UNIT 13 LINEAR PARTIAL DIFFERENTIAL EQUATIONS

Structure

13.1 INTRODUCTION

In Units 10-12 we have presented the necessary tools required for the study of partial differential equations (PDEs). Such equations arise in geometry, physics and many other areas when the number of independent variables in the problem under discussion is two or more. In such cases, any dependent variable is likely to be a function of more than one variable, so that it possesses not the ordinary derivatives with respect to a single variable, but partial derivatives with respect to several variables. In this unit we begin with the origin of partial differential equations, restricting ourselves to one dependent variable z and two independent variables (x and y).

Unlike ordinary differential equations, where equations are either linear or non-linear (refer Unit 1 of Block 1), partial differential equations of the first order have further classifications of linear equations, which we have taken up in this unit. Also, the classification of integrals of partial differential equations of first order, as made by Lagrange (1736-1813) in 1769 and the Lagrange's method of finding integral of linear first order partial differential equation, with its geometrical interpretation, has been discussed here.

Later, in the investigations made by Cauchy between the years 1819-1842, he affirmed the existence of integral of first order partial differential equation in a theorem called Cauchy's theorem. In the year 1875, the Russian mathematician Madam Kowalevsky proved the existence of this integral subject to assigned conditions.

In this unit we have also touched upon Cauchy's theorem and the notion of characteristics for first order linear partial differential equations.

Objectives

After studying this unit you should be able to

- describe the origin of partial differential equations;
- identify linear, semi-linear, quasi-linear and non-linear PDEs of first order;
- distinguish the integrals of first order PDEs into the complete integral, the general integral, the singular integral and the special integral;
- use Lagrange's method for solving the first order linear PDEs;
- state and appreciate the Cauchy's problem;
- outline the role of characteristics.

13.2 ORIGIN OF THE FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

Partial differential equations, as we have already mentioned in Sec.13.1, can arise in a variety of ways in geometry, physics, mathematics, etc. We begin by examining the interesting question of how they arise. We consider the various situations one by one.

Surfaces of revolution

Let us consider the equation

$$z = f(r),\ r = (x^2+y^2)^{1/2}, \qquad \ldots (1)$$

where f is an arbitrary function of class C^1 on some domain D. Then Eqn. (1) represents all surfaces of revolution with z-axis as the axis of revolution; for example, sphere, cone, etc.

On differentiating Eqn. (1), with respect to x and y respectively, we obtain

$$\frac{\partial z}{\partial x} = p = f'(r)\frac{\partial r}{\partial x} \text{ and } \frac{\partial z}{\partial y} = q = f'(r)\frac{\partial r}{\partial y}$$

where

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r}$$

and

$$\frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\Rightarrow\ p = \frac{x}{r} f'(r) \text{ and } q = \frac{y}{r} f'(r) \qquad \ldots (2)$$

Eliminating the function f(r) from Eqns.(2), we get

$$yp - xq = 0, \qquad \ldots (3)$$

which is a partial differential equation of the first order.

Note that throughout the discussion of partial differential equations with z as dependent variable and x and y as independent variables, we shall be denoting the partial derivatives of z with respect to x and y by p and q respectively.

Two-parameter family of surfaces

Consider an equation of a two-parameter family of surfaces

$$z = F(x,y,a,b), \qquad \ldots (4)$$

where a and b are two parameters. If we differentiate Eq.(4) with respect to x and y respectively, we get

$$p = F_x\,(x,y,a,b) \qquad \ldots (5)$$

and

$$q = F_y\,(x,y,a,b) \qquad \ldots (6)$$

We can solve any two of these three Eqs.(4), (5) and (6) to find a and b in terms of x, y, p, q. This would be possible provided that

$$\left.\begin{array}{ll} \text{a)} & F_a F_{xb} - F_b F_{xa} \neq 0, \quad \text{(if Eqns.(4) and (5) are chosen)} \\ \text{b)} & F_{xa}\ F_{yb} - F_{xb}\ F_{ya} \neq 0 \quad \text{(if Eqns.(5) and (6) are chosen)} \\ \text{c)} & F_a F_{yb} - F_b F_{ya} \neq 0 \quad \text{(if Eqns.(4) and (6) are chosen)} \end{array}\right\} \qquad \ldots (7)$$

Thus, to solve for a and b, we require either (7a) or (7b) or (7c).

Substituting the values of a and b so obtained from any two equations of (7) into the third equation, we will get a relation of the form

$$f(x,y,z,p,q) = 0, \qquad \ldots (8)$$

which is a general **partial differential equation of first order.**

Example 1: Obtain the PDE corresponding to

$$z^2(1+a^3) = 8(x+ay+b)^3$$

Solution : The given two-parameter family of surfaces is

$$z^2(1+a^3) = 8(x+ay+b)^3 \qquad \dots (9)$$

Differentiating Eqn. (9) w.r.t. x and y respectively, we obtain

$$zp(1+a^3) = 12(x+ay+b)^2 \qquad \dots (10)$$

and $$zq(1+a^3) = 12a(x+ay+b)^2 \qquad \dots (11)$$

In order to obtain the PDE we have to eliminate a and b from Eqns. (10) and (11). Taking the cube of Eqns. (10) and (11) and adding, we get

$$z^3(1+a^3)^3(p^3+q^3) = 12^3(x+ay+b)^6(1+a^3)$$

$$= \frac{12^3}{8^2} z^4 (1+a^3)^3 \text{ (using Eqn. (9))}$$

$$\Rightarrow (P^3+q^3) = 27z, \qquad \dots (12)$$

which is the required PDE.

Surfaces of the form F(u,v) = 0

Let us consider surfaces of the form

$$F(u,v) = 0, \qquad \dots (13)$$

where $u = u(x,y,z)$, $v = v(x,y,z)$ are known functions of x, y and z, and F is an **arbitrary function** of u and v. If we differentiate relation (13) with respect to x and y, respectively, we obtain

$$\frac{\partial F}{\partial u}(u_x + pu_z) + \frac{\partial F}{\partial v}(v_x + pv_z) = 0 \qquad \dots (14)$$

$$\frac{\partial F}{\partial u}(u_y + qu_z) + \frac{\partial F}{\partial v}(v_y + qv_z) = 0 \qquad \dots (15)$$

To eliminate $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$ from Eqns. (14) and (15), we calculate $\frac{\partial F/\partial u}{\partial F/\partial v}$ from Eqns. (14) and (15) and equate them to obtain

$$\frac{(v_x + pv_z)}{(u_x + pu_z)} = \frac{(v_y + qv_z)}{(u_y + qu_z)}$$

$$\Rightarrow \quad p(v_z u_y - u_z v_y) + q(u_z v_x - u_x v_z) = u_x v_y - v_x u_y$$

$$\Rightarrow \quad p\frac{\partial(u,v)}{\partial(y,z)} + q\frac{\partial(u,v)}{\partial(z,x)} = \frac{\partial(u,v)}{\partial(x,y)}, \qquad \dots (16)$$

which is a partial differential equation of first order.

Example 2 : Eliminate the arbitrary function F from the equation $F(z-x, xy) = 0$ and obtain the PDE.

