Solutions of Some Exercises

In this section the formulas are numbered (S1), (S2), etc, in order to avoid any confusion with formulas from the previous sections.

$\boxed{1.1}$

1. The equality $\langle f, x\rangle = \|x\|^2$ implies that $\|x\| \le \|f\|$. Corollary 1.3 implies that $F(x)$ is nonempty. It is clear from the second form of $F(x)$ that $F(x)$ is closed and convex.
2. In a strictly convex normed space any nonempty convex set that is contained in a sphere is reduced to a single point.
3. Note that
$$\langle f, y\rangle \le \|f\|\ \|y\| \le \frac{1}{2}\|f\|^2 + \frac{1}{2}\|y\|^2.$$
Conversely, assume that f satisfies
$$\text{(S1)} \qquad \frac{1}{2}\|y\|^2 - \frac{1}{2}\|x\|^2 \ge \langle f, y - x\rangle \ \forall y \in E.$$
First choose $y = \lambda x$ with $\lambda \in \mathbb{R}$ in (S1); by varying λ one sees that $\langle f, x\rangle = \|x\|^2$. Next choose y in (S1) such that $\|y\| = \delta > 0$; it follows that
$$\langle f, y\rangle \le \frac{1}{2}\delta^2 + \frac{1}{2}\|x\|^2.$$
Therefore we obtain
$$\delta\|f\| = \sup_{\substack{y\in E\\ \|y\|=\delta}} \langle f, y\rangle \le \frac{1}{2}\delta^2 + \frac{1}{2}\|x\|^2.$$
The conclusion follows by choosing $\delta = \|x\|$.
4. If $f \in F(x)$ one has

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$$\frac{1}{2}\|y\|^2 - \frac{1}{2}\|x\|^2 \geq \langle f, y - x\rangle$$

and if $g \in F(y)$ one has

$$\frac{1}{2}\|x\|^2 - \frac{1}{2}\|y\|^2 \geq \langle g, x - y\rangle.$$

Adding these inequalities leads to $\langle f - g, x - y\rangle \geq 0$. On the other hand, note that

$$\begin{aligned}\langle f - g, x - y\rangle &= \|x\|^2 + \|y\|^2 - \langle f, y\rangle - \langle g, x\rangle \\ &\geq \|x\|^2 + \|y\|^2 - 2\|x\|\ \|y\|.\end{aligned}$$

5. By question 4 we already know that $\|x\| = \|y\|$. On the other hand, we have

$$\langle F(x) - F(y), x - y\rangle = [\|x\|^2 - \langle F(x), y\rangle] + [\|y\|^2 - \langle F(y), x\rangle],$$

and both terms in brackets are ≥ 0. It follows that $\|x\|^2 = \|y\|^2 = \langle F(x), y\rangle = \langle F(y), x\rangle$, which implies that $F(x) \in F(y)$ and thus $F(x) = F(y)$ by question 2.

$\boxed{1.2}$

1(a).

$$\|f\|_{E^\star} = \max_{1\leq i\leq n} |f_i|.$$

1(b). $f \in F(x)$ iff for every $1 \leq i \leq n$ one has

$$f_i = \begin{cases} (\operatorname{sign} x_i)\|x\|_1 & \text{if } x_i \neq 0, \\ \text{anything in the interval } [-\|x\|_1, +\|x\|_1] & \text{if } x_i = 0. \end{cases}$$

2(a).

$$\|f\|_{E^\star} = \sum_{i=1}^{n} |f_i|.$$

2(b). Given $x \in E$ consider the set

$$I = \{1 \leq i \leq n;\ |x_i| = \|x\|_\infty\}.$$

Then $f \in F(x)$ iff one has

(i) $f_i = 0 \quad \forall i \notin I$,
(ii) $f_i x_i \geq 0\ \forall i \in I$ and $\sum_{i\in I} |f_i| = \|x\|_\infty$.

3.

$$\|f\|_{E^\star} = \left(\sum_{i=1}^{n} |f_i|^2\right)^{1/2}$$

and $f \in F(x)$ iff one has $f_i = x_i\ \forall i = 1, 2, \ldots, n$. More generally,

$$\|f\|_{E^\star} = \left(\sum_{i=1}^{n} |f_i|^{p'}\right)^{1/p'},$$

where $1/p + 1/p' = 1$, and $f \in F(x)$ iff one has $f_i = |x_i|^{p-2}x_i/\|x\|_p^{p-2}$ $\forall i = 1, 2, \ldots, n$.

$\boxed{1.3}$

1. $\|f\|_{E^\star} = 1$ (note that $f(t^\alpha) = 1/(1+\alpha) \quad \forall \alpha > 0$).
2. If there exists such a u we would have $\int_0^1 (1-u)dt = 0$ and thus $u \equiv 1$; absurd.

$\boxed{1.5}$

1. Let P denote the family of all linearly independent subsets of E. It is easy to see that P (ordered by the usual inclusion) is inductive. Zorn's lemma implies that P has a maximal element, denoted by $(e_i)_{i\in I}$, which is clearly an algebraic basis. Since $e_i \neq 0$ $\forall i \in I$, one may assume, by normalization, that $\|e_i\| = 1$ $\forall i \in I$.
2. Since E is infinite-dimensional one may assume that $\mathbb{N} \subset I$. There exists a (unique) linear functional on E such that $f(e_i) = i$ if $i \in \mathbb{N}$ and $f(e_i) = 0$ if $i \in I\backslash\mathbb{N}$.
3. Assume that I is countable, i.e., $I = \mathbb{N}$. Consider the vector space F_n spanned by $(e_i)_{i\leq i\leq n}$. F_n is closed (see Section 11.1) and, moreover, $\bigcup_{n=1}^\infty F_n = E$. It follows from the Baire category theorem that there exists some n_0 such that $\operatorname{Int}(F_{n_0}) \neq \emptyset$. Thus $E = F_{n_0}$; absurd.

$\boxed{1.7}$

1. Let $x, y \in \overline{C}$, so that $x = \lim x_n$ and $y = \lim y_n$ with $x_n, y_n \in C$. Thus $tx + (1-t)y = \lim[tx_n + (1-t)y_n]$ and therefore $tx + (1-t)y \in \overline{C}$ $\forall t \in [0,1]$. Assume $x, y \in \operatorname{Int} C$, so that there exists some $r > 0$ such that $B(x,r) \subset C$ and $B(y,r) \subset C$. It follows that
$$tB(x,r) + (1-t)B(y,r) \subset C \;\forall t \in [0,1].$$
But $tB(x,r) + (1-t)B(y,r) = B(tx + (1-t)y, r)$ (why?).
2. Let $r > 0$ be such that $B(y,r) \subset C$. One has
$$tx + (1-t)B(y,r) \subset C \quad \forall t \in [0,1],$$
and therefore $B(tx+(1-t)y, (1-t)r) \subset C$. It follows that $tx+(1-t)y \in \operatorname{Int} C$ $\forall t \in [0,1)$.
3. Fix any $y_0 \in \operatorname{Int} C$. Given $x \in C$ one has $x = \lim_{n\to\infty}[(1-\frac{1}{n})x + \frac{1}{n}y_0]$. But $(1-\frac{1}{n})x + \frac{1}{n}y_0 \in \operatorname{Int} C$ and therefore $x \in \overline{\operatorname{Int} C}$. This proves that $C \subset \overline{\operatorname{Int} C}$ and hence $\overline{C} \subset \overline{\operatorname{Int} C}$.

$\boxed{1.8}$

1. We already know that

$$p(\lambda x) = \lambda p(x)\ \forall\lambda > 0,\ \forall x \in E \quad \text{and} \quad p(x+y) \le p(x) + p(y)\ \forall x, y \in E.$$

It remains to check that

(i) $p(-x) = p(x)\ \forall x \in E$, which follows from the symmetry of C.
(ii) $p(x) = 0 \Rightarrow x = 0$, which follows from the fact that C is bounded. More precisely, let $L > 0$ be such that $\|x\| \le L\ \forall x \in C$. It is easy to see that

$$p(x) \ge \frac{1}{L}\|x\|\ \forall x \in E.$$

2. C is *not* bounded. Consider for example the sequence $u_n(t) = \sqrt{n}/(1+nt)$ and check that $u_n \in C$, while $\|u_n\| = \sqrt{n}$. Here $p(u) = \left(\int_0^1 |u(t)|^2 dt\right)^{1/2}$ is a norm that is not equivalent to $\|u\|$.

$\boxed{1.9}$

1. Let

$$P = \left\{\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n;\ \lambda_i \ge 0\ \forall i \text{ and } \sum_{i=1}^{n} \lambda_i = 1\right\},$$

so that P is a compact subset of $\mathbb{R}^n$ and C_n is the image of P under the continuous map $\lambda \mapsto \sum_{i=1}^n \lambda_i x_i$.
2. Apply Hahn–Banach, second geometric form, to C_n and $\{0\}$. Normalize the linear functional associated to the hyperplane that separates C_n and $\{0\}$.
4. Apply the above construction to $C = A - B$.

$\boxed{1.10}$

(A) $\Rightarrow$ (B) is obvious.
(B) $\Rightarrow$ (A). Let G be the vector space spanned by the x_i's ($i \in I$). Given $x \in G$ write $x = \sum_{i\in J} \beta_i x_i$ and set $g(x) = \sum_{i \in J} \beta_i \alpha_i$. Assumption (B) implies that this definition makes sense and that $|g(x)| \le M\|x\|\ \forall x \in G$. Next, extend g to all of E using Corollary 1.2.

$\boxed{1.11}$

(A) $\Rightarrow$ (B) is again obvious.
(B) $\Rightarrow$ (A). Assume first that the f_i's are linearly independent ($1 \le i \le n$). Set $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{R}^n$. Consider the map $\varphi : E \to \mathbb{R}^n$ defined by

$$\varphi(x) = (\langle f_1, x\rangle, \dots, \langle f_n, x\rangle).$$

Let $C = \{x \in E;\ \|x\| \le M + \varepsilon\}$. One has to show that $\alpha \in \varphi(C)$. Suppose, by contradiction, that $\alpha \notin \varphi(C)$ and separate $\varphi(C)$ and $\{\alpha\}$ (see Exercise 1.9). Hence, there exists some $\beta = (\beta_1, \beta_2, \dots, \beta_n) \in \mathbb{R}^n$, $\beta \ne 0$, such that

$$\beta \cdot \varphi(x) \le \beta \cdot \alpha \quad \forall x \in C, \text{ i.e.,} \left\langle \sum \beta_i f_i, x \right\rangle \le \sum \beta_i \alpha_i \quad \forall x \in C.$$

It follows that $(M+\varepsilon)\|\sum \beta_i f_i\| \le \sum \beta_i \alpha_i$. Using asumption (B) one finds that $\sum \beta_i f_i = 0$. Since the f_i's are linearly independent one concludes that $\beta = 0$; absurd.

In the general case, apply the above result to a maximal linearly independent subset of $(f_i)_{1\le i\le n}$.

$\boxed{1.15}$

1. It is clear that $C \subset C^{\star\star}$ and that $C^{\star\star}$ is closed. Conversely, assume that $x_0 \in C^{\star\star}$ and $x_0 \notin \overline{C}$. One may strictly separate $\{x_0\}$ and $\overline{C}$, so that there exist some $f_0 \in E^\star$ and some $\alpha_0 \in \mathbb{R}$ such that
$$\langle f_0, x\rangle < \alpha_0 < \langle f_0, x_0\rangle \ \forall x \in \overline{C}.$$
Since $0 \in C$ it follows that $\alpha_0 > 0$; letting $f = (1/\alpha_0) f_0$, one has
$$\langle f, x\rangle < 1 < \langle f, x_0\rangle \ \forall x \in C.$$
Thus $f \in C^\star$ and we are led to a contradiction, since $x_0 \in C^{\star\star}$.
2. If C is a linear subspace then
$$C^\star = \{f \in E^\star; \langle f, x\rangle = 0 \ \forall x \in C\} = C^\perp.$$

$\boxed{1.18}$

(a) $$\varphi^\star(f) = \begin{cases} -b & \text{if } f = a, \\ +\infty & \text{if } f \ne a. \end{cases}$$

(b) $$\varphi^\star(f) = \begin{cases} f\log f - f & \text{if } f > 0, \\ 0 & \text{if } f = 0, \\ +\infty & \text{if } f < 0. \end{cases}$$

(c) $$\varphi^\star(f) = |f|.$$

(d) $$\varphi^\star(f) = 0.$$

(e) $$\varphi^\star(f) = \begin{cases} +\infty & \text{if } f \ge 0, \\ -1 - \log|f| & \text{if } f < 0. \end{cases}$$

(f) $$\varphi^\star(f) = (1+f^2)^{1/2}.$$

(g) $$\varphi^\star(f) = \begin{cases} \frac{1}{2} f^2 & \text{if } |f| \le 1, \\ +\infty & \text{if } |f| > 1. \end{cases}$$

(h) $$\varphi^\star(f) = \frac{1}{p'}|f|^{p'} \quad \text{with } \frac{1}{p} + \frac{1}{p'} = 1.$$

(i) $$\varphi^\star(f) = \begin{cases} 0 & \text{if } 0 \le f \le 1, \\ +\infty & \text{otherwise.} \end{cases}$$

(j) $$\varphi^\star(f) = \begin{cases} \frac{1}{p'} f^{p'} & \text{if } f \geq 0, \\ 0 & \text{if } f < 0. \end{cases}$$

(k) $$\varphi^\star(f) = \begin{cases} +\infty & \text{if } f \geq 0, \\ -\frac{1}{p'}|f|^{p'} & \text{if } f < 0. \end{cases}$$

(l) $$\varphi^\star(f) = |f| + \frac{1}{p'}|f|^{p'}.$$

$\boxed{1.20}$ The conjugate functions are defined on $\ell^{p'}$ with $\frac{1}{p} + \frac{1}{p'} = 1$ by

(a) $$\varphi^\star(f) = \begin{cases} \frac{1}{4}\sum_{k=1}^{\infty} \frac{1}{k}|f_k|^2 & \text{if } \sum_{k=1}^{\infty} \frac{1}{k}|f_k|^2 < +\infty, \\ +\infty & \text{otherwise.} \end{cases}$$

(b) $$\varphi^\star(f) = \begin{cases} \sum_{k=2}^{+\infty} a_k|f_k|^{k/(k-1)} & \text{if } \sum_{k=2}^{+\infty} a_k|f_k|^{k/(k-1)} < +\infty, \\ +\infty & \text{otherwise,} \end{cases}$$

with $a_k = \dfrac{(k-1)}{k^{k/(k-1)}}$.

(c) $$\varphi^\star(f) = \begin{cases} 0 & \text{if } \|f\|_{\ell^\infty} \leq 1, \\ +\infty & \text{otherwise.} \end{cases}$$

$\boxed{1.21}$

2. $\varphi^\star = I_A$, where $A = \{[f_1, f_2];\ f_1 \leq 0,\ f_2 \leq 0, \text{ and } 4f_1 f_2 \geq 1\}$.
3. One has
$$\inf_{x\in E}\{\varphi(x) + \psi(x)\} = 0$$
and
$$\varphi^\star = I_{D^\perp}, \quad \text{where} \quad D^\perp = \{[f_1, f_2];\ f_2 = 0\}.$$
It follows that
$$\varphi^\star(-f) + \psi^\star(f) = +\infty \quad \forall f \in E^\star,$$
and thus
$$\sup_{f\in E^\star}\{-\varphi^\star(-f) - \psi^\star(f)\} = -\infty.$$
4. The assumptions of Theorem 1.12 are not satisfied: there is *no* element $x_0 \in E$ such that $\varphi(x_0) < +\infty$, $\psi(x_0) < +\infty$, and φ is continuous at x_0.

$\boxed{1.22}$

1. Write that
$$\|x - a\| \leq \|x - y\| + \|y - a\|.$$
Taking $\inf_{a\in A}$ leads to $\varphi(x) \leq \|x - y\| + \varphi(y)$. Then exchange x and y.

2. Let $x, y \in E$ and $t \in [0, 1]$ be fixed. Given $\varepsilon > 0$ there exist some $a \in A$ and some $b \in A$ such that

$$\|x - a\| \leq \varphi(x) + \varepsilon \quad \text{and} \quad \|y - b\| \leq \varphi(y) + \varepsilon.$$

Therefore

$$\|tx + (1-t)y - [ta + (1-t)b]\| \leq t\varphi(x) + (1-t)\varphi(y) + \varepsilon.$$

But $ta + (1-t)b \in A$, so that

$$\varphi(tx + (1-t)y) \leq t\varphi(x) + (1-t)\varphi(y) + \varepsilon \quad \forall \varepsilon > 0.$$

3. Since A is closed, one has $A = \{x \in E;\, \varphi(x) \leq 0\}$, and therefore A is convex if φ is convex.
4. One has

$$\begin{aligned}
\varphi^\star(f) &= \sup_{x \in E}\{\langle f, x\rangle - \inf_{a \in A}\|x - a\|\} \\
&= \sup_{x \in E}\sup_{a \in A}\{\langle f, x\rangle - \|x - a\|\} \\
&= \sup_{a \in A}\sup_{x \in E}\{\langle f, x\rangle - \|x - a\|\} \\
&= (I_A)^\star(f) + I_{B_{E^\star}}(f).
\end{aligned}$$

$\boxed{1.23}$

1. Let $f \in D(\varphi^\star) \cap D(\psi^\star)$. For every $x, y \in E$ one has

$$\begin{aligned}
\langle f, x - y\rangle - \varphi(x - y) &\leq \varphi^\star(f), \\
\langle f, y\rangle - \psi(y) &\leq \psi^\star(f).
\end{aligned}$$

Adding these inequalities leads to

$$(\varphi \nabla \psi)(x) \geq \langle f, x\rangle - \varphi^\star(f) - \psi(f).$$

In particular, $(\varphi \nabla \psi)(x) > -\infty$. Also, we have

$$\begin{aligned}
(\varphi \nabla \psi)^\star(f) &= \sup_{x \in E}\{\langle f, x\rangle - \inf_{y \in E}[\varphi(x - y) + \psi(y)]\} \\
&= \sup_{x \in E}\sup_{y \in E}\{\langle f, x\rangle - \varphi(x - y) - \psi(y)\} \\
&= \sup_{y \in E}\sup_{x \in E}\{\langle f, x\rangle - \varphi(x - y) - \psi(y)\} \\
&= \varphi^\star(f) + \psi^\star(f).
\end{aligned}$$

2. One has to check that $\forall f, g \in E^\star$ and $\forall x \in E$,

$$\langle f, x\rangle - \varphi(x) - \psi(x) \leq \varphi^\star(f - g) + \psi^\star(g).$$

This becomes obvious by writing

$$\langle f, x\rangle = \langle f - g, x\rangle + \langle g, x\rangle.$$

3. Given $f \in E^\star$, one has to prove that

$$\text{(S1)} \qquad \sup_{x\in E}\{\langle f, x\rangle - \varphi(x) - \psi(x)\} = \inf_{g\in E^\star}\{\varphi^\star(f - g) + \psi^\star(g)\}.$$

Note that

$$\sup_{x\in E}\{\langle f, x\rangle - \varphi(x) - \psi(x)\} = -\inf_{x\in E}\{\tilde{\varphi}(x) + \psi(x)\}$$

with $\tilde{\varphi}(x) = \varphi(x) - \langle f, x\rangle$. Applying Theorem 1.12 to the functions $\tilde{\varphi}$ and ψ leads to

$$\inf_{x\in E}\{\tilde{\varphi}(x) + \psi(x)\} = \sup_{g\in E^\star}\{-\tilde{\varphi}^\star(-g) - \psi^\star(g)\},$$

which corresponds precisely to (S1).

4. Clearly one has

$$\begin{aligned}(\varphi^\star \nabla \psi^\star)^\star(x) &= \sup_{f\in E^\star}\{\langle f, x\rangle - \inf_{g\in E^\star}[\varphi^\star(f - g) + \psi^\star(g)]\}\\ &= \sup_{f\in E^\star}\sup_{g\in E^\star}\{\langle f, x\rangle - \varphi^\star(f - g) - \psi^\star(g)\}\\ &= \sup_{g\in E^\star}\sup_{f\in E^\star}\{\langle f, x\rangle - \varphi^\star(f - g) - \psi^\star(g)\}\\ &= \varphi^{\star\star}(x) + \psi^{\star\star}(x).\end{aligned}$$

$\boxed{1.24}$

1. One knows (Proposition 1.10) that there exist some $f \in E^\star$ and a constant C such that $\varphi(y) \ge \langle f, y\rangle - C \quad \forall y \in E$. Choosing $n \ge \|f\|$, one has $\varphi_n(x) \ge -n\|x\| - C > -\infty$.
2. The function φ_n is the inf-convolution of two convex functions; thus φ_n is convex (see question 7 in Exercise 1.23). In order to prove that $|\varphi_n(x_1) - \varphi_n(x_2)| \le n\|x_1 - x_2\|$, use the same argument as in question 1 of Exercise 1.22.
3. $(\varphi_n)^\star = I_{nB_{E^\star}} + \varphi^\star$ (by question 1 of Exercise 1.23).
5. By question 1 we have $\varphi(y) \ge -\|f\|\,\|y\| - C \;\; \forall y \in E$, which leads to

$$n\|x - y_n\| \le \|f\|\,\|y_n\| + C + \varphi(x) + 1/n.$$

It follows that $\|y_n\|$ remains bounded as $n \to \infty$, and therefore $\lim_{n\to\infty} \|x - y_n\| = 0$. On the other hand, we have $\varphi_n(x) \ge \varphi(y_n) - 1/n$, and since φ is l.s.c. we conclude that $\liminf_{n\to\infty} \varphi_n(x) \ge \varphi(x)$.
6. Suppose, by contradiction, that there exists a constant C such that $\varphi_n(x) \le C$ along a subsequence still denoted by $\varphi_n(x)$. Choosing y_n as in question 5 we see

that $y_n \to x$. Moreover, $\varphi(y_n) \le C + 1/n$ and thus $\varphi(x) \le \liminf_{n\to\infty} \varphi(y_n) \le C$; absurd.

$\boxed{1.25}$

4. For each fixed $t > 0$ the function

$$y \mapsto \frac{1}{2t}\left[\|x+ty\|^2 - \|x\|^2\right]$$

is convex. Thus the function $y \mapsto [x, y]$ is convex as a *limit* of convex functions. On the other hand, $G(x, y) = \sup_{t>0}\{-\frac{1}{2t}[\|x+ty\|^2 - \|x\|^2]\}$ is l.s.c. as a *supremum* of continuous functions.

5. One already knows (see question 3 of Exercise 1.1) that

$$\frac{1}{2}\|x+ty\|^2 - \frac{1}{2}\|x\|^2 \ge \langle f, ty\rangle$$

and therefore

$$[x, y] \ge \langle f, y\rangle \quad \forall x, y \in E,\ \forall f \in F(x).$$

On the other hand, one has

$$\varphi^\star(f) = \frac{1}{2}\|f\|^2 - \langle f, x\rangle + \frac{1}{2}\|x\|^2$$

and

$$\psi^\star(f) = \begin{cases} 0 & \text{if } \langle f, y\rangle + \alpha \le 0,\\ +\infty & \text{if } \langle f, y\rangle + \alpha > 0.\end{cases}$$

It is easy to check that $\inf_{z\in E}\{\varphi(z) + \psi(z)\} = 0$. It follows from Theorem 1.12 that there exists some $f_0 \in E^\star$ such that $\varphi^\star(f_0) + \psi^\star(-f_0) = 0$, i.e., $\langle f_0, y\rangle \ge \alpha$ and $\frac{1}{2}\|f_0\|^2 - \langle f_0, x\rangle + \frac{1}{2}\|x\|^2 = 0$. Consequently, we have $\|f_0\| = \|x\|$ and $\langle f_0, x\rangle = \|x\|^2$, i.e., $f_0 \in F(x)$.

6. (a) $1 < p < \infty$, $[x, y] = \frac{\sum |x_i|^{p-2} x_i y_i}{\|x\|_p^{p-2}}$.
   (b) $p = 1$, $[x, y] = \|x\|_1 \left[\sum_{x_i \ne 0} (\operatorname{sign} x_i) y_i + \sum_{x_i = 0} |y_i|\right]$.
   (c) $p = \infty$, $[x, y] = \max_{i\in I}\{x_i y_i\}$, where $I = \{1 \le i \le n;\ |x_i| = \|x\|_\infty\}$.

$\boxed{1.27}$ Let $\widetilde{T} : E \to F$ be a continuous linear extension of T. It is easy to check that $E = N(\widetilde{T}) + G$ and $N(\widetilde{T}) \cap G = \{0\}$, so that $N(\widetilde{T})$ is a complement of G; absurd.

$\boxed{2.1}$ Without loss of generality we may assume that $x_0 = 0$.

1. Let $X = \{x \in E;\ \|x\| \le \rho\}$ with $\rho > 0$ small enough that $X \subset D(\varphi)$. The sets F_n are closed and $\bigcup_{n=1}^\infty F_n = X$. By the Baire category theorem there is some n_0 such that $\operatorname{Int}(F_{n_0}) \ne \emptyset$. Let $x_1 \in E$ and $\rho_1 > 0$ be such that $B(x_1, \rho_1) \subset F_{n_0}$. Given any $x \in E$ with $\|x\| < \rho_1/2$ write $x = \frac{1}{2}(x_1 + 2x) + \frac{1}{2}(-x_1)$ to conclude that $\varphi(x) \le \frac{1}{2}n_0 + \frac{1}{2}\varphi(-x_1)$.

2. There exist some $\xi \in E$ and some constant $t \in [0,1]$ such that $\|\xi\| = R$ and $x_2 = tx_1 + (1-t)\xi$. It follows that

$$\varphi(x_2) \le t\varphi(x_1) + (1-t)M$$

and consequently $\varphi(x_2) - \varphi(x_1) \le (1-t)[M - \varphi(x_1)]$. But $x_2 - x_1 = (1-t)(\xi - x_1)$ and thus $\|x_2 - x_1\| \ge (1-t)(R-r)$. Hence we have

$$\varphi(x_2) - \varphi(x_1) \le \frac{\|x_2 - x_1\|}{R-r}[M - \varphi(x_1)].$$

On the other hand, if $x_2 = 0$ one obtains $t\|x_1\| = (1-t)R$ and therefore

$$(1-t) = \frac{\|x_1\|}{\|x_1\| + R} \le \frac{1}{2}.$$

It follows that $\varphi(0) - \varphi(x_1) \le \frac{1}{2}[M - \varphi(x_1)]$, so that $M - \varphi(x_1) \le 2[M - \varphi(0)]$.

$\boxed{2.2}$ We have $p(0) \le p(x_n) + p(-x_n) \to 0$, so that $p(0) \le 0$. On the other hand $p(0) \le 2p(0)$ by (i). Thus $p(0) = 0$.

Next we prove that $p(\alpha_n x_n) \to 0$. Argue by contradiction and assume that $|p(\alpha_n x_n)| > 2\varepsilon$ along a subsequence, for some $\varepsilon > 0$. Passing to a further subsequence we may assume that $\alpha_n \to \alpha$ for some $\alpha \in \mathbb{R}$. For simplicity we still denote (x_n) and (α_n) the corresponding sequences.

The sets F_n are closed and $\bigcup_{n \ge 1} F_n = \mathbb{R}$. Applying the Baire category theorem, we find some n_0 such that Int $F_{n_0} \ne \emptyset$. Hence, there exist some $\lambda_0 \in \mathbb{R}$ and some $\delta > 0$ such that $|p((\lambda_0 + t)x_k)| \le \varepsilon \ \forall k \ge n_0$, $\forall t$ with $|t| < \delta$. On the other hand, note that

$$p(\alpha_k x_k) \le p((\lambda_0 + \alpha_k - \alpha)x_k) + p((\alpha - \lambda_0)x_k),$$
$$-p(\alpha_k x_k) \le -p((\lambda_0 + \alpha_k - \alpha)x_k) + p((\lambda_0 - \alpha)x_k).$$

Hence we obtain $|p(\alpha_k x_k)| \le 2\varepsilon$ for k large enough. A contradiction.

Finally, write

$$p(\alpha_n x_n) - p(\alpha x) \le p(\alpha_n(x_n - x)) + p(\alpha_n x) - p(\alpha x) \to 0$$

and

$$p(\alpha_n x) \le p(\alpha_n(x - x_n)) + p(\alpha_n x_n),$$

so that

$$p(\alpha_n x_n) - p(\alpha x) \ge -p(\alpha_n(x_n - x)) + p(\alpha_n x) - p(\alpha x) \to 0.$$

$\boxed{2.4}$ By (i) there exists a linear operator $T : E \to F^\star$ such that $a(x,y) = \langle Tx, y\rangle_{F^\star,F}$ $\forall x, y$. The aim is to show that T is a bounded operator, i.e., $T(B_E)$ is bounded in $F^\star$. In view of Corollary 2.5 it suffices to *fix* $y \in F$ and to check that $\langle T(B_E), y\rangle$ is bounded. This follows from (ii).

$\boxed{2.6}$

1. One has $\langle Ax_n - A(x_0+x), x_n - x_0 - x\rangle \geq 0$ and thus $\langle Ax_n, x\rangle \leq \varepsilon_n \|Ax_n\| + C(x)$ with $\varepsilon_n = \|x_n - x_0\|$ and $C(x) = \|A(x_0+x)\|(1+\|x\|)$ (assuming $\varepsilon_n \leq 1\ \forall n$). It follows from Exercise 2.5 that (Ax_n) is bounded; absurd.
2. Assume that there is a sequence (x_n) in $D(A)$ such that $x_n \to x_0$ and $\|Ax_n\| \to \infty$. Choose $r > 0$ such that $B(x_0, r) \subset \operatorname{conv} D(A)$. For every $x \in E$ with $\|x\| < r$ write
$$x_0 + x = \sum_{i=1}^m t_i y_i \text{ with } t_i \geq 0\ \forall i, \quad \sum_{i=1}^m t_i = 1, \quad \text{and} \quad y_i \in D(A)\ \forall i$$
(of course t_i, y_i, and m depend on x). We have
$$\langle Ax_n - Ay_i, x_n - y_i\rangle \geq 0$$
and thus $t_i\langle Ax_n, x_n - y_i\rangle \geq t_i\langle Ay_i, x_n - y_i\rangle$. It follows that
$$\langle Ax_n, x_n - x_0 - x\rangle \geq \sum_{i=1}^m t_i\langle Ay_i, x_n - y_i\rangle,$$
which leads to
$$\langle Ax_n, x\rangle \leq \varepsilon_n\|Ax_n\| + C(x)$$
with $\varepsilon_n = \|x_n - x_0\|$ and $C(x) = \sum_{i=1}^m t_i\|Ay_i\|(1+\|x_0 - y_i\|)$.
3. Let $x_0 \in \operatorname{Int} D(A)$. Following the same argument as in question 1, one shows that there exist two constants $R > 0$ and C such that
$$\|f\| \leq C \quad \forall x \in D(A) \text{ with } \|x - x_0\| < R \text{ and } \forall f \in Ax.$$

$\boxed{2.7}$ For every $x \in \ell^p$ set $T_n x = \sum_{i=1}^n \alpha_i x_i$, so that $T_n x$ converges to a limit for every $x \in \ell^p$. It follows from Corollary 2.3 that there exists a constant C such that
$$|T_n x| \leq C\|x\|_{\ell^p} \quad \forall x \in \ell^p, \quad \forall n.$$
Choosing x appropriately, one sees that $\alpha \in \ell^{p'}$ and $\|\alpha\|_{p'} \leq C$.

$\boxed{2.8}$ Method (ii). Let us check that the graph of T is closed. Let (x_n) be a sequence in E such that $x_n \to x$ and $Tx_n \to f$. Passing to the limit in the inequality $\langle Tx_n - Ty, x_n - y\rangle \geq 0$ leads to
$$\langle f - Ty, x - y\rangle \geq 0 \quad \forall y \in E.$$
Choosing $y = x + tz$ with $t \in \mathbb{R}$ and $z \in E$, one sees that $f = Tx$.

$\boxed{2.10}$

1. If $T(M)$ is closed then $M + N(T) = T^{-1}(T(M))$ is also closed. Conversely, assume that $M+N(T)$ is closed. Since T is surjective, one has $T((M+N(T))^c) =$

$(T(M))^c$. The open mapping theorem implies that $T((M+N(T))^c)$ is open and thus $T(M)$ is closed.
2. If M is any closed subspace and N is any finite-dimensional space then $M+N$ is closed (see Section 11.1).

$\boxed{2.11}$ By the open mapping theorem there is a constant $c>0$ such that $T(B_E)\supset cB_F$. Let (e_n) denote the canonical basis of ℓ^1, i.e.,

$$e_n=(0,0,\dots,0,\underset{(n)}{1},0,\dots).$$

There exists some $u_n\in E$ such that $\|u_n\|\le 1/c$ and $T(u_n)=e_n$. Given $y=(y_1,y_2,\dots,y_n,\dots)\in\ell^1$, set $Sy=\sum_{i=1}^{\infty}y_ie_i$. Clearly the series converges and S has all the required properties.

$\boxed{2.12}$ Without loss of generality we may assume that T is *surjective* (otherwise, replace E by $R(T)$). Assume by contradiction that there is a sequence (x_n) in E such that

$$\|x_n\|_E=1 \quad\text{and}\quad \|Tx_n\|_F+|x_n|<1/n.$$

By the open mapping theorem there is a constant $c>0$ such that $T(B_E)\supset cB_F$. Since $\|Tx_n\|_F<1/n$, there exists some $y_n\in E$ such that

$$Tx_n=Ty_n \quad\text{and}\quad \|y_n\|_E<1/nc.$$

Write $x_n=y_n+z_n$ with $z_n\in N(T)$, $\|y_n\|_E\to 0$ and $\|z_n\|_E\to 1$. On the other hand, $|x_n|<1/n$; hence $|z_n|<(1/n)+|y_n|\le(1/n)+M\|y_n\|_E$, and consequently $|z_n|\to 0$. This is impossible, since the norms $\|\ \|_E$ and $|\ |$ are equivalent on the finite-dimensional space $N(T)$.

$\boxed{2.13}$ First, let $T\in\mathcal{O}$ so that $T^{-1}\in\mathcal{L}(F,E)$ (by Corollary 2.7). Then $T+U\in\mathcal{O}$ for every $U\in\mathcal{L}(E,F)$ with $\|U\|$ small enough. Indeed, the equation $Tx+Ux=f$ may be written as $x=T^{-1}(f-Ux)$; it has a unique solution (for every $f\in F$) provided $\|T^{-1}\|\,\|U\|<1$ (by Banach's fixed-point theorem; see Theorem 5.7).

Next, let $T\in\Omega$. In view of Theorem 2.13, $R(T)$ is closed and has a complement in F. Let $P:F\to R(T)$ be a continuous projection. The operator PT is bijective from E onto $R(T)$ and hence the above analysis applies. Let $U\in\mathcal{L}(E,F)$ be such that $\|U\|<\delta$; the operator $(PT+PU):E\to R(T)$ is bijective if δ is small enough and thus $(PT+PU)^{-1}$ is well-defined as an element of $\mathcal{L}(R(T),E)$. Set $S=(PT+PU)^{-1}P$. Clearly $S\in\mathcal{L}(F,E)$ and $S(T+U)=I_E$.

$\boxed{2.14}$

1. Consider the quotient space $\widetilde{E}=E/N(T)$ and the canonical surjection $\pi:E\to\widetilde{E}$, so that $\|\pi x\|_{\widetilde{E}}=\operatorname{dist}(x,N(T))\ \forall x\in E$. T induces an injective operator $\widetilde{T}$ on $\widetilde{E}$. More precisely, write $T=\widetilde{T}\circ\pi$ with $\widetilde{T}\in\mathcal{L}(\tilde{E},F)$, so that $R(T)=R(\widetilde{T})$.

   On the other hand, Corollary 2.7 shows that $R(\widetilde{T})$ is closed iff there is a constant C such that

$$\|y\|_{\widetilde{E}} \le C\|\widetilde{T}y\| \quad \forall y \in \widetilde{E},$$

or equivalently

$$\|\pi x\|_{\widetilde{E}} \le C\|\widetilde{T}\pi x\| \quad \forall x \in E.$$

The last inequality reads

$$\operatorname{dist}(x, N(T)) \le C\|Tx\| \quad \forall x \in E.$$

$\boxed{2.15}$ The operator $T : E_1 \times E_2 \to F$ is linear, bounded, and surjective. Moreover, $N(T) = N(T_1) \times N(T_2)$ (since $R(T_1) \cap R(T_2) = \{0\}$). Applying Exercise 2.10 with $M = E_1 \times \{0\}$, one sees that $T(M) = R(T_1)$ is closed provided $M + N(T)$ is closed. But $M + N(T) = E_1 \times N(T_2)$ is indeed closed.

$\boxed{2.16}$ Let π denote the canonical surjection from E onto E/L (see Section 11.2). Consider the operator $T : G \to E/L$ defined by $Tx = \pi x$ for $x \in G$. We have

$$\operatorname{dist}(x, N(T)) = \operatorname{dist}(x, G \cap L) \le C \operatorname{dist}(x, L) = C\|Tx\| \quad \forall x \in G.$$

It follows (see Exercise 2.14) that $R(T) = \pi(G)$ is closed. Therefore $\pi^{-1}[\pi(G)] = G + L$ is closed.

$\boxed{2.19}$ Recall that $N(A^\star) = R(A)^\perp$.

1. Let $u \in N(A)$ and $v \in D(A)$; we have

$$\langle A(u + tv), u + tv\rangle \ge -C\|A(u + tv)\|^2 \quad \forall t \in \mathbb{R},$$

   which implies that $\langle Av, u\rangle = 0$. Thus $N(A) \subset R(A)^\perp$.
2. $D(A)$ equipped with the graph norm is a Banach space. $R(A)$ equipped with the norm of $E^\star$ is a Banach space. The operator $A : D(A) \to R(A)$ satisfies the assumptions of the open mapping theorem. Hence there is a constant C such that

$$\forall f \in R(A), \exists v \in D(A) \text{ with } Av = f \text{ and } \|v\|_{D(A)} \le C\|f\|.$$

   In particular, $\|v\| \le C\|f\|$. Given $u \in D(A)$, the above result applied to $f = Au$ shows that there is some $v \in D(A)$ such that $Au = Av$ and $\|v\| \le C\|Au\|$. Since $u - v \in N(A) \subset R(A)^\perp$, we have

$$\langle Au, u\rangle = \langle Av, u\rangle = \langle Av, v\rangle \ge -\|Av\|\,\|v\| \ge -C\|Au\|^2.$$

$\boxed{2.21}$

1. Distinguish two cases:

   Case (i): $f(a) = 1$. Then $N(A) = \mathbb{R}a$ and $R(A) = N(f)$.
   Case (ii): $f(a) \ne 1$. Then $N(A) = \{0\}$ and $R(A) = E$.
2. A is not closed. Otherwise the closed graph theorem would imply that A is bounded and consequently that f is continuous.
3. $D(A^\star) = \{u \in E^\star; \langle u, a\rangle = 0\}$ and $A^\star u = u \quad \forall u \in D(A^\star)$.

4. $N(A^\star) = \{0\}$ and $R(A^\star) = \{u \in E^\star; \langle u, a\rangle = 0\}$.
5. $R(A)^\perp = \{0\}$ and $R(A^\star)^\perp = \mathbb{R}a$ (note that $N(f)$ is dense in E; see Exercise 1.6). It follows that $N(A^\star) = R(A)^\perp$ and $N(A) \subset R(A^\star)^\perp$.
   Observe that in Case (ii), $N(A) \neq R(A^\star)^\perp$.
6. If A is not closed it may happen that $N(A) \neq R(A^\star)^\perp$.

$\boxed{2.22}$

1. Clearly $D(A)$ is dense in E. In order to check that A is closed let (u^j) be a sequence in $D(A)$ such that $u^j \to u$ in E and $Au^j \to f$ in E. It follows that

$$u_n^j \underset{j\to\infty}{\longrightarrow} u_n \quad \forall n \quad \text{and} \quad nu_n^j \underset{j\to\infty}{\longrightarrow} f_n \quad \forall n.$$

   Thus $nu_n = f_n\ \forall n,$ so that $u \in D(A)$ and $Au = f$.
2.
$$D(A^\star) = \{v = (v_n) \in \ell^\infty; (nv_n) \in \ell^\infty\},$$
$$A^\star v = (nv_n) \text{ and } \overline{D(A^\star)} = c_0.$$

$\boxed{2.24}$

1. We have $D(B^\star) = \{v \in G^\star; T^\star v \in D(A^\star)\}$ and $B^\star = A^\star T^\star$.
2. If $D(A) \neq E$ and $T = 0$, then B is not closed. Indeed, let (u_n) be a sequence in $D(A)$ such that $u_n \to u$ with $u \notin D(A)$. Then $Bu_n \to 0$ but $u \notin D(B)$.

$\boxed{2.25}$

2. By Corollary 2.7, $T^{-1} \in \mathcal{L}(F, E)$. Since $T^{-1}T = I_E$ and $TT^{-1} = I_F$, it follows that $T^\star(T^{-1})^\star = I_{E^\star}$ and $(T^{-1})^\star T^\star = I_{F^\star}$.

$\boxed{2.26}$ We have

$$\varphi^\star(T^\star f) = \sup_{x\in E}\{\langle T^\star f, x\rangle - \varphi(x)\} = \sup_{y\in R(T)}\{\langle f, y\rangle - \psi(y)\} = -\inf_{y\in F}\{\psi(y) + \zeta(y)\},$$

where $\zeta(y) = -\langle f, y\rangle + I_{R(T)}(y)$. Applying Theorem 1.12, we obtain

$$\varphi^\star(T^\star f) = \min_{g\in F^\star}\{\zeta^\star(g) + \psi^\star(-g)\}.$$

But

$$\zeta^\star(g) = \begin{cases} 0 & \text{if } f + g \in R(T)^\perp, \\ +\infty & \text{if } f + g \notin R(T)^\perp, \end{cases}$$

and thus

$$\varphi^\star(T^\star f) = \min_{f+g\in N(T^\star)} \psi^\star(-g) = \min_{h\in N(T^\star)} \psi^\star(f - h).$$

$\boxed{2.27}$ Let $G = E \times X$ and consider the operator

$$S(x, y) = Tx + y : G \to F.$$

Applying the open mapping theorem, we know that S is an open map, and thus $S(E \times (X \setminus \{0\})) = R(T) + (X \setminus \{0\})$ is open in F. Hence its complement, $R(T)$, is closed.

$\boxed{3.1}$ Apply Corollary 2.4.

$\boxed{3.2}$ Note that $\langle f, \sigma_n \rangle = \frac{1}{n} \sum_{i=1}^{n} \langle f, x_i \rangle \quad \forall f \in E^\star$. Since $\langle f, x_n \rangle \to \langle f, x \rangle$, it follows that $\langle f, \sigma_n \rangle \to \langle f, x \rangle$.

$\boxed{3.4}$

1. Set $G_n = \operatorname{conv}\left(\bigcup_{i=n}^{\infty} \{x_i\}\right)$. Since $x_n \rightharpoonup x$ for the topology $\sigma(E, E^\star)$ it follows that $x \in \overline{G_n}^{\sigma(E,E^\star)} \quad \forall n$. On the other hand, G_n being convex, its closure for the weak topology $\sigma(E, E^\star)$ and that for the strong topology are the same (see Exercise 3.3). Hence $x \in \overline{G_n}$ $\forall n$ (the strong closure of G_n) and there exists a sequence (y_n) such that $y_n \in G_n$ $\forall n$ and $y_n \to x$ strongly.
2. There exists a sequence (u_k) in E such that $u_k \to x$ and $u_k \in \operatorname{conv}\left(\bigcup_{i=1}^{\infty} \{x_i\}\right)$ $\forall k$. Hence there exists an increasing sequence of integers (n_k) such that
$$u_k \in \operatorname{conv}\left(\bigcup_{i=1}^{n_k} \{x_i\}\right) \quad \forall k.$$
The sequence (z_n) defined by $z_n = u_k$ for $n_k \le n < n_{k+1}$ (and $z_n = x_1$ for $1 \le n < n_1$) has all the required properties.

$\boxed{3.7}$

1. Let $x \notin A + B$. We shall construct a neighborhood W of 0 for $\sigma(E, E^\star)$ such that
$$(x + W) \cap (A + B) = \emptyset.$$
For every $y \in B$ there exists a convex neighborhood $V(y)$ of 0 such that
$$(x + V(y)) \cap (A + y) = \emptyset$$
(since $A + y$ is closed and $x \notin A + y$).
Clearly
$$B \subset \bigcup_{y \in B} \left(y - \frac{1}{2} V(y)\right),$$
and since B is compact, there is some finite set I such that
$$B \subset \bigcup_{i \in I} \left(y_i - \frac{1}{2} V(y_i)\right) \text{ with } y_i \in B.$$
Set
$$W = \frac{1}{2} \bigcap_{i \in I} V(y_i).$$

We claim that $(x + W) \cap (A + B) = \emptyset$. Indeed, suppose by contradiction that there exists some $w \in W$ such that $x + w \in (A + B)$. Hence there is some $i \in I$ such that

$$x + w \in A + y_i - \frac{1}{2}V(y_i).$$

Since $V(y_i)$ is convex it follows that there exists some $w' \in V(y_i)$ such that $x + w' \in A + y_i$. Consequently $(x + V(y_i)) \cap (A + y_i) \neq \emptyset$; absurd.

Remark. If $E^\star$ is *separable* and A is *bounded* one may use *sequences* in order to prove that $A + B$ is closed, since the weak topology is metrizable on bounded sets (see Theorem 3.29). This makes the argument somewhat easier. Indeed, let $x_n = a_n + b_n$ be a sequence such that $x_n \rightharpoonup x$ weakly $\sigma(E, E^\star)$ with $a_n \in A$ and $b_n \in B$. Since B is weakly compact (and metrizable), there is a subsequence such that $b_{n_k} \rightharpoonup b$ weakly $\sigma(E, E^\star)$ with $b \in B$. Thus $a_{n_k} \rightharpoonup x - b$ weakly $\sigma(E, E^\star)$. But A is weakly closed and therefore $x - b \in A$, i.e., $x \in A + B$.

2. By question 1, $(A - B)$ is weakly closed and therefore it is strongly closed. Hence one may strictly separate $\{0\}$ and $(A - B)$.

$\boxed{3.8}$

1. Since V_k is a neighborhood of 0 for $\sigma(E, E^\star)$, one may assume (see Proposition 3.4) that V_k has the form

$$V_k = \{x \in E; |\langle f, x\rangle| < \varepsilon_k \ \forall f \in F_k\},$$

where $\varepsilon_k > 0$ and F_k is a *finite* subset of $E^\star$. Hence the set $F = \bigcup_{k=1}^{\infty} F_k$ is countable. We claim that any $g \in E^\star$ can be written as a finite linear combination of elements in F. Indeed, set

$$V = \{x \in E; |\langle g, x\rangle| < 1\}.$$

Since V is neighborhood of 0 for $\sigma(E, E^\star)$, there exists some integer m such that $\{x \in E; d(x, 0) < 1/m\} \subset V$ and consequently $V_m \subset V$. Suppose $x \in E$ is such that $\langle f, x\rangle = 0 \ \forall f \in F_m$. Then $tx \in V_m \ \forall t \in \mathbb{R}$ and thus $tx \in V \ \forall t \in \mathbb{R}$, i.e., $\langle g, x\rangle = 0$. Applying Lemma 3.2, we see that g is a linear combination of elements in F_m.
2. Use the same method as in question 3 of Exercise 1.5.
3. If $\dim E^\star < \infty$, then $\dim E^{\star\star} < \infty$; consequently $\dim E < \infty$ (since there is a canonical injection from E into $E^{\star\star}$).
4. Apply the following lemma (which is an easy consequence of Lemma 3.2): Assume that $x_1, x_2, \dots, x_k, y \in E$ satisfy

$$[f \in E^\star; \langle f, x_i\rangle = 0 \ \forall i] \Rightarrow [\langle f, y\rangle = 0].$$

Then there exist constants $\lambda_1, \lambda_2, \dots, \lambda_k$ such that $y = \sum_{i=1}^{k} \lambda_i x_i$.

$\boxed{3.9}$

1. Apply Theorem 1.12 with

$$\varphi(x) = \langle f_0, x\rangle + I_{B_E}(x) \quad \text{and} \quad \psi(x) = I_M(x).$$

2. Note that $B_{E^\star}$ is compact for $\sigma(E^\star, E)$, while $M^\perp$ is closed for $\sigma(E^\star, E)$ (why?).

$\boxed{3.11}$ It suffices to argue on sequences (why?). Assume $x_n \to x$ strongly in E and $Ax_n \not\rightharpoonup Ax$ for $\sigma(E^\star, E)$, i.e., there exists some $y \in E$ such that $\langle Ax_n, y\rangle \not\to \langle Ax, y\rangle$. We already know (by Exercise 2.6) that (Ax_n) is bounded. Hence, there is a subsequence such that $\langle Ax_{n_k}, y\rangle \to \ell \neq \langle Ax, y\rangle$. Applying the monotonicity of A, we have

$$\langle Ax_{n_k} - A(x + ty), x_{n_k} - x - ty\rangle \geq 0.$$

Passing to the limit, we obtain

$$-t\ell + t\langle A(x + ty), y\rangle \geq 0,$$

which implies that $\ell = \langle Ax, y\rangle$; absurd.

$\boxed{3.12}$

1. Assumption (A) implies that $\varphi^\star(f) \geq R\|f\| + \langle f, x_0\rangle - M \quad \forall f \in E^\star$. Conversely, assume that (B) holds and set $\psi(f) = \varphi^\star(f) - \langle f, x_0\rangle$. We claim that there exist constants $k > 0$ and C such that

   (S1) $$\psi(f) \geq k\|f\| - C \quad \forall f \in E^\star.$$

   After a translation we may always assume that $\psi(0) < \infty$ (see Proposition 1.10). Fix $\alpha > \psi(0)$. Using assumption (B) we may find some $r > 0$ such that

   $$\psi(g) \geq \alpha \quad \forall g \in E^\star \text{ with } \|g\| \geq r.$$

   Given $f \in E^\star$ with $\|f\| \geq r$ write

   $$\psi(tf) \leq t\psi(f) + (1 - t)\psi(0) \text{ with } t = r/\|f\|.$$

   Since $\|tf\| = r$, this leads to $\alpha - \psi(0) \leq \frac{r}{\|f\|}(\psi(f) - \psi(0))$, which establishes claim (S1). Passing to the conjugate of (S1) we obtain (A).
2. The function ψ is convex and l.s.c. for the weak* topology (why?). Assumption (B) says that for every $\lambda \in \mathbb{R}$ the set $\{f \in E^\star;\ \psi(f) \leq \lambda\}$ is bounded. Hence, it is weak* compact (by Theorem 3.16), and thus $\inf_{E^\star} \psi$ is achieved. On the other hand,

   $$\inf_{E^\star} \psi = - \sup_{f\in E^\star} \{\langle f, x_0\rangle - \varphi^\star(f)\} = -\varphi^{\star\star}(x_0) = -\varphi(x_0).$$

   Alternatively, one could also apply Theorem 1.12 to the functions φ and $I_{\{x_0\}}$ (note that φ is continuous at x_0; see Exercise 2.1).

$\boxed{3.13}$

1. For every fixed p we have $x_{p+n} \in K_p$ $\forall n$. Passing to the limit (as $n \to \infty$) we see that $x \in K_p$ since K_p is weakly closed (see Theorem 3.7). On the other hand,

let V be a convex neighborhood of x for the topology $\sigma(E, E^\star)$. There exists an integer N such that $x_n \in V$ $\forall n \geq N$. Thus $K_n \subset \overline{V}$ $\forall n \geq N$ and consequently $\bigcap_{n=1}^\infty K_n \subset \overline{V}$. Since this is true for any convex neighborhood V of x, it follows that $\bigcap_{n=1}^\infty K_n \subset \{x\}$ (why?).

2. Let V be an open neighborhood of x for the topology $\sigma(E, E^\star)$. Set $K_n' = K_n \cap (V^c)$. Since K_n is compact for $\sigma(E, E^\star)$ (why?), it follows that K_n' is also compact for $\sigma(E, E^\star)$. On the other hand, $\bigcap_{n=1}^\infty K_n' = \emptyset$, and hence there is some integer N such that $K_N' = \emptyset$, i.e., $K_N \subset V$.
3. We may assume that $x = 0$. Consider the recession cone

$$C_n = \bigcap_{\lambda > 0} \lambda K_n.$$

Since $C_n \subset K_n$ we deduce that $\bigcap_{n=1}^\infty C_n = \{0\}$. Let $S_E = \{x \in E;\ \|x\| = 1\}$. The sequence $(C_n \cap S)$ is decreasing and $\bigcap_{n=1}^\infty (C_n \cap S) = \emptyset$. Thus, by compactness, $C_{n_0} \cap S = \emptyset$ for some n_0. Therefore $C_{n_0} = \{0\}$ and consequently K_{n_0} is bounded (why?). Hence (x_n) is bounded and we are reduced to question 2.
4. Consider the sequence $x_n = (0, 0, \dots, \underset{(n)}{n}, 0, \dots)$, when n is odd, and $x_n = 0$ when n is even.

$\boxed{3.18}$

2. Suppose, by contradiction, that $e^{n_k} \underset{k\to\infty}{\rightharpoonup} a$ in ℓ^1 for the topology $\sigma(\ell^1, \ell^\infty)$. Thus we have $\langle \xi, e^{n_k}\rangle \underset{k\to\infty}{\longrightarrow} \langle \xi, a\rangle$ $\forall \xi \in \ell^\infty$. Consider the element $\xi \in \ell^\infty$ defined by

$$\xi = (0, 0, \dots, \underset{(n_1)}{-1}, 0, \dots, \underset{(n_2)}{1}, 0, \dots, \underset{(n_3)}{-1}, 0, \dots).$$

Note that $\langle \xi, e^{n_k}\rangle = (-1)^k$ does not converge as $k \to \infty$; a contradiction.
3. Let $E = \ell^\infty$, so that $\ell^1 \subset E^\star$. Set $f_n = e^n$, considered as a sequence in $E^\star$. We claim that (f_n) has no subsequence that converges for $\sigma(E^\star, E)$. Suppose, by contradiction, that $f_{n_k} \overset{\star}{\rightharpoonup} f$ in $E^\star$ for $\sigma(E^\star, E)$, i.e., $\langle f_{n_k}, \eta\rangle \to \langle f, \eta\rangle$ $\forall \eta \in E$. Choosing $\eta = \xi$ as in question 2, we see that $\langle f_{n_k}, \xi\rangle = (-1)^k$ does not converge; a contradiction. Here, the set $B_{E^\star}$, equipped with the topology $\sigma(E^\star, E)$ is compact (by Theorem 3.16), but it is *not metrizable*. Applying Theorem 3.28, we may also say that $E = \ell^\infty$ is *not separable* (for another proof see Remark 8 in Chapter 4 and Proposition 11.17).

$\boxed{3.19}$

1. Note that if $x^n \to x$ strongly in ℓ^p, then

$$\forall \varepsilon > 0 \quad \exists I \text{ such that } \sum_{i=I}^\infty |x_i^n|^p \leq \varepsilon^p \ \forall n.$$

2. Apply Exercise 3.17.

3. The space B_E is metrizable for the topology $\sigma(E, E^\star)$ (by Theorem 3.29). Thus it suffices to check the continuity of A on *sequences*.

$\boxed{3.20}$

1. Consider the map $T : E \to C(K)$ defined by

$$(Tx)(t) = \langle t, x\rangle \text{ with } x \in E \text{ and } t \in B_{E^\star} = K.$$

Clearly $\|Tx\| = \sup_{t\in K} |(Tx)(t)| = \|x\|$.

2. Since $K = B_{E^\star}$ is metrizable and compact for $\sigma(E^\star, E)$, there is a dense countable subset (t_n) in K. Consider the map $S : E \to \ell^\infty$ defined by

$$Sx = (\langle t_1, x\rangle, \langle t_2, x\rangle, \ldots, \langle t_n, x\rangle, \ldots).$$

Check that $\|Sx\|_{\ell^\infty} = \|x\|$.

$\boxed{3.21}$ Let (a_i) be a dense countable subset of E. Choose a first subsequence such that $\langle f_{n_k}, a_1\rangle$ converges to a limit as $k \to \infty$. Then, pick a subsequence out of (n_k) such that $\langle f_{n'_k}, a_2\rangle$ converges, etc.

By a standard diagonal process we may extract a sequence (g_k) out of the sequence (f_n) such that $\langle g_k, a_i\rangle \underset{k\to\infty}{\longrightarrow} \ell_i \ \forall i$. Since the set (a_i) is dense in E, we easily obtain that $\langle g_k, a\rangle \to \ell_a \ \forall a \in E$. It follows that g_k converges for $\sigma(E^\star, E)$ to some g (see Exercise 3.16).

$\boxed{3.22}$

(a) B_E is metrizable for $\sigma(E, E^\star)$ (by Theorem 3.29) and 0 belongs to the closure of $S = \{x \in E;\ \|x\| = 1\}$ for $\sigma(E, E^\star)$ (see Remark 2 in Chapter 3).

(b) Since $\dim E = \infty$ there is a closed subspace E_0 in E that is separable and such that $\dim E_0 = \infty$ (why?). Note that E_0 is reflexive and apply Case (a) (in conjunction with Corollary 3.27).

$\boxed{3.25}$ Suppose, by contradiction, that $C(K)$ is reflexive. Then $E = \{u \in C(K);\ u(a) = 0\}$ is also reflexive and $\sup_{u\in B_E} f(u)$ is achieved.

On the other hand, we claim that $\sup_{u\in B_E} f(u) = 1$. Indeed, $\forall N, \exists u \in E$ such that $0 \le u \le 1$ on K and $u(a_i) = 1\ \forall i = 1, 2, \ldots, N$. (Apply, for example, the Tietze–Urysohn theorem; see, e.g., J. Munkres [1].) Hence there exists some $u \in B_E$ such that $f(u) = 1$. This leads to $u(a_n) = 1\ \forall n$ and $u(a) = 0$; absurd.

$\boxed{3.26}$

1. Given $y \in B_F$, there is some integer n_1 such that $\|y - a_{n_1}\| < 1/2$. Since the set $\frac{1}{2}(a_i)_{i>n_1}$ is dense in $\frac{1}{2}B_F$, there is some $n_2 > n_1$ such that

$$\left\| y - a_{n_1} - \frac{1}{2}a_{n_2} \right\| < \frac{1}{4}.$$

Construct by induction an increasing sequence $n_k \uparrow \infty$ of integers such that

$$y = a_{n_1} + \frac{1}{2}a_{n_2} + \frac{1}{4}a_{n_3} + \cdots + \frac{1}{2^{k-1}}a_{n_k} + \cdots .$$

2. Suppose, by contradiction, that $S \in \mathcal{L}(F, \ell^1)$ is such that $TS = I_F$. Let (y_n) be any sequence in F such that $\|y_n\| = 1$ $\forall n$ and $y_n \rightharpoonup 0$ weakly $\sigma(F, F^\star)$. Thus $Sy_n \rightharpoonup 0$ for $\sigma(\ell^1, \ell^\infty)$ and consequently $Sy_n \to 0$ strongly in ℓ^1 (see Problem 8). It follows that $y_n = TSy_n \to 0$; absurd.
3. Use Theorem 2.12.
4. $T^\star : F^\star \to \ell^\infty$ is defined by

$$T^\star v = (\langle v, a_1\rangle, \langle v, a_2\rangle, \ldots, \langle v, a_n\rangle, \ldots).$$

$\boxed{3.27}$ $B_{E^\star}$ is compact and metrizable for $\sigma(E^\star, E)$. Hence there exists a countable subset of $B_{E^\star}$ that is dense for $\sigma(E^\star, E)$.

1. Clearly $\|f\| \le \|f\|_1 \le \sqrt{2}\|f\|$ $\forall f \in E^\star$.
2. Set $|f|^2 = \sum_{n=1}^\infty \frac{1}{2^n}|\langle f, a_n\rangle|^2$. Note that the *norm* $|\ |$ is associated to a scalar product (why?), and thus it is strictly convex, i.e., the *function* $f \mapsto |f|^2$ is strictly convex. More precisely, we have $\forall t \in [0, 1]$, $\forall f, g \in E^\star$,

$$\text{(S1)} \qquad |tf + (1-t)g|^2 + t(1-t)|f-g|^2 = t|f|^2 + (1-t)|g|^2.$$

   Consequently, the function $f \mapsto \|f\|^2 + |f|^2$ is also strictly convex.
3. Same method as in question 2. Note that if $\langle b_n, x\rangle = 0$ $\forall n$, then $x = 0$ (why?).
4. Given $x \in E$ set $[x] = \{\sum_{n=1}^\infty \frac{1}{2^n}|\langle b_n, x\rangle|^2\}^{1/2}$, and let $[f]$ denote the dual norm of $[\ \ ]$ on $E^\star$. Note that $[f]$ also satisfies the identity (S1). Indeed, we have

$$\frac{1}{2}[tf + (1-t)g]^2 = \sup_{x\in E}\left\{\langle tf + (1-t)g, x\rangle - \frac{1}{2}[x]^2\right\},$$
$$\frac{1}{2}[f-g]^2 = \sup_{y\in E}\left\{\langle f-g, y\rangle - \frac{1}{2}[y]^2\right\},$$

   and thus

$$\frac{1}{2}[tf + (1-t)g]^2 + \frac{1}{2}t(1-t)[f-g]^2$$
$$= \sup_{x,y}\left\{\langle tf + (1-t)g, x\rangle + t(1-t)\langle f-g, y\rangle - \frac{1}{2}[x]^2 - \frac{1}{2}t(1-t)[y]^2\right\}.$$

   We conclude that (S1) holds by a change of variables $x = t\xi + (1-t)\eta$ and $y = \xi - \eta$. Applying question 3 of Exercise 1.23, we see that

$$\|f\|_2^2 = \inf_{h\in E^\star}\left\{\|f-h\|_1 + [h]^2\right\} = \min_{h\in E^\star}\left\{\|f-h\|_1^2 + [h]^2\right\}.$$

   We claim that the function $f \mapsto \|f\|_2^2$ is strictly convex. Indeed, given $f, g \in E^\star$, fix $h_1, h_2 \in E^\star$ such that

$$\|f\|_2^2 = \|f-h_1\|_2^2 + [h_1]^2,$$

$$\|g\|_2^2 = \|g - h_2\|_1^2 + [h_2]^2.$$

For every $t \in (0, 1)$ we have

$$\begin{aligned}\|tf + (1-t)g\|_2^2 &\le \|tf + (1-t)g - (th_1 + (1-t)h_2)\|_1^2 + [th_1 + (1-t)h_2]^2 \\ &< t\|f\|_2^2 + (1-t)\|g\|_2^2,\end{aligned}$$

unless $f - h_1 = g - h_2$ and $h_1 = h_2$, i.e., $f = g$.

$\boxed{3.28}$ Since E is reflexive, $\sup_{x \in B_E} \langle f, x\rangle$ is achieved by some unique point $x_0 \in B_E$. Then $x = x_0\|f\|$ satisfies $f \in F(x)$.

Alternatively, we may also consider the duality map $F^\star$ from $E^\star$ into $E^{\star\star}$. The set $F^\star(f)$ is nonempty (by Corollary 1.3). Fix any $\xi \in F^\star(f)$. Since E is reflexive there exists some $x \in E$ such that $Jx = \xi$ (J is the canonical injection from E into $E^{\star\star}$). We have

$$\|\xi\| = \|f\| = \|x\| \quad \text{and} \quad \langle \xi, f\rangle = \|f\|^2 = \langle f, x\rangle.$$

Thus $f \in F(x)$.

Uniqueness. Let x_1 and x_2 be such that $f \in F(x_1)$ and $f \in F(x_2)$. Then $\|x_1\| = \|x_2\| = \|f\|$, and therefore, if $x_1 \ne x_2$ we have

$$\left\| \frac{x_1 + x_2}{2} \right\| < \|f\|.$$

On the other hand, $\langle f, x_1\rangle = \langle f, x_2\rangle = \|f\|^2$ and hence

$$\|f\|^2 = \left\langle f, \frac{x_1 + x_2}{2} \right\rangle < \|f\|^2 \quad \text{if } x_1 \ne x_2.$$

$\boxed{3.29}$

1. Assume, by contradiction, that there exist $M_0 > 0$, $\varepsilon_0 > 0$, and two sequences (x_n), (y_n) such that

$$\|x_n\| \le M, \|y_n\| \le M, \quad \|x_n - y_n\| > \varepsilon_0,$$

and

$$\text{(S1)} \qquad \left\| \frac{x_n + y_n}{2} \right\|^2 > \frac{1}{2}\|x_n\|^2 + \frac{1}{2}\|y_n\|^2 - \frac{1}{n}.$$

Consider subsequences, still denoted by (x_n) and (y_n), such that $\|x_n\| \to a$ and $\|y_n\| \to b$. We find that $a + b \ge \varepsilon_0$ and $\frac{1}{2}a^2 + \frac{1}{2}b^2 \le \left(\frac{a+b}{2}\right)^2$. Therefore $a = b \ne 0$.
Set

$$x_n' = \frac{x_n}{\|x_n\|} \quad \text{and} \quad y_n' = \frac{y_n}{\|y_n\|}.$$

For n large enough we have $\|x_n' - y_n'\| \geq (\varepsilon_0/a) + o(1)$ (as usual, we denote by $o(1)$ various quantities—positive or negative—that tend to zero as $n \to \infty$). By uniform convexity there exists $\delta_0 > 0$ such that

$$\left\| \frac{x_n' + y_n'}{2} \right\| \leq 1 - \delta_0.$$

Thus

$$\left\| \frac{x_n + y_n}{2} \right\| \leq a(1 - \delta_0) + o(1).$$

By (S1) we have

$$\left\| \frac{x_n + y_n}{2} \right\|^2 \geq a^2 + o(1).$$

Hence $a^2 \leq a^2(1 - \delta_0)^2 + o(1)$; absurd.

$\boxed{3.32}$

1. The infimum is achieved since E is reflexive and we may apply Corollary 3.23. The uniqueness comes from the fact that the *space* E is strictly convex and thus the *function* $y \mapsto \|y - x\|^2$ is strictly convex.
2. Let (y_n) be a minimizing sequence; set $d_n = \|x - y_n\|$ and $d = \inf_{y \in C} \|x - y\|$, so that $d_n \to d$. Let (y_{n_k}) be a sequence such that $y_{n_k} \rightharpoonup z$ weakly. Thus $z \in C$ and $\|x - z\| \leq d$ (why?). It follows that

$$x - y_{n_k} \rightharpoonup x - z \text{ weakly} \quad \text{and} \quad \|x - y_{n_k}\| \to d = \|x - z\|,$$

and therefore (see Proposition 3.32) $y_{n_k} \to z$ strongly. The uniqueness of the limit implies that the *whole sequence* (y_n) converges strongly to $P_C x$. The argument is standard and we will use it many times. We recall it for the convenience of the reader. Assume, by contradiction, that (y_n) does *not* converge to $y = P_C x$. Then there exist $\varepsilon > 0$ and a subsequence, (y_{m_j}), such that $\|y_{m_j} - y\| \geq \varepsilon \quad \forall j$. From (y_{m_j}) we extract (by the argument above) a further subsequence, denoted by (y_{n_k}), such that $y_{n_k} \to P_C x$. Since (y_{n_k}) is a subsequence of (y_{m_j}), we have $\|y_{n_k} - y\| \geq \varepsilon \quad \forall k$ and thus $\|P_C x - y\| \geq \varepsilon$. Absurd.

3 and 4. Assume, by contradiction, that there exist some $\varepsilon_0 > 0$ and sequences (x_n) and (y_n) such that

$$\|x_n\| \leq M, \quad \|y_n\| \leq M, \quad \|x_n - y_n\| \to 0, \quad \text{and} \quad \|P_C x_n - P_C y_n\| \geq \varepsilon_0 \quad \forall n.$$

We have

$$\|x_n - P_C x_n\| \leq \left\| x_n - \frac{P_C x_n + P_C y_n}{2} \right\| \leq \left\| \frac{x_n + y_n}{2} - \frac{P_C x_n + P_C y_n}{2} \right\| + o(1),$$

and similarly

$$\|y_n - P_C y_n\| \le \left\| \frac{x_n + y_n}{2} - \frac{P_C x_n + P_C y_n}{2} \right\| + o(1).$$

It follows that

$$\text{(S1)} \quad \frac{1}{2}\|x_n - P_C x_n\|^2 + \frac{1}{2}\|y_n - P_C y_n\|^2 \le \left\| \frac{x_n + y_n}{2} - \frac{P_C x_n + P_C y_n}{2} \right\|^2 + o(1).$$

On the other hand, if we set $a_n = x_n - P_C x_n$ and $b_n = y_n - P_C y_n$, then $\|a_n - b_n\| \ge \varepsilon_0 + o(1)$ and $\|a_n\| \le M'$, $\|b_n\| \le M'$. Using Exercise 3.29, we know that there is some $\delta_0 > 0$ such that

$$\left\| \frac{a_n + b_n}{2} \right\|^2 \le \frac{1}{2}\|a_n\|^2 + \frac{1}{2}\|b_n\|^2 - \delta_0,$$

that is,

$$\text{(S2)} \quad \left\| \frac{x_n + y_n}{2} - \frac{P_C x_n + P_C y_n}{2} \right\|^2 \le \frac{1}{2}\|x_n - P_C x_n\|^2 + \frac{1}{2}\|y_n - P_C y_n\|^2 - \delta_0.$$

Combining (S1) and (S2) leads to a contradiction.

5. Same argument as in question 1.
6. We have

$$\text{(S3)} \qquad n\|y_n - x\|^2 + \varphi(y_n) \le n\|y - x\|^2 + \varphi(y) \quad \forall y \in D(\varphi).$$

Since φ is bounded below by an affine continuous function (see Proposition 1.10), we see that (y_n) remains bounded as $n \to \infty$ (check the details). Let (y_{n_k}) be a subsequence such that $y_{n_k} \rightharpoonup z$ weakly. Note that $z \in \overline{D(\varphi)}$ (why?). From (S3) we obtain $\|z - x\| \le \|y - x\|$ $\forall y \in D(\varphi)$, and thus $\forall y \in \overline{D(\varphi)}$. Hence $z = P_C x$, where $C = \overline{D(\varphi)}$. Using (S3) once more leads to

$$\limsup_{n\to\infty} \|y_n - x\| \le \|y - x\| \ \forall y \in \overline{D(\varphi)}, \text{ and in particular for } y = z.$$

We conclude that $y_{n_k} \to z$ strongly, and finally the uniqueness of the limit shows (as above) that the *whole sequence* (y_n) converges strongly to $P_C x$.

4.3

2. Note that $h_n = \frac{1}{2}\left(|f_n - g_n| + f_n + g_n\right)$.
3. Note that $f_n g_n - fg = (f_n - f)g_n + f(g_n - g)$ and that $f(g_n - g) \to 0$ in $L^p(\Omega)$ by dominated convergence.

4.5

1. Recall that $L^1(\Omega) \cap L^\infty(\Omega) \subset L^p(\Omega)$ and more precisely $\|f\|_p^p \le \|f\|_\infty^{p-1}\|f\|_1$. Since Ω is σ-finite, we may write $\Omega = \bigcup_n \Omega_n$ with $|\Omega_n| < \infty$ $\forall n$. Given $f \in L^p(\Omega)$, check that $f_n = \chi_{\Omega_n} T_n f \in L^1(\Omega) \cap L^\infty(\Omega)$ and that $f_n \underset{n\to\infty}{\longrightarrow} f$ in $L^p(\Omega)$.

2. Let (f_n) be sequence in $L^p(\Omega) \cap L^q(\Omega)$ such that $f_n \to f$ in $L^p(\Omega)$ and $\|f_n\|_q \le 1$. We assume (by passing to a subsequence) that $f_n \to f$ a.e. (see Theorem 4.9). It follows from Fatou's lemma that $f \in L^q(\Omega)$ and that $\|f\|_q \le 1$.
3. We already know, by question 2, that $f \in L^q(\Omega)$ and thus $f \in L^r(\Omega)$ for every r between p and q. On the other hand, we may write $\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}$ with $0 < \alpha \le 1$, and we obtain

$$\|f_n - f\|_r \le \|f_n - f\|_p^\alpha \|f_n - f\|_q^{1-\alpha} \le \|f_n - f\|_p^\alpha (2C)^{1-\alpha}.$$

$\boxed{4.6}$

1. We have $\|f\|_p \le \|f\|_\infty |\Omega|^{1/p}$ and thus $\limsup_{p\to\infty} \|f\|_p \le \|f\|_\infty$. On the other hand, fix $0 < k < \|f\|_\infty$, and let

$$A = \{x \in \Omega; |f(x)| > k\}.$$

Clearly $|A| \ne 0$ and $\|f\|_p \ge k|A|^{1/p}$. It follows that $\liminf_{p\to\infty} \|f\|_p \ge k$ and therefore $\liminf_{p\to\infty} \|f\|_p \ge \|f\|_\infty$.
2. Fix $k > C$ and let A be defined as above. Then $k^p|A| \le \|f\|_p^p \le C^p$ and thus $|A| \le (C/k)^p \ \forall p \ge 1$. Letting $p \to \infty$, we see that $|A| = 0$.
3. $f(x) = \log |x|$.

