Exercise problems

Q1) Show that the real line is a metric space

Ans) The Real line 'R' is the set of all real numbers taken with the metric defined by

$$d(x,y) = |x-y|$$

Because of the modulus (absolute value), $d(x,y) \geq 0$ and will lie on R and finite.

$d(x,y) = 0 \iff x = y$

$|x-x| = 0$ or $|y-y| = 0$

$d(x,y) = d(y,x) \longrightarrow$ because of the modulus.

$d(x,y) \leq d(x,z) + d(z,y)$

$d(x,y) = |x-y| = |x-y+z-z| = |x-z+z-y|$

$\leq |x-z| + |z-y|$

$\leq d(x,z) + d(z,y)$

(R,d) is a metric

$\therefore$ R is a metric space.

Q2) Does $d(x,y) = (x-y)^2$ define a metric on the set of all real nos?

Ans) $d(x,y) = (x-y)^2$

$d(x,y)$ will ≥ 0 (square power), will lie on R and finite

$d(x,y) = 0 \iff x = y$

$(x-y)^2 = 0 \Rightarrow x^2 + y^2 - 2xy = 0$

$\Rightarrow x^2 + y^2 = 2xy$

$\Rightarrow \frac{x}{y} + \frac{y}{x} = 2$ $\Rightarrow n + \frac{1}{n} = 2$ $\Rightarrow n = 1$

$\therefore \frac{x}{y} = 1 \Rightarrow x = y$ ✓

$d(x,y) = d(y,x)$ (Because of the sq power)

$x^2 + y^2 - 2xy \rightarrow y^2 + x^2 - 2yx$

$\rightarrow (y-x)^2$

$\rightarrow d(y,x)$

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$d(x,y) \le d(x,z) + d(z,y)$

$(x-y)^2 = (x-y+z-z)^2$

$= ((x-z)+(z-y))^2$

$= (x-z)^2 + (z-y)^2 + 2(x-z)\cancel{(z-y)}$

$= d(x,z) + d(z,y) + 2(x-z)(z-y)$

Depending on the values of x, y & z the $\triangle$le inequality may hold or may not hold. $d(x,y) = (x-y)^2$ does not define a metric on $\mathbb{R}$.

Q3) Show that $d(x,y) = \sqrt{|x-y|}$ defines a metric on the set of all real numbers.

Ans) $d(x,y) \ge 0$ (modulus, +ve sq root), finite and real.

$d(x,y) = 0 \iff x = y$

$\sqrt{|x-y|} = 0 \Rightarrow |x-y| = 0 \quad \Rightarrow x-y=0$ or $y-x=0$

$\Rightarrow x = y$ ✓

$d(x,y) = d(y,x)$ (modulus)

$\sqrt{|x-y|} = \sqrt{|y-x|}$

$= d(y,x)$

$d(x,y) \le d(x,z) + d(z,y)$.

$\sqrt{|x-y|} = \sqrt{|x-y+z-z|} = \sqrt{|x-z+z-y|}$

We know

$|x-y| = |x-y+z-z| = |x-z+z-y|$

$\le |x-z| + |z-y|$

$\le (|x-z|^{1/2} + |z-y|^{1/2})^2$

$|x-y| \le (|x-z|^{1/2} + |z-y|^{1/2})^2$

$\sqrt{|x-y|} \le \sqrt{|x-z|} + \sqrt{|z-y|}$

$\le d(x,z) + d(z,y)$

$\therefore$ $\triangle$le inequality also holds.

$\therefore d(x,y) = \sqrt{|x-y|}$ is a metric on $\mathbb{R}$

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Q8) Show that another metric $\tilde{d}$ on the set X in 1.1-7 is defined by $\tilde{d}(x,y) = \int_a^b |x(t)-y(t)|\,dt$.

Ans) $\tilde{d}(x,y) = \int_a^b |x(t)-y(t)|\,dt$

$|x(t)-y(t)| \rightarrow$ well defined since it is a closed interval.

