0,0,2)$$
$$= (\alpha, \alpha, \alpha + 2\beta) \quad \forall \alpha, \beta \in \mathbb{R}$$

Q) Which of the following subsets of $\mathbb{R}^3$ constitute a subspace of $\mathbb{R}^3$?
[Here $x = (\xi_1, \xi_2, \xi_3)$]
a) All x with $\xi_1 = \xi_2$ & $\xi_3 = 0$
b) All x with $\xi_1 = \xi_2 + 1$
c) All x with positive ξ_1, ξ_2, ξ_3
d) All x with $\xi_1 - \xi_2 + \xi_3 = k$ (fixed) → const.

Ans a) let $\left.\begin{aligned} x_1 &= (\xi_1, \xi_1, 0) \\ x_2 &= (\xi_3, \xi_3, 0) \end{aligned}\right\} \in Y$ $\quad X = \mathbb{R}^3$

$\alpha x_1 + \beta x_2$ must also $\in Y$
$\alpha x_1 + \beta x_2 = (\alpha \xi_1, \alpha\xi_1, 0) + (\beta \xi_3, \beta \xi_3, 0)$

$$= (\underbrace{\alpha \xi_1 + \beta \xi_3, \alpha \xi_1 + \beta \xi_3}_{\text{Both are same}}, 0)$$

$\therefore \alpha x_1 + \beta x_2 \in Y$.

The subspace Y represents a line : $X = Y$ ($\xi\ z = 0$)

b) let $\left.\begin{aligned} x_1 &= (\xi_2 + 1, \xi_2, \xi_3) \\ x_2 &= (\eta_2 + 1, \eta_2, \eta_3) \end{aligned}\right\} \in Y, \quad X = \mathbb{R}^3$

$\alpha x_1 + \beta x_2 \in Y$

$\alpha x_1 + \beta x_2 = (\alpha(\xi_2+1), \alpha\xi_2, \alpha\xi_3) + (\beta(\eta_2+1), \beta\eta_2, \beta\eta_3)$

$= (\underbrace{\alpha\xi_2 + \beta\eta_2 + \alpha + \beta, \alpha\xi_2 + \beta\eta_2}, \alpha\xi_3 + \beta\eta_3)$

Not equivallent to $\alpha\xi_2 + \beta\eta_2 + 1$

$\therefore \alpha x_1 + \beta x_2 \notin Y$

$\therefore$ Doesnt form a subspace.

c) let $x_1 = (\xi_1, \xi_2, \xi_3)$, $x_2 = (\eta_1, \eta_2, \eta_3)$ $\} \in Y$ $\qquad X = \mathbb{R}^3$

$\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3 \in \mathbb{R}^+$

If $x_1, x_2 \in Y$ then $\alpha x_1 + \beta x_2 \in Y$ for any arbitrary $\alpha, \beta \in \mathbb{R}$ if Y is a subspace.

$\alpha x_1 = (\alpha\xi_1, \alpha\xi_2, \alpha\xi_3)$

$\beta x_2 = (\beta\eta_1, \beta\eta_2, \beta\eta_3)$

$\alpha x_1 + \beta x_2 = (\alpha\xi_1 + \beta\eta_1, \alpha\xi_2 + \beta\eta_2, \alpha\xi_3 + \beta\eta_3)$

Depending on α, β, we might not always get positive values.

$\therefore \alpha x_1 + \beta x_2 \notin Y$.

d) let $x_1 = (\xi_1, \xi_2, \xi_3)$, $x_2 = (\eta_1, \eta_2, \eta_3)$ $\} \in Y$ $\qquad X = \mathbb{R}^3$

$\xi_1 - \xi_2 + \xi_3 = k$ $\qquad \xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3 \in \mathbb{R}$

$\eta_1 - \eta_2 + \eta_3 = k$

If $x_1, x_2 \in Y$, then $\alpha x_1 + \beta x_2$ must also $\in Y$ if Y is a subspace.