Solution : Let $u = z-x$ and $v = xy$, then

$$F(z-x, xy) = F(u,v) = 0$$

From Eqn. (16), we have

$$p\frac{\partial(u,v)}{\partial(y,z)} + q\frac{\partial(u,v)}{\partial(z,x)} = \frac{\partial(u,v)}{\partial(x,y)}$$

$$\Rightarrow xp - yq - x = 0,$$

Integrating factor

The problem of finding an integrating factor for a particular form of the Pfaffian equation

$$M(x,y)dx + N(x,y)dy = 0$$

consists in determining a function $\mu(x,y)$ for which (μ Mdx + μ Ndy) is an exact differential. This leads to

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N) \qquad \ldots (17)$$

which is a partial differential equation of first order for the function $\mu(x,y)$ and for known M and N.

Thus, there are many situations which lead to partial differential equations of first order. You can yourself obtain a few PDEs while doing the following exercises.

E 1) Eliminate the arbitrary constants a and b from the following equations and obtain the partial differentiable equations.

a) $z = ax + by + a^2 + b^2$

b) $(x-a)^2 + (y-b)^2 + z^2 = 1$,

c) $z = x + ax^2y^2 + b$

E 2) Eliminate the arbitrary function F from the following equations and obtain the PDEs.

a) $z = F(\frac{xy}{z})$

b) $F(x^2+y^2+z^2, z^2-2xy) = 0$.

You may recall that in Sec.1.2 of Unit 1, Block 1, we classified the differential equations depending upon the degree of dependent variables and its derivatives into two classes, namely, linear and non-linear. We term the ODE which is not linear as the non- linear one. But, in the case of PDEs of the first order, as we have already mentioned in the introduction of this unit, these equations have further classifications. If a partial differential equation of the first order is not linear, it can be quasi-linear, semi-linear or non-linear. We now take up this classification.

13.3 CLASSIFICATION OF THE FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

Consider the general form of first order PDE given by Eqn. (8), viz.,

$$f(x,y,z,p,q) = 0,$$

with $p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}$

We can classify Eqn. (8) into the following types

(a) If Eqn. (8) can be written in the form

$$P(x,y)p + Q(x,y)q = R(x,y)z, \qquad \ldots (18)$$

it is called a **linear** PDE of first order. Here P, Q and R are functions only of independent variables x and y.

For instance, the equation

$$x^2p + y^2q = (x+y)z \qquad \ldots (19)$$

is a linear equation. Also, Eqn. (3), discussed earlier is a linear equation.

(b) If we can write Eqn.(8) in the form

$$P(x,y)P + Q(x,y)q = R(x,y,z) \qquad \dots (20)$$

it is called a **semi-linear** PDE of first order. Here P and Q are functions of only independent variables whereas R is an arbitrary function of both dependent and independent variables.

Equation

$$px(x+y) = qy(x+y) - (x-y)(2x+2y+z^2) \qquad \dots (21)$$

is semi-linear.

(c) If it is possible to express Eqn. (8) in the form

$$P(x,y,z)p + Q(x,y,z)q = R(x,y,z), \qquad \dots (22)$$

then it is called a **quasi-linear PDE** of first order.

Equations

$$z(xp-yq) = y^2-x^2, \qquad \dots (23)$$

$$(y+zx)p - (x+yz)q = x^2 - y^2 \qquad \dots (24)$$

and

$$x(y^2+z)p - y(x^2+z)q = (x^2-y^2) \qquad \dots (25)$$

are quasi-linear equations. You may **note** that a quasi-linear equation is linear in its highest derivative. In Eqns. (23)-(25), the highest derivative is of order one and its power throughout is one.

You might have also noticed here that the linear and semi-linear equations are special cases of the quasi-linear equation.

If Eqn. (8) is none of the types mentioned above, we call it a **non-linear** PDE of the first order.

For instance, we cannot put equations

$$z^2(1+p^2+q^2) = 1 \qquad \dots (26)$$

$$\text{and } 2(y+zp) = q(xp+yq) \qquad \dots (27)$$

in any of the forms discussed above. Eqns. (26) and (27) are non-linear equations. Eqn. (12), which we obtained in Example 1, is also non-linear.

Note that unlike ODEs, linearity in PDEs does not depend on the degree of the dependent variable z.

You may now try the following exercise.

E 3) Classify the following equations into linear, semi-linear, quasi-linear and non-linear ones.

a) $xp + yq = zx + x^2 + y^2$

b) $9x^2y^2z = 3px^3y^2 + 3qx^2y^3 + pq$

c) $xp + 2yq = x^2/y$

d) $py - qx = x^2-y^2$

e) $(x^2-y^2-z^2)p + 2xyq = 2xz$

f) $(xz+y^2)p + (yz-2x^2)q + 2xy + z^2 = 0$

g) $\cos(x+y)p + \sin(x+y)q = z + \dfrac{1}{z}$

h) $p^2+q^2+2px+2qy+1 = 0$

Having done the classification of PDEs of first order, you can now classify any given equation of the first order. Next, your natural curiosity may lead you to enquire about its solution. Before embarking on a discussion of the methods of solution and giving such classes of equations which are easily integrable, we first define what we mean by a solution of a PDE of first order in the next section. We shall also classify different types of integrals that might arise and give the relation between them.

13.4 SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

Consider a general PDE of the first order as given by Eqn. (8), viz.,

$$f(x,y,z,p,q) = 0$$

By a solution of this equation we mean a function $\phi(x,y)$ of class C^1 defined for all x,y on some domain D such that

$$f(x,y,\phi(x,y), \phi_x(x,y), \phi_y(x,y)) = 0 \qquad \ldots.(28)$$

If we write $z = \phi(x,y)$, then this solution represents a family of surfaces in the xyz-space.

For example, equation

$$x^2+y^2+(z-c)^2 = a^2, \qquad \ldots.(29)$$

which represents the set of all spheres whose centres lie along the z-axis, is the solution of the PDE

$$yp - xq = 0, \qquad \ldots.(30)$$

which is of the first order.

We now take up different types of integrals of a PDE of the first order.

Classification of Integrals

After going through Example 1, and doing E 2), you must have observed that a two-parameter family of surfaces can give rise to a linear or non-linear partial differential equation. We shall, therefore, assume that the partial differential Eqn. (8), viz.,

$$f(x,y,z,p,q) = 0$$

can have a solution

$$z = F(x,y,a,b), \qquad \ldots.(31)$$

which depends on two parameters a, b.

Depending on the number of parameters, we classify the solutions of first order PDE as follows.

(a) The Complete Integral

A two-parameter family of solutions (31), i.e., $z = F(x,y,a,b)$ is called a **complete integral** of Eqn. (8) if, in the region considered, F satisfies any of the Eqns. (7(a,b,c)).

(b) The General Integral

In Eqn. (31) if we take $b = \phi(a)$, we obtain

$$z = F(x,y,a,\phi(a)), \qquad \ldots.(32)$$

which is a one-parameter family of solutions of Eqn. (8) and is a subsystem of the two-parameter family given by Eqn. (31). We can obtain the envelope of Eqn. (32) by eliminating a between relation (32) and relation

$$F_a + F_b\phi'(a) = 0 \qquad \ldots.(33)$$

In fact we can solve relation (33) for a, then

$$a = a(x,y)$$

and substituting this value of a in relation (32), we obtain the **general integral** of Eqn. (8) as

$$z = F(x,y, a(x,y), \phi(a(x,y))) \qquad \ldots (34)$$

The surface given by Eqn. (34), being the envelope of the one-parameter family of surfaces (32), touches every member of the family along the characteristic curve and has the same values of p,q all along the curve as Eqn. (32). Hence, it is a set of solutions of Eqn. (8) depending on an arbitrary function.