$d(x,y)$ func will represent the area bounded by $|x(t)-y(t)|$ within the limit $x=a$, $x=b$ & x-axis.

We know Area ≥ 0, finite (intervals a & b) and real.

If Area $=0 \Rightarrow a=b$ or $x(t)=y(t)$

$a=b \downarrow$ Not possible

$x(t)=y(t) \downarrow$ only other explanation

$d(x,y) = d(y,x) \rightarrow$ modulus

$$\int_a^b |x(t)-y(t)|\,dt = \int_a^b |x(t)-z(t)+z(t)-y(t)|\,dt$$

$$\leq \int_a^b |x(t)-z(t)|\,dt + \int_a^b |z(t)-y(t)|\,dt$$

$$\leq d(x,z) + d(z,y)$$

Q14) Axioms of a metric (M1) to (M4) could be replaced by other axioms (without changing the definitions). For instance show that (M3) and (M4) could be obtained from (M2) and $d(x,y) \leq d(z,x)+d(z,y)$

Ans) $M(2) \rightarrow d(x,y)=0 \iff x=y$

$M(3) \rightarrow d(x,y) = d(y,x)$

$M(4) \rightarrow d(x,y) \leq d(x,z)+d(z,y)$

Given:

$$d(x,y) \leq d(z,x)+d(z,y) \;\; \& \;\; M(2).$$

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assume $z = y$

$\therefore d(z,y) = 0$

$d(x,y) \le d(\cancel{y},\cancel{x}) + d(y,y)$

$d(x,y) \le d(y,x) + 0$

assume $z = x$

$\therefore d(z,x) = 0$

$d(x,y) \le d(x,x) + d(x,y)$

$\le d(x,y)$

from these 2

$d(x,y) = d(y,x)$

$\downarrow$

M(3)

To prove M(4)

$d(x,y) \le d(z,x) + d(z,y)$ M(3)

$\le d(x,z) + d(z,y)$

$\longrightarrow$ M(4) or Δe inequality

Q15) Show that non-negetivity of a metric follows from M(2) to M(4)

Ans) M(2) $\rightarrow d(x,y) = 0 \iff x = y$

M(3) $\rightarrow d(x,y) = d(y,x)$

M(4) $\rightarrow d(x,y) \le d(x,z) + d(z,y)$

If we take $x = y$

$d(x,x) \le d(x,z) + d(z,x)$ M(2)

$0 \le d(x,z) + d(z,x)$ M(3)

$0 \le 2 \cdot d(x,z)$

$\Rightarrow d(x,z) \ge 0$

$\longrightarrow$ Non negetive.

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Q2) Using (6), show that the geometric mean of 2 +ve nos doesnt exceed the arithmetic mean

Ans) Based on the results of the auxillary inequality,

$$\alpha\beta \le \frac{\alpha^p}{p} + \frac{\beta^q}{q} \quad \text{given } \frac{1}{p} + \frac{1}{q} = 1$$

To prove AM ≥ GM, lets take $p = q = 2$

$\frac{1}{2} + \frac{1}{2} = 1$ (satisfies a conjugate exponents condition)

$$\therefore \alpha\beta \le \frac{\alpha^2}{2} + \frac{\beta^2}{2}$$

$$\alpha\beta + \alpha\beta \le \frac{\alpha^2}{2} + \frac{\beta^2}{2} + \alpha\beta \quad \text{(Adding } \alpha\beta \text{ on both sides)}$$

sign doesnt change because $\alpha\beta$ is +ve.

$$2\alpha\beta \le \frac{\alpha^2 + \beta^2 + 2\alpha\beta}{2}$$

$$\alpha\beta \le \frac{(\alpha+\beta)^2}{4}$$

$$\Rightarrow \sqrt{\alpha\beta} \le \frac{\alpha+\beta}{2}$$

$\sqrt{\alpha\beta} \rightarrow$ G.M of 2 elements

$\frac{\alpha+\beta}{2} \rightarrow$ A.M of 2 elements

$\therefore$ A.M ≥ G.M

Q3) Show that the cauchy-schwarz inequality (11) implies
$$(|\xi_1| + |\xi_2| + \cdots |\xi_n|)^2 \le n(|\xi_1|^2 + |\xi_2|^2 \cdots |\xi_n|^2)$$

