

Chapter 7

Lagrangian and Hamiltonian Mechanics

Abstract Chapter 7 is devoted to problems solved by Lagrangian and Hamiltonian mechanics.

7.1 Basic Concepts and Formulae

Newtonian mechanics deals with force which is a vector quantity and therefore difficult to handle. On the other hand, Lagrangian mechanics deals with kinetic and potential energies which are scalar quantities while Hamilton's equations involve generalized momenta, both are easy to handle. While Lagrangian mechanics contains n differential equations corresponding to n generalized coordinates, Hamiltonian mechanics contains $2n$ equations, that is, double the number. However, the equations for Hamiltonian mechanics are linear.

The symbol q is a generalized coordinate used to represent an arbitrary coordinate x , θ , φ , etc.

If T is the kinetic energy, V the potential energy then the Lagrangian L is given by

$$L = T - V \quad (7.1)$$

Lagrangian Equation:

$$\frac{d}{dt} \left(\frac{dL}{dq_K} \right) - \frac{\partial L}{\partial q_K} = 0 \quad (K = 1, 2, \dots) \quad (7.2)$$

where it is assumed that V is not a function of the velocities, i.e. $\frac{\partial V}{\partial \dot{q}_K} = 0$. Eqn (2) is applicable to all the conservative systems.

When n independent coordinates are required to specify the positions of the masses of a system, the system is of n degrees of freedom.

Hamilton
$$H = \sum_{s=1}^r p_s \dot{q}_s - L \quad (7.3)$$

where p_s is the generalized momentum and \dot{q}_K is the generalized velocity.

Hamiltonian's Canonical Equations

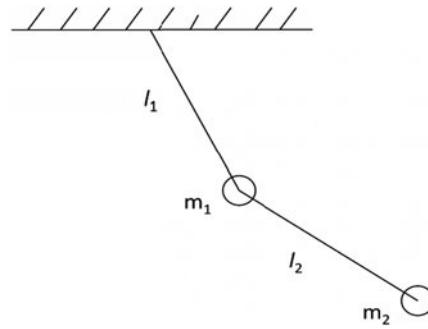
$$\frac{\partial H}{\partial p_r} = \dot{q}_r, \quad \frac{\partial H}{\partial q_r} = -\dot{p}_r \quad (7.4)$$

7.2 Problems

- 7.1** Consider a particle of mass m moving in a plane under the attractive force $\mu m/r^2$ directed to the origin of polar coordinates r, θ . Determine the equations of motion.
- 7.2 (a)** Write down the Lagrangian for a simple pendulum constrained to move in a single vertical plane. Find from it the equation of motion and show that for small displacements from equilibrium the pendulum performs simple harmonic motion.
- (b)** Consider a particle of mass m moving in one dimension under a force with the potential $U(x) = k(2x^3 - 5x^2 + 4x)$, where the constant $k > 0$. Show that the point $x = 1$ corresponds to a stable equilibrium position of the particle. Find the frequency of a small amplitude oscillation of the particle about this equilibrium position.
- [University of Manchester 2007]
- 7.3** Determine the equations of motion of the masses of Atwood machine by the Lagrangian method.
- 7.4** Determine the equations of motion of Double Atwood machine which consists of one of the pulleys replaced by an Atwood machine. Neglect the masses of pulleys.
- 7.5** A particular mechanical system depending on two coordinates u and v has kinetic energy $T = v^2\dot{u}^2 + 2\dot{v}^2$, and potential energy $V = u^2 - v^2$. Write down the Lagrangian for the system and deduce its equations of motion (do not attempt to solve them).
- [University of Manchester 2008]
- 7.6** Write down the Lagrangian for a simple harmonic oscillator and obtain the expression for the time period.
- 7.7** A particle of mass m slides on a smooth incline at an angle α . The incline is not permitted to move. Determine the acceleration of the block.
- 7.8** A block of mass m and negligible size slides on a frictionless inclined plane of mass M at an angle θ with the horizontal. The plane itself rests on a smooth horizontal table. Determine the acceleration of the block and the inclined plane.
- 7.9** A bead of mass m is free to slide on a smooth straight wire of negligible mass which is constrained to rotate in a vertical plane with constant angular speed ω about a fixed point. Determine the equation of motion and find the distance x from the fixed point at time t . Assume that at $t = 0$ the wire is horizontal.

- 7.10** Consider a pendulum consisting of a small mass m attached to one end of an inextensible cord of length l rotating about the other end which is fixed. The pendulum moves on a spherical surface. Hence the name spherical pendulum. The inclination angle φ in the xy -plane can change independently.
- (a) Obtain the equations of motion for the spherical pendulum.
 (b) Discuss the conditions for which the motion of a spherical pendulum is converted into that of (i) simple pendulum and (ii) conical pendulum.
- 7.11** Two blocks of mass m and M connected by a massless spring of spring constant k are placed on a smooth horizontal table. Determine the equations of motion using Lagrangian mechanics.
- 7.12** A double pendulum consists of two simple pendulums of lengths l_1 and l_2 and masses m_1 and m_2 , with the cord of one pendulum attached to the bob of another pendulum whose cord is fixed to a pivot, Fig. 7.1. Determine the equations of motion for small angle oscillations using Lagrange's equations.

Fig. 7.1



- 7.13** Use Hamilton's equations to obtain the equations of motion of a uniform heavy rod of mass M and length $2a$ turning about one end which is fixed.
- 7.14** A one-dimensional harmonic oscillator has Hamiltonian $H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2$. Write down Hamiltonian's equation and find the general solution.
- 7.15** Determine the equations for planetary motion using Hamilton's equations.
- 7.16** Two blocks of mass m_1 and m_2 coupled by a spring of force constant k are placed on a smooth horizontal surface, Fig. 7.2. Determine the natural frequencies of the system.

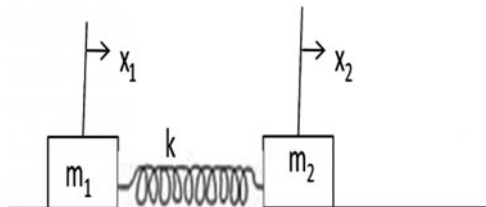
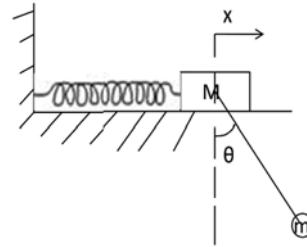


Fig. 7.2

- 7.17** A simple pendulum of length l and mass m is pivoted to the block of mass M which slides on a smooth horizontal plane, Fig. 7.3. Obtain the equations of motion of the system using Lagrange's equations.

Fig. 7.3



- 7.18** Determine the equations of motion of an insect of mass m crawling at a uniform speed v on a uniform heavy rod of mass M and length $2a$ which is turning about a fixed end. Assume that at $t = 0$ the insect is at the middle point of the rod and it is crawling downwards.
- 7.19** A uniform rod of mass M and length $2a$ is attached at one end by a cord of length l to a fixed point. Calculate the inclination of the string and the rod when the string plus rod system revolves about the vertical through the pivot with constant angular velocity ω .
- 7.20** A particle moves in a horizontal plane in a central force potential $U(r)$. Derive the Lagrangian in terms of the polar coordinates (r, θ) . Find the corresponding momenta p_r and p_θ and the Hamiltonian. Hence show that the energy and angular momentum of the particle are conserved.
[University of Manchester 2007]
- 7.21** Consider the system consisting of two identical masses that can move horizontally, joined with springs as shown in Fig. 7.4. Let x, y be the horizontal displacements of the two masses from their equilibrium positions.

- (a) Find the kinetic and potential energies of the system and deduce the Lagrangian.
- (b) Show that Lagrange's equation gives the coupled linear differential equations

$$\begin{cases} m\ddot{x} = -4kx + 3ky \\ m\ddot{y} = 3kx - 4ky \end{cases}$$

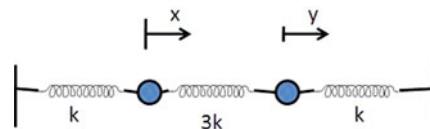
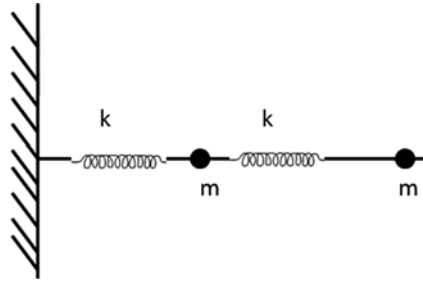


Fig. 7.4

- (c) Find the normal modes of oscillation of this system and their period of oscillation.

7.22 Two identical beads of mass m each can move without friction along a horizontal wire and are connected to a fixed wall with two identical springs of spring constant k as shown in Fig. 7.5.

Fig. 7.5



- (a) Find the Lagrangian for this system and derive from it the equations of motion.
 (b) Find the eigenfrequencies of small amplitude oscillations.
 (c) For each normal mode, sketch the system when it is at maximum displacement.

Note: Your sketch should indicate the relative sizes as well as the directions of the displacements.

[University of Manchester 2007]

7.23 Two beads of mass $2m$ and m can move without friction along a horizontal wire. They are connected to a fixed wall with two springs of spring constants $2k$ and k as shown in Fig. 7.6:

- (a) Find the Lagrangian for this system and derive from it the equations of motion for the beads.
 (b) Find the eigenfrequencies of small amplitude oscillations.
 (c) For each normal mode, sketch the system when it is at the maximum displacement.

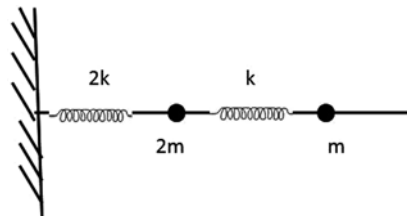


Fig. 7.6

7.24 Three identical particles of mass m , M and m with M in the middle are connected by two identical massless springs with a spring constant k . Find the normal modes of oscillation and the associated frequencies.

7.25 (a) A bead of mass m is constrained to move under gravity along a planar rigid wire that has a parabolic shape $y = x^2/l$, where x and y are, respectively, the horizontal and the vertical coordinates. Show that the Lagrangian for the system is

$$L = \frac{m(\dot{x})^2}{2} \left(1 + \frac{4x^2}{l^2} \right) - \frac{mgx^2}{l}$$

(b) Derive the Hamiltonian for a single particle of mass m moving in one dimension subject to a conservative force with a potential $U(x)$.

[University of Manchester 2006]

7.26 A pendulum of length l and mass m is mounted on a block of mass M . The block can move freely without friction on a horizontal surface as shown in Fig. 7.7.

(a) Show that the Lagrangian for the system is

$$L = \left(\frac{M+m}{2} \right) (\dot{x})^2 + ml \cos \theta \dot{x} \dot{\theta} + \frac{m}{2} l^2 (\dot{\theta})^2 + mgl \cos \theta$$

(b) Show that the approximate form for this Lagrangian, which is applicable for a small amplitude swinging of the pendulum, is

$$L = \left(\frac{M+m}{2} \right) (\dot{x})^2 + ml \dot{x} \dot{\theta} + \frac{m}{2} l^2 (\dot{\theta})^2 + mgl \left(1 - \frac{\theta^2}{2} \right)$$

(c) Find the equations of motion that follow from the simplified Lagrangian obtained in part (b),

(d) Find the frequency of a small amplitude oscillation of the system.

[University of Manchester 2006]

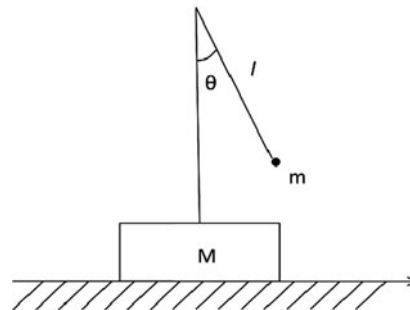


Fig. 7.7

- 7.27 (a)** A particle of mass m slides down a smooth spherical bowl, as in Fig. 7.8. The particle remains in a vertical plane (the xz -plane). First, assume that the bowl does not move. Write down the Lagrangian, taking the angle ϑ with respect to the vertical direction as the generalized coordinate. Hence, derive the equation of motion for the particle.

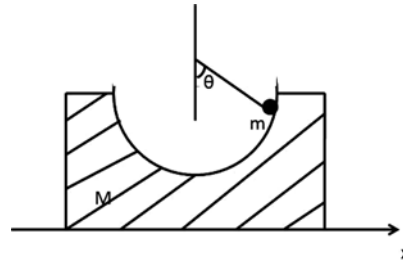


Fig. 7.8

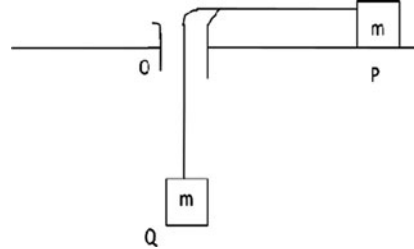
- (b)** Assume now that the bowl rests on a smooth horizontal table and has a mass M , the bowl can slide freely along the x -direction.
- Write down the Lagrangian in terms of the angle θ and the x -coordinate of the bowl, x .
 - Starting from the corresponding Lagrange's equations, obtain an equation giving \ddot{x} in terms of θ , $\dot{\theta}$ and $\ddot{\theta}$ and an equation giving $\ddot{\theta}$ in terms of \ddot{x} and θ .
 - Hence, and assuming that $M \gg m$, show that for small displacements about equilibrium the period of oscillation of the particle is smaller by a factor $[M/(M + m)]^{1/2}$ as compared to the case where the bowl is fixed. [You may neglect the terms in $\theta^2 \ddot{\theta}$ or $\theta \dot{\theta}^2$ compared to terms in $\ddot{\theta}$ or θ .]