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$\alpha x_1 + \beta x_2 = (\alpha \xi_1, \alpha \xi_2, \alpha \xi_3) + (\beta \eta_1, \beta \eta_2, \beta \eta_3)$

$= (\underbrace{\alpha \xi_1 + \beta \eta_1}_{\text{new } \xi_1}, \underbrace{\alpha \xi_2 + \beta \eta_2}_{\text{new } \xi_2}, \underbrace{\alpha \xi_3 + \beta \eta_3}_{\text{new } \xi_3})$

$\alpha \xi_1 + \beta \eta_1 + \alpha \xi_3 + \beta \eta_3 - (\alpha \xi_2 + \beta \eta_2)$

$= \alpha(\xi_1 + \xi_3 - \xi_2) + \beta(\eta_1 + \eta_3 - \eta_2)$

$= \alpha(k) + \beta(k) = (\alpha + \beta)(k)$

If $\alpha + \beta = 1$ then $\alpha x_1 + \beta x_2 \in Y$, for all other values of $\alpha + \beta$, $\alpha x_1 + \beta x_2 \notin Y$. $\therefore$ It doesnt form a subspace.

Q) Show that we may replace (N2) by

$\|x\| = 0 \implies x = 0$

without altering the concept of a norm. Show that non-negetivity of a norm follows (N3) and (N4).

Ans) The norm of a vector space X is a real valued function on X where value at $x \in X$ is denoted by $\|x\|$.

The properties N(1) to N(4) of a norm are suggested and motivated by the length $|x|$ of a vector x in the vector space.

$d(x, 0) = \|x\|$

Given $\|x\| = 0$ only when $x =$ '0' vector

$\|x\| = 0 \Rightarrow x = 0$

Let us assume $y = \alpha x_1 + \beta x_2 \in X$ (since X is a vector space)

$x_1, x_2 \in X$.

$\|y\| < 0$ (Assumption)

$$\|y\| = \|\alpha x_1 + \beta x_2\|$$

$$\leq \|\alpha x_1\| + \|\beta x_2\| \quad (\Delta\text{le inequality}) \ (N4)$$

$$\leq |\alpha| \, \|x_1\| + \|\beta\| \, \|x_2\| \quad (N3)$$

$$\leq (+\text{ve const})(\text{length } 1) + (+\text{ve const}) \cdot (\text{Length } 2)$$

$$\leq \underline{\underline{+\text{ve constant}}}, \quad \|y\| = 0 \text{ if } y = 0$$

$\therefore \|y\| < 0 \Rightarrow$ length is a -ve value which is not possible

$\therefore$ The assumption $\|y\| < 0$ is false

$\therefore \|y\| \geq 0$

Q) There are several norms of practical importance on the vector space of ordered n tuples of numbers, notably defined by:

a) $\|x\|_1 = |\xi_1| + |\xi_2| + - - - + |\xi_n|$

b) $\|x\|_p = (|\xi_1|^p + |\xi_2|^p + - - |\xi_n|^p)^{1/p}$

c) $\|x\|_\infty = \max\{|\xi_1|, |\xi_2|, - - - |\xi_n|\}$

In each case verify (N1) to (N4) are satisfied.

Ans) $\|x\|_1 = |\xi_1| + |\xi_2| + - - |\xi_n|$

$x = (\xi_1, \xi_2, - - \xi_n)$

(N1) : $\|x\|_1 \geq 0$

Since $\|x_1\| = \sum |\xi_i| \rightarrow \geq 0$ ✓

(summation of +ve values)

(N2) : $\|x\| = 0 \iff x = 0$

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We know $x = (\xi_1, \xi_2, \dots \xi_n)$

for $\|x\|$ must be equal to zero

$$|\xi_1| = |\xi_2| = \dots = |\xi_n| = 0$$

$\Rightarrow x = (0, 0, \dots 0)$

N_3 $\|ax\| = |a| \, \|x\|$

$x = (\xi_1, \xi_2, \dots \xi_n)$

$ax = (a\xi_1, a\xi_2, \dots a\xi_n)$

$\|ax\| = |a\xi_1| + |a\xi_2| + \dots |a\xi_n|$

$= |a| \left(|\xi_1| + |\xi_2| + \dots |\xi_n| \right)$

$= |a| \, \|x\|$

N_4 (Δle inequality)