Ans) The Holders inequality for sums states that

$$\sum_{j=1}^{\infty} |\xi_j \eta_j| \le \left(\sum_{k=1}^{\infty} |\xi_k|^p\right)^{1/p} \left(\sum_{m=1}^{\infty} |\eta_m|^q\right)^{1/q}$$

where $p > 1$ & $\frac{1}{p} + \frac{1}{q} = 1$.

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If we take $p=q=2$

$$\sum_{j=1}^{\infty} |\xi_j \eta_j| \le \left(\sqrt{\sum_{k=1}^{\infty} |\xi_k|^2}\right) \cdot \left(\sqrt{\sum_{m=1}^{\infty} |\eta_m|^2}\right)$$

$\downarrow$

Cauchy-Schwarz inequality.

Let us assume for $i = 1$ to n $\eta_n = 1$

$i > n$ $\eta_n = 0$

$$\sum_{j=1}^{\infty} |\xi_j \eta_j| \le \left[\left(\sum_{k=1}^{\infty} |\xi_k|^2\right) \cdot \left(\sum_{m=1}^{\infty} |\eta_m|^2\right)\right]^{1/2}$$

$$(|\xi_1| + |\xi_2| + \dots |\xi_n| + 0 + 0 \dots)$$

$$\le \left[(|\xi_1|^2 + |\xi_2|^2 + \dots)(\underbrace{1+1+\dots 1}_{n \text{ times}} + 0 + 0 \dots)\right]^{1/2}$$

$$|\xi_1| + |\xi_2| + \dots |\xi_n| \le \left[n \cdot (|\xi_1|^2 + |\xi_2|^2 + \dots)\right]^{1/2}$$

$$(|\xi_1| + |\xi_2| + |\xi_3| + \dots |\xi_n|)^2 \le n(|\xi_1|^2 + |\xi_2|^2 + \dots)$$

$$\le n(|\xi_1|^2 + |\xi_2|^2 + \dots |\xi_n|^2)$$

Q4) Find a seq which converges to zero but is not in any space l^p, where $1 \le p < \infty$

Ans) Each element in space l^p must be a sequence $x = (\xi_j) = (\xi_1, \xi_2, \dots)$ of numbers such that $|\xi_1|^p + |\xi_2|^p + \dots$ converges.

$$\therefore \sum_{j=1}^{\infty} |\xi_j|^p < \infty$$

and the metric is defined by:

$$d(x,y) = \left(\sum_{j=1}^{\infty} |\xi_j - \eta_j|^p \right)^{1/p}$$

$\therefore$ according to our question, we need a sequence which converges to 0 but not in any l^p space.

$$x = (\xi_1, \xi_2, \ldots \xi_n, \ldots)$$

$$\sum_{j=1}^{\infty} |\xi_j|^p < \infty \rightarrow \text{convergent condition} = 0 \checkmark$$

$$|\xi_1|^p + |\xi_2|^p + \ldots |\xi_n|^p + \ldots \longrightarrow 0$$

If it is not in l^p then $d(x,y)$ metric doesnt hold.

We can take the seq $(x_n) = \frac{1}{2^p}$ $i = 1, 2, \ldots$

(x_n) converges to zero but doesnt belong to l^p.

Q5) Find a sequence x which is in l^p with $p>1$ but $x \notin l^1$.