[University of Durham 2004]

- 7.28** A system is described by the single (generalized) coordinate q and the Lagrangian $L(q, \dot{q})$. Define the generalized momentum associated with q and the corresponding Hamiltonian, $H(q, p)$. Derive Hamilton's equations from Lagrange's equations of the system. For the remainder of the question, consider the system whose Lagrangian, $L(q, \dot{q})$. Find the corresponding Hamiltonian and write down Hamilton's equations.
- 7.29** Briefly explain what is the "generalized (or canonical) momentum conjugate to a generalized coordinate". What characteristic feature should the Lagrangian function have for a generalized momentum to be a constant of motion? A particle P can slide on a frictionless horizontal table with a small opening at O. It is attached, by a string of length l passing through the opening, to a particle Q hanging vertically under the table (see Fig. 7.9). The two particles have equal mass, m . Let τ denote the distance of P to the opening, θ the angle between OP and some fixed line through O and g the acceleration of

gravity. Initially, $r = a$, Q does not move, and P is given an initial velocity of magnitude $(ag)^{1/2}$ at right angles to OP.

Fig. 7.9



- (a) Write the Lagrangian in terms of the coordinates r and θ and derive the corresponding equations of motion.
- (b) Using these equations of motion and the initial conditions, show that $\ddot{r} = a^3 g / r^3 - g$.
- (c) Hence, (i) show that the trajectory of P is the circle $r = a$, (ii) show that P describes small oscillations about this circle if it is slightly displaced from it and (iii) calculate the period of these oscillations:

$$[v_p^2 = \dot{r}^2 + r^2 \dot{\theta}^2, \text{ where } v_p \text{ is the speed of P}]$$

- 7.30** A particle of mass m is constrained to move on an ellipse E in a vertical plane, parametrized by $x = a \cos \theta$, $y = b \sin \theta$, where $a, b > 0$ and $a \neq b$ and the positive y -direction is the upward vertical. The particle is connected to the origin by a spring, as shown in the diagram, and is subject to gravity. The potential energy in the spring is $\frac{1}{2}kr^2$ where r is the distance of the point mass from the origin (Fig. 7.10).

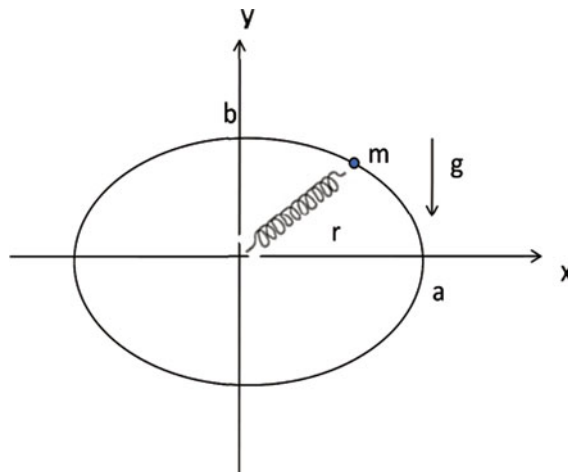


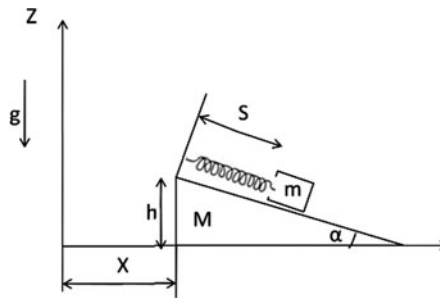
Fig. 7.10

- (i) Using θ as a coordinate, find the kinetic and potential energies of the particle when moving on the ellipse. Write down the Lagrangian and show that Lagrange's equation becomes $m(a^2 \sin^2 \theta + b^2 \cos^2 \theta) \ddot{\theta} = (a^2 - b^2)(k - m\dot{\theta}^2) \sin \theta \cos \theta - mgb \cos \theta$.
- (ii) Show that $\theta = \pm\pi/2$ are two equilibrium points and find any other equilibrium points, giving carefully the conditions under which they exist. You may either use Lagrange's equation or proceed directly from the potential energy.
- (iii) Determine the stabilities of each of the two equilibrium points $\theta = \pm\pi/2$ (it may help to consider the cases $a > b$ and $a < b$ separately).
- (iv) When the equilibrium point at $\theta = -\pi/2$ is stable, determine the period of small oscillations.

[University of Manchester 2008]

- 7.31** In prob. (7.12) on double pendulum if $m_1 = m_2 = m$ and $l_1 = l_2 = l$, obtain the frequencies of oscillation.
- 7.32** Use Lagrange's equations to obtain the natural frequencies of oscillation of a coupled pendulum described in prob. (6.46).
- 7.33** A bead of mass m slides freely on a smooth circular wire of radius r which rotates with constant angular velocity ω . On a horizontal plane about a point fixed on its circumference, show that the bead performs simple harmonic motion about the diameter passing through the fixed point as a pendulum of length $r = g/\omega^2$.
- [with permission from Robert A. Becker, Introduction to theoretical mechanics, McGraw-Hill Book Co., Inc., 1954]
- 7.34** A block of mass m is attached to a wedge of mass M by a spring with spring constant k . The inclined frictionless surface of the wedge makes an angle α to the horizontal. The wedge is free to slide on a horizontal frictionless surface as shown in Fig. 7.11.

Fig. 7.11



- (a) Show that the Lagrangian of the system is

$$L = \frac{(M+m)}{2} \dot{X}^2 + \frac{1}{2} m \dot{s}^2 + m \dot{X} \dot{s} \cos \alpha - \frac{k}{2} (s-l)^2 - mg(h - s \sin \alpha),$$

where l is the natural length of the spring, x is the coordinate of the wedge and s is the length of the spring.

- (b) By using the Lagrangian derived in (a), show that the equations of motion are as follows:

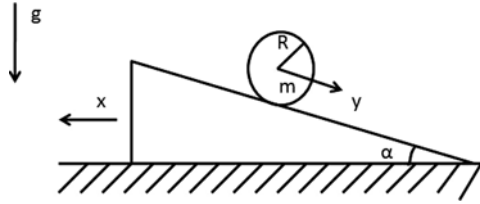
$$\begin{aligned}(m + M)\ddot{x} + m\ddot{s} \cos \alpha &= 0, \\ m\ddot{x} \cos \alpha + m\ddot{s} + k(s - s_0) &= 0, \\ \text{where } s_0 &= l + (mg \sin \alpha)/k.\end{aligned}$$

- (c) By using the equations of motion in (b), derive the frequency for a small amplitude oscillation of this system.

[University of Manchester 2008]

- 7.35** A uniform spherical ball of mass m rolls without slipping down a wedge of mass M and angle α , which itself can slide without friction on a horizontal table. The system moves in the plane shown in Fig. 7.12. Here g denotes the gravitational acceleration.

Fig. 7.12



- (a) Find the Lagrangian and the equations of motion for this system.
 (b) For the special case of $M = m$ and $\alpha = \pi/4$ find
 (i) the acceleration of the wedge and
 (ii) the acceleration of the ball relative to the wedge.

[Useful information: Moment of inertia of a uniform sphere of mass m and radius R is $I = \frac{2}{5}mR^2$.]

[University of Manchester 2007]

7.3 Solutions

- 7.1** This is obviously a two degree of freedom dynamical system. The square of the particle velocity can be written as

$$v^2 = \dot{r}^2 + (r\dot{\theta})^2 \quad (1)$$

Formula (1) can be derived from Cartesian coordinates

$$\begin{aligned}x &= r \cos \theta, & y &= r \sin \theta \\ \dot{x} &= \dot{r} \cos \theta - r\dot{\theta} \sin \theta, & \dot{y} &= \dot{r} \sin \theta + r\dot{\theta} \cos \theta\end{aligned}$$

We thus obtain

$$\dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\theta}^2$$

The kinetic energy, the potential energy and the Lagrangian are as follows:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \quad (2)$$

$$V = -\frac{\mu m}{r} \quad (3)$$

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{\mu m}{r} \quad (4)$$

We take r, θ as the generalized coordinates q_1, q_2 . Since the potential energy V is independent of \dot{q}_i , Lagrangian equations take the form

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \quad (i = 1, 2) \quad (5)$$

$$\text{Now } \frac{\partial L}{\partial \dot{r}} = m\dot{r}, \quad \frac{\partial L}{\partial r} = mr\dot{\theta}^2 - \frac{\mu m}{r^2} \quad (6)$$

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}, \quad \frac{\partial L}{\partial \theta} = 0 \quad (7)$$

Equation (5) can be explicitly written as

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{r}} \right) - \frac{\partial L}{\partial r} = 0 \quad (8)$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0 \quad (9)$$

Using (6) in (8) and (7) in (9), we get

$$m\ddot{r} - mr\dot{\theta}^2 + \frac{\mu m}{r^2} = 0 \quad (10)$$

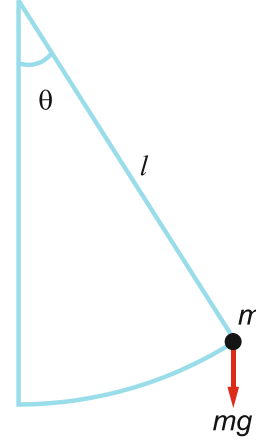
$$m \frac{d(r^2\dot{\theta})}{dt} = 0 \quad (11)$$

Equations (10) and (11) are identical with those obtained for Kepler's problem by Newtonian mechanics. In particular (11) signifies the constancy of areal velocity or equivalently angular momentum (Kepler's second law of planetary motion). The solution of (10) leads to the first law which asserts that the path of a planet describes an ellipse.

This example shows the simplicity and power of Lagrangian method which involves energy, a scalar quantity, rather than force, a vector quantity in Newton's mechanics.

7.2 The position of the pendulum is determined by a single coordinate θ and so we take $q = \theta$. Then (Fig. 7.13)

Fig. 7.13



$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 l^2 = \frac{1}{2}ml^2\dot{\theta}^2 \quad (1)$$

$$V = mgl(1 - \cos \theta) \quad (2)$$

$$L = T - V = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1 - \cos \theta) \quad (3)$$

$$\frac{\partial T}{\partial \dot{\theta}} = ml^2\dot{\theta}, \quad \frac{\partial T}{\partial \theta} = 0 \quad (4)$$

$$\frac{\partial V}{\partial \dot{\theta}} = 0, \quad \frac{\partial V}{\partial \theta} = mgl \sin \theta \quad (5)$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0 \quad (6)$$

$$\frac{d}{dt} \left(\frac{\partial}{\partial \dot{\theta}} (T - V) \right) - \frac{\partial}{\partial \theta} (T - V) = 0$$

$$\frac{d}{dt} (ml^2\dot{\theta}) + mgl \sin \theta = 0$$

$$\text{or } l\ddot{\theta} + g \sin \theta = 0 \quad (\text{equation of motion})$$

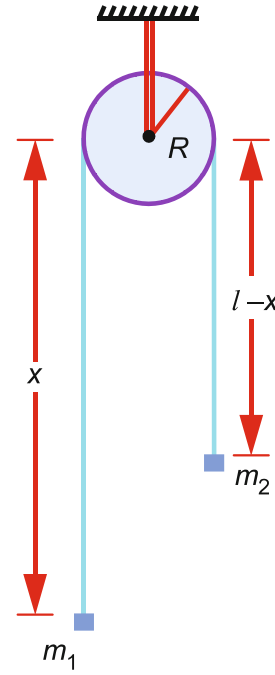
For small oscillation angles $\sin \theta \rightarrow \theta$

$$\ddot{\theta} = -\frac{g\theta}{l} \quad (\text{equation for angular SHM})$$

$$\therefore \omega^2 = \frac{g}{l} \text{ or time period } T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}$$

7.3 In this system there is only one degree of freedom. The instantaneous configuration is specified by $q = x$. Assuming that the cord does not slip, the angular velocity of the pulley is \dot{x}/R , Fig. 7.14.

Fig. 7.14



The kinetic energy of the system is given by

$$T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{x}^2 + \frac{1}{2}I\frac{\dot{x}^2}{R^2} \quad (1)$$

The potential energy of the system is

$$V = -m_1gx - m_2g(l - x) \quad (2)$$

And the Lagrangian is

$$L = \frac{1}{2}\left(m_1 + m_2 + \frac{I}{R^2}\right)\dot{x}^2 + (m_1 - m_2)gx + m_2gl \quad (3)$$

The equation of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \quad (4)$$

yields

$$\left(m_1 + m_2 + \frac{I}{R^2}\right)\ddot{x} - g(m_1 - m_2) = 0$$

or $\ddot{x} = \frac{(m_1 - m_2)g}{m_1 + m_2 + \frac{I}{R^2}}$ (5)

which is identical with the one obtained by Newton's mechanics.