$\|x+y\| \leq \|x\| + \|y\|$

$x = (\xi_1, \xi_2, \dots, \xi_n)$

$y = (\eta_1, \eta_2, \dots, \eta_n)$

$x+y = (\xi_1 + \eta_1, \xi_2 + \eta_2, \dots, \xi_n + \eta_n)$

$\|x+y\| = |\xi_1 + \eta_1| + |\xi_2 + \eta_2| + \dots |\xi_n + \eta_n|$

$\leq |\xi_1| + |\eta_1| + |\xi_2| + |\eta_2| + \dots + |\xi_n| + |\eta_n|$

$\leq \|x\| + \|y\|$

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b) $\|x\|_p = \left(|\xi_1|^p + |\xi_2|^p + \cdots |\xi_n|^p\right)^{1/p}$

$x = (\xi_1, \xi_2, \cdots \xi_n)$

N1 $\Rightarrow \|x\| \geq 0$

$|\xi_1|^p + |\xi_2|^p + \cdots |\xi_n|^p \rightarrow$ sum of +ve nos powered p

≥ 0

$\therefore \|x\|_p = \left(|\xi_1|^p + |\xi_2|^p + \cdots |\xi_n|^p\right)^{1/p} \geq 0.$

N2 $\Rightarrow \|x\| = 0 \Leftrightarrow |x| = 0$

for $\|x\| = 0$, every individual term must be zero.

$|\xi_1|^p = |\xi_2|^p = \cdots |\xi_n|^p = 0$

$\Rightarrow |\xi_1| = |\xi_2| = \cdots |\xi_n| = 0$

$\Rightarrow \underline{x = 0}$ ✓

N3) $\|ax\| = |a| \|x\|$

$ax = (a\xi_1, a\xi_2, \cdots, a\xi_n)$

$\|ax\| = \left(|a\xi_1|^p + |a\xi_2|^p + \cdots |a\xi_n|^p\right)^{1/p}$

$= \left(|a|^p (\|x\|^p)\right)^{1/p}$

$= |a| \|x\|$

N4) $\|x+y\| \leq \|x\| + \|y\|$

$x = (\xi_1, \xi_2, \cdots \xi_n)$

$y = (\eta_1, \eta_2, \cdots \eta_n)$

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$$x+y = (\xi_1+\eta_1, \xi_2+\eta_2, \dots \xi_n+\eta_n)$$

$$\|x+y\| = \left(|\xi_1+\eta_1|^p + |\xi_2+\eta_2|^p + \dots + |\xi_n+\eta_n|^p\right)^{1/p}$$

$$\leq \left(\sum_{k=1}^{n} |\xi_k|^p\right)^{1/p} + \left(\sum_{k=1}^{n} |\eta_k|^p\right)^{1/p}$$ Minkowski inequality

$$\leq \|x\| + \|y\|$$

c) $\|x\|_\infty = \max\{|\xi_1|, |\xi_2|, \dots |\xi_n|\}$

N1) $\|x\| \geq 0$

$\|x\| \longrightarrow \max\{+ve\ no\ 1, +ve\ no\ 2, \dots +ve\ no\ n\}$

$\longrightarrow$ $+ve\ no_k$ $\quad k \in [1,n] \in \mathbb{I}$

$\Rightarrow \|x\| \geq 0$

N2) $\|x\| = 0 \iff x = 0$

$\max\{|\xi_1|, |\xi_2|, \dots |\xi_n|\} = 0$

$\Rightarrow |\xi_1| = |\xi_2| = \dots |\xi_n| = 0$

$\Rightarrow x = 0$

N3) $\|ax\| = |a|\,\|x\|$

$ax = (a\xi_1, a\xi_2, \dots, a\xi_n)$

$\|ax\| = \max\{|a\xi_1|, |a\xi_2|, \dots, |a\xi_n|\}$

$= |a| \max\{|\xi_1|, |\xi_2|, \dots |\xi_n|\}$

$= |a|\,\|x\|$

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N4) $\|x+y\| = \|x\| + \|y\|$