Ans) For a seq to be in l^1, each element in l^1 is a sequence $x_n = (\xi_1, \xi_2, \ldots)$ of numbers such that:

$$\sum_{j=1}^{\infty} |\xi_j| < \infty \quad \text{(converges)}$$

Metric:

$$d(x,y) = \left(\sum_{j=1}^{n} |\xi_j - \eta_j|^p \right)^{1/p} = \sum_{j=1}^{n} |\xi_j - \eta_j|$$

We can take the seq $x_n = (1/n)$

We know that $1 + \frac{1}{2} + \frac{1}{3} + \ldots \frac{1}{n}$ $n \to \infty$ is not convergent

even as each term is progressively decreasing (H.P)

$\therefore \frac{1}{n} \in l^1$ but $\left(\frac{1}{n}\right) \in l^p$ since $\Sigma n^{-p} < \infty$ if $p > 1$

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Q2) Let X be the space of all ordered n-tuples $x = (\xi_1, \xi_2, \ldots \xi_n)$ of real nos and
$d(x,y) = \max_i |\xi_i - \eta_i|$ where $y = (\eta_i)$.
Show that (X,d) is complete.

Ans) given $d(x,y) = \max_i |\xi_i - \eta_i|$

Let us consider any cauchy seq (x_m) in X. Writing $(x_m) = (\xi_1^{(m)}, \xi_2^{(m)}, \ldots \xi_n^{(m)})$. Since (x_m) is cauchy for every $\epsilon > 0$, there is an N such that
$d(x_m, x_r) < \epsilon$ $\quad m, r > N$

$\Rightarrow \max |\xi_i^{(m)} - \eta_i^{(r)}| < \epsilon$

As $m \to \infty$, $\xi_i^{(m)} \to \xi_k^{(m)}$

$\Rightarrow \max |\xi_k^{(m)} - \xi_k^{(n)}| < \epsilon$

This shows that for each each $(\xi_k^{(m)} - \xi_k^{(n)})$ pair for a fixed k will be $< \epsilon$. $(1 \le k \le n)$. This shows that it is a cauchy seq for real no
$\therefore d(x_m, x) < \epsilon$ $\quad x \to$ limit of (x_m)
and $\mathbb{R} \to$ complete space.
$\therefore$ We can say (X,d) is also a complete space, since (x_m) is arbitrary.

Q7) Let X be the set of all +ve integers and $d(m,n) = |m^{-1} - n^{-1}|$. Show that (X,d) is not complete.

Ans) Since X is the set of all +ve integers, it doesnt include 0.
$X = \{1, 2, 3, \ldots\}$
as $m, n \to \infty$
$d(m,n) = \left|\frac{1}{m} - \frac{1}{n}\right|$ tends towards zero.

classmate $d(m,n)$ is also a cauchy seq
$d(x_m, x_r) = \left|\frac{1}{x_m} - \frac{1}{x_r}\right| < \epsilon$ $\quad m, r > N$

They converge at 0.

$\therefore$ It is not complete because it is a non convergent Cauchy sequence.

Q8) Show that the subspace $Y \subset C[a,b]$ consisting of all $x \in C[a,b]$ such that $x(a) = x(b)$ is complete.

Ans) ~~To show Y is a complete subspace, we must~~

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Q11) Show that the space s we have $x_n \to x$ if and only if $\xi_i^{(n)} \to \xi_i$ for all $j = 1, 2, \ldots$ $x_n = (\xi_i^{(n)})$ and $x = (\xi_i)$.

Ans) Metric on 's' is given as

$$d(x,y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\xi_i - \eta_i|}{1 + |\xi_i - \eta_i|}$$

for any $|\xi_i - \eta_i|$, $\frac{|\xi_i - \eta_i|}{1 + |\xi_i - \eta_i|} < 1$.

The deciding term is $\frac{1}{2^i}$.

