Tutorial - 3

Information & Communication

Solutions

PROBLEM - I (Chain Rule and Mutual Information)

The conditional mutual information of Random variables X and Y given Z is defined by

$$I(X;Y|Z) = H(X|Z) - H(X|Y,Z)$$

(a) Prove that (Chain Rule for Mutual Information)

$$I(X_1, X_2, \dots X_n ; Y) = \sum_{i=1}^{n} I(X_i ; Y | X_{i-1}, X_{i-2}, \dots X_1)$$

(b) Prove that

$$I(X;Y|Z) \geq 0$$

with equality if and only if

$$p(x|z) \cdot p(y|z) = p(x,y|z)$$

(c) Suppose if

$$p(x,y,z) = p(x) \cdot p(y|x) \cdot p(z|y)$$

then prove that

$$I(X;Y) \geq I(X;Z)$$

Solution - I

$$I(X;Y|Z) = H(X|Z) - H(X|Y,Z)$$

(a) $\text{L.H.S.} = I(\overbrace{X_1, X_2, \dots X_n}; Y)$

Chain Rule For Entropy

$$= H(X_1 \dots X_n) - H(X_1, \dots X_n | Y)$$

$$= \sum_{i=1}^{n} H(X_i | X_1 \dots X_{i-1}) - \sum_{i=1}^{n} H(X_i | Y, X_1, \dots X_{i-1})$$

Conditional Mutual Information

$$= \sum_{i=1}^{n} H(X_i | \underbrace{X_1 \dots X_{i-1}}) - H(X_i | Y, \underbrace{X_1, \dots X_{i-1}})$$

$$= \sum_{i=1}^{n} I(X_i ; Y, X_1, \dots X_{i-1}) = \text{R.H.S.}$$

Hence proved

(b) $I(X;Y|Z) \geq 0$

$$\text{L.H.S.} = H(X|Z) - H(X|Y,Z)$$

{Lecture Exercise}

$$I(X;Y) = D\left(p(x,y) \,\|\, p(x).p(y)\right) \geq 0$$

Hint: Write Condition Mutual Information in terms of Relative Entropy

$p(y) \cdot p(y|x)$

$$I(X;Y) = H(X) - H(X|Y)$$

$$= H(X,Y) - H(Y|X) - H(X|Y)$$

$$H(A|B) = \sum_{a,b} p(a,b) \cdot \log \frac{1}{p(a|b)}$$

$$= \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x,y) \left(\log \frac{1}{p(x,y)} - \log \frac{1}{p(x|y)} - \log \frac{1}{p(y|x)} \right)$$

$$= \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x,y) \log \left(\frac{p(x|y)\, p(y|x)}{p(x,y)} \right)$$

Valid p.d. in X, Y

$$= \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x,y) \log \left(\frac{p(x,y)}{p(x) \cdot p(y)} \right)$$

$$= D\left(p(x,y) \,\|\, p(x) \cdot p(y) \right) \geq 0$$

$$I(X;Y|Z) = H(X|Z) - H(X|Y,Z)$$

$$= H(X,Y|Z) - H(Y|X,Z) - H(X|Y,Z)$$

$$H(A|B) = \sum_{a,b} p(a,b) \cdot \log \frac{1}{p(a|b)}$$

$$= \sum_{x,y,z} p(x,y,z) \left(\log \frac{1}{p(x,y|z)} - \log \frac{1}{p(y|x,z)} - \log \frac{1}{p(x|y,z)} \right)$$

$$= \sum_{x,y,z} p(x,y,z) \log \left(\frac{p(y|x,z)\, p(x|y,z)}{p(x,y|z)} \right)$$

$$= \sum_{x,y,z} p(x,y,z) \log \left(\frac{p(x,y,z) \cdot \cancel{p(x,y,z)} \cdot p(z)}{p(x,z) \cdot p(y,z) \cdot \cancel{p(x,y,z)}} \right)$$

$$= \sum_{x,y,z} p(x,y,z) \log \left(\frac{p(x,y,z)}{p(x|z) \cdot p(y,z)} \right)$$

Valid p.d. in X, Y, Z

$$= D\left(p(x,y,z) \,||\, p(x|z) \cdot p(y|z) \right) \geq 0$$

Equality holds iff

$$p(x,y,z) = p(x|z) \cdot p(y,z)$$

$$\Rightarrow \boxed{p(x,y|z) = p(x|z) \cdot p(y|z)}$$

Therefore

$$I(X;Y|Z) \geq 0$$

and equality holds iff

$$p(x|z) \cdot p(y|z) = p(x,y|z)$$

Hence proved

(c) Given: $p(x,y,z) = p(x).\, p(y|x).\, p(z|y)$ — (i)

To prove: $I(X;Y) \geq I(X;Z)$

Chain Rule | $p(x,y,z) = p(x).\, p(y|x).\, p(z|y,x)$ — (ii)