$x = (\xi_1, \xi_2, \ldots, \xi_n)$

$y = (\eta_1, \eta_2, \ldots, \eta_n)$

$x+y = (\xi_1+\eta_1, \xi_2+\eta_2, \ldots, \xi_n+\eta_n)$

$\|x+y\| = \max\{|\xi_1+\eta_1|, |\xi_2+\eta_2|, |\xi_3+\eta_3| \ldots |\xi_n+\eta_n|\}$

$\leq \max\{|\xi_1|, |\xi_2|, \ldots\} + \max\{|\eta_1|, |\eta_2| + \ldots |\eta_n|\}$

$\leq \|x\| + \|y\|$

Q) If 2 norms $\|\cdot\|$ & $\|\cdot\|_0$ on a vector space are equivalent show that

$$\|x_n - x\| \to 0 \iff \|x_n - x\|_0 \to 0$$

Ans) A norm $\|\cdot\|$ on a vector space is said to be equivalent to a norm $\|\cdot\|_0$ on X if there are +ve numbers a & b such that for all $x \in X$ we have

$a\|x\|_0 \leq \|x\| \leq b\|x\|_0$

Assume (x_n) in X to be convergent.

$\lim_{n\to\infty} \|x_n - x\| = 0$ or $x_n - x \to 0$

$\lim_{n\to\infty} \|x_n - x\| \leq \lim_{n\to\infty} b\|x_n - x\|_0$

$\lim_{n\to\infty} a\|x_n - x\|_0 \leq \lim_{n\to\infty} \|x_n - x\| \leq \lim_{n\to\infty} b\|x_n - x\|_0$

$\lim_{n\to\infty} a\|x_n - x\|_0 \leq 0 = \lim_{n\to\infty} b\|x_n - x\|_0$

$a > 0$ $\qquad$ $b > 0$

$\Rightarrow \|x_n - x\|_0 \to 0$ $\qquad$ $\Rightarrow \|x_n - x\| \to 0$

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Q) If $\|\cdot\|$ & $\|\cdot\|_0$ are equivalent norms on X, show that the cauchy sequences in $(X, \| \|)$ & $(X, \| \|_0)$ are the same.

Ans) $a\|x\|_0 \leq \|x\| \leq b\|x\|_0 \qquad x \in X.$

→ Equivalence of norms condition.

Let the sequence (x_n) in X be cauchy. Then for every $\varepsilon > 0$ there is an N such that

$\|x_m - x_n\| < \epsilon \qquad$ for all $m, n > N$.

$$a\|x_m - x_n\|_0 \leq \|x_m - x_n\| \leq b\|x_m - x_n\|_0$$

$$a\|x_m - x_n\|_0 \leq \|x_m - x_n\|$$

$a\|x_m - x_n\|_0 < \epsilon \qquad$ for $m, n \geq N$

$\|x_m - x_n\|_0 < \epsilon/a \qquad \epsilon > 0, a > 0$

$$\|x_m - x_n\| \leq b\|x_m - x_n\|_0$$

$$\epsilon < b\|x_m - x_n\|_0$$

$$\|x_m - x_n\|_0 > \epsilon/b$$

∴ for all $m, n > N$

$\|x_m - x_n\|_0$ lies b/w $(\varepsilon/b, \varepsilon/a)$

$\Rightarrow \|x_m - x_n\|_0 < \epsilon \qquad$ for every $\epsilon > 0$ & $m, n \geq N$

∴ Cauchy sequences in $(X, \| \|)$ & $(X, \| \|_0)$ are same.

Q) Show that the operators T_1, T_2, T_3 & T_4 from $\mathbb{R}^2$ to $\mathbb{R}^2$ defined by:

$T_1 : (\xi_1, \xi_2) \rightarrow (\xi_1, 0)$
$T_2 : (\xi_1, \xi_2) \rightarrow (0, \xi_2)$
$T_3 : (\xi_1, \xi_2) \rightarrow (\xi_2, \xi_1)$
$T_4 : (\xi_1, \xi_2) \rightarrow (\gamma\xi_1, \xi_2)$

respectively are linear & interpret these operators geometrically.

a) What are the domain, range & null space of T_1, T_2, T_3 and what is the null space of T_4.

b) (commutativity) Let X be any vector space and $S : X \rightarrow X$ and $T : X \rightarrow X$ any operators. S & T are said to commute if $S \cdot T = T \cdot S$ that is $(ST)x = (TS)x$ for all $x \in X$. Do T_1 & T_3 commute.