Let $x_n \to x$. For any i; there is every $\epsilon > 0$ there is an N such that

$$d(x_m, x_n) < \epsilon \quad \text{for every } m, n \geq N$$

$$\sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\xi_j^{(m)} - \xi_j^{(n)}|}{1 + |\xi_j^{(m)} - \xi_j^{(n)}|} < \epsilon$$

from this

$$\Rightarrow \frac{1}{2^i} \frac{|\xi_i^{(m)} - \xi_j|}{1 + |\xi_i^{(m)} - \xi_i|} \leq d(x_m, x) < \frac{\epsilon}{2^j(1+\epsilon)}$$

Hence $|\xi_j^{(m)} - \xi_j| < \epsilon \quad (n > N)$

For every fixed j, the sequence $(\xi_j^{(1)}, \xi_j^{(2)}, \ldots)$ is a cauchy sequence. We can say $\xi_j^{(m)} \to \xi_j$ as $m \to \infty$, and show $x \in s$ and $x_m \to x$.

Q12) Using prob-11 show that the sequence space 's' is complete.

Ans) from the above question we got

$$|\xi_i^{(m)} - \xi_i| < \epsilon$$

write it as

$$|\xi_i^{(m)} - \xi_i| \leq \epsilon$$

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Since $(x_m) = (\xi_j^{(m)}) \in s$, there is a real number y such that $|\xi_j^{(m)}| < y$ for all j

$$\therefore |\xi_j| = |\xi_j - \xi_j^{(m)} + \xi_j^{(m)}|$$
$$\leq |\xi_j - \xi_j^{(m)}| + |\xi_j^{(m)}|$$
$$\leq \epsilon + k_m$$

Inequality holds for all j
(ξ_j) is a bounded seq of nos. This means $x = (\xi_j) \in s$.

$$d(x_m, x) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\xi_i^{(m)} - \xi_i|}{1 + |\xi_i^{(m)} - \xi_i|} \leq \epsilon$$

$x_m \to x$
Since (x_m) was arbitrary, $s \to$ complete.

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Q15) Let X be the metric space of all real sequences $x = (\xi_i)$ each of which has finitely many non zeros terms, and $d(x,y) = \sum |\xi_i - \eta_i|$ where $y = (\eta_i)$

Note that this is a finite sum but the number of terms depend on x & y. Show that $(x_n) = (\xi_i^{(n)})$, $\xi_j^{(n)} = j^{-2}$ for $j = 1, \ldots n$ and $\xi_j^{(n)} = 0$ for $j > n$

is a Cauchy but not converge

Ans) $x_n = (\xi_1^{(n)}, \xi_2^{(n)}, \ldots \xi_n^{(n)}, 0\ 0\ 0 \ldots)$

$x_r = (\xi_1^{(r)}, \xi_2^{(r)}, \ldots \xi_r^{(r)}, 0\ 0 \ldots)$

Assume $n > r$

$x_n = \left(1, \frac{1}{4}, \frac{1}{9}, \ldots \frac{1}{n^2}, 0\ 0\ 0 \ldots\right)$

$x_r = \left(1, \frac{1}{4}, \frac{1}{9}, \ldots \frac{1}{r^2}, 0\ 0\ 0 \ldots\right)$

$$\therefore d(x_n, x_r) = \sum_{i=1}^{r} |0| + \sum_{i=r+1}^{n} \left|\frac{1}{i^2}\right| + \sum_{i=n+1}^{\infty} |0|$$

$$= 0 + \sum_{i=r+1}^{n} \left|\frac{1}{i^2}\right| + 0$$

$$= \sum_{i=r+1}^{n} \left|\frac{1}{i^2}\right|$$

If it is a cauchy seq

$$d(x_n, x_r) < \epsilon \Rightarrow \sum_{i=r+1}^{n} \left|\frac{1}{i^2}\right| < \epsilon$$

$n > r > N$

Let $x_n = (\xi_j) \in X$

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$$d(x_n, x) = |1 - \xi_1| + \left|\frac{1}{4} - \xi_2\right| + \cdots + \left|\frac{1}{r^2} - \xi_r\right|$$

$$+ \frac{1}{(r+1)^2} + \frac{1}{(r+2)^2} + \cdots \frac{1}{n^2},$$

$\xi_j = 0$ for $j > r$ (r is fixed)

x_n doesnt converge to any x as $d(x_n, x) \to 0$ is not possible as r is fixed.

$\therefore$ It is cauchy but not convergent.