$\boxed{4.7}$ Consider the operator $T : L^p(\Omega) \to L^q(\Omega)$ defined by $Tu = au$. We claim that the graph of T is closed. Indeed, let (u_n) be a sequence in $L^p(\Omega)$ such that $u_n \to u$ in $L^p(\Omega)$ and $au_n \to f$ in $L^q(\Omega)$. Passing to a subsequence we may assume that $u_n \to u$ a.e. and $au_n \to f$ a.e. Thus $f = au$ a.e., and so $f = Tu$. It follows from the closed graph theorem (Theorem 2.9) that T is bounded and so there is a constant C such that

$$\text{(S1)} \qquad \|au\|_q \le C\|u\|_p \quad \forall u \in L^p(\Omega).$$

Case 1: $p < \infty$. It follows from (S1) that

$$\int |a|^q |v| \le C^q \|v\|_{p/q} \quad \forall v \in L^{p/q}(\Omega).$$

Therefore the map $v \mapsto \int |a|^q v$ is a continuous linear functional on $L^{p/q}(\Omega)$ and thus $|a|^q \in L^{(p/q)'}(\Omega)$.

Case 2: $p = \infty$. Choose $u \equiv 1$ in (S1).

$\boxed{4.8}$

1. X equipped with the norm $\| \ \|_1$ is a Banach space. For every n, X_n is a closed subset of X (see Exercise 4.5). On the other hand, $X = \bigcup_n X_n$. Indeed, for every $f \in X$ there is some $q > 1$ such that $f \in L^q(\Omega)$. Thus $f \in L^{1+1/n}(\Omega)$ provided $1 + (1/n) \le q$, and, moreover,

$$\|f\|_{1+1/n} \le \|f\|_1^{\alpha_n} \|f\|_q^{1-\alpha_n} \quad \text{with } \frac{1}{1+(1/n)} = \frac{\alpha_n}{1} + \frac{1-\alpha_n}{q}.$$

It follows from the Baire category theorem that there is some integer n_0 such that $\operatorname{Int} X_{n_0} \neq \emptyset$. Thus $X \subset L^{1+1/n_0}(\Omega)$.

2. The identity map $I\colon X \to L^p(\Omega)$ is a linear operator whose graph is closed. Thus it is a bounded operator.

$\boxed{4.9}$ For every $t \in \mathbb{R}$ we have

$$f(x)t \le j(f(x)) + j^\star(t) \text{ a.e. on } \Omega,$$

and by integration we obtain

$$\left(\frac{1}{|\Omega|}\int_\Omega f\right)t \le \frac{1}{|\Omega|}\int_\Omega j(f) + j^\star(t).$$

Therefore

$$j\left(\frac{1}{|\Omega|}\int_\Omega f\right) = \sup_{t\in\mathbb{R}}\left\{\left(\frac{1}{|\Omega|}\int_\Omega f\right)t - j^\star(t)\right\} \le \frac{1}{|\Omega|}\int_\Omega j(f).$$

$\boxed{4.10}$

1. Let $u_1, u_2 \in D(J)$ and let $t \in [0,1]$. The function $x \mapsto j(tu_1(x)+(1-t)u_2(x))$ is measurable (since j is continuous). On the other hand, $j(tu_1 + (1-t)u_2) \le tj(u_1) + (1-t)j(u_2)$. Recall that there exist constants a and b such that $j(s) \ge as + b$ $\forall s \in \mathbb{R}$ (see Proposition 1.10). It follows that $j(tu_1 + (1+t)u_2) \in L^1(\Omega)$ and that $J(tu_1 + (1-t)u_2) \le tJ(u_1) + (1-t)J(u_2)$.
2. Assume first that $j \ge 0$. We claim that for every $\lambda \in \mathbb{R}$ the set $\{u \in L^p(\Omega);\ J(u) \le \lambda\}$ is closed. Indeed, let (u_n) be a sequence in $L^p(\Omega)$ such that $u_n \to u$ in $L^p(\Omega)$ and $\int j(u_n) \le \lambda$. Passing to a subsequence we may assume that $u_n \to u$ a.e. It follows from Fatou's lemma that $j(u) \in L^1(\Omega)$ and that $\int j(u) \le \lambda$. Therefore J is l.s.c. In the general case, let $\tilde{j}(s) = j(s) - (as+b) \ge 0$. We already know that $\widetilde{J}$ is l.s.c., and so is $J(u) = \tilde{J}(u) + a\int u + b|\Omega|$.
3. We first claim that

$$J^\star(f) \le \int j^\star(f) \quad \forall f \in L^{p'}(\Omega) \text{ such that } j^\star(f) \in L^1(\Omega).$$

Indeed, we have $fu - j(u) \le j^\star(f)$ a.e. on Ω, $\forall u \in L^p(\Omega)$, and thus

$$\sup_{u\in D(J)}\left\{\int fu - J(u)\right\} \le \int j^\star(f).$$

The proof of the reverse inequality is more delicate and requires some "regularization" process. Assume first that $1 < p < \infty$ and set

$$j_n(t) = j(t) + \frac{1}{n}|t|^p, \quad t \in \mathbb{R}.$$

We claim that

$$\text{(S1)} \qquad J_n^\star(f) = \int j_n^\star(f) \quad \forall f \in L^{p'}(\Omega).$$

Indeed, let $f \in L^{p'}(\Omega)$. For a.e. fixed $x \in \Omega$,

$$\sup_{u\in\mathbb{R}} \left\{ f(x)u - j(u) - \frac{1}{n}|u|^p \right\}$$

is achieved by some unique element $u = u(x)$. Clearly we have

$$j(u(x)) + \frac{1}{n}|u(x)|^p - f(x)u(x) \le j(0).$$

It follows that $u \in L^p(\Omega)$ and that $j(u) \in L^1(\Omega)$ (why?).
We conclude that

$$J_n^\star(f) = \sup_{v\in D(J)} \left\{ \int fv - J_n(v) \right\} \ge \int \left\{ fu - j(u) - \frac{1}{n}|u|^p \right\} = \int j_n^\star(f).$$

Since we have already established the reverse inequality, we see that (S1) holds. Next we let $n \uparrow \infty$. Clearly, $J \le J_n$, so that $J_n^\star \le J^\star$, i.e., $\int j_n^\star(f) \le J^\star(f)$. We claim that for every $s \in \mathbb{R}$, $j_n^\star(s) \uparrow j^\star(s)$ as $n \uparrow \infty$. Indeed, we know that $j_n^\star = j^\star \nabla \left(\frac{1}{n}|\ |^p\right)^\star$ (see Exercise 1.23), and we may then argue as in Exercise 1.24. We conclude by monotone convergence that if $f \in D(J^\star)$, then

$$j^\star(f) \in L^1(\Omega) \quad \text{and} \quad \int j^\star(f) \le J^\star(f).$$

Finally, if $p = 1$, the above method can be modified using, for example, $j_n(t) = j(t) + \frac{1}{n}t^2$.

4. Assuming first that $f(x) \in \partial j(u(x))$ a.e. on Ω, we have

$$j(v) - j(u(x)) \ge f(x)(v - u(x)) \ \ \forall v \in \mathbb{R}, \ \text{a.e. on } \Omega.$$

Choosing $v = 0$, we see that $j(u) \in L^1(\Omega)$ and thus

$$J(v) - J(u) \ge \int f(v - u) \quad \forall v \in D(J).$$

Conversely, assume that $f \in \partial J(u)$. Then we have $J(u) + J^\star(f) = \int fu$. Thus $j(u) \in L^1(\Omega)$, $j^\star(f) \in L^1(\Omega)$, and $\int \{j(u) + j^\star(f) - fu\} = 0$. Since $j(u) + j^\star(f) - fu \ge 0$ a.e., we find that $j(u) + j^\star(f) - fu = 0$ a.e., i.e., $f(x) \in \partial j(u(x))$ a.e.

$\boxed{4.11}$ Set $f = u^\alpha,\ g = v^\alpha$, and $p = 1/\alpha$. We have to show that

$$\text{(S1)} \qquad \left(\int f\right)^p + \left(\int g\right)^p \le \left[\int \left(f^p + g^p\right)^{1/p}\right]^p .$$

Set $a = \int f$ and $b = \int g$, so that we have

$$\begin{aligned} a^p + b^p &= \int a^{p-1} f + b^{p-1} g \le \int (a^p + b^p)^{1/p'} (f^p + g^p)^{1/p} \\ &= (a^p + b^p)^{1/p'} \int (f^p + g^p)^{1/p}. \end{aligned}$$

It follows that $(a^p + b^p)^{1/p} \le \int (f^p + g^p)^{1/p}$, i.e., (S1).

$\boxed{4.12}$

1. It suffices to show that

$$\inf_{t\in[-1,+1]} \left\{ \frac{(|t|^p + 1)^{1-s}(|t|^p + 1 - 2\left|\frac{t+1}{2}\right|^p)^s}{|t-1|^p} \right\} > 0,$$

or equivalently that

$$\inf_{t\in[-1,+1]} \left\{ \frac{|t|^p + 1 - 2\left|\frac{t+1}{2}\right|^2}{|t-1|^2} \right\} > 0.$$

But the function $\varphi(t) = |t|^p + 1 - 2\left|\frac{t+1}{2}\right|^p$ satisfies

$$\varphi(t) > 0 \quad \forall t \in [-1, +1), \quad \varphi(1) = \varphi'(1) = 0 \quad \text{and} \quad \varphi''(1) > 0.$$

$\boxed{4.14}$

1 and 2. Let $g_n = \chi_{S_n(\alpha)}$, so that $g_n \to 0$ a.e. and $|g_n| \le 1$. It follows—by dominated convergence (since $|\Omega| < \infty$)—that $\int g_n \to 0$, i.e., $|S_n(\alpha)| \to 0$.

3. Given any integer $m \ge 1$, we may apply question 2 with $\alpha = 1/m$ to find an integer N_m such that $|S_{N_m}(1/m)| < \delta/2^m$. Letting $\Sigma_m = S_{N_m}(1/m)$, we obtain

$$|f_k(x) - f(x)| \le \frac{1}{m} \quad \forall k \ge N_m, \quad \forall x \in \Omega\backslash\Sigma_m.$$

Finally, set $A = \bigcup_{m=1}^\infty \Sigma_m$, so that $|A| < \delta$. We claim that $f_n \to f$ uniformly on $\Omega\backslash A$. Indeed, given $\varepsilon > 0$, fix an integer m_0 such that $m_0 > 1/\varepsilon$. Clearly,

$$|f_k(x) - f(x)| < \varepsilon \quad \forall k \ge N_{m_0}, \quad \forall x \in \Omega\backslash\Sigma_{m_0},$$

and consequently

$$|f_k(x) - f(x)| < \varepsilon \quad \forall k \ge N_{m_0}, \quad \forall x \in \Omega\backslash A.$$

4. Given $\varepsilon > 0$, first fix some $\delta > 0$ using (i) and then fix some A using question 3. We obtain that $\int_A |f_n|^p \le \varepsilon\ \forall n$ and $f_n \to f$ uniformly on $\Omega\backslash A$. It follows from Fatou's lemma that $\int_A |f|^p \le \varepsilon$ and thus

$$\int_\Omega |f_n - f|^p = \int_A |f_n - f|^p + \int_{\Omega\backslash A} |f_n - f|^p \le 2^p\varepsilon + |\Omega|\ \|f_n - f\|^p_{L^\infty(\Omega\backslash A)}.$$

$\boxed{4.15}$

1(iv). Note that $\int f_n\varphi \to 0\ \forall\varphi \in C_c(\Omega)$. Suppose, by contradiction, that $f_{n_k} \rightharpoonup f$ weakly $\sigma(L^1, L^\infty)$. It follows that $\int f\varphi = 0\ \forall\varphi \in C_c(\Omega)$ and thus (by Corollary 4.24) $f = 0$ a.e. Also $\int f_{n_k} \to \int f = 0$; but $\int f_{n_k} = \int_0^{n_k} e^{-t}dt \to 1$; a contradiction.

2(iv). Note that $\int g_n\varphi \to 0\ \forall\varphi \in C_c(\Omega)$ and use the fact that $C_c(\Omega)$ is dense in $L^{p'}(\Omega)$ (since $p' < \infty$).

$\boxed{4.16}$

1. Let us first check that if a sequence (f_n) satisfies

(S1) $$f_n \rightharpoonup \tilde{f} \text{ weakly } \sigma(L^p, L^{p'})$$

and

(S2) $$f_n \to f \text{ a.e.}$$

then $f = \tilde{f}$ a.e.

Indeed, we know from Exercise 3.4 that there exists a sequence (g_n) in $L^p(\Omega)$ such that

(S3) $$g_n \in \text{ conv}\,\{f_n, f_{n+1}, \dots\},$$

and

(S4) $$g_n \to \tilde{f} \text{ strongly in } L^p(\Omega).$$

It follows from (S2) and (S3) that $g_n \to f$ a.e. On the other hand (by Theorem 4.9), there is a subsequence $\left(g_{n_k}\right)$ such that $g_{n_k} \to \tilde{f}$ a.e. Therefore $f = \tilde{f}$ a.e.

Let us now check, under the asumptions (i) and (ii), that $f_n \rightharpoonup f$ weakly $\sigma(L^p, L^{p'})$. There exists a subsequence $\left(f_{n_k}\right)$ converging weakly $\sigma(L^p, L^{p'})$ to some limit, say $\tilde{f}$. From the preceding discussion we know that $f = \tilde{f}$ a.e. The "uniqueness of the limit" implies that the *whole* sequence (f_n) converges weakly to f (fill in the details using a variant of the argument in Exercise 3.32).

3. *First method*. Write

$$\text{(S5)}\qquad \|f_n - f\|_q \le \|f_n - T_k f_n\|_q + \|T_k f_n - T_k f\|_q + \|T_k f - f\|_q.$$

Note that for every $k > 0$ we have

$$\int |f_n - T_k f_n|^q \le \int_{[|f_n|\ge k]} |f_n|^q.$$

On the other hand, we have $\int |f_n|^p \le C^p$ and thus $k^{p-q} \int_{[|f_n|\ge k]} |f_n|^q \le C^p$.

It follows that

$$\text{(S6)}\qquad \|f_n - T_k f_n\|_q \le \left(\frac{C^p}{k^{p-q}}\right)^{1/q} \quad \forall n.$$

Passing to the limit (as $n \to \infty$), with the help of Fatou's lemma we obtain

$$\text{(S7)}\qquad \|f - T_k f\|_q \le \left(\frac{C^p}{k^{p-q}}\right)^{1/q}.$$

Given $\varepsilon > 0$, *fix* k large enough that $(C^p/k^{p-q})^{1/q} < \varepsilon$. It is clear (by dominated convergence) that $\|T_k f_n - T_k f\|_q \underset{n\to\infty}{\longrightarrow} 0$, and hence there is some integer N such that

$$\text{(S8)}\qquad \|T_k f_n - T_k f\|_q < \varepsilon \quad \forall n \ge N.$$

Combining (S5), (S6), (S7), and (S8), we see that $\|f_n - f\|_q < 3\varepsilon\ \forall n \ge N$.

Second method. By Egorov's theorem we know that given $\delta > 0$ there exists some $A \subset \Omega$ such that $|A| < \delta$ and $f_n \to f$ uniformly on $\Omega\backslash A$. Write

$$\begin{aligned}\int_\Omega |f_n - f|^q &= \int_{\Omega\backslash A} + \int_A \\ &\le \|f_n - f\|^q_{L^\infty(\Omega\backslash A)}|\Omega| + \|f_n - f\|_p^q |A|^{1-(q/p)} \\ &\le \|f_n - f\|^q_{L^\infty(\Omega\backslash A)}|\Omega| + (2C)^q \delta^{1-(q/p)},\end{aligned}$$

which leads to

$$\limsup_{n\to\infty} \int |f_n - f|^q \le (2C)^q \delta^{1-(q/p)} \quad \forall \delta > 0.$$

$\boxed{4.17}$

1. By homogeneity it suffices to check that

$$\sup_{t\in[-1,+1]} \left\{ \frac{\big||t+1|^p - |t|^p - 1\big|}{|t|^{p-1} + |t|} \right\} < \infty.$$

$\boxed{4.18}$

1. First, it is easy to check that $\int_a^b u_n(t)dt \to (b-a)\overline{f}$ (for every $a, b \in (0, 1)$). This implies that $u_n \rightharpoonup \overline{f}$ weakly $\sigma(L^p, L^{p'})$ whenever $1 < p \le \infty$ (since $p' < \infty$, step functions are dense in $L^{p'}$). When $p = 1$, i.e., $f \in L^1_{\text{loc}}(\mathbb{R})$, there is a T-periodic function $g \in L^\infty(\mathbb{R})$ such that $\frac{1}{T}\int_0^T |f - g| < \varepsilon$ (where $\varepsilon > 0$ is fixed arbitrarily).

   Set $v_n(x) = g(nx)$, $x \in (0, 1)$ and let $\varphi \in L^\infty(0, 1)$. We have

   $$\left|\int u_n\varphi - \overline{f}\int \varphi\right| \le 3\varepsilon\|\varphi\|_\infty + \left|\int v_n\varphi - \overline{g}\int \varphi\right|$$

   and thus $\limsup_{n\to\infty}\left|\int u_n\varphi - \overline{f}\int\varphi\right| \le 3\varepsilon\|\varphi\|_\infty\ \forall\varepsilon > 0$. It follows that $u_n \rightharpoonup \overline{f}$ weakly $\sigma(L^1, L^\infty)$.
2. $\lim_{n\to\infty}\|u_n - \overline{f}\|_p = \left[\frac{1}{T}\int_0^T |f - \overline{f}|^p\right]^{1/p}$.
3. (i) $u_n \overset{\star}{\rightharpoonup} 0$ for $\sigma(L^\infty, L^1)$.
   (ii) $u_n \overset{\star}{\rightharpoonup} \frac{1}{2}(\alpha + \beta)$ for $\sigma(L^\infty, L^1)$.

$\boxed{4.20}$

1. Let (u_n) be a sequence in $L^p(\Omega)$ such that $u_n \to u$ strongly in $L^p(\Omega)$. There exists a subsequence such that $u_{n_k}(x) \to u(x)$ a.e. and $|u_{n_k}| \le v\ \forall k$ with $v \in L^p(\Omega)$ (see Theorem 4.9). It follows by dominated convergence that $Au_{n_k} \to Au$ strongly in $L^q(\Omega)$. The "uniqueness of the limit" implies that the *whole* sequence (Au_n) converges to Au strongly in $L^q(\Omega)$ (as in the solution to Exercise 3.32).
2. Consider the sequence (u_n) defined in Exercise 4.18, question 3(ii). Note that $u_n \rightharpoonup \frac{1}{2}(\alpha + \beta)$, while $Au_n \rightharpoonup \frac{1}{2}(a(\alpha) + a(\beta))$. It follows that

   $$a\left(\frac{\alpha + \beta}{2}\right) = \frac{1}{2}(a(\alpha) + a(\beta)) \quad \forall\alpha, \beta \in \mathbb{R},$$

   and thus a must be an *affine* function.

$\boxed{4.21}$

1. Check that $\int_I u_n(t)dt \to 0$ for every bounded interval I. Then use the density of step functions (with compact support) in $L^{p'}(\mathbb{R})$.
2. We claim once more that $\int_I u_n(t)dt \to 0$ for every bounded interval I. Indeeed, given $\varepsilon > 0$, fix $\delta > 0$ such that $\delta(\|u_0\|_\infty + |I|) < \varepsilon$. Set $E = [|u_0| > \delta]$ and write

   $$\int_I u_n(t)dt = \int_{(I+n)} u_0(t)dt = \int_{(I+n)\cap E} u_0 + \int_{(I+n)\cap E^c} u_0.$$

   Choose N large enough that $|(I + n) \cap E| < \delta\ \forall n > N$ (why is it possible?).

We obtain

$$\left|\int_I u_n(t)dt\right| \le \delta\|u_0\|_\infty + \delta|I| < \varepsilon \quad \forall n \ge N.$$

Then use the density of step functions (with compact support) in $L^1(\mathbb{R})$.

3. Suppose, by contradiction, that $u_{n_k} \rightharpoonup u$ weakly $\sigma(L^1, L^\infty)$. Consider the function $f \in L^\infty(\mathbb{R})$ defined by

$$f = \sum_i (-1)^i \chi_{(-n_i, -n_i+1)}.$$

Note that $\int u_{n_k} f = (-1)^k$ does not converge.

$\boxed{4.22}$

1. In order to prove that (B) $\Rightarrow$ (A) use the fact that the vector space spanned by the functions χ_E with E measurable and $|E| < \infty$ is dense in $L^{p'}(\Omega)$ provided $p' < \infty$ (why?).
2. Use the fact that the vector space spanned by the functions χ_E (with $E \subset \Omega$ and E measurable) is dense in $L^\infty(\Omega)$ (why?).
4. Given $\varepsilon > 0$, fix some measurable subset $\omega \subset \Omega$ such that $|\omega| < \infty$ and

(S1) $$\int_{\omega^c} f < \varepsilon.$$

We have

$$\int_{\omega^c} f_n = \int_{\omega^c} f + \left(\int_\omega f - \int_\omega f_n\right) + \left(\int_\Omega f_n - \int_\Omega f\right)$$

and therefore

(S2) $$\int_{\omega^c} f_n = \int_{\omega^c} f + o(1) \quad \text{(by (b) and (c))}.$$

On the other hand, we have

$$\int_F f_n = \int_{F\cap\omega} f_n + \int_{F\cap(\omega^c)} f_n = \int_{F\cap\omega} f + \int_{F\cap(\omega^c)} f_n + o(1)$$

and thus

(S3) $$\int_F f_n - \int_F f = \int_{F\cap(\omega^c)} (f_n - f) + o(1).$$

Combining (S1), (S2), and (S3), we obtain

$$\left|\int_F f_n - \int_F f\right| \le 2\varepsilon + o(1).$$

It follows that $\int_F f_n \to \int_F f$. Finally, we use the fact that the vector space spanned by the functions χ_F with $F \subset \Omega$, F measurable and $|F| \leq \infty$, is dense in $L^\infty(\Omega)$ (why?).

$\boxed{4.23}$

1. Let (u_n) be a sequence in C such that $u_n \to u$ strongly in $L^p(\Omega)$. There exists a subsequence $\left(u_{n_k}\right)$ such that $u_{n_k} \to u$ a.e. Thus $u \geq f$ a.e.
2. Assume that $u \in L^\infty(\Omega)$ satisfies

$$\int u\varphi \geq \int f\varphi \quad \forall \varphi \in L^1(\Omega) \text{ such that } f\varphi \in L^1(\Omega) \text{ and } \varphi \geq 0.$$

We claim that $u \geq f$ a.e. Indeed, write $\Omega = \bigcup_n \Omega_n$ with $|\Omega_n| < \infty$ and set $\Omega_n' = \Omega_n \cap [|f| < n]$, so that $\bigcup_n \Omega_n' = \Omega$. Let $A = [u < f]$. Choosing $\varphi = \chi_{A\cap\Omega_n'}$, we find that $\int_{A\cap\Omega_n'} |f - u| \leq 0$ and thus $|A \cap \Omega_n'| = 0$ $\forall n$. It follows that $|A| = 0$.
3. Note that if $\varphi \in L^1(\Omega)$ is *fixed* with $f\varphi \in L^1(\Omega)$ then the set $\{u \in L^\infty(\Omega);\ \int u\varphi \geq \int f\varphi\}$ is closed for the topology $\sigma(L^\infty, L^1)$.

$\boxed{4.24}$

1. For every $\varphi \in L^1(\mathbb{R}^N)$ we have

$$\int v_n\varphi = \int u\zeta_n(\check{\rho}_n \star \varphi) = \int u\zeta_n(\check{\rho}_n \star \varphi - \varphi) + \int u\zeta_n\varphi$$

and thus

$$\left|\int v_n\varphi - \int v\varphi\right| \leq \|u\|_\infty \|\check{\rho}_n \star \varphi - \varphi\|_1 + \|u\|_\infty \|(\zeta_n - \zeta)\varphi\|_1.$$

The first term on the right side tends to zero by Theorem 4.22, while the second term on the right side tends to zero by dominated convergence.
2. Let $B = B(x_0, R)$ and let χ denote the characteristic function of $B(x_0, R+1)$. Set $\tilde{v}_n = \rho_n \star (\zeta_n \chi u)$. Note that $\tilde{v}_n = v_n$ on $B(x_0, R)$, since

$$\operatorname{supp}(\tilde{v}_n - v_n) \subset \overline{B(0, 1/n) + B(x_0, R+1)^c}.$$

On the other hand, we have

$$\begin{aligned}
\int_B |v_n - v| = \int_B |\tilde{v}_n - \chi v| &\leq \int_{\mathbb{R}^N} |\tilde{v}_n - \chi v| \\
&\leq \int_{\mathbb{R}^N} |\rho_n \star (\zeta_n - \zeta)\chi\, u| + \int_{\mathbb{R}^N} |(\rho_n \star \chi v) - \chi v| \\
&\leq \int_{\mathbb{R}^N} |(\zeta_n - \zeta)\chi u| + \int_{\mathbb{R}^N} |(\rho_n \star \chi v) - \chi v| \to 0.
\end{aligned}$$

$\boxed{4.25}$

1. Let $\overline{u}$ denote the extension of u by 0 outside Ω. Let

$$\Omega_n = \{x \in \Omega; \operatorname{dist}(x, \partial\Omega) > 2/n \text{ and } |x| < n\}.$$

Let ζ_n (resp. ζ) denote the characteristic function of Ω_n (resp. Ω), so that $\zeta_n \to \zeta$ on $\mathbb{R}^N$. Let $v_n = \rho_n \star (\zeta_n \overline{u})$. We know that $v_n \in C_c^\infty(\Omega)$ and that $\int_B |v_n - \overline{u}| \to 0$ for every ball B (by Exercise 4.24). Thus, for every ball B there is a subsequence (depending on B) that converges to $\overline{u}$ a.e. on B. By a *diagonal process* we may construct a subsequence (v_{n_k}) that converges to $\overline{u}$ a.e. on $\mathbb{R}^N$.

$\boxed{4.26}$

1. Assume that $A < \infty$. Let us prove that $f \in L^1(\Omega)$ and that $\|f\|_1 \le A$. We have

$$\left|\int f\varphi\right| \le A\|\varphi\|_\infty \quad \forall \varphi \in C_c(\Omega).$$

Let $K \subset \Omega$ be any compact subset and let $\psi \in C_c(\Omega)$ be a function such that $0 \le \psi \le 1$ and $\psi = 1$ on K. Let u be any function in $L^\infty(\Omega)$. Using Exercise 4.25 we may construct a sequence (u_n) in $C_c(\Omega)$ such that $\|u_n\|_\infty \le \|u\|_\infty$ and $u_n \to u$ a.e. on Ω. We have

$$\left|\int f\psi u_n\right| \le A\|u\|_\infty.$$

Passing to the limit as $n \to \infty$ (by dominated convergence) we obtain

$$\left|\int f\psi u\right| \le A\|u\|_\infty \quad \forall u \in L^\infty(\Omega).$$

Choosing $u = \operatorname{sign}(f)$ we find that $\int_K |f| \le A$ for every compact subset $K \subset \Omega$. It follows that $f \in L^1(\Omega)$ and that $\|f\|_1 \le A$.
2. Assume that $B < \infty$. We have

$$\int f\varphi \le B\|\varphi\|_\infty \quad \forall \varphi \in C_c(\Omega),\ \varphi \ge 0.$$

Using the same method as in question 1, we obtain

$$\int f\psi u \le B\|u\|_\infty \quad \forall u \in L^\infty(\Omega),\ u \ge 0.$$

Choosing $u = \chi_{[f>0]}$ we find that $\int_K f^+ \le B$.

$\boxed{4.27}$ Let us first examine an abstract setting. Let E be a vector space and let f, g be two linear functionals on E such that $f \not\equiv 0$. Assume that

$$[\varphi \in E \text{ and } f(\varphi) > 0] \Rightarrow [g(\varphi) \geq 0].$$

We claim that there exists a constant $\lambda \geq 0$ such that $g = \lambda f$. Indeed, fix any $\varphi_0 \in E$ such that $f(\varphi_0) = 1$. For every $\varphi \in E$ and every $\varepsilon > 0$, we have

$$f(\varphi - f(\varphi)\varphi_0 + \varepsilon\varphi_0) = \varepsilon > 0$$

and thus $g(\varphi - f(\varphi)\varphi_0 + \varepsilon\varphi_0) \geq 0$. It follows that $g(\varphi) \geq \lambda f(\varphi)\ \forall \varphi \in E$, and thus $g = \lambda f$, with $\lambda = g(\varphi_0) \geq 0$.

Application. $E = C_c^\infty(\Omega)$, $f(\varphi) = \int u\varphi$, and $g(\varphi) = \int v\varphi$.

$\boxed{4.30}$

1 and 2. Note that $\frac{1}{p'} + \frac{1}{q'} + \frac{1}{r} = 1$ and that $(1-\alpha)r = p$, $(1-\beta)r = q$. For a.e. $x \in \mathbb{R}^N$ write

$$|f(x-y)g(y)| = \varphi_1(y)\varphi_2(y)\varphi_3(y)$$

with $\varphi_1(y) = |f(x-y)|^\alpha$, $\varphi_2(y) = |g(y)|^\beta$, and $\varphi_3(y) = |f(x-y)|^{1-\alpha}|g(y)|^{1-\beta}$. Clearly, $\varphi_1 \in L^{q'}(\mathbb{R}^N)$ and $\varphi_2 \in L^{p'}(\mathbb{R}^N)$. On the other hand, $|\varphi_3(y)|^r = |f(x-y)|^p|g(y)|^q$. We deduce from Theorem 4.15 that for a.e. $x \in \mathbb{R}^N$ the function $y \mapsto |\varphi_3(y)|^r$ is integrable. It follows from Hölder's inequality (see Exercise 4.4) that for a.e. $x \in \mathbb{R}^N$, the function $y \mapsto |f(x-y)g(y)|$ is integrable and that

$$\int |f(x-y)|\,|g(y)|dy \leq \|f\|_p^\alpha \|g\|_q^\beta \left(\int |f(x-y)|^p|g(y)|^q dy\right)^{1/r}.$$

Thus

$$|(f \star g)(x)|^r \leq \|f\|_p^{\alpha r}\|g\|_q^{\beta r} \int |f(x-y)|^p|g(y)|^q dy,$$

and consequently

$$\int |(f \star g)(x)|^r dx \leq \|f\|_p^{\alpha r}\|g\|_q^{\beta r}\|f\|_p^p\|g\|_q^q = \|f\|_p^r\|g\|_q^r.$$

3. If $1 < p < \infty$ and $1 < q < \infty$, there exist sequences (f_n) and (g_n) in $C_c(\mathbb{R}^N)$ such that $f_n \to f$ in $L^p(\mathbb{R}^N)$ and $g_n \to g$ in $L^q(\mathbb{R}^N)$. Then $f_n \star g_n \in C_c(\mathbb{R}^N)$, and, moreover, $\|(f_n \star g_n) - (f \star g)\|_\infty \to 0$. It follows that $(f \star g)(x) \to 0$ as $|x| \to \infty$.

$\boxed{4.34}$ Given any $\varepsilon > 0$ there is a finite covering of $\mathcal{F}$ by balls of radius ε in $L^p(\mathbb{R}^N)$, say $\mathcal{F} \subset \bigcup_{i=1}^k B(f_i, \varepsilon)$.

2. For each i there is some $\delta_i > 0$ such that

$$\|\tau_h f_i - f_i\|_{L^p(\mathbb{R}^N)} < \varepsilon \quad \forall h \in \mathbb{R}^N \text{ with } |h| < \delta_i$$

(see Lemma 4.3). Set $\delta = \min_{1\le i\le k} \delta_i$. It is easy to check that

$$\|\tau_h f - f\|_p < 3\varepsilon \quad \forall f \in \mathcal{F},\ \forall h \in \mathbb{R}^N \text{ with } |h| < \delta.$$

3. For each i there is some bounded open set $\Omega_i \subset \mathbb{R}^N$ such that

$$\|f_i\|_{L^p(\mathbb{R}^N\setminus\Omega_i)} < \varepsilon.$$

Set $\Omega = \bigcup_{i=1}^k \Omega_i$ and check that $\|f\|_{L^p(\mathbb{R}^N\setminus\Omega)} < 2\varepsilon \quad \forall f \in \mathcal{F}$.

$\boxed{4.37}$

1. Write

$$\begin{aligned}\int_I u_n(x)\varphi(x)dx &= \int_{-n}^{+n} f(t)\left(\varphi\left(\frac{t}{n}\right) - \varphi(0)\right)dt + \varphi(0)\int_{-n}^{+n} f(t)dt\\ &= A_n + B_n;\end{aligned}$$

$A_n \to 0$ by Lebesgue's theorem and $B_n \to 0$ since $\int_{-\infty}^{+\infty} f(t)dt = 0$.
2. Note that, for all $\delta > 0$,

$$\int_0^\delta |u_n(x)|dx = \int_0^{n\delta} |f(t)|dt \to \int_0^\infty |f(t)|dt > 0.$$

3. Argue by contradiction. We would have

$$\int_I u\varphi = 0 \quad \forall\varphi \in C([-1,+1])$$

and thus $u \equiv 0$ (by Corollary 4.24). On the other hand, if we choose $\varphi = \chi_{(0,1)}$ we obtain

$$\int_I u_n\varphi = \int_0^n f(t)dt \to \int_0^{+\infty} f(t)dt > 0.$$

Impossible.

$\boxed{4.38}$

2. Check that, $\forall\varphi \in C^1([0,1])$,

$$\int_I u_n\varphi = \int_I \varphi + O\left(\frac{1}{n}\right), \text{ as } n \to \infty.$$

Then use the facts that $\|u_n\|_1$ is bounded and $C^1([0,1])$ is dense in $C([0,1])$.
3. The sequence (u_n) cannot be equi-integrable since $|\operatorname{supp} u_n| \to 0$ and

$$1 = \int_I u_n = \int_{\operatorname{supp} u_n} |u_n|.$$

4. If $u_{n_k} \rightharpoonup u$ weakly $\sigma(L^1, L^\infty)$ we would have, by question 2 and Corollary 4.24, $u \equiv 1$. Choose a further subsequence $(u_{n'_k})$ such that $\sum_k |\operatorname{supp} u_{n'_k}| < 1$. Let $\varphi = \chi_A$ where
$$A = I \setminus \left(\bigcup_k \operatorname{supp} u_{n'_k} \right),$$
so that $|A| > 0$. We have
$$\int_I u_{n'_k} \varphi = 0 \quad \forall k$$
and thus $0 = \int_I \varphi = |A|$. Impossible.
5. Consider a subsequence (u_{n_k}) such that
$$\sum_k |\operatorname{supp} u_{n_k}| < \infty.$$
Let $B_k = \bigcup_{j \geq k} (\operatorname{supp} u_{n_j})$ and $B = \bigcap_k B_k$. Clearly $|B_k| \to 0$ as $k \to \infty$, and thus $|B| = 0$. If $x \notin B$ there exists some k_0 such that $u_{n_k}(x) = 0\ \forall k \geq k_0$.

$\boxed{5.1}$

1. Using the parallelogram law with $a = u + v$ and $b = v$ leads to $(u, 2v) = 2(u, v)$.
2. Compute (i) − (ii) + (iii).
3. Note that by definition of (,), the map $\lambda \in \mathbb{R} \mapsto (\lambda u, v)$ is continuous.

$\boxed{5.2}$ Let A be a measurable set such that $0 < |A| < |\Omega|$, and choose a measurable set B such that $A \cap B = \emptyset$ and $0 < |B| < |\Omega|$. Let $u = \chi_A$ and $v = \chi_B$. Assume first that $1 \leq p < \infty$. We have $\|u + v\|_p^p = \|u - v\|_p^p = |A| + |B|$ and thus $\|u + v\|_p^2 + \|u - v\|_p^2 = 2(|A| + |B|)^{2/p}$. On the other hand, we have $2(\|u\|_p^2 + \|v\|_p^2) = 2(|A|^{2/p} + |B|^{2/p})$. Finally, note that
$$(\alpha + \beta)^{2/p} > \alpha^{2/p} + \beta^{2/p} \quad \forall \alpha, \beta > 0 \text{ if } p < 2,$$
$$(\alpha + \beta)^{2/p} < \alpha^{2/p} + \beta^{2/p} \quad \forall \alpha, \beta > 0 \text{ if } p > 2.$$
Examine the case $p = \infty$ with the same functions u and v.

$\boxed{5.3}$ Check that

$$\text{(S1)}\quad 2(t_n u_n - t_m u_m, u_n - u_m) = (t_n + t_m)|u_n - u_m|^2 + (t_n - t_m)(|u_n|^2 - |u_m|^2),$$

which implies that
$$(t_n - t_m)(|u_n|^2 - |u_m|^2) \leq 0 \quad \forall m, n.$$

1. Let $n > m$, so that $t_n \geq t_m$ and thus $|u_n| \leq |u_m|$. (Note that if $t_n = t_m$, then $u_n = u_m$ in view of (S1)). On the other hand, we have for $n > m$,

$$(t_n + t_m)|u_n - u_m|^2 \leq (t_n - t_m)(|u_m|^2 - |u_n|^2) \leq t_n(|u_m|^2 - |u_n|)^2)$$

and thus

$$|u_n - u_m|^2 \leq |u_m|^2 - |u_n|^2.$$

It follows that $|u_n| \downarrow \ell$ as $n \uparrow \infty$ and that (u_n) is a Cauchy sequence.

2. Let $n > m$, so that $t_m \geq t_n$ and $|u_m| \leq |u_n|$. For $n > m$ we have

$$(t_n + t_m)|u_n - u_m|^2 \leq (t_m - t_n)(|u_n|^2 - |u_m|^2) \leq t_m(|u_n|^2 - |u_m|^2)$$

and thus

$$|u_n - u_m|^2 \leq |u_n|^2 - |u_m|^2.$$

We now have the following alternative:

(i) either $|u_n| \uparrow \infty$ as $n \uparrow \infty$,
(ii) or $|u_n| \uparrow \ell < \infty$ as $n \uparrow \infty$ and then (u_n) is a Cauchy sequence.

On the other hand, letting $v_n = t_n u_n$ and $s_n = 1/t_n$, we obtain

$$(s_n v_n - s_m v_m, v_n - v_m) \leq 0,$$

and thus (v_n) converges to a limit by question 1. It follows that if $t_n \to t > 0$ then (u_n) also converges to a limit. Finally if $t_n \to 0$, both cases (i) and (ii) may occur. Take, for example, $H = \mathbb{R}$, $u_n = C/t_n$ for (i), $u_n = C$ for (ii).

$\boxed{5.4}$ Note that

$$|v - u|^2 = |v - f|^2 - |u - f|^2 + 2(f - u, v - u).$$

$\boxed{5.5}$

1. Let $K = \bigcap_n K_n$. We claim that $u_n \to u = P_K f$. First, note that the sequence $d_n = |f - u_n| = \text{dist}(f, K_n)$ is nondecreasing and bounded above. Thus $d_n \uparrow \ell < \infty$ as $n \uparrow \infty$. Next, using the parallelogram law (with $a = f - u_n$ and $b = f - u_m$), we obtain

$$\left| f - \frac{u_n + u_m}{2} \right|^2 + \left| \frac{u_n - u_m}{2} \right|^2 = \frac{1}{2} \left(|f - u_n|^2 + |f - u_m|^2 \right).$$

It follows that $|u_n - u_m|^2 \leq 2(d_m^2 - d_n^2)$ if $m \geq n$. Thus (u_n) converges to a limit, say u, and clearly $u \in K$. On the other hand, we have $|f - u_n| \leq |f - v| \; \forall v \in K_n$ and in particular $|f - u_n| \leq |f - v| \; \forall v \in K$. Passing to the limit, we obtain $|f - u| \leq |f - v| \; \forall v \in K$.

2. Clearly $K = \overline{\bigcup_n K_n}$ is convex (why?). We claim that $u_n \to u = P_K f$. First, note that the sequence $d_n = |f - u_n| = \text{dist}(f, K_n)$ is nonincreasing and

thus $d_n \to \ell$. Next, we have (with the same method as above) $|u_n - u_m|^2 \le 2\left(d_n^2 - d_m^2\right)$ if $m \ge n$. Thus (u_n) converges to a limit, say u, and clearly $u \in K$. Finally, note that $|f - u_m| \le |f - v|\ \forall v \in K_n$ provided $m \ge n$. Passing to the limit (as $m \to \infty$) leads to $|f - u| \le |f - v|\ \forall v \in \bigcup_n K_n$, and by density $\forall v \in K$.

The sequence (α_n) is nonincreasing and thus it converges to a limit, say α. We claim that $\alpha = \inf_K \varphi$. First, it is clear that $\inf_K \varphi \le \alpha_n$ and thus $\inf_K \varphi \le \alpha$. On the other hand, let u be any element in K and let $u_n = P_{K_n} u$. Passing to the limit in the inequality $\alpha_n \le \varphi(u_n)$, we obtain $\alpha \le \varphi(u)$ (since $u_n \to u$). It follows that $\alpha \le \inf_K \varphi$.

$\boxed{5.6}$

1. Consider, for example, the case that $\|u\| \ge 1$ and $\|v\| \le 1$. We have

$$\begin{aligned}\|Tu - Tv\| &= \left\| \frac{u}{\|u\|} - v \right\| = \frac{\|(u-v) + (v - v\|u\|)\|}{\|u\|} \\ &\le \|u - v\| + \|u\| - 1 \le 2\|u - v\|,\end{aligned}$$

since $\|u\| \le \|u - v\| + \|v\| \le \|u - v\| + 1$.
2. Let $u = (1, 0)$ and $v = (1, \alpha)$. Then we have $\|Tu - Tv\| = 2|\alpha|/(1 + |\alpha|)$, while $\|u - v\| = |\alpha|$. We conclude by choosing $\alpha \ne 0$ and arbitrarily small.
3. T coincides with P_{B_E}. Just check that if $\|u\| \ge 1$. then

$$\left(u - \frac{u}{\|u\|}, v - \frac{u}{\|u\|}\right) \le 0 \quad \forall v \in B_E.$$

$\boxed{5.10}$

(i) $\Rightarrow$ (ii). Write that

$$F(u) \le F\left((1-t)u + tv\right) \quad \forall t \in (0,1),\ \forall v \in K,$$

which implies that

$$\frac{1}{t}\left[F\left(u + t(v-u)\right) - F(u)\right] \ge 0.$$

Passing to the limit as $t \to 0$ we obtain (ii).

(ii) $\Rightarrow$ (i). We claim that

$$F(v) - F(u) \ge \left(F'(u), v - u\right)\ \forall u, v \in H.$$

Indeed, the function $t \in \mathbb{R} \mapsto \varphi(t) = F(u + t(v-u))$ is of class C^1 and convex. Thus $\varphi(1) - \varphi(0) \ge \varphi'(0)$.

$\boxed{5.12}$ T is surjective iff E is complete.

1. Transfer onto $R(T)$ the scalar product of E by letting

$$((T(u), T(v))) = (u, v) \quad \forall u, v \in E.$$

Note that $|((f, g))| \le \|f\|_{E^\star}\|g\|_{E^\star}$ $\forall f, g \in R(T)$. The scalar product $((\ ,\ ))$ can be extended by continuity and density to $\overline{R(T)}$, which is now equipped with the structure of a *Hilbert space*.

2. Fix any $f \in E^\star$. The map $g \in R(T) \mapsto \langle f, T^{-1}(g)\rangle$ is a continuous linear functional on $R(T)$. It may be extended (by continuity) to $\overline{R(T)}$. Using the Riesz–Fréchet representation theorem in $\overline{R(T)}$ we obtain some element $h \in \overline{R(T)}$ such that $((h, g)) = \langle f, T^{-1}(g)\rangle$ $\forall g \in R(T)$. Thus we have $((h, T(v))) = \langle f, v\rangle$ $\forall v \in E$. On the other hand, we have $((h, Tv)) = \langle h, v\rangle$ $\forall h \in \overline{R(T)}$, $\forall v \in E$ (this is obvious when $h \in R(T)$). It follows that $f = h$ and consequently $f \in \overline{R(T)}$, i.e., $\overline{R(T)} = E^\star$.
3. We have constructed an isometry $T : E \to E^\star$ with $R(T)$ dense in $E^\star$. Since $E^\star$ is complete, we conclude that (up to an isomorphism) $E^\star$ is the completion of E.

$\boxed{5.13}$

1. We claim that the parallelogram law holds. Indeed, let $f \in F(u)$ and let $g \in F(v)$. Then $f \pm g \in F(u \pm v)$ and so we have

$$\langle f + g, u + v\rangle = \|u + v\|^2 \quad \text{and} \quad \langle f - g, u - v\rangle = \|u - v\|^2.$$

Adding these relations leads to

$$2(\|u\|^2 + \|v\|^2) = \|u + v\|^2 + \|u - v\|^2.$$

2. Let $T : E \to E^\star$ be the map introduced in Exercise 5.12. We claim that $F(u) = \{T(u)\}$. Clearly, $T(u) \in F(u)$. On the other hand, we know that $E^\star$ is a Hilbert space for the dual norm $\|\ \|_{E^\star}$. In particular, $E^\star$ is strictly convex and thus (see Exercise 1.1) $F(u)$ is reduced to a single element.

$\boxed{5.14}$ The convexity inequality $a(tu + (1 - t)v, tu + (1 - t)v) \le ta(u, u) + (1 - t)a(v, v)$ is equivalent to $t(1 - t)a(u - v, u - v) \ge 0$.

Consider the operator $A \in \mathcal{L}(H)$ defined by $a(u, v) = (Au, v)$ $\forall u, v \in H$. Then $F'(u) = Au + A^\star u$, since we have

$$F(u + h) - F(u) = (Au + A^\star u, h) + a(h, h).$$

$\boxed{5.15}$ First, extend S by continuity into an operator $\widetilde{S} : \overline{G} \to F$. Next, let $T = \widetilde{S} \circ P_{\overline{G}}$, where $P_{\overline{G}}$ denotes the projection from H onto $\overline{G}$.

$\boxed{5.18}$

(ii) $\Rightarrow$ (i). Assumption (ii) implies that T is injective and that $R(T)$ is closed. Thus $R(T)$ has a complement (since H is a Hilbert space). We deduce from Theorem 2.13 that T has a left inverse.

(i) $\Rightarrow$ (ii). Assumption (i) implies that T is injective and that $R(T)$ is closed. Then, use Theorem 2.21.

$\boxed{5.19}$ Note that $\limsup_{n\to\infty} |u_n - u|^2 = \limsup_{n\to\infty}(|u_n|^2 - 2(u_n, u) + |u|^2) \le 0$.

$\boxed{5.20}$

1. If $u \in N(S)$ we have $(Sv, v - u) \ge 0\ \forall v \in H$; replacing v by tv, we see that $(Sv, u) = 0\ \forall v \in H$. Conversely, if $u \in R(S)^\perp$ we have $(Sv - Su, v) \ge 0\ \forall v \in H$; replacing v by tv, we see that $(Su, v) = 0\ \forall v \in H$. (See also Problem 16.)
2. Apply Corollary 5.8 (Lax–Milgram).
3. *Method* (a). Set $u_t = (I + tS)^{-1} f$.

   If $f \in N(S)$, then $u_t = f\ \forall t > 0$.

   If $f \in R(S)$, write $f = Sv$, so that $u_t + S(tu_t - v) = 0$. It follows that

   $(u_t, tu_t - v) \le 0$ and thus $|u_t| \le (1/t)|v|$. Consequently $u_t \to 0$ as $t \to \infty$. By density, one can still prove that $u_t \to 0$ as $t \to \infty$ for every $f \in \overline{R(S)}$ (fill in the details).

   In the general case $f \in H$, write $f = f_1 + f_2$ with $f_1 = P_{N(S)} f$ and $f_2 = P_{\overline{R(S)}} f$.

   Method (b). We have $u_t + tSu_t = f$ and thus $|u_t| \le |f|$. Passing to a subsequence $t_n \to \infty$ we may assume that $u_{t_n} \rightharpoonup u$ weakly and that $Su = 0$ (why?), i.e., $u \in N(S)$. From question 1 we know that $(Su_t, v) = 0\ \forall v \in N(S)$ and thus $(f - u_t, v) = 0\ \forall v \in N(S)$. Passing to the limit, we find that $(f - u, v) = 0\ \forall v \in N(S)$. Thus $u = P_{N(S)} f$ and the "uniqueness of the limit" implies that $u_t \rightharpoonup u$ weakly as $t \to \infty$. On the other hand, we have $(Su_t, u_t) \ge 0$, i.e., $(f - u_t, u_t) \ge 0$ and consequently $\limsup_{t\to\infty} |u_t|^2 \le (f, u) = |u|^2$. It follows that $u_t \to u$ strongly as $t \to \infty$.

$\boxed{5.21}$

1. Set $S = I - T$ and apply question 1 of Exercise 5.20.
2. Write $f = u - Tu$ and note that $\sigma_n(f) = \frac{1}{n}(u - T^n u)$.
3. First, check that $\lim_{n\to\infty} \sigma_n(f) = 0\ \forall f \in \overline{R(I - T)}$. Next, split a general $f \in H$ as $f = f_1 + f_2$ with $f_1 \in N(I - T)$ and $f_2 \in N(I - T)^\perp = \overline{R(I - T)}$. We then have $\sigma_n(f) = \sigma_n(f_1) + \sigma_n(f_2) = f_1 + \sigma_n(f_2)$.
4. Apply successively inequality (1) to $u, Su, S^2u, \dots, S^i u, \dots$, and add the resulting inequalities. Note that

$$|S^n u - S^{n+1} u| \le |S^i u - S^{i+1} u| \quad \forall i = 0, 1, \dots, n.$$

5. Writing $f = u - Tu = 2(u - Su)$, we obtain $|\mu_n(f)| \le 2|u|/\sqrt{n+1}$.
6. Use the same method as in question 3.

$\boxed{5.25}$

2. Let $m > n$. Applying Exercise 5.4 with $f = u_m$ and $v = P_K u_n$, one obtains

$$\begin{aligned}|P_K u_n - P_K u_m|^2 &\le |P_K u_n - u_m|^2 - |P_K u_m - u_m|^2 \\ &\le |P_K u_n - u_n|^2 - |P_K u_m - u_m|^2.\end{aligned}$$

   Therefore, $(P_K u_n)$ is a Cauchy sequence.
3. We may assume that $u_{n_k} \rightharpoonup \overline{u}$ weakly. Recall now that $(u_n - P_K u_n, v - P_K u_n) \le 0 \ \forall v \in K$. Passing to the limit (along the sequence n_k) leads to $(\overline{u} - \ell, v - \ell) \le 0 \ \forall v \in K$. Since $\overline{u} \in K$, we may take $v = \overline{u}$ and conclude that $\overline{u} = \ell$. Once more, the "uniqueness of the limit" implies that $u_n \rightharpoonup \ell$ weakly.
4. For every $v \in K$, $\lim_{n\to\infty} |u_n - v|^2$ exists and thus $\lim_{n\to\infty}(u_n, v - w)$ also exists for every $v, w \in K$. It follows that $\varphi(z) = \lim_{n\to\infty}(u_n, z)$ exists for every $z \in H$. Using the Riesz–Fréchet representation theorem we may write $\varphi(z) = (u, z)$ for some $u \in H$. Finally, note that $(u - \ell, v - \ell) \le 0 \ \forall v \in K$ and thus $\ell = P_K u$.
5. By translation and dilation we may always assume that $K = B_H$. Thus $|u_n| \downarrow \alpha$.

   If $\alpha < 1$, then $u_n = P_K u_n$ for n large enough (and we already know that $P_K u_n$ converges strongly).

   If $\alpha \ge 1$, then $P_K u_n = u_n / |u_n|$ converges strongly and so does u_n.
6. Recall that $(u_n - P_K u_n, v - P_K u_n) \le 0 \ \forall v \in K$ and thus $(u_n - \ell, v - \ell) \le \varepsilon_n$ $\forall v \in K$, with $\varepsilon_n \to 0$ (ε_n depends on v). Adding these inequalities leads to $(\sigma_n - \ell, v - \ell) \le \varepsilon_n' \ \forall v \in K$, with $\varepsilon_n' \to 0$. Assuming that $\sigma_{n_k} \rightharpoonup \overline{\sigma}$ weakly, then $\overline{\sigma} \in K$ satisfies $(\overline{\sigma} - \ell, v - \ell) \le 0 \ \forall v \in K$. Therefore $\overline{\sigma} = \ell$ and the "uniqueness of the limit" implies that $\sigma_n \rightharpoonup \ell$ weakly.

$\boxed{5.26}$

3. Note that $\sqrt{n} u_n$ is bounded, and that for each fixed j, $(\sqrt{n} u_n, e_j) \to 0$ as $n \to \infty$.

$\boxed{5.27}$ Let F be the closure of the vector space spanned by the E_n's. We know (see the proof of Theorem 5.9) that $\sum_{n=1}^{\infty} |P_{E_n} u|^2 = |P_F u|^2 \ \forall u \in H$, and thus $|P_F u| = |u| \ \forall u \in D$. It follows that $|P_F u|^2 = |u|^2 \ \forall u \in D$ and therefore $P_{F^\perp} u = 0 \ \forall u \in D$. Consequently $P_{F^\perp} u = 0 \ \forall u \in H$, i.e., $F^\perp = \{0\}$, and so $F = H$.

$\boxed{5.28}$

1. V is separable by Proposition 3.25. Consider a dense countable subset (v_n) of V and conclude as in the proof of Theorem 5.11.

$\boxed{5.29}$

2. If $2 < p < \infty$ use the inequality $\|u\|_p \le \|u\|_\infty^{1-2/p} \|u\|_2^{2/p}$. Note that *every* infinite-dimensional Hilbert space (separable or not) admits an infinite orthonormal sequence.
6. Integrating over Ω, we find that $k \le M^2 |\Omega|$, which provides an upper bound for the dimension of E.

$\boxed{5.30}$

1. For every fixed $t \in [0,1]$ consider the function $u_t(s) = p(s)\chi_{[0,t]}(s)$ and write that $\sum_{n=1}^{\infty} |(u_t, e_n)| \le \|u_t\|_2^2$.
2. Equality in (2) implies equality in (1) for a.e. $t \in [0,1]$. Thus $u_t = \sum_{n=1}^{\infty}(u_t, e_n)e_n$ for a.e. $t \in [0,1]$, and hence $u_t \in E =$ the closure of the vector space spanned by the e_n's. It remains to check that the space spanned by the functions (u_t) is dense in L^2. Let $f \in L^2$ be such that $\int_0^1 f u_t = 0$ for a.e. t. It follows that $\int_0^t fp = 0$ $\forall t \in [0,1]$, and so $fp = 0$ a.e.

$\boxed{5.31}$ It is easy to check that $(\varphi_i, \varphi_j) = 0$ for $i \neq j$. Let $n = 2^{p+1} - 1$. Let E denote the space spanned by $\{\varphi_0, \varphi_1, \dots, \varphi_n\}$ and let F denote the space spanned by the characteristic functions of the intervals $(\frac{i}{2^{p+1}}, \frac{i+1}{2^{p+1}})$, where i is an integer with $0 \le i \le 2^{p+1} - 1$. Clearly $E \subset F$, $\dim E = n + 1 = 2^{p+1}$, and $\dim F = 2^{p+1}$. Thus $E = F$.

$\boxed{5.32}$

2. The function $u = r_1 r_2$ is orthogonal to all the functions $(r_i)_{i\ge 0}$ and $u \neq 0$. Thus $(r_i)_{i\ge 0}$ is not a basis.
3. It is easy to check that $(w_n)_{n\ge 0}$ is an orthonormal system and that $w_0 = r_0$, $w_{2^\ell} = r_{\ell+1}$ $\forall \ell \ge 0$. In order to prove that $(w_n)_{n\ge 0}$ is a basis one can use the same argument as in Exercise 5.31.

$\boxed{6.2}$

3. Consider the sequence of functions defined on $[0,1]$ by

$$u_n(t) = \begin{cases} 0 & \text{if} \quad 0 \le t \le \frac{1}{2}, \\ n(t - \frac{1}{2}) & \text{if} \quad \frac{1}{2} < t \le \frac{1}{2} + \frac{1}{n}, \\ 1 & \text{if} \quad \frac{1}{2} + \frac{1}{n} < t \le 1. \end{cases}$$

Note that $T(u_n) \to f$, but $f \notin T(B_E)$, since $f \notin C^1([0,1])$.

$\boxed{6.3}$ Argue by contradiction. If the conclusion fails, there exists some $\delta > 0$ such that $\|Tu\|_F \ge \delta\|u\|_E \quad \forall u \in E$. Hence $R(T)$ is closed. Consider the operator $T_0 : E \to R(T)$ defined by $T_0 = T$. Clearly T_0 is bijective. By Corollary 2.6, $T_0^{-1} \in \mathcal{L}(R(T), E)$. On the other hand, $T_0 \in \mathcal{K}(E, R(T))$. Hence B_E is compact and $\dim E < \infty$.

$\boxed{6.5}$ Let $T : V \to \ell^2$ be the operator defined by

$$Tu = (\sqrt{\lambda_1}u_1, \sqrt{\lambda_2}u_2, \dots, \sqrt{\lambda_n}u_n, \dots).$$

Clearly $|Tu|_{\ell^2} = \|u\|_V \quad \forall u \in V$, and T is surjective from V onto ℓ^2. Since ℓ^2 is complete, it follows that V is also complete.

Consider the operator $J_n : V \to \ell^2$ defined by

$$J_n u = (u_1, u_2, \ldots, u_n, 0, 0, \ldots).$$

It is easy to check that $\|J_n - I\|_{\mathcal{L}(V,\ell^2)} \to 0$ and thus the canonical injection from V into ℓ^2 is compact.

$\boxed{6.7}$

1. Assume that T is continuous from E weak into F strong. Then for every $\varepsilon > 0$ there exists a neighborhood V of 0 in E weak such that $x \in V \Rightarrow \|Tx\| < \varepsilon$. We may assume that V has the form

$$V = \{x \in E; |\langle f_i, x\rangle| < \delta \quad \forall i = 1, 2, \ldots, n\},$$

where $f_1, f_2, \ldots, f_n \in E^\star$ and $\delta > 0$.
Let $M = \{x \in E; \langle f_i, x\rangle = 0 \ \forall i = 1, 2, \ldots, n\}$, so that $Tx = 0 \ \forall x \in M$. On the other hand, M has finite codimension (see Example 2 in Section 2.4). Thus $E = M+N$ with $\dim N < \infty$. It follows that $R(T) = T(N)$ is finite-dimensional.
2. Note that if $u_n \rightharpoonup u$ weakly in E then $Tu_n \rightharpoonup Tu$ weakly in F. On the other hand, (Tu_n) has compact closure in F (for the strong topology). Thus $Tu_n \to Tu$ (see, e.g., Exercise 3.5).
6. Note that $T^\star \in \mathcal{L}(E^\star, (c_0)^\star)$. But $(c_0)^\star = \ell^1$ (see Section 11.3). Since $E^\star$ is reflexive, it follows from question 5 that $T^\star$ is compact. Hence (by Theorem 6.4) T is compact.

$\boxed{6.8}$

1. There is a constant c such that $B_{R(T)} \subset cT(B_E)$ and thus the unit ball of $R(T)$ is compact.
2. Let E_0 be a complement of $N(T)$. Then $T_0 = T_{|E_0}$ is bijective from E_0 onto $R(T)$. Thus $\dim E_0 = \dim R(T) < \infty$.

$\boxed{6.9}$

1. (A) $\Rightarrow$ (B):
Let E_0 be a complement of $N(T)$ and let $P : E \to N(T)$ be an associated projection operator. Then $T_0 = T_{|E_0}$ is bijective from E_0 onto $R(T)$. By the open mapping theorem there exists a constant C such that

$$\|u\|_E \le C\|Tu\|_F \quad \forall u \in E_0.$$

It follows that $\forall u \in E$,

$$\|u\|_E \le \|u - Pu\|_E + \|Pu\|_E \le C\|Tu\|_F + \|Pu\|_E.$$

(C) $\Rightarrow$ (A):
 (i) To check that the unit ball in $N(T)$ is compact, let (u_n) be a sequence in $N(T)$ such that $\|u_n\|_E \le 1$. Since $(Q(u_n))$ has compact closure in G, one may extract a subsequence $(Q(u_{n_k}))$ converging in G. Applying (C), we see that (u_{n_k}) is Cauchy.

(ii) Introducing a complement of $N(T)$ we may assume in addition that T is injective. Let (u_n) be a sequence in E such that $Tu_n \to f$. Let us first check that (u_n) is bounded. If not, set $v_n = u_n/\|u_n\|$. Applying (C), we see that a subsequence (v_{n_k}) is Cauchy. Let $v_{n_k} \to v$ with $v \in N(T)$ and $\|v\| = 1$; impossible. Therefore (u_n) is bounded and we may extract a subsequence $(Q(u_{n_k}))$ converging in G. Applying (C) once more, we find that (u_{n_k}) is Cauchy.

To recover the result in Exercise 2.12 write

$$\|u\|_E \le C(\|Tu\|_F + \|Pu\|_E) \le C(\|Tu\|_F + |Pu|),$$

since all norms on $N(T)$ are equivalent. Moreover,

$$|Pu| \le |u - Pu| + |u| \le C\|u - Pu\|_E + |u| \le C\|Tu\|_F + |u|.$$

2. Note that

$$\|u\|_E \le C(\|Tu\|_F + \|Pu\|_E) \le C(\|(T + S)u\|_F + \|Pu\|_E + \|Su\|_F)$$

and consider the compact operator $Q : E \to E \times F$ defined by $Qu = [Pu, Su]$.

$\boxed{6.10}$

1. Note that $\forall u \in E$,

$$\begin{aligned} |Q(1)|\|u\| &\le \|Q(1)u - Q(T)u\| + \|Q(T)u\| \\ &= \|\widetilde{Q}(T)(u - Tu)\| + \|Q(T)u\| \\ &\le C(\|u - Tu\| + \|Q(T)u\|). \end{aligned}$$

2. Proof of the implication $N(I - T) = \{0\} \Rightarrow R(I - T) = E$. Suppose by contradiction that $R(I - T) = E_1 \neq E$. Set $E_n = (I - T)^n E$. Then (E_n) is a decreasing sequence of closed subspaces. Choose $u_n \in E_n$ such that $\|u_n\| = 1$ and $\operatorname{dist}(u_n, E_{n+1}) \ge 1/2$. Write

$$Q(T)u_n - Q(T)u_m = Q(T)u_n - Q(1)u_n + Q(1)u_n - Q(1)u_m + Q(1)u_m - Q(T)u_m.$$

Thus, for $m > n$, we have

$$\|Q(T)u_n - Q(T)u_m\| \ge |Q(1)|/2,$$

and this is impossible.
For the converse, follow the argument described in the proof of Theorem 6.6.
3. Using the same notation as in the proof of Theorem 6.6, write $S = T + \Lambda \circ P$. Here $S \notin \mathcal{K}(E)$, but $\Lambda \circ P \in \mathcal{K}(E)$. Thus $Q(S) \in \mathcal{K}(E)$ (why?). Then continue as in the proof of Theorem 6.6.

$\boxed{6.11}$

1. There exists an integer $n_0 \geq 1$ such that $\text{Int}\, F_{n_0} \neq \emptyset$ and thus $B(u_0, \rho) \subset F_{n_0}$. For every $u \in F$ and $|\lambda| < \rho/\|u\|$ we have $u_0 + \lambda u \in F_{n_0}$. Therefore

$$|\lambda|\; |u(x) - u(y)| \leq |u_0(x) - u_0(y)| + n_0 d(x, y)^{1/n_0} \leq 2n_0 d(x, y)^{1/n_0}.$$

It follows that

$$|u(x) - u(y)| \leq \frac{2n_0}{\rho} \|u\| d(x, y)^{1/n_0} \quad \forall x, y \in K.$$

2. The theorem of Ascoli–Arzelà implies that B_F is compact.

$\boxed{6.13}$ Suppose, by contradiction, that there exist some $\varepsilon_0 > 0$ and a sequence (u_n) such that $\|u_n\|_E = 1$ and $\|Tu_n\|_F \geq \varepsilon_0 + n|u_n|$. Then $|u_n| \to 0$ and we may assume that $u_{n_k} \rightharpoonup u$ weakly. But the function $u \mapsto |u|$ is convex and continuous. Thus it is l.s.c. for the weak topology and hence $u = 0$. It follows that $Tu_n \to 0$. Impossible.

$\boxed{6.15}$

1. If $u = f + \lambda(T - \lambda I)^{-1} f$, we have $\lambda u = T(u - f)$ and hence $|\lambda|\; \|u\| \leq \|T\|(\|u\| + \|f\|)$.
2. By the proof of Proposition 6.7 we know that if $\mu \in \mathbb{R}$ is such that $|\mu - \lambda|\; \|(T - \lambda I)^{-1}\| < 1$, then $\mu \in \rho(T)$. Thus $\text{dist}(\lambda, \sigma(T)) \geq 1/\|(T - \lambda I)^{-1}\|$.
4. $(U - I)^{-1} = \frac{1}{2}(T - I)$.
6. Note that the relation $Uu - \lambda u = f$ is equivalent to

$$Tu - \frac{(\lambda + 1)}{(\lambda - 1)} u = \frac{1}{(\lambda - 1)}(f - Tf).$$

$\boxed{6.16}$

2. $(T - \lambda I)^{-1} = \frac{1}{1 - \lambda^n} \sum_{i=0}^{n-1} \lambda^{n-i-1} T^i$.
3. $(T - \lambda I)^{-1} = -\sum_{i=0}^{n-1} \lambda^{-i-1} T^i$.
4. $(I - T)^{-1} = (I - T^n)^{-1} \sum_{i=0}^{n-1} T^i$.

$\boxed{6.18}$

1. $\|S_r\| = \|S_\ell\| = 1$. Note that $S_\ell \circ S_r = I$ and thus $S_r \notin \mathcal{K}(E)$, $S_\ell \notin \mathcal{K}(E)$.
3. For every $\lambda \in [-1, +1]$ the operator $(S_r - \lambda I)$ is not surjective: for example, if $f = (-1, 0, 0, \dots)$ the equation $S_r x - \lambda x = f$ has no solution $x \in \ell^2$.
4. $N(S_\ell - \lambda I) = \mathbb{R}(1, \lambda, \lambda^2, \dots)$.
6. $S_r^\star = S_\ell$ and $S_\ell^\star = S_r$.
7. Writing $S_r x - \lambda x = f$, we have

$$|x| = |S_r x| = |\lambda x + f| \leq |\lambda||x| + |f|.$$

Thus

$$|S_r x - \lambda x| \geq (1 - |\lambda|)|x|,$$

and hence $R(S_r - \lambda I)$ is closed. Applying Theorem 2.19 yields

$$R(S_r - \lambda I) = N(S_\ell - \lambda I)^\perp = \left\{ x \in \ell^2; \ \sum_{i=1}^{\infty} \lambda^{i-1} x_i = 0 \right\}$$

and

$$R(S_\ell - \lambda I) = N(S_r - \lambda I)^\perp = E.$$

8. We have $\overline{R(S_r \pm I)} = N(S_\ell \pm I)^\perp = E$ and $\overline{R(S_\ell \pm I)} = N(S_r \pm I)^\perp = E$. We already know (see question 3) that $R(S_r \pm I) \neq E$. On the other hand, $R(S_\ell \pm I) \neq E$; otherwise, since $S_\ell \pm I$ is injective, we would have $\pm 1 \in \rho(S_\ell)$. Impossible.
9. $EV(S_r \circ M) = \emptyset$ if $\alpha_n \neq 0 \quad \forall n$ and $EV(S_r \circ M) = \{0\}$ if $\alpha_n = 0$ for some n.
10. We may always assume that $\alpha \neq 0$; otherwise $S_r \circ M$ is compact and the conclusion is obvious.
Let us show that $(T - \lambda I)$ is bijective for every λ with $|\lambda| > |\alpha|$. Note that $M = \alpha I + K$, where K is a compact operator. Letting $T = S_r \circ M$, we obtain $T = \alpha S_r + K_1$ and $(T - \lambda I) = (\alpha S_r - \lambda I) + K_1 = J \circ (I + K_2)$, where $J = (\alpha S_r - \lambda I)$ is bijective and K_1, K_2 are compact. Applying Theorem 6.6 (c), it suffices to check that $N(T - \lambda I) = \{0\}$. This has already been established in question 9.
Let us show that $(T - \lambda I)$ is not bijective for $|\lambda| \leq |\alpha|$. Assume by contradiction that $(T - \lambda I)$ is bijective. Write $(S_r - \frac{\lambda}{\alpha} I) = \frac{1}{\alpha}(T - \lambda I) - \frac{1}{\alpha} K_1 = J' \circ (I + K_3)$, where J' is bijective and K_3 is compact. Applying once more Theorem 6.6 (c), we see that

$$\left(S_r - \frac{\lambda}{\alpha} I \right) \text{ injective } \Leftrightarrow \left(S_r - \frac{\lambda}{\alpha} I \right) \text{ surjective.}$$

But we already know (from questions 2 and 3) that $(S_r - \frac{\lambda}{\alpha} I)$ is injective and not surjective, for $|\lambda| \leq |\alpha|$. Impossible.
11. $\sigma(S_r \circ M) = \left[-\sqrt{|ab|}, +\sqrt{|ab|} \right]$. Indeed, if $|\lambda| \leq \sqrt{|ab|}$, the operator $(S_r \circ M - \lambda I)$ is not surjective, since (for example)

$$f = (-1, 0, 0, \dots) \notin R(S_r \circ M - \lambda I).$$

On the other hand, if $|\lambda| > \sqrt{|ab|}$, the operator $(S_r \circ M - \lambda I)$ is bijective, since $(S_r \circ M)^2 = ab S_r^2$. Thus $\|(S_r \circ M)^2\| \leq |ab|$ and we may apply Exercise 6.16, question 4.

$\boxed{6.20}$

1. Note that

$$|Tu(x) - Tu(y)| \leq |x - y|^{1/p'} \|u\|_p.$$

If $1 < p < \infty$ we may apply Ascoli-Arzelà to conclude that $T(B_E)$ has compact closure in $C([0, 1])$ and a fortiori in $L^p(0, 1)$. If $p = 1$, apply Theorem 4.26.
2. $EV(T) = \emptyset$. Note first that $0 \notin EV(T)$. Indeed, the equation $Tu = 0$ implies

$$\int_0^1 u\chi_{[a,b]} = 0 \quad \forall a, b \in [0, 1].$$

If $1 < p < 1$ we may use the density of step functions in $L^{p'}$ to conclude that $u \equiv 0$. When $p = 1$, we prove that

$$\int_0^1 u\varphi = 0 \quad \forall \varphi \in C([0, 1])$$

by approximating uniformly φ by step functions. We conclude with the help of Corollary 4.24 that $u \equiv 0$.

3. For $\lambda \neq 0$ and for $f \in C([0, 1])$, set $u = (T - \lambda I)^{-1} f$. Then $v(x) = \int_0^x u(t)dt$ satisfies:

$$v \in C^1([0, 1]) \quad \text{and} \quad v - \lambda v' = f \quad \text{with } v(0) = 0.$$

Therefore
$$u(x) = -\frac{1}{\lambda} f(x) - \frac{1}{\lambda^2} \int_0^x e^{(x-t)/\lambda} f(t)dt.$$

The same formula remains valid for $f \in L^p$ (argue by density).

4. $(T^\star v)(x) = \int_x^1 v(t)dt.$

$\boxed{6.22}$

2. Suppose, by contradiction, that there exists some $\mu \in Q(\sigma(T))$ such that $\mu \notin \sigma(Q(T))$. Then $\mu = Q(\lambda)$ with $\lambda \in \sigma(T)$, and $Q(T) - Q(\lambda)I = S$ is bijective. We may write
$$Q(t) - Q(\lambda) = (t - \lambda)\overline{Q}(t) \quad \forall t \in \mathbb{R},$$
and thus
$$(T - \lambda I)\overline{Q}(T) = \overline{Q}(T)(T - \lambda I) = S.$$
Hence $T - \lambda I$ is bijective and $\lambda \in \rho(T)$; impossible.

3. Take $E = \mathbb{R}^2$, $T = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $Q(t) = t^2$.

   Then $EV(T) = \sigma(T) = \emptyset$ and $EV(T^2) = \sigma(T^2) = \{-1\}$.

4. $T^2 + I$ is bijective by Lax–Milgram. Every polynomial of degree 2 without real roots may be written (modulo a nonzero factor) as

$$Q(t) = t^2 + at + b = \left(t + \frac{a}{2}\right)^2 + b - \frac{a^2}{4}$$

with $b - a^2/4 > 0$, and we may apply Lax–Milgram once more.
If a polynomial $Q(t)$ has no real root, then its roots are complex conjugates. We may then write $Q(t) = Q_1(t)Q_2(t) \ldots Q_\ell(t)$, where each $Q_i(t)$ is a polynomial of degree 2 without real roots. Since $Q_i(T)$ is bijective, the same holds for $Q(T)$.

5. (i) Suppose, by contradiction, that $\mu \in EV(Q(T))$ and $\mu \notin Q(EV(T))$. Then there exists $u \neq 0$ such that $Q(T)u = \mu u$. Write

$$Q(t) - \mu = (t - t_1)(t - t_2) \cdots (t - t_q)\overline{Q}(t),$$

where the t_i's are the real roots of the polynomial $Q(t) - \mu$ and $\overline{Q}$ has no real root. Then $t_i \notin EV(T)\ \forall i$, since $\mu \notin Q(EV(T))$. We have

$$(T - t_1 I)(T - t_2 I) \cdots (T - t_k I)\overline{Q}(T)u = 0.$$

Since each factor in this product is injective, we conclude that $u = 0$. Impossible.

(ii) Argue as in (i).