Ans) Let $T_1 : R^2 \rightarrow R^2$ such that $T : (\xi_1, \xi_2) \rightarrow (\xi_1, 0)$

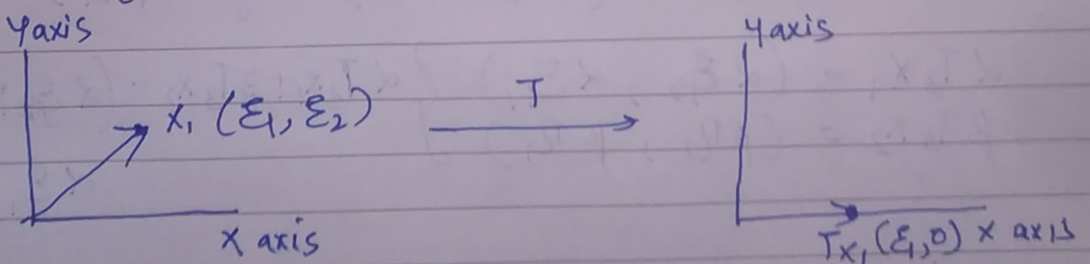
take $x_1, x_2 = (\xi_1, \xi_2), (\eta_1, \eta_2)$ respectively where $\xi_1, \xi_2, \eta_1, \eta_2 \in \mathbb{R}$ and are arbitrary.

$$\alpha x_1 + \beta x_2 = \alpha(\xi_1, \xi_2) + \beta(\eta_1, \eta_2)$$
$$= (\alpha\xi_1 + \beta\eta_1, \alpha\xi_2 + \beta\eta_2)$$

$$T(\alpha x_1 + \beta x_2) = T : (\alpha\xi_1 + \beta\eta_1, \alpha\xi_2 + \beta\eta_2) = (\alpha\xi_1 + \beta\eta_1, 0)$$

$$\left.\begin{aligned} \alpha T x_1 &= (\alpha\xi_1, 0) \\ \beta T x_2 &= (\beta\eta_1, 0) \end{aligned}\right\} \quad \alpha T x_1 + \beta T x_2 = (\alpha\xi_1 + \beta\eta_1, 0)$$

$\therefore T(\alpha x_1 + \beta x_2) = \alpha T x_1 + \beta T x_2 \qquad \therefore T_1$ is linear.



T is equivalent to finding x-intercept of the vector.

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Let $T_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $T_2 : (\xi_1, \xi_2) \rightarrow (0, \xi_2)$

Let $x_1, x_2 \in \mathbb{R}^2$ where $x_1 = (\xi_1, \xi_2)$, $x_2 = (\eta_1, \eta_2)$

$\xi_1, \xi_2, \eta_1, \eta_2$ are arbitrary $\in \mathbb{R}$, $\alpha, \beta \in \mathbb{R}$

$\alpha x_1 + \beta x_2 = (\alpha \xi_1 + \beta \eta_1, \alpha \xi_2 + \beta \eta_2)$

$T_2(\alpha x_1 + \beta x_2) = T_2(\alpha \xi_1 + \beta \eta_1, \alpha \xi_2 + \beta \eta_2)$

$= (0, \alpha \xi_2 + \beta \eta_2)$

$$\left.\begin{array}{l}\alpha T_2 x_1 = (0, \alpha \xi_2) \\ \beta T_2 x_2 = (0, \beta \eta_2)\end{array}\right\} \alpha T_2 x_1 + \beta T_2 x_2 = (0, \alpha \xi_2 + \beta \eta_2)$$

$\therefore T_2(\alpha x_1 + \beta x_2) = \alpha T_2 x_1 + \beta T_2 x_2$ $\therefore T_2$ is linear

Similar to T_1, T_2 is equivalent to finding the y-intercept of the vector.

