1) $f(x(t), y(t), z(t), x'(t), y'(t), z'(t), t)$

$= x^3 y^2 + y\cos^2 x + x'^3 + y'^2 z'^3$

Independent variable $= t$

Dependent variable $= x(t), y(t), z(t)$

In terms of $x(t)$

$f(x(t), x'(t), t) = x^3 y^2 + y\cos^2 x + x'^3$

$\dfrac{\partial f}{\partial x} - \dfrac{\partial}{\partial t}\left(\dfrac{\partial f}{\partial x'}\right)$

$\downarrow$

$\dfrac{\partial f}{\partial x} = 3x^2 y^2 + 2y\cos x \cdot -\sin x$

$= 3x^2 y^2 - y\sin 2x$

$\dfrac{\partial f}{\partial x'} = 3x'^2$

$\dfrac{\partial}{\partial t}\left(\dfrac{\partial f}{\partial x'}\right) = 3\cdot 2x'\cdot x'' = 6x'\cdot x''$

$\therefore\ 3x^2 y^2 - y\sin 2x - 6x'\cdot x'' = 0$

$$f(y(t), y'(t), t) = x^3 y^2 + y\cos^2 x + y'^2 * z'^3$$

$$\frac{\partial f}{\partial y} - \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial y'}\right)$$

$\downarrow$

$$\frac{\partial f}{\partial y} = 2yx^3 + \cos^2 x$$

$$\frac{\partial f}{\partial y'} = 2y' z'^3$$

$$\frac{\partial}{\partial t}\left(\frac{\partial f}{\partial y'}\right) = 2y'' z'^3$$

$$\therefore \quad 2yx^3 + \cos^2 x - 2y'' z'^3$$

$$f(z(t), z'(t), t) = y'^2 z'^3$$

$$\frac{\partial f}{\partial z} - \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial z'}\right)$$

$\downarrow$

$$\frac{\partial f}{\partial z} = 0 \qquad \frac{\partial f}{\partial z'} = 3y'^2 \cdot (z')^2$$

$$\frac{\partial}{\partial t}\left(\frac{\partial f}{\partial z'}\right) = 3(y')^2 \cdot 2z' \cdot z''$$

$$= 6(y')^2 z' \cdot z''$$

$\therefore \; 0 - 6(y')^2 \cdot z' \cdot z'' = 0$

which implies $3(y')^2 \cdot (z')^2 = \text{const}$

$\Rightarrow (y' \cdot z')^2 = \text{const.}$

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2) $(y + zx) z_x - (x+yz) z_y = x^2 - y^2$

$$\frac{dx}{y+zx} = \frac{dy}{-(x+yz)} = \frac{dz}{x^2-y^2}$$

$P_1 = x \qquad Q_1 = y$

$x(y+zx) + y(-)(x+yz)$

$= \cancel{xy} + x^2 z - \cancel{xy} - y^2 z$

$= (x^2 - y^2) z$

$= z \cdot dx$

$\Rightarrow x\,dx + y\,dz - z\,dx = 0$

$\Rightarrow x^2 + y^2 - z^2 = c_1$

$P_1 = y$, $Q_1 = -x$

$y(y+xz) + x(-)(x+yz)$

$= y^2 + \cancel{xyz} - x^2 - \cancel{xyz} = -dz$

$\Rightarrow y\,dx + x\,dy + dz = 0$

$\Rightarrow d(xy) + dz = 0 \Rightarrow xy + z = c_2$

$f(x^2+y^2-z^2, xy+z) = 0.$

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3) $z_x \tan x + z_y \tan y = \tan z$

$$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}$$

$\Rightarrow \cot x\,dx = \cot y\,dy$

$\Rightarrow \ln|\sin x| = \ln|\sin y| + c$

$\Rightarrow \ln\left|\frac{\sin x}{\sin y}\right| = c$

$\Rightarrow \frac{\sin x}{\sin y} = c_1$

$\Rightarrow \cot y\,dy = \cot z\,dz$

$\Rightarrow \frac{\sin y}{\sin z} = c_2$ $\quad \therefore f\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0$

4) $f(x, x', t) = x'^2 \sin x + \sqrt{x^2 + x'^2}$

$$\frac{\partial f}{\partial x} - \frac{d}{dt}\left(\frac{\partial f}{\partial x'}\right) = 0$$

$$\Rightarrow \frac{\partial f}{\partial x} = x'^2 \cos x + \frac{1}{2\sqrt{x^2 + x'^2}} \cdot 2x$$

$$= x'^2 \cos x + \frac{x}{\sqrt{x^2 + x'^2}}$$

$$\frac{\partial f}{\partial x'} = 2x' \sin x + \frac{1}{2\sqrt{x^2 + x'^2}} \cdot 2x'$$

$$= 2x' \sin x + \frac{x'}{\sqrt{x^2 + x'^2}}$$

$$\frac{d}{dt}\left(\frac{\partial f}{\partial x'}\right) = 2\sin x \cdot x'' + 2x'^2 \cos x$$

$$+ x'\left(-\frac{(2x \cdot x' + 2x' \cdot x'')}{2\sqrt{x^2 + x'^2}}\right)$$

$$+ \frac{1}{\sqrt{x^2 + x'^2}} \cdot x''$$

$$\frac{\partial}{\partial t}\left(\frac{\partial f}{\partial x'}\right) = 2\sin x \cdot x'' + 2x'^2 \cos x$$

$$- \frac{x x'^2 - x'^2 x''}{\sqrt{(x^2 + x'^2)^3}} + \frac{x''}{\sqrt{x^2+x'^2}}$$

$$= x''\left[2\sin x - \frac{x'^2}{(x^2+x'^2)^{3/2}} + \frac{1}{(x^2+x'^2)^{3/2}}\right]$$

$$+ x'^2\left(2\cos x - \frac{x}{(x^2+x'^2)^{3/2}}\right)$$

$$\therefore \frac{\partial f}{\partial x} - \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial x'}\right)$$

$$= 2x'\sin x + \frac{x'}{\sqrt{x^2+x'^2}} - x''\left[2\sin\right.$$

5) $x(x+y)z_x - y(x+y)z_y$

$$= -(2x+2y+z)(x-y)$$

$$\Rightarrow \frac{dx}{x(x+y)} = \frac{dy}{-y(x+y)} = \frac{dz}{-(2x+2y+z)(x-y)}$$

$$\frac{dx}{x(x+y)} = \frac{dy}{-y(x+y)}$$

$$\log x = -\log y + c \quad \Rightarrow c_1 = xy \rightarrow (1)$$

$$dx+dy+dz = (x-y)(x+y+z)x^{-1}$$

$$= -(x-y)(x+y+z)$$

$$dx+dy = (x-y)(x+y)$$

$$\therefore \frac{dx+dy+dz}{-(x-y)(x+y+z)} = \frac{dx+dy}{(x-y)(x+y)}$$

$$\Rightarrow \frac{dx+dy+dz}{x+y+z} = -\frac{dx+dy}{x+y}$$

$$\Rightarrow \log(x+y+z) = -\log(x+y) + c$$

$\Rightarrow c_2 = (x+y)(x+y+z)$

$$f\left(xy,\ (x+y)(x+y+z)\right) = 0$$