7.4 By problem the masses of pulleys are negligible. The double machine is an Atwood machine in which one of the weights is replaced by a second Atwood machine, Fig. 7.15. The system now has two degrees of freedom, and its instantaneous configuration is specified by two coordinates x and x' . l and l' denote the length of the vertical parts of the two strings. Mass m_1 is at depth x below the centre of pulley A, m_2 at depth $l - x + x'$ and m_3 at depth $l + l' - x - x'$. The kinetic energy of the system is given by

$$T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2(-\dot{x} + \dot{x}')^2 + \frac{1}{2}m_3(-\dot{x} - \dot{x}')^2 \quad (1)$$

while the potential energy is given by

$$V = -m_1gx - m_2g(l - x + x') - m_3g(l - x + l' - x') \quad (2)$$

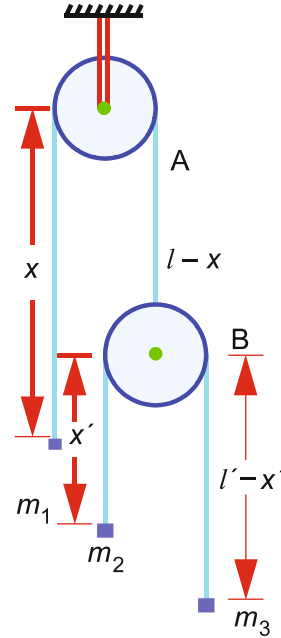


Fig. 7.15

The Lagrangian of the system takes the form

$$L = T - V = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2(-\dot{x} + \dot{x}')^2 + \frac{1}{2}m_3(-\dot{x} - \dot{x}')^2 \\ + (m_1 - m_2 - m_3)gx + (m_2 - m_3)gx + m_2gl + m_3g(l + l') \quad (3)$$

The equations of motion are then

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0 \quad (4)$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}'} \right) - \frac{\partial L}{\partial x'} = 0 \quad (5)$$

which yield

$$(m_1 + m_2 + m_3)\ddot{x} + (m_3 - m_2)\ddot{x}' = (m_1 - m_2 - m_3)g \quad (6)$$

$$(m_3 - m_2)\ddot{x} + (m_2 + m_3)\ddot{x}' = (m_2 - m_3)g \quad (7)$$

Solving (6) and (7) we obtain the equations of motion.

$$\mathbf{7.5} \quad T = v^2\dot{u}^2 + 2\dot{v}^2 \quad (1)$$

$$V = u^2 - v^2 \quad (2)$$

$$L = T - V = v^2\dot{u}^2 + 2\dot{v}^2 - u^2 + v^2 \quad (3)$$

$$\frac{\partial L}{\partial \dot{u}} = 2v^2\dot{u}, \quad \frac{\partial L}{\partial u} = -2u \quad (4)$$

$$\frac{\partial L}{\partial \dot{v}} = 4\dot{v}, \quad \frac{\partial L}{\partial v} = 2v(\dot{u}^2 + 1) \quad (5)$$

The equations of motion

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{u}} \right) - \frac{\partial L}{\partial u} = 0 \quad (6)$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{v}} \right) - \frac{\partial L}{\partial v} = 0 \quad (7)$$

yield

$$2\frac{d}{dt}(v^2\dot{u}) + 2u = 0 \quad (8)$$

$$\text{or } v^2\ddot{u} + 2\dot{u}\dot{v} + 2u = 0$$

$$2\ddot{v} + v(\dot{u}^2 + 1) = 0 \quad (9)$$

7.6 Here we need a single coordinate $q = x$:

$$T = \frac{1}{2}m\dot{x}^2, \quad V = \frac{1}{2}kx^2 \quad (1)$$

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \quad (2)$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}, \quad \frac{\partial L}{\partial x} = -kx \quad (3)$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0 \quad (4)$$

$$m\ddot{x} + kx = 0 \quad (5)$$

$$\text{Let } x = A \sin \omega t \quad (6)$$

$$\ddot{x} = -A\omega^2 \sin \omega t \quad (7)$$

Inserting (6) and (7) and simplifying

$$-m\omega^2 + k = 0$$

$$\omega = \sqrt{\frac{k}{m}} \quad \text{or} \quad T_0 = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{m}{k}}$$

where T_0 is the time period.

7.7 Only one coordinate $q = x$ (distance on the surface of the incline) is adequate to describe the motion:

$$T = \frac{1}{2}m\dot{x}^2, \quad V = -mgx \sin \alpha, \quad L = \frac{1}{2}m\dot{x}^2 + mgx \sin \alpha$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}, \quad \frac{\partial L}{\partial x} = mg \sin \alpha$$

Equation of motion

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0$$

yields

$$\frac{d}{dt}(m\dot{x}) - \frac{\partial}{\partial x}(mgx \sin \alpha) = 0$$

$$\text{or } \ddot{x} = g \sin \alpha$$

7.8 This is a two degree of freedom system because both mass m and M are moving. The coordinate on the horizontal axis is described by x' for the inclined plane and x for the block of mass m on the incline. The origin of the coordinate

system is fixed on the smooth table, Fig. 7.16. The x - and y -components of the velocity of the block are given by

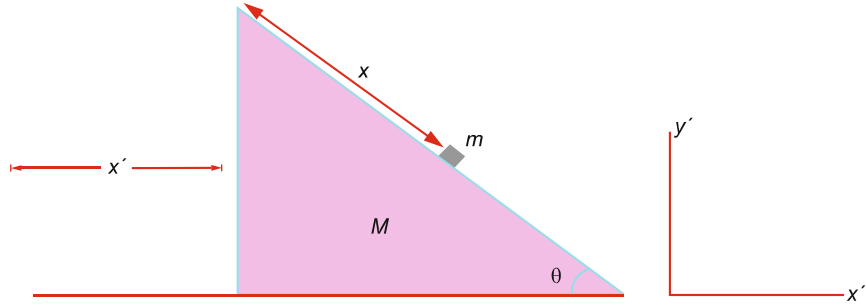


Fig. 7.16

$$v_x = \dot{x}' + \dot{x} \cos \theta \quad (1)$$

$$v_y = -\dot{x} \sin \theta \quad (2)$$

$$\therefore v^2 = v_x^2 + v_y^2 = \dot{x}'^2 + 2\dot{x}\dot{x}' \cos \theta + \dot{x}^2 \quad (3)$$

Hence, the kinetic energy of the system will be

$$T = \frac{1}{2} M \dot{x}'^2 + \frac{1}{2} m (\dot{x}'^2 + 2\dot{x}\dot{x}' \cos \theta + \dot{x}^2) \quad (4)$$

while the potential energy takes the form

$$V = -mgx \sin \theta \quad (5)$$

and the Lagrangian is given by

$$L = \frac{1}{2} M \dot{x}'^2 + \frac{1}{2} m (\dot{x}'^2 + 2\dot{x}\dot{x}' \cos \theta + \dot{x}^2) + mgx \sin \theta \quad (6)$$

The equations of motion

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0 \quad (7)$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}'} \right) - \frac{\partial L}{\partial x'} = 0 \quad (8)$$

yield

$$m(\ddot{x}' \cos \theta + \ddot{x}) - mg \sin \theta = 0 \quad (9)$$

$$M\ddot{x}' + m(\ddot{x}' + \ddot{x} \cos \theta) = 0 \quad (10)$$

Solving (9) and (10)

$$\ddot{x} = \frac{g \sin \theta}{1 - m \cos^2 \theta / (M + m)} = \frac{(M + m)g \sin \theta}{M + m \sin^2 \theta} \quad (11)$$

$$\ddot{x}' = -\frac{g \sin \theta \cos \theta}{(M + m)/m - \cos^2 \theta} = -\frac{mg \sin \theta \cos \theta}{M + m \sin^2 \theta} \quad (12)$$

which are in agreement with the equations of prob. (2.14) derived from Newtonian mechanics.

$$7.9 \quad T = \frac{1}{2}m(\dot{x}^2 + \omega^2 x^2) \quad (1)$$

because the velocity of the bead on the wire is at right angle to the linear velocity of the wire:

$$V = mgx \sin \omega t \quad (2)$$

because $\omega t = \theta$, where θ is the angle made by the wire with the horizontal at time t , and $x \sin \theta$ is the height above the horizontal position:

$$L = \frac{1}{2}m(\dot{x}^2 + \omega^2 x^2) - mgx \sin \omega t \quad (3)$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}, \quad \frac{\partial L}{\partial x} = m\omega^2 x - mg \sin \omega t \quad (4)$$

Lagrange's equation

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} \quad (5)$$

then becomes

$$\ddot{x} - \omega^2 x + g \sin \omega t = 0 \quad (\text{equation of motion}) \quad (6)$$

which has the solution

$$x = Ae^{\omega t} + Be^{-\omega t} + \frac{g}{2\omega^2} \sin \omega t \quad (7)$$

where A and B are constants of integration which are determined from initial conditions.

$$\text{At } t = 0, x = 0 \text{ and } \dot{x} = 0 \quad (8)$$

$$\text{Further } \dot{x} = \omega(Ae^{\omega t} - Be^{-\omega t}) + \frac{g}{2\omega} \cos \omega t \quad (9)$$

Using (8) in (7) and (9) we obtain

$$0 = A + B \quad (10)$$

$$0 = \omega(A - B) + \frac{g}{2\omega} \quad (11)$$

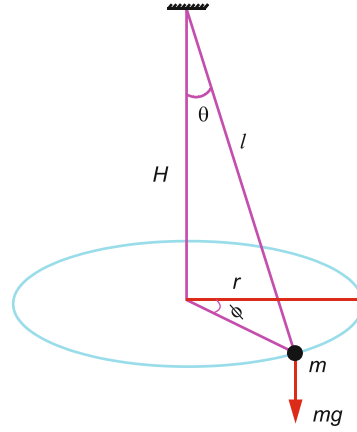
$$\text{Solving (10) and (11) we get } A = -\frac{g}{4\omega^2}, \quad B = \frac{g}{4\omega^2} \quad (12)$$

The complete solution for x is

$$x = \frac{g}{4\omega^2} (e^{-\omega t} - e^{\omega t} + 2 \sin \omega t)$$

7.10 Let the length of the cord be l . The Cartesian coordinates can be expressed in terms of spherical polar coordinates (Fig. 7.17)

Fig. 7.17



$$\begin{aligned} x &= l \sin \theta \cos \phi \\ y &= l \sin \theta \sin \phi \\ z &= -l \cos \theta \\ V &= mgz = -mgl \cos \theta \end{aligned} \quad (1)$$

$$v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = l^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) \quad (2)$$

$$T = \frac{1}{2} ml^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) \quad (3)$$

$$L = \frac{1}{2} ml^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) + mgl \cos \theta \quad (4)$$

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2 \dot{\theta}, \quad \frac{\partial L}{\partial \theta} = ml^2 \sin \theta \cos \theta \dot{\phi}^2 - mgl \sin \theta \quad (5)$$

$$\frac{\partial L}{\partial \dot{\phi}} = ml^2 \sin^2 \theta \dot{\phi}, \quad \frac{\partial L}{\partial \phi} = 0$$

The Lagrangian equations of motion

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0 \text{ and } \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\phi}} \right) - \frac{\partial L}{\partial \phi} = 0 \quad (6)$$

$$\text{give } \ddot{\theta} - \sin \theta \cos \theta \dot{\phi}^2 + \frac{g}{l} \sin \theta = 0 \quad (7)$$

$$ml^2 \frac{d}{dt} (\sin^2 \theta \dot{\phi}) = 0 \quad (8)$$

$$\text{Hence } \sin^2 \theta \dot{\phi} = \text{constant} = C \quad (9)$$

and eliminating $\dot{\phi}$ in (7) with the use of (9) we get a differential equation in θ only.

$$\ddot{\theta} + \frac{g}{l} \sin \theta - C^2 \frac{\cos \theta}{\sin^3 \theta} = 0 \quad (10)$$

$$\text{The quantity } P_\phi = \frac{\partial L}{\partial \dot{\phi}} = ml^2 \sin^2 \theta \dot{\phi} \quad (5)$$

is a constant of motion and is recognized as the angular momentum of the system about the z -axis. It is conserved because torque is not produced either by gravity or the tension of the cord about the z -axis. Thus conservation of angular momentum is reflected in (5).

Two interesting cases arise. Suppose $\phi = \text{constant}$. Then $\dot{\phi} = 0$ and $C = 0$. In this case (10) reduces to

$$\ddot{\theta} + \frac{g}{l} \sin \theta = 0 \quad (11)$$

which is appropriate for simple pendulum in which the bob oscillates in the vertical plane.

Suppose $\theta = \text{constant}$, then from (9) $\dot{\phi} = \text{constant}$. Putting $\ddot{\theta} = 0$ in (7) we get

$$\omega = \dot{\phi} = \sqrt{\frac{g}{l \cos \theta}} = \sqrt{\frac{g}{H}} \quad (12)$$

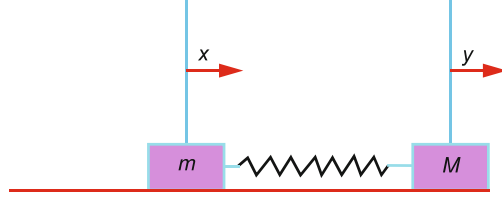
and time period

$$T = \frac{2\pi}{\omega} = \sqrt{\frac{H}{g}} \quad (13)$$

appropriate for the conical pendulum in which the bob rotates on horizontal plane with uniform angular velocity with the cord inclined at constant angle θ with the vertical axis.