$\boxed{6.23}$

3. In $E = \mathbb{R}^2$ take $T(u_1, u_2) = (u_2, 0)$. Then $T^2 = 0$, so that $r(T) = 0$, while $\|T\| = 1$.
5. In $E = \mathbb{R}^3$ take $T(u_1, u_2, u_3) = (u_2, -u_1, 0)$. Then $\sigma(T) = \{0\}$. Using the fact that $T^3 = -T$ it is easy to see that $r(T) = 1$.

   Comment. If we work in Banach spaces over $\mathbb{C}$ the situation is totally different; see Section 11.4. There, we always have $r(T) = \max\{|\lambda|; \lambda \in \sigma(T)\}$. Taking $E = \mathbb{C}^3$ in the current example we have $\sigma(T) = \{0, +i, -i\}$ and then $r(T) = \max\{|\lambda|; \lambda \in \sigma(T)\} = 1$.
6. Assuming that the formula holds for T^n, we have

$$\begin{aligned}(T^{n+1}u)(t) &= \frac{1}{(n-1)!}\int_0^t ds \int_0^s (s-\tau)^{n-1}u(\tau)d\tau \\ &= \frac{1}{(n-1)!}\int_0^t u(\tau)\left[\int_\tau^t (s-\tau)^{n-1}ds\right]d\tau \\ &= \frac{1}{n!}\int_0^t (t-\tau)^n u(\tau)d\tau.\end{aligned}$$

7. Consider the functions f and g defined on $\mathbb{R}$ by

$$f(t) = \begin{cases} \frac{1}{(n-1)!}t^{n-1} & \text{if } 0 \le t \le 1, \\ 0 & \text{otherwise,}\end{cases}$$

$$g(t) = \begin{cases} u(t) & \text{if } 0 \le t \le 1, \\ 0 & \text{otherwise,}\end{cases}$$

   so that for $0 \le t \le 1$, we have

$$(f \star g)(t) = \int_0^1 (t-\tau)u(\tau)d\tau = (T^n u)(t).$$

   We deduce that

$$\|f \star g\|_{L^p(0,1)} \le \|f \star g\|_{L^p(\mathbb{R})} \le \|f\|_{L^1(\mathbb{R})}\|g\|_{L^p(\mathbb{R})} = \frac{1}{n!}\|u\|_{L^p(0,1)}.$$

8. Apply Stirling's formula.

$\boxed{6.24}$

2. (v) $\Rightarrow$ (vi). For every $\varepsilon > 0$, $T_\varepsilon = T + \varepsilon I$ is bijective and $\sigma(T_\varepsilon) \subset [\varepsilon, 1+\varepsilon]$. Thus $\sigma(T_\varepsilon^{-1}) \subset [\frac{1}{1+\varepsilon}, \frac{1}{\varepsilon}]$. Applying Proposition 6.9 to T_ε^{-1} yields

$$(T_\varepsilon^{-1}v, v) \geq \frac{1}{1+\varepsilon}|v|^2 \quad \forall v \in H,$$

i.e.,

$$(T_\varepsilon u, u) \geq \frac{1}{1+\varepsilon}|T_\varepsilon u|^2 \quad \forall u \in H.$$

3. Set $U = 2T - I$. Clearly (vii) is equivalent to

(vii′) $$|u| \leq |Uu| \quad \forall u \in H.$$

Applying Theorem 2.20, we see that (vii) $\Rightarrow$ $(-1, +1) \subset \rho(U) = 2\rho(T) - 1$. Thus (vii) $\Rightarrow$ (viii).

Conversely, (viii) $\Rightarrow$ $(-1, +1) \subset \rho(U)$. Thus $\sigma(U) \subset (-\infty, -1] \cup [1, +\infty)$ and $\sigma(U^{-1}) \subset [-1, +1]$. By Proposition 6.9 we know that $\|U^{-1}\| \leq 1$, i.e., (vii′) holds.

$\boxed{6.25}$ By construction we have

$$\begin{aligned} M \circ (I + K) &= I \quad \text{on } X, \\ (I + K) \circ M &= I \quad \text{on } R(I + K). \end{aligned}$$

Given any $x \in E$, write $x = x_1 + x_2$ with $x_1 \in X$ and $x_2 \in N(I + K)$. Then

$$M \circ (I + K)(x) = M \circ (I + K)(x_1) = x_1 = x - Px$$

where P is a projection onto $N(I + K)$.

For any $x \in E$ we have

$$(I + K) \circ \widetilde{M}(x) = (I + K) \circ M \circ Q(x) = Qx = x - \widetilde{P}x,$$

where $\widetilde{P}$ is a finite-rank projection onto a complement of $R(I + K)$ in E.

$\boxed{8.8}$

4. We have

$$u_n' = \zeta_n u' + \zeta_n' u.$$

Clearly $\zeta_n u' \to u'$ in L^p by dominated convergence. It remains to show that $\zeta_n' u \to 0$ in L^p. Note that

$$\|\zeta_n' u\|_p^p \le C \int_{1/n}^{2/n} n^p |u(x)|^p dx,$$

where $C = \|\zeta'\|_{L^\infty}^p$.
When $p = 1$ we have, since $u \in C([0, 1])$ and $u(0) = 0$,

$$n \int_{1/n}^{2/n} |u(x)| dx \le \max_{x \in [\frac{1}{n}, \frac{2}{n}]} |u(x)| \to 0 \text{ as } n \to \infty.$$

When $p > 1$ we have

$$n^p \int_{1/n}^{2/n} |u(x)|^p dx = n^p \int_{1/n}^{2/n} x^p \frac{|u(x)|^p}{x^p} dx \le 2^p \int_{1/n}^{2/n} \frac{|u(x)|^p}{x^p} dx \to 0$$

by question 1.

$\boxed{8.9}$

1. By question 1 in Exercise 8.8 we know that $\frac{u'(x)}{x} \in L^p$. On the other hand,

$$u(x) = \int_0^x u'(t)dt = xu'(x) - \int_0^x u''(t)tdt,$$

and thus

$$\frac{u(x)}{x^2} = \frac{u'(x)}{x} - \frac{1}{x^2} \int_0^x u''(t)tdt.$$

But

$$\frac{1}{x^2} \left| \int_0^x u''(t)tdt \right| \le \frac{1}{x} \int_0^x |u''(t)|dt \in L^p,$$

as above.
2. We have $v \in C^1((0, 1))$ and

$$v'(x) = -\frac{u(x)}{x^2} + \frac{u'(x)}{x} \in L^p,$$

by question 1.

Moreover,

$$v(x) = \frac{u(x)}{x} = \frac{1}{x} \int_0^x u'(t)dt \to 0 \quad \text{as } x \to 0,$$

since $u \in C^1([0, 1])$ and $u'(0) = 0$.
3. We need only to show that

$$\|\zeta_n' u'\|_p + \|\zeta_n'' u\|_p \to 0 \text{ as } n \to 0.$$

But

$$\|\zeta_n' u'\|_p^p \le Cn^p \int_{1/n}^{2/n} |u'(x)|^p dx \le 2^p C \int_{1/n}^{2/n} \frac{|u'(x)|^p}{x^p} dx$$

and

$$\|\zeta_n'' u\|_p^p \le Cn^{2p} \int_{1/n}^{2/n} |u(x)|^p dx \le 4^p C \int_{1/n}^{2/n} \frac{|u(x)|^p}{x^{2p}} dx,$$

and the conclusion follows since $\frac{u'(x)}{x} \in L^p$ and $\frac{u(x)}{x^2} \in L^p$.

4. Let $u \in X_m$. Then $u' \in X_{m-1}$ and $\frac{u'(x)}{x^{m-1}} \in L^p(I)$ by the induction assumption. Next, observe that

$$\frac{u(x)}{x^m} = \frac{1}{x^m} \int_0^x \frac{u'(t)}{t^{m-1}} t^{m-1} dt.$$

Applying once more Hardy's inequality (see Problem 34, part C) we obtain

$$\frac{|u(x)|}{x^m} \le \frac{1}{x} \int_0^x \frac{|u'(t)|}{t^{m-1}} dt \in L^p(I).$$

In order to prove that $\frac{u(x)}{x^{m-1}} \in X_1$, note that

$$D\left(\frac{u(x)}{x^{m-1}}\right) = \frac{Du(x)}{x^{m-1}} - (m-1)\frac{u(x)}{x^m} \in L^p(I),$$

and that

$$\frac{|u(x)|}{x^{m-1}} \le \frac{1}{x^{m-1}} \int_0^x \frac{|u'(t)|}{t^{m-1}} t^{m-1} dt \le \int_0^x \frac{|u'(t)|}{t^{m-1}} dt \to 0 \quad \text{as } x \to 0,$$

since $\frac{u'(t)}{t^{m-1}} \in L^p(I)$.

5. It suffices to check that $D^\ell v \in X_1$ for every integer ℓ such that $0 \le \ell \le k-1$. But $D^\ell v$ is a linear combination of functions of the form $\frac{D^{j+\alpha}u(x)}{x^{m-j-k+\ell-\alpha}}$, where α is an integer such that $0 \le \alpha \le \ell$. Then use question 4.

6. It suffices to show that $(D^\alpha \zeta_n)(D^\beta u) \to 0$ in $L^p(I)$ when $\alpha + \beta = m$ and $1 \le \alpha \le m$. But $|D^\alpha \zeta_n(x)| \le Cn^\alpha$ and thus

$$\begin{aligned}\int_0^1 |D^\alpha \zeta_n(x)|^p |D^\beta u(x)|^p dx &\le Cn^{\alpha p} \int_{1/n}^{2/n} \left|\frac{D^\beta u(x)}{x^\alpha}\right|^p x^{\alpha p} dx \\ &\le C \int_{1/n}^{2/n} \left|\frac{D^\beta u(x)}{x^\alpha}\right|^p dx \to 0\end{aligned}$$

since $\frac{D^\beta u(x)}{x^\alpha} \in L^p(I)$ by question 4.

8. To prove that $v \in C([0,1])$, note that $v(x) = \frac{1}{x}\int_0^x u'(t)dt$ and that $u' \in C([0,1])$ with $u'(0) = 0$.
Next, we prove that $v \in W^{1,1}(I)$. Integrating by parts, we see that

$$v'(x) = \frac{1}{x^2}\int_0^x u''(t)t\,dt,$$

and a straightforward computation gives

$$\|v'\|_1 \le \int_0^1 |u''(t)|(1-t)dt \le \|u''\|_1.$$

9. Set

$$u(x) = \int_0^x (1+|\log t|)^{-1}dt.$$

It is clear that $u \in W^{2,1}(I)$ with $u(0) = u'(0) = 0$, and, moreover, $\frac{u'(x)}{x} \notin L^1(I)$. The relation

$$\frac{u(x)}{x^2} = \frac{u'(x)}{x} - v'(x),$$

combined with question 8 shows that $\frac{u(x)}{x^2} \notin L^1(I)$.

$\boxed{8.10}$

4. Clearly, as $n \to \infty$,

$$v_n'(x) = G'(nu(x))u'(x) \to f(x) \quad \text{a.e.},$$

where

$$f(x) = \begin{cases} 0 & \text{if} \quad u(x) \neq 0, \\ u'(x) & \text{if} \quad u(x) = 0. \end{cases}$$

6. We have

$$\int_0^1 v_n\varphi' = -\int_0^1 v_n'\varphi \quad \forall\varphi \in C_c^1(I).$$

Passing to the limit as $n \to \infty$ yields

$$\int_0^1 f\varphi = 0 \quad \forall\varphi \in C_c^1(I),$$

and therefore $f = 0$ a.e. on I, i.e., $u'(x) = 0$ a.e. on $[u = 0]$.

$\boxed{8.12}$

1. Use Exercise 8.2 and the fact that

$$\liminf_{n\to\infty} \|u_n'\|_{L^p} \ge \|u'\|_{L^p}.$$

2. Consider the sequence (u_n) in Exercise 8.2. We have $\|u_n\|_{L^1} \le \frac{1}{2}$ and $\|u_n'\|_{L^1} = 1$. Thus $\frac{2}{3}u_n \in B_1$. On the other hand, $\frac{2}{3}u_n \to \frac{2}{3}u$ in L^1, where

$$u(x) = \begin{cases} 0 & \text{if} \quad x \in (0, 1/2), \\ 1 & \text{if} \quad x \in (1/2, 1). \end{cases}$$

But $u \notin W^{1,1}$. Thus B_1 is not closed in L^1.

$\boxed{8.16}$

2. $R(A) = L^p(0, 1)$ and $N(A) = \{0\}$.
3. $v \in D(A^\star)$ iff $v \in L^{p'}$ and there is a constant C such that

$$\left| \int_0^1 vu' \right| \le C\|u\|_p \quad \forall u \in D(A).$$

In particular, $v \in D(A^\star) \Rightarrow v \in W^{1,p'}$, and then

$$\text{(S1)} \qquad \left| u(1)v(1) - \int_0^1 uv' \right| \le C\|u\|_p \quad \forall u \in D(A).$$

We deduce from (S1) that

$$|u(1)|\,|v(1)| \le (C + \|v'\|_{p'})\|u\|_p \quad \forall u \in D(A).$$

It follows that $v(1) = 0$, since there exists a sequence (u_n) in $D(A)$ such that $u_n(1) = 1$ and $\|u_n\|_p \to 0$. Hence we have proved that

$$v \in D(A^\star) \Rightarrow v \in W^{1,p'} \quad \text{and} \quad v(1) = 0.$$

It follows easily that

$$D(A^\star) = \{v \in W^{1,p'} \text{ and } v(1) = 0\},$$

with $A^\star v = -v'$.
4. We have

$$N(\widetilde{A}) = \{0\}, \quad R(\widetilde{A}) = \left\{ f \in L^p;\ \int_0^1 f(t)dt = 0 \right\},$$

and

$$(\widetilde{A})^\star v = -v' \quad \text{with } D((\widetilde{A})^\star) = W^{1,p'}.$$

$\boxed{8.17}$ In the determination of $D(A^\star)$ it is useful to keep in mind the following fact. Let $I = (0, 1)$ and $1 < p \le \infty$. Assume that $u \in L^p(I)$ satisfies

$$\text{(S1)} \qquad \left| \int_I u\varphi' \right| \le C\|\varphi\|_{p'} \quad \forall \varphi \in C_c^1(I) \text{ such that } \int_I \varphi = 0,$$

then $u \in W^{1,p}(I)$.

Indeed, fix a function $\psi_0 \in C_c^1(I)$ such that $\int_I \psi_0 = 1$. Let ζ be any function in $C_c^1(I)$. Inserting $\varphi = \zeta - (\int_I \zeta)\psi_0$ into (S1), we obtain

$$\left|\int_I u\zeta'\right| \le C\|\zeta\|_{p'} + C'\left|\int_I \zeta\right|,$$

where C' depends only on u and ψ_0. Therefore $u \in W^{1,p}(I)$ by Proposition 8.3.

When $Au = u'' - xu'$ we have

$$A^\star v = v'' + xv' + v.$$

Note the following identity

$$A^\star(e^{-\frac{x^2}{2}}u) = e^{-\frac{x^2}{2}}Au \quad \forall u \in H^2(I),$$

which allows to compute $N(A^\star)$ under the various boundary conditions.

$\boxed{8.19}$ Given $f \in L^2(0,1)$, set $F(x) = \int_0^x f(t)dt$. Then

$$\varphi^\star(f) = \begin{cases} \frac{1}{2}\int_0^1 F^2(x)dx & \text{if } \int_0^1 f(t)dt = 0, \\ +\infty & \text{otherwise.} \end{cases}$$

Indeed, if $\int_0^1 f(t)dt = 0$, then $\int_0^1 fv = \int_0^1 F'v = -\int_0^1 Fv' \quad \forall v \in H^1(0,1)$, and

$$\begin{aligned} \varphi^\star(f) &= \sup_{v\in H^1}\left\{\int_0^1 fv - \frac{1}{2}\int_0^1 v'^2\right\} = \sup_{v\in H^1}\left\{-\int_0^1 Fv' - \frac{1}{2}\int_0^1 v'^2\right\} \\ &= \sup_{w\in L^2}\left\{-\int_0^1 Fw - \frac{1}{2}\int_0^1 w^2\right\} = \frac{1}{2}\int_0^1 F^2. \end{aligned}$$

$\boxed{8.21}$

2. Let U be any function satisfying

$$\begin{cases} -(pU')' + qU = f & \text{on } (0,1), \\ U(1) = 0. \end{cases}$$

Then

$$\int_0^1 fv_0 = p(0)(U'(0) - k_0U(0)).$$

Therefore, if $\int_0^1 fv_0 = 0$, any such function U satisfies $U'(0) = k_0U(0)$. Since $U(0)$ can be chosen arbitrarily we see that the set of solutions is one-dimensional.

$\boxed{8.22}$

1. The function $\rho(x) = x$ belongs to $H^1(0,1)$, but $\sqrt{\rho(x)} = \sqrt{x} \notin H^1(0,1)$.

2. For every $\rho \in H^1(0,1)$, with $\rho \geq 0$ on $(0,1)$, set $\gamma_\varepsilon = \sqrt{\rho+\varepsilon}$. Since the function $t \mapsto \sqrt{t+\varepsilon}$ is C^1 on $[0,+\infty)$, we deduce that $\gamma_\varepsilon \in H^1(0,1)$ and, moreover,
$$\gamma'_\varepsilon = \frac{1}{2}\frac{\rho'}{\sqrt{\rho+\varepsilon}},$$
so that $|\gamma'_\varepsilon| \leq \mu$ on the set $[\rho > 0]$. On the other hand, we know that $\rho' = 0$ a.e. on the set $[\rho = 0]$ (see Exercise 8.10) and thus $|\gamma'_\varepsilon| \leq \mu$ a.e. on $[\rho = 0]$. Therefore $|\gamma'_\varepsilon| \leq \mu$ a.e. on $(0,1)$.
Consequently, if $\mu \in L^2$ we deduce that $\|\gamma'_\varepsilon\|_{L^2} \leq C$ as $\varepsilon \to 0$. Since $\gamma_\varepsilon \to \sqrt{\rho}$, as $\varepsilon \to 0$, in $C([0,1])$ and $\gamma'_\varepsilon \rightharpoonup \mu$ in $L^2(0,1)$, we conclude (see Exercise 8.2) that $\sqrt{\rho} \in H^1(0,1)$ and $(\sqrt{\rho})' = \mu$.
Conversely, if $\sqrt{\rho} \in H^1(0,1)$, set $\gamma = \sqrt{\rho}$, so that $\rho = \gamma^2$ and $\rho' = 2\gamma\gamma'$. Hence $\mu = \gamma'$ a.e. on $[\rho > 0]$ and, moreover, $\mu = \gamma'$ a.e. on $[\rho = 0]$ since $\gamma' = 0$ a.e. on $[\gamma = 0] = [\rho = 0]$.

$\boxed{8.24}$

1. One may choose $C_\varepsilon = 1 + 1/\varepsilon$.
2. The weak formulation is
$$\begin{cases} u \in H^1(I), \\ a(u,v) = \int_0^1 (u'v' + kuv) - u(1)v(1) = \int_0^1 fv \quad \forall v \in H^1(I). \end{cases}$$
Clearly $a(u,v)$ is a continuous bilinear form on $H^1(0,1)$. By question 1 it is coercive, e.g., if $k > 2$.

The corresponding minimization problem is
$$\min_{v \in H^1}\left\{\frac{1}{2}\int_0^1 (v'^2 + kv^2) - \frac{1}{2}v(1)^2 - \int_0^1 fv\right\}.$$

3. Let $g \in L^2(I)$ and let $v \in H^2(I)$ be the corresponding solution of (1) (with f replaced by g). We have
$$\begin{aligned}(Tf,g)_{L^2} &= \int_0^1 ug = \int_0^1 u(-v'' + kv) \\ &= -u(1)v'(1) + u(0)v'(0) + u'(1)v(1) - u'(0)v(0) \\ &\quad + \int_0^1 (-u'' + ku)v \\ &= -u(1)v(1) + u(1)v(1) + \int_0^1 fv = (f,Tg)_{L^2}.\end{aligned}$$
Therefore T is self-adjoint. It is compact since it is a bounded operator from $L^2(I)$ into $H^1(I)$, and $H^1(I) \subset L^2(I)$ with compact injection (see Theorem 8.8).

4. By the results of Section 6.4 we know that there exists a sequence (u_n) in $L^2(I)$ satisfying $Tu_n = \mu_n u_n$ with $\|u_n\|_{L^2} = 1$, $\mu_n > 0 \quad \forall n$, and $\mu_n \to 0$. Thus we have $-u_n'' + ku_n = \frac{1}{\mu_n} u_n$, so that $-u_n'' = (\frac{1}{\mu_n} - k)u_n$ on I.
5. The value $\lambda = 0$ is excluded (why?). If $\lambda > 0$ we have $u(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$, where the constants A and B are adjusted to satisfy the boundary condition, i.e., $B = 0$ and $A(\cos\sqrt{\lambda} + \sqrt{\lambda}\sin\sqrt{\lambda}) = 0$, so that $A \neq 0$ iff $\sqrt{\lambda}$ is a solution of the equation $\tan t = -1/t$ (which has an infinite sequence of positive solutions $t_n \to \infty$, as can be seen by inspection of the graphs). If $\lambda < 0$ we have $u(x) = Ae^{\sqrt{|\lambda|}x} + Be^{-\sqrt{|\lambda|}x}$. Putting this together with the boundary conditions gives $A = B$ and $A\sqrt{|\lambda|}e^{\sqrt{|\lambda|}} - B\sqrt{|\lambda|}e^{-\sqrt{|\lambda|}} = Ae^{\sqrt{|\lambda|}} + Be^{-\sqrt{|\lambda|}}$. In order to have some $u \not\equiv 0$, λ must satisfy $\sqrt{|\lambda|}(e^{\sqrt{|\lambda|}} - e^{-\sqrt{|\lambda|}}) = e^{\sqrt{|\lambda|}} + e^{-\sqrt{|\lambda|}}$, i.e., $t = \sqrt{|\lambda|}$ is a solution of the equation $e^{2t} = \frac{t+1}{t-1}$. An inspection of the graphs shows that there is a unique solution $t_0 > 1$ and then $\lambda = -t_0^2$.

$\boxed{8.25}$

2. Assume by contradiction that there is a sequence (u_n) in $H^1(I)$ such that $a(u_n, u_n) \to 0$ and $\|u_n\|_{H^1(I)} = 1$. Passing to a subsequence (u_{n_k}) we may assume that $u_{n_k}' \rightharpoonup u'$ weakly in L^2 and $u_n \to u$ strongly in L^2. By lower semicontinuity (see Proposition 3.5) we have $\liminf \int_I (u_{n_k}')^2 \geq \int_I (u')^2$ and therefore $a(u,u) = 0$, so that $u = 0$. But $\int_I (u_{n_k}')^2 = 1 - \int_I u_{n_k}^2$ and thus $a(u_{n_k}, u_{n_k}) = \int_I (u_{n_k}')^2 + (\int_0^1 u_{n_k})^2 = 1 - \int_I u_{n_k}^2 + (\int_0^1 u_{n_k})^2 \to 1$. Impossible.
4. We have
$$\int_I u'v' = \int_I gv \quad \forall v \in H^1(I),$$
where $g = f - (\int_0^1 u)\chi_{(0,1)}$. Therefore $u \in H^2(I)$ and satisfies
$$\begin{cases} -u'' + (\int_0^1 u)\chi_{(0,1)} = f & \text{on } I, \\ u'(0) = u'(2) = 0. \end{cases}$$
5. We have $u \in C^2(\bar{I})$ iff $\int_0^1 u = 0$. This happens iff $\int_I f = 0$.
8. The eigenvalues of T are positive and if $1/\lambda$ is an eigenvalue, we must have a function $u \not\equiv 0$ satisfying
$$\begin{cases} -u'' + \int_0^1 u = \lambda u & \text{on } (0,1), \\ -u'' = \lambda u & \text{on } (1,2), \\ u'(0) = u'(2) = 0, \\ u(1-) = u(1+) \text{ and } u'(1-) = u'(1+). \end{cases}$$
Therefore
$$u(x) = \frac{k}{\lambda} + A\cos(\sqrt{\lambda}x) \qquad \text{on } (0,1),$$

$$u(x) = A' \cos(\sqrt{\lambda}(x-2)) \quad \text{ on } (1,2),$$

where the constants k, A and A' are determined using the relations

$$\begin{cases} u(1-) = u(1+) \text{ and } u'(1-) = u(1+), \\ k = \int_0^1 u. \end{cases}$$

We conclude that either $\sin(\sqrt{\lambda}) = 0$, i.e., $\lambda = n^2\pi^2$ with $n = 1, 2, \dots$, or λ is a solution of the equation $\tan(\sqrt{\lambda}) = 2\sqrt{\lambda}(1-\lambda)$.

$\boxed{8.26}$

3. Set $a(v,v) = \int_I pv'^2 + qv^2$. We have $(S_N f - S_D f, f) = \int_I f(u_N - u_D)$. We already know that $\frac{1}{2}a(u_N, u_N) - \int_I f u_N \leq \frac{1}{2}a(u_D, u_D) - \int_I f u_D$. On the other hand, $a(u_N, u_N) = \int_I f u_N$ and $a(u_D, u_D) = \int_I f u_D$. Therefore $\int_I f(u_N - u_D) \geq 0$.
6. Set $a_i(v,v) = a(v,v) + k_i v^2(0)$, $i = 1, 2$, and $u_{k_1} = u_1$, $u_{k_2} = u_2$. Since u_i is a minimizer of $\left(\frac{1}{2}a_i(v,v) - \int_I fv\right)$ on $V = \{v \in H^1(I);\, v(1) = 0\}$, we have

$$\frac{1}{2}a(u_2,u_2) + \frac{1}{2}k_2 u_2^2(0) - \int_I f u_2 \leq \frac{1}{2}a(u_1,u_1) + \frac{1}{2}k_2 u_1^2(0) - \int_I f u_1.$$

On the other hand, we have

$$a(u_1,u_1) + k_1 u_1^2(0) = \int_I f u_1,$$

and

$$a(u_2,u_2) + k_2 u_2^2(0) = \int_I f u_2.$$

Therefore

$$-\frac{1}{2}\int_I f u_2 \leq \frac{1}{2}\int_I f u_1 + \frac{1}{2}(k_2 - k_1)u_1^2(0) - \int_I f u_1,$$

so that

$$\left(S_{k_2} f - S_{k_1} f, f\right) = \int_I f(u_2 - u_1) \geq (k_1 - k_2)u_1^2(0) \geq 0.$$

$\boxed{8.27}$

4. The solution φ of

$$\begin{cases} -\varphi'' + \varphi = 1 \text{ on } I, \\ \varphi(-1) = \varphi(1) = 0, \end{cases}$$

is given by $\varphi(x) = 1 + A(e^x + e^{-x})$, where $A = -e/(e^2 + 1)$. By uniqueness of φ we must have $u = \lambda u(0)\varphi$. Therefore

$$\lambda_0 = \frac{1}{\varphi(0)} = \frac{e^2 + 1}{(e - 1)^2}.$$

5. Equation (1) becomes $u = S(f + \lambda u(0)) = Sf + \lambda u(0)S1 = Sf + \lambda u(0)\varphi$. Thus $u(0)(1 - \lambda\varphi(0)) = (Sf)(0)$, i.e., $u(0) = \frac{\lambda_0(Sf)(0)}{\lambda_0 - \lambda}$ and $u = Sf + \frac{\lambda\lambda_0(Sf)(0)\varphi}{\lambda_0 - \lambda}$ is the desired solution.

6. When $\lambda = \lambda_0$, the existence of a solution u implies $(Sf)(0) = 0$ (just follow the computation in question 5). Conversely, assume that $(Sf)(0) = 0$. A solution of (1) must have the form $u = Sf + A\varphi$ for some constant A. A direct computation shows that any such u satisfies $-u'' + u = f + A$. But $u(0) = (Sf)(0) + \frac{A}{\lambda_0} = \frac{A}{\lambda_0}$. Thus we have $-u'' + u = f + \lambda_0 u(0)$, i.e., (1) holds for any A. Therefore the set of all solutions of (1) when $\lambda = \lambda_0$ is $Sf + \mathbb{R}\varphi$.

8.29

2. The existence and uniqueness of a solution $u \in H^1(0, 1)$ comes from Lax–Milgram. In particular, u satisfies

$$\int_0^1 u'v' = \int_0^1 (f - u)v \quad \forall v \in H_0^1(0, 1),$$

and therefore $u' \in H^1(0, 1)$, i.e., $u \in H^2(0, 1)$; moreover, $-u'' + u = f$ on $(0, 1)$. Using the information that $u \in H^2(0, 1)$, we may now write

$$\begin{aligned} a(u, v) = &\int_0^1 (-u'' + u)v + u'(1)v(1) - u'(0)v(0) \\ &+ (u(1) - u(0))(v(1) - v(0)) \\ = &\int_0^1 fv \quad \forall v \in H^1(0, 1). \end{aligned}$$

Consequently,

$$(u'(1) + u(1) - u(0))v(1) - (u'(0) + u(1) - u(0))v(0) = 0 \quad \forall v \in H^1(0, 1).$$

Since $v(0)$ and $v(1)$ are arbitrary, we conclude that

$$u'(1) + u(1) - u(0) = 0 \quad \text{and} \quad u'(0) + u(1) - u(0) = 0.$$

5. Using the same function G as in the proof of Theorem 8.19 we have, taking $v = G(-u)$, $a(u, G(-u)) = \int_0^1 fG(-u) \geq 0$ since $f \geq 0$ and $G \geq 0$. On the other hand,

$$
\begin{aligned}
a(u, G(-u)) &= -\int_0^1 u'^2 G'(-u) - \int_0^1 (-u)G(-u) \\
&\quad + (u(1) - u(0))(G(-u(1)) - G(-u(0))) \\
&\le -\int_0^1 (-u)G(-u),
\end{aligned}
$$

since G is nondecreasing. It follows that

$$\int_0^1 (-u)G(-u) \le 0,$$

and consequently $-u \le 0$.

7. Let $1/\lambda$ be an eigenvalue and let u be a corresponding eigenfunction. Then

$$
\begin{cases}
-u'' + u = \lambda u & \text{on } (0,1), \\
u'(0) = u(0) - u(1), \\
u'(1) = u(0) - u(1).
\end{cases}
$$

Since $a(u,u) = \lambda \int_0^1 u^2 \ge \int_0^1 u^2$, we see that $\lambda \ge 1$. Moreover, $\lambda = 1$ is an eigenvalue corresponding to $u = \text{const}$. Assume now $\lambda > 1$ and set $\alpha = \sqrt{\lambda - 1}$. We must have

$$u(x) = A\cos\alpha x + B\sin\alpha x.$$

In order to satisfy the boundary condition we need to impose

$$
\begin{cases}
B\alpha = A - A\cos\alpha - B\sin\alpha, \\
-A\alpha\sin\alpha + B\alpha\cos\alpha = A - A\cos\alpha - B\sin\alpha.
\end{cases}
$$

This system admits a nontrivial solution iff $2(1-\cos\alpha) + \alpha\sin\alpha = 0$, i.e., $\sin(\alpha/2) = 0$ or $(\alpha/2) + \tan(\alpha/2) = 0$.

$\boxed{8.34}$

1. Let u be a classical solution. Then we have

$$-u'(1)v(1) + u'(0)v(0) + \int_0^1 (u'v' + uv) = \int_0^1 fv \quad \forall v \in H^1(0,1).$$

Let $V = \{v \in H^1(0,1);\ v(0) = v(1)\}$. If $v \in V$ we obtain

$$a(u,v) = \int_0^1 (u'v' + uv) = \int_0^1 fv + kv(0).$$

The weak formulation is

$$u \in V \quad \text{and} \quad a(u,v) = \int_0^1 fv + kv(0) \quad \forall v \in V.$$

2. By Lax–Milgram there exists a unique weak solution $u \in V$, and the corresponding minimization problem is

$$\min_{v \in V} \left\{ \frac{1}{2} \int_0^1 (v'^2 + v^2) - \int_0^1 f v - k v(0) \right\}.$$

3. Clearly, any weak solution u belongs to $H^2(0, 1)$ and satisfies

$$\begin{aligned} -u'' + u &= f \text{ a.e. on } (0, 1), \\ u'(1)v(1) - u'(0)v(0) &= kv(0) \quad \forall v \in V, \end{aligned}$$

i.e.,

$$u'(1) - u'(0) = k.$$

5. The eigenvalues of T are given by $\lambda_k = 1/\mu_k$, where μ_k corresponds to a nontrivial solution of

$$\begin{cases} -u'' + u = \mu_k u & \text{a.e. on } (0, 1), \\ u(1) = u(0), \ u'(1) = u'(0). \end{cases}$$

Therefore $\mu_k \geq 1$ and u is given by

$$u(x) = A \sin\left(\sqrt{\mu_k - 1}x\right) + B \cos\left(\sqrt{\mu_k - 1}x\right)$$

with $\sqrt{\mu_k - 1} = 2\pi k, k = 0, 1, \ldots.$

$\boxed{8.38}$

2. Suppose that $Tu = \lambda u$ with $u \in H^2(\mathbb{R})$ and $u \not\equiv 0$. Clearly $\lambda \neq 0$ and u satisfies

$$-u'' + u = \frac{1}{\lambda} u \quad \text{on } \mathbb{R}.$$

If $\lambda = 1$, we have $u(x) = Ax + B$ for some constants A, B. Since $u \in L^2(\mathbb{R})$ we deduce that $A = B = 0$. Therefore $1 \notin EV(T)$.
If $(\frac{1}{\lambda} - 1) > 0$ we have $u(x) = A \sin \alpha x + B \cos \alpha x$, with $\alpha = \sqrt{\frac{1}{\lambda} - 1}$. The condition $u \in L^2(\mathbb{R})$ yields again $A = B = 0$. Similarly, if $(\frac{1}{\lambda} - 1) < 0$ we have no solution, except $u \equiv 0$. Hence $EV(T) = \emptyset$. T cannot be a compact operator. Otherwise we would have $\sigma(T) = \{0\}$ by Theorem 6.8 and then $T \equiv 0$ by Corollary 6.10. But obviously $T \not\equiv 0$ (otherwise any f in $L^2(\mathbb{R})$ would be $\equiv 0$).
3. If $\lambda < 0$, $(T - \lambda I)$ is bijective from $H = L^2(\mathbb{R})$ onto itself, for example by Lax–Milgram and the fact that $(Tf, f) \geq 0 \ \forall f \in H$. Thus $\lambda \in \rho(T)$.
4. If $\lambda > 1 \geq \|T\|$ we have $\lambda \in \rho(T)$ by Proposition 6.7.
6. T is not surjective, since $R(T) \subset H^2(\mathbb{R})$.

7. $T - I$ is not surjective. Indeed, if we try to solve $Tf - f = \varphi$ for a given φ in $L^2(\mathbb{R})$ we are led to $-u'' + u = f$ (letting $u = Tf$) and $u = f + \varphi$. Therefore $u'' = \varphi$ admits a solution $u \in H^2$. Suppose, for example, that $\operatorname{supp}\varphi \subset [0, 1]$. An immediate computation yields $u(x) = 0\ \forall x \le 0$ and $u(x) = 0\ \forall x \ge 1$. Thus $u'(0) = u'(1) = 0$. It follows that $0 = u'(1) - u'(0) = \int_0^1 \varphi$. Therefore the equation $Tf - f = \varphi$ has no solution $f \in L^2(\mathbb{R})$ when $\int_0^1 \varphi \neq 0$. Hence $T - I$ cannot be surjective.
8. $T - \lambda I$ is not surjective. Indeed, if we try to solve $Tf - \lambda f = \varphi$ we are led to $-u'' + u = f$ (letting $u = Tf$) and $u = \lambda f + \varphi$. Therefore $-u'' + u = \frac{1}{\lambda}(u - \varphi)$. Assume again that $\operatorname{supp}\varphi \subset [0, 1]$. We would have $u'' = -\mu^2 u$ outside $[0, 1]$, with $\mu = \sqrt{\frac{1}{\lambda} - 1}$. Therefore $u \equiv 0$ outside $[0, 1]$ and consequently $u(0) = u'(0) = u(1) = u'(1) = 0$. The equation $-u'' + (1 - \frac{1}{\lambda})u = -\frac{1}{\lambda}\varphi$ implies that $\int_0^1 \varphi v = 0$ for any solution v of $-v'' = \mu^2 v$ on $(0, 1)$; for example $\int_0^1 \varphi(x) \sin \mu x = 0$. Therefore the equation $Tf - \lambda f = \varphi$ has no solution $f \in L^2(\mathbb{R})$ when $\int_0^1 \varphi(x) \sin \mu x \neq 0$. Consequently $(T - \lambda I)$ is not surjective.

$\boxed{8.39}$

2. We have $v^2 \le \frac{1}{2}v^4 + \frac{1}{2} \quad \forall v \in \mathbb{R}$, and thus

$$\varphi(v) \ge \frac{1}{2}\|v\|_{H^1}^2 - \frac{1}{4} - \|f\|_{L^2}\, \|v\|_{H^1}.$$

   Therefore $\varphi(v) \to \infty$ as $\|v\|_{H^1} \to \infty$.
3. The uniqueness follows from the fact that φ is strictly convex on $H^1(0, 1)$; this is a consequence of the strict convexity of the function $t \mapsto t^4$ on $\mathbb{R}$.
4. We have

$$\begin{aligned}\varphi(u + \varepsilon v) = &\frac{1}{2}\int_0^1 (u'^2 + 2\varepsilon u'v' + \varepsilon^2 v'^2)\\ &+ \frac{1}{4}\int_0^1 (u^4 + 4\varepsilon u^3 v + 6\varepsilon^2 u^2 v^2 + 4\varepsilon^3 u v^3 + \varepsilon^4 v^4)\\ &- \int_0^1 f(u + \varepsilon v).\end{aligned}$$

   Writing that $\varphi(u) \le \varphi(u + \varepsilon v)$ gives

$$\int_0^1 (u'v' + u^3 v - f v) + A_\varepsilon \ge 0,$$

   where $A_\varepsilon \to 0$ as $\varepsilon \to 0$. Passing to the limit as $\varepsilon \to 0$ and choosing $\pm v$ yields

$$\int_0^1 (u'v' + u^3 v - f v) = 0 \quad \forall v \in H^1(0, 1).$$

6. From the convexity of the function $t \mapsto t^4$ we have

$$\frac{1}{4}v^4 - \frac{1}{4}u^4 \geq u^3(v-u) \quad \forall u, v \in \mathbb{R}.$$

On the other hand, we clearly have

$$\frac{1}{2}v'^2 - \frac{1}{2}u'^2 \geq u'(v'-u') \text{ a.e. on } (0,1) \quad \forall u, v \in H^1(0,1).$$

Thus $\quad \forall u, v \in H^1(0,1)$

$$\varphi(v) - \varphi(u) \geq \int_0^1 u'(v'-u') + \int_0^1 u^3(v-u) - \int_0^1 f(v-u).$$

If u is a solution of (3) we have

$$\int_0^1 u'(v'-u') + \int_0^1 u^3(v-u) = \int_0^1 f(v-u) \quad \forall v \in H^1(0,1),$$

and therefore $\varphi(u) \leq \varphi(v)\ \forall v \in H^1(0,1)$.

9. We claim that $\psi(v) \to +\infty$ as $\|v\|_{H^1} \to \infty$. Indeed, this boils down to showing that for every constant C the set $\{v \in H^1(0,1);\ \psi(v) \leq C\}$ is bounded in $H^1(0,1)$. If $\psi(v) \leq C$ write

$$\int_0^1 fv = \int_0^1 f(v - v(0)) + v(0) \leq \|f\|_{L^2}\left(\|v'\|_{L^2} + |v(0)|\right),$$

so that $\|v'\|_{L^2}$ and $|v(0)|$ are bounded (why?). Hence $\|v\|_{L^2} \leq \|v'\|_{L^2} + |v(0)|$ is also bounded, so that $\|v\|_{H^1}$ is bounded. For the uniqueness of the minimizer check that $\psi(\frac{u_1+u_2}{2}) \leq \frac{1}{2}(\psi(u_1) + \psi(u_2))$, and equality holds iff $u_1' = u_2'$, and $u_1(0) = u_2(0)$, i.e., $u_1 = u_2$.
We have

$$\begin{aligned}\psi(u+\varepsilon v) = \frac{1}{2}\int_0^1 (u'^2 + 2\varepsilon u'v' + \varepsilon^2 v'^2) \\ + \frac{1}{4}\left(u^4(0) + 4\varepsilon u^3(0)v(0) + \cdots + \varepsilon^4 v^4(0)\right) - \int_0^1 f(u+\varepsilon v).\end{aligned}$$

If u is a minimizer of ψ we write $\psi(u) \leq \psi(u+\varepsilon v)$, and obtain

$$\int_0^1 (u'v' - fv) + u^3(0)v(0) + B_\varepsilon \geq 0,$$

where $B_\varepsilon \to 0$ as $\varepsilon \to 0$. Passing to the limit as $\varepsilon \to 0$, and choosing $\pm v$ yields

$$\text{(S1)} \qquad \int_0^1 (u'v' - fv) + u^3(0)v(0) = 0 \quad \forall v \in H^1(0,1).$$

Consequently, $u \in H^2(0,1)$ satisfies

(S2) $$-u'' = f \quad \text{a.e. on } (0,1).$$

Returning to (S1) and using (S2) yields

$$u'(1)v(1) - u'(0)v(0) + u^3(0)v(0) = 0 \quad \forall v \in H^1(0,1),$$

so that

(S3) $$u'(1) = 0, \quad u'(0) = u^3(0).$$

Conversely, any function u satisfying (S2) and (S3) is a minimizer of ψ: the argument is the same as in question 6. In this case we have an explicit solution. The general solution of (S2) is given by

$$u(x) = -\int_0^x (x-t)f(t)dt + Ax + B,$$

and then (S3) is equivalent to

$$A = \int_0^1 f(t)dt, \quad \text{with } A = B^3.$$

$\boxed{8.42}$

2. Differentiating the equation

(S1) $$v(x) = p^{1/4}(t)u(t)$$

with respect to t gives

$$v'(x)p^{-1/2}(t) = \frac{1}{4}p^{-3/4}(t)p'(t)u(t) + p^{1/4}(t)u'(t).$$

Thus

(S2) $$\begin{aligned} p(t)u'(t) &= v'(x)p^{1/4}(t) - \frac{1}{4}p'(t)u(t) \\ &= v'(x)p^{1/4}(t) - \frac{1}{4}p'(t)p^{-1/4}(t)v(x). \end{aligned}$$

Differentiating (S2) with respect to t gives

(S3) $$(pu')' = v''(x)p^{-1/4}(t) - \frac{1}{4}p''(t)p^{-1/4}(t)v(x) + \frac{1}{16}p'(t)^2p^{-5/4}(t)v(x).$$

Combining (S3) with the equation $-(pu')' + qu = \mu u$ on $(0,1)$ yields

$$v''(x)p^{-1/4}(t) - \frac{1}{4}p''(t)p^{-1/4}(t)v(x) + \frac{1}{16}p'(t)^2p^{-5/4}(t)v(x) = (q(t) - \mu)p^{-1/4}(t)v(x). \tag{S4}$$

Hence v satisfies

$$-v'' + a(x)v = \mu v \text{ on } (0, L),$$

where

$$a(x) = q(t) + \frac{1}{4}p''(t) - \frac{1}{16}p'(t)^2p^{-1}(t).$$

Problems

The numbers in parentheses refer to the chapters in the book whose knowledge is needed to solve the problem.

PROBLEM 1 (1, 4 only for question 9)

Extreme points; the Krein–Milman theorem

Let E be an n.v.s. and let $K \subset E$ be a convex subset. A point $a \in K$ is said to be an *extreme point* if

$$tx + (1-t)y \neq a \quad \forall t \in (0,1), \quad \forall x, y \in K \text{ with } x \neq y.$$

1. Check that $a \in K$ is an extreme point iff the set $K \setminus \{a\}$ is convex.
2. Let a be an extreme point of K. Let $(x_i)_{1 \le i \le n}$ be a finite sequence in K and let $(\alpha_i)_{1 \le i \le n}$ be a finite sequence of real numbers such that $\alpha_i > 0\ \forall i$, $\sum \alpha_i = 1$, and $\sum \alpha_i x_i = a$. Prove that $x_i = a\ \forall i$.

   In what follows we assume that $K \subset E$ is a nonempty compact convex subset of E. A *subset* $M \subset K$ is said to be an *extreme set* if M is nonempty, closed, and whenever $x, y \in K$ are such that $tx + (1-t)y \in M$ for some $t \in (0,1)$, then $x \in M$ and $y \in M$.

3. Let $a \in K$. Check that a is an extreme point iff $\{a\}$ is an extreme set.

   Our first goal is to show that every extreme set contains at least one extreme point.

4. Let $A \subset K$ be an extreme set and let $f \in E^\star$. Set

$$B = \left\{ x \in A;\ \langle f, x \rangle = \max_{y \in A} \langle f, y \rangle \right\}.$$

   Prove that B is an extreme subset of K.

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5. Let $M \subset K$ be an extreme set of K. Consider the collection $\mathcal{F}$ of all the extreme sets of K that are contained in M; $\mathcal{F}$ is equipped with the following ordering:

$$A \le B \quad \text{if} \quad B \subset A.$$

   Prove that $\mathcal{F}$ has a maximal element M_0.

6. Prove that M_0 is reduced to a single point.

   [**Hint**: Use Hahn–Banach and question 4.]

7. Conclude.

8. Prove that K coincides with the closed convex hull of all its extreme points.

   [**Hint**: Argue by contradiction and use Hahn–Banach.]

9. Determine the set $\mathcal{E}$ of all the extreme points of B_E (= the closed unit ball of E) in the following cases:

   (a) $E = \ell^\infty$,
   (b) $E = c$,
   (c) $E = c_0$,
   (d) $E = \ell^1$,
   (e) $E = \ell^p$ with $1 < p < \infty$,
   (f) $E = L^1(\mathbb{R})$.

   [For the notation see Section 11.3].

PROBLEM 2 (1, 2 only for question B4)

Subdifferentials of convex functions

Let E be an n.v.s. and let $\varphi : E \to (-\infty, +\infty]$ be a convex function such that $\varphi \not\equiv +\infty$. For every $x \in E$ the *subdifferential* of φ is defined by

$$\begin{cases} \partial\varphi(x) = \{f \in E^\star;\, \varphi(y) - \varphi(x) \ge \langle f, y - x\rangle \quad \forall y \in E\} & \text{if } x \in D(\varphi), \\ \partial\varphi(x) = \emptyset & \text{if } x \notin D(\varphi), \end{cases}$$

and we set

$$D(\partial\varphi) = \{x \in E;\ \partial\varphi(x) \ne \emptyset\},$$

so that $D(\partial\varphi) \subset D(\varphi)$. Construct an example for which this inclusion is strict.

- A -

1. Show that $\partial\varphi(x)$ is a closed convex subset of $E^\star$.

2. Let $x_1, x_2 \in D(\partial\varphi)$, $f_1 \in \partial\varphi(x_1)$, and $f_2 \in \partial\varphi(x_2)$. Prove that

$$\langle f_1 - f_2, x_1 - x_2\rangle \ge 0.$$

3. Prove that
$$f \in \partial\varphi(x) \Longleftrightarrow \varphi(x) + \varphi^\star(f) = \langle f, x\rangle.$$

4. Determine $\partial\varphi$ in the following cases:

   (a) $\varphi(x) = \frac{1}{2}\|x\|^2$,
   (b) $\varphi(x) = \|x\|$,
   (c) $\varphi(x) = I_K(x)$ (the indicator function of K), where $K \subset E$ is a nonempty convex set (resp. a linear subspace),
   (d) $\varphi(x)$ is a differentiable convex function on E.

   [**Hint**: In the cases (a), (b), $\partial\varphi$ is related to the duality map F defined in Remark 2 of Chapter 1; see also Exercise 1.1.]

5. Let $\psi : E \to (-\infty, +\infty]$ be another convex function such that $\psi \not\equiv +\infty$. Assume that $D(\varphi) \cap D(\psi) \neq \emptyset$. Prove that
$$\partial\varphi(x) + \partial\psi(x) \subset \partial(\varphi + \psi)(x) \quad \forall x \in E$$
   (with the convention that $A + B = \emptyset$ if either $A = \emptyset$ or $B = \emptyset$). Construct an example for which this inclusion is strict.

- B -

Throughout part B we assume that $x_0 \in E$ satisfies the assumption

(1) $\quad \exists M \in \mathbb{R}$ and $\exists R > 0$ such that $\varphi(x) \le M \quad \forall x \in E$ with $\|x - x_0\| \le R$.

1. Prove that $\partial\varphi(x_0) \neq \emptyset$.

   [**Hint**: Use Hahn–Banach in $E \times \mathbb{R}$.]

2. Prove that $\|f\| \le \frac{1}{R}(M - \varphi(x_0)) \quad \forall f \in \partial\varphi(x_0)$.

3. Deduce that $\forall r < R$, $\exists L \ge 0$ such that
$$|\varphi(x_1) - \varphi(x_2)| \le L\|x_1 - x_2\| \quad \forall x_1, x_2 \in E \text{ with } \|x_i - x_0\| \le r,\ i = 1, 2.$$

   [See also Exercise 2.1 for an alternative proof.]

4. Assume here that E is a Banach space and that φ is l.s.c. Prove that
$$\operatorname{Int} D(\partial\varphi) = \operatorname{Int} D(\varphi).$$

5. Prove that for every $y \in E$ one has
$$\lim_{t\downarrow 0} \frac{\varphi(x_0 + ty) - \varphi(x_0)}{t} = \max_{f \in \partial\varphi(x_0)} \langle f, y\rangle.$$

[**Hint**: Look at Exercise 1.25, question 5.]

6. Let $\psi : E \to (-\infty, +\infty]$ be a convex function such that $x_0 \in D(\psi)$. Prove that

$$\partial\varphi(x) + \partial\psi(x) = \partial(\varphi + \psi)(x) \quad \forall x \in E.$$

[**Hint**: Given $f_0 \in \partial(\varphi + \psi)(x)$, apply Theorem 1.12 to the functions $\widetilde{\varphi}(y) = \varphi(y) - \varphi(x) - \langle f_0, y - x\rangle$ and $\widetilde{\psi}(y) = \psi(y) - \psi(x)$.]

- C -

1. Let $\varphi : E \to \mathbb{R}$ be a convex function such that $\varphi(x) \leq k\|x\| + C$ $\forall x \in E$, for some constants $k \geq 0$ and C. Prove that

$$|\varphi(x_1) - \varphi(x_2)| \leq k\|x_1 - x_2\| \quad \forall x_1, x_2 \in E.$$

What can one say about $D(\varphi^\star)$?

2. Let $A \subset \mathbb{R}^n$ be open and convex. Let $\varphi : A \to \mathbb{R}$ be a convex function. Prove that φ is continuous on A.

- D -

Let $\varphi : E \to \mathbb{R}$ be a continuous convex function and let

$$C = \{x \in E;\ \varphi(x) \leq 0\}.$$

Assume that there exists some $x_0 \in E$ such that $\varphi(x_0) < 0$. Given $x \in C$ prove that $f \in \partial I_C(x)$ iff there exists some $\lambda \in \mathbb{R}$ such that $f \in \lambda\partial\varphi(x)$ with $\lambda = 0$ if $\varphi(x) < 0$, and $\lambda \geq 0$ if $\varphi(x) = 0$.

PROBLEM 3 (1)

The theorems of Ekeland, Brönsted–Rockafellar,
and Bishop–Phelps; the ε-subdifferential

- A -

Let M be a nonempty complete metric space equipped with the distance $d(x, y)$. Let $\psi : M \to (-\infty, +\infty]$ be an l.s.c. function that is bounded below and such that $\psi \not\equiv +\infty$. Our goal is to prove that there exists some $a \in M$ such that

$$\psi(x) - \psi(a) + d(x, a) \geq 0 \quad \forall x \in M.$$

Given $x \in M$ set

$$S(x) = \{y \in M;\ \psi(y) - \psi(x) + d(x, y) \leq 0\}.$$

1. Check that $x \in S(x)$, and that $y \in S(x) \Rightarrow S(y) \subset S(x)$.

2. Fix any sequence of real numbers (ε_n) with $\varepsilon_n > 0 \quad \forall n$ and $\varepsilon_n \to 0$. Given $x_0 \in M$, one constructs by induction a sequence (x_n) as follows: once x_n is known, pick any element x_{n+1} satisfying

$$\begin{cases} x_{n+1} \in S(x_n), \\ \psi(x_{n+1}) \leq \inf\limits_{x \in S(x_n)} \psi(x) + \varepsilon_{n+1}. \end{cases}$$

Check that $S(x_{n+1}) \subset S(x_n) \quad \forall n$ and that

$$\psi(x_{n+p}) - \psi(x_n) + d(x_n, x_{n+p}) \leq 0 \quad \forall n, \quad \forall p.$$

Deduce that (x_n) is a Cauchy sequence, and so it converges to a limit, denoted by a.

3. Prove that a satisfies the required property.

[**Hint**: Given $x \in M$, consider two cases: either $x \in S(x_n)\ \forall n$, or $\exists N$ such that $x \notin S(x_N)$.]

4. Give a geometric interpretation.

- B -

Let E be a Banach space and let $\varphi : E \to (-\infty, +\infty]$ be a convex l.s.c. function such that $\varphi \not\equiv +\infty$. Given $\varepsilon > 0$ and $x \in D(\varphi)$, set

$$\partial_\varepsilon \varphi(x) = \{f \in E^\star;\ \ \varphi(x) + \varphi^\star(f) - \langle f, x\rangle \leq \varepsilon\}.$$

Check that $\partial_\varepsilon \varphi(x) \neq \emptyset$.

Our purpose is to show that given any $x_0 \in D(\varphi)$ and any $f_0 \in \partial_\varepsilon \varphi(x_0)$ the following property holds:

$$\begin{cases} \forall \lambda > 0,\ \exists x_1 \in D(\varphi) \text{ and } \exists f_1 \in E^\star \text{ with } f_1 \in \partial\varphi(x_1) \\ \text{such that } \|x_1 - x_0\| \leq \varepsilon/\lambda \text{ and } \|f_1 - f_0\| \leq \lambda. \end{cases}$$

(The *subdifferential* $\partial\varphi$ is defined in Problem 2; it is recommended to solve Problem 2 before this one.)

1. Consider the function ψ defined by

$$\psi(x) = \varphi(x) + \varphi^\star(f_0) - \langle f_0, x\rangle.$$

Prove that there exists some $x_1 \in E$ such that $\|x_1 - x_0\| \leq \varepsilon/\lambda$ and

$$\psi(x) - \psi(x_1) + \lambda\|x - x_1\| \geq 0 \quad \forall x \in E.$$

[**Hint**: Use the result of part A on the set

$$M = \{x \in E;\ \psi(x) \le \psi(x_0) - \lambda\|x - x_0\|\}.]$$

2. Conclude.

   [**Hint**: Use the result of Problem 2, question B6.]

3. Deduce that
$$\overline{D(\partial\varphi)} = \overline{D(\varphi)} \quad \text{and} \quad \overline{R(\partial\varphi)} = \overline{D(\varphi^\star)},$$
   where $R(\partial\varphi) = \{f \in E^\star;\ \exists x \in D(\partial\varphi) \text{ such that } f \in \partial\varphi(x)\}$.

- C -

Let E be a Banach space and let $C \subset E$ be a nonempty closed convex set.

1. Assuming that C is also bounded, prove that the set
$$\left\{ f \in E^\star;\ \sup_{x\in C}\langle f, x\rangle \text{ is achieved} \right\}$$
   is dense in $E^\star$.

   [**Hint**: Apply the results of part B to the function $\varphi = I_C$.]

2. One says that a closed hyperplane H of E is a *supporting hyperplane* to C at a point $x \in C$ if H separates C and $\{x\}$. Prove that the set of points in C that admit a supporting hyperplane is dense in the boundary of $C (= C \setminus \operatorname{Int} C)$.

PROBLEM 4 (1)

Asplund's theorem and strictly convex norms

Let E be an n.v.s. and let $\varphi_0, \psi_0 : E \to [0, \infty)$ be two convex functions such that $\varphi_0(0) = \psi_0(0) = 0$ and $0 \le \psi_0(x) \le \varphi_0(x)$ $\forall x \in E$. Starting with φ_0 and ψ_0 one defines by induction two sequences of functions (φ_n) and (ψ_n) as follows:

$$\varphi_{n+1}(x) = \frac{1}{2}(\varphi_n(x) + \psi_n(x))$$

and
$$\psi_{n+1}(x) = \frac{1}{2}\inf_{y\in E}\{\varphi_n(x+y) + \psi_n(x-y)\} = \frac{1}{2}(\varphi_n \nabla \psi_n)(2x).$$

[Before starting this problem solve Exercise 1.23, which deals with the inf-convolution ∇.]

- A -

1. Check that $0 \le \psi_n(x) \le \varphi_n(x)$ $\forall x \in E,\ \forall n$ and that $\varphi_n(0) = \psi_n(0) = 0$.

2. Check that φ_n and ψ_n are convex.

3. Prove that the sequence (φ_n) is nonincreasing and that the sequence (ψ_n) is nondecreasing. Deduce that (φ_n) and (ψ_n) have a common limit, denoted by θ, with $\psi_0 \le \theta \le \varphi_0$, and that θ is convex.

4. Prove that $\varphi_n^\star \uparrow \theta^\star$.

5. Prove that $\psi_{n+1}^\star = \frac{1}{2}(\varphi_n^\star + \psi_n^\star)$, and deduce that $\psi_n^\star \downarrow \theta^\star$ when $D(\psi_0^\star) = E^\star$.

6. Assume that there exists some $x_0 \in D(\varphi_0)$ such that φ_0 is continuous at x_0. Prove that φ_n and ψ_n are also continuous at x_0.

   [**Hint**: Apply question 2 of Exercise 2.1.]

   Deduce that

$$\varphi_{n+1}^\star(f) = \frac{1}{2} \inf_{g \in E^\star} \{\varphi_n^\star(f+g) + \psi_n^\star(f-g)\}.$$

- B -

Let $\varphi : E \to [0, +\infty)$ be a convex function that is *homogeneous of order two*, i.e., $\varphi(\lambda x) = \lambda^2 \varphi(x)\ \forall \lambda \in \mathbb{R},\ \forall x \in E$. Prove that

$$\varphi(x+y) \le \frac{1}{t}\varphi(x) + \frac{1}{1-t}\varphi(y) \quad \forall x, y \in E, \quad \forall t \in (0,1).$$

Deduce that the function $x \mapsto \sqrt{\varphi(x)}$ is a seminorm and conversely. Establish also that

$$\text{(1)} \qquad 4\varphi(x) \le \frac{1}{t}\varphi(x+y) + \frac{1}{1-t}\varphi(x-y) \quad \forall x, y \in E, \quad \forall t \in (0,1).$$

In what follows we assume, in addition, that φ_0 and ψ_0 are homogeneous of order two and that there is a constant $C > 0$ such that

$$\varphi_0(x) \le (1+C)\psi_0(x) \quad \forall x \in E.$$

1. Check that φ_n, ψ_n, and θ are homogeneous of order two.

2. Prove that for every n, one has

$$\varphi_n(x) \le \left(1 + \frac{C}{4^n}\right)\psi_n(x) \quad \forall x \in E.$$

   [**Hint**: Argue by induction and use (1).]

3. Assuming that either φ_0 or ψ_0 is strictly convex, prove that θ is strictly convex (for the definition of a strictly convex function, see Exercise 1.26).

[**Hint**: Use the inequality established in question B2. It is convenient to split φ_n as $\varphi_n = \theta_n + \frac{1}{2^n}\varphi_0$, where θ_n is some convex function that one should not try to write down explicitly. Note that

$$\theta_n + \left(\frac{1}{2^n} - \frac{C}{4^n}\right)\varphi_0 \leq \theta \leq \theta_n + \frac{1}{2^n}\varphi_0.]$$

- C -

Assume that there exist on E two equivalent norms denoted by $\| \; \|_1$ and $\| \; \|_2$. Let $\| \; \|_1^\star$ and $\| \; \|_2^\star$ denote the corresponding dual norms on $E^\star$. Assume that the norms $\| \; \|_1$ and $\| \; \|_2^\star$ are strictly convex. Using the above results, prove that there exists a third norm $\| \; \|$, equivalent to $\| \; \|_1$ (and to $\| \; \|_2$), that is strictly convex as well as its dual norm $\| \; \|^\star$.

PROBLEM 5 (1, 2)

Positive linear functionals

Let E be an n.v.s. and let P be a convex cone with vertex at 0, i.e., $\lambda x + \mu y \in P$, $\forall x, y \in P, \forall \lambda, \mu > 0$. Set $F = P - P$, so that F is a linear subspace. Consider the following two properties:

(i) Every linear functional f on E such that $f(x) \geq 0 \; \forall x \in P$, is continuous on E.
(ii) F is a closed subspace of finite codimension.

The goal of this problem is to show that (i) $\Rightarrow$ (ii) and that conversely, (ii) $\Rightarrow$ (i) when E is a Banach space and P is closed.

- A -

Throughout part A we assume (i).

1. Prove that F is closed.

   [**Hint**: Given any $x_0 \notin F$, construct a linear functional f on E such that $f(x_0) = 1$ and $f = 0$ on F.]

2. Let M be any linear subspace of E such that $M \cap F = \{0\}$. Prove that $\dim M < +\infty$.

   [**Hint**: Use Exercise 1.5.]

3. Deduce that (i) $\Rightarrow$ (ii).

- B -

Throughout part B we assume that E is a Banach space and that P is closed.

1. Assume here in addition that

   (iii) $$P - P = E.$$

Prove that there exists a constant $C > 0$ such that every $x \in E$ has a decomposition $x = y - z$ with $y, z \in P$, $\|y\| \le C\|x\|$ and $\|z\| \le C\|x\|$.

[**Hint**: Consider the set

$$K = \{x = y - z \text{ with } y, z \in P, \|y\| \le 1 \text{ and } \|z\| \le 1\}$$

and follow the idea of the proof of the open mapping theorem (Theorem 2.6).]

2. Deduce that (iii) $\Rightarrow$ (i).

   [**Hint**: Argue by contradiction and consider a sequence (x_n) in E such that $\|x_n\| \le 1/2^n$ and $f(x_n) \ge 1$. Then, use the result of question B1.]

3. Prove that (ii) $\Rightarrow$ (i).

- C -

In the following examples determine $F = P - P$ and examine whether (i) or (ii) holds:

(a) $E = C([0, 1])$ with its usual norm and

$$P = \{u \in E;\, u(t) \ge 0 \quad \forall t \in [0, 1]\},$$

(b) $E = C([0, 1])$ with its usual norm and

$$P = \{u \in E;\, u(t) \ge 0 \quad \forall t \in [0, 1], \text{ and } u(0) = u(1) = 0\},$$

(c) $E = \{u \in C^1([0, 1]);\, u(0) = u(1) = 0\}$ with its usual norm and

$$P = \{u \in E;\, u(t) \ge 0 \quad \forall t \in [0, 1]\}.$$

PROBLEM 6 (1, 2)

Let E be a Banach space and let $A : D(A) \subset E \to E^\star$ be a closed unbounded operator satisfying

$$\langle Ax, x\rangle \ge 0 \quad \forall x \in D(A).$$

- A -

Our purpose is to show that the following properties are equivalent:

(i) $\forall x \in D(A),\ \exists C(x) \in \mathbb{R}$ such that $\langle Ay, y - x\rangle \ge C(x) \quad \forall y \in D(A)$,

(ii) $\exists k \ge 0$ such that

$$|\langle Ay, x\rangle| \le k(\|x\| + \|Ax\|)\sqrt{\langle Ay, y\rangle} \quad \forall x, y \in D(A).$$

1. Prove that (ii) $\Rightarrow$ (i).

Conversely, assume (i).

2. Prove that there exist two constants $R > 0$ and $M \geq 0$ such that

$$\langle Ay, x - y\rangle \leq M \quad \forall y \in D(A) \quad \text{and} \quad \forall x \in D(A) \text{ with } \|x\| + \|Ax\| \leq R.$$

[**Hint**: Consider the function $\varphi(x) = \sup_{y\in D(A)}\langle Ay, x - y\rangle$ and apply Exercise 2.1.]

3. Deduce that

$$|\langle Ay, x\rangle|^2 \leq 4M\langle Ay, y\rangle \quad \forall y \in D(A) \quad \text{and} \quad \forall x \in D(A) \text{ with } \|x\|+\|Ax\| \leq R.$$

4. Conclude.

- B -

In what follows assume that $D(A) = E$. Let $\alpha > 0$.

1. Prove that the following properties are equivalent:

(iii) $$\|Ay\| \leq \alpha\sqrt{\langle Ay, y\rangle} \quad \forall y \in E,$$

(iv) $$\langle Ay, y - x\rangle \geq -\frac{1}{4}\alpha^2\|x\|^2 \quad \forall x, y \in E.$$

[**Hint**: Use the same method as in part A.]

2. Let $A^\star \in \mathcal{L}(E^{\star\star}, E^\star)$ be the adjoint of A. Prove that (iv) is equivalent to

(iv*) $$\langle A^\star y, y - x\rangle \geq -\frac{1}{4}\alpha^2\|x\|^2 \quad \forall x, y \in E.$$

3. Deduce that (iii) is equivalent to

(iii*) $$\|A^\star y\| \leq \alpha\sqrt{\langle A^\star y, y\rangle} \quad \forall y \in E.$$

PROBLEM 7 (1, 2)

The adjoint of the sum of two unbounded linear operators

Let E be a Banach space. Given two closed linear subspace M and N in E, set

$$\rho(M, N) = \sup_{\substack{x\in M\\ \|x\|\leq 1}} \operatorname{dist}(x, N).$$

- A -

1. Check that $\rho(M, N) \leq 1$; if, in addition, $N \subset M$ with $N \neq M$, prove that $\rho(M, N) = 1$.

[**Hint**: Use Lemma 6.1.]

2. Let L, M, and N be three closed linear subspaces.
Set $a = \rho(M, N)$ and $b = \rho(N, L)$. Prove that $\rho(M, L) \leq a + b + ab$.
Deduce that if $L \subset M$, $a \leq 1/3$, and $b \leq 1/3$, then $L = M$.

3. Prove that $\rho(M, N) = \rho(N^{\perp}, M^{\perp})$.

[**Hint**: Check with the help of Theorem 1.12 that $\forall x \in E$ and $\forall f \in E^{\star}$

$$\operatorname{dist}(x, N) = \sup_{\substack{g \in N^{\perp} \\ \|g\| \leq 1}} \langle g, x \rangle \text{ and } \operatorname{dist}(f, M^{\perp}) = \sup_{\substack{y \in M \\ \|y\| \leq 1}} \langle f, y \rangle.]$$

- B -

Let E and F be two Banach spaces; $E \times F$ is equipped with the norm $\|[u, v]\|_{E \times F} = \|u\|_E + \|v\|_F$. Given two unbounded operators $A : D(A) \subset E \to F$ and $B : D(B) \subset E \to F$ that are densely defined and closed, set

$$\rho(A, B) = \rho(G(A), G(B)).$$

1. Prove that $\rho(A, B) = \rho(B^{\star}, A^{\star})$.

2. Prove that if $D(A) \cap D(B)$ is dense in E, then

$$A^{\star} + B^{\star} \subset (A + B)^{\star}.$$

[Recall that $D(A + B) = D(A) \cap D(B)$ and $D(A^{\star} + B^{\star}) = D(A^{\star}) \cap D(B^{\star})$.]

It may happen that the inclusion $A^{\star} + B^{\star} \subset (A + B)^{\star}$ is strict—construct such an example. Our purpose is to prove that equality holds under some additional assumptions.

3. Assume

$$\text{(H)} \qquad \begin{cases} D(A) \subset D(B) \text{ and there exist constants } k \in [0, 1) \text{ and } C \geq 0 \\ \text{such that } \|Bu\| \leq k\|Au\| + C\|u\| \quad \forall u \in D(A). \end{cases}$$

Prove that $A + B$ is closed and that $\rho(A, A + B) \leq k + C$.

4. In addition to (H) assume also

$$\text{(H}^{\star}\text{)} \qquad \begin{cases} D(A^{\star}) \subset D(B^{\star}) \text{ and there exist constants } k^{\star} \in [0, 1) \text{ and} \\ C^{\star} \geq 0 \text{ such that } \|B^{\star}v\| \leq k^{\star}\|A^{\star}v\| + C^{\star}\|v\| \quad \forall v \in D(A^{\star}). \end{cases}$$

Let $\varepsilon > 0$ be such that $\varepsilon(k + C) \leq 1/3$ and $\varepsilon(k^{\star} + C^{\star}) \leq 1/3$.

Prove that $A + \varepsilon B^{\star} = A^{\star} + \varepsilon B^{\star}$.

5. Assuming (H) and (H*) prove that $(A+B)^\star = A^\star + B^\star$.

   [**Hint**: Use successive steps. Check that the following inequality holds $\forall t \in [0,1]$:

$$\|Bu\| \le \frac{k}{1-k}\|Au + tBu\| + \frac{C}{1-k}\|u\| \quad \forall u \in D(A).]$$

PROBLEM 8 (2, 3, 4 only for question 6)

Weak convergence in ℓ^1. Schur's theorem.

Let $E = \ell^1$, so that $E^\star = \ell^\infty$ (see Section 11.3). Given $x \in E$ write

$$x = (x_1, x_2, \dots, x_i, \dots) \quad \text{and} \quad \|x\|_1 = \sum_{i=1}^{\infty} |x_i|,$$

and given $f \in E^\star$ write

$$f = (f_1, f_2, \dots, f_i, \dots) \quad \text{and} \quad \|f\|_\infty = \sup_i |f_i|.$$

Let (x^n) be a sequence in E such that $x^n \rightharpoonup 0$ weakly $\sigma(E, E^\star)$. Our goal is to show that $\|x^n\|_1 \to 0$.

1. Given $f, g \in B_{E^\star}$ (i.e., $\|f\|_\infty \le 1$ and $\|g\|_\infty \le 1$) set

$$d(f,g) = \sum_{i=1}^{\infty} \frac{1}{2^i}|f_i - g_i|.$$

   Check that d is a metric on $B_{E^\star}$ and that $B_{E^\star}$ is compact for the corresponding topology.

2. Given $\varepsilon > 0$ set

$$F_k = \{f \in B_{E^\star};\ |\langle f, x^n\rangle| \le \varepsilon \quad \forall n \ge k\}.$$

   Prove that there exist some $f^0 \in B_{E^\star}$, a constant $\rho > 0$, and an integer k_0 such that

$$[f \in B_{E^\star} \text{ and } d(f, f^0) < \rho] \Rightarrow [f \in F_{k_0}].$$

   [**Hint**: Use Baire category theorem.]

3. Fix an integer N such that $(1/2^{N-1}) < \rho$. Prove that

$$\|x^n\|_1 \le \varepsilon + 2\sum_{i=1}^{N} |x_i^n| \quad \forall n \ge k_0.$$

4. Conclude.
5. Using a similar method prove that if (x^n) is a sequence in ℓ^1 such that for every $f \in \ell^\infty$ the sequence $(\langle f, x^n\rangle)$ converges to some limit, then (x^n) converges to a limit strongly in ℓ^1.
6. Consider $E = L^1(0, 1)$, so that $E^\star = L^\infty(0, 1)$. Construct a sequence (u^n) in E such that $u^n \rightharpoonup 0$ weakly $\sigma(E, E^\star)$ and such that $\|u^n\|_1 = 1\ \forall n$.

PROBLEM 9 (1, 2, 3)

Hahn–Banach for the weak* topology and applications

Let E be a Banach space.

- A -

1. Let $A \subset E^\star$ and $B \subset E^\star$ be two nonempty convex sets such that $A \cap B = \emptyset$. Assume that A is open in the topology $\sigma(E^\star, E)$. Prove that there exist some $x \in E$, $x \neq 0$, and a constant α such that the hyperplane $\{f \in E^\star;\ \langle f, x\rangle = \alpha\}$ separates A and B.
2. Assume that $A \subset E^\star$ is closed in $\sigma(E^\star, E)$ and $B \subset E^\star$ is compact in $\sigma(E^\star, E)$. Prove that $A + B$ is closed in $\sigma(E^\star, E)$.
3. Let $A \subset E^\star$ and $B \subset E^\star$ be two nonempty convex sets such that $A \cap B = \emptyset$. Assume that A is closed in $\sigma(E^\star, E)$ and B is compact in $\sigma(E^\star, E)$. Prove that there exist some $x \in E, x \neq 0$, and a constant α such that the hyperplane $\{f \in E^\star;$ $\langle f, x\rangle = \alpha\}$ strictly separates A and B.
4. Let $A \subset E^\star$ be convex. Prove that $\overline{A}^{\sigma(E^\star, E)}$, the closure of A in $\sigma(E^\star, E)$, is convex.

- B -

Here are various applications of the above results:

1. Let $N \subset E^\star$ be a linear subspace. Recall that

$$N^\perp = \{x \in E;\ \langle f, x\rangle = 0 \quad \forall f \in N\}$$

and

$$N^{\perp\perp} = \{f \in E^\star;\ \langle f, x\rangle = 0 \quad \forall x \in N^\perp\}.$$

Prove that $N^{\perp\perp} = \overline{N}^{\sigma(E^\star, E)}$.

What can one say if E is reflexive?

Deduce that c_0 is dense in ℓ^∞ in the topology $\sigma(\ell^\infty, \ell^1)$.

2. Let $\varphi : E \to (-\infty, +\infty]$ be a convex l.s.c. function, $\varphi \not\equiv +\infty$. Prove that $\psi = \varphi^\star$ is l.s.c. in the topology $\sigma(E^\star, E)$.

