Q1. (a) Let X denote the number of flips of a fair coin until the first head appears.

Find the entropy of X in bits.

The following expressions may be useful

$$\sum_{n=1}^{\infty} r^n = \frac{r}{1-r} \; ; \quad \boxed{\sum_{n=1}^{\infty} n r^n = \frac{r}{(1-r)^2}}$$

(b) Let Y denote the no. of flips until the second head appears.

Show that $H(Y) \le 2H(X)$

2. Let the joint distribution of Random Variables be given as follows

$\underline{P(X=x, Y=y)}$ (which we write as $p(x,y)$)

is given by the following table

$X \downarrow$ \ $Y \rightarrow$	0	1
0	1/3	1/3
1	0	1/3

$\rightarrow p(X=0, Y=1)$

$p(X=0) = P(X=0, Y=0) + P(X=0, Y=1)$

(a) Find $H(X)$, $H(Y)$ ←

(b) $H(X,Y)$ ←

(c) $H(X|Y)$, $H(Y|X)$ ← $p(Y=0|X=0) = \frac{P(Y=0, X=0)}{P(X=0)}$

(d) $H(Y) - \underline{H(Y|X)}$

(e) $H(X) - H(X|Y)$

(f) $I(X;Y)$ ←

Answers

Q1. Prob distribution of X $\qquad P(H) = p$, $P(T) = 1-p$

$$\mathcal{X} = \{1, \ldots, \} = \mathbb{N}$$

For any $n \in \mathbb{N}$; $P(X=n) = (1-p)^{n-1} p$ [because the losses are independent]

$$= \frac{1}{2^n}$$

$$H(X) = \sum_{n=1}^{\infty} p(X=n) \log_2 \frac{1}{P(X=n)}$$

$$= \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot n = 2. \text{ bits}$$ (because base of log is 2)

(b) $P(Y=n)$ = Prob that the n^{th} toss is the second head we get

Let X_1 & X_2 are identically distributed random variables as X. X_1 denotes the first toss which gives us H.

& $X_2 = Y - X_1$ (X_2 denotes the additional no of tosses we have to wait after getting first H)

$$\boxed{Y = X_1 + X_2}$$

For $n \in \mathbb{N} \setminus \{1\}$,

$$P(Y=n) = P(X_1 + X_2 = n)$$

$$= \sum_{n_1=1}^{n-1} P(X_1 = n_1, X_2 = n - n_1)$$

$$= \sum_{n_1=1}^{n-1} \frac{1}{2^{n_1}} \cdot \frac{1}{2^{n-n_1}} = \sum_{n_1=1}^{n-1} \frac{1}{2^n}$$

X_1, X_2 are independent

$P_{X_1, X_2} = P_{X_1} P_{X_2}$

Independence of X_1, X_2 is not by our intuition but by definition of independence

$$P(Y=n) = \left(\frac{n-1}{2^n}\right)$$

Cross check: (1) $P(Y=n) \in [0,1]$ $\forall n \in \mathbb{N} \setminus \{1\}$

(2) $\sum_{n=2}^{\infty} P(Y=n) = 1$ ✓

$$H(Y) = -\sum_{n=2}^{\infty} p(y=n) \log_2 p(y=n)$$

$$= \sum_{n=2}^{\infty} \left(\frac{n-1}{2^n}\right) \log\left(\frac{2^n}{n-1}\right)$$

$$= \sum_{n=2}^{\infty} \frac{n(n-1)}{2^n} - \sum \frac{n-1}{2^n} \log(n-1)$$

$$H(Y) = 4 - \boxed{\sum_{n=2}^{\infty} \frac{n-1}{2^n} \log_2(n-1)} \rightarrow \text{positive value}$$

$$\boxed{H(X_1+X_2)} < 4$$

$$H(X_1, X_2) = H(X_1) + H(X_2)$$

[as X_1, X_2 are independent] Using lemma showed in class

$$= 2H(X) = 4$$

What this means is

$$H(Y) = \boxed{H(X_1+X_2) < H(X_1, X_2)} = 2H(X)$$