7.11 The two generalized coordinates x and y are indicated in Fig. 7.18. The kinetic energy of the system comes from the motion of the blocks and potential energy from the coupling spring:

Fig. 7.18



$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}M\dot{y}^2 \quad (1)$$

$$V = \frac{1}{2}k(x - y)^2 \quad (2)$$

$$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}M\dot{y}^2 - \frac{1}{2}k(x - y)^2 \quad (3)$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}, \quad \frac{\partial L}{\partial x} = -k(x - y) \quad (4)$$

$$\frac{\partial L}{\partial \dot{y}} = M\dot{y}, \quad \frac{\partial L}{\partial y} = k(x - y) \quad (5)$$

Lagrange's equations are written as

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0, \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{y}} \right) - \frac{\partial L}{\partial y} = 0 \quad (6)$$

Using (4) and (5) in (6) we obtain the equations of motion

$$m\ddot{x} + k(x - y) = 0 \quad (7)$$

$$M\ddot{y} + k(y - x) = 0 \quad (8)$$

7.12 This problem involves two degrees of freedom. The coordinates are θ_1 and θ_2 (Fig. 7.19)

$$T = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 \quad (1)$$

$$v_1^2 = (l_1\dot{\theta}_1)^2 \quad (2)$$

$$v_2^2 = (l_1\dot{\theta}_1)^2 + (l_2\dot{\theta}_2)^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2 \cos(\theta_2 - \theta_1) \text{ (by parallelogram law)} \quad (3)$$

For small angles, $\cos(\theta_2 - \theta_1) \simeq 1$

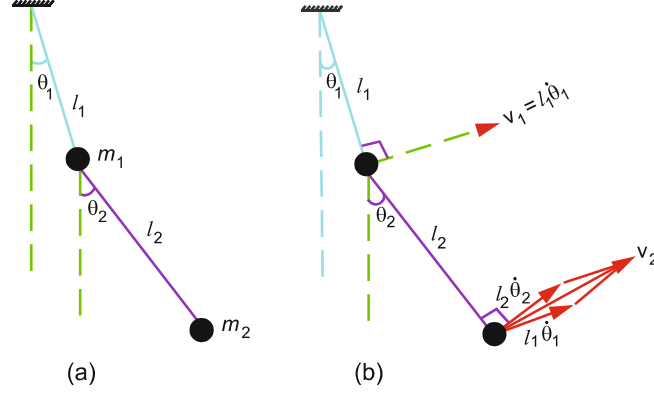


Fig. 7.19

$$T \simeq \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \left[l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \right] \quad (4)$$

$$V = m_1 g l_1 (1 - \cos \theta_1) + m_2 g l_1 (1 - \cos \theta_1) + m_2 g l_2 (1 - \cos \theta_2) \\ \simeq m_1 g l_1 \frac{\theta_1^2}{2} + \frac{m_2 g}{2} \left[l_1 \theta_1^2 + l_2 \theta_2^2 \right] \quad (5)$$

$$L = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \left[l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \right] - m_1 g l_1 \frac{\theta_1^2}{2} \\ - \frac{m_2 g}{2} (l_1 \theta_1^2 + l_2 \theta_2^2) \quad (6)$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = m_1 l_1^2 \dot{\theta}_1 + m_2 l_1^2 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \quad (7)$$

$$\frac{\partial L}{\partial \theta_1} = -m_1 g l_1 \theta_1 - m_2 g l_1 \theta_1 = -(m_1 + m_2) g l_1 \theta_1 \quad (8)$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2 \dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \quad (9)$$

$$\frac{\partial L}{\partial \theta_2} = -m_2 g l_2 \theta_2 \quad (10)$$

Lagrange's equations are

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_1} \right) - \frac{\partial L}{\partial \theta_1} = 0, \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_2} \right) - \frac{\partial L}{\partial \theta_2} = 0, \quad (11)$$

using (7), (8), (9) and (10) in (11) we obtain the equations of motion

$$(m_1 + m_2) l_1 \ddot{\theta}_1 + m_2 l_2 \ddot{\theta}_2 + (m_1 + m_2) g \theta_1 = 0 \quad (12)$$

$$l_2 \ddot{\theta}_2 + g \theta_2 + l_1 \ddot{\theta}_1 = 0 \quad (13)$$

7.13 Writing p_θ and p_ϕ for the generalized momenta, by (4) and (5) and V by (1) of prob. (7.10)

$$\frac{\partial L}{\partial \dot{\theta}} = P_\theta = ml^2 \dot{\theta}, \quad \dot{\theta} = \frac{P_\theta}{ml^2} \quad (1)$$

$$\frac{\partial L}{\partial \dot{\phi}} = p_\phi = ml^2 \sin^2 \theta \dot{\phi}, \quad \dot{\phi} = \frac{p_\phi}{ml^2 \sin^2 \theta} \quad (2)$$

$$H = \Sigma \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L \quad \text{or} \quad H + L = 2T = \Sigma \dot{q}_i \frac{\partial L}{\partial \dot{q}_i}$$

$$2T = \dot{\theta} \frac{\partial L}{\partial \dot{\theta}} + \dot{\phi} \frac{\partial L}{\partial \dot{\phi}} = \frac{1}{ml^2} (p_\theta^2 + \operatorname{cosec}^2 \theta p_\phi^2) \quad (3)$$

$$\therefore H = T + V = \frac{1}{2ml^2} (p_\theta^2 + \operatorname{cosec}^2 \theta p_\phi^2) - mgl \cos \theta \quad (4)$$

The coordinate ϕ is ignorable, and therefore p_ϕ is a constant of motion determined by the initial conditions. We are then left with only two canonical equations to be solved. The canonical equations are

$$\begin{aligned} \dot{q}_j &= \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j} \\ \dot{\theta} &= \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{ml^2} \end{aligned} \quad (5)$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{p_\phi^2 \cos \theta}{ml^2 \sin^3 \theta} - mgl \sin \theta \quad (6)$$

where p_ϕ is a constant of motion. By eliminating p_θ we can immediately obtain a second-order differential equation in θ as in prob. (7.10).

7.14 $H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 \quad (1)$

$$\frac{\partial H}{\partial p} = \dot{q} = p \quad (2)$$

$$\frac{\partial H}{\partial q} = -\dot{p} = \omega^2 q \quad (3)$$

Differentiating (2)

$$\ddot{q} = \dot{p} = -\omega^2 q \quad (4)$$

Let $q = x$, then (4) can be written as

$$\ddot{x} + \omega^2 x = 0 \quad (5)$$

This is the equation for one-dimensional harmonic oscillator. The general solution is

$$x = A \sin(\omega t + \varepsilon) + B \cos(\omega t + \varepsilon) \quad (6)$$

which can be verified by substituting (6) in (5). Here A , B and ε are constants to be determined from initial conditions.

7.15 Let r, θ be the instantaneous polar coordinates of a planet of mass m revolving around a parent body of mass M :

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \quad (1)$$

$$V = GMm \left(\frac{1}{2a} - \frac{1}{r} \right) \quad (2)$$

where G is the gravitational constant and $2a$ is the major axis of the ellipse:

$$p_r = \frac{\partial T}{\partial \dot{r}} = m\dot{r}, \quad \dot{r} = \frac{p_r}{m} \quad (3)$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = mr^2\dot{\theta}, \quad \dot{\theta} = \frac{p_\theta}{mr^2} \quad (4)$$

$$H = \frac{1}{2m} \left(p_r^2 + \frac{p_\theta^2}{r^2} \right) + GMm \left(\frac{1}{2a} - \frac{1}{r} \right) \quad (5)$$

and the Hamiltonian equations are

$$\frac{\partial H}{\partial p_r} = \frac{p_r}{m} = \dot{r}, \quad \frac{\partial H}{\partial r} = -\frac{p_\theta^2}{mr^3} + \frac{GMm}{r^2} = -\dot{p}_r \quad (6)$$

$$\frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2}, \quad \frac{\partial H}{\partial \theta} = 0 = -\dot{p}_\theta \quad (7)$$

Two equations in (7) show that

$$p_\theta = \text{constant} = mr^2\dot{\theta} \quad (8)$$

meaning the constancy of angular momentum or equivalently the constancy of areal velocity of the planet (Kepler's second law of planetary motion).

Two equations in (6) yield

$$\ddot{r} = \frac{\dot{p}_r}{m} = \frac{p_\theta^2}{m^2 r^3} - \frac{GMm}{r^2} = r\dot{\theta}^2 - \frac{GMm}{r^2} \quad (9)$$

Equation (9) describes the orbit of the planet (Kepler's first law of planetary motion)

$$7.16 \quad T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 \quad (1)$$

$$V = \frac{1}{2}k(x_1 - x_2)^2 \quad (2)$$

$$L = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - \frac{1}{2}k(x_1 - x_2)^2 \quad (3)$$

Equations of motion are

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_1} \right) - \frac{\partial L}{\partial x_1} = 0 \quad (4)$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_2} \right) - \frac{\partial L}{\partial x_2} = 0 \quad (5)$$

Using (3) in (4) and (5)

$$m_1\ddot{x}_1 + k(x_1 - x_2) = 0 \quad (6)$$

$$m_2\ddot{x}_2 - k(x_1 - x_2) = 0 \quad (7)$$

It is assumed that the motion is periodic and can be considered as superposition of harmonic components of various amplitudes and frequencies. Let one of these harmonics be represented by

$$x_1 = A \sin \omega t, \quad \ddot{x}_1 = -\omega^2 A \sin \omega t \quad (8)$$

$$x_2 = B \sin \omega t, \quad \ddot{x}_2 = -\omega^2 B \sin \omega t \quad (9)$$

Substituting (8) and (9) in (6) and (7) we obtain

$$\begin{aligned} (k - m_1\omega^2) A - kB &= 0 \\ -kA + (k - m_2\omega^2)B &= 0 \end{aligned}$$

The frequency equation is obtained by equating to zero the determinant formed by the coefficients of A and B :

$$\begin{vmatrix} (k - m_1\omega^2) & -k \\ -k & (k - m_2\omega^2) \end{vmatrix} = 0$$

Expansion of the determinant gives

$$m_1m_2\omega^4 - k(m_1 + m_2)\omega^2 = 0$$

or

$$\omega^2[m_1 m_2 \omega^2 - k(m_1 + m_2)] = 0$$

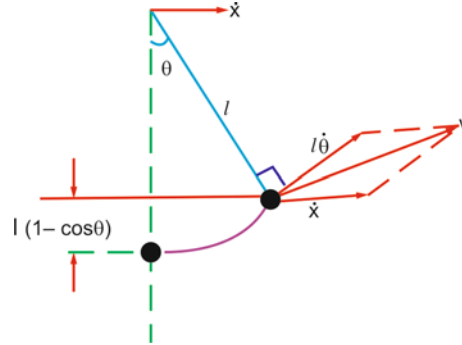
which yields the natural frequencies of the system:

$$\omega_1 = 0 \text{ and } \omega_2 = \sqrt{\frac{k(m_1 + m_2)}{m_1 m_2}} = \sqrt{\frac{k}{\mu}}$$

where μ is the reduced mass. The frequency $\omega_1 = 0$ implies that there is no genuine oscillation of the block but mere translatory motion. The second frequency ω_2 is what one expects for a simple harmonic oscillator with a reduced mass μ .

7.17 Let $x(t)$ be the displacement of the block and $\theta(t)$ the angle through which the pendulum swings. The kinetic energy of the system comes from the motion of the block and the swing of the bob of the pendulum. The potential energy comes from the deformation of the spring and the position of the bob, Fig. 7.20.

