Problem -2 (Tutorial-1)

Tuesday, 1 June 2021 11:57 AM

$$P(X=x, Y=y)$$

X \ Y	0	1
0	1/3	1/3
1	0	1/3

P(X)

$$P(X=0) = \sum_{y \in \mathcal{Y}} P(X=0, Y=y)$$
$$= P(X=0, Y=0) + P(X=0, Y=1)$$
$$= \frac{1}{3} + \frac{1}{3}$$
$$= \boxed{\frac{2}{3}}$$

$$P(X=1) = \sum_{y \in \mathcal{Y}} P(X=1, Y=y)$$
$$= P(X=1, Y=0) + P(X=1, Y=1)$$
$$= 0 + \frac{1}{3}$$
$$= \boxed{\frac{1}{3}}$$

P(Y)

$$P(Y=0) = \sum_{x \in \mathcal{X}} P(X=x, Y=0)$$
$$= P(X=0, Y=0) + P(X=1, Y=0)$$
$$= \frac{1}{3} + 0$$
$$= \boxed{\frac{1}{3}}$$

$$P(Y=1) = \sum_{x \in \mathcal{X}} P(X=x, Y=1)$$
$$= P(X=0, Y=1) + P(X=1, Y=1)$$
$$= \frac{1}{3} + \frac{1}{3} = \boxed{\frac{2}{3}}$$

(a) $H(X)$, $H(Y)$

$$H(X) = \sum_{x \in \mathcal{X}} P(X=x) \log_2\left(\frac{1}{P(X=x)}\right)$$
$$= \frac{1}{3}\log_2 3 + \frac{2}{3}\log_2\left(\frac{3}{2}\right)$$
$$= \frac{\log_2 3}{3} + \frac{2}{3}\left(\log_2 3 - \log_2 2\right)$$
$$= \boxed{\log_2 3 - \frac{2}{3}}$$

$$H(Y) = \sum_{y \in \mathcal{Y}} P(Y=y) \log_2\left(\frac{1}{P(Y=y)}\right)$$
$$= \frac{1}{3}\log 3 + \frac{2}{3}\log_2\left(\frac{3}{2}\right)$$
$$= \boxed{\log_2 3 - \frac{2}{3}}$$

(b) $H(X,Y)$

$$H(X,Y) = \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} P(X=x, Y=y) \log_2\left(\frac{1}{P(X=x, Y=y)}\right)$$
$$= \frac{1}{3}\log_2 3 + \frac{1}{3}\log_2 3 + \frac{1}{3}\log_2 3 + 0$$
$$= \boxed{\log_2 3}$$

$$\lim_{p \to 0^+} p \log(1/p) = 0$$
$$\lim_{p \to 0^+} \frac{-\log(p)}{(1/p)} \quad \left| \ \frac{\infty}{\infty} \text{ form} \right.$$
$$= \lim_{p \to 0^+} \frac{-1/p}{-1/p^2}$$
$$= \lim_{p \to 0^+} p = \underline{\underline{0}}$$

(c) $H(X|Y)$, $H(Y|X)$

$$H(X|Y) = \sum_{y \in \mathcal{Y}} P(Y=y) \cdot H(X|Y=y)$$

$$H(X|Y=y) = \sum_{x \in \mathcal{X}} P(X=x|Y=y) \log_2\left(\frac{1}{P(X=x|Y=y)}\right)$$

$$\Rightarrow H(X|Y) = \sum_{y \in \mathcal{Y}} P(y) \cdot \sum_{x \in \mathcal{X}} P(x|y) \cdot \log_2\left(\frac{1}{P(x|y)}\right)$$

$$\Rightarrow H(X|Y) = \sum_{y \in \mathcal{Y}} \sum_{x \in \mathcal{X}} P(y) \cdot P(x|y) \cdot \log_2\left(\frac{1}{P(x,y)/P(y)}\right)$$

$$\Rightarrow H(X|Y) = \sum_{y \in \mathcal{Y}} \sum_{x \in \mathcal{X}} P(x,y) \log_2\left(\frac{P(y)}{P(x,y)}\right)$$

$$\Rightarrow \boxed{H(X|Y) = \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} P(x,y) \log_2\left(\frac{P(y)}{P(x,y)}\right)}$$

Similarly,

$$\Rightarrow \boxed{H(Y|X) = \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} P(x,y) \log_2\left(\frac{P(x)}{P(x,y)}\right)}$$

$$H(X|Y) = \frac{1}{3}\log_2\left(\frac{2/3}{1/3}\right) + \frac{1}{3}\log_2\left(\frac{2/3}{1/3}\right) + \frac{1}{3}\log_2\left(\frac{1/3}{1/3}\right) + 0 \quad \left| \ \lim_{p \to 0^+} p \log(1/p) = 0 \right.$$
$$= \frac{1}{3} + \frac{1}{3} + 0 = \boxed{2/3}$$

$$H(Y|X) = \frac{1}{3}\log_2\left(\frac{2/3}{1/3}\right) + \frac{1}{3}\log_2\left(\frac{2/3}{1/3}\right) + \frac{1}{3}\log_2\left(\frac{1/3}{1/3}\right) + 0 \quad \left| \ \lim_{p \to 0^+} p \log(1/p) = 0 \right.$$
$$= \frac{1}{3} + \frac{1}{3} + 0 = \boxed{2/3}$$

(d) $H(Y) - H(Y|X)$

$$= \log_2 3 - \frac{2}{3} - \frac{2}{3} = \boxed{\log_2 3 - \frac{4}{3}}$$

(e) $H(X) - H(X|Y)$

$$= \log_2 3 - \frac{2}{3} - \frac{2}{3} = \boxed{\log_2 3 - \frac{4}{3}}$$

(f) $I(X;Y) = H(X) - H(X|Y)$

$$\Rightarrow I(X;Y) = \boxed{\log_2 3 - \frac{4}{3}}$$