Conversely, given a convex function $\psi : E^\star \to (-\infty, +\infty]$ that is l.s.c. for the topology $\sigma(E^\star, E)$ and such that $\psi \not\equiv +\infty$, prove that there exists a convex l.s.c. function $\varphi : E \to (-\infty, +\infty]$, $\varphi \not\equiv +\infty$, such that $\psi = \varphi^\star$.

3. Let F be another Banach space and let $A : D(A) \subset E \to F$ be an unbounded linear operator that is densely defined and closed. Prove that
   (i) $\overline{R(A^\star)}^{\sigma(E^\star,E)} = N(A)^\perp$,
   (ii) $\overline{D(A^\star)}^{\sigma(F^\star,F)} = F^\star$.
   What can one say if E (resp. F) is reflexive?

4. Prove—without the help of Lemma 3.3—that $J(B_E)$ is dense in $B_{E^{\star\star}}$ in the topology $\sigma(E^{\star\star}, E^\star)$ (see Lemma 3.4).

5. Let $A : B_E \to E^\star$ be a monotone map, that is,

$$\langle Ax - Ay, x - y\rangle \geq 0 \quad \forall x, y \in B_E.$$

   Set $S_E = \{x \in E;\ \|x\| = 1\}$. Prove that $A(B_E) \subset \overline{\text{conv}\, A(S_E)}^{\sigma(E^\star,E)}$.

PROBLEM 10 (3)

The Eberlein–Šmulian theorem

Let E be a Banach space and let $A \subset E$. Set $B = \overline{A}^{\sigma(E,E^\star)}$. The goal of this problem is to show that the following properties are equivalent:

(P) B is compact in the topology $\sigma(E, E^\star)$.

(Q) Every sequence (x_n) in A has a weakly convergent subsequence.

Moreover, (P) (or (Q)) implies the following property:

(R) $\begin{cases} \text{For every } y \in B \text{ there exists a } \textit{sequence } (y_n) \subset A \\ \text{such that } y_n \rightharpoonup y \text{ weakly } \sigma(E, E^\star). \end{cases}$

- A -

Proof of the claim (P) $\Rightarrow$ (Q).

1. Prove that (P) $\Rightarrow$ (Q) under the additional assumption that E is separable.

   [**Hint**: Consider a set (b_k) in $B_{E^\star}$ that is countable and dense in $B_{E^\star}$ for the topology $\sigma(E^\star, E)$ (why does such a set exist?). Check that the quantity $d(x, y) = \sum_{k=1}^\infty \frac{1}{2^k}|\langle b_k, x - y\rangle|$ is a metric and deduce that B is metrizable for $\sigma(E, E^\star)$.]

2. Show that (P) $\Rightarrow$ (Q) in the general case.

   [**Hint**: Use question A1.]

- B -

For later purpose we shall need the following:

Lemma. *Let F be an* n.v.s. *and let $M \subset F^\star$ be a finite-dimensional vector space. Then there exists a finite subset $(a_i)_{1\le i\le k}$ in B_F such that*

$$\max_{1\le i\le k} \langle g, a_i\rangle \ge \frac{1}{2}\|g\| \quad \forall g \in M.$$

[**Hint**: First choose points $(g_i)_{1\le i\le k}$ in S_M such that $S_M \subset \bigcup_{i=1}^k B(g_i, 1/4)$, where $S_M = \{g \in M;\ \|g\| = 1\}$.]

- C -

Let $\xi \in E^{\star\star}$ be such that $\xi \in \overline{A}^{\sigma(E^{\star\star},E^\star)}$. Using assumption (Q) we shall prove that $\xi \in B$ and that there exists a sequence $(y_k) \subset A$ such that $y_k \rightharpoonup \xi$ in $\sigma(E, E^\star)$.

1. Set $n_1 = 1$ and fix any $f_1 \in B_{E^\star}$. Prove that there exists some $x_1 \in A$ such that

$$|\langle \xi, f_1\rangle - \langle f_1, x_1\rangle| < 1.$$

2. Let $M_1 = [\xi, x_1]$ be the linear space spanned by ξ and x_1. Prove that there exist $(f_i)_{1<i\le n_2}$ in $B_{E^\star}$ such that

$$\max_{1<i\le n_2} \langle \eta, f_i\rangle \ge \frac{1}{2}\|\eta\| \quad \forall \eta \in M_1.$$

   Prove that there exists some $x_2 \in A$ such that

$$|\langle \xi, f_i\rangle - \langle f_i, x_2\rangle| < \frac{1}{2} \quad \forall i,\ 1 \le i \le n_2.$$

3. Iterating the above construction, we obtain two sequences $(x_k) \subset A$ and $(f_i) \subset B_{E^\star}$, and an increasing sequence of integers (n_k) such that

(a) $$\max_{n_k<i\le n_{k+1}} \langle \eta, f_i\rangle \ge \frac{1}{2}\|\eta\| \quad \forall \eta \in M_k = [\xi, x_1, x_2, \dots, x_k],$$

(b) $$|\langle \xi, f_i\rangle - \langle f_i, x_{k+1}\rangle| < \frac{1}{k+1} \quad \forall i, 1 \le i \le n_{k+1}.$$

4. Deduce from (a) that

$$\sup_{i\ge 1}\langle \eta, f_i\rangle \ge \frac{1}{2}\|\eta\| \qquad \forall \eta \in \bigcup_{k=1}^{\infty} M_k = M$$

   and then that

$$\sup_{i\ge 1}\langle \eta, f_i\rangle \ge \frac{1}{2}\|\eta\| \quad \forall \eta \in \overline{M},$$

where $\overline{M}$ denotes the closure of M in $E^{\star\star}$, in the strong topology.

5. Using (b) and assumption (Q), prove that there exists some $x \in B \cap \overline{M}$ such that

$$\langle \xi, f_i \rangle = \langle f_i, x \rangle \quad \forall i \geq 1.$$

Deduce that $\xi = x$ and conclude.

- D -

1. Prove that (Q) $\Rightarrow$ (P).
2. Prove that (Q) $\Rightarrow$ (R).

PROBLEM 11 (3)

A theorem of Banach–Dieudonné–Krein–Šmulian

Let E be a Banach space and let $C \subset E^\star$ be a convex set. Assume that for each integer n, the set $C \cap (nB_{E^\star})$ is closed in the topology $\sigma(E^\star, E)$. The goal of this problem is to show that C is closed in the topology $\sigma(E^\star, E)$.

- A -

Suppose, in addition, that $0 \notin C$. We shall prove that there exists a sequence (x_n) in E such that

$$\text{(1)} \qquad \|x_n\| \to 0 \quad \text{and} \quad \sup_n \langle f, x_n \rangle > 1 \quad \forall f \in C.$$

Let $d = \text{dist}(0, C)$ and consider a sequence $d_n \uparrow +\infty$ such that $d_1 > d$. Set

$$C_k = \{f \in C; \ \|f\| \leq d_k\}.$$

1. Check that the sets C_k are compact in the topology $\sigma(E^\star, E)$. Prove that there exists some $f_0 \in C$ such that $d = \|f_0\| > 0$.

2. Prove that there exists some $x_1 \in E$ such that

$$\langle f, x_1 \rangle > 1 \quad \forall f \in C_1.$$

[**Hint**: Use Hahn–Banach for the weak* topology; see question A3 of Problem 9.]

3. Set $A_1 = \{x_1\}$. Prove that there exists a finite subset $A_2 \subset E$ such that $A_2 \subset \frac{1}{d_1} B_E$ and $\sup_{x \in A_1 \cup A_2} \langle f, x \rangle > 1 \ \forall f \in C_2$.
[**Hint**: For each finite subset $A \subset E$ such that $A \subset \frac{1}{d_1} B_E$ consider the set

$$Y_A = \left\{ f \in C_2; \ \sup_{x \in A_1 \cup A} \langle f, x \rangle \leq 1 \right\},$$

and prove first that $\cap_A Y_A = \emptyset$.]

4. Construct, by induction, a finite subset $A_k \subset E$ such that

$$A_k \subset \frac{1}{d_{k-1}} B_E \quad \text{and} \quad \sup_{x \in \cup_{i=1}^k A_i} \langle f, x \rangle > 1 \quad \forall f \in C_k.$$

5. Construct a sequence (x_n) satisfying (1).

- B -

1. Assume once more that $0 \notin C$. Prove that there exists some $x \in E$ such that

$$\langle f, x \rangle \geq 1 \quad \forall f \in C.$$

[**Hint**: Let (x_n) be a sequence satisfying (1). Consider the operator $T : E^\star \to c_0$ defined by $T(f) = (\langle f, x_n \rangle)_n$ and separate (in c_0) $T(C)$ and the open unit ball of c_0.]

2. Conclude.

PROBLEM 12 (1, 2, 3)

Before starting this problem it is necessary to solve Exercise 1.23.

Let E be a reflexive Banach space and let $\varphi, \psi : E \to (-\infty, +\infty]$ be convex l.s.c. functions such that $D(\varphi) \cap D(\psi) \neq \emptyset$. Set $\theta = \varphi^\star \nabla \psi^\star$.

- A -

We claim that

$$\overline{D((\varphi + \psi)^\star)} = \overline{D(\varphi^\star) + D(\psi^\star)}.$$

1. Prove that $D(\varphi^\star) + D(\psi^\star) \subset D((\varphi + \psi)^\star)$.
2. Prove that θ maps $E^\star$ into $(-\infty, +\infty]$, θ is convex, $D(\theta) = D(\varphi^\star) + D(\psi^\star)$ and $\theta^\star = \varphi + \psi$.
3. Deduce that $\overline{D((\varphi + \psi)^\star)} = \overline{D(\theta)}$ and conclude.

- B -

Assume, in addition, that φ and ψ satisfy

(H) $$\bigcup_{\lambda \geq 0} \lambda (D(\varphi) - D(\psi)) = E.$$

We claim that

(i) $$(\varphi + \psi)^\star = \varphi^\star \nabla \psi^\star,$$

(ii) $$\inf_{x\in E}\{\varphi(x)+\psi(x)\} = \max_{g\in E^\star}\{-\varphi^\star(-g)-\psi^\star(g)\},$$

(iii) $$D((\varphi+\psi)^\star) = D(\varphi^\star)+D(\psi^\star).$$

1. Prove that for every fixed $f \in E^\star$ and $\alpha \in \mathbb{R}$ the set

$$M = \{g \in E^\star;\ \varphi^\star(f-g)+\psi^\star(g) \le \alpha\}$$

is bounded.
[**Hint**: Use assumption (H) and Corollary 2.5.]

2. Let $\alpha \in \mathbb{R}$ be fixed. Let (f_n) and (g_n) be two sequences in $E^\star$ such that (f_n) is bounded and $\varphi^\star(f_n-g_n)+\psi^\star(g_n) \le \alpha$. Prove that (g_n) is bounded.

3. Deduce that θ is l.s.c.

4. Prove (i), (ii), and (iii).
Compare these results with question 3 of Exercise 1.23 and with Theorem 1.12.

PROBLEM 13 (1, 3)

Properties of the duality map. Uniform convexity. Differentiability of the norm

Let E be a Banach space. Recall the definition of the duality map (see Remark 2 in Chapter 1): For every $x \in E$,

$$F(x) = \{f \in E^\star;\ \|f\| = \|x\| \text{ and } \langle f, x\rangle = \|x\|^2\}.$$

Before starting this problem it is useful to solve Exercises 1.1 and 1.25.

- A -

Assume that $E^\star$ is strictly convex, so that $F(x)$ consists of a single element.

1. Check that

$$\lim_{\lambda\to 0}\frac{1}{2\lambda}(\|x+\lambda y\|^2-\|x\|^2) = \langle Fx, y\rangle \quad \forall x, y \in E.$$

[**Hint**: Apply a result of Exercise 1.25; distinguish the cases $\lambda > 0$ and $\lambda < 0$.]

2. Prove that for every $x, y \in E$, the map $t \in \mathbb{R} \mapsto \langle F(x+ty), y\rangle$ is continuous at $t = 0$.

[**Hint**: Use the inequality $\frac{1}{2}(\|v\|^2-\|u\|^2) \ge \langle Fu, v-u\rangle$ with $u = x+ty$ and $v = x+\lambda y$.]

3. Deduce that F is continuous from E strong into $E^\star$ weak*.

[**Hint**: Use the result of Exercise 3.11.]

Prove the same result by a simple direct method in the case that E is reflexive or separable.

4. Check that

$$\langle Fx + Fy, x + y\rangle + \langle Fx - Fy, x - y\rangle = 2(\|x\|^2 + \|y\|^2) \quad \forall x, y \in E.$$

Deduce that

$$\|Fx + Fy\| + \|x - y\| \geq 2 \quad \forall x, y \in E \text{ with } \|x\| = \|y\| = 1.$$

5. Assume, in addition, that E is reflexive and strictly convex. Prove that F is bijective from E onto $E^\star$. Check that F^{-1} coincides with the duality map of $E^\star$.

- B -

In this part we assume that $E^\star$ is uniformly convex.

1. Prove that F is continuous from E strong into $E^\star$ strong.
2. More precisely, prove that F is uniformly continuous on bounded sets of E.

   [**Hint**: Argue by contradiction and apply question A4.]
3. Deduce that the function $\varphi(x) = \frac{1}{2}\|x\|^2$ is differentiable and that its differential is F, i.e., for every $x_0 \in E$ we have

$$\lim_{\substack{x \to x_0 \\ x \neq x_0}} \frac{\varphi(x) - \varphi(x_0) - \langle Fx_0, x - x_0\rangle}{\|x - x_0\|} = 0.$$

- C -

Conversely, assume that for every $x \in E$, the set Fx consists of a single element and that F is uniformly continuous on bounded sets of E. Prove that $E^\star$ is uniformly convex.

[**Hint**: Prove first the inequality

$$\|f + g\| \leq \frac{1}{2}\|f\|^2 + \frac{1}{2}\|g\|^2 - \langle f - g, y\rangle + \sup_{\substack{x \in E \\ \|x\| \leq 1}} \{\varphi(x + y) + \varphi(x - y)\}$$
$$\forall y \in E, \quad \forall f, g \in E^\star.]$$

PROBLEM 14 (1, 3)

Regularization of convex functions by inf-convolution

Let E be a Banach space such that $E^\star$ is uniformly convex. Assume that $\varphi : E \to (-\infty, +\infty]$ is convex l.s.c. and $\varphi \not\equiv +\infty$. The goal of this problem is to show that there exists a sequence (φ_n) of differentiable convex functions such that $\varphi_n \uparrow \varphi$ as $n \uparrow +\infty$.

- A -

For each fixed $x \in E$ consider the function $\Phi_x : E^\star \to (-\infty, +\infty]$ defined by

$$\Phi_x(f) = \frac{1}{2}\|f\|^2 + \varphi^\star(f) - \langle f, x\rangle, \quad f \in E^\star.$$

1. Check that there exists a unique element $f_x \in E^\star$ such that

$$\Phi_x(f_x) = \inf_{f \in E^\star} \Phi_x(f).$$

   Set $Sx = f_x$.

2. Prove that the map $x \mapsto Sx$ is continuous from E strong into $E^\star$ strong.

   [**Hint**: Prove first that S is continuous from E strong into $E^\star$ weak*.]

- B -

Consider the function $\psi : E \to \mathbb{R}$ defined by

$$\psi(x) = \psi(x) = \inf_{y \in E} \left\{ \frac{1}{2}\|x - y\|^2 + \varphi(y) \right\}.$$

We claim that ψ is convex, differentiable, and that its differential coincides with S.

1. Check that ψ is convex and that

$$\text{(i)} \qquad \psi(x) = -\min_{f \in E^\star} \left\{ \frac{1}{2}\|f\|^2 + \varphi^\star(f) - \langle f, x\rangle \right\} \quad \forall x \in E,$$

$$\text{(ii)} \qquad \psi^\star(f) = \frac{1}{2}\|f\|^2 + \varphi^\star(f) \quad \forall f \in E^\star.$$

   [**Hint**: Apply Theorem 1.12.]

2. Deduce that

$$\psi(x) + \psi^\star(Sx) = \langle Sx, x\rangle \quad \forall x \in E$$

   and that

$$|\psi(y) - \psi(x) - \langle Sx, y - x\rangle| \le \|Sy - Sx\| \; \|y - x\| \quad \forall x, y \in E.$$

3. Conclude.

- C -

For each integer $n \ge 1$ and every $x \in E$ set

$$\varphi_n(x) = \inf_{y \in E} \left\{ \frac{n}{2} \|x - y\|^2 + \varphi(y) \right\}.$$

Prove that φ_n is convex, differentiable, and that for every $x \in E$, $\varphi_n(x) \uparrow \varphi(x)$ as $n \uparrow +\infty$.

[**Hint**: Use the same method as in Exercise 1.24.]

PROBLEM 15 (1, 5 for question B6)

Center of a set in the sense of Chebyshev. Normal structure. Asymptotic center of a sequence in the sense of Edelstein. Fixed points of contractions following Kirk, Browder, Göhde, and Edelstein.

Before starting this problem it is useful to solve Exercise 3.29.

Let E be a uniformly convex Banach space and let $C \subset E$ be a nonempty closed convex set.

- A -

Let $A \subset C$ be a nonempty *bounded set*. For every $x \in E$ define

$$\varphi(x) = \sup_{y \in A} \|x - y\|.$$

1. Check that φ is a convex function and that

$$|\varphi(x_1) - \varphi(x_2)| \le \|x_1 - x_2\| \quad \forall x_1, x_2 \in E.$$

2. Prove that there exists a unique element $c \in C$ such that

$$\varphi(c) = \inf_{x \in C} \varphi(x).$$

The point c is called the *center* of A and is denoted by $c = \sigma(A)$.

3. Prove that if A is not reduced to a single point then

$$\varphi(\sigma(A)) < \operatorname{diam} A = \sup_{x, y \in A} \|x - y\|.$$

- B -

Let (a_n) be a *bounded sequence* in C; set

$$A_n = \bigcup_{i=n}^{\infty} \{a_i\} \quad \text{and} \quad \varphi_n(x) = \sup_{y \in A_n} \|x - y\| \quad \text{for } x \in E.$$

1. For every $x \in E$, consider $\varphi(x) = \lim_{n \to +\infty} \varphi_n(x)$. Prove that this limit exists and that φ is convex and continuous on E.

2. Prove that there exists a unique element $\sigma \in C$ such that

$$\varphi(\sigma) = \inf_{x \in C} \varphi(x).$$

The point σ is called the *asymptotic center* of the sequence (a_n).

3. Let $\sigma_n = \sigma(A_n)$ be the center of the set A_n in the sense of question A2. Prove that

$$\lim_{n\to\infty} \varphi_n(\sigma_n) = \lim_{n\to\infty} \varphi(\sigma_n) = \varphi(\sigma),$$

and that $\sigma_n \rightharpoonup \sigma$ weakly $\sigma(E, E^\star)$.

4. Deduce that $\sigma_n \to \sigma$ strongly.

[**Hint**: Argue by contradiction and apply the result of Exercise 3.29.]

5. Assume $a_n \to a$ strongly. Determine the asymptotic center of the sequence (a_n).

6. Assume here that E is a Hilbert space and that $a_n \rightharpoonup a$ weakly $\sigma(E, E^\star)$. Compute $\varphi(x)$ and determine the asymptotic center of the sequence (a_n).
[**Hint**: Expand squares of norms.]

- C -

Assume that $T : C \to C$ is a contraction, that is,

$$\|Tx - Ty\| \le \|x - y\| \quad \forall x, y \in C.$$

Let $a \in C$ be given and let $a_n = T^n a$ be the sequence of its iterates. Assume that the sequence (a_n) is *bounded*. Let σ be the asymptotic center of the sequence (a_n).

1. Prove that σ is a fixed point of T, i.e., $T\sigma = \sigma$.

2. Check that the set of fixed points of T is closed and convex.

PROBLEM 16 (2, 3)

Characterization of linear maximal monotone operators

Let E be a Banach space and let $A : D(A) \subset E \to E^\star$ be an unbounded linear operator satisfying the *monotonicity* condition

$$\text{(M)} \qquad \langle Au, u\rangle \ge 0 \quad \forall u \in D(A).$$

We denote by (P) the following property:

$$\text{(P)} \qquad \begin{cases} \text{If } x \in E \text{ and } f \in E^\star \text{ are such that} \\ \langle Au - f, u - x\rangle \ge 0 \quad \forall u \in D(A), \\ \text{then } x \in D(A) \text{ and } Ax = f. \end{cases}$$

- A -

1. Prove that if (P) holds then $D(A)$ is dense in E.

   [**Hint**: Show that if $f \in E^\star$ and $\langle f, u \rangle = 0\ \forall u \in D(A)$, then $f = 0$.]

2. Prove that if (P) holds then A is closed.

3. Prove that the function $u \in D(A) \mapsto \langle Au, u \rangle$ is convex.

4. Prove that $N(A) \subset R(A)^\perp$. Deduce that if $D(A)$ is dense in E then $N(A) \subset N(A^\star)$.

5. Prove that if $D(A) = E$, then (P) holds.

   Throughout the rest of this problem we assume, in addition, that

   (i) E is reflexive, E and $E^\star$ are strictly convex,
   (ii) $D(A)$ is dense in E and A is closed,

   so that $A^\star : D(A^\star) \subset E \to E^\star$ and $D(A^\star)$ is dense in E (why?).

The goal of this problem is to establish the equivalence (P) $\Leftrightarrow$ (M*), where (M*) denotes the following property:

(M*) $$\langle A^\star v, v \rangle \geq 0 \quad \forall v \in D(A^\star).$$

- B -

In this section we assume that (P) holds.

1. Prove that
$$\langle A^\star v, v \rangle \geq 0 \quad \forall v \in D(A) \cap D(A^\star).$$

2. Let $v \in D(A^\star)$ with $v \notin D(A)$. Prove that $\forall f \in E^\star$, $\exists u \in D(A)$ such that
$$\langle Au - f, u - v \rangle < 0.$$
   Choosing $f = -A^\star v$, prove that $\langle A^\star v, v \rangle > 0$. Deduce that (M*) holds.

3. Prove that $N(A) = N(A^\star)$ and $\overline{R(A)} = \overline{R(A^\star)}$.

- C -

In this part we assume that (M*) holds.

1. Check that the space $D(A)$ equipped with the graph norm $\|u\|_{D(A)} = \|u\|_E + \|Au\|_{E^\star}$ is reflexive.

2. Given $x \in E$ and $f \in E^\star$, consider the function φ defined on $D(A)$ by
$$\varphi(u) = \frac{1}{2}\|Au - f\|^2 + \frac{1}{2}\|u - x\|^2 + \langle Au - f, u - x \rangle.$$

Prove that φ is convex and continuous on $D(A)$.
Prove that $\varphi(u) \to +\infty$ as $\|u\|_{D(A)} \to \infty$.

3. Deduce that there exists some $u_0 \in D(A)$ such that $\varphi(u_0) \le \varphi(u)$ $\forall u \in D(A)$. What equation (involving A and $A^\star$) does one obtain by choosing $u = u_0 + tv$ with $v \in D(A)$, $t > 0$, and letting $t \to 0$?

   [**Hint**: Apply the result of Problem 13, part A.]

4. Prove that ($M^\star$) $\Rightarrow$ (P).
5. Deduce that $A^\star$ also satisfies property (P).

PROBLEM 17 (1, 3, 4)

- A -

Let E be a reflexive Banach space and let M be a closed linear subspace of E. Let C be a convex subset of $E^\star$. For every $u \in E$ set

$$\varphi(u) = \sup_{g \in C} \langle g, u \rangle.$$

1. Prove that for every $f \in \overline{M^\perp + C}$ we have

$$\varphi(u) \ge \langle f, u \rangle \quad \forall u \in M.$$

   [**Hint**: Start with the case $f \in M^\perp + C$.]

2. Conversely, let $f \in E^\star$ be such that

$$\varphi(u) \ge \langle f, u \rangle \quad \forall u \in M.$$

   Prove that $f \in \overline{M^\perp + C}$.

   [**Hint**: Use Hahn–Banach.]

3. Assuming that C is closed and bounded, prove that $M^\perp + C$ is closed.

- B -

In this section we assume that $E = L^p(\Omega)$ with $1 < p < \infty$,

$$M = \left\{ u \in L^p(\Omega);\ \int ju = 0 \right\},$$

and

$$C = \{ g \in L^{p'}(\Omega);\ |g(x)| \le k(x) \text{ a.e. } x \in \Omega \},$$

where j and $k \ge 0$ are given functions in $L^{p'}(\Omega)$.

1. Check that M is a closed linear subspace and that C is convex, closed, and bounded.

2. Determine $M^{\perp}$.
3. Determine $\varphi(u)$ for every $u \in L^p(\Omega)$.
4. Deduce that if $f \in L^{p'}(\Omega)$ satisfies

$$\int k|u| \geq \int fu \quad \forall u \in M,$$

then there exist a constant $\lambda \in \mathbb{R}$ and a function $g \in C$ such that $f = \lambda j + g$.
5. Prove that the converse also holds.

- C -

Let $M \subset L^1(\Omega)$ be a linear subspace. Let $f, g \in L^\infty(\Omega)$ be such that $f \leq g$ a.e. on Ω. Prove that the following properties are equivalent:

$$\text{(i)} \qquad \exists \varphi \in M^{\perp} \text{ such that } f \leq \varphi \leq g \text{ a.e. on } \Omega,$$

$$\text{(ii)} \qquad \int (fu^+ - gu^-) \leq 0 \quad \forall u \in M,$$

where $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$.

[**Hint**: Assuming (ii), check that $\int (g + f)u \leq \int (g - f)|u| \;\; \forall u \in M$ and apply Theorem I.12 to find some $\psi \in L^\infty(\Omega)$ with $|\psi| \leq g - f$, such that $\psi - (g + f) \in M^{\perp}$. Take $\varphi = \frac{1}{2}(g + f - \psi)$.]

PROBLEM 18 (3, 4)

Let Ω be a measure space with finite measure. Let $1 < p < \infty$. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous nondecreasing function such that

$$|g(t)| \leq C(|t|^{p-1} + 1) \quad \forall t \in \mathbb{R}, \quad \text{for some constant } C.$$

Set $G(t) = \int_0^t g(s)ds$.

1. Check that for every $u \in L^p(\Omega)$, we have $g(u) \in L^{p'}(\Omega)$ and $G(u) \in L^1(\Omega)$.
   Let (u_n) be a sequence in $L^p(\Omega)$ and let $u \in L^p(\Omega)$ be such that

$$\text{(i)} \qquad u_n \rightharpoonup u \quad \text{weakly } \sigma(L^p, L^{p'})$$

and

$$\text{(ii)} \qquad \limsup \int G(u_n) \leq \int G(u).$$

The purpose of this problem is to establish the following properties:

$$\text{(1)} \qquad g(u_n) \to g(u) \textit{ strongly} \text{ in } L^q \text{ for every } q \in [1, p'),$$

$$\text{(2)} \qquad \begin{cases} \text{Assuming, in addition, that } g \text{ is increasing (strictly),} \\ \text{then } u_n \to u \textit{ strongly} \text{ in } L^q \text{ for every } q \in [1, p). \end{cases}$$

2. Check that $G(a) - G(b) - g(b)(a-b) \geq 0 \ \forall a, b \in \mathbb{R}$.

   What can one say if $G(a) - G(b) - g(b)(a-b) = 0$?

3. Let (a_n) be a sequence in $\mathbb{R}$ and let $b \in \mathbb{R}$ be such that

$$\lim[G(a_n) - G(b) - g(b)(a_n - b)] = 0.$$

   Prove that $g(a_n) \to g(b)$.

4. Prove that $\int |G(u_n) - G(u) - g(u)(u_n - u)| \to 0$.
   Deduce that there exists a subsequence (u_{n_k}) such that

$$G(u_{n_k}) - G(u) - g(u)(u_{n_k} - u) \to 0 \text{ a.e. on } \Omega$$

   and therefore $g(u_{n_k}) \to g(u)$ a.e. on Ω.

5. Prove that (1) holds. (Check (1) for the whole sequence and not only for a subsequence.)

6. Prove that (2) holds.

In what follows, we assume, in addition, that there exist constants $\alpha > 0$ and C such that

$$(3) \qquad |g(t)| \geq \alpha |t|^{p-1} - C \quad \forall t \in \mathbb{R}.$$

7. Prove that $g(u_n) \to g(u)$ strongly in $L^{p'}$.

8. Can one reach the same conclusion without assumption (3)?

9. If, in addition, g is increasing (strictly) prove that $u_n \to u$ strongly in L^p.

PROBLEM 19 (3, 4)

Let E be the space $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ equipped with the norm

$$\|u\|_E = \|u\|_1 + \|u\|_2.$$

1. Check that E is a Banach space. Let $f(x) = f_1(x) + f_2(x)$ with $f_1 \in L^\infty(\mathbb{R})$ and $f_2 \in L^2(\mathbb{R})$. Check that the mapping $u \mapsto \int_{\mathbb{R}} f(x)u(x)dx$ is a continuous linear functional on E.

2. Let $0 < \alpha < 1/2$; check that the mapping

$$u \mapsto \int_{\mathbb{R}} \frac{1}{|x|^\alpha} u(x)dx$$

   is a continuous linear functional on E.

   [**Hint**: Split the integral into two parts: $[|x| > M]$ and $[|x| \leq M]$.]

3. Set
$$K = \left\{ u \in E;\ u \geq 0 \text{ a.e. on } \mathbb{R} \text{ and } \int_{\mathbb{R}} u(x)dx \leq 1 \right\}.$$
Check that K is a closed convex subset of E.

4. Let (u_n) be a sequence in K and let $u \in K$ be such that $u_n \rightharpoonup u$ weakly in $L^2(\mathbb{R})$. Check that $u \in K$ and prove that
$$\int_{\mathbb{R}} \frac{1}{|x|^\alpha} u_n(x)dx \to \int_{\mathbb{R}} \frac{1}{|x|^\alpha} u(x)dx.$$

Consider the function J defined, for every $u \in E$, by
$$J(u) = \int_{\mathbb{R}} u^2(x)dx - \int_{\mathbb{R}} \frac{1}{|x|^\alpha} u(x)dx.$$

5. Check that there is a constant C such that $J(u) \geq C\ \forall u \in K$.

We claim that $m = \inf_{u \in K} J(u)$ is achieved.

6. Let (u_n) be a sequence in K such that $J(u_n) \to m$. Prove that $\|u_n\|_E$ is bounded.

7. Let (u_{n_k}) be a subsequence such that $u_{n_k} \rightharpoonup u$ weakly in $L^2(\mathbb{R})$. Prove that $J(u) = m$.

8. Is E a reflexive space?

PROBLEM 20 (4)

Clarkson's inequalities. Uniform convexity of L^p

- A -

In this part we assume that $2 \leq p < \infty$ and we shall establish the following inequalities:

(1) $$|x+y|^p + |x-y|^p \leq 2(|x|^{p'} + |y|^{p'})^{p/p'} \quad \forall x, y \in \mathbb{R},$$

(2) $$2(|x|^{p'} + |y|^{p'})^{p/p'} \leq 2^{p-1}(|x|^p + |y|^p) \quad \forall x, y \in \mathbb{R}.$$

1. Prove (2).

[**Hint**: Use the convexity of the function $g(t) = |t|^{p/p'}$.]

2. Set
$$f(x) = (1 + x^{1/p})^p + (1 - x^{1/p})^p, \quad x \in (0, 1).$$
Prove that
$$f''(x) \leq 0 \quad \forall x \in (0, 1).$$
Deduce that

(3) $$f(x) \leq f(y) + (x-y)f'(y) \quad \forall x, y \in (0,1).$$

3. Prove that
$$f(x) \leq 2(1 + x^{p'/p})^{p/p'} \quad \forall x \in (0,1).$$
[**Hint**: Use (3) with $y = x^{p'}$.]
4. Deduce (1).

In what follows Ω denotes a σ-finite measure space.

- B -

In this part we assume again that $2 \leq p < \infty$.

1. Prove the following inequalities:

(4) $$\|f+g\|_p^p + \|f-g\|_p^p \leq 2(\|f\|_p^{p'} + \|g\|_p^{p'})^{p/p'} \quad \forall f, g \in L^p(\Omega),$$

(5) $$2(\|f\|_p^{p'} + \|g\|_p^{p'})^{p/p'} \leq 2^{p-1}(\|f\|_p^p + \|g\|_p^p) \quad \forall f, g \in L^p(\Omega).$$

2. Deduce *Clarkson's first inequality* (see Theorem 4.10).

- C -

In this part we assume that $1 < p \leq 2$.

1. Establish the following inequality:

(6) $$\|f+g\|_p^{p'} + \|f-g\|_p^{p'} \leq 2(\|f\|_p^p + \|g\|_p^p)^{p'/p} \quad \forall f, g \in L^p(\Omega).$$

Inequality (6) is called *Clarkson's second inequality.*

[**Hint**: There are two different methods:

(i) By duality from (4), observing that
$$\sup_{\varphi, \psi \in L^{p'}} \left\{ \frac{\int (u\varphi + v\psi)}{[\|\varphi\|_{p'}^p + \|\psi\|_{p'}^p]^{1/p}} \right\} = (\|u\|_p^{p'} + \|v\|_p^{p'})^{1/p'}.$$

(ii) Directly from (1) combined with the result of Exercise 4.11.]

2. Deduce that $L^p(\Omega)$ is uniformly convex for $1 < p \leq 2$.

PROBLEM 21 (4)

The distribution function. Marcinkiewicz spaces

Throughout this problem Ω denotes a measure space with finite measure μ. Given a measurable function $f : \Omega \to \mathbb{R}$, we define its *distribution function* α to be

$$\alpha(t) = |[|f| > t]| = \text{meas}\{x \in \Omega;\ |f(x)| > t\} \quad \forall t \geq 0.$$

- A -

1. Check that α is nonincreasing. Prove that $\alpha(t+0) = \alpha(t)\ \ \forall t \geq 0$. Construct a simple example in which $\alpha(t-0) \neq \alpha(t)$ for some $t > 0$.
2. Let (f_n) be a sequence of measurable functions such that $f_n \to f$ a.e. on Ω. Let (α_n) and α denote the corresponding distribution functions. Prove that

$$\alpha(t) \leq \liminf_{n\to\infty} \alpha_n(t) \leq \limsup_{n\to\infty} \alpha_n(t) \leq \alpha(t-0) \quad \forall t > 0.$$

Deduce that $\alpha_n(t) \to \alpha(t)$ a.e.

- B -

1. Let $g \in L^1_{\text{loc}}(\mathbb{R})$ be a function such that $g \geq 0$ a.e. Set

$$G(t) = \int_0^t g(s)ds.$$

Prove that for every measurable function f,

$$\int_\Omega G(|f(x)|)d\mu < \infty \Longleftrightarrow \int_0^\infty \alpha(t)g(t)dt < \infty$$

and that

$$\int_\Omega G(|f(x)|)d\mu = \int_0^\infty \alpha(t)g(t)dt.$$

[**Hint**: Use Fubini and Tonelli.]

2. More generally, prove that

$$\int_{[|f|>\lambda]} G(|f(x)|)d\mu = \alpha(\lambda)G(\lambda) + \int_\lambda^\infty \alpha(t)g(t)dt \quad \forall \lambda \geq 0.$$

3. Deduce that for $1 \leq p < \infty$,

$$f \in L^p(\Omega) \Longleftrightarrow \int_0^\infty \alpha(t)t^{p-1}\,dt < \infty$$

and that

$$\int_{[|f|>\lambda]} |f(x)|^p d\mu = \alpha(\lambda)\lambda^p + p\int_\lambda^\infty \alpha(t)t^{p-1}dt \quad \forall \lambda \geq 0.$$

Check that if $f \in L^p(\Omega)$, then $\lim_{t\to+\infty}\alpha(t)t^p = 0$.

- C -

Let $1 < p < \infty$. For every $f \in L^1(\Omega)$ define

$$[f]_p = \sup \left\{ |A|^{-1/p'} \int_A |f|;\ A \subset \Omega \text{ measurable, } |A| > 0 \right\} \leq \infty,$$

and consider the set

$$M^p(\Omega) = \{f \in L^1(\Omega);\ [f]_p < \infty\},$$

called the *Marcinkiewicz space* of order p. The space M^p is also called the *weak* L^p space, but this terminology is confusing because the word "weak" is already used in connection with the weak topology.

1. Check that $M^p(\Omega)$ is a linear space and that $[\ \ ]_p$ is a norm. Prove that

$$L^p(\Omega) \subset M^p(\Omega) \text{ and that } [f]_p \leq \|f\|_p \text{ for every } f \in L^p(\Omega).$$

2. Prove that $M^p(\Omega)$, equipped with the norm $[\ \ ]_p$, is a Banach space. Check that $M^p(\Omega) \subset M^q(\Omega)$ with continuous injection for $1 < q \leq p$.

   We claim that

$$[f \in M^p(\Omega)] \Longleftrightarrow \left[f \text{ is measurable and } \sup_{t>0} t^p \alpha(t) < \infty \right].$$

3. Prove that if $f \in M^p(\Omega)$ then $t^p \alpha(t) \leq [f]_p^p \ \ \forall t > 0$.
4. Conversely, let f be a measurable function such that

$$\sup_{t>0} t^p \alpha(t) < \infty.$$

   Prove that there exists a constant C_p (depending only on p) such that

$$[f]_p^p \leq C_p \sup_{t>0} t^p \alpha(t).$$

   [**Hint**: Use question B3 and write

$$\int_A |f| = \int_{A \cap [|f| > \lambda]} |f| + \int_{A \cap [|f| \leq \lambda]} |f|;$$

   then vary λ.]
5. Prove that $M^p(\Omega) \subset L^q(\Omega)$ with continuous injection for $1 \leq q < p$.
6. Let $1 < q < r < \infty$ and $\theta \in (0, 1)$; set

$$\frac{1}{p} = \frac{\theta}{q} + \frac{1 - \theta}{r}.$$

Prove that there is a constant C—depending only on q, r, and θ—such that

$$\|f\|_p \leq C[f]_q^{\theta}[f]_r^{1-\theta} \quad \forall f \in M^r(\Omega).$$

7. Set $\Omega = \{x \in \mathbb{R}^N;\ |x| < 1\}$, equipped with the Lebesgue measure, and let $f(x) = |x|^{-N/p}$ with $1 < p < \infty$. Check that $f \in M^p(\Omega)$, while $f \notin L^p(\Omega)$.

PROBLEM 22 (4)

An interpolation theorem (*Schur, Riesz, Thorin, Marcinkiewicz*)

Let Ω be a measure space with finite measure. Let

$$T : L^1(\Omega) \to L^1(\Omega)$$

be a bounded linear operator whose norm is denoted by $N_1 = \|T\|_{\mathcal{L}(L^1,L^1)}$. We assume that

$$T(L^\infty(\Omega)) \subset L^\infty(\Omega).$$

1. Prove that T is a bounded operator from $L^\infty(\Omega)$ into itself. Set

$$N_\infty = \|T\|_{\mathcal{L}(L^\infty,L^\infty)}.$$

The goal of this problem is to show that

$$T(L^p(\Omega)) \subset L^p(\Omega) \quad \text{for every } 1 < p < \infty$$

and that $T : L^p(\Omega) \to L^p(\Omega)$ is a bounded operator whose norm $N_p = \|T\|_{\mathcal{L}(L^p,L^p)}$ satisfies the inequality $N_p \leq 2N_1^{1/p}N_\infty^{1/p'}$.

For simplicity, we assume first that $N_\infty = 1$. Given a function $u \in L^1(\Omega)$, we set, for every $\lambda > 0$,

$$u = v_\lambda + w_\lambda \quad \text{with} \quad v_\lambda = u\chi_{[|u|>\lambda]} \quad \text{and} \quad w_\lambda = u\chi_{[|u|\leq\lambda]},$$

$$f = Tu, \qquad g_\lambda = Tv_\lambda, \qquad \text{and} \qquad h_\lambda = Tw_\lambda, \qquad \text{so that} \qquad f = g_\lambda + h_\lambda.$$

2. Check that

$$\|g_\lambda\|_1 \leq N_1 \int_{[|u|>\lambda]} |u(x)|d\mu \quad \text{and} \quad \|h_\lambda\|_\infty \leq \lambda \quad \forall \lambda > 0.$$

3. Consider the distribution functions

$$\alpha(t) = |[|u| > t]|, \beta(t) = |[|f| > t]|, \gamma_\lambda(t) = [|g_\lambda| > t].$$

Prove that

$$\int_0^\infty \gamma_\lambda(t)dt \le N_1[\alpha(\lambda)\lambda + \int_\lambda^\infty \alpha(t)dt] \quad \forall\lambda > 0,$$

and that

$$\beta(t) \le \gamma_\lambda(t-\lambda) \quad \forall\lambda > 0, \quad \forall t > \lambda.$$

[**Hint**: Apply the results of Problem 21, part B.]

4. Assuming $u \in L^p(\Omega)$, prove that $f \in L^p(\Omega)$ and that

$$\|f\|_p \le 2N_1^{1/p}\|u\|_p.$$

5. Conclude in the general case, in which $N_\infty \neq 1$.

Remark. By a different argument one can prove in fact that $N_p \le N_1^{1/p} N_\infty^{1/p'}$; see, e.g., Bergh–Löfström [1] and the references in the Notes on Chapter 1 of their book.

PROBLEM 23 (3, 4)

Weakly compact subsets of L^1 and equi-integrable families.
The theorems of Hahn–Vitali–Saks, Dunford–Pettis, and de la Vallée-Poussin.

Let Ω be a σ-finite measure space. We recall (see Exercise 4.36) that a subset $\mathcal{F} \subset L^1(\Omega)$ is said to be *equi-integrable* if it satisfies the following properties:

(a) $\qquad \mathcal{F}$ is bounded in $L^1(\Omega)$,

(b) $\qquad \begin{cases} \forall\varepsilon > 0 \ \exists\delta > 0 \quad \text{such that} \int_A |f| < \varepsilon \quad \forall f \in \mathcal{F}, \\ \forall A \subset \Omega \text{ with } A \text{ measurable and } |A| < \delta, \end{cases}$

(c) $\qquad \begin{cases} \forall\varepsilon > 0 \ \exists\omega \subset \Omega \quad \text{measurable with } |\omega| < \infty \\ \text{such that } \int_{\Omega\setminus\omega} |f| < \varepsilon. \end{cases}$

The first goal of this problem is to establish the equivalence of the following properties for a given set $\mathcal{F}$ in $L^1(\Omega)$:

(i) $\mathcal{F}$ is contained in a weakly ($\sigma(L^1, L^\infty)$) compact set of $L^1(\Omega)$,

(ii) $\mathcal{F}$ is equi-integrable.

- A -

The implication (i) $\Rightarrow$ (ii).

1. Let (f_n) be a sequence in $L^1(\Omega)$ such that

$$\int_A f_n \to 0 \quad \forall A \subset \Omega \text{ with } A \text{ measurable and } |A| < \infty.$$

Prove that (f_n) satisfies property (b).

[**Hint**: Consider the subset $X \subset L^1(\Omega)$ defined by

$$X = \{\chi_A \text{ with } A \subset \Omega,\, A \text{ measurable and } |A| < \infty\}.$$

Check that X is closed in $L^1(\Omega)$ and apply the Baire category theorem to the sequence

$$X_n = \left\{ \chi_A \in X;\ \left| \int_A f_k \right| \le \varepsilon \quad \forall k \ge n \right\},$$

where $\varepsilon > 0$ is fixed.]

2. Let (f_n) be a sequence in $L^1(\Omega)$ such that

$$\int_A f_n \to 0 \quad \forall A \subset \Omega \text{ with } A \text{ measurable and } |A| \le \infty.$$

Prove that (f_n) satisfies property (c).

[**Hint**: Let (Ω_i) be a nondecreasing sequence of measurable sets with finite measure such that $\Omega = \bigcup_i \Omega_i$. Consider on $L^\infty(\Omega)$ the metric d defined by

$$d(f, g) = \sum_i \frac{1}{2^i |\Omega_i|} \int_{\Omega_i} |f - g|.$$

Set $Y = \{\chi_A \text{ with } A \subset \Omega, \text{A measurable}\}$. Check that Y is complete for the metric d and apply the Baire category theorem to the sequence

$$Y_n = \left\{ \chi_A \in Y;\ \left| \int_A f_k \right| \le \varepsilon \quad \forall k \ge n \right\},$$

where $\varepsilon > 0$ is fixed.]

3. Deduce that if (f_n) is a sequence in $L^1(\Omega)$ such that $f_n \rightharpoonup f$ weakly $\sigma(L^1, L^\infty)$, then (f_n) is equi-integrable.

4. Prove that (i) $\Rightarrow$ (ii).

[**Hint**: Argue by contradiction and apply the theorem of Eberlein–Šmulian; see Problem 10.]

5. Take up again question 1 (resp. question 2) assuming only that $\int_A f_n$ converges to a finite limit $\ell(A)$ for every $A \subset \Omega$ with A measurable and $|A| < \infty$ (resp. $|A| \le \infty$).

- B -

The implication (ii) $\Rightarrow$ (i).

1. Let E be a Banach space and let $\mathcal{F} \subset E$. Assume that

$$\forall \varepsilon > 0\ \exists \mathcal{F}_\varepsilon \subset E,\ \mathcal{F}_\varepsilon \text{ weakly } (\sigma(E, E^\star)) \text{ compact such that } \mathcal{F} \subset \mathcal{F}_\varepsilon + \varepsilon B_E.$$

Prove that $\mathcal{F}$ is contained in a weakly compact subset of E.

[**Hint**: Consider $\mathcal{G} = \overline{\mathcal{F}}^{\sigma(E^{\star\star}, E^{\star})}$.]

2. Deduce that (ii) $\Rightarrow$ (i).

   [**Hint**: Consider the family $(\chi_\omega T_n f)_{f \in \mathcal{F}}$ with $|\omega| < \infty$ and T_n is the truncation as in the proof of Theorem 4.12.]

- C -

Some applications.

1. Let (f_n) be a sequence in $L^1(\Omega)$ such that $f_n \rightharpoonup f$ weakly $\sigma(L^1, L^\infty)$ and $f_n \to f$ a.e. Prove that $\|f_n - f\|_1 \to 0$.

   [**Hint**: Apply Exercise 4.14.]

2. Let $u_1, u_2 \in L^1(\Omega)$ with $u_1 \le u_2$ a.e. Prove that the set $K = \{f \in L^1(\Omega);\ u_1 \le f \le u_2 \text{ a.e.}\}$ is compact in the weak topology $\sigma(L^1, L^\infty)$.

3. Let (f_n) be an equi-integrable sequence in $L^1(\Omega)$. Prove that there exists a subsequence (f_{n_k}) such that $f_{n_k} \rightharpoonup f$ weakly $\sigma(L^1, L^\infty)$.

4. Let (f_n) be a bounded[1] sequence in $L^1(\Omega)$ such that $\int_A f_n$ converges to a finite limit, $\ell(A)$, for every measurable set $A \subset \Omega$. Prove that there exists some $f \in L^1(\Omega)$ such that $f_n \rightharpoonup f$ weakly $\sigma(L^1, L^\infty)$.

5. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous increasing function such that

$$|g(t)| \le C \quad \forall t \in \mathbb{R}.$$

   Set $G(t) = \int_0^t g(s)ds$. Let (f_n) be a sequence in $L^1(\Omega)$ such that $f_n \rightharpoonup f$ weakly $\sigma(L^1, L^\infty)$ and $\limsup \int G(f_n) \le \int G(f)$. Prove that $\|f_n - f\|_1 \to 0$.

   [**Hint**: Look at Problem 18.]

- D -

In this part, we assume that $|\Omega| < \infty$. Let $\mathcal{F} \subset L^1(\Omega)$.

1. Let $G : [0, +\infty) \to [0, +\infty)$ be a continuous function such that $\lim_{t \to +\infty} G(t)/t = +\infty$. Assume that there exists a constant C such that

$$\int G(|f|) \le C \quad \forall f \in \mathcal{F}.$$

   Prove that $\mathcal{F}$ is equi-integrable.

[1] In fact, it is not necessary to assume that (f_n) is bounded, but then the proof is more complicated; see, e.g., R. Edwards [1] p. 276–277.

2. Conversely, assume that $\mathcal{F}$ is equi-integrable. Prove that there exists a convex increasing function $G : [0, +\infty) \to [0, +\infty)$ such that $\lim_{t\to+\infty} G(t)/t = +\infty$ and $\int G(|f|) \le C \ \forall f \in \mathcal{F}$, for some constant C.

   [**Hint**: Use the distribution function; see Problem 21.]

PROBLEM 24 (1, 3, 4)

Radon measures

Let K be a compact metric space, with distance d, and let $E = C(K)$ equipped with its usual norm

$$\|f\| = \max_{x\in K} |f(x)|.$$

The dual space $E^\star$, denoted by $\mathcal{M}(K)$, is called the space of *Radon measures* on K. The space $\mathcal{M}(K)$ is equipped with the dual norm, denoted by $\| \ \|_{\mathcal{M}}$ or simply $\| \ \|$. The purpose of this problem is to present some properties of $\mathcal{M}(K)$.

- A -

We prove here that $C(K)$ is separable. Given $\delta > 0$, let $\bigcup_{j\in J} B(a_j, \delta/2)$ be a finite covering of K. Set

$$q_j(x) = \max\{0, \delta - d(x, a_j)\}, \quad j \in J, \ x \in K,$$

and

$$q(x) = \sum_{j\in J} q_j(x).$$

1. Check that the functions $(q_j)_{j\in J}$ and q are continuous on K. Show that

$$q(x) \ge \delta/2 \quad \forall x \in K.$$

2. Set

$$\theta_j(x) = \frac{q_j(x)}{q(x)}, \quad j \in J, \ x \in K.$$

   Show that the functions $(\theta_j)_{j\in J}$ are continuous on K,

$$[\theta_j(x) \ne 0] \iff [d(x, a_j) < \delta],$$

   and

$$\sum_{j\in J} \theta_j(x) = 1 \quad \forall x \in K.$$

   The collection of functions $(\theta_j)_{j\in J}$ is called a *partitition of unity* (subordinate to the open covering $\bigcup_{j\in J} B(a_j, 2\delta)$, because $\operatorname{supp}\theta_j \subset B(a_j, 2\delta)$).

3. Given $f \in C(K)$, set

$$\tilde{f}(x) = \sum_{j\in J} f(a_j)\theta_j(x).$$

   Prove that

$$\|f - \tilde{f}\| \le \sup_{\substack{x,y \in K \\ d(x,y) < \delta}} |f(x) - f(y)|.$$

4. Choosing $\delta = 1/n$, the above construction yields a finite set J, now denoted by J_n, a finite collection of points $(a_j)_{j \in J_n}$, and a finite collection of functions $(\theta_j)_{j \in J_n}$. Show that the vector space spanned by the functions (θ_j), $j \in J_n$, $n = 1, 2, 3 \ldots$, is dense in $C(K)$.
5. Deduce that $C(K)$ is separable.

- B -

In this part we assume that $K = \overline{\Omega}$, where Ω is a bounded open set in $\mathbb{R}^N$. It is convenient to identify $L^1(\Omega)$ with a subspace of $\mathcal{M}(\overline{\Omega})$ through the embedding $T : L^1(\Omega) \to \mathcal{M}(\overline{\Omega})$ defined by

$$\langle Tu, f \rangle = \int_\Omega uf \quad \forall u \in L^1(\Omega), \quad \forall f \in C(\overline{\Omega}).$$

1. Check that $\|Tu\|_{\mathcal{M}} = \|u\|_{L^1}$ $\forall u \in L^1(\Omega)$.
   [**Hint**: Use Exercise 4.26.]
2. Let (v_n) be a bounded sequence in $L^1(\Omega)$. Show that there exist a subsequence (v_{n_k}) and some $\mu \in \mathcal{M}(\overline{\Omega})$ such that $v_{n_k} \overset{\star}{\rightharpoonup} \mu$ in $\mathcal{M}(\overline{\Omega})$ and

   $$\|\mu\|_{\mathcal{M}} \le \liminf_{k \to \infty} \|v_{n_k}\|_{L^1}.$$

   [**Hint**: Use Corollary 3.30.]

   The aim is now to prove that given any $\mu_0 \in \mathcal{M}(\overline{\Omega})$ there exists a sequence (u_n) in $C_c^\infty(\Omega)$ such that

   $$\int_\Omega u_n f \to \langle \mu_0, f \rangle \quad \forall f \in C(\overline{\Omega}) \tag{1}$$

   and

   $$\|u_n\|_{L^1} = \|\mu_0\|_{\mathcal{M}} \quad \forall n. \tag{2}$$

   Without loss of generality we may assume that $\|\mu_0\|_{\mathcal{M}} = 1$ (why?).

   Set

   $$A = \{u \in C_c^\infty(\Omega);\ \|u\|_{L^1} \le 1\}.$$

3. Prove that $\mu_0 \in \overline{A}^{\sigma(E^\star, E)}$.
   [**Hint**: Apply Hahn–Banach in $E^\star$ for the weak* topology $\sigma(E^\star, E)$; see Problem 9. Then use Corollary 4.23.]
4. Deduce that there exists a sequence (v_n) in A such that $v_n \overset{\star}{\rightharpoonup} \mu_0$ in $\sigma(E^\star, E)$. Check that $\lim_{n\to\infty} \|v_n\|_{L^1} = 1$.

5. Conclude that the sequence $u_n = v_n/\|v_n\|_{L^1}$ satifies (1) and (2).

   We say that $\mu \geq 0$ if

$$\langle \mu, f\rangle \geq 0 \quad \forall f \in C(\overline{\Omega}),\ f \geq 0 \text{ on } \overline{\Omega}.$$

6. Check that if $\mu \geq 0$, then $\langle \mu, 1\rangle = \|\mu\|$, where 1 denotes the function $f \equiv 1$.
7. Assume $\mu_0 \in \mathcal{M}(\overline{\Omega})$, with $\mu_0 \geq 0$ and $\|\mu_0\| = 1$ (such measures are called *probability measures*). Construct a sequence (u_n) in $C_c^\infty(\Omega)$ satisfying (1), (2), and, moreover,

$$(3) \qquad u_n(x) \geq 0 \quad \forall n,\ \forall x \in \overline{\Omega}.$$

8. Compute $\|u + \delta_a\|_{\mathcal{M}}$, where $u \in L^1$, and δ_a, with $a \in \overline{\Omega}$, is defined below.

- C -

We now return to the general setting and denote by δ_a the *Dirac mass* at a point $a \in K$, i.e., the measure defined by

$$\langle \delta_a, f\rangle = f(a) \quad \forall f \in C(K).$$

Set

$$D = \left\{ \mu = \sum_{j\in J} \alpha_j \delta_{a_j};\ J \text{ is finite}, \alpha_j \in \mathbb{R},\ \text{ and the points } a_j\text{'s are all distinct} \right\}.$$

1. Show that if $\mu \in D$ then

$$\|\mu\| = \sum_{j\in J} |\alpha_j|$$

   and

$$[\mu \geq 0] \Leftrightarrow [\alpha_j \geq 0 \quad \forall j].$$

   Set

$$D_1 = \{\mu \in D;\ \|\mu\| \leq 1\}.$$

2. Show that any measure $\mu_0 \in \mathcal{M}(K)$ with $\|\mu_0\| \leq 1$ belongs to $\overline{D_1}^{\sigma(E^\star, E)}$.

   [**Hint**: Use the same technique as in part B.]
3. Deduce that given any measure $\mu_0 \in \mathcal{M}(\overline{\Omega})$ there exists a sequence (ν_n) in D such that $\nu_n \overset{\star}{\rightharpoonup} \mu_0$ and $\|\nu_n\| = \|\mu_0\|\ \forall n$.
4. Let μ_0 be a probability measure. Prove that there exists a sequence (ν_n) of probability measures in D such that $\nu_n \overset{\star}{\rightharpoonup} \mu_0$.

   Remark. An alternative approach to question 4 is to show that the Dirac masses are the extremal points of the convex set of probability measures; then apply Krein–Milman (see Problem 1) in the weak* topology; for more details see, e.g., R. Edwards [1].

- D -

The goal of this part is to show that every $\mu \in \mathcal{M}(K)$ admits a unique decomposition $\mu = \mu_1 - \mu_2$ with $\mu_1, \mu_2 \in \mathcal{M}(K),\ \mu_1, \mu_2 \geq 0$, and $\|\mu_1\| + \|\mu_2\| = \|\mu\|$. (The measures μ_1 and μ_2 are often denoted by μ^+ and μ^-.)

Given $f \in C(K)$ with $f \geq 0$, set

$$L(f) = \sup\{\langle \mu, g\rangle;\ g \in C(K) \text{ and } 0 \leq g \leq f \text{ on } K\}.$$

1. Check that $0 \leq L(f) \leq \|\mu\|\|f\|,\ L(\lambda f) = \lambda L(f)\ \forall \lambda \geq 0$, and

$$L(f_1 + f_2) = L(f_1) + L(f_2) \quad \forall f_1, f_2 \in C(K) \text{ with } f_1 \geq 0 \text{ and } f_2 \geq 0.$$

Given any $f \in C(K)$, set

$$\mu_1(f) = L(f^+) - L(f^-), \quad \text{where } f^+ = \max\{f, 0\} \text{ and } f^- = \max\{-f, 0\}.$$

2. Show that the mapping $f \mapsto \mu_1(f)$ is linear on $C(K)$ and that $|\mu_1(f)| \leq \|\mu\|\,\|f\|\ \forall f \in C(K)$, so that $\mu_1 \in \mathcal{M}(K)$. Check that $\mu_1 \geq 0$.
3. Set $\mu_2 = \mu_1 - \mu$ and check that $\mu_2 \geq 0$. Show that $\|\mu\| = \|\mu_1\| + \|\mu_2\|$.
4. Let $\nu \in \mathcal{M}(K)$ be such that $\nu \geq 0$ and $\nu \geq \mu$ (i.e., $\nu - \mu \geq 0$). Show that $\nu \geq \mu_1$. Similarly if $\nu \in \mathcal{M}(K)$ and $\nu \geq -\mu$, show that $\nu \geq \mu_2$. Deduce the uniqueness of the decomposition.

- E -

Show that all the above results (except question B6) remain valid when the space $E = C(\overline{\Omega})$ is replaced by the subspace

$$E_0 = \{f \in C(\overline{\Omega});\ f = 0 \text{ on the boundary of } \overline{\Omega}\}.$$

The dual of E_0 is often denoted by $\mathcal{M}(\Omega)$ (as opposed to $\mathcal{M}(\overline{\Omega})$).

- F -

Dunford–Pettis revisited

Let (f_n) be a sequence in $L^1(\Omega)$. Recall that (f_n) is said to be equi-integrable if it satisfies the property

$$\begin{cases} \forall \varepsilon > 0 \quad \exists \delta > 0 \text{ such that } \int_A |f_n| < \varepsilon \quad \forall n, \\ \text{and } \forall A \subset \Omega \text{ with } A \text{ measurable and } |A| < \delta. \end{cases} \tag{4}$$

The goal is to prove that every equi-integrable sequence (f_n) admits a subsequence (f_{n_j}) such that $f_{n_j} \rightharpoonup f$ weakly $\sigma(L^1, L^\infty)$, for some function $f \in L^1(\Omega)$.

1. Show that (f_n) is bounded in $L^1(\Omega)$.
2. Check that

$$\int_{\Omega} |f_n - T_k f_n| \le \int_{\substack{\Omega \\ [|f_n|>k]}} |f_n| \quad \forall n, \forall k,$$

where T_k denotes the truncation operation.

3. Deduce that $\forall \varepsilon > 0\ \exists k > 0$ such that

$$\int_{\Omega} |f_n - T_k f_n| \le \varepsilon \quad \forall n.$$

   [**Hint**: Use (4); see also Exercise 4.36.]

   Passing to a subsequence, still denoted by (f_n), we may assume that $f_n \overset{\star}{\rightharpoonup} \mu$ weak* in $\mathcal{M}(\overline{\Omega})$, for some measure $\mu \in \mathcal{M}(\overline{\Omega})$.

4. Prove that $\forall \varepsilon > 0\ \exists g = g_\varepsilon \in L^\infty(\Omega)$ such that

$$\|\mu - g_\varepsilon\|_{\mathcal{M}} \le \varepsilon.$$

   [**Hint**: For fixed k, a subsequence of $(T_k f_n)$ converges to some limit g weak* in $\sigma(L^\infty, L^1)$.]

5. Deduce that $\mu \in L^1(\Omega)$.

   [**Hint**: Use a Cauchy sequence argument in $L^1(\Omega)$.]

6. Prove that $f_n \rightharpoonup \mu$ weakly $\sigma(L^1, L^\infty)$.

   [**Hint**: Given $u \in L^\infty(\Omega)$, consider a sequence (u_m) in $C_c^\infty(\Omega)$ such that $u_m \to u$ a.e. on Ω and $\|u_m\|_\infty \le \|u\|_\infty\ \forall m$ (see Exercise 4.25); then use Egorov's theorem (see Theorem 4.29 and Exercise 4.14).]

PROBLEM 25 (1, 5)

Let H be a Hilbert space and let $C \subset H$ be a convex cone with vertex at 0, that is, $0 \in C$ and $\lambda u + \mu v \in C\ \forall \lambda, \mu > 0, \forall u, v \in C$. We assume that C is nonempty, open, and that $C \ne H$.

Check that $0 \notin C$ and that $0 \in \overline{C}$. Consider the set

$$\Sigma = \{u \in H;\ (u, v) \le 0 \quad \forall v \in C\}.$$

1. Check that Σ is a convex cone with vertex at 0, Σ is closed, and $0 \in \Sigma$. Prove that $C = \{v \in H; (u, v) < 0\ \forall u \in \Sigma \backslash \{0\}\}$ and deduce that Σ is not reduced to $\{0\}$.

   [**Hint**: Use Hahn–Banach.]

2. Let $\omega \in C$ be fixed and consider the set

$$K = \{u \in \Sigma;\ (u, \omega) = -1\}.$$

   Prove that K is a nonempty, bounded, closed, convex set such that $0 \notin K$ and

$$\Sigma \backslash \{0\} = \bigcup_{\lambda > 0} \lambda K.$$

Draw a figure.

[**Hint**: Consider a ball centered at ω of radius $\rho > 0$ contained in C.]

3. Let $a = P_K 0$. Prove that $a \in (-C) \cap \Sigma$.
4. Prove directly, by a simple argument, that $(-C) \cap \Sigma \neq \emptyset$.
5. Let $D \subset H$ be a nonempty, open, convex set and let $x_0 \notin D$. Prove that there exists some $w_0 \in D$ such that

$$(w_0 - x_0, w - x_0) > 0 \quad \forall w \in D.$$

Give a geometric interpretation.

[**Hint**: Consider the set $C = \cup_{\mu > 0} \mu(D - x_0)$.]

PROBLEM 26 (1, 5)

The Prox map in the sense of Moreau

Let H be a Hilbert space and let $\varphi : H \to (-\infty, +\infty]$ be a convex l.s.c. function such that $\varphi \not\equiv +\infty$.

1. Prove that for every $f \in H$, there exists some $u \in D(\varphi)$ such that

$$\text{(P)} \qquad \frac{1}{2}|f - u|^2 + \varphi(u) = \inf_{v \in H} \left\{ \frac{1}{2}|f - v|^2 + \varphi(v) \right\} \equiv I.$$

[**Hint**: Check first that $I > -\infty$. Then use either a Cauchy sequence argument or the fact that H is reflexive.]

2. Check that u satisfies (P) iff

$$\text{(Q)} \quad u \in D(\varphi) \quad \text{and} \quad (u, v - u) + \varphi(v) - \varphi(u) \geq (f, v - u) \quad \forall v \in D(\varphi).$$

3. Prove that if u and $\bar{u}$ are solutions of (P) corresponding to f and $\bar{f}$, then $|u - \bar{u}| \leq |f - \bar{f}|$. Deduce the uniqueness of the solution of (P).
4. Investigate the special case in which $\varphi = I_K$ is the indicator function of a closed convex set K.
5. Let $\varphi^\star$ be the conjugate function of φ and consider the problem

$$\text{(P}^\star\text{)} \qquad \frac{1}{2}|f - u^\star|^2 + \varphi^\star(u^\star) = \inf_{v \in H} \left\{ \frac{1}{2}|f - v|^2 + \varphi^\star(v) \right\} = I^\star.$$

Prove that the solutions u of (P) and $u^\star$ of (P*) satisfy

$$u + u^\star = f \quad \text{and} \quad I + I^\star = \frac{1}{2}|f|^2.$$

6. Given $f \in H$ and $\lambda > 0$ let u_λ denote the solution of the problem

$$(\mathrm{P}_\lambda) \qquad \frac{1}{2}|f - u_\lambda|^2 + \lambda\varphi(u_\lambda) = \inf_{v \in H} \left\{ \frac{1}{2}|f - v|^2 + \lambda\varphi(v) \right\}.$$

   Prove that $\lim_{\lambda \to 0} u_\lambda = P_{\overline{D(\varphi)}} f =$ the projection of f on $\overline{D(\varphi)}$.

   [**Hint**: Either start with weak convergence or use Exercise 5.3.]

7. Let $K = \{v \in D(\varphi);\ \varphi(v) = \inf_H \varphi\}$ and assume $K \neq \emptyset$.
   Check that K is a closed convex set and prove that $\lim_{\lambda \to +\infty} u_\lambda = P_K f$.
   What happens to (u_λ) as $\lambda \to +\infty$ when $K = \emptyset$?

8. Prove that $\lim_{\lambda \to +\infty} \frac{1}{\lambda} u_\lambda = -P_{\overline{D(\varphi^\star)}} 0$.

   [**Hint**: Start with the case where $f = 0$ and apply questions 5 and 6.]

PROBLEM 27 (5)

Alternate projections

Let H be a Hilbert space and let $K \subset H$ be a nonempty closed convex set. Check that

$$|P_K u - P_K v|^2 \le (P_K u - P_K v,\ u - v) \le |u - v|^2 \quad \forall u, v \in H.$$

Let $K_1 \subset H$ and $K_2 \subset H$ be two nonempty closed convex sets. Set $P_1 = P_{K_1}$ and $P_2 = P_{K_2}$. Given $u \in H$, define by induction the sequence (u_n) as follows:

$$u_0 = u,\ \ u_1 = P_1 u_0,\ \ u_2 = P_2 u_1,\ \ \dots\ \ , u_{2n-1} = P_1 u_{2n-2},\ \ u_{2n} = P_2 u_{2n-1},\ \ \dots$$

- A -

The purpose of this part is to prove that the sequence $(u_{2n} - u_{2n-1})$ converges to $P_K 0$, where $K = \overline{K_2 - K_1}$ (note that K is convex, why?).

1. Given $v \in H$ consider the sequence (v_n) defined by the same iteration as above starting with $v_0 = v$. Check that

$$|u_{2n} - v_{2n}|^2 \le (u_{2n} - v_{2n},\ u_{2n-1} - v_{2n-1}) \le |u_{2n-1} - v_{2n-1}|^2$$

   and that

$$|u_{2n+1} - v_{2n+1}|^2 \le (u_{2n+1} - v_{2n+1},\ u_{2n} - v_{2n}) \le |u_{2n} - v_{2n}|^2.$$

2. Deduce that the sequence $(|u_n - v_n|)$ is nonincreasing and thus converges to a limit, denoted by ℓ.
   Prove that $\lim_{n \to \infty} |(u_{2n} - v_{2n}) - (u_{2n-1} - v_{2n-1})|^2 = 0$.

3. Check that the sequence $(|u_{2n} - u_{2n-1}|)$ is nonincreasing.

   Set $d = \text{dist}(K_1, K_2) = \inf\{|a_1 - a_2|; a_1 \in K_1 \text{ and } a_2 \in K_2\}$.
   We claim that $\lim_{n\to\infty} |u_{2n} - u_{2n-1}| = d$.

4. Given $\varepsilon > 0$, choose $v \in K_2$ such that $\text{dist}(v, K_1) \leq d + \varepsilon$.

   Prove that $|v_{2n} - v_{2n-1}| \leq d + \varepsilon\ \forall n$.

5. Deduce that $\lim_{n\to\infty} |u_{2n} - u_{2n-1}| = d$.

   Set $z = P_K 0$.

6. Check that $|z| = d$ and that $|z|^2 \leq (z, w)\ \forall w \in K_2 - K_1$.

7. Prove that the sequence $(u_{2n} - u_{2n-1})$ converges to z.

   [**Hint**: Estimate $|z - (u_{2n} - u_{2n-1})|^2$ using the above results.]

8. Give a geometric interpretation.

- B -

Throughout the rest of this problem we assume that $z = P_K 0 \in K_2 - K_1$. (This assumption holds, for example, if one of the sets K_1 or K_2 is bounded, why?)

We claim that there exist $a_1 \in K_1$ and $a_2 \in K_2$ with $a_2 - a_1 = z$ such that $u_{2n} \rightharpoonup a_2$ and $u_{2n-1} \rightharpoonup a_1$ weakly. Note that a_1 and a_2 may depend on the choice of $u_0 = u$. Draw a figure.

1. Consider the Hilbert space $\mathcal{H} = H \times H$ equipped with its natural scalar product. Set $\mathcal{K} = \{[b_1, b_2] \in \mathcal{H}\ ; b_1 \in K_1, b_2 \in K_2 \text{ and } b_2 - b_1 = z\}$.
   Check that $\mathcal{K}$ is a nonempty closed convex set.

2. Let $b = [b_1, b_2] \in \mathcal{K}$. Determine the sequence (v_n) corresponding to $v_0 = b_1$. Deduce that the sequences $(|u_{2n-1} - b_1|)$ and $(|u_{2n} - b_2|)$ are nonincreasing.

3. Set $x_n = [u_{2n-1},\ u_{2n}]$ and prove that the sequence (x_n) satisfies the following property:

   (P) $\begin{cases} \text{For every subsequence } (x_{n_k}) \text{ that converges weakly to some} \\ \text{element } \bar{x} \in \mathcal{H}, \text{ then } \bar{x} \in \mathcal{K}. \end{cases}$

4. Apply Opial's lemma (see Exercise 5.25, question 3) and conclude.

PROBLEM 28 (5)

Projections and orthogonal projections

Let H be a Hilbert space. An operator $P \in \mathcal{L}(H)$ such that $P^2 = P$ is called a *projection*. Check that a projection satisfies the following properties:

(a) $I - P$ is a projection,

(b) $N(I-P) = R(P)$ and $N(P) = R(I-P)$,
(c) $N(P) \cap N(I-P) = \{0\}$,
(d) $H = N(P) + N(I-P)$.

- A -

An operator $P \in \mathcal{L}(H)$ is called an *orthogonal projection* if there exists a closed linear subspace M such that $P = P_M$ (where P_M is defined in Corollary 5.4). Check that every orthogonal projection is a projection.

1. Given a projection P, prove that the following properties are equivalent:

   (a) P is an orthogonal projection,
   (b) $P^\star = P$,
   (c) $\|P\| \le 1$,
   (d) $N(P) \perp N(I-P)$,

   where the notation $X \perp Y$ means that $(x, y) = 0 \; \forall x \in X, \forall y \in Y$.

2. Let $T \in \mathcal{L}(H)$ be an operator such that

   $$T^\star = T \quad \text{and} \quad T^2 = I.$$

   Prove that $P = \frac{1}{2}(I - T)$ is an orthogonal projection. Prove the converse.

   Assuming, in addition, that $(Tu, u) \ge 0 \; \forall u \in H$, prove that $T = I$.

- B -

Throughout this part, M and N denote two closed linear subspaces of H. Set $P = P_M$ and $Q = P_N$.

1. Prove that the following properties are equivalent:

   (a) $PQ = QP$,
   (b) PQ is a projection,
   (c) QP is a projection.

   In this case, check that

   (i) PQ is the orthogonal projection onto $M \cap N$,
   (ii) $(P + Q - PQ)$ is the orthogonal projection onto $\overline{M + N}$.

2. Prove that the following properties are equivalent:

   (a) $M \perp N$,
   (b) $PQ = 0$,
   (c) $QP = 0$,
   (d) $|Pu|^2 + |Qu|^2 \le |u|^2 \quad \forall u \in H$,
   (e) $|Pu| \le |u - Qu| \quad \forall u \in H$,
   (f) $|Qu| \le |u - Pu| \quad \forall u \in H$,

(g) $P + Q$ is a projection.

In this case, check that $(P + Q)$ is the orthogonal projection onto $M + N$ (note that $M + N$ is closed; why?).

3. Prove that the following properties are equivalent:

   (a) $M \subset N$,
   (b) $PQ = P$,
   (c) $QP = P$,
   (d) $|Pu| \leq |Qu| \; \forall u \in H$,
   (e) $Q - P$ is a projection.

   In this case, check that $Q - P$ is the orthogonal projection onto $M^\perp \cap N$.

PROBLEM 29 (5)

Iterates of nonlinear contractions.
The ergodic theorems of Opial and Baillon

Let H be a Hilbert space and let $T : H \to H$ be a nonlinear contraction, that is,

$$|Tu - Tv| \leq |u - v| \quad \forall u, v \in H.$$

We assume that the set

$$K = \{u \in H; \; Tu = u\}$$

of fixed points is nonempty. Check that K is closed and convex. Given $f \in H$ set

$$\sigma_n = \frac{1}{n}(f + Tf + T^2 f + \cdots + T^{n-1} f)$$

and

$$\mu_n = \left(\frac{I + T}{2}\right)^n f.$$

The goal of this problem is to prove the following:

(A) Each of the sequences (σ_n) and (μ_n) converges *weakly* to a fixed point of T.
(B) If, in addition, T is *odd*, that is, $T(-v) = -Tv \; \forall v \in H$, then (σ_n) and (μ_n) converge *strongly*.