Fig. 7.20



The velocity v of the bob is obtained by combining vectorially its linear velocity ($l\dot{\theta}$) with the velocity of the block (\dot{x}). The height through which the bob is raised from the equilibrium position is $l(1 - \cos \theta)$, where l is the length of the pendulum:

$$v^2 = \dot{x}^2 + l^2 \dot{\theta}^2 + 2l\dot{x}\dot{\theta} \cos \theta \quad (1)$$

$$T = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m (\dot{x}^2 + l^2 \dot{\theta}^2 + 2l\dot{x}\dot{\theta}) \quad (2)$$

(\because for $\theta \rightarrow 0$, $\cos \theta \rightarrow 1$)

$$\begin{aligned} V &= \frac{1}{2} k x^2 + mgl(1 - \cos \theta) \\ &= \frac{1}{2} k x^2 + mgl \frac{\theta^2}{2} \end{aligned} \quad (3)$$

$$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + l^2\dot{\theta}^2 + 2l\dot{x}\dot{\theta}) - \frac{1}{2}kx^2 - mgl\frac{\theta^2}{2} \quad (4)$$

Applying Lagrange's equations

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0, \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0 \quad (5)$$

we obtain

$$(M + m)\ddot{x} + ml\ddot{\theta} + kx = 0 \quad (6)$$

$$l\ddot{\theta} + \ddot{x} + g\theta = 0 \quad (7)$$

7.18 Considering that at $t = 0$ the insect was in the middle of the rod, the coordinates of the insect x, y, z at time t are given by

$$x = (a + vt) \sin \theta \cos \phi$$

$$y = (a + vt) \sin \theta \sin \phi$$

$$z = (a + vt) \cos \theta$$

and the square of its velocity is

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = v^2 + (a + vt)^2(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta)$$

$$\therefore T = \frac{2}{3}Ma^2(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + \frac{m}{2}\{v^2 + (a + vt)^2(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta)\}$$

$$V = -Mga \cos \theta - mg(a + vt) \cos \theta + \text{constant}$$

$$L = T - V$$

The application of the Lagrangian equations to the coordinates θ and ϕ yields

$$\begin{aligned} \frac{d}{dt} \left[\frac{4}{3}Ma^2\dot{\theta} + m(a + vt)^2\dot{\theta} \right] - \left[\frac{4}{3}Ma^2 + m(a + vt)^2 \right] \dot{\phi}^2 \sin \theta \cos \theta \\ = -\{Ma + m(a + vt)\}g \sin \theta \end{aligned} \quad (1)$$

$$\text{and } \frac{d}{dt} \left[\left\{ \frac{4}{5}Ma^2 + m(a + vt)^2 \right\} \dot{\phi} \sin^2 \theta \right] = 0 \quad (2)$$

Equation (2) can be integrated at once as it is free from ϕ :

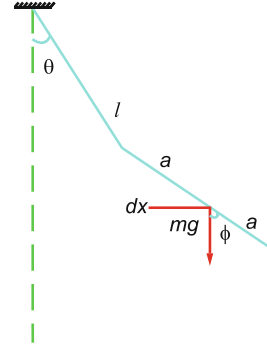
$$\left\{ \frac{4}{5}Ma^2 + m(a + vt)^2 \right\} \dot{\phi} \sin^2 \theta = \text{constant} = C \quad (3)$$

Equation (3) is the equation for the constancy of angular momentum about the vertical axis.

When $\dot{\phi}$ in (2) is eliminated with the aid of (3) we obtain a second-order differential equation in θ .

7.19 Let ρ be the linear density of the rod, i.e. mass per unit length. Consider an infinitesimal element of length of the rod

Fig. 7.21



$$\begin{aligned}
 dT &= \rho \omega^2 (l \sin \theta + x \sin \phi)^2 dx \\
 T &= \int dT = \rho \omega^2 \int_0^{2a} (l^2 \sin^2 \theta + 2lx \sin \theta \sin \phi + x^2 \sin^2 \phi) dx \\
 &= \omega^2 \left(Ml^2 \sin^2 \theta + 2Mla \sin \theta \sin \phi + \frac{4}{3}Ma^2 \sin^2 \phi \right) \quad (1)
 \end{aligned}$$

where we have substituted $\rho = M/2a$:

$$V = -Mg(l \cos \theta + a \cos \phi) \quad (2)$$

$$\begin{aligned}
 L &= \omega^2 \left(Ml^2 \sin^2 \theta + 2Mla \sin \theta \sin \phi + \frac{4}{3}Ma^2 \sin^2 \phi \right) \\
 &\quad + Mg(l \cos \theta + a \cos \phi) \quad (3)
 \end{aligned}$$

$$\frac{\partial L}{\partial \dot{\theta}} = 0, \quad \frac{\partial L}{\partial \theta} = \omega^2 (2Ml^2 \sin \theta \cos \theta + 2Mla \cos \theta \sin \phi) - Mgl \sin \theta \quad (4)$$

$$\frac{\partial L}{\partial \dot{\phi}} = 0, \quad \frac{\partial L}{\partial \phi} = \omega^2 \left(2Mla \sin \theta \cos \phi + \frac{8}{3}Ma^2 \sin \phi \cos \phi \right) - Mga \sin \phi \quad (5)$$

The Lagrange's equations

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0, \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\phi}} \right) - \frac{\partial L}{\partial \phi} = 0 \quad (6)$$

yield

$$2\omega^2(l \sin \theta + a \sin \phi) = g \tan \theta \quad (7)$$

$$2\omega^2(l \sin \theta + \frac{4}{3}a \sin \phi) = g \tan \phi \quad (8)$$

Equation (7) and (8) can be solved to obtain θ and ϕ .

7.20 Express the Cartesian coordinates in terms of plane polar coordinates (r, θ)

$$x = r \cos \theta, \quad y = r \sin \theta \quad (1)$$

$$\dot{x} = \dot{r} \cos \theta - r \dot{\theta} \sin \theta \quad (2)$$

$$\dot{y} = \dot{r} \sin \theta + r \dot{\theta} \cos \theta \quad (3)$$

Square (2) and (3) and add

$$v^2 = \dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2 \dot{\theta}^2 \quad (4)$$

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{r}^2 + r^2 \dot{\theta}^2) \quad (5)$$

$$V = U(r) \quad (6)$$

$$\therefore L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2 \dot{\theta}^2) - U(r) \quad (\text{Lagrangian}) \quad (7)$$

Generalized momenta:

$$p_k = \frac{\partial L}{\partial \dot{q}_k}, \quad p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}, \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2 \dot{\theta} \quad (8)$$

Hamiltonian:

$$H = T + V = \frac{1}{2}m(\dot{r}^2 + r^2 \dot{\theta}^2) + V(r) \quad (9)$$

Conservation of Energy: In general H may contain an explicit time dependence as in some forced systems. We shall therefore write $H = H(q, p, t)$. Then H varies with time for two reasons: first, because of its explicit dependence on t , second because the variable q and p are themselves functions of time. Then the total time derivative of H is

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \sum_{\beta=1}^n \frac{\partial H}{\partial q_\beta} \dot{q}_\beta + \sum_{\beta=1}^n \frac{\partial H}{\partial p_\beta} \dot{p}_\beta \quad (10)$$

Now Hamilton's equations are

$$\frac{\partial H}{\partial p_\beta} = \dot{q}_\beta, \quad \frac{\partial H}{\partial q_\beta} = -\dot{p}_\beta \quad (11)$$

Using (11) in (10), we obtain

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \sum_{\beta=1}^n \left[\frac{\partial H}{\partial q_\beta} \frac{\partial H}{\partial p_\beta} - \frac{\partial H}{\partial p_\beta} \frac{\partial H}{\partial q_\beta} \right] \quad (12)$$

whence

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} \quad (13)$$

Equation (13) asserts that H changes with time only by virtue of its explicit time dependence. The net change is induced by the fact that the variation of q and p with time is zero.

Now in a conservative system, neither T nor V contains any explicit dependence on time.

Hence $\frac{\partial H}{\partial t} = 0$. It follows that

$$\frac{dH}{dt} = 0 \quad (14)$$

which leads to the law of conservation of energy

$$H = T + V = E = \text{constant} \quad (15)$$

The Hamiltonian formalism is amenable for finding various conservation laws.

Conservation of angular momentum: The Hamiltonian can be written as

$$H = \sum_{\beta=1}^n p_\beta \dot{q}_\beta - L \quad (16)$$

Using the polar coordinates (r, θ)

$$H = p_r \dot{r} + p_\theta \dot{\theta} - \left(\frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 - U(r) \right) \quad (17)$$

Using (7) and (8) in (17)

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + U(r) \quad (18)$$

Now the second equation in (11) gives

$$-\dot{p}_\theta = \frac{\partial H}{\partial \theta} = 0 \quad (\because \theta \text{ is absent in (18)}) \quad (19)$$

This leads to the conservation of angular momentum

$$p_\theta = J = \text{constant} \quad (20)$$

7.21 Let each mass be m .

$$\text{(a)} \quad T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \quad (1)$$

$$V = \frac{1}{2}kx^2 + \frac{1}{2}3k(x-y)^2 + \frac{1}{2}ky^2 = k(2x^2 - 3xy + 2y^2) \quad (2)$$

$$L = T - V = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - k(2x^2 - 3xy + 2y^2) \quad (3)$$

$$\text{(b)} \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0 \quad (4)$$

$$\text{and} \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{y}} \right) - \frac{\partial L}{\partial y} = 0 \quad (5)$$

yield

$$m\ddot{x} = -4kx + 3ky \quad (6)$$

$$m\ddot{y} = 3kx - 4ky \quad (7)$$

$$\text{(c)} \quad \text{Let } x = A \sin \omega t \text{ and } y = B \sin \omega t \quad (8)$$

$$\ddot{x} = -A\omega^2 \sin \omega t \text{ and } \ddot{y} = -B\omega^2 \sin \omega t \quad (9)$$

Substituting (8) and (9) in (6) and (7) and simplifying we obtain

$$(4k - \omega^2 m)A - 3kB = 0 \quad (10)$$

$$-3kA + (4k - \omega^2 m)B = 0 \quad (11)$$

The frequency equation is obtained by equating to zero the determinant formed by the coefficients of A and B :

$$\begin{vmatrix} (4k - \omega^2 m) & -3k \\ -3k & (4k - \omega^2 m) \end{vmatrix} = 0 \quad (12)$$

Expanding the determinant

$$(4k - \omega^2 m)^2 - 9k^2 = 0 \quad (13)$$

This gives the frequencies

$$\omega_1 = \sqrt{\frac{k}{m}}, \quad \omega_2 = \sqrt{\frac{7k}{m}} \quad (14)$$

Periods of oscillations are

$$T_1 = \frac{2\pi}{\omega_1} = 2\pi\sqrt{\frac{m}{k}} \quad (15)$$

$$T_2 = \frac{2\pi}{\omega_2} = 2\pi\sqrt{\frac{m}{7k}} \quad (16)$$

If we put $\omega = \omega_1 = \sqrt{\frac{k}{m}}$ in (10) or (11) we get $A = B$ and if we put $\omega = \omega_2 = \sqrt{\frac{7k}{m}}$ in (10) or (11), we get $A = -B$. The first one corresponds to symmetric mode of oscillation and the second one to asymmetric one.

The normal coordinates q_1 and q_2 are formed by the linear combination of x and y :

$$q_1 = x - y, \quad q_2 = x + y \quad (17)$$

$$\therefore x = \frac{q_1 + q_2}{2}, \quad y = \frac{q_2 - q_1}{2} \quad (18)$$

Substituting (18) in (6) and (7)

$$\frac{m}{2}(\ddot{q}_1 + \ddot{q}_2) = -2k(q_1 + q_2) + \frac{3k}{2}(q_2 - q_1) \quad (19)$$

$$\frac{m}{2}(\ddot{q}_2 - \ddot{q}_1) = \frac{3k}{2}(q_1 + q_2) - 2k(q_2 - q_1) \quad (20)$$

$$\text{Adding (19) and (20), } m\ddot{q}_2 = -kq_2 \quad (21)$$

$$\text{Subtracting (20) from (19), } m\ddot{q}_1 = -7kq_1 \quad (22)$$

Equation (21) is a linear equation in q_2 alone, with constant coefficients. Similarly (22) is a linear equation in q_1 with constant coefficients. Since the coefficients on the right sides are positive quantities, we note that both (21) and (22) are differential equations of simple harmonic motion having the frequencies given in (14). It is the characteristic of normal coordinates that when the equa-

tions of motion are expressed in terms of normal coordinates they are linear with constant coefficients, and each contains but one dependent variable.

Another feature of normal coordinates is that both kinetic energy and potential energy will have quadratic terms and the cross-products will be absent. Thus in this example, when (18) is used in (1) and (2) we get the expressions

$$T = \frac{m}{4}(\dot{q}_1^2 + \dot{q}_2^2), \quad V = \frac{k}{4}(7q_1^2 + q_2^2)$$

The two normal modes are depicted in Fig. 7.22.

(a) Symmetrical with $\omega_1 = \sqrt{\frac{k}{m}}$ and (b) asymmetrical with $\omega_2 = \sqrt{\frac{7k}{m}}$

Fig. 7.22



7.22 (a) See Fig. 7.23 : $T = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2)$ (1)

$$V = \frac{1}{2}kx_1^2 + \frac{1}{2}k(x_2 - x_1)^2$$

$$= k\left(x_1^2 - x_1x_2 + \frac{1}{2}x_2^2\right) \quad (2)$$

$$L = T - V = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) - k\left(x_1^2 - x_1x_2 + \frac{1}{2}x_2^2\right) \quad (3)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = 0 \quad (4)$$

$$m\ddot{x}_1 + k(2x_1 - x_2) = 0 \quad (5)$$

$$m\ddot{x}_2 + k(x_1 - x_2) = 0 \quad (6)$$

(5) and (6) are equations of motion

(b) Let the harmonic solutions be $x_1 = A \sin \omega t$, $x_2 = B \sin \omega t$ (7)

Then $\ddot{x}_1 = -A\omega^2 \sin \omega t$, $\ddot{x}_2 = -B\omega^2 \sin \omega t$, (8)

Using (7) and (8) in (5) and (6) we get

$$(2k - m\omega^2)A - kB = 0 \quad (9)$$

$$-kA + (k - m\omega^2)B = 0 \quad (10)$$

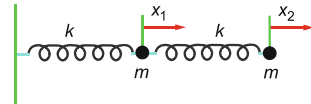


Fig. 7.23

The eigenfrequency equation is obtained by equating to zero the determinant formed by the coefficients of A and B :

$$\begin{vmatrix} (2k - m\omega^2) & -k \\ -k & (k - m\omega^2) \end{vmatrix} = 0 \quad (11)$$

Expanding the determinant, we obtain

$$\begin{aligned} m^2\omega^4 - 3mk\omega^2 + k^2 &= 0 \\ \omega^2 &= \left(\frac{3 \pm \sqrt{5}}{2} \right) \frac{k}{m} \end{aligned} \quad (12)$$

$$\therefore \omega_1 = 1.618\sqrt{\frac{k}{m}}, \quad \omega_2 = 0.618\sqrt{\frac{k}{m}} \quad (13)$$

- (c) Inserting $\omega = \omega_1$ in (10) we find $B = 1.618 A$. This corresponds to a symmetric mode as both the amplitudes have the same sign. Inserting $\omega = \omega_2$ in (10), we find $B = -0.618 A$. This corresponds to asymmetric mode. These two modes of oscillation are depicted in Fig. 7.24 with relative sizes and directions of displacement.