$T_3 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $T_3 : (\xi_1, \xi_2) \longrightarrow (\xi_2, \xi_1)$

Let $x_1, x_2 \in \mathbb{R}^2$ where $x_1 = (\xi_1, \xi_2)$ & $x_2 = (\eta_1, \eta_2)$
$\xi_1, \xi_2, \eta_1, \eta_2 \in \mathbb{R}$ and are arbitrary. $\alpha, \beta \in \mathbb{R}$

$\alpha x_1 + \beta x_2 = (\alpha \xi_1 + \alpha \xi_2) + (\beta \eta_1 + \beta \eta_2)$

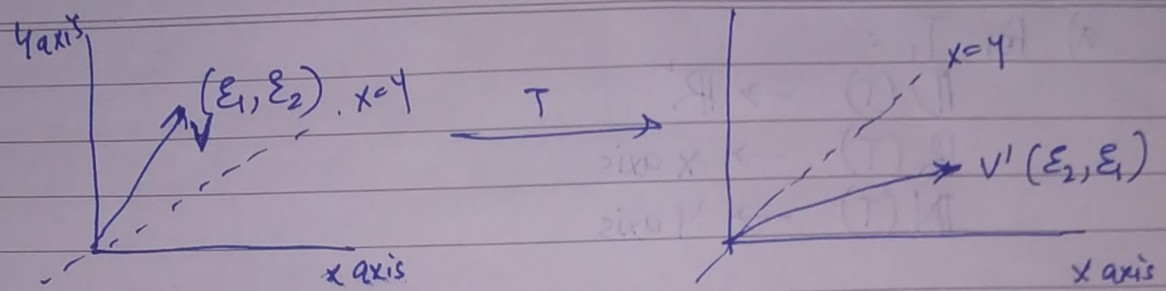
$= (\alpha \xi_1 + \beta \eta_1, \alpha \xi_2 + \beta \eta_2)$

$T_3 : (\alpha x_1 + \beta x_2) = (\alpha \xi_2 + \beta \eta_2, \alpha \xi_1 + \beta \eta_1)$

$$\left.\begin{array}{l}\alpha T_3 x_1 = (\alpha \xi_2, \alpha \xi_1) \\ \beta T_3 x_2 = (\beta \eta_2, \beta \eta_1)\end{array}\right\} \alpha T_3 x_1 + \beta T_3 x_2 = (\alpha \xi_2 + \beta \eta_2, \alpha \xi_1 + \beta \eta_1)$$

$\therefore T_3(\alpha x_1 + \beta x_2) = \alpha T_3 x_1 + \beta T_3 x_2$

$\therefore T_3$ is linear.



T_3 is an operator which reflects the vector over the line $x=y$

$T_4 : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T_4 : (\xi_1, \xi_2) \to (\gamma \xi_1, \gamma \xi_2)$

Let $x_1, x_2 \in \mathbb{R}^2$ where $x_1 = (\xi_1, \xi_2)$, $x_2 = (\eta_1, \eta_2)$
$\xi_1, \xi_2, \eta_1, \eta_2, \alpha, \beta$ are arbitrary, $\in \mathbb{R}$.

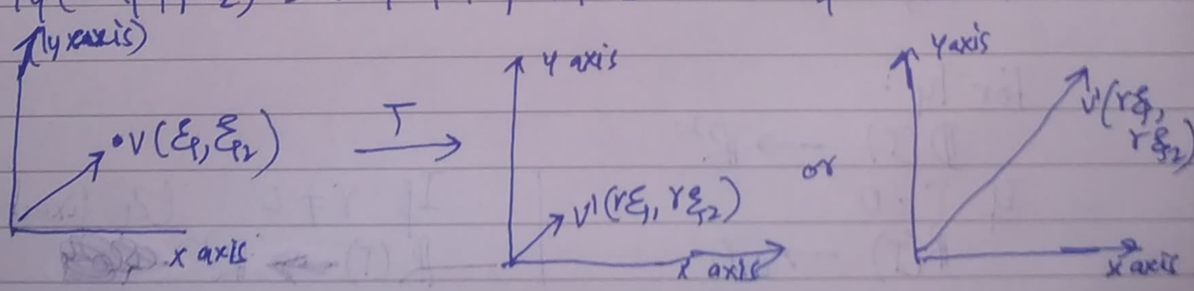
$$\alpha x_1 + \beta x_2 = (\alpha \xi_1, \alpha \xi_2) + (\beta \eta_1, \beta \eta_2)$$
$$= (\alpha \xi_1 + \beta \eta_1, \alpha \xi_2 + \beta \eta_2)$$