It is advisable to solve Exercises 5.22 and 5.25 before starting this problem. In the special case that T is linear, see also Exercise 5.21.

- A -

Set

$$u_n = T^n f.$$

1. Check that for every $v \in K$, the sequence $(|u_n - v|)$ is nonincreasing. Deduce that the sequences (σ_n) and $(T\sigma_n)$ are bounded.

2. Prove that
$$|\sigma_n - T\sigma_n| \le \frac{1}{\sqrt{n}}|f - T\sigma_n| \quad \forall n \ge 1.$$
[**Hint**: Note that $|T\sigma_n - Tu_i|^2 \le |\sigma_n - u_i|^2$ and add these inequalities for $0 \le i \le n-1$.]

3. Deduce that the sequence (σ_n) satisfies property (P) of Exercise 5.25. Conclude that $\sigma_n \rightharpoonup \sigma$ weakly, with $\sigma \in K$.

Set
$$S = \frac{1}{2}(I + T).$$

4. Prove that
$$|(u - Su) - (v - Sv)|^2 + |Su - Sv|^2 \le |u - v|^2 \quad \forall u, v \in H.$$

5. Deduce that for every $v \in K$,
$$\sum_{n=0}^{\infty} |\mu_n - \mu_{n+1}|^2 \le |f - v|^2$$
and consequently
$$|\mu_n - S\mu_n| \le \frac{1}{\sqrt{n+1}}|f - v| \quad \forall n.$$

6. Conclude that $\mu_n \rightharpoonup \mu$ weakly, with $\mu \in K$.

- B -

Throughout the rest of this problem we assume that T is *odd*, that is,
$$T(-v) = -Tv \quad \forall v \in H.$$

1. Prove that for every integer p,
$$2|(u, v) - (T^p u,\ T^p v)| \le |u|^2 + |v|^2 - |T^p u|^2 - |T^p v|^2 \quad \forall u, v \in H.$$
[**Hint**: Start with the inequality $|T^p u - T^p v|^2 \le |u - v|^2 \ \forall u, v \in H$.]

2. Deduce that for every fixed integer $i \ge 0$,
$$\ell(i) = \lim_{n\to\infty} (u_n, u_{n+i}) \quad \text{exists.}$$
Prove that this convergence holds uniformly in i, that is,
$$(1) \qquad |(u_n, u_{n+i}) - \ell(i)| \le \varepsilon_n \quad \forall i \quad \text{and} \quad \forall n, \quad \text{with} \lim_{n\to\infty} \varepsilon_n = 0.$$

3. Similarly, prove that for every fixed integer $i \geq 0$,

$$m(i) = \lim_{n\to\infty} (\mu_n, \mu_{n+i}) \quad \text{exists.}$$

Prove that $m(0) = m(1) = m(2) = \cdots$.

[**Hint**: Use the result of question A5.]

4. Deduce that $\mu_n \to \mu$ strongly.

We now claim that $\sigma_n \to \sigma$ strongly.

5. Set

$$X_p = \frac{1}{p}\sum_{i=0}^{p-1} \ell(i).$$

Prove that

$$|(u_n, \sigma_{n+p}) - X_p| \leq \varepsilon_n + \frac{2n}{p}|f|^2 \quad \forall n, \quad \forall p.$$

[**Hint**: Use (1).]

6. Deduce that

(i) $X = \lim_{p\to\infty} X_p$ exists,
(ii) $|(u_n, \sigma) - X| \leq \varepsilon_n \; \forall n$,
(iii) $|\sigma|^2 = X$.

7. Prove that

$$|\sigma_n|^2 \leq \frac{2}{n^2}\sum_{i=0}^{n-1}(n-i)\ell(i) + \frac{2}{n}\sum_{i=0}^{n-1}\varepsilon_i.$$

8. Deduce that $\limsup_{n\to\infty} |\sigma_n|^2 \leq X$ and conclude.

PROBLEM 30 (3, 5)

Variants of Stampacchia's theorem. The min–max theorem of von Neumann

Let H be a Hilbert space.

- A -

Let $a(u, v) : H \times H \to \mathbb{R}$ be a continuous bilinear form such that

$$a(v, v) \geq 0 \;\; \forall v \in H.$$

Let $K \subset H$ be a nonempty closed convex set. Let $f \in H$. Assume that there exists some $v_0 \in K$ such that the set

$$\{u \in K;\; a(u, v_0 - u) \geq (f,\; v_0 - u)\}$$

is bounded.

1. Prove that there exists some $u \in K$ such that

$$a(u, v - u) \geq (f, v - u) \quad \forall v \in K.$$

[**Hint**: Set $f_\varepsilon = f + \varepsilon v_0$ and consider the bilinear form $a_\varepsilon(u, v) = a(u, v) + \varepsilon(u, v)$, $\varepsilon > 0$. Then, pass to the limit as $\varepsilon \to 0$ using Exercise 5.14.]

2. Recover Stampacchia's theorem.
3. Give a geometric interpretation in the case that K is bounded and $a(u, v) = 0$ $\forall u, v \in H$.

- B -

Let $b(u, v) : H \times H \to \mathbb{R}$ be a bilinear form that is continuous and *coercive*. Let $\varphi : H \to (-\infty, +\infty]$ be a convex l.s.c. function such that $\varphi \not\equiv +\infty$.

1. Prove that there exists a unique $u \in D(\varphi)$ such that

$$b(u, v - u) + \varphi(v) - \varphi(u) \geq 0 \quad \forall v \in D(\varphi).$$

[**Hint**: Apply the result of question A1 in the space $H \times \mathbb{R}$ with $K = \operatorname{epi} \varphi$, $f = [0, -1]$, $a(U, V) = b(u, v)$ with $U = [u, \lambda]$ and $V = [v, \mu]$. Note that a is *not coercive*.]

2. Recover Stampacchia's theorem.

- C -

Let H_1 and H_2 be two Hilbert spaces and let $A \subset H_1$, $B \subset H_2$ be two nonempty, bounded, closed convex sets.

1. Let $F(\lambda, \mu) : H_1 \times H_2 \to \mathbb{R}$ be a continuous bilinear form. Prove that there exist $\overline{\lambda} \in A$ and $\overline{\mu} \in B$ such that

$$(1) \qquad F(\overline{\lambda}, \mu) \leq F(\overline{\lambda}, \overline{\mu}) \leq F(\lambda, \overline{\mu}) \quad \forall \lambda \in A, \quad \forall \mu \in B.$$

[**Hint**: Apply question A1 with $H = H_1 \times H_2$, $K = A \times B$, and $a(u, v) = F(\lambda, \overline{\mu}) - F(\overline{\lambda}, \mu)$, where $u = [\overline{\lambda}, \overline{\mu}]$, $v = [\lambda, \mu]$.]

2. Deduce that

$$(2) \qquad \min_{\lambda \in A} \max_{\mu \in B} F(\lambda, \mu) = \max_{\mu \in B} \min_{\lambda \in A} F(\lambda, \mu).$$

Note that all min and max are achieved (why?).

[**Hint**: Check that without any further assumptions, max min $\leq$ min max; use (1) to prove the reverse inequality.]

3. Prove that (2) implies the existence of some $\overline{\lambda} \in A$ and $\overline{\mu} \in B$ satisfying (1).

- D -

Let E and F be two reflexive Banach spaces; let $A \subset E$ and $B \subset F$ be two nonempty, bounded, closed convex sets. Let $K : E \times F \to \mathbb{R}$ be a function satisfying the following assumptions:

(a) For every fixed $v \in B$ the function $u \mapsto K(u, v)$ is convex and l.s.c.
(b) For every fixed $u \in A$ the function $v \mapsto K(u, v)$ is concave and u.s.c., i.e., the function $v \mapsto -K(u, v)$ is convex and l.s.c.

Our goal is to prove that

$$\min_{u \in A} \max_{v \in B} K(u, v) = \max_{v \in B} \min_{u \in A} K(u, v).$$

We shall argue by contradiction and assume that there exists a constant γ such that

$$\max_{v \in B} \min_{u \in A} K(u, v) < \gamma < \min_{u \in A} \max_{v \in B} K(u, v).$$

1. For every $u \in A$, set

$$B_u = \{v \in B;\ K(u, v) \geq \gamma\}$$

and for every $v \in B$, set

$$A_v = \{u \in A;\ K(u, v) \leq \gamma\}.$$

Check that $\cap_{u \in A} B_u = \emptyset$ and $\cap_{v \in B} A_v = \emptyset$.

2. Choose $u_1, u_2, \ldots, u_n \in A$ and $v_1, v_2, \ldots, v_m \in B$ such that $\cap_{i=1}^n B_{u_i} = \emptyset$ and $\cap_{j=1}^m A_{v_j} = \emptyset$ (justify). Apply the result of C1 with $H_1 = \mathbb{R}^n$, $H_2 = \mathbb{R}^m$,

$$A' = \left\{ \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n);\ \lambda_i \geq 0 \quad \forall i \text{ and } \sum_{i=1}^n \lambda_i = 1 \right\},$$

$$B' = \left\{ \mu = (\mu_1, \mu_2, \ldots, \mu_m);\ \mu_j \geq 0 \quad \forall j \text{ and } \sum_{j=1}^m \mu_j = 1 \right\},$$

and $F(\lambda, \mu) = \sum_{i,j} \lambda_i \mu_j K(u_i, v_j)$. Set $\overline{u} = \sum_i \overline{\lambda}_i u_i$ and $\overline{v} = \sum_j \overline{\mu}_j v_j$. Prove that

$$K(\overline{u}, v_\ell) \leq K(u_k, \overline{v}) \quad \forall k = 1, 2, \ldots, n, \quad \forall \ell = 1, 2, \ldots, m.$$

3. Check that on the other hand,

$$\min_k K(u_k, \overline{v}) < \gamma \quad \text{and} \quad \max_\ell K(\overline{u}, v_\ell) > \gamma.$$

[**Hint**: Argue by contradiction.]

4. Conclude.

PROBLEM 31 (3, 5)

Monotone operators. The theorem of Minty–Browder

Let E be a reflexive Banach space. A (nonlinear) mapping

$$A : D(A) \subset E \to E^\star$$

is said to be *monotone* if it satisfies

$$\langle Au - Av,\ u - v\rangle \geq 0 \quad \forall u, v \in D(A)$$

(here $D(A)$ denotes any subset of E).

- A -

Let $A : D(A) \subset E \to E^\star$ be a monotone mapping and let $K \subset E$ be a nonempty, bounded, closed convex set. Our goal is to prove that there exists some $u \in K$ such that

$$\langle Av, u - v\rangle \geq 0 \quad \forall v \in D(A) \cap K.$$

For this purpose, set, for each $v \in D(A) \cap K$,

$$K_v = \{u \in K;\ \langle Av, v - u\rangle \geq 0\}.$$

We have to prove that $\cap_{v \in D(A) \cap K} K_v \neq \emptyset$; we shall argue by contradiction and assume that

$$\bigcap_{v \in D(A) \cap K} K_v = \emptyset.$$

1. Check that K_v is closed and convex.
2. Deduce that there exist $v_1, v_2, \ldots, v_n \in D(A) \cap K$ such that

$$\bigcap_{i=1}^{n} K_{v_i} = \emptyset.$$

Set $B = \{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n); \lambda_i \geq 0\ \ \forall i$ and $\sum_{i=1}^n \lambda_i = 1\}$, and consider the bilinear form

$$F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

defined by

$$F(\lambda, \mu) = \textstyle\sum_{i,j=1}^n \lambda_i \mu_j \langle Av_j,\ v_i - v_j\rangle.$$

3. Check that $F(\lambda, \lambda) \leq 0\ \forall \lambda \in \mathbb{R}^n$.
4. Prove that there exists some $\overline{\lambda} \in B$ such that $F(\overline{\lambda}, \mu) \leq 0\ \forall \mu \in B$.

   [**Hint**: Apply question C1 of Problem 30.]

5. Set $\overline{u} = \sum_{i=1} \overline{\lambda}_i v_i$ and prove that
$$\langle Av_j, \overline{u} - v_j \rangle \leq 0 \quad \forall j = 1, 2, \ldots, n.$$
6. Conclude.

- B -

Throughout the rest of this problem, we assume that $D(A) = E$, $A : E \to E^\star$ is monotone, and A is continuous.

1. Let $K \subset E$ be a nonempty, bounded, closed convex set. Prove that there exists some $u \in K$ such that $\langle Au, w - u \rangle \geq 0 \; \forall w \in K$.

   [**Hint**: Consider $v_t = (1 - t)u + tw$ with $t \in (0, 1)$ and $w \in K$.]
2. Let K be a closed convex set containing 0 (K need not be bounded). Assume that the set $\{u \in K;\ \langle Au, u \rangle \leq 0\}$ is bounded. Prove that there exists some $u \in K$ such that
$$\langle Au, v - u \rangle \geq 0 \quad \forall v \in K.$$
   [**Hint**: Apply B1 to the set $K_R = \{v \in K;\ \|v\| \leq R\}$ with R large enough.]
3. Assume here that
$$\lim_{\|v\| \to \infty} \frac{\langle Av, v \rangle}{\|v\|} = +\infty.$$
   Prove that A is surjective.
4. Assume here that E is a Hilbert space identified with $E^\star$. Prove that $I + A$ is bijective from E onto itself.

PROBLEM 32 (5)

Extension of contractions. The theorem of Kirszbraun–Valentine via the method of Schoenberg

Let H be a Hilbert space and let I be a finite set of indices.

- A -

Let $(y_i)_{i \in I}$ be elements of H and let $(c_i)_{i \in I}$ be elements of $\mathbb{R}$. Set
$$\varphi(u) = \max_{i \in I}\{|u - y_i|^2 - c_i\}, \quad u \in H,$$
and
$$J(u) = \{i \in I;\ |u - y_i|^2 - c_i = \varphi(u)\}.$$

1. Check that $\inf_{u \in H} \varphi(u)$ is achieved by some unique element $u_0 \in H$.
2. Prove that $\max_{i \in J(u_0)} (v, u_0 - y_i) \geq 0 \; \forall v \in H$.
3. Deduce that

$$u_0 \in \operatorname{conv}\left(\bigcup_{i \in J(u_0)} \{y_i\}\right). \tag{1}$$

4. Conversely, if $u_0 \in H$ satisfies (1), prove that $\varphi(u_0) = \inf_{u \in H} \varphi(u)$.
5. Extend this result to the case in which $\varphi(u) = \max_{i \in I} \{f_i(u)\}$ and each $f_i : H \to \mathbb{R}$ is a convex C^1 function.

- B -

Let $(x_i)_{i \in I}$ and $(y_i)_{i \in I}$ be elements of H such that

$$|y_i - y_j| \le |x_i - x_j| \quad \forall i, j \in I.$$

We claim that given any $p \in H$, there exists some $q \in \operatorname{conv}\left(\bigcup_{i \in I} \{y_i\}\right)$ such that

$$|q - y_i| \le |p - x_i| \quad \forall i \in I.$$

1. Set $P = \{\lambda = (\lambda_i)_{i \in I};\ \lambda_i \ge 0\ \forall i \text{ and } \sum_{i \in I} \lambda_i = 1\}$.
   Prove that for every $p \in H$ and for every $\lambda \in P$,

   $$\sum_{j \in I} \lambda_j \left| \left(\sum_{i \in I} \lambda_i y_i\right) - y_j \right|^2 \le \sum_{j \in I} \lambda_j |p - x_j|^2.$$

   [**Hint**: Check that $\sum_{j \in I} \lambda_j \left| \left(\sum_{i \in I} \lambda_i y_i\right) - y_j \right|^2 = \frac{1}{2} \sum_{i, j \in I} \lambda_i \lambda_j |y_i - y_j|^2$.]

2. Consider the function

   $$\varphi(u) = \max_{i \in I} \{|u - y_i|^2 - |p - x_i|^2\}.$$

   Let $u_0 \in H$ be such that $\varphi(u_0) = \inf_{u \in H} \varphi(u)$. Prove that $\varphi(u_0) \le 0$.

   [**Hint**: Apply questions A3 and B1.]

3. Conclude.

- C -

1. Extend the result of part B to the case that I is an infinite set of indices.
2. Let $D \subset H$ by any subset of H and let $S : D \to H$ be a contraction, i.e.,

   $$|Su - Sv| \le |u - v| \quad \forall u, v \in D.$$

   Prove that there exists a contraction T defined on all of H that extends S and such that

   $$T(H) \subset \overline{\operatorname{conv} S(D)}.$$

   [**Hint**: Use Zorn's lemma and question C1.]

PROBLEM 33 (4, 6)

Multiplication operator in L^p

Let Ω be a measure space (having finite or infinite measure). Set $E = L^p(\Omega)$ with $1 < p < \infty$. Let $a : \Omega \to \mathbb{R}$ be a measurable function. Consider the unbounded linear operator $A : D(A) \subset E \to E$ defined by

$$D(A) = \{u \in L^p(\Omega);\, au \in L^p(\Omega)\} \quad \text{and} \quad Au = au.$$

1. Prove that $D(A)$ is dense in E.

   [**Hint**: Given $u \in E$, consider the sequence $u_n(x) = (1 + n^{-1}|a(x)|)^{-1}u(x)$.]
2. Show that A closed.
3. Prove that $D(A) = E$ iff $a \in L^\infty(\Omega)$.

   [**Hint**: Apply the closed graph theorem.]
4. Determine $N(A)$ and $N(A)^\perp$.
5. Determine $D(A^\star)$, $A^\star$, $N(A^\star)$, and $N(A^\star)^\perp$.
6. Prove that A is surjective iff there exists $\alpha > 0$ such that $|a(x)| \geq \alpha$ a.e. on Ω.

   [**Hint**: Use question 3.]

   In what follows we assume that $a \in L^\infty(\Omega)$.
7. Determine the eigenvalues and the spectrum of A. Check that $\sigma(A) \subset [\inf_\Omega a, \sup_\Omega a]$ and that $\inf_\Omega a \in \sigma(A)$, $\sup_\Omega a \in \sigma(A)$. Here $\inf_\Omega$ and $\sup_\Omega$ refer to the ess $\inf_\Omega$ and ess $\sup_\Omega$ (defined in Section 8.5).
8. In case Ω is an open set in $\mathbb{R}^N$ (equipped with the Lebesgue measure) and $a \in C(\Omega) \cap L^\infty(\Omega)$, prove that $\sigma(A) = \overline{a(\Omega)}$.
9. Prove that $\sigma(A) = \{0\}$ iff $a = 0$ a.e. on Ω.
10. Assume that Ω has no atoms. Prove that A is compact iff $a = 0$ a.e. on Ω.

PROBLEM 34 (4, 6)

Spectral analysis of the Hardy operator $Tu(x) = \frac{1}{x}\int_0^x u(t)dt$

- A -

Let $E = C([0, 1])$ equipped with the norm $\|u\| = \sup_{t\in[0,1]} |u(t)|$. Given $u \in E$ define the function Tu on $[0, 1]$ by

$$Tu(x) = \begin{cases} \frac{1}{x}\int_0^x u(t)dt & \text{if } x \in (0, 1], \\ u(0) & \text{if } x = 0. \end{cases}$$

Check that $Tu \in E$ and that $\|Tu\| \le \|u\|\ \forall u \in E$, so that $T \in \mathcal{L}(E)$.

1. Prove that $EV(T) = (0, 1]$ and determine the corresponding eigenfunctions.
2. Check that $\|T\|_{\mathcal{L}(E)} = 1$. Is T a compact operator from E into itself?
3. Show that $\sigma(T) = [0, 1]$. Give an explicit formula for $(T - \lambda I)^{-1}$ when $\lambda \in \rho(T)$. Prove that $(T - \lambda I)$ is surjective from E onto E for every $\lambda \in \mathbb{R}, \lambda \notin \{0, 1\}$. Check that T and $(T - I)$ are not surjective.
4. In this question we consider T as a bounded operator from $E = C([0, 1])$ into $F = L^q(0, 1)$ with $1 \le q < \infty$. Prove that $T \in \mathcal{K}(E, F)$.

   [**Hint**: Consider the operator $(T_\varepsilon u)(x) = \frac{1}{x+\varepsilon} \int_0^x u(t)dt$ with $\varepsilon > 0$ and estimate $\|T_\varepsilon - T\|_{\mathcal{L}(E,F)}$ as $\varepsilon \to 0$.]

- B -

In this part we set $E = C^1([0, 1])$ equipped with the norm

$$\|u\| = \sup_{t\in[0,1]} |u(t)| + \sup_{t\in[0,1]} |u'(t)|.$$

Given $u \in C^1([0, 1])$ we define Tu as in part A.

1. Check that if $u \in C^1([0, 1])$, then $Tu \in C^1([0, 1])$ and $\|Tu\| \le \|u\|\ \forall u \in E$.
2. Prove that $EV(T) = (0, \frac{1}{2}] \cup \{1\}$.
3. Prove that $\sigma(T) = [0, \frac{1}{2}] \cup \{1\}$.

- C -

In this part we set $E = L^p(0, 1)$ with $1 < p < \infty$. Given $u \in L^p(0, 1)$ define Tu by

$$Tu(x) = \frac{1}{x} \int_0^x u(t)dt \qquad \text{for } x \in (0, 1].$$

Check that $Tu \in C((0, 1])$ and that $Tu \in L^q(0, 1)$ for every $q < p$. Our goal is to prove that $Tu \in L^p(0, 1)$ and that

$$\|Tu\|_{L^p(0,1)} \le \frac{p}{p-1} \|u\|_{L^p(0,1)} \quad \forall u \in E. \tag{1}$$

1. Prove that (1) holds when $u \in C_c((0, 1))$.

   [**Hint**: Set $\varphi(x) = \int_0^x u(t)dt$; check that $|\varphi|^p \in C^1([0, 1])$ and compute its derivative. Estimate $\|Tu\|_{L^p}$ using the formula

   $$\int_0^1 |Tu(x)|^p dx = \int_0^1 |\varphi(x)|^p \frac{dx}{x^p} = \frac{1}{p-1} \int_0^1 |\varphi(x)|^p d\left(-\frac{1}{x^{p-1}}\right)$$

   and integrating by parts.]

2. Prove that (1) holds for every $u \in E$.

   In what follows we consider T as a bounded operator from E into itself.

3. Show that $EV(T) = (0, \frac{p}{p-1})$.
4. Deduce that $\|T\|_{\mathcal{L}(E)} = \frac{p}{p-1}$. Is T a compact operator from E into itself?
5. Prove that $\sigma(T) = [0, \frac{p}{p-1}]$.
6. Determine $T^\star$.
7. In this question we consider T as a bounded operator from $E = L^p(0, 1)$ into $F = L^q(0, 1)$ with $1 \le q < p < \infty$. Show that $T \in \mathcal{K}(E, F)$.

PROBLEM 35 (6)

Cotlar's lemma

Let H be a Hilbert space identified with its dual space.

- A -

Assume $T \in \mathcal{L}(H)$, so that $T^\star \in \mathcal{L}(H)$.

1. Prove that $\|T^\star T\| = \|T\|^2$.
2. Assume in this question that T is self-adjoint.
   Show that
   $$\|T^N\| = \|T\|^N \text{ for every integer } N.$$
3. Deduce that (for a general $T \in \mathcal{L}(H)$),
   $$\|(T^\star T)^N\| = \|T\|^{2N} \text{ for every integer } N.$$

- B -

Let (T_j), $1 \le j \le m$, be a finite collection of operators in $\mathcal{L}(H)$. Assume that $\forall j, k \in \{1, 2, \ldots, m\}$,

$$\|T_j^\star T_k\|^{1/2} \le \omega(j-k), \tag{1}$$

$$\|T_k T_j^\star\|^{1/2} \le \omega(k-j), \tag{2}$$

where $\omega : \mathbb{Z} \to [0, \infty)$.

Set

$$\sigma = \sum_{i=-(m-1)}^{m-1} \omega(i).$$

The goal of this problem is to show that

$$\left\| \sum_{j=1}^{m} T_j \right\| \le \sigma. \tag{3}$$

Set

$$U = \sum_{j=1}^{m} T_j$$

and fix an integer N.

1. Show that

$$\|T_{j_1}^\star T_{k_1} T_{j_2}^\star T_{k_2} \cdots T_{j_N}^\star T_{k_N}\| \\ \le \sigma\omega(j_1 - k_1)\omega(k_1 - j_2)\omega(j_2 - k_2)\cdots\omega(k_{N-1} - j_N)\omega(j_N - k_N),$$

for any choice of the integers $j_1, k_1, \ldots, j_N, k_N \in \{1, 2, \ldots, m\}$.

2. Deduce that

$$\sum_{j_1}\sum_{k_1}\cdots\sum_{j_N}\sum_{k_N} \|T_{j_1}^\star T_{k_1} \cdots T_{j_N}^\star T_{k_N}\| \le m\sigma^{2N},$$

where the summation is taken over all possible choices of the integers $j_i, k_i \in \{1, 2, \ldots, m\}$.

3. Prove that

$$\|(U^\star U)^N\| \le m\sigma^{2N}$$

and deduce that (3) holds.

PROBLEM 36 (6)

More on the Riesz–Fredholm theory

Let E be a Banach space and let $T \in \mathcal{K}(E)$. For every integer $k \ge 1$ set

$$N_k = N((I - T)^k) \quad \text{and} \quad R_k = R((I - T)^k).$$

1. Check that $\forall k \ge 1$, $R_{k+1} \subset R_k$, R_k is closed, $T(R_k) \subset R_k$, and $(I - T)R_k \subset R_{k+1}$.
2. Prove that there exists an integer $p \ge 1$ such that

$$\begin{cases} R_{k+1} \ne R_k & \forall k < p \text{ (no condition if } p = 1), \\ R_{k+1} = R_k & \forall k \ge p. \end{cases}$$

3. Check that $\forall k \ge 1$, $N_k \subset N_{k+1}$, $\dim N_k < \infty$, $T(N_k) \subset N_k$, and $(I - T)N_{k+1} \subset N_k$.

4. Show that
$$\operatorname{codim} R_k = \dim N_k \quad \forall k \geq 1,$$
and deduce that
$$\begin{cases} N_{k+1} \neq N_k & \forall k < p \text{ (no condition if } p = 1), \\ N_{k+1} = N_k & \forall k \geq p. \end{cases}$$

5. Prove that
$$\begin{cases} R_p \cap N_p = \{0\}, \\ R_p + N_p = E. \end{cases}$$

6. Prove that $(I - T)$ restricted to R_p is bijective from R_p onto itself.

7. Assume here in addition that E is a Hilbert space and that T is self-adjoint. Prove that $p = 1$.

PROBLEM 37 (6)

Courant–Fischer min–max principle. Rayleigh–Ritz method

Let H be an infinite-dimensional separable Hilbert space. Let T be a self-adjoint compact operator from H into itself such that $(Tx, x) \geq 0$ $\forall x \in H$. Denote by (μ_k), $k \geq 1$, its eigenvalues, repeated with their multiplicities, and arranged in nonincreasing order:
$$\mu_1 \geq \mu_2 \geq \cdots \geq 0.$$
Let (e_j) be an associated orthonormal basis composed of eigenvectors. Let E_k be the space spanned by $\{e_1, e_2, \dots, e_k\}$. For $x \neq 0$ we define the *Rayleigh quotient*
$$R(x) = \frac{(Tx, x)}{|x|^2}.$$

1. Prove that $\forall k \geq 1$,
$$\min_{\substack{x \in E_k \\ x \neq 0}} R(x) = \mu_k.$$

2. Prove that $\forall k \geq 2$,
$$\max_{\substack{x \in E_{k-1}^{\perp} \\ x \neq 0}} R(x) = \mu_k,$$
and
$$\max_{\substack{x \in H \\ x \neq 0}} R(x) = \mu_1.$$

3. Let Σ be any k-dimensional subspace of H with $k \geq 1$. Prove that
$$\min_{\substack{x \in \Sigma \\ x \neq 0}} R(x) \leq \mu_k.$$

[**Hint**: If $k \geq 2$, show that $\Sigma \cap E_{k-1}^{\perp} \neq \{0\}$ and apply question 2.]

4. Deduce that $\forall k \geq 1$,

$$\max_{\substack{\Sigma \subset H \\ \dim \Sigma = k}} \min_{\substack{x \in \Sigma \\ x \neq 0}} R(x) = \mu_k.$$

5. Let Σ be any $(k-1)$-dimensional subspace of H with $k \geq 2$. Prove that

$$\max_{\substack{x \in \Sigma^{\perp} \\ x \neq 0}} R(x) \geq \mu_k.$$

[**Hint**: Prove that $\Sigma^{\perp} \cap E_k \neq \{0\}$.]

6. Deduce that $\forall k \geq 1$,

$$\min_{\substack{\Sigma \subset H \\ \dim \Sigma = k-1}} \max_{\substack{x \in \Sigma^{\perp} \\ x \neq 0}} R(x) = \mu_k.$$

7. Assume here that $N(T) = \{0\}$, so that $R(x) \neq 0 \ \forall x \neq 0$, or equivalently $\mu_k > 0 \ \forall k$. Show that $\forall k \geq 1$,

$$\min_{\substack{\Sigma \subset H \\ \dim \Sigma = k}} \max_{\substack{x \in \Sigma \\ x \neq 0}} \frac{1}{R(x)} = \frac{1}{\mu_k},$$

and

$$\max_{\substack{\Lambda \subset H \\ \operatorname{codim} \Lambda = k-1}} \min_{\substack{x \in \Lambda \\ x \neq 0}} \frac{1}{R(x)} = \frac{1}{\mu_k},$$

where Σ and Λ are closed subspaces of H.
In particular, for $k = 1$,

$$\min_{\substack{x \in H \\ x \neq 0}} \frac{1}{R(x)} = \frac{1}{\mu_1};$$

and, moreover, $\forall k \geq 2$,

$$\min_{\substack{x \in E_{k-1}^{\perp} \\ x \neq 0}} \frac{1}{R(x)} = \frac{1}{\mu_k}.$$

8. Let V be a closed subspace of H (finite- or infinite-dimensional). Let P_V be the orthogonal projection from H onto V and consider the operator $S : V \to V$ defined by $S = P_V \circ T_{|V}$. Check that S is a self-adjoint compact operator from V into itself such that $(Sx, x) \geq 0 \ \forall x \in V$.

9. Denote by (ν_k), $k \geq 1$, the eigenvalues of S, repeated with their multiplicities and arranged in nonincreasing order. Prove that $\forall k$ with $1 \leq k \leq \dim V$,

$$\max_{\substack{\Sigma \subset V \\ \dim \Sigma = k}} \min_{\substack{x \in \Sigma \\ x \neq 0}} R(x) = \nu_k.$$

Deduce that $\nu_k \le \mu_k$ $\forall k$ with $1 \le k \le \dim V$.

10. Consider now an increasing sequence $V^{(n)}$ of closed subspaces of H such that

$$\overline{\bigcup_n V^{(n)}} = H.$$

Set $S^{(n)} = P_{V^{(n)}} \circ T_{|V^{(n)}}$ and let $(\nu_k^{(n)})$ denote the eigenvalues of $S^{(n)}$ arranged as in question 9. Prove that for each fixed k the sequence $n \mapsto \nu_k^{(n)}$ is nondecreasing and converges, as $n \to \infty$, to μ_k.

PROBLEM 38 (2, 6, 11)

Fredholm–Noether operators

Let E and F be Banach spaces and let $T \in \mathcal{L}(E, F)$.

- A -

The goal of part A is to prove that the following conditions are equivalent:

$$(1) \qquad \begin{cases} \text{(a)} \quad R(T) \text{ is closed and has finite codimension in } F, \\ \text{(b)} \quad N(T) \text{ admits a complement in } E. \end{cases}$$

$$(2) \qquad \begin{cases} \text{There exist } S \in \mathcal{L}(F, E) \text{ and } K \in \mathcal{K}(F, F) \text{ such that} \\ T \circ S = I_F + K. \end{cases}$$

$$(3) \qquad \begin{cases} \text{There exist } U \in \mathcal{L}(F, E) \text{ and a finite-rank} \\ \text{projection } P \text{ in } F \text{ such that } T \circ U = I_F - P. \end{cases}$$

Moreover, one can choose U and P such that $\dim R(P) = \operatorname{codim} R(T)$.

1. Prove that (1) $\Rightarrow$ (3)

   [**Hint**: Let X be a complement of $N(T)$ in E. Then $T_{|X}$ is bijective from X onto $R(T)$. Denote by U_0 its inverse. Let Q be a projection from F onto $R(T)$ and set $U = U_0 \circ Q$.]

2. Prove that (2) $\Rightarrow$ (3).

   [**Hint**: Use Exercise 6.25.]

3. Prove that (3) $\Rightarrow$ (1).

   [**Hint**: To establish part (a) of (1) note that $R(T) \supset R(I_F - P)$ and apply Proposition 11.5. Similarly, show that $R(U^\star)$ is closed and thus $R(U)$ is also closed. Finally, prove that there exist finite-dimensional spaces Σ_1 and Σ_2 in E such that $N(T) + R(U) + \Sigma_1 = E$ and $N(T) \cap R(U) \subset \Sigma_2$. Then apply Proposition 11.7.]

4. Conclude.

- B -

Prove that the following conditions are equivalent:

$$(4) \qquad \begin{cases} \text{(a)} \quad R(T) \text{ is closed and admits a complement,} \\ \text{(b)} \quad \dim N(T) < \infty. \end{cases}$$

$$(5) \qquad \begin{cases} \text{There exists } \widetilde{S} \in \mathcal{L}(F, E) \text{ and } \widetilde{K} \in \mathcal{K}(E, E) \text{ such that} \\ \widetilde{S} \circ T = I_E + \widetilde{K}. \end{cases}$$

$$(6) \qquad \begin{cases} \text{There exist } \widetilde{U} \in \mathcal{L}(F, E) \text{ and a finite-rank} \\ \text{ projection } \widetilde{P} \text{ in } E \text{ such that } \widetilde{U} \circ T = I_E - \widetilde{P}. \end{cases}$$

- C -

One says that an operator $T \in \mathcal{L}(E, F)$ is *Fredholm* (or *Noether*) if it satisfies

$$(\text{FN}) \qquad \begin{cases} \text{(a)} \quad R(T) \text{ is closed and has finite codimension,} \\ \text{(b)} \quad \dim N(T) < \infty. \end{cases}$$

(The property that $R(T)$ is closed can be deduced from the other assumptions; see Exercise 2.27.)

The class of operators satisfying (FN) is denoted by $\Phi(E, F)$. The *index* of T is by definition

$$\operatorname{ind} T = \dim N(T) - \operatorname{codim} R(T).$$

1. Assume that $T \in \Phi(E, F)$. Show that there exist $U \in \mathcal{L}(F, E)$ and finite-rank projections P in F (resp. $\tilde{P}$ in E) such that

   $$(7) \qquad \begin{cases} \text{(a)} \quad T \circ U = I_F - P, \\ \text{(b)} \quad U \circ T = I_E - \widetilde{P}, \end{cases}$$

   with $\dim R(P) = \operatorname{codim} R(T),\ \dim R(\widetilde{P}) = \dim N(T)$.
   [**Hint**: Use the operator U constructed in question A1.]

   An operator $V \in \mathcal{L}(F, E)$ satisfying

   $$(8) \qquad \begin{cases} \text{(a)} \quad T \circ V = I_F + K, \\ \text{(b)} \quad V \circ T = I_E + \widetilde{K}, \end{cases}$$

   with $K \in \mathcal{K}(F)$ and $\widetilde{K} \in \mathcal{K}(E)$, is called a *pseudoinverse* of T (or an *inverse modulo compact operators*).

2. Show that any pseudoinverse V belongs to $\Phi(F, E)$.

3. Prove that an operator $T \in \mathcal{L}(E, F)$ belongs to $\Phi(E, F)$ iff $R(T)$ is closed, $\dim N(T) < \infty$, and $\dim N(T^\star) < \infty$. Moreover,

$$\operatorname{ind} T = \dim N(T) - \dim N(T^\star).$$

[**Hint**: Apply Propositions 11.14 and 2.18.]

4. Let $T \in \Phi(E, F)$. Prove that $T^\star \in \Phi(F^\star, E^\star)$ and that

$$\operatorname{ind} T^\star = -\operatorname{ind} T.$$

[**Hint**: Apply Proposition 11.13 and Theorem 2.19.]

5. Conversely, let $T \in \mathcal{L}(E, F)$ be such that $T^\star \in \Phi(F^\star, E^\star)$. Prove that $T \in \Phi(E, F)$.

6. Assume that $J \in \mathcal{L}(E, F)$ is bijective and $K \in \mathcal{K}(E, F)$. Show that $T = J + K$ belongs to $\Phi(E, F)$ and $\operatorname{ind} T = 0$. Conversely, if $T \in \Phi(E, F)$ and $\operatorname{ind} T = 0$, prove that T can be written as $T = J + K$ with J and K as above (one may even choose K to be of finite rank).

   [**Hint**: Applying Theorem 6.6, prove that $I_E + J^{-1} \circ K$ belongs to $\Phi(E, E)$ and has index zero. For the converse, consider an isomorphism from $N(T)$ onto a complement Y of $R(T)$.]

7. Let $T \in \Phi(E, F)$ and $K \in \mathcal{K}(E, F)$. Prove that $T + K \in \Phi(E, F)$.

8. Under the assumptions of the previous question, show that

$$\operatorname{ind}(T + K) = \operatorname{ind} T.$$

   [**Hint**: Set $\widetilde{E} = E \times Y$, $\widetilde{F} = F \times N(T)$, and $\widetilde{T} : \widetilde{E} \to \widetilde{F}$ defined by $\widetilde{T}(x, y) = (Tx + Kx, 0)$. Show that $\widetilde{T} = \widetilde{J} + \widetilde{K}$, where $\widetilde{J}$ is bijective from $\widetilde{E}$ onto $\widetilde{F}$ and $\widetilde{K} \in \mathcal{K}(\widetilde{E}, \widetilde{F})$. Then apply question 6.]

9. Let $T \in \Phi(E, F)$. Prove that there exists $\varepsilon > 0$ (depending on T) such that for every $M \in \mathcal{L}(E, F)$ with $\|M\| < \varepsilon$, we have $T + M \in \Phi(E, F)$. Show that

$$\operatorname{ind}(T + M) = \operatorname{ind} T.$$

   [**Hint**: Let V be a pseudoinverse of T. Then $W = I_E + (V \circ M)$ is bijective if $\|M\| < \|V\|^{-1}$. Check that $T + M = (T \circ W) +$ compact; then apply the previous question.]

10. Let (H_t), $t \in [0, 1]$, be a family of operators in $\mathcal{L}(E, F)$. Assume that $t \mapsto H_t$ is continuous from $[0, 1]$ into $\mathcal{L}(E, F)$, and that $H_t \in \Phi(E, F)$ $\forall t \in [0, 1]$. Prove that $\operatorname{ind} H_t$ is constant on $[0, 1]$.

11. Let E_1, E_2, and E_3 be Banach spaces and let $T_1 \in \Phi(E_1, E_2)$, $T_2 \in \Phi(E_2, E_3)$. Prove that $T_2 \circ T_1 \in \Phi(E_1, E_3)$.

12. With the same notation as above, show that

$$\operatorname{ind}(T_2 \circ T_1) = \operatorname{ind} T_1 + \operatorname{ind} T_2.$$

[**Hint**: Consider the family of operators $H_t : E_1 \times E_2 \to E_2 \times E_3$ defined in matrix notation, for $t \in [0, 1]$, by

$$H_t = \begin{pmatrix} I & 0 \\ 0 & T_2 \end{pmatrix} \begin{pmatrix} (1-t)I & tI \\ -tI & (1-t)I \end{pmatrix} \begin{pmatrix} T_1 & 0 \\ 0 & I \end{pmatrix},$$

where I is the identity operator in E_2. Check that $t \mapsto H_t$ is continuous from $[0, 1]$ into $\mathcal{L}(E_1 \times E_2, E_2 \times E_3)$. Using the previous question, show that for each t, $H_t \in \Phi(E_1 \times E_2, E_2 \times E_3)$. Compute $\operatorname{ind} H_0$ and $\operatorname{ind} H_1$.]

13. Let $T \in \Phi(E, F)$. Compute the index of any pseudoinverse V of T.

- D -

In this part we study two simple examples.

1. Assume $\dim E < \infty$ and $\dim F < \infty$. Show that any linear operator T from E into F belongs to $\Phi(E, F)$ and compute its index.
2. Let $E = F = \ell^2$. Consider the shift operators S_r and S_ℓ defined in Exercise 6.18. Prove that for every $\lambda \in \mathbb{R}, \lambda \neq +1, \lambda \neq -1,$ we have $S_r - \lambda I \in \Phi(\ell^2, \ell^2)$, and $S_\ell - \lambda I \in \Phi(\ell^2, \ell^2)$. Compute their indices.
   Show that $S_r \pm I$, $S_\ell \pm I$ do not belong to $\Phi(\ell^2, \ell^2)$.

   [**Hint**: Use the results of Exercise 6.18.]

PROBLEM 39 (5, 6)

Square root of a self-adjoint nonnegative operator

Let H be a Hilbert space. Let $S \in \mathcal{L}(H)$; we say that S is *nonnegative*, and we write $S \geq 0$, if $(Sx, x) \geq 0$ $\forall x \in H$. When $S_1, S_2 \in \mathcal{L}(H)$, we write $S_1 \geq S_2$ (or $S_2 \leq S_1$) if $S_1 - S_2 \geq 0$.

- A -

1. Let $S \in \mathcal{L}(H)$ be such that $S^\star = S$ and $0 \leq S \leq I$. Show that $\|S^2\| = \|S\|^2 \leq 1$, and that $0 \leq S^2 \leq S \leq I$.

   [**Hint**: Use Exercise 6.24.]
2. Let $S \in \mathcal{L}(H)$ be such that $S^\star = S$ and $S \geq 0$. Let $P(t) = \sum a_k t^k$ be a polynomial such that $a_k \geq 0$ $\forall k$. Prove that $[P(S)]^\star = P(S)$ and $P(S) \geq 0$.
3. Let (S_n) be a sequence in $\mathcal{L}(H)$ such that $S_n^\star = S_n$ $\forall n$ and $S_{n+1} \leq S_n$ $\forall n$. Assume that $\|S_n\| \leq M$ $\forall n$, for some constant M. Prove that for every $x \in H$, $S_n x$ converges as $n \to \infty$ to a limit, denoted by Sx, and that $S \in \mathcal{L}(H)$ with $S^\star = S$.

[**Hint**: Let $n \geq m$. Use Exercise 6.24 to prove that $|S_n x - S_m x|^2 \leq 2M(S_m x - S_n x, x)$.]

- B -

Assume that $T \in \mathcal{L}(H)$ satisfies $T^\star = T$, $T \geq 0$, and $\|T\| \leq 1$. Consider the sequence (S_n) defined by

$$S_{n+1} = S_n + \frac{1}{2}(T - S_n^2), \quad n \geq 0,$$

starting with $S_0 = I$.

1. Show that $S_n^\star = S_n \ \forall n \geq 0$.
2. Show that
$$I - S_{n+1} = \frac{1}{2}(I - S_n)^2 + \frac{1}{2}(I - T),$$
and deduce that $I - S_n \geq 0 \ \forall n$.
3. Prove that $S_n \geq 0 \ \forall n$.

   [**Hint**: Show by induction that $I - S_n \leq I$ using questions A.1 and B.2.]
4. Deduce that $\|S_n\| \leq 1 \ \forall n$.
5. Prove that
$$S_n - S_{n+1} = \frac{1}{2}\left[(I - S_n) + (I - S_{n-1})\right] \circ (S_{n-1} - S_n) \quad \forall n$$
and deduce that $S_{n-1} - S_n \geq 0 \ \forall n$.

   [**Hint**: Show by induction that $(I - S_n) = P_n(I - T)$ and $(S_{n-1} - S_n) = Q_n(I - T)$, where P_n and Q_n are polynomials with nonnegative coefficients.]
6. Show that $\lim_{n\to\infty} S_n x = Sx$ exists. Prove that $S \in \mathcal{L}(H)$ satisfies $S^\star = S$, $S \geq 0$, $\|S\| \leq 1$, and $S^2 = T$.

- C -

1. Let $U \in \mathcal{L}(H)$ be such that $U^\star = U$ and $U \geq 0$. Prove that there exists $V \in \mathcal{L}(H)$ such that $V^\star = V$, $V \geq 0$, and $V^2 = U$.

   [**Hint**: Apply the construction of part B to $T = U/\|U\|$.]

   Next, we prove the uniqueness of V. More precisely, if W is any operator $W \in \mathcal{L}(H)$ such that $W^\star = W$, $W \geq 0$, and $W^2 = U$, then $W = V$. The operator V is called the *square root* of U and is denoted by $U^{1/2}$.
2. Prove that the operator V constructed above commutes with every operator X that commutes with U (i.e., $X \circ U = U \circ X$ implies $X \circ V = V \circ X$).
3. Prove that W commutes with U and deduce that V commutes with W.

4. Check that $(V - W) \circ (V + W) = 0$ and deduce that $V = W$ on $R(V + W)$. Show that $N(V) = N(W) = N(U) = N(V + W)$. Conclude that $V = W$ on H.

   [**Hint**: Note that $V = W$ on $\overline{R(V + W)} = N(U)^\perp$, and that $V = W = 0$ on $N(U)$.]

5. Show that $\|U^{1/2}\| = \|U\|^{1/2}$.

6. Let $U_1, U_2 \in \mathcal{L}(H)$ be such that $U_1^\star = U_1$, $U_2^\star = U_2$, $U_1 \geq 0$, $U_2 \geq 0$, and $U_1 \circ U_2 = U_2 \circ U_1$. Prove that $U_1 \circ U_2 \geq 0$.

   [**Hint**: Introduce $U_1^{1/2}$ and $U_2^{1/2}$.]

- D -

Let $U \in \mathcal{K}(H)$ be such that $U^\star = U$ and $U \geq 0$. Prove that its square root V belongs to $\mathcal{K}(H)$. Assuming that H is separable, compute V on a Hilbert basis composed of eigenvectors of U. Find the eigenvalues of V.

PROBLEM 40 (4, 5, 6)

Hilbert–Schmidt operators

- A -

Let E and F be separable Hilbert spaces, both identified with their dual spaces. The norms on E and on F are denoted by the same symbol $|\ |$. Let $T \in \mathcal{L}(E, F)$, so that $T^\star \in \mathcal{L}(F, E)$.

1. Let (e_k) (resp. (f_k)) be any orthonormal basis of E (resp. F). Show that $\sum_{k=1}^\infty |T(e_k)|^2 < \infty$ iff $\sum_{k=1}^\infty |T^\star(f_k)|^2 < \infty$, and that

$$\sum_{k=1}^\infty |T(e_k)|^2 = \sum_{k=1}^\infty |T^\star(f_k)|^2.$$

2. Let (e_k) and $(\tilde{e}_k)$ be two orthonormal bases of E. Show that $\sum_{k=1}^\infty |T(e_k)|^2 < \infty$ iff $\sum_{k=1}^\infty |T(\tilde{e}_k)|^2 < \infty$ and that

$$\sum_{k=1}^\infty |T(e_k)|^2 = \sum_{k=1}^\infty |T(\tilde{e}_k)|^2.$$

   One says that $T \in \mathcal{L}(E, F)$ is a *Hilbert–Schmidt* operator and one writes $T \in \mathcal{HS}(E, F)$ if there exists some orthonormal basis (e_k) of E such that

$$\sum_{k=1}^\infty |T(e_k)|^2 < \infty.$$

3. Prove that $\mathcal{HS}(E, F)$ is a linear subspace of $\mathcal{L}(E, F)$ and that

$$\|T\|_{\mathcal{HS}} = \left(\sum_{k=1}^{\infty} |T(e_k)|^2\right)^{1/2}$$

defines a norm on $\mathcal{HS}(E, F)$. Let $\| \ \|$ denote the standard norm or $\mathcal{L}(E, F)$. Show that

$$\|T\| \leq \|T\|_{\mathcal{HS}} \quad \forall T \in \mathcal{HS}(E, F).$$

4. Prove that $\mathcal{HS}(E, F)$ equipped with the norm $\| \ \|_{\mathcal{HS}}$ is a Banach space. Show that in fact, it is a Hilbert space.

5. Show that $\mathcal{HS}(E, F) \subset \mathcal{K}(E, F)$.

   [**Hint**: Given $x \in E$, write $x = \sum_{k=1}^{\infty} x_k e_k$ and set $T_n(x) = \sum_{k=1}^{n} x_k T(e_k)$. Show that $\|T_n - T\| \to 0$ as $n \to \infty$.]

6. Show that any finite-rank operator from E into F belongs to $\mathcal{HS}(E, F)$.

7. Let $T \in \mathcal{L}(E, F)$. Prove that $T \in \mathcal{HS}(E, F)$ iff $T^\star \in \mathcal{HS}(F, E)$ and that

$$\|T^\star\|_{\mathcal{HS}(F,E)} = \|T\|_{\mathcal{HS}(E,F)}.$$

8. Assume that $T \in \mathcal{K}(E, E)$ with $T^\star = T$, and let (λ_k) denote the sequence of eigenvalues of T. Show that $T \in \mathcal{HS}(E, E)$ iff $\sum_{k=1}^{\infty} \lambda_k^2 < \infty$ and that

$$\|T\|_{\mathcal{HS}}^2 = \sum_{k=1}^{\infty} \lambda_k^2.$$

   Construct an example of an operator $T \in \mathcal{K}(E, E)$ with $E = \ell^2$ such that $T \notin \mathcal{HS}(E, E)$.

9. Let G be another separable Hilbert space. Let $T_1 \in \mathcal{L}(E, F)$ and $T_2 \in \mathcal{L}(F, G)$. Show that $T_2 \circ T_1 \in \mathcal{HS}(E, G)$ if either T_1 or T_2 belongs to $\mathcal{HS}$.

10. Let $T \in \mathcal{HS}(E, E)$ and assume $N(I + T) = \{0\}$. Show that $(I + T)$ is bijective and that $(I + T)^{-1} = I + S$ with $S \in \mathcal{HS}(E, E)$.

11. Let (e_k) (resp. (f_k)) be an orthonormal basis of E (resp. F). Consider the operator $T_{k,\ell} : E \to F$ defined by

$$T_{k,\ell}(x) = (x, e_k) f_\ell.$$

    Show that $(T_{k,\ell})$ is an orthonormal basis of $\mathcal{HS}(E, F)$.

- B -

Assume that Ω is an open subset of $\mathbb{R}^N$. In what follows we take $E = F = L^2(\Omega)$. Let $K \in L^2(\Omega \times \Omega)$, and consider the operator

$$(Tu)(x) = \int_\Omega K(x, y) u(y) dy. \tag{1}$$

1. Show that $T \in \mathcal{L}(E, E)$ and that

$$\|T\|_{\mathcal{L}(E,E)} \leq \|K\|_{L^2(\Omega\times\Omega)}.$$

2. Show that $T \in \mathcal{HS}(E, E)$ and that

$$\|T\|_{\mathcal{HS}(E,E)} \leq \|K\|_{L^2(\Omega\times\Omega)}.$$

[**Hint**: Let (e_j) be an orthonormal basis of $L^2(\Omega)$. Check that the family $e_{j,k} = e_j \otimes e_k$, where $(e_j \otimes e_k)(x, y) = e_j(x)e_k(y)$, is an orthonormal basis of $L^2(\Omega \times \Omega)$. Then write

$$\|T(e_k)\|^2_{L^2(\Omega)} = \sum_{j=1}^{\infty} |(T(e_k), e_j)|^2 = \sum_{j=1}^{\infty} |(K, e_j \otimes e_k)|.]$$

3. Conversely, let $T \in \mathcal{HS}(E, E)$. Prove that there exists a unique function $K \in L^2(\Omega \times \Omega)$ such that (1) holds. K is called the *kernel* of T.

[**Hint**: Let $t_{j,k} = (Te_k, e_j)$ and check that $\sum_{j,k=1}^{\infty} |t_{j,k}|^2 < \infty$. Define $K = \sum_{j,k=1}^{\infty} t_{j,k}e_j \otimes e_k$ and prove that (1) holds.]

4. Assume that $\Omega = (0, 1)$, $E = L^2(\Omega)$, and consider the operator

$$(Tu)(x) = \int_0^x u(t)dt.$$

Show that $T \in \mathcal{HS}(E, E)$ and compute $\|T\|_{\mathcal{HS}}$.

PROBLEM 41 (1, 6)

The Krein–Rutman theorem

Let E be a Banach space and let $P \subset E$ be a closed convex set containing 0. Assume that P is a convex cone with vertex at 0, i.e., $\lambda x + \mu y \in P$ $\forall\lambda > 0$, $\mu > 0$, $x \in P$, and $y \in P$.

Assume that

$$\text{Int}\, P \neq \emptyset \tag{1}$$

and

$$P \neq E. \tag{2}$$

Let $T \in \mathcal{K}(E)$ be such that

$$T(P \setminus \{0\}) \subset \text{Int}\, P. \tag{3}$$

- A -

1. Show that (Int P) $\cap (-P) = \emptyset$.

   [**Hint**: Use Exercise 1.7.]

   In what follows we *fix* some $u \in \text{Int } P$.

2. Show that there exists $\alpha > 0$ such that

$$\|x + u\| \geq \alpha \quad \forall x \in P.$$

   [**Hint**: Argue by contradiction and deduce that $-u \in P$.]

3. Check that there exists $r > 0$ such that

$$Tu - ru \in P.$$

4. Assume that some $x \in P$ satisfies

$$T(x + u) = \lambda x \quad \text{for some } \lambda \in \mathbb{R}.$$

   Prove that $\lambda \geq r$.

   [**Hint**: It is convenient to introduce an order relation on E defined by $y \geq z$ if $y - z \in P$. Show by induction that $\left(\frac{\lambda}{r}\right)^n x \geq u,\ n = 1, 2, \dots$.]

5. Consider the nonlinear map

$$F(x) = T\left(\frac{x + u}{\|x + u\|}\right), \quad x \in P.$$

   Show that $F : P \to P$ is continuous and $F(P) \subset K$ for some compact set $K \subset E$. Deduce that there exists some $x_1 \in P$ such that

$$T(x_1 + u) = \lambda_1 x_1$$

   with $\lambda_1 = \|x_1 + u\| \geq r$.

   [**Hint**: Apply the Schauder fixed-point theorem; see Exercise 6.26.]

6. Deduce that for every $\varepsilon > 0$ there exists $x_\varepsilon \in P$ such that

$$T(x_\varepsilon + \varepsilon u) = \lambda_\varepsilon x_\varepsilon$$

   with $\lambda_\varepsilon = \|x_\varepsilon + \varepsilon u\| \geq r$.

7. Prove that there exist $x_0 \in \text{Int } P$ and $\mu_0 > 0$ such that

$$T x_0 = \mu_0 x_0.$$

[**Hint**: Show that (x_ε) is bounded. Deduce that there exists a sequence $\varepsilon_n \to 0$ such that $x_{\varepsilon_n} \to x_0$ and $\lambda_{\varepsilon_n} \to \mu_0$ with the required properties.]

- B -

1. Given two points $a \in \operatorname{Int} P$ and $b \in E$, $b \notin P$, prove that there exists a unique $\sigma \in (0, 1)$ such that

$$\begin{aligned}(1-t)a + tb &\in \operatorname{Int} P \quad \forall t \in [0, \sigma),\\ (1-\sigma)a + \sigma b &\in P,\\ (1-t)a + tb &\notin P \qquad \forall t \in (\sigma, 1].\end{aligned}$$

Then we set $\tau(a, b) = \sigma/(1-\sigma)$, with $0 < \tau(a, b) < \infty$.

2. Let $x \in P \setminus \{0\}$ be such that

$$Tx = \mu x \quad \text{for some } \mu \in \mathbb{R}.$$

Prove that $\mu = \mu_0$ and $x = mx_0$ for some $m > 0$, where μ_0 and x_0 have been constructed in question A7.

[**Hint**: Suppose by contradiction that $x \neq mx_0$, $\forall m > 0$. Show that $\mu > 0$, $x \in \operatorname{Int} P$, and $-x \notin P$. Set $y = x_0 - \tau_0 x$, where $\tau_0 = \tau(x_0, -x)$. Compute Ty and deduce that $\mu < \mu_0$. Then reverse the roles of x_0 and x.]

3. Let $x \in E \setminus \{0\}$ be such that

$$Tx = \mu x \quad \text{for some } \mu \in \mathbb{R}.$$

Prove that *either* $\mu = \mu_0$ and $x = mx_0$ with $m \in \mathbb{R}$, $m \neq 0$, *or* $|\mu| < \mu_0$.

[**Hint**: In view of question 2 one may assume that $x \notin P$ and $-x \notin P$. If $\mu > 0$ consider $\tau(x_0, x)$, and if $\mu < 0$ consider both $\tau(x_0, x)$ and $\tau(x_0, -x)$.]

4. Deduce that $N(T - \mu_0 I) = \mathbb{R}x_0$. In other words, the geometric multiplicity of the eigenvalue μ_0 is one.

5. Prove that $N((T - \mu_0 I)^k) = \mathbb{R}x_0$ for all $k \geq 2$. In other words, the algebraic multiplicity of the eigenvalue μ_0 is also one.

[**Hint**: In view of Problem 36, it suffices to show that $N((T - \mu_0 I)^2) = \mathbb{R}x_0$.]

PROBLEM 42 (6)

Lomonosov's theorem on invariant subspaces

Let E be an infinite-dimensional Banach space and let $T \in \mathcal{K}(E)$, $T \neq 0$. The goal of part A is to prove that there exists a nontrivial, closed, invariant subspace Z of T, i.e., $T(Z) \subset Z$, with $Z \neq \{0\}$, and $Z \neq E$.

- A -

Set

$$\mathcal{A} = \text{span}\{I, T, T^2, \dots\}$$
$$= \left\{ \sum_{i \in I} \lambda_i T^i, \text{ with } \lambda_i \in \mathbb{R} \text{ and } I \text{ is a finite subset of } \{0, 1, 2, \dots\} \right\}.$$

For every $y \in E$, set $\mathcal{A}_y = \{Sy;\ S \in \mathcal{A}\}$. Clearly, $y \in \mathcal{A}_y$ and thus $\mathcal{A}_y \neq \{0\}$ for every $y \neq 0$. Moreover, $\mathcal{A}_y$ is a subspace of E and $T(\mathcal{A}_y) \subset \mathcal{A}_y$, so that $T(\overline{\mathcal{A}_y}) \subset \overline{\mathcal{A}_y}$. If $\overline{\mathcal{A}_y} \neq E$ for some $y \neq 0$, then $\overline{\mathcal{A}_y}$ is a nontrivial, closed, invariant subspace of T. Therefore we can assume that

(1) $$\overline{\mathcal{A}_y} = E \quad \forall y \in E,\ y \neq 0.$$

Since $T \neq 0$, we may *fix* some $x_0 \in E$ such that $Tx_0 \neq 0$, and some r such that

$$0 < r \leq \frac{\|Tx_0\|}{2\|T\|} \leq \frac{\|x_0\|}{2}.$$

Set

$$C = \{x \in E;\ \|x - x_0\| \leq r\}.$$

1. Check that $0 \notin C$ and that

$$\|Tx - Tx_0\| \leq \frac{1}{2}\|Tx_0\| \quad \forall x \in C,$$

so that

$$\|Tx\| \geq \frac{1}{2}\|Tx_0\| \quad \forall x \in C.$$

Deduce that $0 \notin \overline{T(C)}$.

2. Prove that for every $y \in E$, $y \neq 0$, there exists some $S \in \mathcal{A}$, denoted by S_y, such that

$$\|Sy - x_0\| \leq \frac{r}{2}.$$

[**Hint**: Use assumption (1).]

3. Deduce that for every $y \in E$, $y \neq 0$, there exists some $\varepsilon > 0$ (depending on y), denoted by ε_y, such that

$$\|Sz - x_0\| \leq r \quad \forall z \in B(y, \varepsilon),$$

where S is as in question 2.

4. Consider a finite covering of $\overline{T(C)}$ by balls $B(y_j, \frac{1}{2}\varepsilon_{y_j})$ with $j \in J$, J finite. Set, for $j \in J$ and $x \in E$,

$$q_j(x) = \max\{0, \varepsilon_{y_j} - \|Tx - y_j\|\} \quad \text{and} \quad q(x) = \sum_{j \in J} q_j(x).$$

Check that the functions q_j, $j \in J$, and q are continuous on E. Show that $\forall x \in C$,

$$q(x) \geq \min_{j \in J} \left\{ \frac{1}{2}\, \varepsilon_{y_j} \right\} > 0.$$

Set

$$F(x) = \frac{1}{q(x)} \sum_{j \in J} q_j(x) S_{y_j}(Tx), \quad x \in C.$$

5. Prove that F is continuous from C into E and that

$$\|F(x) - x_0\| \leq r \quad \forall x \in C.$$

[**Hint**: Use question 3.]

6. Prove that $F(C) \subset K$, where K is a compact subset of C. Deduce that there exists $\xi \in C$ such that $F(\xi) = \xi$.

[**Hint**: Apply the Schauder fixed-point theorem; see Exercise 6.26.]

7. Set

$$U = \frac{1}{q(x)} \sum_{j \in J} q_j(\xi)(S_{y_j} \circ T),$$

with ξ as in question 6. Show that $U \in \mathcal{K}(E)$. Deduce that $Z = N(I - U)$ is finite-dimensional; check that $\xi \in Z$.

8. Prove that $T(Z) \subset Z$ and conclude.

[**Hint**: Show that $U \in \mathcal{A}$ and deduce that $T \circ U = U \circ T$.]

9. Construct a linear operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ that has no invariant subspaces except the trivial ones.

- B -

We now establish a stronger version of the above result. Assume that $T \in \mathcal{K}(E)$ and $T \neq 0$. Let $R \in \mathcal{L}(E)$ be such that $R \circ T = T \circ R$. Prove that R admits a nontrivial, closed, invariant subspace.

[**Hint**: Set $\mathcal{B} = \text{span}\,\{I, R, R^2, \ldots\}$ and $\mathcal{B}_y = \{Sy;\, S \in \mathcal{B}\}$. Check that all the steps in part A still hold with $\mathcal{A}$ replaced by $\mathcal{B}$ and $\mathcal{A}_y$ by $\mathcal{B}_y$.]

PROBLEM 43 (2, 4, 5, 6)

Normal operators

Let H be a Hilbert space identified with its dual space. An operator $T \in \mathcal{L}(H)$ is said to be *normal* if it satisfies

$$T \circ T^\star = T^\star \circ T.$$

1. Prove that T is normal iff it satisfies

$$|Tu| = |T^\star u| \quad \forall u \in H.$$

[**Hint**: Compute $|T(u+v)|^2$.]

Throughout the rest of this problem we assume that T is normal.

2. Assume that $u \in N(T - \lambda I)$ and $v \in N(T - \mu I)$ with $\lambda \neq \mu$. Show that $(u, v) = 0$.

[**Hint**: Prove, using question 1, that $N(T^\star - \mu I) = N(T - \mu I)$, and compute (Tu, v).]

3. Prove that $\overline{R(T)} = \overline{R(T^\star)} = N(T)^\perp = N(T^\star)^\perp$.

4. Let $f \in R(T)$. Check that there exists $u \in \overline{R(T^\star)}$ satisfying $f = Tu$.

[**Hint**: Note that $H = \overline{R(T)} \oplus N(T)$.]

5. Consider a sequence $u_n \in R(T^\star)$ such that $u_n \to u$ as $n \to \infty$. Write $u_n = T^\star y_n$ for some $y_n \in H$. Show that Ty_n converges as $n \to \infty$ to a limit $z \in H$ that satisfies $T^\star z = f$.

[**Hint**: Use question 1 and a Cauchy sequence argument.]

6. Deduce that $R(T) = R(T^\star)$.

[**Hint**: Use the fact that $N(T) = N(T^\star)$.]

7. Show that $\|T^2\| = \|T\|^2$.

[**Hint**: Write $|Tu|^2 \le |T^\star Tu|\,|u| = |T^2 u|\,|u|$.]

8. Deduce that $\|T^p\| = \|T\|^p$ for every integer $p \ge 1$.

[**Hint**: Consider first the case $p = 2^k$. For a general integer p, choose any k such that $2^k \ge p$ and write $\|T\|^{2^k} = \|T^{2^k}\| = \|T^{2^k - p} T^p\|$.]

9. Prove that $N(T^2) = N(T)$ and deduce that $N(T^p) = N(T)$ for every integer $p \ge 1$.

[**Hint**: Note that if $T^2 u = 0$, then $Tu \in N(T) \cap R(T)$.]

PROBLEM 44 (5, 6)

Isometries and unitary operators. Skew-adjoint operators.
Polar decomposition and Cayley transform.

Let H be a Hilbert space identified with its dual space and let $T \in \mathcal{L}(H)$. One says that

(i) T is an *isometry* if $|Tu| = |u|\ \forall u \in H$,
(ii) T is a *unitary* operator if T is an isometry that is also surjective,
(iii) T is *skew-adjoint* (or *antisymmetric*) if $T^\star = -T$.

- A -

1. Assume that T is an isometry. Check that $\|T\| = 1$.
2. Prove that $T \in \mathcal{L}(H)$ is an isometry iff $T^\star \circ T = I$.
3. Assume that $T \in \mathcal{L}(H)$ is an isometry. Prove that the following conditions are equivalent:

   (a) T is a unitary operator,
   (b) $T^\star$ is injective,
   (c) $T \circ T^\star = I$,
   (d) $T^\star$ is an isometry,
   (e) $T^\star$ is a unitary operator.

4. Give an example of an isometry that is *not* a unitary operator.

   [**Hint**: Use Exercise 6.18.]

5. Assume that T is an isometry. Prove that $R(T)$ is closed and that $T \circ T^\star = P_{R(T)} =$ the orthogonal projection on $R(T)$.
6. Assume that T is an isometry. Prove that
   either T is a unitary operator and then $\sigma(T) \subset \{-1, +1\}$,
   or T is *not* a unitary operator and then $\sigma(T) = [-1, +1]$.
7. Assume that $T \in \mathcal{K}(H)$ is an isometry. Show that $\dim H < \infty$.
8. Prove that $T \in \mathcal{L}(H)$ is skew-adjoint iff $(Tu, u) = 0\ \forall u \in H$.
9. Assume that $T \in \mathcal{L}(H)$ is skew-adjoint. Show that $\sigma(T) \subset \{0\}$.

   [**Hint**: Use Lax–Milgram.]

10. Assume that $T \in \mathcal{L}(H)$ is skew-adjoint. Set

$$U = (T + I) \circ (T - I)^{-1}.$$

Check that U is well defined, that $U = (T - I)^{-1} \circ (T + I)$, and that $U \circ T = T \circ U$. Prove that U is a unitary operator (U is called the *Cayley transform* of T).

11. Conversely, let $T \in \mathcal{L}(H)$ be such that $1 \notin \sigma(T)$. Assume that $U = (T + I) \circ (T - I)^{-1}$ is an isometry. Prove that T is skew-adjoint.

- B -

We will say that an operator $T \in \mathcal{L}(H)$ satisfies property (1) if

(1) there exists an isometry J from $N(T)$ into $N(T^\star)$.

The goal of part B is to prove that every operator $T \in \mathcal{L}(H)$ satisfying property (1) can be factored as

$$T = U \circ P,$$

where $U \in \mathcal{L}(H)$ is an isometry and $P \in \mathcal{L}(H)$ is a self-adjoint nonnegative operator (recall that nonnegative means $(Pu, u) \geq 0\ \forall u \in H$). Such a factorization is called a *polar decomposition* of T. In addition, P is uniquely determined on H, and U is uniquely determined on $N(T)^\perp$ (but not on H).

1. Check that assumption (1) is satisfied in the following cases:

   (i) T is injective,
   (ii) $\dim H < \infty$,
   (iii) T is normal (see Problem 43),
   (iv) $T = I - K$ with $K \in \mathcal{K}(H)$.

2. Give an example in which (1) is *not* satisfied.

   [**Hint**: Use Exercise 6.18.]

3. Assume that we have a polar decomposition $T = U \circ P$. Prove that $P^2 = T^\star \circ T$.

4. Deduce that P is uniquely determined on H.

   [**Hint**: Use Problem 39.]

5. Let $T = U \circ P$ be a polar decomposition of T. Show that U is uniquely determined on $N(T)^\perp$.

6. Assume that T admits a polar decomposition. Show that (1) holds.

   [**Hint**: Set $J = U_{|N(T)}$.]

7. Prove that every operator $T \in \mathcal{L}(H)$ satisfying (1) admits a polar decomposition.

8. Assume that T satisfies the stronger assumption

   (2) there exists an isometry J from $N(T)$ *onto* $N(T^\star)$.

   Show that T admits a polar decomposition $T = U \circ P$, where U is a unitary operator.

9. Deduce that every normal $T \in \mathcal{L}(H)$ admits a polar decomposition $T = U \circ P$ where U is a unitary operator and $U \circ P = P \circ U$.

10. Show that every operator $T \in \mathcal{L}(H)$ satisfying (2) can be factored as $T = P \circ U$, where $U \in \mathcal{L}(H)$ is a unitary operator and $P \in \mathcal{L}(H)$ is a self-adjoint nonnegative operator.

    [**Hint**: Apply question 8 to $T^\star$.]

11. Show that every operator $T \in \mathcal{K}(H)$ satisfying (1) admits a polar decomposition $T = U \circ P$, where $P \in \mathcal{K}(H)$.

12. Assume that H is separable and $T \in \mathcal{K}(H)$ (but T does not necessarily satisfy (1)). Prove that there exist two orthonormal bases (e_n) and (f_n) of H such that

$$Tu = \sum_{n=1}^{\infty} \alpha_n (u, e_n) f_n \quad \forall u \in H,$$

where (α_n) is a sequence such that $\alpha_n \geq 0 \; \forall n$ and $\alpha_n \to 0$ as $n \to \infty$. Compute $T^\star$. Conversely, show that any operator of this form must be compact.

PROBLEM 45 (8)

Strong maximum principle

Consider the bilinear form

$$a(u, v) = \int_0^1 pu'v' + quv,$$

where $p \in C^1([0, 1])$, $p \geq \alpha > 0$ on $(0, 1)$, and $q \in C([0, 1])$. We assume that a is coercive on $H_0^1(0, 1)$ (but we make no sign assumption on q).

Given $f \in L^2(0, 1)$, let $u \in H^2(0, 1)$ be the solution of

$$\begin{cases} -(pu')' + qu = f & \text{on } (0, 1), \\ u(0) = u(1) = 0. \end{cases} \tag{1}$$

Assume that $f \geq 0$ a.e. on $(0, 1)$ and $f \not\equiv 0$. Our goal is to prove that

$$u'(0) > 0, \; u'(1) < 0 \tag{2}$$

and

$$u(x) > 0 \quad \forall x \in (0, 1). \tag{3}$$

1. Assume that $\psi \in H^1(0, 1)$ satisfies

$$\begin{cases} a(\psi, v) \leq 0 & \forall v \in H_0^1(0, 1), v \geq 0 \text{ on } (0, 1), \\ \psi(0) \leq 0, \; \psi(1) \leq 0. \end{cases} \tag{4}$$

   Prove that $\psi \leq 0$ on $(0, 1)$.

[**Hint:** Take $v = \psi^+$ in (4) and use Exercise 8.11.]

Consider the problem

$$(5) \qquad \begin{cases} -(p\zeta')' + q\zeta = 0 & \text{on } (0,1), \\ \zeta(0) = 0, \ \zeta(1) = 1. \end{cases}$$

2. Show that (5) has a unique solution ζ and that $\zeta \geq 0$ on $(0, 1)$.
3. Check that $u \geq 0$ on $(0, 1)$ and deduce that $u'(0) \geq 0$ and $u'(1) \leq 0$.
4. Prove that

$$(6) \qquad p(1)|u'(1)| = \int_0^1 f\zeta.$$

[**Hint:** Multiply (1) by ζ and (5) by u.]

Set $\varphi(x) = \left(e^{Bx} - 1\right)$, $B > 0$.

5. Check that if B is sufficiently large (depending only on p and q), then

$$(7) \qquad -(p\varphi')' + q\varphi \leq 0 \quad \text{on } (0,1).$$

In what follows we fix B such that (7) holds.

6. Let $A = \left(e^B - 1\right)^{-1}$. Prove that

$$\zeta \geq A\varphi \quad \text{on } (0,1).$$

[**Hint:** Apply question 1 to $\psi = A\varphi - \zeta$.]

7. Deduce that $u'(1) < 0$.

[**Hint:** Apply question 4.]

8. Check that $u'(0) > 0$.

[**Hint:** Change t into $(1 - t)$.]

9. Fix $\delta \in (0, \frac{1}{2})$ so small that

$$\frac{u(x)}{x} \geq \frac{1}{2}u'(0) \quad \forall x \in (0,\delta) \quad \text{and} \quad \frac{u(x)}{1-x} \geq \frac{1}{2}|u'(1)| \quad \forall x \in (1-\delta, 1).$$

Why does such δ exist? Let v be the solution of the problem

$$\begin{cases} -(pv')' + qv = 0 & \text{on } (\delta, 1-\delta), \\ v(\delta) = v(1-\delta) = \gamma, \end{cases}$$

where

$$\gamma = \frac{\delta}{2}\min\{u'(0), |u'(1)|\}.$$

Show that $u \geq v \geq 0$ on $(\delta, 1 - \delta)$.

10. Prove that $v > 0$ on $(\delta, 1 - \delta)$.

[**Hint:** Assume by contradiction that $v(x_0) = 0$ for some $x_0 \in (\delta, 1-\delta)$, and apply Theorem 7.3 (Cauchy–Lipschitz–Picard) as in Exercise 8.33.]