(c) Singular Integrals : In addition to the general integral, we can sometimes obtain still another solution by finding the envelope of the two-parameter family (31). This is obtained by eliminating a and b from the equations

$$\left.\begin{aligned} z &= F(x,y,a,b) \\ 0 &= F_a \\ 0 &= F_b \end{aligned}\right\} \qquad \ldots (35)$$

and is called the **singular integral** of Eqn. (8). We can also obtain the singular solution **directly** from the PDE (8) without going through the process described above. This can be done by eliminating p and q from the equations

$$\left.\begin{aligned} &f(x,y,z,p,q) = 0, \\ &f_p = 0, f_q = 0 \end{aligned}\right\} \qquad \ldots (36)$$

In fact, the two processes are equivalent as you can see from the following discussion.

Since z = F(x,y,a,b) is a two-parameter family of solutions of Eqn. (8), the equation

$$f(x,y, F(x,y,a,b), F_x (x,y,a,b), F_y(x,y,a,b)) = 0 \qquad \ldots (37)$$

holds identically in a and b. We can differentiate Eqn. (37) with respect to a and b, and obtain

$$\left.\begin{aligned} f_zF_a + f_pF_{xa} + f_qF_{ya} = 0, \\ f_zF_b + f_pF_{xb} + f_qF_{yb} = 0, \end{aligned}\right\} \qquad \ldots (38)$$

But we know from Eqn. (35) that on the singular integral,

$$F_a = 0, F_b = 0.$$

Therefore, relation (38) reduces to

$$\left.\begin{aligned} f_pF_{xa} + f_qF_{ya} = 0, \\ f_pF_{xb} + f_qF_{yb} = 0 \end{aligned}\right\} \qquad \ldots (39)$$

In order to eliminate a and b from Eqns.(39), we know from Eqn. (7) that F satisfies

$$F_{xa}F_{yb} - F_{xb}F_{ya} \neq 0 \text{ Eqn. (7b))}$$

and hence

$$f_p = 0, f_q = 0.$$

Therefore, we can find the equation of the singular integral from Eqns. (36) by eliminating p and q. Eqns.(36) provide an **alternative characterization** of the singular integral in terms of the given PDE whenever such an integral exists.

(d) Special Integral

Usually (but not always), the three classes (a), (b) and (c) include all the integrals of the first order partial differential equation. Exceptions may arise in special cases for equations of particular forms. These equations have solutions which we call **special integrals** and these cannot be obtained from (a), (b) or (c) above.

For example, if we eliminate the function F from the equation $F(x+y, x-\sqrt{z}) = 0$ we obtain the partial differential equation as $p-q = 2\sqrt{z}$. Therefore $F(x+y, x-\sqrt{z})$ is the general integral of the equation $p-q = 2\sqrt{z}$. But $z = 0$ also satisfies this equation and it cannot be obtained from the general integral. It is, therefore, the special integral of the equation.

We illustrate the ideas presented about different types of integrals of a partial differential equation in the following examples. We shall start with a two-parameter family of surfaces, construct the corresponding partial differential equation and then derive the general integral and the singular integral from the complete integral.

Example 3 : Eliminating a and b from the family of planes

$$z = ax + by + a^2 + b^2,$$

determine the partial differential equation of this family of planes. State the complete integral of the equation and find its singular integral and the general integral.

Solution : The family of planes is

$$f = z - (ax+by+a^2+b^2) = 0 \qquad \ldots. (40)$$

Differentiating Eqn. (40) partially w.r. to x and y, we get

$$\left.\begin{aligned} p - a &= 0 \\ q - b &= 0 \end{aligned}\right\} \qquad \ldots. (41)$$

Eliminating a and b from Eqns. (40) and (41), we get

$$z = px + qy + p^2+q^2, \qquad \ldots. (42)$$

which is the partial differential equation for the given family of planes.

Since Eqn. (40) is a two-parameter family of planes, it represents the complete integral for the PDE (42)

Using relation (35), the singular integral is obtained by eliminating a and b from Eqn. (40) and

$$\frac{\partial f}{\partial a} = 0 \Rightarrow x + 2a = 0$$

$$\frac{\partial f}{\partial b} = 0 \Rightarrow y + 2b = 0$$

Solving, we get the singular integral as

$$z = x.\left(-\frac{x}{2}\right) + y\left(-\frac{y}{2}\right) + \left(-\frac{x}{2}\right)^2 + \left(-\frac{y}{2}\right)^2$$

$$\Rightarrow 4z = -(x^2+y^2), \qquad \ldots. (43)$$

which is a **paraboloid of revolution.**

If we set $b = \phi(a)$, then we get the one-parameter family as

$$z - (ax + \phi(a)y + a^2 + \phi^2(a)) = 0 \qquad \ldots. (44)$$

The envelope of this one-parameter family is obtained by eliminating a from Eqn. (44) and its derivative w.r.t a, viz.,

$$x + \phi'(a)y + 2a + 2\phi(a)\,\phi'(a) = 0, \qquad \ldots. (45)$$

which will be the general integral of the given equation.

Example 4: Find the partial differential equation for the family of spheres of radius 1 in the xyz-space whose centres (a,b,0) lie on the xy-plane. State their complete integral and find the general integral and the singular integral.

Solution : The two-parameter family of spheres of radius 1 in the xyz-space whose centres (a,b,0) lie on the xy-plane is

$$(x-a)^2 + (y-b)^2 + z^2 = 1 \qquad \dots (46)$$

Eliminating a and b from Eqn. (46) and its differentials w.r.t. x and y, we get the partial differential equation

$$z^2(1 + p^2 + q^2) = 1 \text{ (see E 1(b))} \qquad \dots (47)$$

Relation (46) is the **complete integral** of this equation since it involves two arbitrary constants. If we set $b = \phi'(a)$ in Eqn. (46), we get the one-parameter family of spheres whose centres are $(a, \phi(a), 0)$ and which lie on the curve $y = \phi(x)$ in the xy-plane. The envelope of this family is then obtained by eliminating a from the equations

$$(x-a)^2 + (y - \phi(a))^2 + z^2 = 1 \qquad \dots (48)$$

and its derivative w.r.t. a, namely,

$$x - a + \phi'(a)(y - \phi(a)) = 0. \qquad \dots (49)$$

Eqns. (48) and (49) determine a surface whose axis is $y = \phi(x)$ and which is the **general integral** of Eqn. (47). If $\phi(a) = 2a$, then the general integral is

$$(y-2x)^2 + 5z^2 = 5 \qquad \dots (50)$$

The two-parameter family (46) gives yet another envelope which we can obtain by eliminating a and b from equations

$$(x-a)^2 + (y-b)^2 + z^2 = 1$$

$$x - a = 0,$$

$$y - b = 0,$$

$$\Rightarrow \quad z = 1 \text{ and } z = -1 \qquad \dots (51)$$

Thus the envelope of two-parameter family is the pair of planes $z = \pm 1$. Eqn. (51) gives us the **singular integral** of the equation. We can also obtain the singular integral from the PDE directly by using the relation (36).

$$\left.\begin{aligned} &\text{Here} \quad f = z^2(1 + p^2 + q^2) - 1 = 0 \\ &f_p = 2z^2p = 0 \Rightarrow p = 0 \\ &f_q = 2z^2q = 0 \Rightarrow q = 0 \end{aligned}\right\} \qquad \dots (52)$$

Eliminating p, q in Eqns. (52), we obtain $z = 1$ and $z = -1$ which is same as Eqn. (51).

