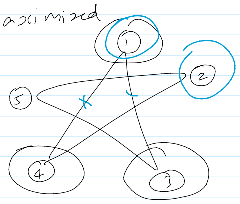
29 September 2020 12:59

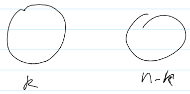
Coping with Intractability

"Approximation"

MAXCUT: Given a graph $G(V,E)$, find $S \subseteq V$ s.t. $|E(S,S^c)|$ is maximized



K_n



$S=\{1\}$, $|E(S,S^c)|=2$

$S=\{1,2\}$, " 4

$S=\{1,2,3\}$

$S=\{1,3\}$, 2

$(k)(n-k)$

Decision version of MAXCUT $\in NP$

$(G(V,E), k)$: Check if $MAXCUT(G) \geq k$

Randomized Algorithm

$A(x, r)$ → random binary string $\in \{0,1\}^{poly}$

$Pr[A(x,r) \text{ finding a good cut}] > 2/3$

Algorithm

For each $v \in V$,

Toss coin and add v to S if Head

Output S

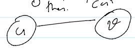
What is the $|E(S,S^c)|$?

$X_{uv} = 1$ if $(u,v) \in E(S,S^c)$

0 otherwise.

$$\boxed{|E(S,S^c)| = \sum_{(u,v)\in E} X_{uv}}$$

$X_{uv} = 1$ if $(u,v) \in E$ and it is part of cut, 0 then



u	v
1	1
0	0
✓ 1	0
✓ 0	1

$\mathbb{E}|E(S,S^c)| = \sum_{(u,v)\in E} \mathbb{E} X_{uv}$ (lin of expectation)

$= \sum_{(u,v)\in E} \frac{1}{2} = \frac{|E|}{2}$

$Pr[|E(S,S^c)| \geq |E|/3]$

Tail Bounds

— Markov's ($X \geq 0$ non negative)

$Pr[X \geq a] \leq \mathbb{E}(X)/a$

MAXCUT value $= c^* \leq m$

$Pr[m - |E(S,S^c)| \leq \frac{2}{3}m]$

$= 1 - Pr[|\overline{E(S,S^c)}| > \frac{2}{3}m]$

$\geq 1 - \frac{m/2}{\frac{2}{3}m}$

$\geq 1 - 3/4$

$\geq 1/4$

$Pr[|E(S,S^c)| \geq m/3] \geq \frac{1}{4}$

$\parallel$

$Pr[c^* \geq |E(S,S^c)| \geq c^*/3]$

$\geq 1/4$

Improving Prob of getting good cut

- Repeat k time and choose the best cut seen so far.

$Pr[\text{getting cut of size} \geq c^*/3]$

$\geq 1 - \left(\frac{3}{4}\right)^k$

$\geq 1 - \varepsilon$

$k = O(\log(1/\varepsilon))$

Find a Cut of size $(\frac{1}{2} - \varepsilon)|E|$

$Pr[|\overline{E(S,S^c)}| > (\frac{1}{2}+\varepsilon)|E|] \leq \frac{\frac{1}{2}m}{(\frac{1}{2}+\varepsilon)m} = \frac{1}{1+2\varepsilon} < 1$

too many edges not part of the cut

$Pr[|E(S,S^c)| \geq (\frac{1}{2}+\varepsilon)m] \geq 1 - \frac{1}{1+2\varepsilon} = 1-(1+2\varepsilon)^{-1}$

$\geq 2\varepsilon \leftarrow 1-(1-2\varepsilon)$

$Pr[\text{getting cut of size} \geq c^*(\frac{1}{2}-\varepsilon)]$

$\geq 1 - (1-2\varepsilon)^k$

$\geq 1 - e$

$$\geq 1 - (1-2\varepsilon)^k$$
$$\geq 1 - \delta$$

$k \log(1-2\varepsilon) = \log(\delta)$

$k \geq \frac{\log(\delta)}{\log(1-2\varepsilon)}$ (does not depend on n or m)

Approximation Algorithm (max opt. problem)

An algo A is an α-approximation if

$$\alpha \, OPT(x) \leq A(x) \leq OPT(x) \qquad \alpha \in (0,1)$$

We saw $(\frac{1}{2}-\varepsilon)$-approximation algo. for MAXCUT.

Randomized Approximation Algorithm

$$\Pr[\alpha \, OPT(x) \leq A(x) \leq OPT(x)] \geq 2/3$$

Combinatorial Optimisation Problem

max size of cut

min vertex cover

maximum matching.

max satisfiability problem

max independent set

min spanning tree

$NP \overset{?}{\subseteq} RP$

$P \subseteq RP \subseteq NP$

if $x \in L$,

large # of certificates (random coins) makes M_L accept

$x \notin L$

Randomized Complexity Classes

RP: is the class of L decided by randomize poly time TMs

$\forall L \in RP$, $\exists M_L$

for $x \in L$,

$$\Pr_r[M(x,r)=1] \geq 1/200 \quad \varepsilon \quad \frac{1}{2^{n^2}}$$

if $x \notin L$,

$$\Pr_r[M(x,r)=1] = 0$$

$RP_{1/2} = RP_{\frac{1}{1000}}$

$\subseteq$ (trivial)

$\supseteq$ (by boosting, repetition)