OPEN QUANTUM SYSTEMS AND QUANTUM THERMODYNAMICS

M20Temp16 Monsoon 2020

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Date: AUGUST 21, 2020

Problem Set - I

Date of Submission: August 31, 2020 23:55 HRS

Q1. Consider a wavefunction $\psi(x) = e^{\frac{-ipx}{\hbar}}$, where x represents position and p represents momentum. Further consider the momentum operator $\hat{p} = -i\hbar\nabla$, and position operator $\hat{x}$ acting on the wave function $\psi(x)$.
Prove the following: $[\hat{x}, \hat{p}] = i\hbar$.

Hint: $\hat{x}\psi(x) = x\psi(x)$

Q2. Using the above commutation relation between $\hat{x}$ and $\hat{p}$, determine the commutation relation $[\hat{x}, \hat{p}^2]$. Further determine the commutation relation $[\hat{x}, \hat{p}^n]$, where n is an arbitrary positive integer.

Q3. Consider a Hermitian operator $\hat{Q}$ having the following spectral decomposition $\hat{Q} = \sum_i q_i |i\rangle\langle i|$.
Then prove the following: $\sin\hat{Q} = \sum_i \sin q_i |i\rangle\langle i|$.

Q4. Prove the following commutation identity (with full working):

$$[[A,C],[B,D]] = [[[A,B],C],D] + [[[B,C],D],A] + [[[C,D],A],B] + [[[D,A],B],C]$$

Q5. Consider a ket space spanned by the eigenkets $\{|a_i\rangle\}$ of a Hermitian operator $\hat{A}$. Assume there is no degeneracy.
Prove that: $\prod_{a_i}(\hat{A} - a_i) = \hat{O} \rightarrow$ null operator. Here, $\prod_{i=1}^{n} A_i = A_1 \cdot A_2 \cdot A_3 \cdot \ldots \cdot A_n$

Q6. Suppose A' and A'' are matrix representations of an operator $\hat{A}$ on a vector space V with respect to two different orthonormal bases, $|v_i\rangle$ and $|w_i\rangle$. Characterize the relationship between A' and A''.

Q7. Consider two noncommuting operators $\hat{A}$ and $\hat{B}$, i.e., $[\hat{A}, \hat{B}] \neq 0$. Show that they cannot have a complete set of common eigenfunctions.

Q8. Consider the Pauli matrices:

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Prove the following:

(i) $[\hat{S}_i, \hat{S}_j] = i\varepsilon_{ijk}\hbar\hat{S}_k$. Here, ε_{ijk} is the Levi-Civita symbol.

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } ijk \text{ is } (x,y,z) \text{ or } (y,z,x) \text{ or } (z,x,y) \\ -1 & \text{if } ijk \text{ is } (z,y,x) \text{ or } (x,z,y) \text{ or } (y,z,x) \\ 0 & \text{if repetition occurs} \end{cases}$$

(ii) $\{\hat{S}_i, \hat{S}_j\} = \frac{\hbar^2}{2}\delta_{ij}\mathbb{I}$. Here, δ_{ij} is the Kronecker delta.

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

(iii) $[\hat{S}^2, \hat{S}_i] = 0$. Here, $\hat{S}^2 = \hat{S} \cdot \hat{S} = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2$