First Fundamental Theorem of Asset Pricing

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Introduction

The fundamental theorems of asset pricing provide necessary and sufficient conditions for a market to be arbitrage-free and for a market to be complete.

Arbitrage

An arbitrage opportunity is a way of making money with no initial investment without any possibility of loss.

Usually, an arbitrage opportunity is present when there is the possibility to instantaneously buy something for a low price and sell it for a higher price.

Completeness

The complete market has two conditions:

1) Negligible transaction costs and therefore also perfect information.

2) There is a price for every asset in every possible state of the world.

One step model

Suppose that today €1=\$1.5. Assume that you know that tomorrow the euro will be worth either \$1.2 with probability $1-p$ or \$1.65 with probability p. Assume also that you can borrow or lend money in dollar currency at a fixed interest rate of 10%. Under these circumstances, the market that you are facing can be modelled by a one-step binomial model.

One step because you are only given information about the eurovalue tomorrow, binomial because there are only two possible values of the euro tomorrow.

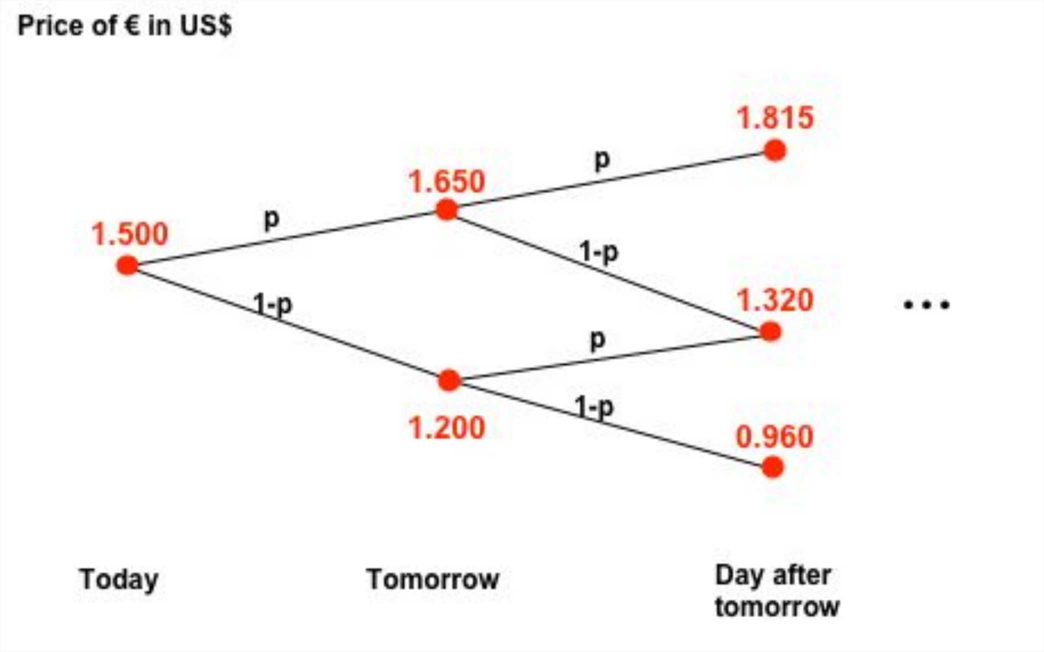
Multi step model

Suppose that today €1=1.5 and for any given day of the month the probability of the price of the euro to go down by 20% is $1-p$ while the probability of the price going up by 10% is p.

As before assume that you can borrow or lend money at a fixed daily interest rate of 10%.

In this case, the market can be modelled by a multi step binomial model and the best way of visualizing such a model is through a binary tree.

Multi step model



The general case

The CRR model with time horizon t involves a risk less bond and a risky asset (e.g. stocks, bonds, commodities such as oil or gold, currency exchange rate, etc).

The price process of the riskless bond is $S^0_t = (1 + r)^t$ for all $t = 0,1,2..T$ with $r > -1$

The price process of the risky asset is denoted by $S_t = S_{t-1}(1 + R_t)$ for $t = 0,1,...,T$ where with R_t the return in the t th trading period. The return R_t can only take two possible values $-1 \leq a \leq b$.

This implies that the price of the riskyasset at any time t, either jumps to the higher value $S_t(1 + b)$ or to the lower value $S_t(1 + a)$

Example of Arbitrage

Assume that the risk less bond is the dollar, the risky asset is the euro, t is the number of days remaining in the month, $r = 0.1$, $a = -0.2$ and $b = 0.1$.

Suppose further that you can borrow euros with no interest, that $p = 0.5$ and that there is only one day left in the month (so that you are facing a onestep situation). You could take advantage of this circumstance by using the following strategy: borrow one euro today, sell it immediately for \$1.5 and lend this money.

Tomorrow you will get for sure \$1.65 since the interest rate is 10%.