11. Deduce that $u(x) > 0 \; \forall x \in (0, 1)$.

Finally, we present a sharper form of the strong maximum principle.

12. Prove that there is a constant $a > 0$ (depending only on p and q) such that

$$u(x) \geq ax(1-x)\int_0^1 f(t)t(1-t)dt.$$

[**Hint:** Start with the case where $p \equiv 1$ and $q \equiv k^2$ is a positive constant; use an explicit solution of (1). Next, consider the case where $p \equiv 1$ and no further assumption is made on q. Finally, reduce the general case to the previous situation, using a change of variable.]

PROBLEM 46 (8)

The method of subsolutions and supersolutions

Let $h(t) : [0, +\infty) \to [0, +\infty)$ be a continuous nondecreasing function. Assume that there exist two functions $v, w \in C^2([0, 1])$ satisfying

$$(1) \qquad \begin{cases} 0 \leq v \leq w & \text{on } I = (0,1), \\ -v'' + v \leq h(v) & \text{on } I, \quad v(0) = v(1) = 0, \\ -w'' + w \geq h(w) & \text{on } I, \quad w(0) \geq 0, w(1) \geq 0, \end{cases}$$

(v is called a *subsolution* and w a *supersolution*). The goal is to prove that there exists a solution $u \in C^2([0, 1])$ of the problem

$$(2) \qquad \begin{cases} -u'' + u = h(u) & \text{on } I, \\ u(0) = u(1) = 0, \\ v \leq u \leq w & \text{on } I. \end{cases}$$

Consider the sequence $(u_n)_{n \geq 1}$ defined inductively by

$$(3) \qquad \begin{cases} -u_n'' + u_n = h(u_{n-1}) & \text{on } I, \; n \geq 1, \\ u_n(0) = u_n(1) = 0, \end{cases}$$

starting from $u_0 = w$.

1. Show that $v \leq u_1 \leq w$ on I.

   [**Hint:** Apply the maximum principle to $(u_1 - w)$ and to $(u_1 - v)$.]

2. Prove by induction that for every $n \geq 1$,

$$v \leq u_n \text{ on } I \quad \text{and} \quad u_{n+1} \leq u_n \text{ on } I.$$

3. Deduce that the sequence (u_n) converges in $L^2(I)$ to a limit u and that $h(u_n) \to h(u)$ in $L^2(I)$.
4. Show that $u \in H_0^1(I)$, and that

$$\int_0^1 u'\varphi' + \int_0^1 u\varphi = \int_0^1 h(u)\varphi \quad \forall \varphi \in H_0^1(I).$$

5. Conclude that $u \in C^2([0, 1])$ is a classical solution of (2).

In what follows we choose $h(t) = t^\alpha$, where $0 < \alpha < 1$. The goal is to prove that there exists a unique function $u \in C^2([0, 1])$ satisfying

$$\begin{cases} -u'' + u = u^\alpha & \text{on} I, \\ u(0) = u(1) = 0, & \\ u(x) > 0 & \forall x \in I. \end{cases} \tag{4}$$

6. Let $v(x) = \varepsilon \sin(\pi x)$ and $w(x) \equiv 1$. Show that if ε is sufficiently small, assumption (1) is satisfied. Deduce that there exists a solution of (4).

We now turn to the question of uniqueness. Let u be the solution of (4) obtained by the above method, starting with $u_0 \equiv 1$. Let $\tilde{u} \in C^2([0, 1])$ be another solution of (4).

7. Show that $\tilde{u} \leq 1$ on I.

[**Hint:** Consider a point $x_0 \in [0, 1]$ where $\tilde{u}$ achieves its maximum.]

8. Prove that the sequence $(u_n)_{n \geq 1}$ defined by (3), starting with $u_0 \equiv 1$, satisfies

$$\tilde{u} \leq u_n \quad \text{on } I,$$

and deduce that $\tilde{u} \leq u$ on I.
9. Show that

$$\int_0^1 (\tilde{u}^\alpha u - u^\alpha \tilde{u}) = 0.$$

10. Conclude that $\tilde{u} = u$ on I.

[**Hint:** Write $\tilde{u}^\alpha u - u^\alpha \tilde{u} = u\tilde{u}(\tilde{u}^{\alpha-1} - u^{\alpha-1})$ and note that $u^{\alpha-1} \leq \tilde{u}^{\alpha-1}$.]

We now present an alternative proof of existence. Set, for every $u \in H_0^1(I)$,

$$F(u) = \frac{1}{2}\int_0^1 (u'^2 + u^2) - \int_0^1 g(u),$$

where $g(t) = \frac{1}{\alpha+1}(t^+)^{\alpha+1}$, $0 < \alpha < 1$, and $t^+ = \max\,(t, 0)$.
11. Prove that there exists a constant C such that

$$F(u) \geq \frac{1}{2}\|u\|_{H^1}^2 - C\|u\|_{H^1}^{\alpha+1} \quad \forall u \in H_0^1(I).$$

12. Deduce that

$$m = \inf_{v \in H_0^1(I)} F(v) > -\infty,$$

and that the infimum is achieved.

[**Hint:** Let (u_n) be a minimizing sequence. Check that a subsequence (u_{n_k}) converges weakly in $H_0^1(I)$ to a limit u and that $\int_0^1 g(u_{n_k}) \to \int_0^1 g(u)$. The reader is warned that the functional F is *not* convex; why?]

13. Show that $m < 0$.

[**Hint:** Prove that $F(\varepsilon v) < 0$ for all $v \in H_0^1(I)$ such that $v^+ \not\equiv 0$ and for all ε sufficiently small.]

14. Check that

$$g(b) - g(a) \geq (a^+)^\alpha (b - a) \quad \forall a, b \in \mathbb{R}.$$

15. Let $u \in H_0^1(I)$ be a minimizer of F on $H_0^1(I)$. Prove that

$$\int_0^1 (u'v' + uv) = \int_0^1 (u^+)^\alpha v \quad \forall v \in H_0^1(I).$$

[**Hint:** Write that $F(u) \leq F(u + tv)$, apply question 14, and let $t \to 0$.]

16. Deduce that $u \in C^2([0, 1])$ is a solution of

$$\text{(5)} \qquad \begin{cases} -u'' + u = (u^+)^\alpha & \text{on } I, \\ u(0) = u(1) = 0. \end{cases}$$

Prove that $u \geq 0$ on I and $u \not\equiv 0$.

17. Conclude that $u > 0$ on I using the strong maximum principle (see Problem 45).

PROBLEM 47 (8)

Poincaré–Wirtinger's inequalities

Let $I = (0, 1)$.

- A -

1. Prove that

$$\text{(1)} \qquad \|u - \overline{u}\|_{L^\infty(I)} \leq \|u'\|_{L^1(I)} \quad \forall u \in W^{1,1}(I), \text{ where } \overline{u} = \int_I u.$$

[**Hint:** Note that $\overline{u} = u(x_0)$ for some $x_0 \in [0, 1]$.]

2. Show that the constant 1 in (1) is optimal, i.e.,

$$\text{(2)} \qquad \sup\{\|u - \overline{u}\|_{L^\infty(I)};\ u \in W^{1,1}(I), \text{ and } \|u'\|_{L^1(I)} = 1\} = 1.$$

[**Hint:** Consider a sequence (u_n) of smooth functions on $[0, 1]$ such that $u_n' \geq 0$ on $(0, 1)$ $\forall n$, $u_n(1) = 1$ $\forall n$, $u_n(x) = 0$ $\forall x \in [0, 1 - \frac{1}{n}]$, $\forall n$.]

3. Prove that the sup in (2) is not achieved, i.e., there exists no function $u \in W^{1,1}(I)$ such that

$$\|u - \overline{u}\|_{L^\infty(I)} = 1 \quad \text{and} \quad \|u'\|_{L^1(I)} = 1.$$

4. Prove that

$$\|u\|_{L^\infty(I)} \leq \frac{1}{2}\|u'\|_{L^1(I)} \quad \forall u \in W_0^{1,1}(I). \tag{3}$$

[**Hint:** Write that $|u(x) - u(0)| \leq \int_0^x |u'(t)|dt$ and $|u(x) - u(1)| \leq \int_x^1 |u'(t)|dt$.]

5. Show that $\frac{1}{2}$ is the best constant in (3). Is it achieved?

[**Hint:** Fix $a \in (0, 1)$ and consider a function $u \in W_0^{1,1}(I)$ increasing on $(0, a)$, decreasing on $(a, 1)$, with $u(a) = 1$.]

6. Deduce that the following inequalities hold:

$$\|u - \overline{u}\|_{L^q(I)} \leq C\|u'\|_{L^p(I)} \quad \forall u \in W^{1,p}(I). \tag{4}$$

and

$$\|u\|_{L^q(I)} \leq C\|u'\|_{L^p(I)} \quad \forall u \in W_0^{1,p}(I) \tag{5}$$

with $1 \leq q \leq \infty$ and $1 \leq p \leq \infty$.
Prove that the best constants in (4) and (5) are achieved when $1 \leq q \leq \infty$ and $1 < p \leq \infty$.

[**Hint:** Minimize $\|u'\|_{L^p(I)}$ in the class $u \in W^{1,p}(I)$ such that $\|u - \overline{u}\|_{L^q(I)} = 1$, resp. $u \in W_0^{1,p}(I)$ and $\|u\|_{L^q(I)} = 1$.]

- B -

The next goal is to find the best constant in (4) when $p = q = 2$, i.e.,

$$\|u - \overline{u}\|_{L^2(I)} \leq C\|u'\|_{L^2(I)} \quad \forall u \in H^1(I). \tag{6}$$

Set $H = \{f \in L^2(I); \int_I f = 0\}$ and $V = \{v \in H^1(I); \int_I v = 0\}$.

1. Check that for every $f \in H$ there exists a unique $u \in V$ such that

$$\int_I u'v' = \int_I fv \quad \forall v \in V.$$

2. Prove that $u \in H^2(I)$ and satisfies

$$\begin{cases} -u'' = f & \text{a.e. on } I, \\ u'(0) = u'(1) = 0. \end{cases}$$

3. Show that the operator $T : H \to H$ defined by $Tf = u$ is self-adjoint, compact, and that $\int_I fTf \geq 0 \; \forall f \in H$.
4. Let λ_1 be the largest eigenvalue of T. Prove that (6) holds with $C = \sqrt{\lambda_1}$ and that $\sqrt{\lambda_1}$ is the best constant in (6).

   [**Hint:** Use Exercise 6.24.]
5. Compute explicitly the best constant in (6).

- C -

1. Prove that

$$(7) \qquad \|u - \overline{u}\|_{L^1(I)} \leq 2 \int_I |u'(t)| t(1-t) dt \quad \forall u \in W^{1,1}(I).$$

2. Deduce that

$$(8) \qquad \|u - \overline{u}\|_{L^1(I)} \leq \frac{1}{2} \|u'\|_{L^1(I)} \quad \forall u \in W^{1,1}(I).$$

3. Show that the constant 1/2 in (8) is optimal, i.e.,

$$(9) \qquad \sup\{\|u - \overline{u}\|_{L^1(I)};\; u \in W^{1,1}(I), \text{ and } \|u'\|_{L^1(I)} = 1\} = \frac{1}{2}.$$

4. Is the sup in (9) achieved?

PROBLEM 48 (8)

A nonlinear problem

Let $j : [-1, +1] \to [0, +\infty)$ be a continuous convex function such that $j \in C^2((-1, +1))$, $j(0) = 0$, $j'(0) = 0$, and

$$\lim_{t \uparrow +1} j'(t) = +\infty, \qquad \lim_{t \downarrow -1} j'(t) = -\infty.$$

(A good example to keep in mind is $j(t) = 1 - \sqrt{1 - t^2}$, $t \in [-1, +1]$.) Given $f \in L^2(0, 1)$, define the function $\varphi : H_0^1(0, 1) \to (-\infty, +\infty]$ by

$$\varphi(v) = \begin{cases} \frac{1}{2} \int_0^1 v'^2 + \int_0^1 j(v) - \int_0^1 fv & \text{if } v \in H_0^1(0,1) \text{ and } \|v\|_{L^\infty} \leq 1, \\ +\infty & \text{otherwise.} \end{cases}$$

1. Check that φ is convex l.s.c. on $H_0^1(0,1)$ and that $\lim_{\|v\|_{H_0^1}\to+\infty} \varphi(v) = +\infty$.
2. Deduce that there exists a unique $u \in H_0^1(0,1)$ such that

$$\varphi(u) = \min_{v\in H^1(0,1)} \varphi(v).$$

The goal is to prove that if $f \in L^\infty(0,1)$ then $\|u\|_{L^\infty(0,1)} < 1$, $u \in H^2(0,1)$, and u satisfies

$$\begin{cases} -u'' + j'(u) = f & \text{on } (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{1}$$

3. Check that

$$j(t) - j(a) \geq j'(a)(t-a) \quad \forall t \in [-1,+1], \quad \forall a \in (-1,+1).$$

[**Hint:** Use the convexity of j.]

Fix $a \in [0,1)$.

4. Set $v = \min(u, a)$. Prove that $v \in H_0^1(0,1)$ and that

$$v' = \begin{cases} u' & \text{a.e. on } [u \leq a], \\ 0 & \text{a.e. on } [u > a]. \end{cases}$$

[**Hint:** Write $v = a - (a-u)^+$ and use Exercise 8.11.]

5. Prove that

$$\frac{1}{2}\int_{[u>a]} u'^2 \leq \int_{[u>a]} (f - j'(a))(u-a).$$

[**Hint:** Write that $\varphi(u) \leq \varphi(v)$, where v is defined in question 4. Then use question 3.]

6. Choose $a \in [0,1)$ such that $f(x) \leq j'(a)$ $\forall x \in [0,1]$ and prove that $u(x) \leq a$ $\forall x \in [0,1]$.

[**Hint:** Show that $\int_0^1 w'^2 = 0$, where $w = (u-a)^+$ belongs to $H_0^1(0,1)$; why?]

7. Conclude that $\|u\|_{L^\infty(0,1)} < 1$.

[**Hint:** Apply the previous argument, replacing u by $-u$, $j(t)$ by $j(-t)$, and f by $-f$.]

8. Deduce that u belongs to $H^2(0,1)$ and satisfies (1).

[**Hint:** Write that $\varphi(u) \leq \varphi(u + \varepsilon v)$ with $v \in H_0^1(0,1)$ and ε small.]

9. Check that $u \in C^2([0,1])$ if $f \in C([0,1])$.

10. Conversely, show that any function $u \in C^2([0,1])$ such that $\|u\|_{L^\infty(0,1)} < 1$, and satisfying (1), is a minimizer of φ on $H_0^1(0,1)$.

    [**Hint:** Use question 3 with $t = v(x)$ and $a = u(x)$.]

Assume now that $f \in L^2(0,1)$. Set $f_n = T_n f$, where T_n is the truncation operation (defined in Chapter 4 after Theorem 4.12). Let u_n be the solution of (1) corresponding to f_n.

11. Prove that $\|j'(u_n)\|_{L^2(0,1)} \le C$ as $n \to \infty$.

    [**Hint:** Multiply (1) by $j'(u_n)$.]

12. Deduce that $\|u_n\|_{H^2(0,1)} \le C$.

13. Show that a subsequence (u_{n_k}) converges weakly in $H^2(0,1)$ to a limit $u \in H^2(0,1)$ with $u_{n_k} \to u$ in $C^1([0,1])$. Prove that $|u(x)| < 1$ a.e. on $(0,1)$, and $j'(u) \in L^2(0,1)$.

    [**Hint:** Apply Fatou's lemma to the sequence by $j'(u_{n_k})^2$.]

14. Show that $j'(u_{n_k})$ converges weakly in $L^2(0,1)$ to $j'(u)$ and deduce that (1) holds.

    [**Hint:** Apply Exercise 4.16.]

15. Deduce that $\|u\|_{L^\infty(0,1)} < 1$ if one assumes, in addition, that

$$\liminf_{t \uparrow 1} j'(t)(1-t)^{1/3} > 0 \quad \text{and} \quad \limsup_{t \downarrow -1} j'(t)(1+t)^{1/3} < 0.$$

    [**Hint:** Assume, by contradiction, that $u(x_0) = 1$ for some $x_0 \in (0,1)$. Check that $|u'(x)| \le |x - x_0|^{1/2}\|u''\|_{L^2}$ $\forall x \in (0,1)$ and $|u(x) - 1| \le \frac{2}{3}|x - x_0|^{3/2}\|u''\|_{L^2}$ $\forall x \in (0,1)$. Deduce that $j'(u) \notin L^2(0,1)$.]

PROBLEM 49 (8)

Min–max principles for the eigenvalues of Sturm–Liouville operators

Consider the Sturm–Liouville operator $Au = -(pu')' + qu$ on $(0,1)$ with Dirichlet boundary condition $u(0) = u(1) = 0$. Assume that $p \in C^1([0,1])$, $p(x) \ge \alpha > 0$ $\forall x \in [0,1]$, and $q \in C([0,1])$. Set

$$a(u,v) = \int_0^1 (pu'v' + quv) \quad \forall u, v \in H_0^1(0,1).$$

Note that we make no further assumption on q, so that the bilinear form a need not be coercive. Fix M sufficiently large that $\tilde{a}(u,v) = a(u,v) + M\int_0^1 uv$ is coercive (e.g., $M > -\min_{x\in[0,1]} q(x)$). Let (λ_k) be the sequence of eigenvalues of A. The space $H = H_0^1(0,1)$ is equipped with the scalar product $\tilde{a}(u,v)$, now denoted by $(u,v)_H$, and the corresponding norm $|u|_H = \tilde{a}(u,u)^{1/2}$. Given any $f \in L^2(0,1)$, let $u \in H_0^1(0,1)$ be the unique solution of the problem

$$\tilde{a}(u, v) = \int_0^1 fv \quad \forall v \in H_0^1(0, 1).$$

Set $u = Tf$ and consider T as an operator from H into itself.

1. Show that T is self-adjoint and compact.

   [**Hint:** Recall that the identity map from H into $L^2(0, 1)$ is compact.]

2. Let (λ_k) be the sequence of eigenvalues of A (in the sense of Theorem 8.22) with corresponding eigenfunctions (e_k), and let (μ_k) be the sequence of eigenvalues of T. Check that $\mu_k > 0$ $\forall k$ and show that

$$\lambda_k = \frac{1}{\mu_k} - M \quad \forall k \quad \text{ and } \quad T(e_k) = \mu_k e_k \quad \forall k.$$

3. Prove that

$$(Tw, w)_H = \int_0^1 w^2 \quad \forall w \in H,$$

   and deduce that

$$\frac{1}{R(w)} = \frac{a(w, w)}{\int_0^1 w^2} + M \quad \forall w \in H, \ w \neq 0,$$

   where R is the Rayleigh quotient associated with T, i.e., $R(w) = \frac{(Tw,w)_H}{|w|_H^2}$ (see Problem 37).

4. Prove that

$$\lambda_1 = \min_{\substack{w \in H_0^1 \\ w \neq 0}} \left\{ \frac{a(w, w)}{\int_0^1 w^2} \right\}, \tag{1}$$

   and $\forall k \geq 2$,

$$\lambda_k =$$

$$\min \left\{ \frac{a(w, w)}{\int_0^1 w^2}; \, w \in H_0^1(0, 1), \, w \neq 0 \text{ and } \int_0^1 we_j = 0 \; \forall j = 1, 2, \dots, k-1 \right\}.$$

   [**Hint:** Apply question 2 in Problem 37 and show that $(w, e_j)_H = 0$ iff $\int_0^1 we_j = 0$.]

5. Prove that $\forall k \geq 1$,

$$\lambda_k = \min_{\substack{\Sigma \subset H_0^1(0,1) \\ \dim \Sigma = k}} \max_{\substack{u \in \Sigma \\ u \neq 0}} \left\{ \frac{a(u, u)}{\int_0^1 u^2} \right\},$$

   and

$$\lambda_k = \max_{\substack{\Lambda \subset H_0^1(0,1) \\ \text{codim}\, \Lambda = k-1}} \min_{\substack{u \in \Lambda \\ u \neq 0}} \left\{ \frac{a(u,u)}{\int_0^1 u^2} \right\},$$

where Σ and Λ are closed subspaces of $H_0^1(0,1)$.

[**Hint:** Apply question 7 in Problem 37.]

6. Prove similar results for the Sturm–Liouville operator with Neumann boundary conditions.

We now return to formula (1) and discuss further properties of the eigenfunctions corresponding to the first eigenvalue λ_1. In particular, we will see that there is a positive eigenfunction generating the eigenspace associated to λ_1.

7. Let $w_0 \in H_0^1(0,1)$ be a minimizer of (1) such that $\int_0^1 w_0^2 = 1$. Show that

$$Aw_0 = \lambda_1 w_0 \quad \text{on } (0,1).$$

8. Set $w_1 = |w_0|$. Check that w_1 is also a minimizer of (1) and deduce that

$$Aw_1 = \lambda_1 w_1 \quad \text{on } (0,1). \tag{2}$$

[**Hint:** Use Exercise 8.11.]

9. Prove that $w_1 > 0$ on $(0,1)$, $w_1'(0) > 0$, and $w_1'(1) < 0$.

[**Hint:** Apply the strong maximum principle to the operator $A + M$; see Problem 45.]

10. Assume that $w \in H_0^1(0,1)$ satisfies

$$Aw = \lambda_1 w \quad \text{on } (0,1).$$

Prove that w is a multiple of w_1.

[**Hint:** Recall that eigenvalues are simple; see Exercise 8.33. Find another proof that does not rely on the simplicity of eigenvalues; use w^2/w_1 as test function in (2).]

11. Show that any function $\psi \in H_0^1(0,1)$ satisfying

$$A\psi = \mu\psi \ \text{ on } (0,1), \quad \psi \geq 0 \ \text{ on } (0,1), \quad \text{and} \quad \int_0^1 \psi^2 = 1,$$

for some $\mu \in \mathbb{R}$, must coincide with w_1.

[**Hint:** If $\mu \neq \lambda_1$, check that $\int_0^1 \psi w_1 = 0$. Deduce that $\mu = \lambda_1$.]

PROBLEM 50 (8)

Another nonlinear problem

Let $q \in C([0,1])$ and consider the bilinear form

$$a(u, v) = \int_0^1 (u'v' + quv), \quad u, v \in H_0^1(0, 1).$$

Assume that there exists $v_1 \in H_0^1(0, 1)$ such that

$$a(v_1, v_1) < 0. \tag{1}$$

1. Check that assumption (1) is equivalent to

   $$\lambda_1(A) < 0, \tag{2}$$

   where $\lambda_1(A)$ is the first eigenvalue of the operator $Au = -u'' + qu$ with zero Dirichlet condition.

2. Verify that

   $$-\infty < m = \inf_{u \in H_0^1(0,1)} \left\{ \frac{1}{2} a(u, u) + \frac{1}{4} \int_0^1 |u|^4 \right\} < 0. \tag{3}$$

   [**Hint:** Use $u = \varepsilon v_1$ with $\varepsilon > 0$ sufficiently small.]

3. Prove that the inf in (3) is achieved by some u_0.

   [*Warning:* The functional in (3) is *not* convex; why?]

   Our goal is to prove that (3) admits precisely two minimizers.

4. Prove that u_0 belongs to $C^2([0, 1])$ and satisfies

   $$\begin{cases} -u'' + qu + u^3 = 0 & \text{on } (0, 1), \\ u(0) = u(1) = 0. \end{cases} \tag{4}$$

5. Set $u_1 = |u_0|$. Show that u_1 is also a minimizer for (3). Deduce that u_1 satisfies (4).

   [**Hint:** Apply Exercise 8.11.]

6. Prove that $u_1(x) > 0 \ \forall x \in (0, 1)$, $u_1'(0) > 0$, and $u_1'(1) < 0$.

   [**Hint:** Choose a constant a so large that $-u_1'' + a^2 u_1 = f \geq 0$, $f \not\equiv 0$. Then use the strong maximum principle.]

7. Let $u_0 \in H_0^1(0, 1)$ be again any minimizer in (3). Prove that either $u_0(x) > 0$ $\forall x \in (0, 1)$, or $u_0(x) < 0 \ \forall x \in (0, 1)$.

   [**Hint:** Check that $|u_0(x)| > 0 \ \forall x \in (0, 1)$.]

8. Let U_1 be any solution of (4) satisfying $U_1 \geq 0$ on $[0, 1]$, and $U_1 \not\equiv 0$. Set $\rho_1 = U_1^2$. Consider the functional

$$\Phi(\rho) = \int_0^1 \left(|(\sqrt{\rho})'|^2 + q\rho + \frac{1}{2}\rho^2 \right)$$

defined on the set

$$K = \left\{ \rho \in H_0^1(0, 1);\ \rho \geq 0 \text{ on } (0, 1) \text{ and } \sqrt{\rho} \in H_0^1(0, 1) \right\}.$$

Prove that

$$\text{(5)} \qquad \Phi(\rho) - \Phi(\rho_1) \geq \frac{1}{2} \int_0^1 (\rho - \rho_1)^2 \quad \forall \rho \in K.$$

[**Hint:** Let $u \in C_c^1((0, 1))$. Note that

$$2\frac{U_1' u u'}{U_1} \leq u'^2 + \frac{U_1'^2 u^2}{U_1^2} \quad \text{on } (0, 1),$$

and deduce (using integration by parts) that

$$\int_0^1 (u'^2 - U_1'^2) \geq -\int_0^1 \frac{U_1''}{U_1}(u^2 - U_1^2) \quad \forall u \in H_0^1(0, 1).$$

Then apply equation (4) to establish (5).]

9. Deduce that there exists exactly one nontrivial solution u of (4) such that $u \geq 0$ on $[0, 1]$. Denote it by U_0.

[*Comment:* There exist in general many sign-changing solutions of (4).]

10. Prove that there exist exactly two minimizers for (3): U_0 and $-U_0$.

PROBLEM 51 (8)

Harmonic oscillator. Hermite polynomials.

Let $p \in C(\mathbb{R})$ be such that $p \geq 0$ on $\mathbb{R}$. Consider the space

$$V = \left\{ v \in H^1(\mathbb{R});\ \int_{-\infty}^{+\infty} pv^2 < \infty \right\}$$

equipped with the scalar product

$$(u, v)_V = \int_{-\infty}^{+\infty} (u'v' + uv + puv),$$

and the corresponding norm $|u|_V = (u, u)_V^{1/2}$.

1. Check that V is a separable Hilbert space.
2. Show that $C_c^\infty(\mathbb{R})$ is dense in V.

[**Hint:** Let ζ_n be a sequence of cut-off functions as in the proof of Theorem 8.7. Given $u \in V$, consider $\zeta_n u$ and then use convolution.]

Consider the bilinear form

$$a(u, v) = \int_{-\infty}^{+\infty} u'v' + puv, \quad u, v \in V.$$

In what follows we assume that there exist constants $\delta > 0$ and $A > 0$ such that

$$p(x) \geq \delta \quad \forall x \in \mathbb{R} \text{ with } |x| \geq A. \tag{1}$$

3. Prove that a is coercive on V. Deduce that for every $f \in L^2(\mathbb{R})$ there exists a unique solution $u \in V$ of the problem

$$a(u, v) = \int_{-\infty}^{+\infty} fv \quad \forall v \in V. \tag{2}$$

4. Assuming that $f \in L^2(\mathbb{R}) \cap C(\mathbb{R})$, show that u satisfies

$$\begin{cases} u \in C^2(\mathbb{R}), \\ -u'' + pu = f \quad \text{on } \mathbb{R}, \\ u(x) \to 0 \quad \text{as } |x| \to \infty. \end{cases} \tag{3}$$

5. Conversely, prove that any solution u of (3) belongs to V and satisfies (2).

   [**Hint:** Multiply the equation $-u'' + pu = f$ by $\zeta_n^2 u$ and use the fact that a is coercive.]

In what follows we assume that

$$\lim_{|x| \to \infty} p(x) = +\infty. \tag{4}$$

6. Given $f \in L^2(\mathbb{R})$, set $u = Tf$, where u is the solution of (2). Prove that $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is self-adjoint and compact.

   [**Hint:** Using Corollary 4.27 check that $V \subset L^2(\mathbb{R})$ with compact injection.]

7. Deduce that there exist a sequence (λ_n) of positive numbers with $\lambda_n \to \infty$ as $n \to \infty$, and a Hilbert basis (e_n) of $L^2(\mathbb{R})$ satisfying

$$\begin{cases} e_n \in V \cap C^2(\mathbb{R}), \\ -e_n'' + pe_n = \lambda_n e_n \quad \text{on } \mathbb{R}. \end{cases} \tag{5}$$

In what follows we take $p(x) = x^2$.

8. Check that (5) admits a solution of the form $e_n(x) = e^{-x^2/2} P_n(x)$, where $\lambda_n = (2n + 1)$ and $P_n(x)$ is a polynomial of degree n.

Partial Solutions of the Problems

Problem 1

5. In view of Zorn's lemma (Lemma 1.1) it suffices to check that $\mathcal{F}$ is inductive. Let $(A_i)_{i\in I}$ be a totally ordered subset of $\mathcal{F}$. Set $A = \bigcap_{i\in I} A_i$ and check that A is nonempty, A is an extreme set of K, $A \in \mathcal{F}$, and A is an upper bound for $(A_i)_{i\in I}$.
6. Suppose not, that there are two distinct points $a, b \in M_0$. By Hahn–Banach (Theorem 1.7) there exists some $f \in E^\star$ such that $\langle f, a\rangle \neq \langle f, b\rangle$. Set

$$M_1 = \left\{x \in M_0;\ \langle f, x\rangle = \max_{y\in M_0} \langle f, y\rangle\right\}.$$

Clearly $M_1 \in \mathcal{F}$ and $M_0 \leq M_1$. Since M_0 is maximal, it follows that $M_1 = M_0$. This is absurd, since the points a and b cannot both belong to M_1.
8. Let K_1 be the closed convex hull of all the extreme points of K. Assume, by contradiction, that there exists some point $a \in K$ such that $a \notin K_1$. Then there exists some hyperplane strictly separating $\{a\}$ and K_1. Let $f \in E^\star$ be such that

$$\langle f, x\rangle < \langle f, a\rangle \quad \forall x \in K_1.$$

Note that

$$B = \left\{x \in K;\ \langle f, x\rangle = \max_{y\in K} \langle f, y\rangle\right\}$$

is an extreme set of K such that $B \cap K_1 = \emptyset$. But B contains at least one extreme point of K; absurd.
9. (a) $\mathcal{E} = \{x = (x_i);\ |x_i| = 1\ \forall i\}$,
   (b) $\mathcal{E} = \{x = (x_i);\ |x_i| = 1\ \forall i$, and x_i is stationary for large $i\}$,
   (c) $\mathcal{E} = \emptyset$,
   (d) $\mathcal{E} = \{x = (x_i);\ \exists j$ such that $|x_j| = 1$, and $x_i = 0\ \forall i \neq j\}$,
   (e) $\mathcal{E} = \{x = (x_i);\ \sum |x_i|^p = 1\}$,
   (f) $\mathcal{E} = \emptyset$.

   To see that $\mathcal{E} = \emptyset$ in the case (f) let $f \in L^1(\mathbb{R})$ be any function such that $\int_{\mathbb{R}} |f| = 1$. By a translation we may always assume that $\int_{-\infty}^0 |f| = \int_0^\infty |f| = 1/2$. Then

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write $f = (g+h)/2$ with

$$g = \begin{cases} 2f & \text{on } (-\infty, 0), \\ 0 & \text{on } (0, +\infty), \end{cases} \quad \text{and} \quad h = \begin{cases} 0 & \text{on } (-\infty, 0), \\ 2f & \text{on } (0, +\infty). \end{cases}$$

Problem 2

Determine $\partial\varphi(x)$ for the function φ defined by $\varphi(x) = -\sqrt{x}$ for $x \geq 0$ and $\varphi(x) = +\infty$ for $x < 0$.

-A-

4. (a) $\partial\varphi(x) = F(x)$,
   (b) $\partial\varphi(x) = \frac{1}{\|x\|}F(x)$ if $x \neq 0$ and $\partial\varphi(0) = B_{E^\star}$,
   (c) $\partial\varphi(x) = \begin{cases} 0 & \text{if } x \in \operatorname{Int} K, \\ \text{outward normal cone at } x & \text{if } x \in \text{ Boundary of } K, \end{cases}$
   $\partial\varphi(x) = K^\perp$ if K is a linear subspace,
   (d) $\partial\varphi(x) = D\varphi(x) =$ differential of φ at x.
5. Study the following example: In $E = \mathbb{R}^2$ (equipped with the Euclidean norm),

$$\varphi = I_C \quad \text{with } C = \{[x_1, x_2];\, (x_1 - 1)^2 + x_2^2 \leq 1\},$$

   and

$$\psi = I_D \quad \text{with } D = \{[x_1, x_2];\, x_1 = 0\}.$$

- B -

1. Let $C = \operatorname{epi}\varphi$. Apply Hahn–Banach (first geometric form) with $A = \operatorname{Int} C$ and $B = [x_0, \varphi(x_0)]$. Note that $A \neq \emptyset$ (why?). Hence there exist some $f \in E^\star$ and some constants k and a such that $\|f\| + |k| \neq 0$ and

$$\langle f, x\rangle + k\lambda \geq a \geq \langle f, x_0\rangle + k\varphi(x_0) \quad \forall x \in D(\varphi), \forall \lambda \geq \varphi(x).$$

   Check that $k > 0$ and deduce that $-\frac{1}{k}f \in \partial\varphi(x_0)$.
6. Note that $\inf_E(\tilde\varphi + \tilde\psi) = 0$, and so there exists some $g \in E^\star$ such that $\tilde\varphi^\star(-g) + \tilde\psi^\star(g) = 0$. Check that $f_0 - g \in \partial\varphi(x)$, and that $g \in \partial\psi(x)$; thus $f_0 \in \partial\varphi(x) + \partial\psi(x)$.

-C-

1. For every $R > 0$ and every $x_0 \in E$ we have

$$\varphi(x) \leq k(\|x_0\| + R) + C \equiv M(R) \quad \forall x \in E \text{ with } \|x - x_0\| \leq R.$$

   Thus

$$\|f\| \leq \frac{1}{R}(k\|x_0\| + kR + C - \varphi(x_0)) \quad \forall f \in \partial\varphi(x_0).$$

Letting $R \to \infty$ we see that $\|f\| \le k \; \forall f \in \partial\varphi(x_0)$ and consequently

$$\varphi(x) - \varphi(x_0) \ge -k\|x - x_0\| \quad \forall x, x_0 \in E.$$

We have $D(\varphi^\star) \subset kB_{E^\star}$. Indeed, if $f \in D(\varphi^\star)$, write

$$\langle f, x\rangle \le \varphi(x) + \varphi^\star(f) \le k\|x\| + C + \varphi^\star(f).$$

Choosing $\|x\| = R$, we obtain

$$R\|f\| \le kR + C + \varphi^\star(f) \quad \forall R > 0$$

and the conclusion follows by letting $R \to \infty$.

2. Check, with the help of a basis of $\mathbb{R}^n$, that every point $x_0 \in A$ satisfies assumption (1).

-D-

The main difficulty is to show that if $f \in \partial I_C(x)$ with $\varphi(x) = 0$ and $f \neq 0$, then there exists some $\lambda > 0$ such that $f \in \lambda\partial\varphi(x)$. Apply Hahn–Banach (first geometric form) in $E \times \mathbb{R}$ to the convex sets $A = \operatorname{Int}(\operatorname{epi}\varphi)$ and $B = \{[y, 0] \in E \times \mathbb{R};\ \langle f, y - x\rangle \ge 0\}$ (check that $A \cap B = \emptyset$). Thus, there exist some $g \in E^\star$ and some constant k such that $\|g\| + |k| \neq 0$ and

$$\langle g, y\rangle + k\mu \ge \langle g, z\rangle \quad \forall [y, \mu] \in \operatorname{epi}\varphi, \quad \forall [z, 0] \in B.$$

It follows, in particular, that $k \ge 0$ and that

$$\langle g, y\rangle + k\varphi(y) \ge \langle g, x\rangle \quad \forall y \in E.$$

In fact, $k \neq 0$ (since $k = 0$ would imply $g = 0$). Thus $-\frac{g}{k} \in \partial\varphi(x)$ (since $\varphi(x) = 0$). Moreover, $g \neq 0$ (why?). Finally, we have $\langle g, x\rangle \ge \langle g, z\rangle \; \forall [z, 0] \in B$ and consequently $\langle g, u\rangle \le 0 \; \forall u \in E$ such that $\langle f, u\rangle \ge 0$. It follows that $g = 0$ on the set $f^{-1}(\{0\})$. We conclude that there is a constant $\theta < 0$ such that $g = \theta f$ (see Lemma 3.2).

Problem 3

-A-

3. *Either* $x \in S(x_n) \; \forall n$ and then we have $\psi(x_{n+1}) \le \psi(x) + \varepsilon_{n+1} \; \forall n$. Passing to the limit one obtains $\psi(a) \le \psi(x)$ and a fortiori $\psi(x) - \psi(a) + d(x, a) \ge 0$. *Or* $\exists N$ such that $x \notin S(x_N)$ and then $x \notin S(x_n) \; \forall n \ge N$. It follows that

$$\psi(x) - \psi(x_n) + d(x, x_n) > 0 \quad \forall n \ge N.$$

Passing to the limit also yields $\psi(x) - \psi(a) + d(x, a) \ge 0$.

-B-

1. The set M equipped with the distance $d(x, y) = \lambda\|x - y\|$ is complete (since ψ is l.s.c.) and nonempty ($x_0 \in M$). Note that $\psi \geq 0$. By the result of part A there exists some $x_1 \in M$ such that
$$\psi(x) - \psi(x_1) + \lambda\|x - x_1\| \geq 0 \quad \forall x \in M.$$
If $x \notin M$ we have $\psi(x) > \psi(x_0) - \lambda\|x_0 - x\|$ (by definition of M), while $\psi(x_0) - \lambda\|x_0 - x\| \geq \psi(x_1) - \lambda\|x - x_1\|$ (since $x_1 \in M$).
Combining the two cases, we see that
$$\psi(x) - \psi(x_1) + \lambda\|x - x_1\| \geq 0 \quad \forall x \in E.$$
On the other hand, since $x_1 \in M$, we have $\psi(x_1) \leq \psi(x_0) - \lambda\|x_0 - x_1\|$. But $\psi(x_0) \leq \varepsilon$ and $\psi(x_1) \geq 0$. Consequently $\|x_0 - x_1\| \leq \varepsilon/\lambda$.
2. Consider the functions $\omega(x) = \psi(x) - \psi(x_1)$ and $\theta(x) = \lambda\|x - x_1\|$, so that $0 \in \partial(\omega + \theta)(x_1)$. We know that $\partial(\omega + \theta) = \partial\omega + \partial\theta$ and that $\partial\theta(x_1) = \lambda B_{E^\star}$. It follows that $0 \in \partial\varphi(x_1) - f + \lambda B_{E^\star}$.
3. Let us check that $D(\varphi) \subset \overline{D(\partial\varphi)}$. Given any $x_0 \in D(\varphi)$, we know, from the previous questions, that $\forall\varepsilon > 0, \forall\lambda > 0, \exists x_1 \in D(\partial\varphi)$ such that $\|x_1 - x_0\| < \varepsilon/\lambda$. Clearly $R(\partial\varphi) \subset D(\varphi^\star)$. Conversely, let us check that $D(\varphi^\star) \subset \overline{R(\partial\varphi)}$. Given any $f_0 \in D(\varphi^\star)$ we know that $\forall\varepsilon > 0, \exists x_0 \in D(\varphi)$ such that $f_0 \in \partial_\varepsilon\varphi(x_0)$, and thus $\forall\lambda > 0, \exists f_1 \in R(\partial\varphi)$ such that $\|f_1 - f_0\| < \lambda$.

-C-

1. Let $f_0 \in E^\star$. Since $(I_C)^\star(f_0) < \infty$, we know that $\forall\varepsilon > 0$, $\exists x_0 \in C$ such that $f_0 \in \partial_\varepsilon I_C(x_0)$. It follows that $\forall\lambda > 0$, $\exists x_1 \in C, \exists f_1 \in \partial I_C(x_1)$ with $\|f_1 - f_0\| \leq \lambda$. Clearly we have $\sup_{x\in C}\langle f_1, x\rangle = \langle f_1, x_1\rangle$.
2. Let x_0 be a boundary point of C. Then $\forall\varepsilon > 0$, $\exists a \in E, a \notin C$, such that $\|a - x_0\| < \varepsilon$. Separating C and $\{a\}$ by a closed hyperplane we obtain some $f_0 \in E^\star$ such that $f_0 \neq 0$ and $\langle f_0, x - a\rangle \leq 0$ $\forall x \in C$. Of course, we may assume that $\|f_0\| = 1$. Thus, we have $\langle f_0, x - x_0\rangle \leq \varepsilon$ $\forall x \in C$ and consequently $f_0 \in \partial_\varepsilon I_C(x_0)$. Applying the result of part B with $\lambda = \sqrt{\varepsilon}$ we find some $x_1 \in C$ and some $f_1 \in \partial I_C(x_1)$ such that $\|x_1 - x_0\| \leq \sqrt{\varepsilon}$ and $\|f_1 - f_0\| \leq \sqrt{\varepsilon}$. Since $f_1 \neq 0$ (provided $\varepsilon < 1$), we see that there exists a supporting hyperplane to C at x_1.

Problem 4

2. Argue by induction and apply question 7 of Exercise 1.23.
3. Note that $x = \frac{1}{2}[(x + y) + (x - y)]$, and so by convexity,
$$\psi_n(x) \leq \left[\frac{1}{2}\psi_n(x + y) + \psi_n(x - y)\right] \leq \frac{1}{2}\left[\varphi_n(x + y) + \psi_n(x - y)\right] \quad \forall x, y.$$

Thus $\psi_n(x) \le \psi_{n+1}(x)$. We have $\varphi_n \downarrow \theta$, $\psi_n \uparrow \tilde{\theta}$ and $\varphi_{n+1} = \frac{1}{2}(\varphi_n + \psi_n)$. Therefore $\theta = \tilde{\theta}$.

4. The sequence $(\varphi_n^\star)$ is nondecreasing and converges to a limit, denoted by ω. Since $\theta \le \varphi_n$, it follows that $\varphi_n^\star \le \theta^\star$ and $\omega \le \theta^\star$. On the other hand, we have $\langle f, x\rangle - \varphi_n(x) \le \varphi_n^\star(f)\ \forall x \in E, \forall f \in E^\star$. Thus $\langle f, x\rangle - \theta(x) \le \omega(f)\ \forall x \in E$, $\forall f \in E^\star$, that is, $\theta^\star \le \omega$. We conclude that $\omega = \theta^\star$.

5. Applying question 1 of Exercise 1.23, we see that $\psi_{n+1}^\star = \frac{1}{2}(\varphi_n^\star + \psi_n^\star)$. The sequence $(\psi_n^\star)$ is nonincreasing and thus it converges to a limit ζ such that $\zeta = \frac{1}{2}(\theta^\star + \zeta)$. It follows that $\zeta = \theta^\star$ (since $\zeta < \infty$).

-B-

From the convexity and the homogeneity of φ we obtain

$$\begin{aligned}\varphi(x+y) &= \varphi\left(t\frac{x}{t} + (1-t)\frac{y}{1-t}\right)\\ &\le t\varphi\left(\frac{x}{t}\right) + (1-t)\varphi\left(\frac{y}{1-t}\right) = \frac{1}{t}\varphi(x) + \frac{1}{1-t}\varphi(y).\end{aligned}$$

In order to establish (1) choose $x = \frac{1}{2}(X+Y)$ and $y = \frac{1}{2}(X-Y)$.

2. Using (1) we find that $\forall x, y \in E$, $\forall t \in (0,1)$,

$$\begin{aligned}\varphi_{n+1}(x) &= \frac{1}{2}\{\varphi_n(x) + \psi_n(x)\}\\ &\le \frac{1}{2}\left\{\frac{1}{4t}\varphi_n(x+y) + \frac{1}{4(1-t)}\varphi_n(x-y)\right.\\ &\qquad \left. + \frac{1}{4t}\psi_n(x+y) + \frac{1}{4(1-t)}\psi_n(x-y)\right\}.\end{aligned}$$

Applying A1 and the induction assumption we have $\forall x, y \in E$ and $\forall t \in (0,1)$,

$$\varphi_{n+1}(x) \le \frac{1}{2}\left\{\frac{2}{4t}\varphi_{n+1}(x+y) + \frac{1}{4(1-t)}\left(2 + \frac{C}{4^n}\right)\psi_n(x-y)\right\}.$$

Choosing t such that $\frac{2}{4t} = \frac{1}{4(1-t)}(2+\frac{C}{4^n})$, that is, $t = 1/2(1+\frac{C}{4^{n+1}})$, we conclude that

$$\varphi_{n+1}(x) \le \frac{1}{2}\left(1 + \frac{C}{4^{n+1}}\right)\{\varphi_n(x+y) + \psi_n(x-y)\} \quad \forall x, y \in E.$$

It follows that $\varphi_{n+1}(x) \le \frac{1}{2}(1 + \frac{C}{4^{n+1}})\psi_{n+1}(x)\ \forall x \in E$.

3. With $x \ne y$ and $t \in (0,1)$ write

$$\begin{aligned}
\theta(tx+(1-t)y) &\le \theta_n(tx+(1-t)y) + \frac{1}{2^n}\varphi_0(tx+(1-t)y) \\
&\le t\theta_n(x) + (1-t)\theta_n(y) + \frac{1}{2^n}\varphi_0(tx+(1-t)y) \\
&\le t\theta(x) + (1-t)\theta(y) \\
&\qquad + \frac{1}{2^n}\Bigg[\varphi_0(tx+(1-t)y) - t\varphi_0(x) - (1-t)\varphi_0(y) \\
&\qquad + \frac{C}{2^n}(t\varphi_0(x) + (1-t)\varphi_0(y))\Bigg] \\
&< t\theta(x) + (1-t)\theta(y),
\end{aligned}$$

for n large enough, since φ_0 is strictly convex.

-C-

Take $\varphi_0(x) = \frac{1}{2}\|x\|_1^2$ and $\psi_0(x) = \frac{1}{2}\alpha^2\|x\|_2^2$, with $\alpha > 0$ sufficiently small. The norm $\| \;\|$ is defined through the relation $\theta(x) = \frac{1}{2}\|x\|^2$.

Problem 5

-B-

1. It suffices to prove that there is a constant $c > 0$ such that $B(0, c) \subset K$. By (iii) we have $\bigcup_{n=1}^{\infty}(nK) = E$ and thus $\bigcup_{n=1}^{\infty}(n\overline{K}) = E$. Applying Baire's theorem, one sees that $\operatorname{Int}(\overline{K}) \neq \emptyset$, and hence there exist some $y_0 \in E$ and a constant $c > 0$ such that $B(y_0, 4c) \subset \overline{K}$. Since K is convex and symmetric it follows that $B(0, 2c) \subset \overline{K}$.

   We claim that $B(0, c) \subset K$. Fix $x \in E$ with $\|x\| < c$. There exist $y_1, z_1 \in P$ such that $\|y_1\| \le 1/2$, $\|z_1\| \le 1/2$ and $\|x - (y_1 - z_1)\| < c/2$. Next, there exist $y_2, z_2 \in P$ such that $\|y_2\| \le 1/4$, $\|z_2\| \le 1/4$, and

   $$\|x - (y_1 - z_1) - (y_2 - z_2)\| < c/4.$$

   Iterating this construction, one obtains sequences (y_n) and (z_n) in P such that $\|y_n\| \le 1/2^n$, $\|z_n\| \le 1/2^n$, and

   $$\left\| x - \sum_{i=1}^{n}(y_i - z_i) \right\| < c/2^n.$$

   Then write $x = y - z$ with $y = \sum_{i=1}^{\infty} y_i$ and $z = \sum_{i=1}^{\infty} z_1$, so that $x \in K$.
2. Write $x_n = y_n - z_n$ with $y_n, z_n \in P$, $\|y_n\| \le C/2^n$, and $\|z_n\| \le C/2^n$. Then $1 \le f(x_n) \le f(y_n)$. Set $u_n = \sum_{i=1}^{n} y_i$ and $u = \sum_{i=1}^{\infty} y_i$. On the one hand, $f(u_n) \ge n$, and on the other hand, $f(u - u_n) \ge 0$. It follows that $f(u) \ge n \;\forall n$; absurd.
3. Consider a complement of F (see Section 2.4).

-C-

(a) One has $F = P - P = E$; one can also check (i) directly: if $f \geq 0$ on P, then $|f(u)| \leq \|u\|_\infty f(1)\ \forall u \in E$.

(b) Here $F = \{u \in E;\ u(0) = u(1) = 0\}$ is a closed subspace of finite codimension.

(c) One has $F = E$. Indeed, if $u \in E$ there is a constant $c > 0$ such that $|u(t)| \leq ct(1-t)\ \forall t \in [0, 1]$ and one can write $u = v - w$ with $w = ct(1-t)$ and $v = u + ct(1-t)$.

Problem 7

-A-

2. Fix $x \in M$ with $\|x\| \leq 1$. Let $\varepsilon > 0$. Since $\operatorname{dist}(x, N) \leq a$, there exists some $y \in N$ such that $\|x - y\| \leq a + \varepsilon$, and thus $\|y\| \leq 1 + a + \varepsilon$. On the other hand, $\operatorname{dist}(\frac{y}{\|y\|+\varepsilon}, L) \leq b$ and so $\operatorname{dist}(y, L) \leq b(\|y\| + \varepsilon) \leq b(1 + a + 2\varepsilon)$. It follows that $\operatorname{dist}(x, L) \leq a + \varepsilon + b(1 + a + 2\varepsilon)\ \forall \varepsilon > 0$.

-B-

In order to construct an example such that $A^\star + B^\star \neq (A + B)^\star$ it suffices to consider any unbounded operator $A : D(A) \subset E \to F$ that is densely defined, closed, and such that $D(A) \neq E$. Then take $B = -A$. We have $(A + B)^\star = 0$ with $D((A + B)^\star) = F^\star$, while $A^\star + B^\star = 0$ with $D(A^\star + B^\star) = D(A^\star)$. [Note that $D(A^\star) \neq F^\star$; why?].

3. $A + B$ is closed; indeed, let (u_n) be a sequence in E such that $u_n \to u$ in E and $(A + B)u_n \to f$ in F. Note that

$$\|Bu\| \leq k\|Au + Bu\| + k\|Bu\| + C\|u\| \quad \forall u \in D(A)$$

and thus

$$\|Bu\| \leq \frac{k}{1-k}\|Au + Bu\| + \frac{C}{1-k}\|u\| \quad \forall u \in D(A).$$

It follows that (Bu_n) is a Cauchy sequence. Let $Bu_n \to g$, and so $u \in D(B)$ with $Bu = g$. On the other hand, $Au_n \to f - Bu$, and so $u \in D(A)$ with $Au + Bu = f$. Clearly one has

$$\begin{aligned}\rho(A, A + B) &= \sup_{\substack{u \in D(A) \\ \|u\|+\|Au\| \leq 1}} \inf_{v \in D(A)} \{\|u - v\| + \|Au - (Av + Bv)\|\} \\ &\leq \sup_{\substack{u \in D(A) \\ \|u\|+\|Au\| \leq 1}} \|Bu\| \leq k + C.\end{aligned}$$

4. The same argument shows that under assumption (H*), one has

$$\rho(A^\star, A^\star + B^\star) \leq k^\star + C^\star.$$

[There are some minor changes, since the dual norm on $E^\star \times F^\star$ is given by $\|[f, g]\|_{E^\star\times F^\star} = \max\{\|f\|_{E^\star}, \|g\|_{E^\star}\}$.]

5. Let $t \in [0, 1]$. For every $u \in D(A)$ one has

$$\|Bu\| \leq k\|Au\| + C\|u\| \leq k(\|Au + tBu\| + t\|Bu\|) + C\|u\|,$$

and thus

$$\|Bu\| \leq \frac{k}{1-k}\|Au + tBu\| + \frac{C}{1-k}\|u\|.$$

Fix any $\varepsilon > 0$ such that $1/\varepsilon = n$ is an integer, $\frac{\varepsilon(k+C)}{1-k} \leq \frac{1}{3}$, and $\frac{\varepsilon(k^\star+C^\star)}{1-k^\star} \leq \frac{1}{3}$. Set $A_1 = A + \varepsilon B$, so that $A_1^\star = A^\star + \varepsilon B^\star$ and, moreover,

$$\|Bu\| \leq \frac{k}{1-k}\|A_1 u\| + \frac{C}{1-k}\|u\| \quad \forall u \in D(A),$$

and also

$$\|B^\star v\| \leq \frac{k^\star}{1-k^\star}\|A_1^\star v\| + \frac{C^\star}{1-k^\star}\|v\| \quad \forall v \in D(A^\star).$$

It follows that $(A_1 + \varepsilon B)^\star = A_1^\star + \varepsilon B^\star$, i.e., $(A + 2\varepsilon B)^\star = A^\star + 2\varepsilon B^\star$, and so on, step by step with $A_j = A + j\varepsilon B$ and $j \leq n - 1$.

Problem 8

1. Let $\mathcal{T}$ be the topology corresponding to the metric d. Since $B_{E^\star}$ equipped with the topology $\sigma(E, E^\star)$ is compact, it suffices to check that the canonical injection $(B_{E^\star}, \sigma(E^\star, E)) \to (B_{E^\star}, \mathcal{T})$ is continuous. This amounts to proving that for every $f_0 \in B_{E^\star}$ and for every $\varepsilon > 0$ there exists a neighborhood $V(f^0)$ of f^0 for $\sigma(E^\star, E)$ such that

$$V(f^0) \cap B_{E^\star} \subset \{f \in B_{E^\star}; d(f, f^0) < \varepsilon\}.$$

Let (e^i) be the canonical basis of ℓ^1. Choose

$$V(f^0) = \{f \in E^\star; |\langle f - f^0, e^i\rangle| < \delta\ \forall i = 1, 2, \ldots, n\}$$

with $\delta + (1/2^{n-1}) < \varepsilon$.

2. Note that $(B_{E^\star}, d)$ is a complete metric space (since it is compact). The sets F_k are closed for the topology $\mathcal{T}$, and, moreover, $\bigcup_{k=1}^\infty F_k = B_{E^\star}$ (since $\langle f, x^n\rangle \to 0$ for every $f \in E^\star$). Baire's theorem says that there exists some integer k_0 such that $\operatorname{Int}(F_{k_0}) \neq \emptyset$.
3. Write $f^0 = (f_1^0, f_2^0, \ldots, f_i^0, \ldots)$ and consider the elements $f \in B_{E^\star}$ of the form

$$f = (f_1^0, f_2^0, \ldots, f_N^0, \pm 1, \pm 1, \pm 1, \ldots),$$

so that

$$d(f, f^0) \leq \sum_{i=N+1}^{\infty} \frac{2}{2^i} < \rho.$$

Such f's belong to F_{k_0} and one has, for every $n \geq k_0$,

$$\left|\langle f, x^n \rangle\right| = \left|\sum_{i=1}^{\infty} f_i x_i^n\right| = \left|\sum_{i=1}^{N} f_i^0 x_i^n + \sum_{i=N+1}^{\infty} (\pm x_i^n)\right| \leq \varepsilon.$$

It follows that

$$\sum_{i=N+1}^{\infty} |x_i^n| \leq \varepsilon + \sum_{i=1}^{N} |f_i^0|\, |x_i^n| \leq \varepsilon + \sum_{i=1}^{N} |x_i^n|,$$

and thus

$$\sum_{i=1}^{\infty} |x_i^n| \leq \varepsilon + 2\sum_{i=1}^{N} |x_i^n| \quad \forall n \geq k_0.$$

4. The conclusion is clear, since for each fixed i the sequence x_i^n tends to 0 as $n \to \infty$.
5. Given $\varepsilon > 0$, set

$$F_k = \{f \in B_{E^\star};\ |\langle f, x^n - x^m \rangle| \leq \varepsilon\ \forall m, n \geq k\}.$$

By the same method as above one finds integers k_0 and N such that

$$\|x^n - x^m\|_1 \leq \varepsilon + 2\sum_{i=1}^{N} |x_i^n - x_i^m| \quad \forall m, n \geq k_0.$$

It follows that (x^n) is a Cauchy sequence in ℓ^1.
6. See Exercises 4.18 and 4.19.

Problem 9

-A-

1. A is open for the strong topology (since it is open for the topology $\sigma(E^\star, E)$). Thus (by Hahn–Banach applied in $E^\star$) there exist some $\xi \in E^{\star\star}$, $\xi \neq 0$, and a constant α such that

$$\langle \xi, f \rangle \leq a \leq \langle \xi, g \rangle \quad \forall f \in A, \quad \forall g \in B.$$

Fix $f_0 \in A$ and a neighborhood V of 0 for the topology $\sigma(E^\star, E)$ such that $f_0 + V \subset A$. We may always assume that V is symmetric; otherwise, consider $V \cap (-V)$. We have $\langle \xi, f_0 + g \rangle \leq \alpha\ \forall g \in V$, and hence there exists a constant C such that $|\langle \xi, g \rangle| \leq C\ \forall g \in V$. Therefore $\xi : E^\star \to \mathbb{R}$ is continuous for the

topology $\sigma(E^\star, E)$. In view of Proposition 3.14 there exists some $x \in E$ such that $\langle \xi, f\rangle = \langle f, x\rangle \ \forall f \in E^\star$.
2. See the solution of Exercise 3.7.
3. Let V be an open set for the topology $\sigma(E^\star, E)$ that is convex, and such that $0 \in V$ and $V \cap (A - B) = \emptyset$. Separating V and $(A - B)$, we find some $x \in E$, $x \neq 0$, and a constant α such that

$$\langle f, x\rangle \leq \alpha \leq \langle g - h, x\rangle \quad \forall f \in V,\ \forall g \in A,\ \forall h \in B.$$

Since V is also a neighborhood of 0 for the strong topology, there exists some $r > 0$ such that $rB_{E^\star} \subset V$. Thus $\alpha \geq r\|x\| > 0$, which leads to a strict separation of A and B.
4. Let $f, g \in \overline{A}^{\sigma(E^\star, E)}$ and let V be a convex neighborhood of 0 for $\sigma(E^\star, E)$. Then $(f + V) \cap A \neq \emptyset$ and $(g + V) \cap A \neq \emptyset$. Thus $(tf + (1 - t)g + V) \cap A \neq \emptyset$ $\forall t \in [0, 1]$.

- B-

1. If E is reflexive, then $\overline{N}^{\sigma(E^\star, E)} = \overline{N} =$ the closure of N for the strong topology, since $\sigma(E^\star, E) = \sigma(E^\star, E^{\star\star})$ and N is convex. Let $E = \ell^1$, so that $E^\star = \ell^\infty$; taking $N = c_0$ we have $N^\perp = \{0\}$ and $N^{\perp\perp} = \ell^\infty$.
2. For every $x \in E$, set $\varphi(x) = \sup_{f \in E^\star}\{\langle f, x\rangle - \psi(f)\}$. Then $\varphi : E \to (-\infty, +\infty]$ is convex and l.s.c. In order to show that $\varphi \not\equiv +\infty$ and that $\varphi^\star = \psi$, one may follow the same arguments as in Proposition 1.10 and Theorem 1.11, except that here one uses question A3 instead of the usual Hahn–Banach theorem.
3. (i) One knows (Corollary 2.18) that $N(A) = R(A^\star)^\perp$ and thus $N(A)^\perp = R(A^\star)^{\perp\perp} = \overline{R(A^\star)}^{\sigma(E^\star, E)}$. If E is reflexive, then $N(A)^\perp = \overline{R(A^\star)}$.
(ii) Argue as in the proof of Theorem 3.24 and apply question A3.
4. Suppose, by contradiction, that there exists some $\xi \in B_{E^{\star\star}}$ such that $\xi \notin \overline{J(B_E)}^{\sigma(E^{\star\star}, E^\star)}$. Applying question A3 in $E^{\star\star}$, we may find some $f \in E^\star$ and a constant α such that

$$\langle f, x\rangle < \alpha < \langle \xi, f\rangle \quad \forall x \in B_E.$$

Thus $\|f\| \leq \alpha < \langle \xi, f\rangle \leq \|f\|$; absurd.
5. Assume, by contradiction, that there exists some $u_0 \in E$ with $\|u_0\| < 1$ and $Au_0 \notin \overline{\text{conv}\, A(S_E)}^{\sigma(E^\star, E)}$. Applying question A3, we may find some $x_0 \in E$ and a constant α such that

$$\langle Au, x_0\rangle \leq \alpha < \langle Au_0, x_0\rangle \ \forall u \in S_E;$$

thus $\langle Au - Au_0, x_0\rangle < 0 \ \forall u \in S_E$. On the other hand, there is some $t > 0$ such that $\|u_0 + tx_0\| = 1$, and by monotonicity, we have $\langle A(u_0 + tx_0) - Au_0, x_0\rangle \geq 0$; absurd.

Problem 10

- A -

1. $B_{E^\star}$ is compact and metrizable for the topology $\sigma(E^\star, E)$ (see Theorem 3.28). It follows, by a standard result in point-set topology, that there exists a subset in $B_{E^\star}$ that is countable and dense for $\sigma(E^\star, E)$. Let $\mathcal{T}$ denote the topology associated to the metric d. It is easy to see that the canonical injection $i\colon (B_E, \sigma(E, E^\star)) \to (B_E, \mathcal{T})$ is continuous (see part (b) in the proof of Theorem 3.28). [Note that in general, i^{-1} is not continuous; otherwise, B_E would be metrizable for the topology $\sigma(E, E^\star)$ and $E^\star$ would be separable (see Exercise 3.24). However, there are examples in which E is separable and $E^\star$ is not, for instance $E = L^1(\Omega)$ and $E^\star = L^\infty(\Omega)$.]
   Since B is compact for $\sigma(E, E^\star)$, it follows (by Corollary 2.4) that B is bounded. Thus B is a compact (metric) space for the topology $\mathcal{T}$ and, moreover, the topologies $\sigma(E, E^\star)$ and $\mathcal{T}$ coincide on B.
2. Consider the closed linear space spanned by the x_n's.

- B -

For each i choose $a_1 \in B_F$ such that $\langle g_i, a_i\rangle \geq 3/4$.

- C -

4. For each $\eta \in E^{\star\star}$ set $h(\eta) = \sup_{i\geq 1}\langle \eta, f_i\rangle$; the function $h : E^{\star\star} \to \mathbb{R}$ is continuous for the strong topology on $E^{\star\star}$, since we have $|h(\eta_1) - h(\eta_2)| \leq \|\eta_1 - \eta_2\|\ \forall \eta_1, \eta_2 \in E^{\star\star}$.
5. A subsequence of the sequence (x_n) converges to x for $\sigma(E, E^\star)$ (by assumption (Q) and we have $\langle \xi, f_i\rangle = \langle f_i, x\rangle\ \forall i \geq 1$.

   On the other hand, x belongs to the closure of $[x_1, x_2, \ldots, x_k, \ldots]$ for the topology $\sigma(E, E^\star)$ and thus also for the strong topology (by Theorem 3.7). In particular, $x \in \overline{M}$ and consequently $\xi - x \in \overline{M}$. It follows that $\xi = x$ since

$$0 = \sup_{i\geq 1}\langle \xi - x, f_i\rangle \geq \frac{1}{2}\|\xi - x\|.$$

-D-

1. A is bounded by assumption (Q) and Corollary 2.4. It follows that $\overline{A}^{\sigma(E^{\star\star}, E^\star)}$ is compact for the topology $\sigma(E^{\star\star}, E^\star)$ by Theorem 3.16. But the result of part C shows that $B = \overline{A}^{\sigma(E^{\star\star}, E^\star)}$, or more precisely that $J(B) = \overline{J(A)}^{\sigma(E^{\star\star}, E^\star)}$. Consequently $J(B)$ is compact for the topology $\sigma(E^{\star\star}, E^\star)$. Since the map $J^{-1} : J(E) \to E$ is continuous from $\sigma(E^{\star\star}, E^\star)$ to $\sigma(E, E^\star)$, it follows that B is compact for $\sigma(E, E^\star)$.
2. Already established in question C4.

Problem 11

-A-

2. Separating $\{0\}$ and C_1 we find some $x_1 \in E$ and a constant α such that $0 < \alpha < \langle f, x_1 \rangle \ \forall f \in C_1$. If needed, replace x_1 by a multiple of x_1.
3. One has to find a finite subset $A \subset E$ such that $A \subset (1/d_1)B_E$ and $Y_A = \emptyset$. We first claim that $\bigcap_{A \in \mathcal{F}} Y_A = \emptyset$, where $\mathcal{F}$ denotes the family of all finite subsets A in $(1/d_1)B_E$. Assume, by contradiction, that $f \in \bigcap_{A \in \mathcal{F}} Y_A$; we have

$$\langle f, x_1 \rangle \leq 1 \quad \text{and} \quad \langle f, x \rangle \leq 1 \ \ \forall x \in (1/d_1)B_E.$$

   Thus $\|f\| \leq d_1$ and so $f \in C_1$; it follows that $\langle f, x_1 \rangle > 1$; absurd.
   By compactness there is a finite sequence $A'_1, A'_2, \dots, A'_j$ such that $\bigcap_{i=1}^{j} Y_{A'_i} = \emptyset$. Set $A' = A'_1 \cup A'_2, \dots \cup A'_j$. It is easy to check that $Y_{A'} = \emptyset$.
4. For every finite subset A in $(1/d_{k-1})B_E$ consider the set

$$Y_A = \left\{ f \in C_k;\ \sup \left\{ \langle f, x \rangle;\ x \in \left(\bigcup_{i=1}^{k-1} A_i \right) \cup A \right\} \leq 1 \right\}.$$

   One proves, as in question 3, that there is some A such that $Y_A = \emptyset$.
5. Write the set $\bigcup_{k=1}^{\infty} A_k$ as a sequence (x_n) that tends to 0.

- B -

1. Applying Hahn–Banach in c_0, there exist some $\theta \in \ell^1 (= (c_0)^\star$; see Chapter 11) with $\theta \neq 0$, and a constant α such that

$$\langle \theta, \xi \rangle \leq \alpha \leq \langle \theta, T(f) \rangle \quad \forall \xi \in c_0 \text{ with } \|\xi\| < 1, \quad \forall f \in C.$$

   It follows that

$$0 < \|\theta\|_{\ell^1} \leq \alpha \leq \sum \theta_n \langle f, x_n \rangle \quad \forall f \in C.$$

   Letting $x = \sum \theta_n x_n$, we obtain

$$\langle f, x \rangle \geq \alpha > 0 \quad \forall f \in C.$$

   If needed, replace x by a multiple of x and conclude.
2. Fix any $f_0 \notin C$; set $\widetilde{C} = C - f_0$. Then $0 \notin \widetilde{C}$ and for each integer n the set $\widetilde{C} \cap (nB_{E^\star})$ is closed for $\sigma(E^\star, E)$. Hence, there is some $x \in E$ such that $\langle f, x \rangle \geq 1 \ \forall f \in \widetilde{C}$. The set $V = \{f \in E^\star;\ \langle f - f_0, x \rangle < 1\}$ is a neighborhood of f_0 for $\sigma(E^\star, E)$ and $V \cap C = \emptyset$.

Problem 12

- A -

2. Apply the results of questions 1, 7, and 4 in Exercise 1.23 to the functions $\varphi^\star$ and $\psi^\star$.
3. We have $\theta^{\star\star} = (\varphi + \psi)^\star$. Following the same argument as in the proof of Theorem 1.11, it is easy to see that $\operatorname{epi}\theta^{\star\star} = \overline{\operatorname{epi}\theta}$ (warning: in general, θ need not be l.s.c.).

   Therefore we obtain $D(\theta^{\star\star}) \subset \overline{D(\theta)}$, i.e., $D((\varphi + \psi)^\star) \subset \overline{D(\varphi^\star) + D(\psi^\star)}$.

-B-

1. It suffices to check that for every fixed $x \in E$ the set $\langle M, x\rangle$ is bounded. In fact, it suffices to check that $\langle M, x\rangle$ is bounded below (choose $\pm x$). Given $x \in E$, $x \neq 0$, write $x = \lambda(a - b)$ with $\lambda > 0$, $a \in D(\varphi)$, and $b \in D(\psi)$. We have

$$\langle f - g, a\rangle \leq \varphi(a) + \varphi^\star(f - g),$$
$$\langle g, b\rangle \leq \psi(b) + \psi^\star(g),$$

   and thus
$$-\left\langle g, \frac{x}{\lambda}\right\rangle \leq -\langle f, a\rangle + \varphi(a) + \psi(b) + \alpha \quad \forall g \in M.$$

   Consequently $\langle M, x\rangle \geq C$, where C depends only on x, f, and α.
2. Use the same method as above.
3. Let $\alpha \in \mathbb{R}$ be fixed and let (f_n) be a sequence in $E^\star$ such that $\theta(f_n) \leq \alpha$ and $f_n \to f$. Thus, there is a sequence (g_n) in $E^\star$ such that $\varphi^\star(f_n - g_n) + \psi^\star(g_n) \leq \alpha + (1/n)$. Consequently, (g_n) is bounded and we may assume that $g_{n_k} \rightharpoonup g$ for $\sigma(E^\star, E)$. Since $\varphi^\star$ and $\psi^\star$ are l.s.c. for $\sigma(E^\star, E)$, it follows that $\varphi^\star(f - g) + \psi^\star(g) \leq \alpha$, and so $\theta(f) \leq \alpha$.
4. (i) We have $\theta = \theta^{\star\star} = (\varphi + \psi)^\star$.

   (ii) Write that $(\varphi + \psi)^\star(0) = (\varphi^\star \nabla \psi^\star)(0)$ and note that

$$\inf_{g \in E^\star} \{\varphi^\star(-g) + \psi^\star(g)\}$$

   is achieved by the result of question B1.

   (iii) This is a direct consequence of (i).

Remark. Assumption (H) holds if there is some $x_0 \in D(\varphi) \cap D(\psi)$ such that φ is continuous at x_0.

Problem 13

- A -

1. By question 5 of Exercise 1.25 we know that

$$\lim_{\substack{\lambda \to 0 \\ \lambda > 0}} \frac{1}{2\lambda} \left(\|x + \lambda y\|^2 - \|x\|^2 \right) = \langle Fx, y \rangle.$$

If $\lambda < 0$ set $\mu = -\lambda$ and write

$$\frac{1}{2\lambda} \left(\|x + \lambda y\|^2 - \|x\|^2 \right) = -\frac{1}{2\mu} \left(\|x + \mu(-y)\|^2 - \|x\|^2 \right).$$

2. Let $t_n \to 0$ be such that $\langle F(x + t_n y), y \rangle \to \ell$. We have

$$\frac{1}{2} \left(\|x + \lambda y\|^2 - \|x + t_n y\|^2 \right) \geq \langle F(x + t_n y), (\lambda - t_n) y \rangle.$$

Passing to the limit (with $\lambda \in \mathbb{R}$ fixed) we obtain $\frac{1}{2} \left(\|x + \lambda y\|^2 - \|x\|^2 \right) \geq \lambda \ell$. Dividing by λ (distinguish the cases $\lambda > 0$ and $\lambda < 0$) and letting $\lambda \to 0$ leads to $\langle Fx, y \rangle = \ell$. The uniqueness of the limit allows us to conclude that

$$\lim_{t \to 0} \langle F(x + ty), y \rangle = \langle Fx, y \rangle$$

(check the details).

3. Recall that F is monotone by question 4 of Exercise 1.1.

Alternative proof. It suffices to show that if $x_n \to x$ then $Fx_n \rightharpoonup Fx$ for $\sigma(E^\star, E)$. Assume $x_n \to x$. If E is reflexive or separable there is a subsequence such that $Fx_{n_k} \rightharpoonup f$ for $\sigma(E^\star, E)$. Recall that $\langle Fx_n, x_n \rangle = \|x_n\|^2$ and $\|Fx_n\| = \|x_n\|$. Passing to the limit we obtain $\langle f, x \rangle = \|x\|^2$ and $\|f\| \leq \|x\|$. Thus $f = Fx$; the uniqueness of the limit allows us to conclude that $Fx_n \rightharpoonup Fx$ for $\sigma(E^\star, E)$ (check the details).

- B -

1. If $x_n \to x$, then $Fx_n \rightharpoonup Fx$ for $\sigma(E^\star, E)$ and $\|Fx_n\| = \|x_n\| \to \|x\| = \|Fx\|$. It follows from Proposition 3.32 that $Fx_n \to Fx$.
2. Assume, by contradiction, that there are two sequences (x_n), and (y_n) such that $\|x_n\| \leq M$, $\|y_n\| \leq M$, $\|x_n - y_n\| \to 0$, and $\|Fx_n - Fy_n\| \geq \varepsilon > 0$. Passing to a subsequence we may assume that $\|x_n\| \to \ell$, and $\|y_n\| \to \ell$ with $\varepsilon \leq 2\ell$, so that $\ell \neq 0$. Set $a_n = x_n / \|x_n\|$ and $b_n = y_n / \|y_n\|$. We have $\|a_n\| = \|b_n\| = 1$, $\|a_n - b_n\| \to 0$, and $\|Fa_n - Fb_n\| \geq \varepsilon' > 0$ for n large enough. Since $E^\star$ is uniformly convex there exists $\delta > 0$ such that

$$\left\| \frac{Fa_n + Fb_n}{2} \right\| \leq 1 - \delta.$$

On the other hand, the inequality of question A4 leads to

$$2 \leq \|Fa_n + Fb_n\| + \|a_n - b_n\|;$$

this is impossible.