It is easy to verify that the general solution (50) and the singular solutions (51) satisfy the PDE (47). Also, the singular solutions (51) touch the general solution along : $y-2x = 0$, $z = 1$ and $y-2x = 0$, $z = -1$. We leave it for you to verify it yourself.

E 4) Verify that

a) the general solution (50) and the singular solution (51) satisfy the PDE (47).

b) the singular solutions (51) touch the general solution along, $y-2x = 0$, $z = 1$ and $y - 2x = 0$, $z = -1$.

Let us take up another example.

Example 5: Given that

$$z = ax + by + a^2 + b^2, \qquad \dots (53)$$

is the complete integral of the PDE.

$$z = px + qy + p^2 + q^2,$$

determine its general integral.

Solution : We can find any number of general integrals starting with the complete integral. Here, we give two general integrals. Take $b = \pm\sqrt{1-a^2}$ in the complete integral (53). This means we have to take only a subsystem of planes from relation (53) that pass through the point (0,0,1). The equation of this one-parameter family of planes is

$$F = z - ax \pm \sqrt{1-a^2}\, y - 1 = 0 \qquad \ldots. (54)$$

Here $$\frac{\partial F}{\partial a} = 0 \Rightarrow -x \mp \frac{ay}{\sqrt{1-a^2}} = 0 \qquad \ldots. (55)$$

The intersections of these planes define a one-parameter family of characteristics (straight lines) which generate the right circular cone (also the envelope of the family of planes (54)) whose equation is

$$(z-1)^2 = x^2 + y^2 \qquad \ldots. (56)$$

Eqn. (54) is the general integral of the PDE (53). In order to find one more general integral of Eqn. (53) we make another choice of $\phi(a)$, say,

$b = \phi(a) = a$.

Then

$$F = z = ax + ay + 2a^2 \qquad \ldots. (57)$$

and

$$\frac{\partial F}{\partial a} = 0 \Rightarrow x + y = -4a \qquad \ldots. (58)$$

By eliminating a between Eqns. (57) and (58), we obtain the envelope

$$8z = -(x+y)^2 \qquad \ldots. (59)$$

which is a **parabolic cylinder** and constitutes another general integral of the PDE (53).

How about doing an exercise now?

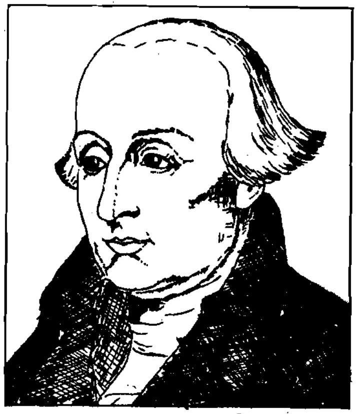
E 5) Given that $z = ax + by + ab$ is the complete integral of the PDE

$z = px + qy + pq$

obtain its general integral and singular integral.

We have so far been talking quite frequently about the integrals of the first order PDEs. We have also classified the types of such integrals. But, we have not given the methods of finding the integral of first order PDEs. We now give a theorem which gives the solution on some domain D in the case of linear PDEs of the first order. The problem of finding the solution of non-linear PDEs of the first order shall be taken up in the next unit.

13.5 LINEAR EQUATION OF FIRST ORDER


Lagrange (1736-1813)

Consider the quasi-linear Eqn. (22), viz.,

$P(x,y,z)p + Q(x,y,z)q = R(x,y,z)$,

where P, Q and R are given functions of x,y,z of class C^1 on some domain containing x,y,z and these functions do not involve p or q. This equation was studied by the French mathematician Lagrange and is called the **Lagrange's equation**. The method of solution of this equation is based on the following theorem which guarantees its **general solution**.

Theorem 1: The general solution of the quasi-linear Eqn. (22) (or Lagrange's equation)

$P(x,y,z)p + Q(x,y,z)q = R(x,y,z)$

is

$$F(u,v) = 0, \tag{60}$$

where F is an arbitrary function of u and v, and $u(x,y,z) = c_1$, $v(x,y,z) = c_2$ are the solutions of the system of simultaneous equations

$$\frac{dx}{P(x,y,z)} = \frac{dy}{Q(x,y,z)} = \frac{dz}{R(x,y,z)} \qquad \ldots (61)$$

This theorem can be proved either by purely analytical methods or by geometrical method. We shall first take up an analytical proof of this theorem.

Analytical Proof : If the two families of surfaces

$$u(x,y,z) = c_1 \text{ and } v(x,y,z) = c_2 \qquad \ldots (62)$$

form a solution of the system of Eqns. (61), viz.,

$$\frac{dx}{P(x,y,z)} = \frac{dy}{Q(x,y,z)} = \frac{dz}{R(x,y,z)},$$

then along any curve, given by Eqn. (61) we have

$$\left.\begin{aligned} u_x dx + u_y dy + u_z dz = 0 \\ v_x dx + v_y dy + v_z dz = 0 \end{aligned}\right\} \qquad \ldots (63)$$

Solving Eqns.(63) for dx, dy, dz, we get

$$\frac{dx}{u_y v_z - u_z v_y} = \frac{dy}{u_z v_x - u_x v_z} = \frac{dz}{u_x v_y - u_y v_x}, \qquad \ldots (64)$$

which are the differential equations of the curves given by surfaces (62). Hence the system of Eqns. (61) and (64) must be compatible, i.e., both system should represent the same integral curves. Comparing Eqns. (61) and (64), we get

$$P = K(u_y v_z - u_z v_y),\ Q = K(u_z v_x - u_x c_z),\ R = K(u_x v_y - u_y v_x), \qquad \ldots (65)$$

where K is a function of x,y,z.

Now, consider the relation (60), viz.,

$$F(u,v) = 0$$

where u and v are known functions of x,y,z and F is an arbitrary function of u and v.

Differentiating relation (60) with respect to x and y, respectively, we obtain

$$\frac{\partial F}{\partial u}\left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}p\right] + \frac{\partial F}{\partial v}\left[\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}p\right] = 0 \qquad \ldots (66)$$

$$\frac{\partial F}{\partial u}\left[\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}q\right] + \frac{\partial F}{\partial v}\left[\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}q\right] = 0 \qquad \ldots (67)$$

Eliminating $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$ from Eqns. (66) and (67), we obtain the equation

$$p(u_y v_z - u_z v_y) + q(u_z v_x - u_x v_z) = (u_x v_y - u_y v_x), \qquad \ldots (68)$$

which is a partial differential equation of the type (22).

Substituting from Eqns.(65) in Eqn. (22) and using the relation (68), we find that Eqn. (22) is satisfied identically. Thus $F(u,v) = 0$ is a solution of Eqn. (22), if u and v are given by relations (62), and is called its general solution.

This completes the proof.

We next take up geometrical proof of Theorem 1.

Geometrical proof : We shall prove this theorem in two stages : (a) We shall show that all integral surfaces of Eqn. (22) are generated by the integral curves of Eqns.(61).

(b) Then we shall prove that all surfaces generated by integral curves of Eqns.(61) are the integral surfaces of Eqn. (22).

Let us first prove (a).

(a) If z $\phi(x,y)$ be a solution surface S of Eqn. (22), then Eqn. (22) implies that at every point M(x,y,z) on S the vector field (P,Q,R) is perpendicular to the normal, with direction ratios ($p = \phi_x$, $q = \phi_y$, -1) of S (see Fig.1). In other words, the direction defined by the direction ratios (P,Q,R) is tangential to the integral surface $z = \phi(x,y)$.