Example of Arbitrage

If the price of the euro goes up to \$1.65 you use your money to buy a euro and pay your debt obtaining a net gain of \$0. If the price of the euro goes down to \$1.2, you buy a euro to pay your debt but in this case, your ending balance is $\$1.65 - \$1.2 = \$0.45$.

Hence by following this strategy you can make \$0.45 with a probability of 0.5 and no risk. Of course, you could do the same with an arbitrary amount of euros in the beginning, which generates even greater gains.

Such a strategy is commonly known as an arbitrage opportunity.

Martingale

In probability theory, a martingale is a sequence of random variables (i.e., a stochastic process) for which,at a particular time, the conditional expectation of the next value in the sequence, regardless of all prior values, is equal to the present value.

A martingale is a process that models a fair game

Risk-neutral measure

A risk-neutral measure is a probability measure such that each share price is exactly equal to the discounted expectation of the share price under this measure.

1) The probability measure of a transformed random variable.

Risk-neutral measure

2) An implied probability measure, that is one implied from the current observable/posted/traded prices of the relevant instruments.

3) It is the implied probability measure (solves a kind of inverse problem) that is defined using a linear(risk-neutral) utility in the payoff, assuming some known model for the payoff.

Risk-neutral measure

This means that you try to find the risk-neutral measure by solving the equation where current prices are the expected present value of the future payoffs under the risk-neutral measure.

The concept of a unique risk-neutral measure is most useful when one imagines making prices across a number of derivatives that would make a unique risk-neutral measure since it implies a kind of consistency in ones hypothetical untraded prices and, theoretically points to arbitrage opportunities in markets where bid/ask prices are visible

Hahn–Banach separation theorems

Let X be a real topological vector space and choose A, B convex non-empty disjoint subsets of X.

1) If A is open then A and B are separated by a (closed) hyper-plane.

Explicitly, this means that there exists a continuous linear map $f : X \to K$ and $s \in R$ such that $f(a) < s \leq f(b)$ for all $a \in A$, $b \in B$. If bothA and B are open then the right-hand side may be taken strict as well.

2) If X is locally convex, A is compact, and B closed, then A and B are strictly separated: there exists a continuous linear map $f : X \to K$ and $s, t \in R$ such that $f(a) < t < s < f(b)$ for all $a \in A, b \in B$.

Hahn–Banach separation theorems

The key element of the Hahn–Banach theorem is fundamentally a result about the separation of two convex sets: $\{ -p(-x-n)-f(n) : n \in M\}$, and $\{p(m+x)-f(m) : m \in M\}$.

Let be a real locally convex topological vector space and let A and B be non-empty convex subsets. If Int A != φ and B ∩ IntA = φ , then there exists a continuous linear functional f on X such that sup f(A) ≤ inf f(B) and f(a) < int f(B) for all a ∈ IntA (such is f is necessarily non-zero).

Other Applications of Hahn-Banach Theorem

Farkas Lemma

It states that: Let A $\in$ R^{mxn} and b $\in$ R^m. Then exactly one of the two following assertions is true:

1. There exists an x$\in$ R^n such that Ax=b and x$\geq$0. or
2. There exists a y $\in$ R^m such that $A^T y \geq 0$ and $b^T y < 0$

First Fundamental Theorem of Asset Pricing (FFTAP)

The first version of this theorem was proven by M.Harrison and D. Kreps in 1979. More general versions of the theorem were proven in 1981 by M. Harrison and S.Pliska and in 1994 by F. Delbaen and W.Schachermayer.

The Theorem states that: A discrete market, on a discrete probability space (Ω, F, P), is arbitrage-free if, and only if, there exists at least one risk neutral probability measure that is equivalent to the original probability measure,P.

First Fundamental Theorem of Asset Pricing (FFTAP)

A financial market with time horizon t and price processes of the risky asset and risk less bond given by S_1, S_2, . . ., S_t and S^0_1, . . . ,S^0_t, respectively, is arbitrage-free under the probability P if and only if there exists another probability measure Q such that

1) For any event A,P(A) = 0 if and only if Q(A) = 0. we say in this case that P and Q are equivalent probability measures.

2) The discounted price process, $x_0 := S_0/S^0_0$, … ,$x_t := S_t/S^0_t$ is a martingale under Q.

A measure Q that satisfies 1) and 2) is known as a risk-neutral measure

First Fundamental Theorem of Asset Pricing (FFTAP)

This can be explained by the following reasoning: suppose that the price of the risky asset is measured in dollars. If at time t the price of the risky asset is \$ x, the fair price the seller should charge at time 0 for the asset should be \$ $x/(1 + r)^t$, with r the interest rate.

If he charges \$ $y \leq$ \$ $x/(1 + r)^t$ then the buyer could take advantage of the situation by borrowing \$y at time 0 to buy the asset and then selling at time t to repay his debt of \$$y(1 + r)^t$, obtaining a positive profit of \$$(x - y(1 + r)^t)$.