3. We have

$$\varphi(x) - \varphi(x_0) \geq \langle Fx_0, x - x_0\rangle$$

and

$$\varphi(x_0) - \varphi(x) \geq \langle Fx, x_0 - x\rangle.$$

It follows that

$$0 \leq \varphi(x) - \varphi(x_0) - \langle Fx_0, x - x_0\rangle \leq \langle Fx - Fx_0, x - x_0\rangle$$

and therefore

$$|\varphi(x) - \varphi(x_0) - \langle Fx_0, x - x_0\rangle| \leq \|Fx - Fx_0\| \, \|x - x_0\|.$$

The conclusion is derived easily with the help of question B1.

-C -

Write

$$\begin{aligned}\|f + g\| &= \sup_{\substack{x\in E\\ \|x\|\leq 1}} \langle f + g, x\rangle \\ &= \sup_{\substack{x\in E\\ \|x\|\leq 1}} \{\langle f, x + y\rangle + \langle g, x - y\rangle + \langle g, x - y\rangle - \langle f - g, y\rangle\} \\ &\leq \frac{1}{2}\|f\|^2 + \frac{1}{2}\|g\|^2 - \langle f - g, y\rangle + \sup_{\substack{x\in E\\ \|x\|\leq 1}} \{\varphi(x + y) + \varphi(x - y)\}.\end{aligned}$$

From the computation in question B3 we see that for every $x, y \in E$,

$$|\varphi(x + y) - \varphi(x) - \langle Fx, y\rangle| \leq \|F(x + y) - F(x)\| \, \|y\|$$

and

$$|\varphi(x - y) - \varphi(x) + \langle Fx, y\rangle| \leq \|F(x - y) - F(x)\| \, \|y\|.$$

It follows that for every $x, y \in E$,

$$\varphi(x + y) + \varphi(x - y) \leq 2\varphi(x) + \|y\|(\|F(x + y) - F(x)\| + \|F(x - y) - F(x)\|).$$

Therefore, if $\|f\| \leq 1$ and $\|g\| \leq 1$, we obtain for every $y \in E$,

$$\|f + g\| \leq 2 - \langle f - g, y\rangle + \|y\| \sup_{\substack{x\in E\\ \|x\|\leq 1}} \{\|F(x + y) - F(x)\| + \|F(x - y) - F(x)\|\}.$$

Fix $\varepsilon > 0$ and assume that $\|f - g\| > \varepsilon$. Since F is uniformly continuous, there exists some $\alpha > 0$ such that for $\|y\| \leq \alpha$ we have

$$\sup_{\substack{x\in E\\ \|x\|\leq 1}} \{\|F(x + y) - F(x)\| + \|F(x - y) - F(x)\|\} < \varepsilon/2.$$

On the other hand, there exists some $y_0 \in E$, $y_0 \neq 0$, such that $\langle f-g, y_0\rangle \geq \varepsilon\|y_0\|$, and we may assume that $\|y_0\| = \alpha$. We conclude that

$$\|f+g\| \leq 2 - \varepsilon\|y_0\| + \frac{\varepsilon}{2}\|y_0\| = 2 - \frac{\varepsilon}{2}\alpha.$$

Problem 14

- A -

2. Assume that $x_n \to x$ in E and set $f_n = Sx_n$, so that $\forall f \in E^\star$,

$$\text{(S1)} \qquad \frac{1}{2}\|f_n\|^2 + \varphi^\star(f_n) - \langle f_n, x_n\rangle \leq \frac{1}{2}\|f\|^2 + \varphi^\star(f) - \langle f, x_n\rangle.$$

It follows that the sequence (f_n) is bounded (why?) and thus there is a subsequence such that $f_{n_k} \overset{\star}{\rightharpoonup} g$ for $\sigma(E^\star, E)$. Passing to the limit in (S1) (note that the function $f \mapsto \frac{1}{2}\|f\|^2 + \varphi^\star(f)$ is l.s.c. for $\sigma(E^\star, E)$), we find that

$$\frac{1}{2}\|g\|^2 + \varphi^\star(g) - \langle g, x\rangle \leq \frac{1}{2}\|f\|^2 + \varphi^\star(f) - \langle f, x\rangle \quad \forall f \in E^\star$$

(one uses also Proposition 3.13). Thus $g = Sx$; the uniqueness of the limit implies that $f_n \overset{\star}{\rightharpoonup} Sx$ (check the details). Returning to (S1) and choosing $f = Sx$, we obtain $\limsup\|f_n\|^2 \leq \|Sx\|^2$. We conclude with the help of Proposition 3.32 that $f_n \to Sx$.

- B -

1. The convexity of ψ follows from question 7 of Exercise 1.23. Equality (i) is a consequence of Theorem 1.12, and equality (ii) follows from question 1 of Exercise 1.24.
2. We have

$$\langle Sx, y\rangle \leq \psi(y) + \psi^\star(Sx) = \psi(y) + \langle Sx, x\rangle - \psi(x)$$

and thus

$$0 \leq \psi(y) - \psi(x) - \langle Sx, y-x\rangle \quad \forall x, y \in E.$$

Changing x into y and y into x, we obtain

$$0 \leq \psi(x) - \psi(y) - \langle Sy, x-y\rangle \quad \forall x, y \in E.$$

We conclude that

$$0 \leq \psi(y) - \psi(x) - \langle Sx, y-x\rangle \leq \langle Sy - Sx, y-x\rangle.$$

Problem 15

- A -

2. Note that $\psi(x) \geq \|x\| - \|a\|$ with $a \in A$ being fixed and thus $\psi(x) \to +\infty$ as $\|x\| \to \infty$; therefore c exists. In order to establish the uniqueness it suffices to check that

$$\varphi^2\left(\frac{c_1 + c_2}{2}\right) < \frac{1}{2}\varphi^2(c_1) + \frac{1}{2}\varphi^2(c_2) \quad \forall c_1, c_2 \in E \text{ with } c_1 \neq c_2.$$

Let $c_1, c_2 \in E$ with $c_1 \neq c_2$. Fix some $0 < \varepsilon < \|c_1 - c_2\|$. In view of Exercise 3.29, and because A is bounded, there exists some $\delta > 0$ such that

$$\left\|\frac{(c_1 - y) + (c_2 - y)}{2}\right\| \leq \frac{1}{2}\|c_1 - y\|^2 + \frac{1}{2}\|c_2 - y\|^2 - \delta \quad \forall y \in A,$$

since $\|(c_1 - y) - (c_2 - y)\| > \varepsilon$. Taking $\sup_{y \in A}$ leads to

$$\varphi^2\left(\frac{c_1 + c_2}{2}\right) \leq \frac{1}{2}\varphi^2(c_1) + \frac{1}{2}\varphi^2(c_2) - \delta.$$

3. We know that $\varphi(\sigma(A)) < \varphi(x)$ $\forall x \in C,\ x \neq \sigma(A)$. If A is not reduced to a single point there exists some $x_0 \in A,\ x_0 \neq \sigma(A)$, and we have

$$\varphi(\sigma(A)) < \varphi(x_0) = \sup_{y \in A} \|x_0 - y\| \leq \operatorname{diam} A.$$

- B -

1. Note that the sequence $(\varphi_n(x))$ is nonincreasing.
3. We have

$$\varphi(\sigma) \leq \varphi(\sigma_n) \leq \varphi_n(\sigma_n) \leq \varphi_n(x) \quad \forall x \in C.$$

Taking $x = \sigma$, we find that all the limits are equal. It is easy to see that the sequence (σ_n) is bounded, and thus for a subsequence, $\sigma_{n_k} \rightharpoonup \tilde{\sigma}$ weakly $\sigma(E, E^\star)$. Hence we have

$$\varphi(\tilde{\sigma}) \leq \liminf \varphi(\sigma_{n_k}) \leq \varphi(x) \quad \forall x \in C.$$

It follows that $\varphi(\tilde{\sigma}) = \inf_C \varphi$ and, by uniqueness, $\tilde{\sigma} = \sigma$. The uniqueness of the limit implies that $\sigma_n \rightharpoonup \sigma$ (check the details).

4. Assume, by contradiction, that there exist some $\varepsilon > 0$ and a subsequence (σ_{n_k}) such that $\|\sigma_{n_k} - \sigma\| > \varepsilon$ $\forall k$. Using once more Exercise 3.29 we obtain some $\delta > 0$ such that

$$\varphi_{n_k}^2\left(\frac{1}{2}(\sigma_{n_k} + \sigma)\right) \leq \frac{1}{2}\varphi_{n_k}^2(\sigma_{n_k}) + \frac{1}{2}\varphi_{n_k}^2(\sigma) - \delta \quad \forall k,$$

and since $\varphi \leq \varphi_{n_k}$, we deduce that

$$\varphi^2\left(\frac{1}{2}(\sigma_{n_k}+\sigma)\right) \le \frac{1}{2}\varphi_{n_k}^2(\sigma_{n_k}) + \frac{1}{2}\varphi_{n_k}^2(\sigma) - \delta \quad \forall k.$$

This leads to a contradiction, since φ is l.s.c.

5. Note that $\varphi(x) = \|x-a\|$ and thus $\sigma = a$.
6. Write

$$|x-a_n|^2 = |x-a+a-a_n|^2 = |x-a|^2 + 2(x-a, a-a_n) + |a-a_n|^2,$$

and thus

$$\varphi^2(x) = \limsup_{n\to\infty} |x-a_n|^2 = |x-a|^2 + \limsup_{n\to\infty} |a_n-a|^2 = |x-a|^2 + \varphi^2(a).$$

It follows that $\sigma = a$.

- C -

1. We have $\|a_{n+1} - Tx\| \le \|a_n - x\|$ $\forall n, \forall x \in C$, and therefore $\varphi_{n+1}(Tx) \le \varphi_n(x)$ $\forall x \in C$. Passing to the limit leads to $\varphi(Tx) \le \varphi(x)$ $\forall x \in C$. In particular $\varphi(T\sigma) \le \varphi(\sigma)$ and thus $T\sigma = \sigma$.
2. Let $x, y \in C$ be fixed points of T; set $z = tx + (1-t)y$ with $t \in [0, 1]$. We have

$$\|Tz - x\| \le (1-t)\|y-x\| \quad \text{and} \quad \|Tz - y\| \le t\|y-x\|$$

and therefore $\|Tz - x\| = (1-t)\|y-x\|$, $\|Tz - y\| = t\|y-x\|$. The conclusion follows from the fact that E is strictly convex. (Recall that uniform convexity implies strict convexity; see Exercise 3.31).

Problem 16

- A -

1. We have $\langle Au - f, u\rangle \ge 0$ $\forall u \in D(A)$ and using (P) we see that $f = A0 = 0$.
2. Let (u_n) be a sequence in $D(A)$ such that $u_n \to x$ in E and $Au_n \to f$ in $E^\star$. We have $\langle Au - Au_n, u - u_n\rangle \ge 0$ $\forall u \in D(A)$. Passing to the limit we obtain $\langle Au - f, u - x\rangle \ge 0$ $\forall u \in D(A)$. From (P) we deduce that $x \in D(A)$ and $Ax = f$.
3. It is easy to check that if $t \in (0, 1)$, the convexity inequality

$$\langle A(tu + (1-t)v), tu + (1-t)v\rangle \le t\langle Au, u\rangle + (1-t)\langle Av, v\rangle$$

is equivalent to $\langle Au - Av, u - v\rangle \ge 0$.

4. Let $u \in N(A)$; we have $\langle Av, v - u\rangle \ge 0$, $\forall v \in D(A)$. Replacing v by λv, we see that $\langle Av, u\rangle = 0$ $\forall v \in D(A)$; that is, $u \in R(A)^\perp$.

- B -

1. Note that $\langle A^\star v, v\rangle = \langle Av, v\rangle$ $\forall v \in D(A) \cap D(A^\star)$.

2. The first claim is a direct consequence of (P) and the assumption that $v \notin D(A)$. Choosing $f = -A^\star v$, we have some $u \in D(A)$ such that $\langle Au + A^\star v, u - v\rangle < 0$ and consequently $\langle A^\star v, v\rangle > \langle Au, u\rangle \geq 0$.
3. Applying question A4 to $A^\star$ (this is permissible since $A^\star$ is monotone), we see that $N(A^\star) \subset N(A^{\star\star}) = N(A)$; therefore $N(A) = N(A^\star)$. We always have $\overline{R(A)} = N(A^\star)^\perp$ (see Corollary 2.18), and since E is reflexive, we also have $\overline{R(A^\star)} = N(A)^\perp$.

- C -

1. The map $u \in D(A) \mapsto [u, Au]$ is an isometry from $D(A)$, equipped with the graph norm, onto $G(A)$, which is a closed subspace of $E \times E^\star$.
2. Note that

$$\langle Au - f, u - x\rangle \geq -\|Au\|\,\|x\| - \|f\|\,\|u\| + \langle f, x\rangle.$$

3. Using the properties below (see Problem 13)

$$\lim_{t\to 0} \frac{1}{2t}(\|x + ty\|^2 - \|x\|^2) = \langle Fx, y\rangle \qquad \forall x, y \in E,$$
$$\lim_{t\to 0} \frac{1}{2t}(\|f + tg\|^2 - \|f\|^2) = \langle g, F^{-1}f\rangle \quad \forall f, g \in E^\star,$$

we find that for all $v \in D(A)$,

$$\langle Av, F^{-1}(Au_0 - f)\rangle + \langle F(u_0 - x), v\rangle + \langle Au_0 - f, v\rangle + \langle Av, u_0 - x\rangle = 0.$$

It follows that $F^{-1}(Au_0 - f) + u_0 - x \in D(A^\star)$ and

$$\text{(S1)} \qquad A^\star[F^{-1}(Au_0 - f) + u_0 - x] + (Au_0 - f) + F(u_0 - x) = 0.$$

4. Let $x \in E$ and $f \in E^\star$ be such that $\langle Au - f, u - x\rangle \geq 0\ \forall u \in D(A)$. One has to prove that $x \in D(A)$ and $Ax = f$. We know that there exists some $u_0 \in D(A)$ satisfying (S1). Applying (M*) leads to

$$\langle Au_0 - f + F(u_0 - x), F^{-1}(Au_0 - f) + u_0 - x\rangle \leq 0,$$

that is,

$$\|Au_0 - f\|^2 + \|u_0 - x\|^2 + \langle Au_0 - f, u_0 - x\rangle + \langle F(u_0 - x), F^{-1}(Au_0 - f)\rangle \leq 0.$$

It follows that

$$\|Au_0 - f\|^2 + \|u_0 - x\|^2 \leq \|u_0 - x\|\,\|Au_0 - f\|;$$

therefore $u_0 = x$ and $Au_0 = f$.
5. Apply to the operator $A^\star$ the implication (M*) $\Rightarrow$ (P).

Problem 18

2. $[G(a) - G(b) - g(b)(a-b) = 0] \Leftrightarrow [g(a) = g(b)]$.
3. Passing to a subsequence we may assume that $a_{n_k} \to a$ (possibly $\pm\infty$). We have $\int_b^a (g(t)-g(b))dt = 0$ and therefore $g(a) = g(b)$. It follows that $g(a_{n_k}) \to g(b)$.
4. Note that

$$0 \le \int |G(u_n) - G(u) - g(u)(u_n - u)| = \int G(u_n) - \int G(u) - \int g(u)(u_n - u)$$

and use assumption (ii). Then apply Theorem 4.9.
5. Since $g(u_n)$ is bounded in $L^{p'}$ (why?), we deduce (see Exercise 4.16) that $g(u_{n_k}) \to g(u)$ strongly in L^q for every $q \in [1, p')$. The uniqueness of the limit implies that $g(u_n) \to g(u)$.
6. If g is increasing then $u_{n_k} \to u$ a.e., and using once more Exercise 4.16 we see that $u_{n_k} \to u$ strongly in L^q for every $q \in [1, p)$.

7 and 9. Applying question 4 and Theorem 4.9, we know that there exists some function $f \in L^1$ such that

$$\text{(S1)} \qquad |G(u_{n_k}) - G(u) - g(u)(u_{n_k} - u)| \le f \quad \forall k.$$

From (S1) and (3) we deduce that $|u_{n_k}|^p \le \tilde{f}$ for some other function $\tilde{f} \in L^1$. The conclusion follows by dominated convergence.

8. I don't know.

Problem 19

4. Note that the set $\widetilde{K} = \{u \in L^2(\mathbb{R}); u \ge 0 \text{ a.e.}\}$ is a closed convex subset of $L^2(\mathbb{R})$. Thus, it is also closed for the weak L^2 topology. It remains to check that $u \in L^1(\mathbb{R})$ and that $\int_{\mathbb{R}} u \le 1$. Let $A \subset \mathbb{R}$ be any measurable set with finite measure. We have $\int_A u_n \to \int_A u$ since $\chi_A \in L^2(\mathbb{R})$ and thus $\int_A u \le 1$. It follows that $u \in L^1(\mathbb{R})$ and that $\int_{\mathbb{R}} u \le 1$. Next, write

$$\begin{aligned}\left|\int \frac{1}{|x|^\alpha}(u_n - u)\right| &\le \int_{[|x|>M]} \frac{1}{|x|^\alpha}|u_n - u| + \left|\int_{[|x|\le M]} \frac{1}{|x|^\alpha}(u_n - u)\right| \\ &\le \frac{2}{M^\alpha} + \left|\int_{[|x|\le M]} \frac{1}{|x|^\alpha}(u_n - u)\right|.\end{aligned}$$

For each fixed M the last integral tends to 0 as $n \to \infty$ (since $u_n \rightharpoonup u$ weakly in $L^2(\mathbb{R})$). We deduce that

$$\limsup_{n\to\infty} \left|\int \frac{1}{|x|^\alpha}(u_n - u)\right| \le \frac{2}{M^\alpha} \quad \forall M > 0.$$

5. Write

$$\int \frac{1}{|x|^\alpha} u(x)dx = \int_{[|x|>1]} \frac{1}{|x|^\alpha} u(x)dx + \int_{[|x|\le 1]} \frac{1}{|x|^\alpha} u(x)dx$$
$$\le \int u(x)dx + C\|u\|_2 \le 1 + C\|u\|_2 \quad \forall u \in K.$$

8. E is not reflexive. Assume, by contradiction, that E is reflexive and consider the sequence $u_n = \chi_{[n,n+1]}$. Since (u_n) is bounded in E, there is a subsequence u_{n_k} such that $u_{n_k} \rightharpoonup u$ weakly $\sigma(E, E^\star)$. In particular, $\int fu_{n_k} \to \int fu \, \forall f \in L^\infty(\mathbb{R})$ and therefore $\int fu = 0$ for every $f \in L^\infty(\mathbb{R})$ with compact support. It follows that $u = 0$ a.e. On the other hand, if we choose $f \equiv 1$ we see that $\int u = 1$; absurd.

Problem 20

- A -

2. Note that

$$f''(x) = \left(1 - \frac{1}{p}\right) x^{-2+(1/p)}[(1 - x^{1/p})^{p-2} - (1 + x^{1/p})^{p-2}] \le 0.$$

- B -

1. Replacing x by $f(x)$ and y by $g(x)$ in (1) and integrating over Ω, we obtain

$$\|f + g\|_p^p + \|f - g\|_p^p \le 2 \int (|f(x)|^{p'} + |g(x)|^{p'})^{p/p'}.$$

On the other hand, letting $u(x) = |f(x)|^{p'}$ and $v(x) = |g(x)|^{p'}$ and using the fact that $p/p' \ge 1$, we obtain

$$\int (u + v)^{p/p'} = \|u + v\|_{p/p'}^{p/p'} \le (\|u\|_{p/p'} + \|v\|_{p/p'})^{p/p'}$$
$$= (\|f\|_p^{p'} + \|g\|_p^{p'})^{p/p'}.$$

Applying (2) with $x = \|f\|_p$ and $y = \|g\|_p$ leads to (5).

- C -

1. *Method* (i). By Hölder's inequality we have

$$\int u\varphi + v\psi \le \|u\|_p \|\varphi\|_{p'} + \|v\|_p \|\psi\|_{p'}$$
$$\le (\|u\|_p^{p'} + \|v\|_p^{p'})^{1/p'} (\|\varphi\|_{p'}^p + \|\psi\|_{p'}^p)^{1/p}.$$

Moreover, equality holds when $\varphi = |u|^{p-2}u\|u\|_p^\alpha$ and $\psi = |v|^{p-2}v\|v\|_p^\alpha$ with $\alpha = p' - p$. Applying the above inequality to $u = f + g$ and $v = f - g$, we obtain

$$(\|f+g\|_p^{p'}+\|f-g\|_p^{p'})^{1/p'} = \sup_{\varphi,\psi\in L^{p'}} \left\{ \frac{\int f(\varphi+\psi)+g(\varphi-\psi)}{[\|\varphi\|_{p'}^p+\|\psi\|_{p'}^p]^{1/p}} \right\}.$$

Using Hölder's inequality we obtain

$$\begin{aligned}\int f(\varphi+\psi)+g(\varphi-\psi) &\le \|f\|_p\|\varphi+\psi\|_{p'}+\|g\|_p\|\varphi-\psi\|_{p'}\\ &\le (\|f\|_p^p+\|g\|_p^p)^{1/p}(\|\varphi+\psi\|_{p'}^{p'}+\|\varphi-\psi\|_{p'}^{p'})^{1/p'}.\end{aligned}$$

On the other hand, inequality (4) applied with p' in place of p says that

$$\|\varphi+\psi\|_{p'}^{p'}+\|\varphi-\psi\|_{p'}^{p'} \le 2(\|\varphi\|_{p'}^p+\|\psi\|_{p'}^p)^{p'/p},$$

and (6) follows.

Method (ii). Applying (1) with $x\to f(x)$, $y\to g(x)$ and $p\to p'$, we obtain

$$|f(x)+g(x)|^{p'}+|f(x)-g(x)|^{p'} \le 2(|f(x)|^p+|g(x)|^p)^{p'/p}$$

and thus

$$(|f(x)+g(x)|^{p'}+|f(x)-g(x)|^{p'})^{p/p'} \le 2^{p/p'}(|f(x)|^p+|g(x)|^p).$$

Integrating over Ω, we obtain, with the notation of Exercise 4.11,

$$[|f+g|^{p'}+|f-g|^{p'}]_{p/p'} \le 2(\|f\|_p^p+\|g\|_p^p)^{p'/p}.$$

The conclusion follows from the fact that $[u+v]_{p/p'} \ge [u]_{p/p'}+[v]_{p/p'}$ (since $p/p' \le 1$).

Problem 21

- A -

1. Use monotone convergence to prove that $\alpha(t+0)=\alpha(t)$. Note that if $f=\chi_\omega$ with $\omega\subset\Omega$ measurable, then $\alpha(1-0)=|\omega|$, while $\alpha(1)=0$.
2. Given $t>0$, let $\omega_n=[|f_n|>t]$, $\omega=[|f|>t]$, $\chi_n=\chi_{\omega_n}$, and $\chi=\chi_\omega$. It is easy to check that $\chi(x)\le\liminf\chi_n(x)$ for a.e. $x\in\Omega$ (distinguish the cases $x\in\omega$ and $x\notin\omega$). Applying Fatou's lemma, we see that

$$\alpha(t)=\int_\Omega\chi\le\liminf\int_\Omega\chi_n=\liminf\alpha_n(t).$$

On the other hand, let $\delta\in(0,t)$ and write

$$\int_\Omega \chi_n = \int_{[|f|\le t-\delta]} \chi_n + \int_{[|f|>t-\delta]} \chi_n \le \int_{[|f|\le t-\delta]} \chi_n + \alpha(t-\delta).$$

Since $\chi_n \to 0$ a.e. on the set $[|f| \le t-\delta]$, we have, by dominated convergence, $\int_{[|f|\le t-\delta]} \chi_n \to 0$. It follows that $\limsup \int_\Omega \chi_n \le \alpha(t-\delta)\ \forall \delta \in (0,t)$.

- B -

1. Consider the measurable function $H : \Omega \times (0,\infty) \to \mathbb{R}$ defined by

$$H(x,t) = \begin{cases} g(t) & \text{if } |f(x)| > t, \\ 0 & \text{if } |f(x)| \le t. \end{cases}$$

Note that

$$\int_\Omega H(x,t)d\mu = \alpha(t)g(t) \quad \text{for a.e. } t \in (0,\infty),$$

while

$$\int_0^\infty H(x,s)ds = \int_0^{|f(x)|} g(s)ds = G(|f(x)|) \quad \text{for a.e. } x \in \Omega.$$

Then use Fubini and Tonelli.

2. Given $\lambda > 0$ consider the function $\tilde{f} : \Omega \to \mathbb{R}$ defined by

$$\tilde{f}(x) = \begin{cases} f(x) & \text{on } [|f| > \lambda], \\ 0 & \text{on } [|f| \le \lambda], \end{cases}$$

so that its distribution function $\tilde{\alpha}$ is given by

$$\tilde{\alpha}(t) = \begin{cases} \alpha(\lambda) & \text{if } t \le \lambda, \\ \alpha(t) & \text{if } t > \lambda. \end{cases}$$

Apply to $\tilde{f}$ the result of question B1.

- C -

3. Use the inequality $\int_A |f| \le |A|^{1/p'}[f]_p$ with $A = [|f| > t]$ and note that $\int_A |f| \ge t\alpha(t)$.
4. Let $C = \sup_{t>0} t^p\alpha(t)$. We have

$$\int_A |f| \le \alpha(\lambda)\lambda + \int_\lambda^\infty \alpha(t)dt + \lambda|A| \le C\left(1 + \frac{1}{p-1}\right)\lambda^{1-p} + \lambda|A| \quad \forall \lambda > 0.$$

Choose $\lambda = |A|^{-1/p}$.

6. Write

$$\|f\|_p^p = p\int_0^\infty \alpha(t)t^{p-1}dt = p\int_0^\lambda \alpha(t)t^{p-1}dt + p\int_\lambda^\infty \alpha(t)t^{p-1}dt$$
$$\leq p[f]_q^q \int_0^\lambda \frac{t^{p-1}}{t^q}dt + p[f]_r^r \int_\lambda^\infty \frac{t^{p-1}}{t^r}dt$$

and choose λ appropriately.

Problem 22

1. Apply the closed graph theorem.
3. We know, by Problem 21, question B3, that $\|g_\lambda\|_1 = \int_0^\infty \gamma_\lambda(t)dt$. Applying question 2 and once more question B3 of Problem 21, we see that $\|g_\lambda\|_1 \leq N_1[\alpha(\lambda)\lambda + \int_\lambda^\infty \alpha(t)dt]$. On the other hand, since $\|f - g_\lambda\|_\infty \leq \lambda$, we have $[|f| > t] \subset [|g_\lambda| > t - \lambda]$.
4. By question 3 we know that

$$\int_\lambda^\infty \beta(s)ds \leq N_1\left[\alpha(\lambda)\lambda + \int_\lambda^\infty \alpha(t)dt\right] \quad \forall\lambda > 0.$$

Multiplying this inequality by λ^{p-2} and integrating leads to

$$\int_0^\infty \lambda^{p-2}d\lambda \int_\lambda^\infty \beta(s)ds$$
$$\leq N_1\left[\int_0^\infty \alpha(\lambda)\lambda^{p-1}d\lambda + \int_0^\infty \lambda^{p-2}d\lambda \int_\lambda^\infty \alpha(t)dt\right],$$

that is,

$$\frac{1}{p-1}\int_0^\infty \beta(s)s^{p-1}ds \leq N_1\left(1 + \frac{1}{p-1}\right)\int_0^\infty \alpha(\lambda)\lambda^{p-1}d\lambda.$$

From question B3 of Problem 21 we deduce that $\|f\|_p^p \leq pN_1\|u\|_p^p$; finally, we note that $p^{1/p} \leq e^{1/e} \leq 2\ \forall p \geq 1$.

Problem 23

- A -

1. The sets X_n are closed and $\bigcup_n X_n = X$. Hence, there is some integer n_0 such that $\mathrm{Int}(X_{n_0}) \neq \emptyset$. Thus, there exists $A_0 \subset \Omega$ measurable with $|A_0| < \infty$, and there exists some $\rho > 0$ such that

$$\left[\chi_B \in X \text{ and } \int_\Omega |\chi_B - \chi_{A_0}| < \rho\right] \Rightarrow \left[\left|\int_B f_k\right| \leq \varepsilon \quad \forall k \geq n_0\right].$$

We first claim that

$$\text{(S1)} \qquad \int_A |f_k| \leq 4\varepsilon \quad \forall A \subset \Omega \text{ measurable with } |A| < \rho, \text{ and } \forall k \geq n_0.$$

Indeed, let $A \subset \Omega$ be measurable with $|A| < \rho$; consider the sets

$$B_1 = A_0 \cup A \quad \text{and} \quad B_2 = B_1 \backslash A.$$

We have

$$\int_\Omega |\chi_{B_1} - \chi_{A_0}| \le |A| < \rho \quad \text{and} \quad \int_\Omega |\chi_{B_2} - \chi_{A_0}| \le |A| < \rho,$$

and therefore

$$\left|\int_{B_1} f_k\right| \le \varepsilon \quad \text{and} \quad \left|\int_{B_2} f_k\right| \le \varepsilon \quad \forall k \ge n_0.$$

It follows that

$$\left|\int_A f_k\right| = \left|\int_{B_1} f_k - \int_{B_2} f_k\right| \le 2\varepsilon \quad \forall k \ge n_0.$$

Applying the preceding inequality with A replaced by $A \cap [f_k > 0]$ and by $A \cap [f_k < 0]$, we are led to (S1). The conclusion of question 1 is obvious, since there exists some $\rho' > 0$ such that

$$\int_A |f_k| \le 4\varepsilon \quad \forall A \subset \Omega \text{ measurable with } |A| < \rho', \quad \forall k = 1, 2, \ldots, n_0.$$

2. There is some integer n_0 such that $\operatorname{Int}(Y_{n_0}) \ne \emptyset$. Thus, there exists $A_0 \subset \Omega$ measurable and there exists some $\rho > 0$ such that

$$[\chi_B \in Y \text{ and } d(\chi_B, \chi_A) < \rho] \Rightarrow \left[\left|\int_B f_k\right| \le \varepsilon \ \forall k \ge n_0\right].$$

Fix an integer j such that $2^{-j} < \rho$. We claim that

$$\text{(S2)} \qquad \int_A |f_k| \le 4\varepsilon \ \forall A \subset \Omega \text{ measurable with } A \cap \Omega_j = \emptyset, \ \forall k \ge n_0.$$

Indeed, let $A \subset \Omega$ be measurable with $A \cap \Omega_j = \emptyset$; consider the sets

$$B_1 = A_0 \cup A \quad \text{and} \quad B_2 = B_1 \backslash A.$$

We have $d(\chi_{B_1}, \chi_{A_0}) \le 2^{-j} < \rho$ and $d(\chi_{B_2}, \chi_{A_0}) \le 2^{-j} < \rho$; therefore $|\int_{B_1} f_k| \le \varepsilon$ and $|\int_{B_2} f_k| \le \varepsilon$, $\forall k \ge n_0$. We then proceed as in question 1.

4. Let us prove, for example, that (i) $\Rightarrow$ (b). Suppose, by contradiction, that (b) fails. There exist some $\varepsilon_0 > 0$, a sequence (A_n) of measurable sets in Ω, and a sequence (f_n) in $\mathcal{F}$ such that $|A_n| \to 0$ and $\int_{A_n} |f_n| \ge \varepsilon_0 \ \forall n$. By the Eberlein–Šmulian theorem there exists a subsequence such that $f_{n_k} \rightharpoonup f$ weakly $\sigma(L^1, L^\infty)$. Thus (see question 3) (f_{n_k}) is equi-integrable and we obtain a contradiction.

5. Assume, for example, that $\int_A f_n \to \ell(A)$ for every $A \subset \Omega$ with A measurable and $|A| < \infty$. We claim that (b) holds.

   Indeed, consider the sequence

$$X_n = \left\{ \chi_A \in X; \left| \int_A f_j - \int_A f_k \right| \le \varepsilon \ \ \forall j \ge n, \ \ \forall k \ge n \right\}.$$

   In view of the Baire category theorem there exist n_0, $A_0 \subset \Omega$ measurable with $|A_0| < \infty$, and $\rho > 0$ such that

$$\left[\chi_B \in X \text{ and } \int_\Omega |\chi_B - \chi_{A_0}| < \rho \right] \Rightarrow \left[\left| \int_B f_k - \ell(B) \right| \le \varepsilon \ \ \forall k \ge n_0 \right].$$

   Let $A \subset \Omega$ be measurable with $|A| < \rho$; with the same method as in question 1 one obtains

$$\left| \int_A f_k \right| \le 2\varepsilon + |\ell(B_2) - \ell(B_1)| \le 4\varepsilon + \left| \int_A f_{n_0} \right| \quad \forall k \ge n_0.$$

   It follows that

$$\int_A |f_k| \le 8\varepsilon + 2 \int_A |f_{n_0}| \quad \forall A \text{ measurable with } |A| < \rho, \ \forall k \ge n_0,$$

   and the conclusion is easy.

- B -

1. We have $\mathcal{F} \subset \mathcal{F}_\varepsilon + \varepsilon B_E \subset \mathcal{F}_\varepsilon + \varepsilon B_{E^{\star\star}}$. But $\mathcal{F}_\varepsilon + \varepsilon B_{E^{\star\star}}$ is compact for the topology $\sigma(E^{\star\star}, E^\star)$ (since it is a sum of two compact sets). It follows that $\mathcal{G}$ is compact for $\sigma(E^{\star\star}, E^\star)$. Also, since $\mathcal{G} \subset E + \varepsilon B_{E^{\star\star}}$ $\forall \varepsilon > 0$, we deduce that $\mathcal{G} \subset E$. These properties imply that $\mathcal{G}$ is compact for $\sigma(E, E^\star)$.
2. Given $\varepsilon > 0$ choose $\omega \subset \Omega$ measurable with $|\omega| < \infty$ such that $\int_{\Omega\setminus\omega} |f| \le \varepsilon/2$ $\forall f \in \mathcal{F}$, and choose n such that $\int_{[|f|>n]} |f| \le \varepsilon/2$ $\forall f \in \mathcal{F}$ (see Exercise 4.36). Set $\mathcal{F}_\varepsilon = (\chi_\omega T_n(f))_{f\in\mathcal{F}}$. Clearly, $\mathcal{F}_\varepsilon$ is bounded in $L^\infty(\omega)$ and thus it is contained in a compact subset of $L^1(\Omega)$ for $\sigma(L^1, L^\infty)$. On the other hand, for every $f \in \mathcal{F}$, we have

$$\int_\Omega |f - \chi_\omega T_n(f)| \le \int_\omega |f - T_n f| + \int_{\Omega\setminus\omega} |f| \le \int_{[|f|>n]} |f| + \int_{\Omega\setminus\omega} |f| \le \varepsilon.$$

   Thus, $\mathcal{F} \subset \mathcal{F}_\varepsilon + \varepsilon B_E$ with $E = L^1(\Omega)$.

- C -

4. Applying A5 we know that (f_n) satisfies (b) and (c). In view of B2 the set (f_n) has a compact closure in the topology $\sigma(L^1, L^\infty)$. Thus (by Eberlein–Šmulian) there is a subsequence such that $f_{n_k} \rightharpoonup f$ weakly $\sigma(L^1, L^\infty)$. It follows that

$\ell(A) = \int_A f \ \forall A$ measurable. The uniqueness of the limit implies that $f_n \rightharpoonup f$ weakly $\sigma(L^1, L^\infty)$ (check the details).

- D -

1. Apply Exercise 4.36.
2. Set

$$\Phi(t) = \sup_{f \in \mathcal{F}} \int_{[|f|>t]} |f|,$$

so that $\Phi \geq 0$, Φ is nonincreasing, and $\lim_{t\to\infty}\Phi(t) = 0$. We may always assume that $\Phi(t) > 0 \ \forall t > 0$; otherwise, there exists some T such that $\|f\|_\infty \leq T$, for all $f \in \mathcal{F}$, and the conclusion is obvious. Consider a function $g : [0, \infty) \to (0, \infty)$ such that g is nondecreasing and $\lim_{t\to\infty} g(t) = \infty$. Set $G(t) = \int_0^t g(s)ds$, $t \geq 0$, so that G is increasing, convex, and $\lim_{t\to\infty} G(t)/t = +\infty$. We recall (see Problem 21) that for every f,

$$\int G(|f|) = \int_0^\infty \alpha(t)g(t)dt \quad \text{and} \quad \int_{[|f|>t]} |f| = \alpha(t)t + \int_t^\infty \alpha(s)ds.$$

Set $\beta(t) = \int_t^\infty \alpha(s)ds$, so that $\beta(t) \leq \Phi(t)$ and $\beta'(t) = -\alpha(t)$. We claim that if we choose $g(t) = [\Phi(t)]^{-1/2}$, then the corresponding function G has the required property. Indeed, for every $f \in \mathcal{F}$, we have

$$\begin{aligned}\int G(|f|) &= \int_0^\infty \alpha(t)g(t)dt \leq \int_0^\infty -\beta'(t)[\beta(t)]^{-1/2}dt \\ &= 2[\beta(0)]^{1/2} = 2\left[\int_0^\infty \alpha(s)ds\right]^{1/2} = 2\left[\int |f|\right]^{1/2} \leq C.\end{aligned}$$

Problem 24

- B -

3. Clearly A is convex, and so is $\overline{A}^{\sigma(E^\star,E)}$ (see Problem 9, question A4). Suppose by contradiction that $\mu_0 \notin \overline{A}^{\sigma(E^\star,E)}$. By Hahn–Banach (applied in $E^\star$ with the weak* topology) there exist $f_0 \in C(\overline{\Omega})$ and $\beta \in \mathbb{R}$ such that

(S1)
$$\int_\Omega u f_0 < \beta < \langle \mu_0, f_0 \rangle \quad \forall u \in A.$$

On the other hand, we have

(S2)
$$\sup_{u\in A} \int_\Omega u f_0 = \|f_0\|_\infty;$$

indeed, A is dense in the unit ball of $L^1(\Omega)$ (by Corollary 4.23) and L^∞ is the dual of L^1 (see Theorem 4.14). Combining (S1) and (S2) yields

$$\|f_0\|_\infty \le \beta < \langle \mu_0, f_0\rangle \le \|f_0\|_\infty,$$

since $\|\mu_0\| \le 1$. This is impossible.

4. $B_{E^\star}$ is metrizable because $E = C(\overline{\Omega})$ is separable (see Theorem 3.28). Since $\mu_0 \in \overline{A}^{\sigma(E^\star,E)} \subset B_{E^\star}$ there exists a sequence (v_n) in A such that $v_n \overset{*}{\rightharpoonup} \mu_0$. Then apply Proposition 3.13.

6. Clearly $\langle \mu, 1\rangle \le \|\mu\|\ \forall \mu$. On the other hand, if $\|f\|_\infty \le 1$ and $\mu \ge 0$ we have $\langle \mu, f\rangle \le \langle \mu, 1\rangle$ and thus $\|\mu\| = \sup_{\|f\|_\infty \le 1}\langle \mu, f\rangle \le \langle \mu, 1\rangle$.
7. Set $A^+ = \{u \in A;\, u(x) \ge 0\ \forall x \in \overline{\Omega}\}$. Repeat the same proof as in question 3 with A being replaced by A^+; check that

$$\sup_{u\in A^+} \int_\Omega u f_0 = \|f_0^+\|_\infty$$

and that

$$\langle \mu_0, f_0\rangle \le \|f_0^+\|_\infty.$$

8. We claim that $\|u + \delta_a\|_{\mathcal{M}} = \|u\|_{L^1} + 1$. Clearly $\|u + \delta_a\|_{\mathcal{M}} \le \|u\|_{L^1} + 1$. To prove the reverse inequality, fix any $\varepsilon > 0$ and choose $r > 0$ sufficiently small that $\int_{B(a,r)} |u| < \varepsilon$. Let $\omega = \Omega \backslash \overline{B(a,r)}$ and pick $\varphi \in C_c(\omega)$ with $\|\varphi\|_{L^\infty(\omega)} \le 1$ and

$$\int_\omega u\varphi \ge \|u\|_{L^1(\omega)} - \varepsilon.$$

Then let $\theta \in C_c(B(a,r))$ be such that $\theta(a) = 1$ and $\|\theta\|_{L^\infty(\Omega)} \le 1$. Check that $\|\varphi + \theta\|_{L^\infty(\Omega)} \le 1$ and

$$\langle u + \delta_a, \varphi + \theta\rangle \ge \|u\|_{L^1(\Omega)} - 2\varepsilon + 1.$$

-D-

1. Clearly $L(f_1) + L(f_2) \le L(f_1 + f_2)$. For the reverse inequality, note that if $0 \le g \le f_1 + f_2$, then one can write $g = g_1 + g_2$ with $0 \le g_1 \le f_1$ and $0 \le g_2 \le f_2$; take, for example, $g_1 = \max\{g - f_2, 0\}$ and $g_2 = g - g_1$.
2. If $f = h + k$ with $h, k \in C(K)$, we have

$$f^+ - f^- = h^+ - h^- + k^+ - k^-,$$

so that

$$f^+ + h^- + k^- = h^+ + k^+ + f^-,$$

and thus

$$L(f^+) + L(h^-) + L(k^-) = L(h^+) + L(k^+) + L(f^-),$$

i.e.,

$$\mu_1(f) = \mu_1(h) + \mu_1(k).$$

Note that $L(f^+) \leq \|\mu\| \, \|f^+\|$ and $L(f^-) \leq \|\mu\| \, \|f^-\|$. Thus $|\mu_1(f)| \leq \|\mu\| \, \|f\|$. If $f \geq 0$ we have $\mu_1(f) = L(f) \geq 0$, so that $\mu_1 \geq 0$.

3. If $f \geq 0$, we have (taking $g = f$) $L(f) \geq \langle \mu, f \rangle$, so that $\langle \mu_1, f \rangle = L(f) \geq \langle \mu, f \rangle$, i.e., $\mu_2 = \mu_1 - \mu \geq 0$. Next, note that if $g \in C(K)$ and $0 \leq g \leq 1$, we have $-1 \leq 2g - 1 \leq 1$ and thus

$$\langle \mu, 2g - 1 \rangle \leq \|\mu\|.$$

Therefore

$$L(1) = \sup\{\langle \mu, g \rangle; 0 \leq g \leq 1\} \leq \frac{1}{2}(\langle \mu, 1 \rangle + \|\mu\|),$$

i.e.,

$$2\langle \mu_1, 1 \rangle = 2L(1) \leq \langle \mu_1, 1 \rangle - \langle \mu_2, 1 \rangle + \|\mu\|.$$

Thus

$$\|\mu_1\| + \|\mu_2\| = \langle \mu_1 + \mu_2, 1 \rangle \leq \|\mu\|$$

and consequently $\|\mu\| = \|\mu_1\| + \|\mu_2\|$.

-E-

One can repeat all the above proofs without modification. The only change occurs in question D3, where we have used the function 1, which is no longer admissible. We introduce, instead of 1, a sequence (θ_n) in E_0 such that $\theta_n \uparrow 1$ as $n \uparrow \infty$. Note that for every $\nu \in \mathcal{M}(\Omega)$, $\nu \geq 0$, we have $\langle \nu, \theta_n \rangle \uparrow \|\nu\|$.

If $g \in E_0$ and $0 \leq g \leq \theta_n$ we have $-\theta_n \leq 2g - \theta_n \leq \theta_n$ and thus $\langle \mu, 2g - \theta_n \rangle \leq \|\mu\|$. Hence

$$L(\theta_n) = \sup\{\langle \mu, g \rangle; 0 \leq g \leq \theta_n\} \leq \frac{1}{2}(\langle \mu, \theta_n \rangle + \|\mu\|)$$

i.e.,

$$2\langle \mu_1, \theta_n \rangle = 2L(\theta_n) \leq \langle \mu_1, \theta_n \rangle - \langle \mu_2, \theta_n \rangle + \|\mu\|.$$

Letting $n \to \infty$ yields $\|\mu_1\| + \|\mu_2\| \leq \|\mu\|$.

Problem 25

1. Let $v_0 \in C$ and let $u \in \Sigma \backslash \{0\}$; if $B(v_0, \rho) \subset C$ then $(u, v_0 + \rho z) \leq 0 \; \forall z \in H$ with $|z| < 1$. It follows that $(u, v_0) + \rho |u| \leq 0$. Conversely, let $v_0 \in H$ be such that $(u, v_0) < 0 \; \forall u \in \backslash \{0\}$. In order to prove that $v_0 \in C$, assume by contradiction that $v_0 \notin C$ and separate C and $\{v_0\}$.
2. If $u \in \Sigma$, then $(u, \omega) + \rho |u| \leq 0$; therefore $\rho |u| \leq 1$ for every $u \in K$.
4. If $(-C) \cap \Sigma = \emptyset$ separate $(-C)$ and Σ, and obtain a contradiction.
5. Since $a \in (-C) \cap \Sigma$ we may write $-a = \mu(w_0 - x_0)$ with $\mu > 0$ and $w_0 \in D$. On the other hand, since $a \in \Sigma \backslash \{0\}$ we have $(a, v) < 0 \; \forall v \in C$ and thus $(a, w - x_0) < 0 \; \forall w \in D$. It follows that $(x_0 - w_0, w - x_0) < 0 \; \forall w \in D$.

Problem 26

1. By Proposition 1.10 there exist some $g \in H$ and some constant C such that $\varphi(v) \geq (g, v) + C \ \forall v \in H$; therefore $I > -\infty$. Let (u_n) be a minimizing sequence, that is, $\frac{1}{2}|f - u_n|^2 + \varphi(u_n) = I_n \to I$. Using the parallelogram law we obtain
$$\left|f - \frac{u_n + u_m}{2}\right|^2 + \left|\frac{u_n - u_m}{2}\right|^2 = \frac{1}{2}\left(|f - u_n|^2 + |f - u_m|^2\right)$$
$$= I_n + I_m - \varphi(u_n) - \varphi(u_m) \leq I_n + I_m - 2\varphi\left(\frac{u_n + u_m}{2}\right).$$
It follows that $\left|\frac{u_n - u_m}{2}\right|^2 \leq I_n + I_m - 2I$.
2. If u satisfies (Q) we have
$$\frac{1}{2}|f - v|^2 + \varphi(v) \geq \frac{1}{2}|f - u|^2 + \varphi(u) + \frac{1}{2}|u - v|^2 \quad \forall v \in H.$$
Conversely, if u satisfies (P) we have
$$\frac{1}{2}|f - u|^2 + \varphi(u) \leq \frac{1}{2}|f - v|^2 + \varphi(v) \quad \forall v \in H;$$
choose $v = (1 - t)u + tw$ with $t \in (0, 1)$ and note that
$$\frac{1}{2}|f - v|^2 = \frac{1}{2}|f - u|^2 + t(f - u, u - w) + \frac{t^2}{2}|u - w|^2,$$
and
$$\varphi(v) \leq (1 - t)\varphi(u) + t\varphi(w).$$
3. Choose $v = \bar{u}$ in (Q), $v = u$ in ($\bar{\text{Q}}$), and add.
5. By (Q) we have
$$(f - u, v) - \varphi(v) \leq (f - u, u) - \varphi(u) \quad \forall v \in H$$
and thus $\varphi^\star(f - u) = (f - u, u) - \varphi(u)$. It follows that
$$\frac{1}{2}|u|^2 + \varphi^\star(f - u) = -\frac{1}{2}|f - u|^2 - \varphi(u) + \frac{1}{2}|f|^2.$$
Letting $u^\star = f - u$, one obtains
$$\frac{1}{2}|f - u^\star|^2 + \varphi^\star(u^\star) = -\frac{1}{2}|f - u|^2 - \varphi(u) + \frac{1}{2}|f|^2,$$
and one checks easily that
$$-\frac{1}{2}|f - u|^2 - \varphi(u) + \frac{1}{2}|f|^2 \leq \frac{1}{2}|f - v|^2 + \varphi^\star(v) \quad \forall v \in H.$$

(Recall that $(u, v) \le \varphi(u) + \varphi^\star(v)$.)

6. We have

$$(\mathrm{P}_\lambda) \qquad \frac{1}{2}|f - u_\lambda|^2 + \lambda\varphi(u_\lambda) \le \frac{1}{2}|f - v|^2 + \lambda\varphi(v) \quad \forall v \in H.$$

Using that fact that $\varphi(u_\lambda) \ge (g, u_\lambda) + C$, it is easy to see that $|u_\lambda|$ remains bounded as $\lambda \to 0$. We may therefore assume that $u_{\lambda_n} \rightharpoonup u_0$ weakly ($\lambda_n \to 0$) with $u_0 \in \overline{D(\varphi)}$ (why?). Passing to the limit in (P_{λ_n}) (how?), we obtain

$$\frac{1}{2}|f - u_0|^2 \le \frac{1}{2}|f - v|^2 \quad \forall v \in D(\varphi),$$

and we deduce that $u_0 = P_{\overline{D(\varphi)}} f$. The uniqueness of the limit implies that $u_\lambda \rightharpoonup u_0$ weakly as $\lambda \to 0$. To see that $u_\lambda \to u_0$ we note that

$$\frac{1}{2}\limsup_{\lambda\to 0}|f - u_\lambda|^2 \le \frac{1}{2}|f - v|^2 \quad \forall v \in D(\varphi),$$

which implies that $\limsup_{\lambda\to 0}|f - u_\lambda| \le |f - u_0|$ and the strong convergence follows.

Alternative proof. Combining (Q_λ) and (Q_μ) we obtain

$$\left(\frac{1}{\lambda}(u_\lambda - f) - \frac{1}{\mu}(u_\mu - f), u_\lambda - u_\mu\right) \le 0 \quad \forall \lambda, \mu > 0.$$

We deduce from Exercise 5.3, question 1, that $(u_\lambda - f)$ converges strongly as $\lambda \to 0$ to some limit. In order to identify the limit one may proceed as above.

7. We have $\frac{1}{2}|f - u_\lambda|^2 + \lambda\varphi(u_\lambda) \le \frac{1}{2}|f - v|^2 + \lambda\varphi(v)\ \forall v \in H$, and in particular, $|f - u_\lambda| \le |f - v|\ \forall v \in K$. We may therefore assume that $u_{\lambda_n} \rightharpoonup u_\infty$ weakly ($\lambda_n \to +\infty$) and we obtain $|f - u_\infty| \le |f - v|\ \forall v \in K$. On the other hand, we have

$$\varphi(u_\lambda) \le \frac{1}{2\lambda}|f - v|^2 + \varphi(v) \quad \forall v \in H,$$

and passing to the limit, we obtain $\varphi(u_\infty) \le \varphi(v)\ \forall v \in H$. Thus, $u_\infty \in K$, $u_\infty = P_K f$ (why?), and $u_\lambda \rightharpoonup u_\infty$ weakly as $\lambda \to +\infty$ (why?). Finally, note that $\limsup_{\lambda\to+\infty}|f - u_\lambda| \le |f - u_\infty|$.
If $K = \emptyset$, then $|u_\lambda| \to \infty$ as $\lambda \to +\infty$ (argue by contradiction).

8. If $f = 0$ check that $(1/\lambda)u_\lambda = -u^\star_{1/\lambda}\ \forall\lambda > 0$. In the general case (in which $f \ne 0$) denote by u_λ and $\overline{u_\lambda}$ the solutions of (P_λ) corresponding respectively to f and to 0. We know, by question 3, that $|u_\lambda - \overline{u_\lambda}| \le |f|$ and thus $|\frac{1}{\lambda}u_\lambda - \frac{1}{\lambda}\overline{u}_\lambda| \to 0$ as $\lambda \to +\infty$.

Problem 27

- A -

3. By definition of the projection we have

$$|u_{2n+2} - u_{2n+1}| = |P_2 u_{2n+1} - u_{2n+1}| \le |u_{2n} - u_{2n+1}|$$

(since $u_{2n} \in K_2$), and similarly

$$|u_{2n+1} - u_{2n}| = |P_1 u_{2n} - u_{2n}| \le |u_{2n-1} - u_{2n}|.$$

It follows that

$$|u_{2n+2} - u_{2n+1}| \le |u_{2n} - u_{2n-1}|.$$

- B -

To see that a_1 and a_2 may depend on u_0 take convex sets K_1 and K_2 as shown in Figure 9.

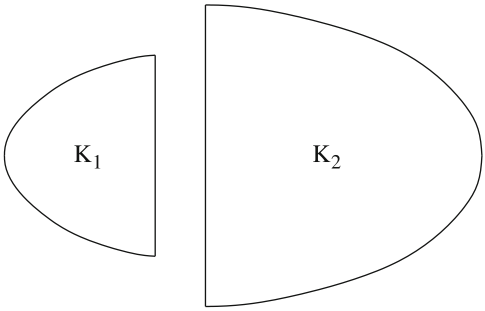

Fig. 9

Problem 28

-A -

1. (a) $\Rightarrow$ (b). Note that $(u, Pv) = (Pu, Pv) = (Pu, v)\ \forall u, v \in H$.

   (b) $\Rightarrow$ (c). We have $|Pu|^2 = (Pu, Pu) = (u, P^2u) = (u, Pu)\ \forall u \in H$.

   (c) $\Rightarrow$ (d). From (c) we have $((u - Pu) - (v - Pv), u - v) \ge 0\ \forall u, v \in H$ and therefore $(u, u - v) \ge 0\ \forall u \in N(P)$ and $\forall v \in N(I - P)$. Replacing u by λu, we obtain (d).

   (d) $\Rightarrow$ (a). Set $M = N(I - P)$ and check that $P = P_M$.

- B -

1. (b) $\Rightarrow$ (c). Note that $(PQ)^2 = PQ$ and pass to the adjoints.
   (c) $\Rightarrow$ (a). QP is a projection operator and $\|QP\| \le 1$. Thus QP is an orthogonal projection and therefore $(QP)^\star = QP$, that is, $PQ = QP$.

(i) Check that $N(I - PQ) = M \cap N$.

(ii) Applying (i) to $(I - P)$ and $(I - Q)$, we see that $(I - P)(I - Q)$ is the orthogonal projection onto $M^\perp \cap N^\perp$. Therefore $I - (I - P)(I - Q) = P + Q - PQ$ is the orthogonal projection onto $(M^\perp \cap N^\perp)^\perp = \overline{M + N}$.

2. It is easy to check that

$$\text{(a)} \Rightarrow \text{(b)} \Leftrightarrow \text{(c)} \Rightarrow \text{(d)} \Leftrightarrow \text{(e)} \Leftrightarrow \text{(f)} \Rightarrow \text{(a)}.$$

Clearly (b) + (c) $\Rightarrow$ (g). Conversely, we claim that (g) $\Rightarrow$ (b) + (c). Indeed, we have $PQ + QP = 0$. Multiplying this identity on the left and on the right by P, we obtain $PQ - QP = 0$; thus, $PQ = 0$. Finally, apply case (ii) of question B1.

3. Replace N by $N^\perp$ and apply question B2.

Problem 29

- A-

5. Note that $\sum_{i=0}^{n} |\mu_i - \mu_{i+1}|^2 \leq |f - v|^2$ and that $|\mu_n - \mu_{n+1}| \leq |\mu_i - \mu_{i+1}|$ $\forall i = 0, 1, \dots, n$.

- B -

2. Since $0 \in K$, the sequence $(|u_n|)$ is nonincreasing and thus it converges to some limit, say a. Applying the result of B1 with $u = u_n$ and $v = u_{n+i}$, we obtain

$$\begin{aligned} 2|(u_n, u_{n+i}) - (u_{n+p}, u_{n+p+i})| &\leq 2(|u_n|^2 - |u_{n+p+i}|^2) \\ &\leq 2(|u_n|^2 - a^2). \end{aligned}$$

Therefore $\ell(i) = \lim_{n\to\infty}(u_n, u_{u+i})$ exists and we have

$$|(u_n, u_{n+i}) - \ell(i)| \leq |u_n|^2 - a^2 \equiv \varepsilon_n.$$

3. Applying to S the above result, we see that

$$|\,(\mu_n, \mu_{n+i}) - m(i)| \leq \varepsilon_n' \quad \forall i,\ \forall n.$$

In particular, we have

$$||\mu_n|^2 - m(0)| \leq \varepsilon_n' \quad \text{and} \quad |\,(\mu_n, \mu_{n+1}) - m(1)| \leq \varepsilon_n'$$

and therefore

$$|m(0) - m(1)| \leq 2\varepsilon_n' + |\mu_n||\mu_n - \mu_{n+1}| \to 0 \quad \text{as } n \to \infty.$$

It follows that $m(0) = m(1)$ and similarly, $m(1) = m(2)$, etc.

4. We have established that $|(\mu_n, \mu_{n+i}) - m(0)| \le \varepsilon_n' \ \forall i, \forall n$. Passing to the limit as $i \to \infty$ we obtain $|(\mu_n, \mu) - m(0)| \le \varepsilon_n'$ and then, as $n \to \infty$, we obtain $|\mu|^2 = m(0)$. Thus, $|\mu_n| \to |\mu|$ and consequently $\mu_n \to \mu$ strongly.
5. Applying (1) and adding the corresponding inequalities for $i = 0, 1, \dots, p-1$, leads to
$$\left|\left(u_n, \frac{(n+p)}{p}\sigma_{n+p} - \frac{n}{p}\sigma_n\right) - X_p\right| \le \varepsilon_n.$$
We deduce that
$$\left|\left(u_n, \sigma_{n+p}\right) - X_p\right| \le \varepsilon_n + \frac{n}{p}|u_n|\left(|\sigma_{n+p}| + |\sigma_n|\right) \le \varepsilon_n + \frac{2n}{p}|f|^2$$
(since $|u_n| \le |f| \ \forall n$).
6. We have
$$|X_p - X_q| \le 2\varepsilon_n + 2n\left(\frac{1}{p} + \frac{1}{q}\right)|f|^2 + |(u_n, \sigma_{n+p} - \sigma_{n+q})|$$
and thus $\limsup_{p,q\to\infty} |X_p - X_q| \le 2\varepsilon_n \ \forall n$.
7. Write that
$$n^2|\sigma_n|^2 = \sum_{i=0}^{n-1}|u_i|^2 + 2\sum_{i=1}^{n-1}\sum_{j=0}^{n-i-1}(u_j, u_{j+i})$$
and apply (1).
8. Note that $\sum_{i=0}^{n-1}(n-i)\ell(i) = \sum_{j=1}^{n} jX_j$ and use the fact that $X_j \to X$ as $j \to \infty$.

Problem 30

- C -

3. Choose $\bar{\lambda} \in A$ and $\bar{\mu} \in B$ such that
$$\min_{\lambda\in A}\max_{\mu\in B} F(\lambda, \mu) = \max_{\mu\in B} F(\bar{\lambda}, \mu) \quad \text{and} \quad \max_{\mu\in B}\min_{\lambda\in A} F(\lambda, \mu) = \min_{\lambda\in A} F(\lambda, \bar{\mu}).$$

- D -

2. The sets B_u and A_v are compact for the weak topology. Applying the convexity of K in u and the concavity of K in v, we obtain
$$K\left(\sum_i \lambda_i u_i, v_j\right) \le \sum_i \lambda_i K(u_i, v_j)$$
and

$$K\left(u_i, \sum_j \mu_j v_j\right) \geq \sum_j \mu_j K(u_i, v_j).$$

It follows that

$$\sum_j \mu_j K\left(\sum_i \lambda_i u_i, v_j\right) \leq F(\lambda, \mu) \leq \sum_i \lambda_i K\left(u_i, \sum_j \mu_j v_j\right)$$

and in particular

$$\sum_j \mu_j K(\bar{u}, v_j) \leq F(\bar{\lambda}, \mu) \quad \forall \mu \in B',$$
$$\sum_i \lambda_i K(u_i, \bar{v}) \geq F(\lambda, \bar{\mu}) \quad \forall \lambda \in A'.$$

Applying (1), we see that

$$\sum_j \mu_j K(\bar{u}, v_j) \leq \sum_i \lambda_i K(u_i, \bar{v}) \quad \forall \lambda \in A',\ \forall \mu \in B'.$$

Finally, choose λ and μ to be the elements of the canonical basis.

Problem 31

- A -

3. Note that $\sum_{i,j} \lambda_i \lambda_j \langle Av_j, v_i - v_j\rangle = \frac{1}{2}\sum_{i,j} \lambda_i \lambda_j \langle Av_j - Av_i, v_i - v_j\rangle$.

- B -

2. For every $R > 0$ there exists some $u_R \in K_R$ such that $\langle Au_R, v - u_R\rangle \geq 0\, \forall v \in K_R$. Choosing $v = 0$ we see that there exists a constant M (independent of R) such that $\|u_R\| \leq M\ \forall R$. Fix any $R > M$. Given $w \in K$, take $v = (1-t)u_R + tw$ with $t > 0$ sufficiently small (so that $v \in K_R$).
3. Take $K = E$. First, prove that there exists some $u \in E$ such that $Au = 0$. Then, replace A by the map $v \mapsto Av - f$ ($f \in E^\star$ being fixed).

Problem 32

2. For $\varepsilon > 0$ small enough we have

$$\begin{aligned}\varphi(u_0) \leq \varphi(u_0 + \varepsilon v) &= \max_{i \in I}\{|u_0 - y_i|^2 - c_i + 2\varepsilon(v, u_0 - u_i)\} + O(\varepsilon^2)\\ &= \max_{i \in J(u_0)}\{|u_0 - y_i|^2 - c_i + 2\varepsilon(v, u_0 - y_i)\} + O(\varepsilon^2)\\ &\leq \varphi(u_0) + 2\varepsilon \max_{i \in J(u_0)}\{(v, u_0 - y_i)\} + O(\varepsilon^2).\end{aligned}$$

3. Argue by contradiction and apply Hahn–Banach.
4. Note that for every $u, v \in H$, we have

$$\varphi(v) - \varphi(u) \geq \max_{i \in J(u)} \{|v - y_i|^2 - |u - y_i|^2\} \geq 2 \max_{i \in J(u)} \{(u - y_i, v - u)\}.$$

5. Condition (1) is replaced by $0 \in \text{conv}\,(\bigcup_{i \in J(u_0)} \{f'(u_0)\})$.

- B -

1. Letting $\sigma_x = \sum_{i \in I} \lambda_i x_i$ and $\sigma_y = \sum_{i \in I} \lambda_i y_i$, we obtain

$$\begin{aligned}\sum_{j \in I} \lambda_j |\sigma_y - y_j|^2 = -|\sigma_y|^2 + \sum_{j \in I} \lambda_j |y_j|^2 &= \frac{1}{2} \sum_{i,j \in I} \lambda_i \lambda_j |y_i - y_j|^2 \\ &\leq \frac{1}{2} \sum_{i,j \in I} \lambda_i \lambda_j |x_i - x_j|^2 = -|\sigma_x|^2 + \sum_{j \in I} \lambda_j |x_j|^2 \\ &= -|\sigma_x - p|^2 + \sum_{j \in I} \lambda_j |x_j - p|^2.\end{aligned}$$

2. Write $u_0 = \sum_{i \in J(u_0)} \lambda_j y_j$. By the result of B1 we have

$$\sum_{j \in J(u_0)} \lambda_j |u_0 - y_j|^2 \leq \sum_{j \in J(u_0)} \lambda_j |p - x_j|^2.$$

   It follows that $\sum_{j \in J(u_0)} \lambda_j \varphi(u_0) \leq 0$ and thus $\varphi(u_0) \leq 0$.

Remark. One could also establish the existence of q by applying the von Neumann min–max theorem (see Problem 30, part D) to the function

$$K(\lambda, \mu) = \sum_{j \in I} \mu_j \left| \left(\sum_{i \in I} \lambda_i y_i \right) - y_j \right|^2 - \sum_{j \in I} \mu_j |p - x_j|^2.$$

- C -

1. Set $K_i = \{z \in H; |z - y_i| \leq |p - x_i|\}$ and $K = \overline{\text{conv}\left(\bigcup_{i \in I} \{y_i\}\right)}$. One has to show that $\left(\bigcap_{i \in I} K_i\right) \neq \emptyset$. This is done by contradiction and reduction to a finite set I.
2. Consider the ordered set of all contractions $T : D(T) \subset H \to H$ that extend S and such that $T(D(T)) \subset \overline{\text{conv}\ S(D)}$. By Zorn's lemma it has a maximal element T_0 and $D(T_0) = H$ (why?).

Problem 33

1. Note that $|au_n| \leq n|u|$, so that $u_n \in D(A)$. Moreover, $|u_n| \leq |u|$ and $u_n \to u$ a.e.
2. Let $u_n \to u$ and $au_n \to f$ in L^p. Passing to a subsequence, we may assume that $u_n \to u$ a.e. and $au_n \to f$ a.e. Thus $au = f$.
3. If $D(A) = E$, the closed graph theorem (Theorem 2.9) implies the existence of a constant C such that
$$\int_\Omega |au|^p \leq C \int_\Omega |u|^p \quad \forall u \in L^p.$$
Hence the mapping $v \mapsto \int_\Omega |a|^p v$ is a continuous linear functional on L^1. By Theorem 4.14 there exists $f \in L^\infty$ such that
$$\int_\Omega |a|^p v = \int_\Omega f v \quad \forall v \in L^1.$$
Thus $a \in L^\infty$.
4. $N(A) = \{u \in L^p;\, u = 0 \text{ a.e. on } [a \neq 0]\}$ and $N(A)^\perp = \{f \in L^{p'};\, f = 0 \text{ a.e. on } [a = 0]\}$.
To verify the second assertion, let $f \in N(A)^\perp$. Then $\int_\Omega fu = 0\ \forall u \in N(A)$. Taking $u = |f|^{p'-2} f \chi_{[a=0]}$, we see that $f = 0$ a.e. on $[a = 0]$.
5. $D(A^\star) = \{v \in L^{p'};\, av \in L^{p'}\}$ and $A^\star v = av$.

Indeed, if $v \in D(A^\star)$, there exists a constant C such that
$$\left| \int_\Omega v(au) \right| \leq C\|u\|_p \quad \forall u \in D(A).$$
The linear functional $u \in D(A) \mapsto \int_\Omega v(au)$ can be extended by Hahn–Banach (or by density) to a continuous linear functional on all of L^p. Hence, by Theorem 4.11, there exists some $f \in L^{p'}$ such that
$$\int_\Omega v(au) = \int_\Omega fu \quad \forall u \in D(A).$$
Given any $\varphi \in L^p$, take $u = (1 + |a|)^{-1}\varphi$, so that
$$\int_\Omega \frac{av}{1+|a|}\varphi = \int_\Omega \frac{f}{1+|a|}\varphi.$$
Thus $f = av \in L^{p'}$.
6. Assume that there exists $\alpha > 0$ such that $|a(x)| \geq \alpha$ a.e. Then A is surjective, since any $f \in L^p$ can be written as $au = f$, where $u = a^{-1}f \in D(A)$. Conversely, assume that A is surjective. Then $a \neq 0$ a.e. Moreover, $\forall f \in L^p$, $a^{-1}f \in L^p$. Applying question 3 to the function a^{-1}, we see that $a^{-1} \in L^\infty$.

7. $$EV(A) = \{\lambda \in \mathbb{R};\ |[a = \lambda]| > 0\},$$
$$\rho(A) = \{\lambda \in \mathbb{R};\ \exists \varepsilon > 0 \text{ such that } |a(x) - \lambda| \geq \varepsilon \text{ a.e. on } \Omega\},$$

and

$$\sigma(A) = \{\lambda \in \mathbb{R};\quad \forall \varepsilon > 0, |[|a - \lambda| < \varepsilon]| > 0\}.$$

Set $M = \sup_\Omega a$ and let us show that $M \in \sigma(A)$. By definition of M we know that $a \leq M$ a.e. on Ω and $\forall \varepsilon > 0$ $|[a > M - \varepsilon]| > 0$. Thus, $\forall \varepsilon > 0$ $|[|a - M| < \varepsilon]| > 0$ and therefore $M \in \sigma(A)$.

Note that $\sigma(A)$ coincides with the smallest closed set $F \subset \mathbb{R}$ such that $a(x) \in F$ a.e. in Ω. (The existence of a smallest such set can be established as in Proposition 4.17.)

10. Let us show that $\sigma(A) = \{0\}$. Let $\lambda \in \sigma(A)$ with $\lambda \neq 0$. Then $\lambda \in EV(A)$ (by Theorem 6.8) and thus $\big|[a = \lambda]\big| > 0$. Set $\omega = [a = \lambda]$. Then $N(A - \lambda I)$ is a finite-dimensional space not reduced to $\{0\}$. On the other hand, $N(A - \lambda I)$ is clearly isomorphic to $L^p(\omega)$. Then ω consists of a finite number of atoms (see Remark 6 in Chapter 4) and it has at least one atom, since $L^p(\omega)$ is not reduced to $\{0\}$. Impossible.

Problem 34

- A -

1. Clearly $0 \notin EV(T)$. Assume that $\lambda \in EV(T)$ and $\lambda \neq 0$. Let u be the corresponding eigenfunction, so that

$$\frac{1}{x} \int_0^x u(t)dt = \lambda u(x).$$

Thus $u \in C^1((0, 1])$ and satisfies

$$u = \lambda u + \lambda x u'.$$

Integrating this ODE, we see that $u(x) = Cx^{-1+1/\lambda}$, for some constant C. Since $u \in C([0, 1])$, we must have $0 < \lambda \leq 1$. Conversely, any $\lambda \in (0, 1]$ is an eigenvalue with corresponding eigenspace $Cx^{-1+1/\lambda}$.

3. We already know that $[0, 1] \subset \sigma(T) \subset [-1, +1]$. We will now prove that for any $\lambda \in \mathbb{R}$, $\lambda \notin \{0, 1\}$, the equation

$$\text{(S1)} \qquad Tu - \lambda u = f \in E$$

admits at least one solution $u \in E$.

Assuming that we have a solution u, set $\varphi(x) = \int_0^x u(t)dt$. Then

$$\varphi - \lambda x \varphi' = xf,$$

and hence we must have

$$\varphi(x) = \frac{1}{\lambda} x^{1/\lambda} \int_x^1 t^{-1/\lambda} f(t)dt + Cx^{1/\lambda},$$

for some constant C. Therefore

$$\text{(S2)} \quad u(x) = \varphi'(x) = \frac{1}{\lambda^2} x^{-1+1/\lambda} \int_x^1 t^{-1/\lambda} f(t)dt - \frac{1}{\lambda} f(x) + \frac{C}{\lambda} x^{-1+1/\lambda}.$$

If $\lambda < 0$ or if $\lambda > 1$ we must choose

$$\text{(S3)} \qquad C = -\frac{1}{\lambda} \int_0^1 t^{-1/\lambda} f(t)dt$$

in order to make u continuous at $x = 0$, and then the unique solution u of (S1) is given by

$$\text{(S4)} \qquad u(x) = (T - \lambda I)^{-1} f = -\frac{1}{\lambda^2} x^{-1+1/\lambda} \int_0^x t^{-1/\lambda} f(t)dt - \frac{1}{\lambda} f(x),$$

with

$$u(0) = \frac{1}{1-\lambda} f(0).$$

It follows that $\sigma(T) = [0, 1]$ and $\rho(T) = (-\infty, 0) \cup (1, \infty)$.

When $0 < \lambda < 1$, the function u given by

$$\text{(S5)} \qquad u(x) = \frac{1}{\lambda^2} x^{-1+1/\lambda} \int_x^1 t^{-1/\lambda} f(t)dt - \frac{1}{\lambda} f(x),$$

with

$$u(0) = \frac{1}{1-\lambda} f(0),$$

is still a solution of (S1). But the solution of (S1) is not unique, since we can add to u any multiple of $x^{-1+1/\lambda}$. Hence, for $\lambda \in (0, 1)$, the operator $(T - \lambda I)$ is surjective but not injective.

When $\lambda = 0$, the operator T is injective but not surjective. Indeed for every $u \in E$, $Tu \in C^1((0, 1])$.

When $\lambda = 1$, $(T - I)$ is not injective and is not surjective. We already know that $N(T - I)$ consists of constant functions. Suppose now that u is a solution of $Tu - u = f$. Then $f(0) = u(0) - u(0) = 0$ and therefore $(T - I)$ is not surjective.

4. A direct computation gives

$$\|T_\varepsilon u - Tu\|_{L^q(0,1)} \le \frac{\varepsilon^{1/q}}{(q-1)^{1/q}} \|u\|_{L^\infty(0,1)} \quad \text{if } q > 1,$$

and

$$\|T_\varepsilon u - Tu\|_{L^1(0,1)} \le \varepsilon \log(1+1/\varepsilon)\|u\|_{L^\infty(0,1)}.$$

Thus $\|T_\varepsilon - T\|_{\mathcal{L}(E,F)} \to 0$. Clearly $T_\varepsilon \in \mathcal{K}(E,F)$ (why?), and we may then apply Theorem 6.1 to conclude that $T \in \mathcal{K}(E,F)$.

- B -

1. It is convenient to write

$$Tu(x) = \frac{1}{x}\int_0^x u(t)dt = \int_0^1 u(xs)ds$$

and therefore

$$(Tu)'(x) = \int_0^1 u'(xs)sds.$$

2. Assume that $\lambda \in EV(T)$. By question A1 the corresponding eigenfunction must be $u(x) = Cx^{-1+1/\lambda}$. This function belongs to $C^1([0,1])$ only when $0 < \lambda \le 1/2$ or $\lambda = 1$.
3. We will show that if $\lambda \notin [0, \frac{1}{2}] \cup \{1\}$, then $(T - \lambda I)$ is bijective. Consider the equation

$$Tu - \lambda u = f \in C^1([0,1]).$$

When $\lambda < 0$ or $\lambda > 1$ we know, by part A, that if a solution exists, it must be given by (S4). Rewrite it as

$$u(x) = -\frac{1}{\lambda^2}\int_0^1 s^{-1/\lambda} f(xs)ds - \frac{1}{\lambda}f(x),$$

and thus $u \in C^1([0,1])$.