$$T_4(\alpha x_1 + \beta x_2) = (\gamma(\alpha \xi_1 + \beta \eta_1), \gamma(\alpha \xi_2 + \beta \eta_2))$$

$$\alpha T_4 x_1 = \alpha(\gamma \xi_1, \gamma \xi_2) = \gamma(\alpha \xi_1, \alpha \xi_2)$$
$$\beta T_4 x_2 = \beta(\gamma \eta_1 + \gamma \eta_2) = \gamma(\beta \eta_1, \beta \eta_2)$$

$$\alpha T_4 x_1 + \beta T_4 x_2 = (\gamma(\alpha \xi_1 + \beta \eta_1), \gamma(\alpha \xi_2 + \beta \eta_2))$$

$$T_4(\alpha x_1 + \beta x_2) = \alpha T_4 x_1 + \beta T_4 x_2 \quad \therefore T_4 \text{ is linear}$$



The operator T_4 will keep the direction of the vector but alter its length (incr or decr)

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a) for T_1 :

$D(T) \rightarrow R^2$
$R(T) \rightarrow$ x axis
$N(T) \rightarrow$ y axis

$T_1(\xi_1, \xi_2) \rightarrow (\xi_1, 0) \longrightarrow (0,0) \Rightarrow \xi_1 = 0$.

For any ξ_2, if $\xi_1 = 0$, $T(\xi_1, \xi_2) = (0,0)$

$\therefore N(T_1) =$ y axis.
T_1^{-1} doesnt exist.

for T_2 :

$D(T) \rightarrow R^2$
$R(T) \rightarrow$ y axis
$N(T) \rightarrow$ x axis

$T_2(\xi_1, \xi_2) \rightarrow (0, \xi_2) \longrightarrow (0,0) \Rightarrow \xi_2 = 0$

for any ξ_1, if $\xi_2 = 0$, $T(\xi_1, \xi_2) = (0,0)$

$\therefore N(T_2) =$ ~~y~~ x axis
T_2^{-1} doesnt exist.

For T_3 :

$D(T) \rightarrow R^2$
$R(T) \rightarrow R^2$
$N(T) \rightarrow$ Zero vector
T_3^{-1} exists.

for T_4 :

$D(T) \rightarrow R^2$

If $r = 0$
$R(T) \rightarrow 0$
$N(T) \rightarrow R^2$
T_4^{-1} doent exist

If $r \neq 0$ (& fixed value)
$R(T) \rightarrow R^2$
$N(T) \rightarrow$ Zero vector.
T_4^{-1} exists

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b) $T_1 : (\xi_1, \xi_2) \longrightarrow (\xi_1, 0) \quad (T_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2)$

$T_3 : (\xi_1, \xi_2) \longrightarrow (\xi_2, \xi_1) \quad (T_3 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2)$

Let $x_1 \in \mathbb{R}^2$ be (ξ_1, ξ_2) where $\xi_1, \xi_2 \in \mathbb{R}$ and are arbitrary.

$$(T_1 \cdot T_3) \cdot x_1 = T_1 (T_3 : (\xi_1, \xi_2)) = T_1 : (\xi_2, \xi_1)$$
$$= (\xi_2, 0)$$

$$(T_3 \cdot T_1) \cdot x_1 = T_3 \cdot (T_1 : (\xi_1, \xi_2)) = T_3 : (\xi_1, 0)$$
$$= (0, \xi_1)$$

$T_1 T_3 x_1 \neq T_3 T_1 x_1$

$\therefore$ T_1 & T_3 are non-commuteable operators.