(i) and (ii) imply

$p(z|y,x) = p(z|y)$

$\Rightarrow p(x|y,z) = p(x|y)$

<u>Conditional Independence</u>

Given Y, X and Z are independent

Intuitively: $I(X;Z|Y) = 0$

Given Y, Z contains no additional information about X

$H(X|Y) - H(X|Z,Y)$

$= 0$ <u>True</u>

$I(X;Y) - I(X;Z)$

$= H(\cancel{X}) - H(X|Y) - H(\cancel{X}) + H(X|Z)$

$= H(X|Z) - H(X|Y)$

$= H(X|Z) - H(X|Z,Y)$

$= I(X;Y|Z) \geq 0$

<u>Hence proved</u>

Problem - I (Functions of Random Variables)

(a) Let X and Y be two random variables on the set of non-negative integers. Show that if $Y = 2X$ then

$$H(X \mid Y) = H(Y \mid X) = 0$$

(b) Let $Y = g(X)$ for some function g. Show that $H(Y \mid X) = 0$. Under what conditions on g is $H(X \mid Y)$ also $= 0$? Analyze

$$\mathrm{Supp}\,(P_{X,Y}) = \mathbb{Z}_+$$

(c) Let X and Y be two random variables. Let E be a random variable defined as

$$E = \begin{cases} 0 & \text{if } Y = X \\ 1 & \text{if } Y \neq X \end{cases}$$

Prove that

$$H(E \mid X, Y) = 0$$

(a) Given : $Y = 2X$

To Show : $H(Y|X) = H(X|Y) = 0$

Intuitively : Given X
Y is completely known

$Y = 2X$
$\Rightarrow X = 2Y$

$\therefore$ Given Y
X is completely known

$Y = 2X$ | X completely determines Y

$$p(Y=y \mid X=x) = \begin{cases} 1 & y = 2x \\ 0 & y \neq 2x \end{cases}$$

$$H(Y|X) = \sum_{x \in \mathbb{Z}_+} p(X=x) \sum_{y \in \mathbb{Z}_+} p(Y=y \mid X=x) \log \frac{1}{p(Y=y \mid X=x)} = 0$$

$X = 2Y$ | Y completely determines X

$$p(X=x \mid Y=y) = \begin{cases} 1 & x = y/2 \\ 0 & x \neq y/2 \end{cases}$$

$$H(X|Y) = \sum_{y \in \mathbb{Z}_+} p(Y=y) \sum_{x \in \mathbb{Z}_+} p(X=x \mid Y=y) \log \frac{1}{p(X=x \mid Y=y)} = 0$$

(b) Given : $Y = g(X)$

To Show : $H(Y|X) = 0$

Intuitively: Given X
Y is completely known

$Y = g(X)$ | X completely determines Y

$$p(Y=y \mid X=x) = \begin{cases} 1 & y = g(x) \\ 0 & y \neq g(x) \end{cases}$$

$$H(Y|X) = \sum_{x,y \in \mathrm{Supp}(P_{Y,X})} p(X=x) \cdot p(Y=y|X=x) \log \frac{1}{p(Y=y|X=x)} = 0$$

Conditions on g such that $H(X|Y) = 0$

Intuitively : g should be a bijective function

To show : $H(X|Y) = 0$ if and only if g is a bijective Function

If part : easy to check

Only if part :

$$H(X|Y) = \sum_{y \in \mathbb{Z}_+} p(Y=y) H(X \mid Y=y)$$

$\forall\, y$ s.t. $P(Y=y) > 0$

$H(X|Y) = 0$

$\Leftrightarrow \quad p(x|y) \log p(x|y) = 0$

$\Leftrightarrow \quad p(x|y) = \{0, 1\}$

Whenever $p(x|y) = 1$

Let $x = f(y)$

If $x' \neq f(y)$

then $p(x'|y) = 0$

$\sum_{x \in \mathcal{X}} p(x|y) = 1$

$\therefore$ For each y there will be only one such x

Whenever $p(y) = 0$ assign arbitrary value in $\mathcal{X}$ to $f(y)$.

$\therefore X$ is completely determined by Y

$\Leftrightarrow \quad X = f(Y)$

$\Leftrightarrow \quad X = f(g(X))$

$\therefore \quad g^{-1}$ exist if $H(X|Y) = 0$

$\therefore \quad g$ is a bijective function if $H(X|Y) = 0$

Problem - 3 (Chain Rule)

Let $\{X_i, 1 \le i \le 3\}$ be uniform random variables over $\{0,1\}$. Let Y be a uniform random variable over $\{0,1\}$

X_1,X_2 \ X_3,Y	$X_1=0$, $X_2=0$	$X_1=0$, $X_2=1$	$X_1=1$, $X_2=0$	$X_1=1$, $X_2=1$
$X_3=0$, $Y=0$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{32}$	$\frac{1}{64}$
$X_3=0$, $Y=1$	$\frac{1}{32}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$
$X_3=1$, $Y=0$	$\frac{1}{64}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$
$X_3=1$, $Y=1$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{4}$

check if

(a) $H(X_1,X_2,X_3) = H(X_1) + H(X_2|X_1) + H(X_3|X_2,X_1)$

(b) $H(X_1,X_2,X_3|Y) = H(X_1|Y) + H(X_2|X_1,Y) + H(X_3|X_2,X_1,Y)$