When $1 > \lambda > 1/2$, we know from part A that (S1) admits solutions $u \in C([0,1])$. Moreover, all solutions u are given by (S2). We will see that there is a (unique) choice of the constant C in (S2) such that $u \in C^1([0,1])$.
Write

$$u(x) = x^{-1+1/\lambda}\left[\frac{1}{\lambda^2}\int_x^1 t^{-1/\lambda}(f(t) - f(0))dt + \frac{f(0)}{\lambda^2 - \lambda} + \frac{C}{\lambda}\right] - \frac{f(x)}{\lambda} - \frac{f(0)}{\lambda^2 - \lambda}.$$

A natural choice for C is such that

$$\frac{1}{\lambda^2}\int_0^1 t^{-1/\lambda}(f(t) - f(0))dt + \frac{f(0)}{\lambda^2 - \lambda} + \frac{C}{\lambda} = 0,$$

and then u becomes

$$u(x) = -\frac{1}{\lambda^2} x^{-1+1/\lambda} \int_0^x t^{-1/\lambda}(f(t) - f(0))dt - \frac{f(x)}{\lambda} - \frac{f(0)}{\lambda^2 - \lambda}.$$

Changing variables yields

$$u(x) = -\frac{1}{\lambda^2} \int_0^1 s^{-1/\lambda}(f(xs) - f(0))ds - \frac{f(x)}{\lambda} - \frac{f(0)}{\lambda^2 - \lambda}.$$

Direct inspection shows that indeed $u \in C^1([0, 1])$ with

$$u'(x) = -\frac{1}{\lambda^2} \int_0^1 s^{1-1/\lambda} f'(xs)ds - \frac{f'(x)}{\lambda}.$$

- C -

1. We have

$$\int_0^1 |Tu(x)|^p dx = -\frac{1}{p-1}|\varphi(1)|^p + \frac{p}{p-1}\int_0^1 |\varphi(x)|^{p-1}(\operatorname{sign}\varphi(x))\varphi'(x)\frac{dx}{x^{p-1}},$$

and therefore, by Hölder,

$$\int_0^1 |Tu(x)|^p dx \le \frac{p}{p-1}\left[\int_0^1 \left|\frac{\varphi(x)}{x}\right|^p dx\right]^{\frac{p-1}{p}} \left[\int_0^1 |\varphi'(x)|^p dx\right]^{\frac{1}{p}},$$

i.e.,

$$\|Tu\|_p^p \le \frac{p}{p-1}\|Tu\|_p^{p-1}\|u\|_p.$$

3. Clearly $0 \notin EV(T)$. Suppose that $\lambda \in EV(T)$ and $\lambda \neq 0$. As in part A we see that the corresponding eigenfunction is $u = Cx^{-1+1/\lambda}$. This function belongs to $L^p(0, 1)$ iff $0 < \lambda < p/(p-1)$.
5. Assume that $\lambda < 0$. Let us prove that $\lambda \in \rho(T)$. For $f \in C([0, 1])$, let Sf be the right-hand side in (S4). Clearly

$$|Sf(x)| \le \frac{1}{\lambda^2}\frac{1}{x}\int_0^x |f(t)|dt + \frac{1}{\lambda}|f(x)|.$$

Therefore S can be extended as a bounded operator from $L^p(0, 1)$ into itself. Since we have

$$(T - \lambda I)S = S(T - \lambda I) = I \quad \text{on } C([0, 1]),$$

the same holds on $L^p(0, 1)$. Consequently $\lambda \in \rho(T)$.

Suggestion for further investigation: prove that for $\lambda \in (0, \frac{p}{p-1})$ the operator $(T - \lambda I)$ is surjective from $L^p(0, 1)$ onto itself. *Hint*: start with formula (S5) and show that $\|u\|_p \le C\|f\|_p$ using the same method as in questions C1 and C2.

6. $(T^\star v)(x) = \int_x^1 \frac{v(t)}{t} dt$.
7. Check that T_ε is a compact operator from $L^p(0,1)$ into $C([0,1])$ with the help of Ascoli's theorem. Then prove that

$$\|T_\varepsilon - T\|_{\mathcal{L}(L^p, L^q)} \leq C\varepsilon^{\frac{1}{q}-\frac{1}{p}}.$$

Problem 35

- A -

1. Clearly $\|T^\star T\| \leq \|T\|^2$. On the other hand,

$$|Tx|^2 = (Tx, Tx) = (T^\star Tx, x) \leq \|T^\star T\| \, |x|^2.$$

Thus $\|T\|^2 \leq \|T^\star T\|$.
2. By induction we have

$$\|T^{2^k}\| = \|T\|^{2^k} \quad \forall \text{ integer } k.$$

Given any integer N, fix k such that $N \leq 2^k$.
Then

$$\|T\|^{2^k} = \|T^{2^k}\| = \|T^N T^{2^k - N}\| \leq \|T^N\| \, \|T\|^{2^k - N},$$

and thus

$$\|T\|^N \leq \|T^N\|.$$

- B -

Set

$$X = \|T_{j_1}^\star T_{k_1} T_{j_2}^\star T_{k_2} \cdots T_{j_N}^\star T_{k_N}\|.$$

By assumption (1) we have

$$X \leq \omega^2(j_1 - k_1)\omega^2(j_2 - k_2) \cdots \omega^2(j_N - k_N),$$

and by assumption (2),

$$\begin{aligned} X &\leq \|T_{j_1}^\star\| \omega^2(k_1 - j_2)\omega^2(k_2 - j_3) \cdots \omega^2(k_{N-1} - j_N)\|T_{k_N}\| \\ &\leq \omega^2(0)\omega^2(k_1 - j_2)\omega^2(k_2 - j_3) \cdots \omega^2(k_{N-1} - j_N), \end{aligned}$$

since $\|T_i\| = \|T_i^\star T_i\|^{1/2} \leq \omega(0)$.

Multiplying the above estimates, we obtain

$$X \leq \omega(0)\omega(j_1 - k_1)\omega(k_1 - j_2) \cdots \omega(j_{N-1} - k_{N-1})\omega(k_{N-1} - j_N)\omega(j_N - k_N).$$

Summing over k_N, then over j_N, then over k_{N-1}, then over $j_{N-1}, \ldots$, then over k_2, then over j_2, then over k_1, yields a bound by σ^{2N}. Finally, summing over j_1 gives the bound $m\sigma^{2N}$.

3. We have
$$\|(U^\star U)^N\| \leq \sum_{j_1}\sum_{k_1}\sum_{j_2}\sum_{k_2}\cdots\sum_{j_N}\sum_{k_N}\|T^\star_{j_1}T_{k_1}T^\star_{j_2}T_{k_2}\cdots T^\star_{j_N}T_{k_N}\| \leq m\sigma^{2N}.$$
Therefore
$$\|U\| \leq m^{1/2N}\sigma,$$
and the desired conclusion follows by letting $N \to \infty$.

Problem 36

1. To see that R_k is closed, note that $(I - T)^k = I - S$ for some $S \in \mathcal{K}(E)$ and apply Theorem 6.6.
2. Suppose $R_{q+1} = R_q$ for some $q \geq 1$. Then $R_{k+1} = R_k \ \forall k \geq q$. On the other hand, we cannot have $R_{k+1} \neq R_k \ \forall k \geq 1$ (see part (c) in the proof of Theorem 6.6).

4. From Theorem 6.6(b) and (d) we have
$$R_k = N((I - T^\star)^k)^\perp$$
and thus
$$\text{codim } R_k = \dim N((I - T^\star)^k) = \dim N((I - T)^k) = \dim N_k.$$

5. Let $x \in R_p \cap N_p$. Then $x = (I - T)^p\xi$ for some $\xi \in E$ and $(I - T)^p x = 0$. It follows that $\xi \in N_{2p} = N_p$ and thus $x = 0$. On the other hand,
$$\text{codim } R_p = \dim N_p;$$
combining this with the fact that $R_p \cap N_p = \{0\}$, we conclude that $E = R_p + N_p$.
6. $(I - T)R_p = R_{p+1} = R_p$. Theorem 6.6(c) applied in the space R_p allows us to conclude that $(I - T)$ is also injective on R_p.
7. It suffices to show that $N_2 = N_1$. Let $x \in N_2$. Then $(I - T)^2 x = 0$ and thus $|(I - T)x|^2 = ((I - T)x, (I - T)x) = ((I - T)^2 x, x) = 0$.

Problem 37

10. From question 9 we know that $\nu_k^{(n)}$ is nondecreasing in n and $\nu_k^{(n)} \leq \mu_k \ \forall k \geq 1$ and $\forall n$. Thus it suffices to prove that
$$\text{(S1)} \qquad \liminf_{n\to\infty} \nu_k^{(n)} \geq \mu_k.$$
In fact, using question 9, one has, $\forall k < n$,
$$\max_{\substack{\Sigma\subset V^{(n)}\\ \dim \Sigma = k}} \min_{\substack{x\in\Sigma\\ x\neq 0}} R(x) = \nu_k^{(n)}.$$

Note that from the assumption on $V^{(n)}$,

(S2) $$\lim_{n\to\infty} P_{V^{(n)}}(x) = x \quad \forall x \in H.$$

Thus $P_{V^{(n)}}(e_1), \dots, P_{V^{(n)}}(e_k)$ are linearly independent for $n \geq N_k$ sufficiently large (depending on k, but recall that k is fixed); this implies

(S3) $$\nu_k^{(n)} \geq \min_{\substack{x\in E(n,k)\\ x\neq 0}} R(x),$$

where $E(n,k)$ is the space spanned by $\{P_{V^{(n)}}(e_1), \dots, P_{V^{(n)}}(e_k)\}$. However, it is clear from (S2) that

(S4) $$\lim_{n\to\infty} \min_{\substack{x\in E(n,k)\\ x\neq 0}} R(x) = \min_{\substack{x\in E_k\\ x\neq 0}} R(x).$$

Inequality (S1) follows from (S3), (S4), and question 1.

Problem 38

- A -

2. Use Exercise 6.25 or apply question 1 to the operator $(I_F + K)$, that satisfies (1) (why?). Then write
$$T \circ (S \circ M) = I_F - P.$$

3. Clearly $R(I_F - P)$ is closed and codim $R(I_F - P)$ is finite. By Proposition 11.5 we know that any space $X \supset R(I_F - P)$ is also closed and has finite codimension. In particular, (1)(a) holds.

   Next, we have
$$U^\star \circ T^\star = I_{F^\star} - P^\star,$$
   where $P^\star$ is a compact operator (since P is). Thus we may argue as above and conclude that $R(U^\star)$ is closed. From Theorem 2.19 we infer that $R(U)$ is also closed.

   We now prove that $N(T)+R(U)+\Sigma_1 = E$ for some finite-dimensional space Σ_1. Given any $x \in E$, write $x = x_1+x_2$ with $x_1 = x-U(Tx)$ and $x_2 = U(Tx)$. Note that $Tx_1 = Tx - (T \circ U)(Tx) = P(Tx)$ by (3). Therefore $x_1 \in T^{-1}(R(P)) = N(T) + \Sigma_1$, where Σ_1 is finite-dimensional, since $R(P)$ is. Consequently, any $x \in E$ belongs to $N(T) + R(U) + \Sigma_1$.

   Finally, we prove that $N(T) \cap R(U) \subset \Sigma_2$ with Σ_2 finite-dimensional. Indeed, let $x \in N(T) \cap R(U)$. Then $x = Uy$ for some $y \in F$ and $Tx = (T \circ U)(y) = 0$. Thus, by (3), $y - Py = 0$ and therefore $y \in R(P)$. Consequently $x \in U(R(P)) = \Sigma_2$, which is finite-dimensional, since $R(P)$ is. Applying Proposition 11.7, we conclude that $N(T)$ admits a complement in E.

- B -

(4) $\Rightarrow$ (6). Let U_0 be as in question 1 of part A. Then $U_0 \circ T = I$ on X. Given any $x \in E$ write $x = x_1 + x_2$ with $x_1 \in X$ and $x_2 \in N(T)$. Then

$$(U_0 \circ T)(x) = (U_0 \circ T)(x_1) = x_1 = x - x_2 = x - \widetilde{P}x,$$

where $\widetilde{P}$ is a finite-rank projection onto $N(T)$.

(5) $\Rightarrow$ (6). Use Exercise 2.26.

(6) $\Rightarrow$ (4). From (6) it is clear that $\dim N(T) < \infty$. Also, since $T^\star \circ (\widetilde{U})^\star = I_{E^\star} - (\widetilde{P})^\star$, we may apply part A ((2) $\Rightarrow$ (1)) to $T^\star$ in $E^\star$ and deduce that $R(T^\star)$ is closed in $E^\star$. Therefore $R(T)$ is closed in F.

As in question 3 of part A, we construct finite-dimensional spaces Σ_3 and Σ_4 in F such that

$$N(\widetilde{U}) + R(T) + \Sigma_3 = F,$$
$$N(\widetilde{U}) \cap R(T) \subset \Sigma_4,$$

and we conclude (using Proposition 11.7) that $R(T)$ admits a complement.

- C -

1. Note that $Q \circ T = T$ and thus $U \circ T = U_0 \circ Q \circ T = U_0 \circ T = I - \widetilde{P}$.
2. Use (2) $\Rightarrow$ (1) and (5) $\Rightarrow$ (4).
3. Let $Z \subset F$ be a closed subspace. From Proposition 11.13 we know that Z has finite codimension iff $Z^\perp$ is finite-dimensional, and then $\operatorname{codim} Z = \dim Z^\perp$. Apply this to $Z = R(T)$, with $Z^\perp = N(T^\star)$ (by Proposition 2.18).
4. We already know that $\dim N(T^\star) = \operatorname{codim} R(T) < \infty$. Next, we have $\dim N(T) < \infty$, and thus $\operatorname{codim} N(T)^\perp < \infty$ (by Proposition 11.13). But $N(T)^\perp = R(T^\star)$ (by Theorem 2.19). Therefore $\operatorname{codim} R(T^\star) < \infty$ and, moreover, $\operatorname{codim} R(T^\star) = \dim N(T)$.
5. From Theorem 2.19 we know that $R(T)$ is closed. Since $N(T^\star) = R(T)^\perp$ is finite-dimensional, Proposition 11.11 yields that $\operatorname{codim} R(T) < \infty$. Since $R(T^\star) = N(T)^\perp$ and $\operatorname{codim} R(T^\star) < \infty$, we deduce from Proposition 11.11 that $\dim N(T) < \infty$.
6. Write $T = J(I_E + J^{-1} \circ K)$. By Theorem 6.6 we know that $(I_E + J^{-1} \circ K) \in \Phi(E, E)$ and $\operatorname{ind}(I_E + J^{-1} \circ K) = 0$. Thus $T \in \Phi(E, F)$ and $\operatorname{ind} T = 0$, since J is an isomorphism.
   Conversely, assume that $T \in \Phi(E, F)$ and $\operatorname{ind} T = 0$. Let X be a complement of $N(T)$ in E and let Y be a complement of $R(T)$ in F. Since $\operatorname{ind} T = 0$, we have $\dim N(T) = \dim Y$. Let Λ be an isomorphism from $N(T)$ onto Y. Given $x \in E$, write $x = x_1 + x_2$ with $x_1 \in X$ and $x_2 \in N(T)$. Set $Jx = Tx_1 + \Lambda x_2$. Clearly J is bijective and $Tx = Tx_1 = Jx - \Lambda x_2$ is a desired decomposition.
7. Use a pseudoinverse.
8. Let X and Y be as in question 6. Set $\widetilde{E} = E \times Y$ and $\widetilde{F} = F \times N(T)$. Consider the operator $\widetilde{T} : \widetilde{E} \to \widetilde{F}$ defined by

$$\widetilde{T}(x, y) = (Tx + Kx, 0).$$

Clearly

$$R(\widetilde{T}) = R(T + K) \times \{0\},$$
$$N(\widetilde{T}) = N(T + K) \times Y.$$

Thus $\widetilde{T} \in \Phi(\widetilde{E}, \widetilde{F})$ and

$$\operatorname{codim} R(\widetilde{T}) = \operatorname{codim} R(T + K) + \dim N(T),$$
$$\dim N(\widetilde{T}) = \dim N(T + K) + \dim Y = \dim N(T + K) + \operatorname{codim} R(T).$$

We claim that $\widetilde{T} = \widetilde{J} + \widetilde{K}$, where $\widetilde{J}$ is bijective from $\widetilde{E}$ onto $\widetilde{F}$, and $\widetilde{K} \in \mathcal{K}(\widetilde{E}, \widetilde{F})$. Indeed, writing $x = x_1 + x_2$, with $x_1 \in X$ and $x_2 \in N(T)$, we have

$$\widetilde{T}(x, y) = (Tx_1 + Kx, 0) = \widetilde{J}(x, y) + \widetilde{K}(x, y),$$

where

$$\widetilde{J}(x, y) = (Tx_1 + y, x_2)$$

and

$$\widetilde{K}(x, y) = (Kx, 0) - (y, x_2).$$

Clearly $\widetilde{J}$ is bijective and $\widetilde{K}$ is compact (since y and x_2 are finite-dimensional variables). Applying question 6, we see that

$$\operatorname{ind} \widetilde{T} = 0 = \dim N(\widetilde{T}) - \operatorname{codim} R(\widetilde{T}).$$

It follows that

$$\operatorname{ind}(T + K) = \dim N(T + K) - \operatorname{codim} R(T + K) = \dim N(T) - \operatorname{codim} R(T).$$

9. Let V be a pseudoinverse of T and set $\varepsilon = \|V\|^{-1}$ (any $\varepsilon > 0$ if $V = 0$). From (8)(b) we have

$$V \circ (T + M) = I_E + (V \circ M) + \widetilde{K}.$$

If $\|M\| < \varepsilon$ we see that $\|V \circ M\| < 1$, and thus $W = I_E + (V \circ M)$ is bijective from E onto E (see Proposition 6.7). Multiplying the equation

$$V \circ (T + M) = W + \widetilde{K}$$

on the left by T and using (8)(a) yields

$$T + M = (T \circ W) + (T \circ \widetilde{K}) - K \circ (T + M).$$

Since W is bijective, it is clear (from the definition of $\Phi(E, F)$) that $T \circ W \in \Phi(E, F)$ and $\operatorname{ind}(T \circ W) = \operatorname{ind} T$. Applying the previous question, we conclude that $T + M \in \Phi(E, F)$ and $\operatorname{ind}(T + M) = \operatorname{ind}(T \circ W) = \operatorname{ind} T$.

11. Check that $V_1 \circ V_2$ is a pseudoinverse for $T_2 \circ T_1$.

12. Note that $H_0(x_2, x_2) = (T_1x_1, T_2x_2)$, so that $\operatorname{ind} H_0 = \operatorname{ind} T_1 + \operatorname{ind} T_2$. On the other hand, $H_1(x_2, x_2) = (x_2, -T_2(T_1x_1))$, so that $\operatorname{ind} H_1 = \operatorname{ind}(T_2 \circ T_1)$.
13. $\operatorname{ind} V = -\operatorname{ind} T$ by (8), questions 6 and 12.

- D -

1. $\operatorname{ind} T = \dim E - \dim F$, since $\dim R(T) = \dim E - \dim N(T)$ and $\operatorname{codim} R(T) = \dim F - \dim R(T)$.
2. When $|\lambda| < 1$, $\operatorname{ind}(S_r - \lambda I) = -1$ and $\operatorname{ind}(S_\ell - \lambda I) = +1$. When $|\lambda| > 1$, $(S_r - \lambda I)$ and $(S_\ell - \lambda I)$ are bijective; thus $\operatorname{ind}(S_r - \lambda I) = 0$ and $\operatorname{ind}(S_\ell - \lambda I) = 0$.

Problem 41

- A -

1. Assume by contradiction that $a \in (\operatorname{Int} P) \cap (-P)$. From Exercise 1.7 we have $0 = \frac{1}{2}a + \frac{1}{2}(-a) \in \operatorname{Int} P$ and this implies $P = E$.
2. Suppose not; then there exists a sequence (x_n) in P such that $x_n + u \to 0$. Since $(x_n + u) - u = x_n \in P$, we obtain at the limit $-u \in P$. This contradicts question 1.
3. Clearly $u \neq 0$ (since $0 \notin \operatorname{Int} P$ by (2)). From (3) we have $Tu \in \operatorname{Int} C$ and thus $B(Tu, \rho) \subset C$ for some $\rho > 0$. Then choose $0 < r < \rho/\|u\|$.
4. Since $\lambda x = T(x + u) \geq Tu \geq ru$, we have $\frac{\lambda}{r}x \geq u$. Assuming $(\frac{\lambda}{r})^n x \geq u$, we obtain $(\frac{\lambda}{r})^n Tx \geq Tu$ and thus $(\frac{\lambda}{r})^n(\lambda x - Tu) \geq Tu \geq ru$. Hence $(\frac{\lambda}{r})^n \lambda x \geq ru$, i.e., $(\frac{\lambda}{r})^{n+1} x \geq u$. On the other hand, $\lambda x = T(x + u) \in \operatorname{Int} P$, which implies that $\lambda > 0$ (by question 1). If we had $0 < \lambda < r$ we could pass to the limit as $n \to \infty$ and obtain $-u \in P$, which is impossible (again by question 1).
5. The map $x \mapsto (x + u)/\|x + u\|$ is clearly continuous on P (by question 2). $F(P) \subset T(B_E) \subset K$ since $T \in \mathcal{K}(E)$.
6. When replacing u by εu, the constant α in question 2 may change, but the constant r in question 3 remains unchanged.
7. We have $\lambda_\varepsilon \|x_\varepsilon\| = \|T(x_\varepsilon + \varepsilon u)\| \leq \|T\| \, \|x_\varepsilon + \varepsilon u\|$ and therefore $\|x_\varepsilon\| \leq \|T\|$. Hence $\lambda_\varepsilon \leq \|T\| + \varepsilon\|u\|$. Passing to a subsequence $\varepsilon_n \to 0$, we may assume that $\lambda_{\varepsilon_n} \to \mu_0$ and $Tx_{\varepsilon_n} \to \ell$ (since $T \in \mathcal{K}(E)$). Hence $x_{\varepsilon_n} \to x_0$ with $x_0 \in P$, $\mu_0 = \|x_0\| \geq r$ and $Tx_0 = \mu_0 x_0$, so that $x_0 \in \operatorname{Int} P$ by (3).

- B -

1. The set $\Sigma = \{s \in [0, 1]; (1 - s)a + sb \in P\}$ is a closed interval (since P is convex and closed). Then $\sigma = \max\{s; s \in \Sigma\}$ has the required properties by Exercise 1.7.
2. We cannot have $\mu = 0$ (otherwise, $0 \in \operatorname{Int} P$) and we cannot have $\mu < 0$ (otherwise, $-x \in (\operatorname{Int} P) \cap (-P)$). Thus $\mu > 0$, and then $x \in \operatorname{Int} P$, which implies $-x \notin P$. Note that x_0 and x play symmetric roles: $x_0, x \in \operatorname{Int} P$, $-x_0 \notin P$, $-x \notin P$, $Tx_0 = \mu_0 x_0$ with $\mu_0 > 0$, and $Tx = \mu x$ with $\mu > 0$. Set $y = x_0 - \tau_0 x$, where $\tau_0 = \tau(x_0, -x)$. Then $y \in P$ (from the definition of σ and τ). Moreover, $y \neq 0$ (otherwise $x = mx_0$ with $m = 1/\tau_0$). Thus $Ty \in \operatorname{Int} P$. But

$$Ty = Tx_0 - \tau_0 Tx = \mu_0 x_0 - \tau_0 \mu x.$$

Hence $x_0 + \frac{\tau_0 \mu}{\mu_0}(-x) \in \operatorname{Int} P$. From the definition of τ_0 we deduce that $\frac{\tau_0 \mu}{\mu_0} < \tau_0$ and therefore $\mu < \mu_0$. Reversing the roles of x_0 and x yields $\mu_0 < \mu$. Hence we obtain a contradiction. Therefore $x = mx_0$ for some $m > 0$ and then $\mu = \mu_0$.

3. If $x \in P$ or $-x \in P$ we deduce the first part of the alternative from question 2. We may thus assume that $x \notin P$ and $-x \notin P$. We will then show that $|\mu| < \mu_0$. If $\mu = 0$ we are done. Suppose that $\mu > 0$ and let $\tau_0 = \tau(x_0, x)$. Set $y = x_0 + \tau_0 x$, so that $y \in P$. We have $y \neq 0$ (otherwise $-x \in P$) and thus

$$Ty = \mu_0 x_0 + \tau_0 \mu x \in \operatorname{Int} P.$$

Hence $x_0 + \frac{\tau_0 \mu}{\mu_0} x \in \operatorname{Int} P$. From the definition of τ_0 we deduce that $\frac{\tau_0 \mu}{\mu_0} < \tau_0$, and thus $\mu < \mu_0$.
Suppose now that $\mu < 0$. Let $\tau_0 = \tau(x_0, x)$ and $\tilde{\tau}_0 = \tau(x_0, -x)$. Set $y = x_0 + \tau_0 x$ and $\tilde{y} = x_0 - \tilde{\tau}_0 x$, so that $y, \tilde{y} \in P$ and $y \neq 0$, $\tilde{y} \neq 0$. As above, we obtain

$$x_0 + \frac{\tau_0 \mu}{\mu_0} x \in \operatorname{Int} P \quad \text{and} \quad x_0 - \frac{\tilde{\tau}_0 \mu}{\mu_0} x \in \operatorname{Int} P.$$

Thus

$$x_0 + \frac{\tau_0 |\mu|}{\mu_0}(-x) \in \operatorname{Int} P \quad \text{and} \quad x_0 + \frac{\tilde{\tau}_0 |\mu|}{\mu_0} x \in \operatorname{Int} P.$$

From the definition of τ_0 and $\tilde{\tau}_0$ we deduce that

$$\frac{\tau_0 |\mu|}{\mu_0} < \tilde{\tau}_0 \quad \text{and} \quad \frac{\tilde{\tau}_0 |\mu|}{\mu_0} < \tau_0.$$

Therefore

$$\frac{|\mu|}{\mu_0} < \min \left\{ \frac{\tilde{\tau}_0}{\tau_0}, \frac{\tau_0}{\tilde{\tau}_0} \right\} \leq 1.$$

4. Using question 3 with $\mu = \mu_0$ yields $N(T - \mu_0 I) \subset \mathbb{R} x_0$.
5. In view of the results in Problem 36 it suffices to show that $N((T - \mu_0 I)^2) = N(T - \mu_0 I)$. Let $x \in E$ be such that $(T - \mu_0 I)^2 x = 0$. Using question 4 we may write $Tx - \mu_0 x = \alpha x_0$ for some $\alpha \in \mathbb{R}$. We need to prove that $\alpha = 0$. Suppose not, that $\alpha \neq 0$. Set $y = \frac{x}{\alpha}$, so that $Ty - \mu_0 y = x_0$. Then $T^2 y = \mu_0 Ty + Tx_0 = \mu_0^2 y + 2\mu_0 x_0$. By induction we obtain $T^n y = \mu_0^n y + n\mu_0^{n-1} x_0$ for all $n \geq 1$, which we may write as

$$T^n \left(x_0 - \frac{\mu_0 y}{n} \right) = -\frac{\mu_0^{n+1}}{n} y.$$

Since $x_0 \in \operatorname{Int} P$, we may choose n sufficiently large that $x_0 - \frac{\mu_0 y}{n} \in P$. Since $T^n(P) \subset P$, we conclude that $-y \in P$. Thus $T^n(-y) \in P$. Returning to the equation $T^n y = \mu_0^n y + n\mu_0^{n-1} x_0 \ \forall n \geq 1$, we obtain $-y - \frac{n}{\mu_0} x_0 \in P$, i.e., $-x_0 - \frac{\mu_0}{n} y \in P$. As $n \to +\infty$ we obtain $-x_0 \in P$. Impossible.

Thus we have established that the geometric multiplicity of μ_0 (i.e., $\dim(N - \mu_0 I)$) is one, but also that the algebraic multiplicity is one.

Problem 42

1. We have, $\forall x \in C$,

$$\|Tx - Tx_0\| \le \|T\| \, \|x - x_0\| \le \|T\| r \le \frac{1}{2} \, \|Tx_0\|,$$

and by the triangle inequality,

$$\|Tx_0\| - \|Tx\| \le \|Tx - Tx_0\| \le \frac{1}{2} \, \|Tx_0\|.$$

Thus $\|y\| \ge \frac{1}{2} \, \|Tx_0\| \ \forall y \in T(C)$, and therefore also $\forall y \in \overline{T(C)}$. Since $Tx_0 \neq 0$, we see that $0 \notin \overline{T(C)}$.

2. By assumption (1), $\mathcal{A}_y$ is dense in E, and consequently $\mathcal{A}_y \cap B(x_0, r/2) \neq \emptyset$, i.e, there exists $S \in \mathcal{A}$ such that $\|Sy - x_0\| < r/2$.
3. We have, $\forall z \in B(y, \varepsilon)$,

$$\|Sz - x_0\| \le \|S(z - y)\| + \|Sy - x_0\| \le \|S\|\varepsilon + \frac{r}{2}.$$

Then choose $\varepsilon = \frac{r}{2\|S\|}$.

4. If $x \in C$, then $Tx \in B(y_j, \frac{1}{2} \, \varepsilon_{y_j})$ for some $j \in J$. Therefore $q_j(x) \ge \frac{1}{2} \, \varepsilon_{y_j}$ and thus $q(x) \ge \min_{j \in J}\{\frac{1}{2} \, \varepsilon_{y_j}\}$.
5. The functions q_j are continuous on E and the function $1/q$ is continuous on C. Thus F is continuous on C. Write

$$F(x) - x_0 = \frac{1}{q(x)} \sum_{j \in J} q_j(x) \left[S_{y_j}(Tx) - x_0 \right].$$

Note that $q_j(x) \ge 0 \ \forall x \in E$ and $q_j(x) > 0$ implies $\|Tx - y_j\| < \varepsilon_{y_j}$. Using the result of question 2 with $z = Tx$ and $y = y_j$ yields

$$\|S_{y_j}(Tx) - x_0\| \le r.$$

Therefore

$$q_j(x)\|S_{y_j}(Tx) - x_0\| \le q_j(x) r \quad \forall x \in E, \quad \forall j \in J,$$

and thus

$$\|F(x) - x_0\| \le r.$$

6. Let $Q = \overline{T(C)}$, so that Q is compact. Thus $R_j = S_{y_j}(Q)$ is compact, and so is $[0,\,1]R_j$ (since it is the image of $[0,\,1] \times R_j$ under the continuous map

$(t, x) \mapsto tx$). Finally, $\sum_{j\in J}[0,1]R_j$ is also compact (being the image under the map $(x_1, x_2, \dots) \mapsto \sum_{j\in J} x_j$ of a product of compact sets).

7. Each operator $S_{y_j} \circ T$ is compact by Proposition 6.3. Since $\mathcal{K}(E)$ is a subspace (see Theorem 6.1), we see that $U \in \mathcal{K}(E)$ ($q_j(\xi)$ is a constant). From Theorem 6.6 we know that $F = N(I - U)$ is finite-dimensional.
   Writing that $F(\xi) = \xi$ gives
$$\frac{1}{q(\xi)} \sum_{j\in J} q_j(\xi) S_{y_j}(T\xi) = \xi,$$
   and by definition of U,
$$U(\xi) = \frac{1}{q(\xi)} \sum_{j\in J} q_j(\xi) S_{y_j}(T\xi) = \xi.$$

8. We need to show that $Ta = U(Ta)\ \forall a \in Z$. Note that $S_{y_j} \in \mathcal{A}$ (by the construction of question 2; thus $S_{y_j} \circ T \in \mathcal{A}$ and $U \in \mathcal{A}$. From the definition of $\mathcal{A}$ it is clear that $U \circ T = T \circ U$. Let $a \in Z$, so that $a = Ua$. Then $Ta = T(Ua) = U(Ta)$.

   The space Z is finite-dimensional and thus $Z \neq E$ (this is the only place where we use the fact that E is infinite-dimensional). Clearly Z is closed and $Z \neq \{0\}$, since $\xi \in Z$ (and $\xi \in C$ implies $\xi \neq 0$ by question 1). Thus Z is a nontrivial closed invariant subspace of T.
9. Nontrivial subspaces have dimension one. Thus the only nontrivial invariant subspaces are of the form $\mathbb{R}x_0$ with $x_0 \neq 0$ and $Tx_0 = \alpha x_0$ for some $\alpha \in \mathbb{R}$. Therefore it suffices to choose any T with no real eigenvalue, for example a rotation by $\pi/2$.

Problem 43

1. $|T(u+v)|^2 = |Tu|^2 + |Tv|^2 + 2(T^\star Tu, v)$ and $|T^\star(u+v)|^2 = |T^\star u|^2 + |T^\star v|^2 + 2(TT^\star u, v)$.
3. By Corollary 2.18 (and since H is reflexive) we always have $\overline{R(T)} = N(T^\star)^\perp$ and $\overline{R(T^\star)} = N(T)^\perp$.
4. Since $f \in R(T)$, we have $f = Tv$ for some $v \in H$. Using question 3 we may decompose $v = v_1 + v_2$ with $v_1 \in \overline{R(T)}$ and $v_2 \in N(T)$. Then $f = Tv = Tv_1$ and we choose $u = v_1$.
5. We have by question 1 $|u_n - u_m| = |T^*(y_n - y_m)| = |T(y_n - y_m)| \to 0$ as $m, n \to \infty$. Thus Ty_n is a Cauchy sequence; let $z = \lim_{n\to\infty} Ty_n$. Then $T^\star Ty_n = TT^\star y_n$ with $TT^\star y_n = Tu_n \to Tu = f$ and $T^\star Ty_n \to T^\star z$. Thus $T^\star z = f$.
6. In question 5 we have proved that $R(T) \subset R(T^\star)$. Applying this inclusion to $T^\star$ (which is also normal) gives $R(T^\star) \subset R(T)$.
7. Clearly $\|T^2\| \leq \|T^2\|$. For the reverse inequality write $|Tu|^2 = (T^\star Tu, u) \leq |T^\star Tu|\,|u|$. Since T is normal, we have $|T^\star Tu| = |TTu| \leq \|T^2\|\,|u|$. Therefore $\|T\|^2 = \sup_{u\neq 0} \frac{|Tu|^2}{|u|^2} \leq \|T^2\|$.

8. When $p = 2^k$ we argue by induction on k. Indeed, $\|T^{2^{k+1}}\| = \|S^2\|$, where $S = T^{2^k}$. Since S is normal, we have $\|S^2\| = \|S\|^2$. But $\|S\| = \|T\|^{2^k}$ from the induction assumption. Therefore $\|T^{2^{k+1}}\| = \|T\|^{2^{k+1}}$.

   For a general integer p, choose any k such that $2^k \geq p$. We have

$$\|T\|^{2^k} = \|T^{2^k}\| = \|T^{2^k-p}T^p\| \leq \|T^{2^k-p}\|\,\|T^p\| \leq \|T\|^{2^k-p}\|T^p\|.$$

   Thus $\|T\|^p \leq \|T^p\|$, and since $\|T^p\| \leq \|T\|^p$, we obtain $\|T^p\| = \|T\|^p$.
9. Let $u \in N(T^2)$. Then $Tu \in N(T) \cap R(T) \subset N(T) \cap N(T)^\perp$ by question 2. Therefore $Tu = 0$ and $u \in N(T)$. The same argument shows that $N(T^p) \subset N(T^{p-1})$ for $p \geq 2$, and thus $N(T^p) \subset N(T)$. Clearly $N(T) \subset N(T^p)$ and therefore $N(T^p) = N(T)$.

Problem 44

- A -

2. Clearly $T^\star \circ T = I$ implies $|Tu| = |u|\ \forall u \in H$. Conversely, write $|T(u+v)|^2 = |u+v|^2$ and deduce that $(Tu, Tv) = (u, v)\ \forall u, v \in H$, so that $T^\star \circ T = I$.
3. (a) $\Rightarrow$ (b). $T^\star \circ T = I$ and T bijective imply that $T^\star = T^{-1}$, so that $T^\star$ is also bijective.
   (b) $\Rightarrow$ (c). $T^\star \circ T = I$ implies that $T^\star$ is surjective. If $T^\star$ is also injective, then $T^\star$ is bijective and $T = (T^\star)^{-1}$. Hence $T \circ T^\star = I$.
   (c) $\Rightarrow$ (d). Obvious.
   (d) $\Rightarrow$ (e). $T^\star \circ T = I$ implies that $T^\star$ is surjective. If $T^\star$ is an isometry, it must be a unitary operator.
   (e) $\Rightarrow$ (a). Apply (a) $\Rightarrow$ (e) to $T^\star$.
4. In $H = \ell^2$ the right shift S_r defined by $S_r(x_1, x_2, x_3, \dots) = (0, x_1, x_2, \dots)$ is an isometry that is not surjective.
5. Let $f_n \in R(T)$ with $f_n \to f$. Write $f_n = Tu_n$ and $|u_n - u_m| = |f_n - f_m|$, so that (u_n) is Cauchy sequence and $u_n \to u$ with $f = Tu$. Given $v \in H$, set $g = TT^\star v$. Then $g \in R(T)$ and we have $\forall x \in H$,

$$(v - g, Tx) = (v, Tx) - (TT^\star v, Tx) = (v, Tx) - (v, TT^\star Tx) = 0,$$

   since $T^\star \circ T = I$. Thus $v - g \in R(T)^\perp$ and consequently $g = P_{R(T)}v$.
6. Assume that T is an isometry. Write $(T - \lambda I) = (I - \lambda T^*) \circ T$. Assume $|\lambda| < 1$. Then $\|\lambda T^\star\| < 1$ and thus $(I - \lambda T^\star)$ is bijective. When T is a unitary operator we deduce that $(T - \lambda I)$ is bijective; therefore $(-1, +1) \subset \rho(T)$ and hence $\sigma(T) \subset (-\infty, -1] \cup [+1, +\infty)$, so that $\sigma(T) \subset \{-1, +1\}$ (since $\sigma(T) \subset [-1, +1]$). On the other hand, if T is *not* a unitary operator and $|\lambda| < 1$, we see that $(T - \lambda I)$ cannot be bijective; therefore $(-1, +1) \subset \sigma(T)$, so that $\sigma(T) = [-1, +1]$ (since $\sigma(T)$ is closed and $\sigma(T) \subset [-1, +1]$).

7. T is an isometry from H onto $T(H)$. If $T \in \mathcal{K}(H)$ then $T(B_H) = B_{T(H)}$ is compact. Hence $\dim T(H) < \infty$ by Theorem 6.5. Since T is bijective from H onto $T(H)$, it follows that $\dim H < \infty$.
8. If T is skew-adjoint then $(Tu, u) = (u, T^\star u) = -(u, Tu)$ and thus $(Tu, u) = 0$. Conversely, write $0 = (T(u + v), (u + v)) = (Tu, v) + (Tv, u)\ \forall u, v \in H$, so that $Tu + T^\star u = 0\ \forall u \in H$.
9. Assume $\lambda \neq 0$. Then $(T - \lambda I) = -\lambda(I - \frac{1}{\lambda}T)$, and the operator $(I - \frac{1}{\lambda}T)$ satisfies the conditions of the Lax–Milgram theorem (Corollary 5.8). Thus $(T - \lambda I)$ is bijective.
10. From question 9 we know that $1 \notin \sigma(T)$, and thus $(T - I)^{-1}$ is well defined. From the relation $(T - I) \circ (T + I) = (T + I) \circ (T - I)$ we deduce that $U = (T - I)^{-1} \circ (T + I)$. Similarly $U \circ T = (T - I)^{-1} \circ (T + I) \circ T = T \circ (T + I) \circ (T - I)^{-1} = T \circ U$ because $(T + I) \circ T \circ (T - I) = (T - I) \circ T \circ (T + I)$. Next, we have $U^\star = (T^\star - I)^{-1} \circ (T^\star + I)$ and thus $U^\star \circ U = (T^\star - I)^{-1} \circ (T^\star + I) \circ (T + I) \circ (T - I)^{-1} = I$, since $(T^\star + I) \circ (T + I) = (T^\star - I) \circ (T - I)$ because $T^\star + T = 0$.
    Thus U is an isometry. On the other hand, $U = (T + I) \circ (T - I)^{-1}$ is bijective since $-1 \in \rho(T)$ by question 9.
11. By assumption we have $U^\star \circ U = I$. Thus $(T^\star - I)^{-1} \circ (T^\star + I) \circ (T + I) \circ (T - I)^{-1} = I$. This implies $(T^\star + I) \circ (T + I) = (T^\star - I) \circ (T - I)$, i.e., $T^\star + T = 0$.

- B -

1. (i) Trivial.
   (ii) If $\dim H < \infty$, standard linear algebra gives $\dim N(T) = \dim N(T^\star)$.
   (iii) If T is normal, then $N(T) = N(T^\star)$.
   (iv) $\dim N(T) = \dim N(T^\star) < \infty$ by Theorem 6.6.
2. If $T = S_\ell$, a left shift, then $\dim N(T) = 1$ and $T^\star = S_r$ satisfies $N(T^\star) = \{0\}$.
3. We have $T^\star = P \circ U^\star$ and thus $T^\star \circ T = P \circ U^\star \circ U \circ P = P^2$ by question A.2.
4. From the results of Problem 39 we know that P must be a square root of $T^\star \circ T$, and that P is unique.
5. Suppose that $T = U \circ P = V \circ P$ are two polar decompositions. Then $U = V$ on $R(P)$ and by continuity $U = V$ on $\overline{R(P)}$. But $P^2 = T^\star \circ T$ implies $N(P) = N(T)$. Thus $\overline{R(P)} = N(P^\star)^\perp = N(P)^\perp = N(T)^\perp$ (since $P^\star = P$).
6. From the relation $T = U \circ P$ we see that $U(R(P)) \subset R(T)$. In fact, we have $U(R(P)) = R(T)$; indeed, given $f \in R(T)$ write $f = Tx$ for some $x \in H$, and then $U(Px) = f$, so that $f \in U(R(P))$.

   By continuity U maps $\overline{R(P)} = N(T)^\perp$ into $\overline{R(T)} = N(T^\star)^\perp$. Since U is an isometry, the space $U(N(T)^\perp)$ is closed (by the standard Cauchy sequence argument). But $U(N(T)^\perp) \supset R(T)$ and therefore $U(N(T)^\perp) = \overline{R(T)} = N(T^\star)^\perp$. Using the property $(Ux, Uy) = (x, y)\ \forall x, y \in H$ we find that $(Ux, Uy) = 0\ \forall x \in N(T)^\perp,\ \forall y \in N(T)$. Thus $U(y) \in N(T^\star)^{\perp\perp} = N(T^\star)\ \forall y \in N(T)$. Consequently $J = U_{|N(T)}$ is an isometry from $N(T)$ into $N(T^\star)$.

7. Let P be the square root of $T^\star \circ T$. We now construct the isometry U. First define $U_0 : R(P) \to R(T)$ as follows. Given $f \in R(P)$, there exists some $u \in H$ (not necessarily unique) such that $f = Pu$. We set

$$U_0 f = Tu.$$

This definition makes sense; indeed, if $f = Pu = Pu'$, then $u - u' \in N(P) = N(T)$, so that $Tu = Tu'$. Moreover,

$$|U_0 f| = |Tu| = |Pu| = |f| \quad \forall f \in R(P).$$

In addition we have $U_0(R(P)) = R(T)$. Indeed, we already know that $U_0(R(P)) \subset R(T)$. The reverse inclusion follows from the identity $U_0(Pu) = Tu \ \forall u \in H$.

Let $\widetilde{U}_0$ be the extension by continuity of U_0 to $\overline{R(P)}$. Then $\widetilde{U}_0$ is an isometry from $\overline{R(P)} = N(T)^\perp$ into $\overline{R(T)} = N(T^\star)^\perp$. But $R(\widetilde{U}_0) \supset R(U_0) = R(T)$ and therefore (as above) $R(\widetilde{U}_0) \supset \overline{R(T)} = N(T^\star)^\perp$. Hence $\widetilde{U}_0$ is an isometry from $N(T)^\perp$ *onto* $N(T^\star)^\perp$.

Finally, we extend $\widetilde{U}_0$ to all of H as follows. Given $x \in H$, write

$$x = x_1 + x_2$$

with $x_1 \in N(T)^\perp$ and $x_2 \in N(T)$. Set

$$Ux = \widetilde{U}_0 x_1 + J x_2.$$

Then

$$|Ux|^2 = |\widetilde{U}_0 x_1|^2 + 2(\widetilde{U}_0 x_1, Jx_2) + |Jx_2|^2 = |x_1|^2 + |x_2|^2 = |x|^2,$$

since $\widetilde{U}_0 x_1 \in N(T^\star)^\perp$ and $x_2 \in N(T^\star)$ (by (1)).

Clearly $U(Pu) = U_0(Pu) = Tu \ \forall u \in H$, and therefore we have constructed a polar decomposition of T.

8. The construction of question 7 shows that $R(U) = N(T^\star)^\perp \oplus R(J)$. Thus $R(U) = H$ if $R(J) = N(T^\star)$, and then U is a unitary operator.
9. If T is a normal operator then $N(T) = N(T^\star)$ (see Problem 43). Thus (2) is satisfied and we may apply question 8. Next, we have $T^\star = P \circ U^\star$, and since T is normal we can write

$$(P \circ U^\star) \circ (U \circ P) = T^\star \circ T = T \circ T^\star = (U \circ P) \circ (P \circ U^\star),$$

which implies that

$$P^2 = U \circ P^2 \circ U^\star,$$

and thus

$$P^2 \circ U = U \circ P^2.$$

Applying the result of question C2 in Problem 39 we deduce that $P \circ U = U \circ P$.

10. We have $P^2 = T^\star \circ T \in \mathcal{K}(H)$. This implies that $P \in \mathcal{K}(H)$. Indeed, let (u_n) be a sequence in H with $|u_n| \le 1$. Passing to a subsequence (still denoted by u_n), we may assume that $u_n \rightharpoonup u$ and $P^2 u_n \to P^2 u$. Then $|P(u_n - u)|^2 = (P^2(u_n - u), u_n - u) \to 0$, so that $Pu_n \to Pu$. Hence $P \in \mathcal{K}(H)$.
11. We have $T^\star \circ T \in \mathcal{K}(H)$, since $T \in \mathcal{K}(H)$ and its square root P is compact (see part D in Problem 39).
12. Let (e_n) be an orthonormal basis of H consisting of eigenvectors of $T^\star T$, with corresponding eigenvalues (λ_n), so that $\lambda_n \ge 0\ \forall n$ and $\lambda_n \to 0$ as $n \to \infty$. Let $I = \{n \in \mathbb{N};\ \lambda_n > 0\}$. Consider the isometry U_0 defined on $R(P)$ with values in $R(T)$ constructed in question 7; we have $U_0 \circ P = T$ on H.
Set $f_n = U_0(e_n)$ for $n \in I$; this is well defined, since $Pe_n = \sqrt{\lambda_n} e_n$, so that $e_n \in R(P)$ when $n \in I$. Then $(f_n)_{n \in I}$ is an orthonormal system in H (but it is not a basis of H, since $f_n \in R(U_0) \subset \overline{R(T)} \neq H$ in general). Choose any basis of H, still denoted by $(f_n)_{n \in \mathbb{N}}$, containing the system $(f_n)_{n \in I}$. For $u \in H$, write
$$u = \sum_{n \in \mathbb{N}} (u, e_n) e_n,$$
so that
$$Pu = \sum_{n \in \mathbb{N}} \sqrt{\lambda_n}(u, e_n) e_n = \sum_{n \in I} \sqrt{\lambda_n}(u, e_n) e_n,$$
and then
$$Tu = U_0(Pu) = \sum_{n \in I} \sqrt{\lambda_n}(u, e_n) f_n = \sum_{n \in \mathbb{N}} \sqrt{\lambda_n}(u, e_n) f_n.$$
Clearly
$$T^\star v = \sum_{n \in \mathbb{N}} \sqrt{\lambda_n}(v, f_n) e_n.$$
Set
$$T_N u = \sum_{n=1}^{N} \alpha_n (u, e_n) f_n,$$
so that $T_N \in \mathcal{K}(H)$ (since it is a finite-rank operator). Then $\|T_N - T\| \le \max_{n \ge N+1} |\alpha_n|$, so that $\|T_N - T\| \to 0$ as $N \to \infty$, provided $\alpha_n \to 0$ as $n \to \infty$; thus $T \in \mathcal{K}(H)$ by Corollary 6.2.

Problem 45

12. Consider the equation
$$\begin{cases} -u'' + k^2 u = f & \text{on } (0, 1), \\ u(0) = u(1) = 0. \end{cases}$$
The solution is given by

$$u(x) = \frac{\sinh(kx)}{k \sinh k} \int_0^1 f(s) \sinh(k(1-s))ds - \frac{1}{k} \int_0^x f(s) \sinh(k(x-s))ds.$$

A tedious computation shows that

$$u(x) \geq \frac{k}{\sinh k} x(1-x) \int_0^1 f(s)s(1-s)ds.$$

Next, suppose that $p \equiv 1$ and u satisfies

$$\begin{cases} -u'' + qu = f & \text{on } (0, 1), \\ u(0) = u(1) = 0. \end{cases}$$

Write

$$-u'' + k^2 u = f + (k^2 - q)u.$$

We already know that $u \geq 0$. Choosing the constant k sufficiently large we have $f + (k^2 - q)u \geq f$, and we are reduced to the previous case.

In the general case, consider the new variable

$$y = \frac{1}{L} \int_0^x \frac{1}{p(t)} dt, \text{ where } L = \int_0^1 \frac{1}{p(t)} dt.$$

Set $v(y) = u(x)$. Then

$$u_x(x) = v_y(y) \frac{1}{Lp(x)}$$

and

$$(p(x)u_x)_x = v_{yy}(y) \frac{1}{L^2 p(x)}.$$

Therefore the problem

$$\begin{cases} -(pu')' + qu = f & \text{on } (0, 1), \\ u(0) = u(1) = 0 \end{cases}$$

becomes

$$\begin{cases} -v_{yy}(y) + L^2 p(x) q(x) v(y) = L^2 p(x) f(x) & \text{on } (0, 1), \\ v(0) = v(1) = 0, \end{cases}$$

and we are reduced to the previous case, noting that $x(1-x) \sim y(1-y)$.

Problem 46

12. Let (u_n) be a minimizing sequence i.e., $F(u_n) \to m$. We have

$$F(u_n) = \frac{1}{2}\int_0^1 ({u'_n}^2 + u_n^2) - \int_0^1 g(u_n) \le C.$$

On the other hand we may use Young's inequality (see (2) in Chapter 4, and the corresponding footnote) with $a = (t^+)^{\alpha+1}$ and $p = 2/(\alpha+1)$, so that $p > 1$ since $\alpha < 1$. We obtain

$$g(u_n) \le \varepsilon u_n^2 + C_\varepsilon \quad \forall \varepsilon > 0.$$

Choosing, e.g., $\varepsilon = 1/4$ we see that (u_n) is bounded in $H_0^1(I)$. Therefore we may extract a subsequence (u_{n_k}) converging weakly in $H_0^1(I)$, and strongly in $C(\overline{I})$ (by Theorem 8.8), to some limit $u \in H_0^1(I)$. Therefore

$$\liminf_{k\to\infty} \int_0^1 ({u'_{n_k}}^2 + u_{n_k}^2) \ge \int_0^1 (u'^2 + u^2)$$

and

$$\lim_{k\to\infty} \int_0^1 g(u_{n_k}) = \int_0^1 g(u).$$

Consequently $F(u) \le m$, and thus $F(u) = m$.

Problem 47

- A -

2. Choose a sequence (u_n) proposed in the hint. We have

$$\overline{u}_n = \int_{1-\frac{1}{n}}^1 u_n(x)dx$$

and thus $|\overline{u}_n| \le 1/n$. On the other hand

$$\|u_n - \overline{u}_n\|_{L^\infty(I)} \ge u_n(1) - \overline{u}_n \ge 1 - \frac{1}{n},$$

and

$$\|u'_n\|_{L^1(I)} = \int_0^1 u'_n(x)dx = u_n(1) - u_n(0) = 1.$$

3. Suppose, by contradiction, that the sup is achieved by some function $u \in W^{1,1}(I)$, i.e.,

$$\|u - \overline{u}\|_{L^\infty(I)} = 1 \text{ and } \|u'\|_{L^1(I)} = 1.$$

We may assume, e.g., that there exists some $x_0 \in [0, 1]$ such that

(S1) $$u(x_0) - \overline{u} = +1.$$

On the other hand,

$$\text{(S2)} \qquad \overline{u} = \int_0^1 \left(u(x) - \min_{[0,1]} u \right) dx + \min_{[0,1]} u \geq \min_{[0,1]} u = u(y_0),$$

for some $y_0 \in [0, 1]$. Combining (S1) and (S2) we obtain

$$u(x_0) - u(y_0) \geq 1.$$

But

$$u(x_0) - u(y_0) \leq \int_0^1 |u'(x)| dx = 1.$$

Therefore all the inequalities become equalities, and in particular $u \equiv \min_{[0,1]} u$. This contradicts (S1).

6. Set

$$m = \inf\{\|u'\|_{L^p(I)}; u \in W^{1,p}(I) \text{ and } \|u - \overline{u}\|_{L^q(I)} = 1\},$$

and let (u_n) be a minimizing sequence, i.e., $\|u_n'\|_{L^p(I)} \to m$ and $\|u_n - \overline{u}_n\|_{L^q(I)} = 1$. Without loss of generality we may assume that $\overline{u}_n = 0$. Therefore (u_n) is bounded in $W^{1,p}(I)$. We may extract a subsequence (u_{n_k}) converging weakly in $W^{1,p}(I)$ when $p < \infty$ (and (u_{n_k}') converges weak* in $L^\infty(I)$ when $p = \infty$) to some limit $u \in W^{1,p}(I)$. By Theorem 8.8 we may also assume that $u_{n_k} \to u$ in $C(\overline{I})$ (since $p > 1$). Clearly we have

$$\|u'\|_{L^p(I)} \leq m, \quad \overline{u} = 0, \text{ and } \|u\|_{L^q(I)} = 1.$$

- B -

1. Apply Lax–Milgram in V equipped with the H^1-norm, to the bilinear form $a(u, v) = \int_I u'v'$. Note that a is coercive (e.g., by question A6).
2. Let $w \in C_c^1(I)$. Choosing $v = (w - \overline{w})$ we obtain

$$\int_I u'w' = \int_I f(w - \overline{w}) = \int_I fw \quad \forall w \in C_c^1(I).$$

We deduce that $u \in H^2(I)$ and $-u'' = f$. Similarly we have

$$\int_I u'w' = \int_I fw \quad \forall w \in H^1(I)$$

and thus $u'(0) = u'(1) = 0$ (since $w(0)$ and $w(1)$ are arbitrary).

4. We have $\sigma(T/\lambda_1) \subset [0, 1]$. Applying Exercise 6.24 ((v) $\Rightarrow$ (vi)) we know that

$$\lambda_1(Tf, f) \geq |Tf|^2 \quad \forall f \in H$$

and we deduce that

$$\lambda_1 \int_0^1 u'^2 \geq \int_0^1 u^2 \quad \forall u \in W,$$

where

$$W = \left\{ u \in H^2(0,1);\, u'(0) = u'(1) = 0 \text{ and } \int_0^1 u = 0 \right\}.$$

On the other hand, given any $u \in V$, there exists a sequence $u_n \in W$ such that $u_n \to u$ in H^1. (Indeed let $\varphi_n \in C_c^1(I)$ be a sequence such that $\varphi_n \to u'$ in $L^2(I)$ and set $u_n(x) = \int_0^x \varphi_n(t)dt + c_n$, where the constant c_n is adjusted so that $\int_0^1 u_n = 0$.) Therefore we obtain

$$\|u\|_{L^2(I)} \le \sqrt{\lambda_1}\|u'\|_{L^2(I)} \quad \forall u \in V.$$

Choosing an eigenfunction e_1 of T corresponding to λ_1, and letting $u_1 = Te_1$ we obtain

$$\|u_1\|_{L^2(I)} = \sqrt{\lambda_1}\|u_1'\|_{L^2(I)}.$$

The eigenvalues of T are given by $\lambda_k = \frac{1}{k^2\pi^2}$, $k = 1, 2, \dots$. Therefore the best constant in (6) is $1/\pi$.

- C -

1. Write, for $u \in W^{1,1}(I)$,

$$\begin{aligned}\int_0^1 |u(x) - \overline{u}|dx &= \int_0^1 \left|u(x) - \int_0^1 u(y)dy\right|dx \le \int_0^1\int_0^1 |u(x) - u(y)|dxdy \\ &\le \int_0^1 dx \int_0^x dy \int_y^x |u'(t)|dt + \int_0^1 dx \int_x^1 dy \int_x^y |u'(t)|dt \\ &= 2\int_0^1 |u'(t)|t(1-t)dt\end{aligned}$$

by Fubini.

3. Choose a function $u \in W^{1,1}(I)$ such that $u(x) = -\frac{1}{2}$ $\forall x \in \left(0, \frac{1}{2} - \varepsilon\right)$, $u(x) = +\frac{1}{2}$ $\forall x \in \left(\frac{1}{2} + \varepsilon, 1\right)$, $\overline{u} = 0$ and $u' \ge 0$, where $\varepsilon \in \left(0, \frac{1}{2}\right)$ is arbitrary. Then $\|u'\|_{L^1} = 1$ and $\|u\|_{L^1} \ge \frac{1}{2} - \varepsilon$.
4. There is no function $u \in W^{1,1}(I)$ such that $\|u - \overline{u}\|_{L^1(I)} = \frac{1}{2}$ and $\|u'\|_{L^1(I)} = 1$. Suppose, by contradiction, that such a function exists. Then

$$\frac{1}{2} = \|u - \overline{u}\|_{L^1(I)} \le 2\int_I |u'(t)|t(1-t)dt \le \frac{1}{2}\int_I |u'(t)|dt = \frac{1}{2},$$

since $2t(1-t) \le \frac{1}{2}$ $\forall t \in (0,1)$. All the inequalities become equalities and therefore $\left(\frac{1}{4} - t(1-t)\right)|u'(t)| = 0$ a.e. Hence $u' = 0$ a.e. Impossible.

Problem 49

7. We have

$$\lambda_1 = a(w_0, w_0) \le \frac{a(w_0 + tv, w_0 + tv)}{\|w_0 + tv\|_{L^2}^2} \quad \forall v \in H_0^1(0,1), \quad \forall t \text{ sufficiently small.}$$

Therefore we obtain

$$\lambda_1 \left(1 + 2t \int_0^1 w_0 v + t^2 \int_0^1 v^2\right) \le \lambda_1 + 2ta(w_0, v) + t^2 a(v, v),$$

and consequently

$$\lambda_1 \int_0^1 w_0 v = a(w_0, v) \quad \forall v \in H_0^1(0,1),$$

i.e., $Aw_0 = \lambda_1 w_0$ on $(0, 1)$.

8. We know from Exercise 8.11 that $w_1 = |w_0| \in H_0^1(0,1)$ and $|w_1'| = |w_0'|$ a.e. Therefore $a(w_1, w_1) = a(w_0, w_0)$, and thus w_1 is also a minimizer for (1). We may then apply question 7.
10. Here is another proof which does not rely on the fact that all eigenvalues are simple. (This proof can be adapted to elliptic PDE's in dimension > 1.) It is easy to see (using question 9) that w^2/w_1 belongs to $H_0^1(0,1)$. Therefore we have

$$\int_0^1 (Aw_1)\frac{w^2}{w_1} = \lambda_1 \int_0^1 w^2 = \int_0^1 (Aw)w.$$

Integrating by parts we obtain

$$\int_0^1 pw_1' \left(\frac{2ww'}{w_1} - \frac{w^2}{w_1^2} w_1'\right) + qw^2 = \int_0^1 pw'^2 + qw^2,$$

and therefore

$$\int_0^1 p\left(w' - \frac{w_1' w}{w_1}\right)^2 = 0.$$

Consequently $(\frac{w}{w_1})' = \frac{1}{w_1}(w' - \frac{w_1' w}{w_1}) = 0$, and therefore w is a multiple of w_1.

Problem 50

2. Note that

$$\int_0^1 |q|u^2 \le \left(\int_0^1 q^2\right)^{1/2} \left(\int_0^1 u^4\right)^{1/2} \le \varepsilon \int_0^1 u^4 + C_\varepsilon \int_0^1 q^2.$$

Choosing $\varepsilon = 1/8$ we deduce that, $\forall u \in H_0^1(0,1)$,

$$\frac{1}{2}a(u,u) + \frac{1}{4}\int_0^1 u^4 \ge \frac{1}{2}\int_0^1 u'^2 + \frac{1}{8}\int_0^1 u^4 - C.$$

3. Let (u_n) be a minimizing sequence, i.e., $\frac{1}{2}a(u_n, u_n)+\frac{1}{4}\int_0^1 u_n^4 \to m$. Clearly (u_n) is bounded in $H_0^1(0,1)$. Passing to a subsequence, still denoted by u_n, we may assume that $u_n \rightharpoonup u_0$ weakly in $H_0^1(0,1)$ and $u_n \to u_0$ in $C([0,1])$. Therefore $\liminf_{n\to\infty}\int_0^1 u_n'^{\,2} \geq \int_0^1 u_0'^{\,2}$, $\int_0^1 qu_n^2 \to \int_0^1 qu_0^2$ and $\int_0^1 u_n^4 \to \int_0^1 u_0^4$. Consequently $a(u_0,u_0)+\frac{1}{4}\int |u_0|^4 \leq m$, and thus u_0 is a minimizer.
4. We have

$$\begin{aligned}\frac{1}{2}a(u_0,u_0)+\frac{1}{4}\int_0^1 u_0^4 &\leq \frac{1}{2}a(u_0+tv, u_0+tv)+\frac{1}{4}\int_0^1 (u_0+tv)^4\\ &= \frac{1}{2}a(u_0,u_0)+ta(u_0,v)+\frac{1}{4}\int_0^1 (u_0^4+4u_0^3tv)+O(t^2).\end{aligned}$$

Taking $t>0$ we obtain

$$a(u_0,v)+\int_0^1 u_0^3 v \geq O(t).$$

Letting $t\to 0$ and choosing $\pm v$ we are led to

$$a(u_0,v)+\int_0^1 u_0^3 v = 0 \quad \forall v \in H_0^1(0,1).$$

6. Recall that $u_1 \not\equiv 0$ since $\frac{1}{2}a(u_1,u_1)+\frac{1}{4}\int_0^1 u_1^4 = m < 0$. On the other hand we have

$$-u_1''+a^2u_1 = (a^2-q-u_1^2)u_1 = f \geq 0$$

and $f\not\equiv 0$ (provided $a^2-q-u_1^2>0$). We deduce from the strong maximum principle (see Problem 45) that $u_1>0$ on $(0,1)$, $u_1'(0)>0$, and $u_1'(1)<0$.
8. Let $u\in C_c^1((0,1))$; we have, using integration by parts,

$$-\int_0^1 U_1''\frac{u^2}{U_1} = \int_0^1 U_1'\left(\frac{2uu'}{U_1}-\frac{u^2U_1'}{U_1^2}\right) \leq \int_0^1 u'^2,$$

and therefore

$$\text{(S1)} \qquad \int_0^1 u'^2 - U_1'^2 \geq -\int_0^1 \frac{U_1''}{U_1}(u^2-U_1^2) = -\int_0^1 (q+U_1^2)(u^2-U_1^2).$$

By density, inequality (S1) holds for every $u\in H_0^1(0,1)$. Assume $\rho\in K$ and set $u=\sqrt{\rho}$. Then $u\in H_0^1(0,1)$, and we have

$$\Phi(\rho)-\Phi(\rho_1) = \int_0^1 u'^2+qu^2+\frac{1}{2}u^4-U_1'^2-qU_1^2-\frac{1}{2}U_1^4.$$

Using (S1) we see that

$$\Phi(\rho) - \Phi(\rho_1) \geq \int_0^1 -\left(q + U_1^2\right)\left(u^2 - U_1^2\right) + qu^2 + \frac{1}{2}u^4 - qU_1^2 - \frac{1}{2}U_1^4$$
$$= \int_0^1 \frac{1}{2}u^4 + \frac{1}{2}U_1^4 - U_1^2u^2 = \frac{1}{2}\int_0^1 \left(u^2 - U_1^2\right)^2.$$

Problem 51

1. The mapping $v \mapsto Tv = (v', v, \sqrt{p}v)$ is an isometry from V into $L^2(\mathbb{R})^3$. It is easy to check that $T(V)$ is a closed subspace of $L^2(\mathbb{R})^3$, and therefore V is complete. V is separable since $L^2(\mathbb{R})^3$ is separable.
3. Let $u \in C_c^\infty(\mathbb{R})$. We have $\forall x \in [-A, +A]$,

(S1) $$|u(x) - u(-A)| \leq \int_{-A}^{+A} |u'(t)|dt \leq \sqrt{2A}\|u'\|_{L^2(\mathbb{R})}.$$

On the other hand $u^2(-A) = 2\int_{-\infty}^{-A} uu'$ and therefore
(S2)
$$|u(-A)|^2 \leq \int_{-\infty}^{-A} |u|^2 + \int_{-\infty}^{+\infty} |u'|^2 \leq \frac{1}{\delta}\int_{-\infty}^{+\infty} p|u|^2 + \int_{-\infty}^{+\infty} |u'|^2 \leq Ca(u,u).$$

Combining (S1) and (S2) we obtain

(S3) $$|u(x)| \leq Ca(u,u)^{1/2} \quad \forall x \in [-A, +A],$$

and consequently
$$\int_{-A}^{+A} |u|^2 \leq Ca(u,u).$$

Next, write that

$$\int_{-\infty}^{+\infty} |u|^2 \leq \int_{|x|\leq A} |u|^2 + \int_{|x|\geq A} |u|^2 dx \leq Ca(u,u).$$

Since
$$\int_{-\infty}^{+\infty} |u'|^2 \leq a(u,u) \quad \text{and} \quad \int_{-\infty}^{+\infty} p|u|^2 \leq a(u,u),$$

we conclude that $a(u,u) \geq \alpha\|u\|_V^2$ $\forall u \in C_c^\infty(\mathbb{R})$, for some $\alpha > 0$.
4. It is clear that $u \in H^2(I)$ for every bounded open interval I and u satisfies $-u'' + pu = f$ a.e. on I. Since p, u and f are continuous on I we deduce that $u \in C^2(I)$. On the other hand, $u(x) \to 0$ as $|x| \to \infty$ by Corollary 8.9 (recall that $V \subset H^1(\mathbb{R})$).
5. We have

(S4) $$\int_{\mathbb{R}} u'(2\zeta_n'\zeta_n u + \zeta_n^2 u') + \int_{\mathbb{R}} p\zeta_n^2 u^2 = \int_{\mathbb{R}} f\zeta_n^2 u.$$

But

$$a(\zeta_n u, \zeta_n u) = \int_{\mathbb{R}} (\zeta_n u' + \zeta_n' u)^2 + \int_{\mathbb{R}} p\zeta_n^2 u^2 = \int_{\mathbb{R}} f\zeta_n^2 u + \int_{\mathbb{R}} \zeta_n'^2 u^2 \quad \text{by (S4).}$$

Thus (since $|\zeta_n| \le 1$)

$$a(\zeta_n u, \zeta_n u) \le \|f\|_{L^2(\mathbb{R})}\|\zeta_n u\|_{L^2(\mathbb{R})} + \frac{C}{n^2}\int_{n\le|x|\le 2n} u^2.$$

Since $u(x) \to 0$ as $|x| \to \infty$ we see that $\frac{1}{n^2}\int_{n\le|x|\le 2n} u^2 \to 0$ as $n \to \infty$. Using the fact that a is coercive on V we conclude that $\|\zeta_n u\|_V \le C$. It follows easily that $u \in V$. Returning to (S3) we obtain

$$a(u, v) = \int_{-\infty}^{+\infty} fv \quad \forall v \in C_c^\infty(\mathbb{R}),$$

and by density the same relation holds $\forall v \in V$.

6. Let $\mathcal{F} = \{u \in V; \|u\|_V \le 1\}$. We need to show that $\mathcal{F}$ has compact closure in $L^2(\mathbb{R})$. For this purpose we apply Corollary 4.27. Recall (see Proposition 8.5) that

$$\|\tau_h u - u\|_{L^2(\mathbb{R})} \le |h|\|u'\|_{L^2(\mathbb{R})}$$

and therefore

$$\lim_{|h|\to 0} \|\tau_h u - u\|_{L^2(\mathbb{R})} = 0 \text{ uniformly in } u \in \mathcal{F}.$$

On the other hand, given any $\varepsilon > 0$ we may fix a bounded interval I such that $|p(x)| > \frac{1}{\varepsilon^2}$ $\forall x \in \mathbb{R} \setminus I$. Therefore

$$\int_{\mathbb{R}\setminus I} |u|^2 \le \varepsilon^2 \int_{\mathbb{R}} p|u|^2 \le \varepsilon^2\|u\|_V^2 \le \varepsilon^2 \quad \forall u \in \mathcal{F}.$$

Notation

General notations

A^c	complement of the set A
$E^\star$	dual space
$\langle\ ,\ \rangle$	scalar product in the duality $E^\star$, E
$[f = \alpha] = \{x;\ f(x) = \alpha\}$	
$B(x_0, r)$	open ball of radius r centered at x_0
$B_E = \{x \in E;\ \|x\| \leq 1\}$	
$\operatorname{epi}\varphi = \{[x, \lambda];\ \varphi(x) \leq \lambda\}$	
$\varphi^\star$	conjugate function
$\mathcal{L}(E, F)$	space of bounded linear operators from E into F
$M^\perp$	orthogonal of M
$D(A)$	domain of the operator A
$G(A)$	graph of the operator A
$N(A)$	kernel (= null space) of the operator A
$R(A)$	range of the operator A
$\sigma(E, E^\star)$	weak topology on E
$\sigma(E^\star, E)$	weak* topology on $E^\star$
$\rightharpoonup$	weak convergence
J	canonical injection from E into $E^{\star\star}$
p'	conjugate exponent of p, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$
a.e.	almost everywhere
$\|A\|$	measure of the set A
$\operatorname{supp} f$	support of the function f
$f \star g$	convolution product of f with g
ρ_n	sequence of mollifiers
$(\tau_h f)(x) = f(x + h)$	shift of the function f
$\omega \subset\subset \Omega$	ω strongly included in Ω, i.e., $\overline{\omega}$ is compact and $\overline{\omega} \subset \Omega$
P_K	projection onto the closed convex set K

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$\vert\ \vert$	Hilbert norm
$\rho(T)$	resolvent set of the operator T
$\sigma(T)$	spectrum of the operator T
$EV(T)$	the set of eigenvalues of the operator T
$J_\lambda = (I + \lambda A)^{-1}$	resolvent of the operator A
$A_\lambda = AJ_\lambda$	Yosida approximation of the operator A
$\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_N}\right)$	gradient of the function u

$D^\alpha u = \dfrac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_N^{\alpha_N}}$, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$, $|\alpha| = \sum_{i=1}^N \alpha_i$

$\Delta u = \displaystyle\sum_{i=1}^N \frac{\partial^2 u}{\partial x_i^2}$	Laplacian of u

$\mathbb{R}_+^N = \{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}; x_N > 0\}$

$Q = \{x = (x', x_N) \in \mathbb{R}^N \times \mathbb{R}; |x'| < 1 \text{ and } |x_N| < 1\}$

$Q_+ = Q \cap \mathbb{R}_+^N$

$Q_0 = \{x \in Q; x_N = 0\}$

$(D_h u)(x) = \dfrac{1}{|h|}\,(u(x+h) - u(x))$

$\dfrac{\partial u}{\partial n}$	outward normal derivative

Function spaces

$\Omega \subset \mathbb{R}^N$	open set in $\mathbb{R}^N$
$\partial\Omega = \Gamma$	boundary of Ω

$L^p(\Omega) = \{u : \Omega \to \mathbb{R}$: u is measurable and $\int_\Omega |u|^p < \infty\}$, $1 \le p < \infty$

$L^\infty(\Omega) = \{u : \Omega \to \mathbb{R}$: u is measurable and $|u(x)| \le C$ a.e. in Ω for some constant $C\}$

$C_c(\Omega)$	space of continuous functions with compact support in Ω
$C^k(\Omega)$	space of k times continuously differentiable functions on Ω, $k \ge 0$
$C^\infty(\Omega) = \bigcap_{k \ge 0} C^k(\Omega)$	
$C^k(\overline{\Omega})$	functions in $C^k(\Omega)$ such that for every multi-index α with $|\alpha| \le k$, the function $x \mapsto D^\alpha u(x)$ admits a continuous extension to $\overline{\Omega}$
$C^\infty(\overline{\Omega}) = \bigcap_{k \ge 0} C^k(\overline{\Omega})$	

$$C^{0,\alpha}(\overline{\Omega}) = \left\{ u \in C(\overline{\Omega});\ \sup_{\substack{x,y \in \Omega \\ x \ne y}} \frac{|u(x) - u(y)|}{|x-y|^\alpha} < \infty \right\} \text{ with } 0 < \alpha < 1$$

$C^{k,\alpha}(\overline{\Omega}) = \{u \in C^k(\Omega); D^j u \in C^{0,\alpha}(\overline{\Omega}) \quad \forall j, |j| \le k\}$

$W^{1,p}(\Omega)$, $W_0^{1,p}(\Omega)$, $W^{m,p}(\Omega)$, $H^1(\Omega)$, $H_0^1(\Omega)$, $H^m(\Omega)$	Sobolev spaces

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Index

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