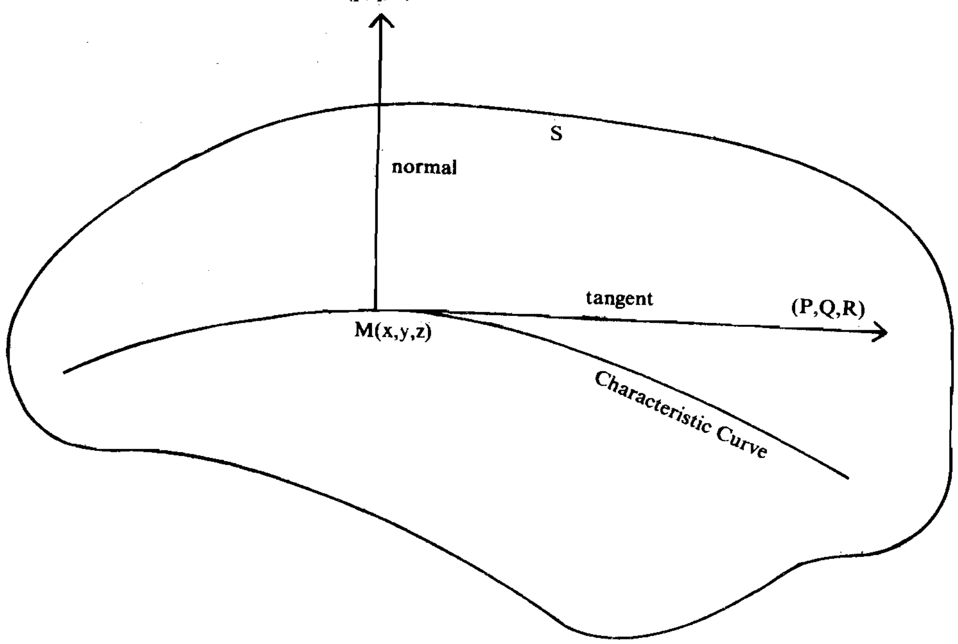


Fig. 1

Thus, if we start from an arbitrary point M(x,y,z) on the surface S and move in such a way that the direction of the motion is always along (P,Q,R) then we trace out an integral curve of Eqns.(61) and since P,Q, and R are assumed to be unique, there will be only one such curve through M. Further, since (P,Q,R) is always tangential to the surface S, we never leave the surface. Thus this integral curve of Eqns.(61) lies completely on the surface S.

Hence the integral surface of Eqn. (22) is generated by the integral curves of Eqn. (61).

(b) Next, if we are given that the surface $z = \phi(x,y)$ is generated by integral curves of Eqns.(61), then you may notice that its normal at a point (x,y,z), which is in the direction (p,q,–1), will be perpendicular to the direction (P,Q,R) of the curves generating the surface.

To complete the proof of the theorem, we shall prove that any surface generated by the integral curves of Eqns.(61) has an equation of the form (60).

Let any curve on the surface S, which is not a particular member of the system (62), viz.,

$u(x,y,z) = c_1$ and $v(x,y,z) = c_2$,

have equations

$$f(x,y,z) = 0 \text{ and } g(x,y,z) = 0 \qquad \ldots.(69)$$

If the curve of intersections of surfaces given by Eqns. (62) is the generating curve of the surface S, then it will intersect the curve given by surfaces (69). The condition that it should do so will be obtained by eliminating x,y,z from the four Eqns. (62) and (69). This will be a relation between constants c_1 and c_2 of the form

$$F(c_1,c_2) = 0 \qquad \ldots.(70)$$

Thus the surface is generated by curves (62), which obey the condition (70), and will, therefore, have an equation of the form

$F(u,v) = 0$

Conversely, any surface of the form (60) is generated by the integral curves (62) of the Eqns.(61), because it is that surface generated by those curves of the system (62) which satisfy the relation (70).

This completes the proof of the theorem.

You may **note** here that the vector field (P,Q,R) defines a two-parameter family of curves which lie on the surfaces S, whose differential equations are

$$\frac{dx}{P(x,y,z)} = \frac{dy}{Q(x,y,z)} = \frac{dz}{R(x,y,z)}$$

This system of equations is called the **Lagrange's auxiliary or subsidiary equations.**

The integral curves, given by the intersection of surfaces

$$u(x,y,z) = c_1 \text{ and } v(x,y,z) = c_2,$$

are called the **characteristic curves** or **characteristics** of the Eqns.(22) written as C-curves and are the solutions of Eqns.(61).

We shall now illustrate this method of finding the solution of the equations of the type (22) by considering the following examples.

Example 6 : Find the general integral of the equation

$$z_t + zz_x = 0$$

Solution : The auxiliary equations are

$$\frac{dt}{1} = \frac{dx}{z} = \frac{dz}{0}. \qquad \ldots.(71)$$

The two integrals of Eqns. (71) are

$$u = z = c_1,\ v = x - c_1 t = c_2 \text{ or } v = x - zt = c_2$$

The general integral is

$$F(z, x-zt) = 0$$

or

$$z(x, t) = \phi(x-zt). \qquad \ldots.(72)$$

You may also check the result obtained by differentiating Eqn. (72) w.r.t to t and x and then substituting it in the given equation. In this case we have

$$z_t = \phi'(x-zt)\,(-z-tz_t)$$

$$= -z\,\phi'\,(x-zt) - t\phi'\,(x-zt)\,z_t$$

$$\Rightarrow \quad z_t = \frac{-z\phi'}{1+t\phi'}$$

Similarly,

$$z_x = \frac{\phi'}{1+t\phi'},$$

and clearly

$$z_t + zz_x = 0.$$

Let us consider another example.

Example 7 : Find the general integral of

$$x^2p + y^2q = (x+y)z \qquad \ldots.(73)$$

Solution : The auxiliary equations are

$$\frac{dx}{x^2}=\frac{dy}{y^2}=\frac{dz}{z(x+y)}=\frac{x^{-1}dx+y^{-1}dy-z^{-1}dz}{0} \quad \ldots . (74)$$

Integrating Eqns.(74), we get

$$u=\frac{1}{x}-\frac{1}{y}=c_1,\ v=\frac{xy}{z}=c_2$$

The general integral is

$$F\left(\frac{1}{x}-\frac{1}{y},\frac{xy}{z}\right)=0$$

The general integral can also be written as

$$z=xy\,F_1\left(\frac{1}{x}-\frac{1}{y}\right),$$

where F_1 is an arbitrary function.

You may now try the following exercise.

E 6) Find the general integrals of the following equations :

a) $z(xp-yq)=y^2-x^2$

b) $y^2p-xyq=x(z-2y)$

c) $(z^2-2yz-y^2)p+x(y+z)q=x(y-z)$

d) $xp+yq=z$

e) $yzp+xzq=xy$

You may recall that in Unit 1, Sec.1.3, we defined an initial value problem for ordinary differential equation of the first order. Therein, we stated the existence theorem for the solution of an ordinary differential equation

$$\frac{dy}{dx}=f(x,y),$$

when $y=y_0$ at $x=x_0$.

We shall, in the next section, take up the initial-value problem for a quasi-linear partial differential equation.

