Linear Partial Differential Equation and Variational Calculus

Srikar Kale , Pradeep Pal

April 2021

1 Question 6

The equation of state of a system is given by

$$PV = \alpha U(T, V),$$

where α is a constant and *U(T, V)* is the specific internal energy. Show that the specific internal energy and specific entropy can be expressed in the form

$$U = V^{-\alpha}\phi(TV^{\alpha})$$

$$S = \psi(TV^{\alpha})$$

where it is given that

$$\phi'(x) = x\phi'(x)$$

Hint : use

$$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_V - P$$

and then finally you have to use Lagrange's method of solving first order PDE

2 Solution

We know,

$$U(T, V) = \frac{PV}{\alpha} \tag{1}$$

We can find the dU by

$$dU = \left(\frac{\partial U}{\partial T}\right)_V dT + \left(\frac{\partial U}{\partial V}\right)_T dV \tag{2}$$

where

$$\left(\frac{\partial U}{\partial T}\right)_V = \frac{V}{\alpha}\left(\frac{\partial P}{\partial T}\right)_V \tag{3}$$

also

$$\left(\frac{\partial U}{\partial V}\right)_T = \frac{P}{\alpha}\left(\frac{\partial V}{\partial V}\right) + \frac{V}{\alpha}\left(\frac{\partial P}{\partial V}\right)_T = \frac{P}{\alpha} + \frac{V}{\alpha}\left(\frac{\partial P}{\partial V}\right)_T \tag{4}$$

So dU can be written as

$$dU = \frac{V}{\alpha}\left(\frac{\partial P}{\partial T}\right)_V dT + \left(\frac{P}{\alpha} + \frac{V}{\alpha}\left(\frac{\partial P}{\partial V}\right)_T\right) dV \tag{5}$$

Using the hint, we equate the 2 parts of $(\frac{\partial U}{\partial V})_T$

$$\frac{P}{\alpha} + \frac{V}{\alpha}\left(\frac{\partial P}{\partial V}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_V - P \tag{6}$$

$$P + \frac{P}{\alpha} = T\left(\frac{\partial P}{\partial T}\right)_V - \frac{V}{\alpha}\left(\frac{\partial P}{\partial V}\right)_T \tag{7}$$

$$P + \frac{P}{\alpha} = TP_T - \frac{V}{\alpha}P_V \tag{8}$$

So, the Lagrange's equation:

$$\frac{\partial V}{\frac{-V}{\alpha}} = \frac{\partial T}{T} = \frac{\partial P}{P\left(\frac{1+\alpha}{\alpha}\right)} \tag{9}$$

Solving the first equality,

$$\frac{\partial V}{\frac{-V}{\alpha}} = \frac{\partial T}{T} \tag{10}$$

$$-\log V^{\alpha} = \log T + C_1 \implies C_1 = V^{\alpha}T \tag{11}$$

Solving the second equality,

$$\frac{\partial T}{T} = \frac{\partial P}{P\left(\frac{1+\alpha}{\alpha}\right)} \tag{12}$$

$$\log T = \frac{\alpha}{1+\alpha}\log P + C_2 \implies C_2 = \frac{T}{P^{\frac{\alpha}{1+\alpha}}} \tag{13}$$

Solving the remaining equality,

$$\frac{\partial V}{\frac{-V}{\alpha}} = \frac{\partial P}{P\left(\frac{1+\alpha}{\alpha}\right)} \tag{14}$$

$$-\alpha\log V = \frac{\alpha}{(1+\alpha)}\log P + C_3 \implies C_3 = VP^{\frac{1}{1+\alpha}} \tag{15}$$

Writing P in terms of T from eq(13),

$$P^{\frac{\alpha}{(1+\alpha)}} = \frac{T}{C_2} \implies P = \frac{T^{\frac{1+\alpha}{\alpha}}}{C_2^{'}} \tag{16}$$

Now substituting P in U(T,V),

$$U(T,V) = \frac{PV}{\alpha} = \frac{T^{\frac{1+\alpha}{\alpha}}V}{\alpha C_2^{`}} \tag{17}$$

then,

$$U(T,V) = V^{-\alpha}.\frac{V^{1+\alpha}.T^{1+\frac{1}{\alpha}}}{\alpha C_2^{'}} \tag{18}$$

$$U(T,V) = V^{-\alpha}.\frac{(V^{\alpha}T)(VT^{\frac{1}{\alpha}})}{\alpha C_2^{'}} \tag{19}$$

Therefore,

$$\phi(TV^{\alpha}) = \frac{TV^{\alpha}T^{\frac{1}{\alpha}}V}{\alpha C_2^{'}} \tag{20}$$

We can write this as,

$$\phi(x) = \frac{x.x^{\frac{1}{\alpha}}}{\alpha C_2^{'}} = \frac{x^{1+\frac{1}{\alpha}}}{\alpha C_2^{'}} \tag{21}$$

$$\phi^{'}(x) = \left(1+\frac{1}{\alpha}\right)\frac{x^{\frac{1}{\alpha}}}{\alpha C_2^{'}} = x\psi^{'}(x) \tag{22}$$

From this, we can write $\psi^{`}(x)$ as

$$\psi^{'}(x) = \left(1+\frac{1}{\alpha}\right)\frac{x^{\frac{1}{\alpha}-1}}{\alpha C_2^{'}} \tag{23}$$

$\psi(x)$ can now be obtained by integrating the above equation,

$$\psi(x) = \left(1+\frac{1}{\alpha}\right)\frac{1}{\alpha C_2^{'}}\frac{x^{\frac{1}{\alpha}}}{\frac{1}{\alpha}} \tag{24}$$

$$= \frac{\alpha+1}{\alpha C_2^{'}}x^{\frac{1}{\alpha}} \tag{25}$$

Therefore by substituting the value of x as TV^{α} we get,

$$\psi(TV^{\alpha}) = \frac{\alpha+1}{\alpha C_2^{'}}T^{\frac{1}{\alpha}}V = S \tag{26}$$

3 Conclusion

We obtained the specific internal energy and specific entropy in the form

$$U = V^{-\alpha}\phi(TV^{\alpha})$$

$$S = \psi(TV^{\alpha})$$

where it is given that

$$\phi'(x) = x\phi'(x)$$