First Fundamental Theorem of Asset Pricing (FFTAP)

If he charges $\$y > \$\, x/(1+r)^t$ then he could take advantage of the situation by selling the asset at time 0 and lending $\$y$ dollars so that at time t he would receive $\$y(1+r)^t$ and after buying back the asset he would make a positive profit of $\$y(1+r)^t - x$).

Basically, when one considers the discounted price process every price is being measured in the unity of the risk less asset S^0 which is also commonly known in the literature as the Numéraire.

If p is not 0 or 1 the market is arbitrage-free if and only if $a < r < b$.

How the Hahn-Banach theorem comes into play

In this context, we have two economically interesting sets.

The first one is the set of all terminal wealth which are attainable through trading in a financial market, starting from zero wealth.

The second set consists of all Non negative terminal wealth. these are the so-called arbitrage opportunities.

How the Hahn-Banach theorem comes into play

Up to some technical conditions, no-arbitrage is formally defined as the non-intersection of these two sets.

As a consequence, the Hahn-Banach guarantees a linear functional which is positive on the positive wealth and non-positive on the wealth attained by trading.

The output of this functional can be thought of as the risk-neutral price of some terminal payoff.

How the Hahn-Banach theorem comes into play

Associated with this functional (by duality theory) isa probability measure so that the original random process modelling asset prices is a martingale under this new measure.

The relationship between no-arbitrage and existence of pricing rules is called the first fundamental theorem of asset pricing.

Mathematical Proof Behind Asset Pricing for a One Period Model

This model is depicted by a vector, $x \in R^m$, speaking to the costs of m instruments toward the start of the period, a set of all potential results over the period, and a limited capacity $X : \Omega \rightarrow R^m$, speaking to the costs of the m instruments toward the finish of the period relying upon the result, $\omega \in \Omega$.

These are the few definitions associated with proofing the theorem.

Arbitrage exists if there is a vector $\eta \in R^m$ such that $\eta.x < 0$ and $\eta.X(\omega) \leq 0$ for all $\omega \in \Omega$. The expense of setting up the position will be $\eta x = \eta_1 x_1 + \cdots + \eta_m x_m$. This being negative methods cash is made by putting on the position. At the point when the position is exchanged toward the finish of the period, the returns are ηX.

One period Model

This being non-negative methods no cash is lost.

An arbitrary probability measure on Ω and use the conditions $\eta \cdot x = 0$ and $\eta \cdot X \geq 0$ with $E[\eta \cdot X] > 0$ to define an arbitrage opportunity.

The realized return for a position, η , by $R_\eta = \eta \cdot X / \eta \cdot x$, whenever $\eta \cdot x \; != 0$.

If there exists $\gamma \in R^m$ with $\gamma \cdot X(\omega) = 1$ for $\omega \in \Omega$ (a zero coupon bond) then the price is $\gamma \cdot x = 1/R_\gamma$.

It can be observed that arbitrage is equivalent to the condition $R_\eta < 0$ on Ω for some $\eta \in R^m$.

Theorem :

Arbitrage exists if and only if x does not belong to the smallest closed cone containing the range of X. If $x_{\in}$ is the nearest point in the cone to x, then $\eta = x_{\in} - x$ is an arbitrage.

1. In the event that x has a place with the cone, it is subjectively near a limited entirety $\sum_j X(j)\pi_j$, where $j \in \Omega$ and $\pi_j > 0$ for all j.
2. On the off chance that $\eta \cdot X(\omega) \geq 0$ for all $\omega \in \Omega$ at that point $\eta \cdot \sum_j X(\omega_j)\, \pi_j \geq 0$, consequently η_x can't be negative.
3. The other heading is an outcome of the accompanying with C being the littlest shut cone containing X (Ω).

Lemma 1: If C $\subset R^m$ is a closed cone and x !$\in$ C , then there exists $\eta \in R^m$ such that $\eta x < 0$ and $\eta y \geq 0$ for all $y \in C$

1. Since C is closed and convex, there exists $x_\in \in C$ such that $\|x_\in - x\| \leq \|y - x\|$ for all $y \in C$.
2. We have $\|x_\in - x\| \leq \| tx_\in - x\|$ for $t \geq 0$, so $0 \leq (t^2 - 1)\|x_\in\|^2 - 2(t-1)x_\in . x = f(t)$.
3. Because f(t) is quadratic in t and vanishes at t = 1, we have $0 = f'(1) = 2\|x_\in\|^2 2x_\in .x$, hence $\eta . x_\in = 0$.
4. Now $0 < \|\eta\|^2 = \eta . x_\in - \eta . x$, so $\eta . x < 0$
5. Since $\|x_\in - x\| \leq \|ty + x_\in - x\|$ for $t \geq 0$ and $y \in C$, we have $0 \leq t^2 \|y\|^2 + 2ty. (x_\in . x)$. Dividing by t and setting t = 0 shows $y.\eta \geq 0$

The End

Thank you.