You have already seen in this unit that solution of a PDE of the first order is either a surface or a curve, which is the intersection of two surfaces. Thus the initial conditions in this case will not be at a point, but the solution surfaces will pass through a curve Γ_0, called the **initial data curve**. Though a complete discussion of existence theorem would be out of place at this stage, it is important that you realize just what is meant by an existence theorem for PDE of the first order. We shall establish through existence theorem the conditions under which you can assert whether or not a given PDE has a solution at all. The conditions to be satisfied in the case of a first order PDE are put in the convenient form in the classic **problem of Cauchy**, which we shall take up now.

13.6 THE CAUCHY PROBLEM

(a) If $x_0(\mu)$, $y_0(\mu)$ and $z_0(\mu)$ are functions which, together with their first derivatives, are continuous in the interval M defined by $\mu_1<\mu<\mu_2$;

and if (b) $F(x,y,z,p,q,)=P(x,y,z)p+Q(x,y,z)q-R(x,y,z)$, is continuous function of x,y,z,p and q in a certain region U of the xyzpq-space,

then we are required to establish the existence of function $\phi(x,y)$ with the following properties:

(i) $\phi(x,y)$ and its partial derivatives with respect to x and y are continuous functions of x and y in region R of the xy-space.

(ii) For all values of x and y in R, the point $(x,y,\phi(x,y), \phi_x(x,y), \phi_y(x,y))$ lies in U and

$F[x,y,\phi(x,y), \phi_x (x,y), \phi_y(x,y)] = 0$

(iii) For all μ belonging to the interval M, the point

$\{x_0 (\mu), y_0 (\mu)\}$ belongs to the region R, and $\phi\{x_0(\mu), y_0(\mu)\} = z_0 (\mu)$

Geometrically we wish to prove that there exists a surface $z = \phi(x,y)$ which passes through a curve Γ_0 whose parametric equations are

$$x = x_0(\mu), y = y_0(\mu), z = z_0(\mu)$$

and at every point of which the direction $(p,q,-1)$ of the normal is such that

$$F(x,y,z,p,q) = 0$$

$$\Rightarrow \quad P(x,y,z)p + Q(x,y,z)q = R(x,y,z)$$

We have given here only one form of the Cauchy problem. The problem can be formulated in many other ways which are equivalent to the formulation given above. But the significant point is that Cauchy problem cannot be proved with the degree of generality as formulated above because it is necessary to make some further assumptions about the form of the function F and the nature of the curve Γ_0.

We shall not be discussing the different forms of Cauchy problem here. However, we shall now indicate how a general solution of Eqn. (22), viz.,

$$Pp + Qq = R,$$

may be used to determine the integral surface which passes through a given curve.

We have already mentioned in Sec.13.5 that the two-parameter family of curves which are the solution of the auxiliary equations

$$\frac{dx}{P(x,y,z)} = \frac{dy}{Q(x,y,z)} = \frac{dz}{R(x,y,z)} = dt \qquad \ldots (75)$$

are called the characteristics of the Eqn. (22). Along a characteristic curve, Eqn. (22) can be written as

$$z_x \frac{dx}{dt} + z_y \frac{dy}{dt} = R(x,y,z),$$

or

$$\frac{dz}{dt} = R(x,y,z). \qquad \ldots (76)$$

In the linear and semi-linear cases, the functions $P = P(x,y)$ and $Q = Q(x,y)$ depend on x,y only and from Eqn. (75), we obtain

$$\frac{dx}{dt} = P(x,y), \frac{dy}{dt} = Q(x,y) \qquad \ldots (77)$$

These are simultaneous equations and it is possible to express them as two single equations in x and t and y and t, which on integration yield

$$x = x(t), y = y(t) \qquad \ldots (78)$$

which constitute a family of curves. We write these curves as the C_0-curves and in the xy-plane we call them the **base characteristics** of the PDE. These curves are the projections of the characteristic curves on the plane $z = 0$. Along a C_0-curve Eqn. (76) becomes,

$$\frac{dz}{dt} = R(x(t), y(t),z) \qquad \ldots (79)$$

This is an ordinary differential equation for z and it determines the variation of z along C_0 as

$$z = z(t) \qquad \ldots (80)$$

In the quasi-linear case we can find the C_0 curves in advance (as above), and hence integrate Eq.(75) simultaneously, to obtain

$$x = x(t),\ y = y(t),\ z = z(t) \qquad \ldots (81)$$

We can use both the Eqs.(80) and (81) to find z(t) if we prescribe the initial values z(0) in an appropriate manner.

However, we can express in another way the characteristics as the intersection of two families of surfaces $u(x,y,z) = c_1$ and $v(x,y,z) = c_2$. These are the two integrals of Eqns.(75).

We shall now take up a few examples and make use of all these concepts while solving specific problems.

Example 8 : Find the solution of the initial-value problem for the quasi-linear equation

$$u_t + uu_x + u = 0,\ x \in R,\ t > 0 \qquad \ldots (82)$$

for the initial data curve.

$$\Gamma_0 : x_0 = s,\ t_0 = 0,\ u_0 = -2s,\ -\infty < s < \infty \qquad \ldots (83)$$

where s is the defining parameter of the curve.

Solution : The characteristics are given by

$$\frac{dt}{1} = \frac{dx}{u} = \frac{du}{-u} = d\xi, \qquad \ldots (84)$$

where ξ is the defining parameter of the C-curves.

Therefore,

$$u = ae^{-\xi},\ x = -ae^{-\xi} + b,\ t = \xi + c,$$

where a, b, c are the constants of integration.

If $u = -2s$, $t = 0$, $x = s$ when $\xi = 0$ then $a = -2s$, $b = -s$, $c = 0$ and we obtain

$$u = -2s\, e^{-\xi},\ x = 2s\, e^{-\xi} - s,\ t = \xi$$

Eliminating the parameters s and ξ, we find the solution surface S as

$$u = \frac{2x}{e^t - 2},\ \ln 2 > t \geq 0$$

The solution breaks down at $t = \ln 2$, because $u \to \infty$ as $t \to \ln 2$ and beyond. For there is no continuously differentiable solution.

Example 9 : Find the general integral of the equation

$$(2xy-1)p + (z-2x^2)q = 2(x-yz) \qquad \ldots (85)$$

and the solution surface which passes through the line $x = 1$, $y = 0$.

Solution : The auxiliary equations corresponding to Eq.(85) are

$$\frac{dx}{2xy-1} = \frac{dy}{z-2x^2} = \frac{dz}{2(x-yz)} \qquad \ldots (86)$$

Each of these fractions is equal to

$$\frac{zdx + dy + xdz}{0} = \frac{xdx + ydy + (dz/2)}{0} \qquad \ldots (87)$$

The integrals of Eqs. (86) and (87) are

$$u = y + xz = c_1,\ v = x^2 + y^2 + z = c_2$$

The general integral is

$$x^2 + y^2 + z = F(y+xz), \qquad \ldots . (88)$$

where F is an arbitrary function. We shall determine the form of F using the initial data :

$$x = 1, y = 0.$$

Substituting this data in Eq.(88), we obtain

$$F(z) = 1 + z$$

and therefore,

$$F(y+xz) = 1 + y + xz \qquad \ldots . (89)$$

Substituting from Eqn.(89) into Eqn. (88), we obtain the solution surface S as

$$x^2 + y^2 - xz - y + z = 1$$

Let us consider another example in which the characteristics are expressed as the intersection of two families of surfaces

$$u = c_1 \text{ and } v = c_2.$$

Example 10 : Find the equation of the integral surface of the equation

$$x^3p + y(3x^2+y)q = z(2x^2+y) \qquad \ldots . (90)$$

which passes through the curve

$$\Gamma_0 : x_0 = 1, y_0 = s, z = s(1+s) \qquad \ldots . (91)$$

Solution : The auxiliary equations corresponding to Eqn. (90) are

$$\frac{dx}{x^3} = \frac{dy}{y(3x^2+y)} = \frac{dz}{z(2x^2+y)} \qquad \ldots . (92)$$