(a) Symmetric mode $\omega_1 = 1.618\sqrt{\frac{k}{m}}$

(b) Asymmetric mode $\omega_2 = 0.618\sqrt{\frac{k}{m}}$



Fig. 7.24

- 7.23 (a)** Let x_1 and x_2 be the displacements of the beads of mass $2m$ and m , respectively.

$$T = \frac{1}{2}(2m)\dot{x}_1^2 + \frac{1}{2}(m)\dot{x}_2^2 \quad (1)$$

$$V = \frac{1}{2} \cdot 2kx_1^2 + \frac{1}{2}k(x_2 - x_1)^2 \quad (2)$$

$$L = m \left(\dot{x}_1^2 + \frac{1}{2}\dot{x}_2^2 \right) - k \left(\frac{3}{2}x_1^2 - x_1x_2 + \frac{1}{2}x_2^2 \right) \quad (3)$$

Lagrange's equations are

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_1} \right) - \frac{\partial L}{\partial x_1} = 0, \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_2} \right) - \frac{\partial L}{\partial x_2} = 0 \quad (4)$$

which yield equations of motion

$$2m\ddot{x}_1 + k(3x_1 - x_2) = 0 \quad (5)$$

$$m\ddot{x}_2 - k(x_1 - x_2) = 0 \quad (6)$$

(b) Let the harmonic solutions be

$$x_1 = A \sin \omega t, \quad x_2 = B \sin \omega t \quad (7)$$

$$\ddot{x}_1 = -A\omega^2 \sin \omega t, \quad \ddot{x}_2 = -B\omega^2 \sin \omega t \quad (8)$$

Substituting (7) and (8) in (5) and (6) we obtain

$$(3k - 2m\omega^2)A - kB = 0 \quad (9)$$

$$kA + (m\omega^2 - k)B = 0 \quad (10)$$

The frequency equation is obtained by equating to zero the determinant formed by the coefficients of A and B :

$$\begin{vmatrix} (3k - 2m\omega^2) & -k \\ k & m\omega^2 - k \end{vmatrix} = 0$$

Expanding the determinant

$$2m^2\omega^4 - 5km\omega^2 + 2k^2 = 0$$

$$\omega_1 = \sqrt{\frac{2k}{m}}, \quad \omega_2 = \sqrt{\frac{k}{2m}}$$

(c) Put $\omega = \omega_1 = \sqrt{\frac{2k}{m}}$ in (9) or (10). We find $B = -A$.

Put $\omega = \omega_2 = \sqrt{\frac{k}{2m}}$ in (9) or (10). We find $B = +2A$.

The two normal modes are sketched in Fig. 7.25.

Fig. 7.25



(a) Asymmetric mode

$$\omega_1 = \sqrt{\frac{2k}{m}} \quad B = -A$$

(b) Symmetric mode $\omega_2 = \sqrt{\frac{k}{2m}}$ $B = +2A$

7.24 There are three coordinates x_1 , x_2 and x_3 , Fig. 7.26:

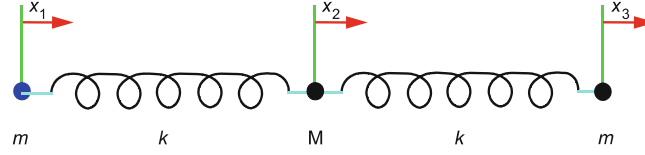


Fig. 7.26

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}M\dot{x}_2^2 + \frac{1}{2}m\dot{x}_3^2 \quad (1)$$

$$V = \frac{1}{2}k[(x_2 - x_1)^2 + (x_3 - x_2)^2]$$

$$= \frac{1}{2}k(x_1^2 - 2x_1x_2 + 2x_2^2 - 2x_2x_3 + x_3^2) \quad (2)$$

$$L = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}M\dot{x}_2^2 + \frac{1}{2}m\dot{x}_3^2 - \frac{1}{2}k(x_1^2 - 2x_1x_2 + 2x_2^2 - 2x_2x_3 + x_3^2) \quad (3)$$

Lagrange's equations

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_1} \right) - \frac{\partial L}{\partial x_1} = 0, \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_2} \right) - \frac{\partial L}{\partial x_2} = 0, \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_3} \right) - \frac{\partial L}{\partial x_3} = 0 \quad (4)$$

yield

$$m\ddot{x}_1 + k(x_1 - x_2) = 0 \quad (5)$$

$$M\ddot{x}_2 + k(-x_1 + 2x_2 - x_3) = 0 \quad (6)$$

$$m\ddot{x}_3 + k(-x_2 + x_3) = 0 \quad (7)$$

Let the harmonic solutions be

$$x_1 = A \sin \omega t, \quad x_2 = B \sin \omega t, \quad x_3 = C \sin \omega t \quad (8)$$

$$\therefore \ddot{x}_1 = -A\omega^2 \sin \omega t, \quad \ddot{x}_2 = -B\omega^2 \sin \omega t, \quad \ddot{x}_3 = -C\omega^2 \sin \omega t \quad (9)$$

Substituting (8) and (9) in (5), (6) and (7)

$$(k - m\omega^2)A - kB = 0 \quad (10)$$

$$-kA + (2k - M\omega^2)B - kC = 0 \quad (11)$$

$$-kB + (k - m\omega^2)C = 0 \quad (12)$$

The frequency equation is obtained by equating to zero the determinant formed by the coefficients of A , B and C

$$\begin{vmatrix} (k - m\omega^2) & -k & 0 \\ -k & (2k - M\omega^2) & -k \\ 0 & -k & (k - m\omega^2) \end{vmatrix} = 0$$

Expanding the determinant we obtain

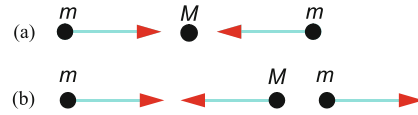
$$\omega^2(k - m\omega^2)(\omega^2 Mm - 2km - Mk) = 0 \quad (13)$$

The frequencies are

$$\omega_1 = 0, \quad \omega_2 = \sqrt{\frac{k}{m}}, \quad \omega_3 = \sqrt{\frac{k(2m + M)}{Mm}} \quad (14)$$

The frequency $\omega_1 = 0$ simply means a translation of all the three particles without vibration. Ratios of amplitudes of the three particles can be found out by substituting ω_2 and ω_3 in (10), (11) and (12). Thus when $\omega = \omega_2 \sqrt{\frac{k}{m}}$ is substituted in (10), we find the amplitude for the central atom $B = 0$. When $B = 0$ is used in (11) we obtain $C = -A$. This mode of oscillation is depicted in Fig. 7.27a.

Fig. 7.27



Substituting $\omega = \omega_3 = \sqrt{\frac{k(2m + M)}{Mm}}$ in (10) and (12) yields

$$B = -\left(\frac{2m}{M}\right) A = -\left(\frac{2m}{M}\right) C$$

Thus in this mode particles of mass m oscillate in phase with equal amplitude but out of phase with the central particle.

This problem has a bearing on the vibrations of linear molecules such as CO_2 . The middle particle represents the C atom and the particles on either side represent O atoms. Here too there will be three modes of oscillations. One will have a zero frequency, $\omega_1 = 0$, and will correspond to a simple translation of the centre of mass. In Fig. 7.27a the mode with $\omega_1 = \omega_2$ is such that the carbon atom is stationary, the oxygen atoms oscillating back and forth in opposite phase with equal amplitude. In the third mode which has frequency ω_3 , the carbon atom undergoes motion with respect to the centre of mass and is in opposite phase from that of the two oxygen atoms. Of these two modes

only ω_3 is observed optically. The frequency ω_2 is not observed because in this mode, the electrical centre of the system is always coincident with the centre of mass, and so there is no oscillating dipole moment Σer available. Hence dipole radiation is not emitted for this mode. On the other hand in the third mode characterized by ω_3 such a moment is present and radiation is emitted.

$$7.25 \text{ (a)} \quad y = \frac{x^2}{l} \quad (1)$$

$$\dot{y} = \frac{2x \cdot \dot{x}}{l} \quad (2)$$

$$v^2 = \dot{x}^2 + \dot{y}^2 = \dot{x}^2 \left(1 + \frac{4x^2}{l^2} \right)$$

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2 \left(1 + \frac{4x^2}{l^2} \right)$$

$$V = mgy = \frac{mgx^2}{l}$$

$$L = T - V = \frac{1}{2}m\dot{x}^2 \left(1 + \frac{4x^2}{l^2} \right) - \frac{mgx^2}{l}$$

$$(b) \quad L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - U(r)$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}, \quad \dot{r} = \frac{p_r}{m}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}, \quad \dot{\theta} = \frac{p_\theta}{mr^2}$$

$$H = \frac{1}{2m} \left(p_r^2 + \frac{p_\theta^2}{r^2} \right) + U(r)$$

7.26 (a) At any instant the velocity of the block is \dot{x} on the plane surface. The linear velocity of the pendulum with respect to the block is $l\dot{\theta}$, Fig. 7.28. The velocity $l\dot{\theta}$ must be combined vectorially with \dot{x} to find the velocity v of the pendulum with reference to the plane:

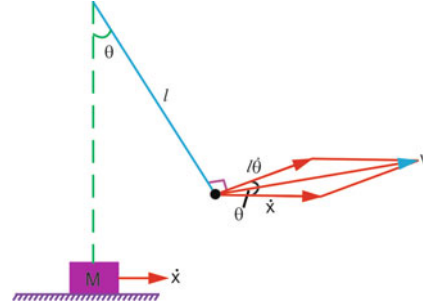
$$v^2 = \dot{x}^2 + l^2\dot{\theta}^2 + 2\dot{x}l\dot{\theta} \cos \theta \quad (1)$$

The total kinetic energy of the system

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + l^2\dot{\theta}^2 + 2\dot{x}l\dot{\theta} \cos \theta) \quad (2)$$

Taking the zero level of the potential energy at the pivot of the pendulum, the potential energy of the system which comes only from the pendulum is

Fig. 7.28



$$V = -mgl \cos \theta \quad (3)$$

$$\therefore L = T - V = \frac{1}{2}(M + m)\dot{x}^2 + ml \cos \theta \dot{x} \dot{\theta} + \frac{1}{2}ml^2 \dot{\theta}^2 + mgl \cos \theta \quad (4)$$

where we have used (2) and (3).

