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ASSIGNMENT 3

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Q1) Show that the subset $M = \{y = (\eta_j) \mid \sum \eta_j = 1\}$ of complex nos $\mathbb{C}^n$ is complete & convex. Find the vector of minimum norm in M.

Ans) Convex part:

Let $y_1 = (\eta_j^1) = (\eta_1^1, \eta_2^1, \dots \eta_n^1)$
such that $\sum_{i=1}^{n} \eta_i^1 = 1$

Let $y_2 = (\eta_j^2) = (\eta_1^2, \eta_2^2, \dots \eta_n^2)$
such that $\sum_{j=1}^{n} \eta_j^2 = 1$

$y_1, y_2 \in Y$.

Let $z = \alpha y_1 + (1-\alpha) y_2$ $\quad \alpha \in [0,1]$

For M to be convex, every point of the line segment between y_1 & y_2 must belong to M.

$$z = (\alpha\eta_1^1, \alpha\eta_2^1, \alpha\eta_3^1, \dots \alpha\eta_n^1) + ((1-\alpha)\eta_1^2, (1-\alpha)\eta_2^2, \dots (1-\alpha)\eta_n^2)$$

$$= (\alpha\eta_1^1 + (1-\alpha)\eta_1^2, \alpha\eta_2^1 + (1-\alpha)\eta_2^2, \dots, \alpha\eta_n^1 + (1-\alpha)\eta_n^2)$$

for z to be in M, $\sum \alpha\eta_i^1 + (1-\alpha)\eta_i^2 = 1$

$\Rightarrow \alpha \sum \eta_i^1 + (1-\alpha) \sum \eta_i^2$

$\Rightarrow \alpha \cdot 1 + (1-\alpha) \cdot 1$

$\Rightarrow 1$

$\therefore$ regardless of α, z will always be in M.

$\therefore$ M is convex.

Completeness:

We know $\mathbb{C}^n$ is a Hilbert space (complete inner product space).

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A subspace M of a Hilbert space $\mathbb{C}^n$ is complete iff M is closed in $\mathbb{C}^n$.

Proof:

Let M be complete, for every $x \in \bar{M}$ there is a sequence (x_n) in M which converges to x. Since (x_n) is cauchy, M is complete (x_n) converges in M with a unique limit value.

Hence $x \in M$.

This proves M is closed because $x \in \bar{M}$ was arbitrary.

As $x_n \to x$, $\langle x_n, y \rangle \longrightarrow \langle x, y \rangle$

Let M be closed and (x_n) is a cauchy in M. Then $x_n \to x \in \mathbb{C}^n \Rightarrow x \in \bar{M}$ and $x \in M$ since $M = \bar{M}$.

This proves completeness of M.

Vector of minimum norm in M:

The vector of minimum norm will be a vector z which is $\perp$ to M.

$$\langle z, y \rangle = 0 \qquad \forall\, y \in M.$$

$z \in \mathbb{C}^n$, $y \in M$

$$z = (\xi_1, \xi_2, \dots \xi_n)$$
$$y = (\eta_1, \eta_2, \dots \eta_n) \qquad \sum_{i=1}^{N} \eta_i = 1$$

$$\langle z, y \rangle = \xi_1 \bar{\eta}_1 + \xi_2 \bar{\eta}_2 + \dots \xi_n \bar{\eta}_n = 0$$

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Q) Show that the vector space X of all real valued functions on $[-1,1]$ is the direct sum of the set of all even continous functions and the set of all odd continous functions on $[-1,1]$.

Ans) We need to prove $C[-1,1] = C_e[-1,1] \oplus C_o[-1,1]$

$C_e \rightarrow$ set of all even continuous fns
$C_o \rightarrow$ set of all odd continuous fns.

Let f be a fn of $C[-1,1]$

$f(x) = g_e(x) + g_o(x)$ $\longleftrightarrow$ goal and the representation must be unique.

$f \in C[-1,1]$, $g_e \in C_e[-1,1]$, and $g_o \in C_o[-1,1]$

Let

$g_e(x) = \dfrac{f(x)+f(-x)}{2}$, $g_o(x) = \dfrac{f(x)-f(-x)}{2}$

$g_e(-x) = g_e(x)$, $g_o(-x) = -g_o(x)$

$\longrightarrow$ Basic conditions satisfied.

$\therefore f(x) = g_e(x) + g_o(x)$

Uniqueness:

Let $h_e(x) \in C_e[-1,1]$ and $h_o(x) \in C_o[-1,1]$ exist as

$f(x) = h_e(x) + h_o(x) = g_e(x) + g_o(x)$

$\Rightarrow h_e(x) + h_o(x) - g_e(x) - g_o(x) = 0$

$\Rightarrow \underbrace{h_e(x) - g_e(x)}_{\text{even fn}} = \underbrace{g_o(x) - h_o(x)}_{\text{odd fn}}$

$\Rightarrow K(x) = K(-x) = -K(x) \Rightarrow K(x) = 0$

$\therefore h_e(x) = g_e(x)$ & $g_o(x) = h_o(x)$

$\therefore$ Uniqueness satisfied.