Each of the fractions of Eqns.(92) is equal to

$$\frac{-x^{-1}dx + y^{-1}dy - z^{-1}dz}{0} \Rightarrow x^{-1}dx + y^{-1}dy - z^{-1}dz = 0$$

Therefore

$$y = c_1xz \qquad \ldots . (93)$$

On solving the pair formed by the first and third fractions of Eqns.(92) and using Eqn. (93), we obtain

$$\frac{2x}{z} - \frac{x^2}{z^2}\frac{dz}{dx} = c_1,$$

$$\Rightarrow \frac{d}{dx}(\frac{x^2}{z}) = c_1,$$

$$\Rightarrow \frac{x^2}{z} = c_1x + c_2,$$

or

$$x^2 = y + c_2z. \qquad \ldots . (94)$$

Substituting the initial data in Eqns. (93) and (94), we get

$$1 = c_1(1+s), 1-s = c_2s(1+s) \qquad \ldots . (95)$$

On eliminating s from Eqn. (95), we get a relation between c_1 and c_2 as

$$c_1(2c_1-1) = c_2(1-c_1). \qquad \ldots . (96)$$

Substituting for c_1 and c_2 from Eqns. (93) and (94) in Eqn. (96), we obtain the solution surface as

$2y^2 - xyz = x(x^2-y)(xz-y)$.

You may now try the following exercises.

E 7) Find the integral surface of the equation

$$x(y^2+z)p - y(x^2+z)q = (x^2-y^2)z$$

passing through the curve $x+y = 0$, $z = 1$.

E 8) Find the integral surface of $yp + xq - z = 0$ passing through the curve $z = x^3$, $y = 0$.

E 9) Find the integral surface of the equation

$$(y-z)p + (z-x)q = x - y$$

which passes through the curve $z = 0$, $xy = 1$.

13.7 SUMMARY

In this unit, we have covered the following :

(1) PDEs can arise in many ways in geometry, physics and mathematics. For instance

(a) on elimination of arbitrary function defining the surfaces of revolution

(b) on elimination of two constants, defining two parameter family of surfaces, between the equation defining the family of surfaces and its partial derivatives w.r.to independent variables.

(c) on elimination of the function F defining the surfaces of the form $F(u,v) = 0$ with $u = u(x,y,z)$ and $v = v(x,y,z)$, with respect to independent variables.

(d) while satisfying the conditions of an equation to be exact.

(2) The general form of first order PDE is $f(x,y,z,p,q) = 0$.

(3) A first order PDE is classified as

(a) **linear** if it can be expressed as

$$p(x,y)p + Q(x,y)q = R(x,y)z$$

(b) **semi-linear** if it can be expressed in the form

$$P(x,y)p + Q(x,y)q = R(x,y,z)$$

(c) **quasi-linear** if it can be expressed in the form

$$P(x,y,z)p + Q(x,y,z)q = R(x,y,z)$$

(d) **non-linear** if it cannot be expressed in any of the forms given in (a), (b) and (c) above.

Further, linearity in PDEs does not depend on the degree of the dependent variable.

(4) The solutions of first-order PDEs are classified as

(a) **complete integral**, which is a relation between the variables involving as many constants as there are independent variables.

(b) **general integral**, which is obtained by eliminating 'a' between the complete integral

$$f(x,y,z,a,b) = 0$$

and the equations

$$b = \phi(a)$$

and $\dfrac{\partial f}{\partial a} + \dfrac{\partial f}{\partial b}\phi'(a) = 0$,

where ϕ is an arbitrary function.

The general integral represents the envelope of a one-parameter family of surfaces.

(c) **Singular Integral**, which is obtained on eliminating a and b between the complete integral f(x,y,z,a,b) = 0,

$$\left.\begin{aligned} &\frac{\partial f}{\partial a} = 0, \\ \text{and } &\frac{\partial f}{\partial b} = 0 \end{aligned}\right\}$$

The singular integral represents the envelope of the two- parameter family of surfaces.

The singular integral can as well be obtained by eliminating p and q from the PDE

$$F(x,y,z,p,q) = 0,$$

$$\text{and } \frac{\partial F}{\partial p} = 0, \frac{\partial F}{\partial q} = 0$$

(d) In the exceptional cases, if there are integrals of the given PDE which are not included in the complete integral, the general integral or the singular integral, these integrals are called **special integrals**.

(5) **Lagrange's method** for solving quasi-linear PDE of first order gives the general integral $\phi(u,v) = 0$ of an equation

$$P(x,y,z)p + Q(x,y,z)q = R(x,y,z),$$

where u(x,y,z) = constant and v(x,y,z) = constant are two independent integrals of auxiliary equations (or subsidiary equations)

$$\frac{dx}{P(x,y,z)} = \frac{dy}{Q(x,y,z)} = \frac{dz}{R(x,y,z)}$$

(6) Cauchy's problem is the existence theorem for the solution of the first order quasi-linear PDE and it states that

if (a) $x_0(\mu)$, $y_0(\mu)$, $z_0(\mu)$ together with their first derivatives are continuous in the interval M given by $\mu_1 < \mu < \mu_2$; and

if (b) F(x,y,z,p,q) = P(x,y,z)p + Q(x,y,z)q – R(x,y,z) is continuous function of x,y,z,p and q in certain region U of xyzpq-space.

then there exists a function $\phi(x,y)$ with the following properties :

(i) $\phi(x,y)$ and its partial derivatives w.r to x and y are continuous functions of x and y in region R of xy-space.

(ii) For all values of x and y in R, the point $(x,y,\phi(x,y), \phi_x(x,y), \phi_y(x,y))$ lies in U and

$$F(x,y,\phi(x,y), \phi_x(x,y), \phi_y(x,y)) = 0$$

(iii) For all μ belonging to the interval M, the point

$\{x_0(\mu), y_0(\mu)\}$ belongs to the region R and

$\phi\{x_0(\mu), y_0(\mu)\} = z_0(\mu)$

Geometrically, Cauchy's problem proves that there exists a surface $z = \phi(x,y)$ which passes through a curve Γ_0, whose parametric equations are $x = x_0(\mu)$, $y = y_0(\mu)$, $z = z_0(\mu)$, and at every point of which the direction (p,q,–1) of the normal is such that

$$P(x,y,z)p + Q(x,y,z)q = R(x,y,z)$$

13.8 SOLUTIONS/ANSWERS

E 1) a) The given equation is

$$z = ax + by + a^2 + b^2 \qquad \ldots (97)$$

Differentiating partially with respect to x and y, we get

$$p = a \qquad \ldots (98)$$

and $q = b$ .... (99)

Eliminating a and b from Eqn. (97) by using Eqns. (98) and (99), we get

$z = px + qy + p^2 + q^2$

which is the required PDE.