- (b) For small angles $\cos \theta \simeq 1 - \frac{\theta^2}{2}$, in the first approximation, and $\cos \theta \simeq 1$, in the second approximation. Thus in this approximation (4) becomes

$$L = \frac{1}{2}(M + m)\dot{x}^2 + ml\dot{x}\dot{\theta} + \frac{1}{2}ml^2\dot{\theta}^2 + mgl \left(1 - \frac{\theta^2}{2}\right) \quad (5)$$

- (c) The Lagrange's equations

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0, \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0 \quad (6)$$

lead to the equations of motion

$$\ddot{x} + l\ddot{\theta} + g\theta = 0 \quad (7)$$

$$(M + m)\ddot{x} + ml\ddot{\theta} = 0 \quad (8)$$

- (d) Eliminating \ddot{x} between (7) and (8) and simplifying

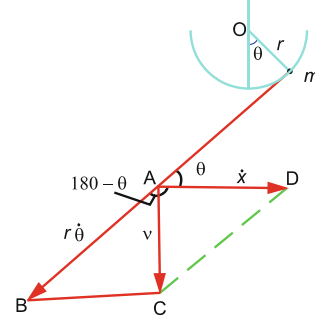
$$\ddot{\theta} + \frac{(M + m)g}{Ml} \theta = 0 \quad (9)$$

This is the equation for angular simple harmonic motion whose frequency is given by

$$\omega = \sqrt{\frac{(M + m)g}{Ml}} \quad (10)$$

- 7.27 (a)** First, we assume that the bowl does not move. Both kinetic energy and potential energy arise from the particle alone. Taking the origin at O, the centre of the bowl, Fig. 7.29, the linear velocity of the particle is $v = r\dot{\theta}$. There is only one degree of freedom:

Fig. 7.29



$$T = \frac{1}{2}mv^2 = \frac{1}{2}mr^2\dot{\theta}^2 \quad (1)$$

$$V = -mgr \cos \theta \quad (2)$$

$$L = \frac{1}{2}mr^2\dot{\theta}^2 + mgr \cos \theta \quad (3)$$

Lagrange's equation

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0 \quad (4)$$

becomes

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0 \quad (5)$$

which yields the equation of motion

$$mr^2\ddot{\theta} + mgr \sin \theta = 0$$

$$\text{or } \ddot{\theta} + \frac{g}{r} \sin \theta = 0 \quad (\text{equation of motion}) \quad (6)$$

For small angles, $\sin \theta \simeq \theta$. Then (6) becomes

$$\ddot{\theta} + \frac{g}{r}\theta = 0 \quad (7)$$

which is the equation for simple harmonic motion of frequency $\omega = \sqrt{\frac{g}{r}}$ or time period

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{r}{g}} \quad (8)$$

- (b) (i) The bowl can now slide freely along the x -direction with velocity \dot{x} . The velocity of the particle with reference to the table is obtained by adding $l\dot{\theta}$ to \dot{x} vectorially, Fig. 7.29. The total kinetic energy then comes from the motion of both the particle and the bowl. The potential energy, however, is the same as in (a):

$$v^2 = r^2\dot{\theta}^2 + \dot{x}^2 - 2r\dot{\theta}\dot{x}\cos(180 - \theta) \quad (9)$$

from the diagonal AC of the parallelogram ABCD

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(r^2\dot{\theta}^2 + \dot{x}^2 - 2r\dot{\theta}\dot{x}\cos\theta) \quad (10)$$

$$V = -mgr\cos\theta \quad (11)$$

$$L = T - V \quad (\text{Lagrangian})$$

$$= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(r^2\dot{\theta}^2 + \dot{x}^2 - 2r\dot{\theta}\dot{x}\cos\theta) + mgr\cos\theta \quad (12)$$

(ii) and (iii).

In the small angle approximation the $\cos\theta$ in the kinetic energy can be neglected as $\cos\theta \rightarrow 1$ but can be retained in the potential energy in order to avoid higher order terms.

Equation (12) then becomes

$$L = \frac{1}{2}(M + m)\dot{x}^2 - mr\dot{\theta}\dot{x} + \frac{1}{2}mr^2\dot{\theta}^2 + mgr\cos\theta \quad (13)$$

We have now two degrees of freedom, x and θ , and the corresponding Lagrange's equations are

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0, \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0 \quad (14)$$

which yield the equations of motion

$$(M + m)\ddot{x} - mr\ddot{\theta} = 0 \quad (15)$$

$$\ddot{x} - r\ddot{\theta} - g\theta = 0 \quad (16)$$

Equations (15) and (16) constitute the equations of motion. Eliminating \ddot{x} we obtain

$$\ddot{\theta} + \left(\frac{M + m}{M} \right) \frac{g}{r} \theta = 0 \quad (17)$$

which is the equation for simple harmonic motion with frequency $\omega = \sqrt{\frac{(M + m)g}{Mr}}$ and time period

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{Mr}{(M + m)g}} \quad (18)$$

On comparing (18) with (8) it is observed that the period of oscillation is smaller by a factor $[M/(M + m)^{1/2}]$ as compared to the case where the bowl is fixed.

7.28 Take the differential of the Lagrangian

$$L(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n, t)$$

$$dL = \sum_{r=1}^n \left(\frac{\partial L}{\partial q_r} dq_r + \frac{\partial L}{\partial \dot{q}_r} d\dot{q}_r \right) + \frac{\partial L}{\partial t} dt \quad (1)$$

Now the Lagrangian equations are

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_r} \right) - \frac{\partial L}{\partial q_r} = 0 \quad (2)$$

and the generalized momenta are defined by

$$\frac{\partial L}{\partial \dot{q}_r} = p_r \quad (3)$$

Using (3) in (2) we have

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_r} \right) = \dot{p}_r \quad (4)$$

Using (2), (3) and (4) in (1)

$$dL = \sum_{r=1}^n (\dot{p}_r dq_r + p_r d\dot{q}_r) + \frac{\partial L}{\partial t} dt \quad (5)$$

Equation (5) can be rearranged in the form

$$d \left(\sum_{r=1}^n p_r \dot{q}_r - L \right) = \sum_{r=1}^n (\dot{q}_r dp_r - \dot{p}_r dq_r) - \frac{\partial L}{\partial t} dt \quad (6)$$

The Hamiltonian function H is defined by

$$H = \sum_{r=1}^n p_r \dot{q}_r - L(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n, t) \quad (7)$$

Equation (6) therefore may be written as

$$dH = \sum_{r=1}^n (\dot{q}_r dp_r - \dot{p}_r dq_r) - \frac{\partial L}{\partial t} dt \quad (8)$$

While the Lagrangian function L is an explicit function of $q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n$ and t , it is usually possible to express H as an explicit function only of $q_1, \dots, q_n, p_1, \dots, p_n, t$, that is, to eliminate the n generalized velocities from (7). The n equations of type (3) are employed for this purpose. Each provides one of the p 's in terms of the \dot{q} 's. Assuming that the elimination of the generalized velocities is possible, we may write

$$H = H(q_1, \dots, q_n, p_1, \dots, p_n, t) \quad (9)$$

H now depends explicitly on the generalized coordinates and generalized momenta together with the time. Therefore, taking the differential dH , we obtain

$$dH = \sum_{r=1}^n \left(\frac{\partial H}{\partial q_r} dq_r + \frac{\partial H}{\partial p_r} dp_r \right) + \frac{\partial H}{\partial t} dt \quad (10)$$

Comparing (8) and (10), we have the relations

$$\frac{\partial H}{\partial p_r} = \dot{q}_r, \quad \frac{\partial H}{\partial q_r} = -\dot{p}_r \quad (11)$$

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \quad (12)$$

Equations (11) are called Hamilton's canonical equations. These are $2n$ is number. For a system with n degrees of freedom the n Lagrangian equations (2) of the second order are replaced by $2n$ Hamiltonian equations of the first order. We note from the second equation of (11) that if any coordinate q_i is not contained explicitly in the Hamiltonian function H , the conjugate momentum p_i is a constant of motion. Such coordinates are called ignorable coordinates.

7.29 The generalized momentum p_r conjugate to the generalized coordinate q_r is defined as $\frac{\partial L}{\partial \dot{q}_r} = p_r$. If the Lagrangian of a dynamical system does not contain a certain coordinate, say q_s , explicitly then p_s is a constant of motion.

(a) The kinetic energy arises only from the motion of the particle P on the table as the particle Q is stationary. The potential energy arises from the particle Q alone.

When P is at distance r from the opening, Q will be at a depth $l-x$ from the opening:

$$T = \frac{1}{2}mv_p^2 = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \quad (1)$$

$$V = -mg(l-r) \quad (2)$$

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + mg(l-r) \quad (3)$$

For the two coordinates r and θ , Lagrange's equations take the form

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{r}} \right) - \frac{\partial L}{\partial r} = 0, \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0 \quad (4)$$

Equations (4) yield

$$\ddot{r} = r\dot{\theta}^2 - g \quad (5)$$

$$\begin{aligned} \frac{d}{dt}(mr^2\dot{\theta}) &= 0 \\ \therefore r^2\dot{\theta} &= C = \text{constant} \end{aligned} \quad (6)$$

Equations (5) and (6) constitute the equations of motion.

(b) Initial conditions: At $r = a$, $r\dot{\theta} = \sqrt{ag}$

$$\therefore \dot{\theta} = \sqrt{\frac{g}{a}} \quad (7)$$

Using (7) in (6) with $r = a$, we obtain

$$C^2 = a^3g \quad (8)$$

Using (6) and (8) in (5)

$$\ddot{r} = \frac{rc^2}{r^4} - g = \frac{a^3 g}{r^3} - g \quad (9)$$

- (c) (i) Since Q does not move, P must be at constant distance $r = a$ from the opening. Therefore P describes a circle of constant radius a .
(ii) Let P be displaced by a small distance x from the stable circular orbit of radius a , that is

$$r = a + x \quad (10)$$

$$\therefore \ddot{r} = \ddot{x} \quad (11)$$

Using (10) and (11) in (9)

$$\begin{aligned} \ddot{x} &= g \left[\frac{a^3}{(a+x)^3} - 1 \right] = g \left[\left(1 + \frac{x}{a} \right)^{-3} - 1 \right] \\ \text{or } \ddot{x} &\simeq -\frac{3gx}{a} \\ \text{or } \ddot{x} + \frac{3gx}{a} &= 0 \end{aligned} \quad (12)$$

which is the equation for simple harmonic motion. Thus the particle P when slightly displaced from the stable orbit of radius a executes oscillations around $r = a$.

This aspect of oscillations has a bearing on the so-called betatron oscillations of ions in circular machines which accelerate charged particles to high energies. If the amplitudes of the betatron oscillations are large then they may hit the wall of the doughnut and be lost, resulting in the loss of intensity of the accelerated particles.

$$\mathbf{7.30} \quad (\text{i}) \quad x = a \cos \theta, \quad y = b \sin \theta, \quad r^2 = a^2 \cos^2 \theta + b^2 \sin^2 \theta \quad (1)$$

$$\dot{x} = -a\dot{\theta} \sin \theta, \quad \dot{y} = b\dot{\theta} \cos \theta \quad (2)$$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m(a^2 \sin^2 \theta + b^2 \cos^2 \theta)\dot{\theta}^2 \quad (3)$$

$$V = mgy + \frac{1}{2}kr^2 = mgb \sin \theta + \frac{1}{2}k(a^2 \cos^2 \theta + b^2 \sin^2 \theta) \quad (4)$$

$$\begin{aligned} L &= \frac{1}{2}m(a^2 \sin^2 \theta + b^2 \cos^2 \theta)\dot{\theta}^2 - mgb \sin \theta \\ &\quad - \frac{1}{2}k(a^2 \cos^2 \theta + b^2 \sin^2 \theta) \end{aligned} \quad (5)$$

Lagrange's equation

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0$$

yields

$$\begin{aligned} & m(a^2 \sin^2 \theta + b^2 \cos^2 \theta) \ddot{\theta} + m(a^2 \sin \theta \cos \theta - b^2 \sin \theta \cos \theta) \dot{\theta}^2 \\ & \quad + mgb \cos \theta + k(-a^2 \sin \theta \cos \theta + b^2 \sin \theta \cos \theta) = 0 \\ \text{or } & m(a^2 \sin^2 \theta + b^2 \cos^2 \theta) \ddot{\theta} - (a^2 - b^2)(k - m\dot{\theta}^2) \sin \theta \cos \theta \\ & \quad + mgb \cos \theta = 0 \end{aligned} \quad (6)$$

- (ii) Equilibrium point is located where the force is zero, or $\partial V / \partial \theta = 0$. Differentiating (4)

$$\frac{\partial V}{\partial \theta} = mgb \cos \theta + k(b^2 - a^2) \sin \theta \cos \theta \quad (7)$$

Clearly the right-hand side of (7) is zero for $\theta = \pm \frac{\pi}{2}$

Writing (7) as

$$[mgb + k(b^2 - a^2) \sin \theta] \cos \theta \quad (8)$$

Another equilibrium point is obtained when

$$\sin \theta = \frac{mgb}{k(a^2 - b^2)} \quad (9)$$

provided $a > b$.

- (iii) An equilibrium point will be stable if $\frac{\partial^2 V}{\partial \theta^2} > 0$ and will be unstable if $\frac{\partial^2 V}{\partial \theta^2} < 0$. Differentiating (8) again we have

$$\frac{\partial^2 V}{\partial \theta^2} = k(b^2 - a^2)(\cos^2 \theta - \sin^2 \theta) - mgb \sin \theta \quad (10)$$

For $\theta = \frac{\pi}{2}$, (10) reduces to

$$k(a^2 - b^2) - mgb \quad (11)$$

Expression (11) will be positive if $a^2 > b^2 + \frac{mgb}{k}$, and $\theta = \frac{\pi}{2}$ will be a stable point.