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Q3) Let $X = \mathbb{R}^2$. Find $M^\perp$ if M is

(a) $\{x\}$, where $x = (\xi_1, \xi_2) \neq 0$

(b) a linearly independent set $\{x_1, x_2\} \subset X$

Ans) a) $x = (\xi_1, \xi_2) \neq 0 \in X$

$y = (\eta_1, \eta_2) \in M^\perp$

Since $y \in M^\perp$, $\langle x, y \rangle = 0$

$\Rightarrow \xi_1 \eta_1 + \xi_2 \eta_2 = 0$

$\Rightarrow -\xi_1 \eta_1 = \xi_2 \eta_2 \quad \Rightarrow -\dfrac{\xi_1}{\xi_2} = \dfrac{\eta_2}{\eta_1}$

$\therefore \eta_2 = k \cdot -\xi_1, \quad \eta_1 = k \cdot \xi_2$

$\therefore y = (k\xi_2, -k\xi_1)$

b) Let $y \in (\eta_1, \eta_2) \in M^\perp$

$y \perp x_1$, $y \perp x_2$ as $y \perp M$. $x_1, x_2 \in M$.

If x_1, x_2 are linearly independent, then y must be the zero vector.

$M^\perp$ contains $\{0\}$.

Ex:

$x_1 \to (1, 0)$, $x_2 \to (0, 1)$

$x_1 \perp x_2$

$y = \{0\}$ satisfies here.

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Q4) Show that $Y = \{x \mid x = (\xi_i) \in l^2, \xi_{2n} = 0, n \in \mathbb{N}\}$ is a closed subspace of l^2 and find $Y^\perp$. What is $Y^\perp$ if $Y = \text{span}\{e_1, e_2, \dots e_n\} \subset l^2$ where $e_j = (\delta_{jk})$?

Ans) $l^2 = (\xi_1, \xi_2, \dots), \sum_{i=1}^{\infty} |\xi_i|^2 < \infty$

$Y = (\xi_1, 0, \xi_3, 0 \dots)$.

Y is a proper subspace of l^2

Clearly the sum of the elements of subspace Y converges at a slower rate compared to l^2.

If $x_n \to x$, iff $\langle x_n, y\rangle \to \langle x, y\rangle \ \forall y \in l^2$

$x_n \in Y \Rightarrow x \in Y$.

$\langle x_n, e_{2k}\rangle = 0 \to \langle x, e_{2k}\rangle = 0$

$e_{2k} = (0, 0, \dots 1, 0, \dots 0)$ $1 \to 2k^{th}$ entry.

Every $2k$th entry of x vanishes.
$\Rightarrow x \in Y$.

In l^2 (Hilbert space),

$$\langle x, y\rangle = \sum_{j=1}^{\infty} \xi_j \overline{\eta_j} \qquad y = (\eta_j)$$

let $y \in Y^\perp$ then $\forall x \in Y, \langle x, y\rangle = 0$

$\Rightarrow \langle x, y\rangle = \xi_1 \overline{\eta_1} + 0 + \xi_2 \overline{\eta_2} + \dots \xi_n \overline{\eta_n}$
$+ \dots$ [illegible] $= 0$

$e_1 = (1, 0, \ldots 0)$

$e_2 = (0, 1, 0 \ldots 0)$

$\vdots$

$e_n = (0, 0, \ldots 1)$

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Q5) Let A and $B \supset A$ be nonempty subsets of an inner product space X. Show that:

a) $A \subset A^{\perp\perp}$

b) $B^{\perp} \subset A^{\perp}$

c) $A^{\perp\perp\perp} = A^{\perp}$

Ans)

a) $x \in A$

$\Rightarrow x \perp A^{\perp}$

$\Rightarrow x \in (A^{\perp})^{\perp} \longrightarrow x \in A^{\perp\perp}$

$\Rightarrow A \subset A^{\perp\perp}$

b) $x \in B^{\perp}$

$\Rightarrow x \perp B(\supset A)$ as A is subset of B, $\Rightarrow x \perp A$

$\Rightarrow x \in A^{\perp}$

$\Rightarrow B^{\perp} \subset A^{\perp}$

c) $A^{\perp\perp\perp} = (A^{\perp})^{\perp\perp} \supset A^{\perp} \longrightarrow (i)$ From (a) $(A \subset A^{\perp\perp})$

$A \subset A^{\perp\perp}$

$\Rightarrow A^{\perp} \supset A^{\perp\perp\perp}$ From (b)

$\longrightarrow (ii)$

$\therefore$ Combing (i) & (ii)

$A^{\perp} = A^{\perp\perp\perp}$

Q6) Show that the annhilator $M^{\perp}$ of a set $M \neq \phi$ in an I.P.S X is a closed subspace of X.

Ans An orthogonal complement is a special annhilator where by definition, the annhilator $M^{\perp}$ of a set $M \neq \phi$ in an I.P.S X is the set

$$M^{\perp} = \{x \in X \mid x \perp M\}$$

$M^{\perp}$ is a vector space.

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Let x be an element of $\overline{M^\perp}$. Hence there is a sequence (x_n) in $M^\perp$ such that $x_n \to x$.

Now for any $y \in M$, we have
$\langle x, y \rangle = \langle \lim_{n\to\infty} x_n, y \rangle = \lim_{n\to\infty} \langle x_n, y \rangle = 0$

$\therefore x \in M^\perp$

$\therefore M^\perp$ is a closed subspace.

$M^\perp = \{x \in H : \langle x, y \rangle = 0 \quad \forall y \in M\}$

For any scalar α, β and $f, g \in M^\perp$

$$(\alpha f + \beta g)x = \alpha f(x) + \beta(g(x))$$
$$= 0 + 0 \quad \forall x \in M.$$

$\therefore \alpha f + \beta g \in M^\perp$ so $M^\perp$ is a vector space of dual of X.

Let $f \in \overline{M^\perp}$. Then there exists a sequence of bounded linear functionals of X which are $\{f_n\} \subset M^\perp$ such that $\lim_{n\to\infty} f_n = f$.

$\therefore \forall x \in M$
$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} 0 = 0.$$

$f \in M^\perp$.

So $M^\perp$ is closed.