(b) $z^2 (1 + p^2 + q^2) = 1$.

c) $xp - yq - x = 0$

E 2) a) The given equation is

$$z = F\left(\frac{xy}{z}\right) \quad \dots (100)$$

Differentiating Eqn. (100) partially w.r.to x and y respectively, we get

$$p = F'.\left(\frac{y}{z} - \frac{xy}{z^2}p\right) \quad \dots (101)$$

$$\text{and } q = F'.\left(\frac{x}{z} - \frac{xy}{z^2}q\right), \quad \dots (102)$$

where F' is derivative of F w.r.to $\left(\frac{xy}{z}\right)$

Eliminating F' from Eqns. (101) and (102), we get

$$\frac{p}{\frac{y}{z} - \frac{xy}{z^2}p} = \frac{q}{\frac{x}{z} - \frac{xy}{z^2}q}$$

$\Rightarrow p(xz - xyq) = q(yz - xyp)$

$\Rightarrow z\,(px - qy) = 0$,

where is the required PDE.

b) The given equation is

$F(x^2 + y^2 + z^2,\ z^2-2xy) = 0$

Let $x^2 + y^2 + z^2 = u,\ z^2-2xy = v$ .... (103)

Then given equation reduces to

$F(u,v) = 0$

Differentiating the above equation w.r. to x and y partially and eliminating $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$, we obtain (following Eqn. (16)).

$$p\frac{\partial(u,v)}{\partial(y,z)} + q\frac{\partial(u,v)}{\partial(z,x)} = \frac{\partial(u,v)}{\partial(x,y)} \quad \dots (104)$$

$$\left.\begin{aligned} \text{Here } \frac{\partial(u,v)}{\partial(y,z)} &= u_y v_z - u_z v_y = 2y.2z - 2z(-2x) = 4(yz+xz) \\ \frac{\partial(u,v)}{\partial(z,x)} &= u_z v_x - u_x v_z = 2z(-2y) - 2x.2z = -4(yz+xz) \\ \frac{\partial(u,v)}{\partial(x,y)} &= u_x v_y - u_y v_x = 2x(-2x) - (2y)(-2y) = 4(y^2-x^2) \end{aligned}\right\} \quad \dots (105)$$

Substituting from Eqns. (105) into Eq.(104), we get

$(yz+zx)(p-q) = (y^2-x^2)$

which is the required PDE

E 3) a) Semi-linear

b) non-linear

c) linear

d) linear

e) quasi-linear

f) quasi-linear

g) semi-linear

h) non-linear

E 5) The complete integral is

$z = ax+by+ab.$

Let $b = \phi(a)$

Then the complete integral yields

$$z = ax + \phi(a)y + a\,\phi(a) \qquad \ldots.(106)$$

Differentiating Eqn. (106) partially w.r.t. a, we get

$$0 = x + \phi'(a)y + \phi(a) + a\,\phi'(a) \qquad \ldots.(107)$$

Elimination of a between Eqns. (106) and (107) gives us the general integral. Singular integral is obtained by eliminating a and b between the equations

$$\left.\begin{aligned} z &= ax + by + ab \\ 0 &= x + b \\ 0 &= y + a \end{aligned}\right\}$$

in the form $z = -xy$

E 6) a) The given equation can be written as

$zxp - zyp = y^2-x^2$

The auxiliary equation are

$$\frac{dx}{xz} = \frac{dy}{-zy} = \frac{dz}{y^2-x^2} = \frac{xdx + ydy + zdz}{0}$$

The integrals of the above system of equations are

$u = xy = c_1$ and $v = x^2+y^2+z^2 = c_2$

Hence the general integral of given PDE is

$F(xy, x^2+y^2+z^2) = 0,$

where F is an arbitrary function.

(b) The given equation is

$y^2p - xyq = x(z-2y)$

The auxiliary equations are

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

Integrating the above system of equations, we get

$u = x^2+y^2 = c_1$ and $v = yz - y^2 = c_2$

Hence the general integral of the given PDE is

$$x^2+y^2 = \phi(yz-y^2),$$

where ϕ is an arbitrary function.

c) The auxiliary equations corresponding to the given equation are

$$\frac{dx}{z^2-2yz-y^2} = \frac{dy}{x(y+z)} = \frac{dz}{x(y-z)} = \frac{xdx + ydy + zdz}{0}$$

Integrating above system of equations, we have

$$u = y^2 - 2yz - z^2 = c_1 \text{ , } v = x^2 + y^2 + z^2 = c_2$$

Hence the general integral of given PDE is

$$x^2 + y^2 + z^2 = f(y^2 - 2yz-z^2),$$

where f is an arbitrary function.

d) $z = x\, F(y/x)$

e) The auxiliary equations are

$$\frac{dx}{yz} = \frac{dy}{xz} = \frac{dz}{xy}$$

Integrating, we get

$$u = y^2-z^2 = c_1 \text{ and } v = x^2-y^2 = c_2$$

Hence the general integral of the given PDE is

$$y^2-z^2 = F(x^2-y^2)$$

$$\Rightarrow y^2 = z^2+F(x^2-y^2),$$

where F is an arbitrary function.

E 7) The auxiliary equations are

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{(x^2-y^2)z}$$

$$= \frac{yzdx + zxdy + xydz}{0} = \frac{xdx + ydy - dz}{0}$$

Integrating, we have

$$xyz = c_1 \text{ and } x^2+y^2-2z = c_2 \qquad \ldots.(108)$$

The given curve is $x+y = 0$, $z = 1$, whose parametric equation is

$$x = t,\ y = -t,\ z = 1$$

Substituting these values in Eqns. (108), we get

$$-t^2 = c_1 \text{ and } 2t^2-2 = c_2$$

Eliminating t from the above equations, we get

$$2c_1 + 2 = -c_2$$

Substituting for c_1 and c_2 from Eqns. (108), we obtain

$$x^2+y^2+2xyz-2z+2 = 0,$$

which is the desired integral surface.

E 8) The auxiliary equations are

$$\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{z} = \frac{dx+dy+dz}{x+y+z}$$

Integrating $\frac{dx}{y} = \frac{dx}{x}$, we get

$$x^2-y^2 = c_1 \qquad \ldots. (109)$$

And integrating, $\frac{dz}{z} = \frac{d(x+y+z)}{x+y+z}$, we obtain

$$\frac{x+y+z}{z} = c_2 \qquad \ldots. (110)$$

The given curve $z = x^3$, $y = 0$ has the parametric equations as

$$x = t, y = 0, z = t^3$$

Substituting these values of x, y, z in Eqns. (109) and (110), we get

$$t^2 = c_1 \text{ and } \frac{t+t^3}{t^3} = c_2$$

Eliminating t from these equations, we get

$$\frac{1+c_1}{c_1} = c_2$$

Substituting in the above equation the values for c_1 and c_2 from Eqns. (109) and (110), we get

$z(1+x^2-y^2) = (x^2-y^2)(x+y+z)$ as the desired integral surface.

E 9) The auxiliary equations are

$$\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y}$$

$$= \frac{dx+dy+dz}{0} = \frac{xdx + ydy + zdz}{0}$$

Integrating, we get

$$x+y+z = c_1 \text{ and } x^2 + y^2+z^2 = c_2 \qquad \ldots. (111)$$

The given curve $z = 0$, $xy = 1$ has parametric equations as

$$x = t, y = \frac{1}{t}, z= 0$$

Substituting these values of x,y,z in Eqns.(111), we obtain

$$t + \frac{1}{t} = c_1 \text{ and } t^2 + \frac{1}{t^2} = c_2$$

Eliminating t from these equations, we get

$$c_1^2 - 2 = c_2$$

In this equation substituting the values of c_1 and c_2 from Eqns. (111), we have

$$xy + yz + zx - 1 = 0$$

$$\Rightarrow z = \frac{1-xy}{x+y},$$

which is the desired integral surface.