For $\theta = -\frac{\pi}{2}$, (10) reduces to

$$k(a^2 - b^2) + mgb \quad (12)$$

Expression (12) will be positive if $a^2 > b^2 - \frac{mgb}{k}$, and $\theta = -\frac{\pi}{2}$ will be a stable point.

$$(iv) \quad T = 2\pi \sqrt{\frac{A(\theta)}{V''(\theta)}}$$

$$V''\left(-\frac{\pi}{2}\right) = k(a^2 - b^2) + mgb, \text{ by (12)}$$

$A(\theta)$ is the coefficient of $\frac{1}{2}\dot{\theta}^2$ in (3)

$$\therefore A(\theta) = m(a^2 \sin^2 \theta + b^2 \cos^2 \theta)$$

$$\therefore A\left(-\frac{\pi}{2}\right) = ma^2$$

$$\therefore T = 2\pi \sqrt{\frac{ma^2}{k(a^2 - b^2) + mgb}}$$

7.31 In prob. (7.12) the following equations were obtained:

$$(m_1 + m_2)l_1\ddot{\theta}_1 + m_2l_2\ddot{\theta}_2 + (m_1 + m_2)g\theta_1 = 0 \quad (1)$$

$$l_2\ddot{\theta}_2 + g\theta_2 + l_1\ddot{\theta}_1 = 0 \quad (2)$$

For $l_1 = l_2 = l$ and $m_1 = m_2 = m$, (1) and (2) become

$$2l\ddot{\theta}_1 + l\ddot{\theta}_2 + 2g\theta_1 = 0 \quad (3)$$

$$l\ddot{\theta}_2 + l\ddot{\theta}_1 + g\theta_2 = 0 \quad (4)$$

The harmonic solutions of (3) and (4) are written as

$$\theta_1 = A \sin \omega t, \quad \theta_2 = B \sin \omega t \quad (5)$$

$$\ddot{\theta}_1 = -A\omega^2 \sin \omega t, \quad \ddot{\theta}_2 = -B\omega^2 \sin \omega t \quad (6)$$

Substituting (5) and (6) in (3) and (4) and simplifying

$$2(l\omega^2 - g)A + l\omega^2 B = 0 \quad (7)$$

$$l\omega^2 A + (l\omega^2 - g)B = 0 \quad (8)$$

The frequency equation is obtained by equating to zero the determinant formed by the coefficients of A and B :

$$\begin{vmatrix} 2(l\omega^2 - g) & l\omega^2 \\ l\omega^2 & (l\omega^2 - g) \end{vmatrix} = 0$$

Expanding the determinant

$$l^2\omega^4 - 4lg\omega^2 + 2g^2 = 0$$

$$\omega^2 = \left(2 \pm \sqrt{2}\right) \frac{g}{l}$$

$$\therefore \omega = \sqrt{\left(2 \pm \sqrt{2}\right) \frac{g}{l}}$$

$$\therefore \omega_1 = 0.76\sqrt{\frac{g}{l}}, \omega_2 = 1.85\sqrt{\frac{g}{l}}$$

7.32 While the method employed in [prob. \(6.46\)](#) was based on forces or torques, that is, Newton's method, the Lagrangian method is based on energy:

$$T = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) \quad (1)$$

For small angles \dot{y}_1 and \dot{y}_2 are negligibly small

$$V = \frac{1}{2}k(x_1 - x_2)^2 + mgb(1 - \cos\theta_1) + mgb(1 - \cos\theta_2)$$

$$\text{For small angles } 1 - \cos\theta_1 = \frac{\theta_1^2}{2} = \frac{x_1^2}{2b^2}.$$

$$\text{Similarly, } 1 - \cos\theta_2 = \frac{x_2^2}{2b^2}$$

$$\therefore V = \frac{1}{2}k(x_1 - x_2)^2 + \frac{mg}{2b}(x_1^2 + x_2^2) \quad (2)$$

$$\therefore L = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2}k(x_1^2 - 2x_1x_2 + x_2^2) - \frac{mg}{2b}(x_1^2 - x_2^2) \quad (3)$$

The Lagrange's equations for the coordinates x_1 and x_2 are

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_1} \right) - \frac{\partial L}{\partial x_1} = 0, \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_2} \right) - \frac{\partial L}{\partial x_2} = 0 \quad (4)$$

Using (3) in (4) we obtain

$$m\ddot{x}_1 + \left(k + \frac{mg}{b}\right)x_1 - kx_2 = 0 \quad (5)$$

$$m\ddot{x}_2 - kx_1 + \left(k + \frac{mg}{b}\right)x_2 = 0 \quad (6)$$

Assuming that x_1 and x_2 are periodic with the same frequency but different amplitudes, let

$$x_1 = A \sin \omega t, \quad \ddot{x}_1 = -A\omega^2 \sin \omega t \quad (7)$$

$$x_2 = B \sin \omega t, \quad \ddot{x}_2 = -B\omega^2 \sin \omega t \quad (8)$$

Substituting (7) and (8) in (5) and (6) and simplifying

$$\left(k + \frac{mg}{b} - m\omega^2\right) A - kB = 0 \quad (9)$$

$$-kA + \left(k + \frac{mg}{b} - m\omega^2\right) B = 0 \quad (10)$$

The frequency equation is obtained by equating to zero the determinant formed by the coefficients of A and B :

$$\begin{vmatrix} \left(k + \frac{mg}{b} - m\omega^2\right) & -k \\ -k & \left(k + \frac{mg}{b} - m\omega^2\right) \end{vmatrix} = 0$$

Expanding the determinant and solving gives

$$\omega_1 = \sqrt{\frac{g}{b}} \text{ and } \omega_2 = \sqrt{\frac{g}{b} + \frac{2k}{m}},$$

In agreement with the results of [prob. \(6.46\)](#).

7.33 Let the origin be at the fixed point O and OB be the diameter passing through C the centre of the circular wire, Fig. 7.30. The position of m is indicated by the angle θ subtended by the radius CP with the diameter OB . Only one general coordinate $q = \theta$ is sufficient for this problem. Let $\phi = \omega t$ be the angle which the diameter OB makes with the fixed x -axis. From the geometry of the diagram (Fig. 7.30) the coordinates of m are expressed as

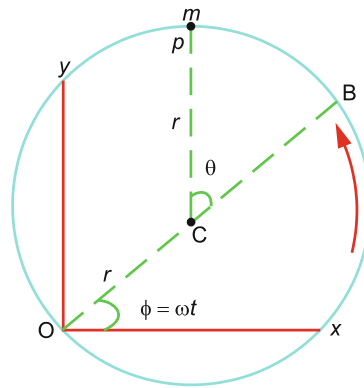


Fig. 7.30

$$x = r \cos \omega t + r \cos (\theta + \omega t) \quad (1)$$

$$y = r \sin \omega t + r \sin (\theta + \omega t) \quad (2)$$

The velocity components are found as

$$\dot{x} = -r \omega \sin \omega t - r(\dot{\theta} + \omega) \sin (\theta + \omega t) \quad (3)$$

$$\dot{y} = r \omega \cos \omega t - r(\dot{\theta} + \omega) \cos (\theta + \omega t) \quad (4)$$

Squaring and adding and simplifying we obtain

$$\dot{x}^2 + \dot{y}^2 = r^2 \omega^2 + r^2 (\dot{\theta} + \omega)^2 + 2r^2 \omega (\dot{\theta} + \omega) \cos \theta \quad (5)$$

$$\therefore T = \frac{1}{2} m r^2 [\omega^2 + (\dot{\theta} + \omega)^2 + 2\omega (\dot{\theta} + \omega) \cos \theta] \quad (6)$$

Here $V = 0$, and so $L = T$. The Lagrange's equation then simply reduces to

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\theta}} \right) - \frac{\partial T}{\partial \theta} = 0 \quad (7)$$

Cancelling the common factor $m r^2$ (7) becomes

$$\frac{d}{dt} (\dot{\theta} + \omega + \omega \cos \theta) + \omega (\dot{\theta} + \omega) \sin \theta = 0 \quad (8)$$

which reduces to

$$\ddot{\theta} + \omega^2 \sin \theta = 0 \quad (9)$$

which is the equation for simple pendulum. Thus the bead oscillates about the rotating line OB as a pendulum of length $r = a/\omega^2$.

7.34 (a) The velocity v of mass m relative to the horizontal surface is given by combining \dot{s} with \dot{x} . The components of the velocity v are

$$v_x = \dot{x} + \dot{s} \cos \alpha \quad (1)$$

$$v_y = -\dot{s} \sin \alpha \quad (2)$$

$$\therefore v^2 = v_x^2 + v_y^2 = \dot{x}^2 + \dot{s}^2 + 2\dot{x} \dot{s} \cos \alpha \quad (3)$$

Kinetic energy of the system

$$T = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m (\dot{s}^2 + \dot{x}^2 + 2\dot{s} \dot{x} \cos \alpha) \quad (4)$$

Potential energy comes exclusively from the mass m (spring energy + gravitational energy)

$$V = \frac{k}{2}(s-l)^2 + mg(h-s \sin \alpha) \quad (5)$$

$$L = T - V = \frac{(M+m)}{2}\dot{x}^2 + \frac{1}{2}m\dot{s}^2 + m\dot{x}\dot{s} \cos \alpha - \frac{k}{2}(s-l)^2 - mg(h-s \sin \alpha) \quad (6)$$

(b) The generalized coordinates are $q_1 = x$ and $q_2 = s$. The Lagrange's equations are

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0, \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{s}} \right) - \frac{\partial L}{\partial s} = 0 \quad (7)$$

Using (6) in (7), equations of motion become

$$(M+m)\ddot{x} + m\ddot{s} \cos \alpha = 0 \quad (8)$$

$$m\ddot{x} \cos \alpha + m\ddot{s} + k(s-s_0) = 0 \quad (9)$$

where $s_0 = l + (mg \sin \alpha)/k$.

$$\text{Let } x = A \sin \omega t \quad \text{and} \quad s - s_0 = B \sin \omega t \quad (10)$$

$$\ddot{x} = -\omega^2 A \sin \omega t, \quad \ddot{s} = -B\omega^2 \sin \omega t \quad (11)$$

Substituting (10) and (11) in (8) and (9) we obtain

$$A(M+m) + B \cos \alpha = 0 \quad (12)$$

$$Am\omega^2 \cos \alpha + B(m\omega^2 - k) = 0 \quad (13)$$

Eliminating A and B , we find

$$\omega = \sqrt{\frac{k(M+m)}{m(M+m \sin^2 \alpha)}} \quad (14)$$

Components of the velocity of the ball as observed on the table are

$$\mathbf{7.35} \quad v_x = \dot{x} + \dot{y} \cos \alpha \quad (1)$$

$$v_y = \dot{y} \sin \alpha \quad (2)$$

$$v^2 = v_x^2 + v_y^2 = \dot{x}^2 + \dot{y}^2 + 2\dot{x}\dot{y} \cos \alpha \quad (3)$$

$$\begin{aligned} T(\text{ball}) &= \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}mv^2 + \frac{1}{2} \times \frac{2}{5}mr^2\omega^2 \\ &= \frac{1}{2}mv^2 + \frac{1}{5}mv^2 = \frac{7}{10}mv^2 \end{aligned} \quad (4)$$

$$T(\text{wedge}) = \frac{1}{2}(M+m)\dot{x}^2 \quad (5)$$

$$\therefore T(\text{system}) = \frac{7}{10}m(\dot{x}^2 + \dot{y}^2 + 2\dot{x}\dot{y}\cos\alpha) + \frac{1}{2}(M+m)\dot{x}^2 \quad (6)$$

$$V(\text{system}) = V(\text{ball}) = -mgy\sin\alpha \quad (7)$$

$$L = \frac{7}{10}m(\dot{x}^2 + \dot{y}^2 + 2\dot{x}\dot{y}\cos\alpha) + \frac{1}{2}(M+m)\dot{x}^2 + mgy\sin\alpha \quad (8)$$

Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0 \quad (9)$$

become

$$\frac{7m}{5}\ddot{x} + \frac{7}{5}m\ddot{y}\cos\alpha + (M+m)\ddot{x} = 0 \quad (10)$$

$$\frac{7m}{5}\ddot{y} + \frac{7}{5}m\ddot{x}\cos\alpha - mg\sin\alpha = 0 \quad (11)$$

Solving (10) and (11) and simplifying

$$\ddot{x} = -\frac{5mg\sin\alpha\cos\alpha}{5M + (5 + 7\sin^2\alpha)m} \quad (\text{for the wedge})$$

$$\ddot{y} = \frac{5(5M + 12m)g\sin\alpha}{7(5M + (5 + 7\sin^2\alpha)m)} \quad (\text{for the ball})$$

For $M = m$ and $\alpha = \pi/4$

$$\ddot{x} = \frac{5g}{27}$$

$$\ddot{y} = \frac{85\sqrt{2}}{189}$$

Chapter 8

Waves

Abstract Chapter 8 deals with waves. The topics covered are wave equation, progressive and stationary waves, vibration of strings, wave velocity in solids, liquids and gases, capillary waves and gravity waves, the Doppler effect, shock wave, reverberation in buildings, stationary waves in pipes and intensity level.

8.1 Basic Concepts and Formulae

The travelling wave: The simple harmonic progressive wave travelling in the positive x -direction can be variously written as

$$\begin{aligned}y &= A \sin \frac{2\pi}{\lambda}(vt - x) \\&= A \sin 2\pi \left(\frac{t}{T} - \frac{x}{\lambda} \right) \\&= A \sin(\omega t - kx) \\&= A \sin 2\pi f \left(t - \frac{x}{v} \right)\end{aligned}\tag{8.1}$$

where A is the amplitude, f the frequency and v the wave velocity, λ the wavelength, $\omega = 2\pi f$ the angular frequency and $k = 2\pi/\lambda$, the wave number.

Similarly, the wave in the negative x -direction can be written as

$$y = A \sin \frac{2\pi}{\lambda}(vt + x)\tag{8.2}$$

and so on.

The superposition principle states that when two or more waves traverse the same region independently, the displacement of any particle at a given time is given by the vector addition of the displacement due to the individual waves.

Interference of waves: Interference is the physical effect caused by the superposition of two or more wave trains crossing the same region simultaneously. The wave trains must have a constant phase difference.