DATE

Q1) Consider the wave function $\psi(x) = e^{-ipx/\hbar}$, where x represents position and p represents momentum. Further consider the momentum operator $\hat{p} = -i\hbar\nabla$, and the position operator $\hat{x}$ acting on the wave function $\psi(x)$. Prove $[\hat{x}, \hat{p}] = i\hbar$

Hint: $\hat{x}\psi(x) = x\psi(x)$

Ans) $[\hat{x}, \hat{p}] = i\hbar$

$[\hat{x}, \hat{p}] = \hat{x}\cdot\hat{p} - \hat{p}\cdot\hat{x}$

$$(\hat{x}\cdot\hat{p} - \hat{p}\cdot\hat{x})\psi(x) = \hat{x}\cdot\hat{p}\cdot\psi(x) - \hat{p}\cdot\hat{x}\cdot\psi(x)$$

$$= \hat{x}\cdot -i\hbar\frac{\partial}{\partial x}(\psi(x)) - \hat{p}\cdot x\cdot\psi(x)$$

$$\frac{\partial}{\partial x}\psi(x) = \frac{\partial}{\partial x} e^{-ipx/\hbar} = e^{-ipx/\hbar} \times -ip/\hbar = \frac{-ip}{\hbar}\psi(x)$$

$$\frac{\partial}{\partial x}(x\cdot\psi(x)) = \psi(x)\cdot\frac{\partial x}{\partial x} + x\cdot\frac{\partial}{\partial x}(\psi(x))$$

$$= \psi(x) + x\cdot -\frac{ip}{\hbar}\psi(x) = \psi(x) - \frac{xip}{\hbar}\psi(x)$$

$$[\hat{x}, \hat{p}]\psi(x) = \hat{x}\, -i\cancel{\hbar} \times -\frac{ip}{\cancel{\hbar}}\psi(x) + i\hbar\frac{\partial}{\partial x}(x\cdot\psi(x))$$

$$= -xp\psi(x) + i\hbar\psi(x) - i\hbar\frac{xip}{\hbar}\psi(x)$$

$$= -xp\psi(x) + i\hbar\psi(x) + xp\psi(x)$$

$$= \underline{\underline{i\hbar\psi(x)}}$$

$\therefore [\hat{x}, \hat{p}] = \underline{\underline{i\hbar}}$

Q2) Using the above communication relation b/w $\hat{x}$ & $\hat{p}$, determine the communication relation $[\hat{x}, \hat{p}^2]$. Further determine the communication relation $[\hat{x}, \hat{p}^n]$, where n is an arbitrary positive integer.

Ans) $[x, \hat{p}^2] = [x, \hat{p}]\hat{p} + \hat{p}[x, \hat{p}]$

from this.

$[\hat{A}, \hat{B}\hat{C}] = [\hat{A}, \hat{B}]\hat{C} + \hat{B}[\hat{A}, \hat{C}]$

$$[\hat{x}, \hat{p}^2] = [\hat{x}, \hat{p}]\cdot\hat{p} + \hat{p}[\hat{x}, \hat{p}]$$
$$= i\hbar\hat{p} + \hat{p}\cdot i\hbar$$
$$= 2i\hbar\hat{p}$$

$$[\hat{x}, \hat{p}^3] = [\hat{x}, \hat{p}^2]\cdot\hat{p} + \hat{p}^2[\hat{x}, \hat{p}]$$

$$= 2i\hbar\hat{p}^2 + \hat{p}^2[i\hbar]$$
$$= 3i\hbar\hat{p}^2$$

$$[\hat{x}, \hat{p}^n] = \underline{\underline{n i\hbar p^{n-1}}}$$

classmate

PAGE

DATE

Q3) Consider a Hermitian operator $\hat{Q}$ which has a spectral decomposition $\hat{Q} = \sum_i q_i |i\rangle\langle i|$. Then prove $\sin\hat{Q} = \sum_i \sin q_i \, |i\rangle\langle i|$

Ans) $\sin\theta = \frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots$

$$\sin\hat{Q} = \frac{\hat{Q}}{1!} - \frac{1}{3!}\hat{Q}\cdot\hat{Q}\cdot\hat{Q} + \frac{1}{5!}\underbrace{\hat{Q}\cdot\hat{Q}\cdots\hat{Q}}_{5\text{times}} - \dots$$

$$= \sum_i q_i|i\rangle\langle i| - \frac{1}{3!}\left(\sum_i q_i|i\rangle\langle i|\right)\left(\sum_j q_j|j\rangle\langle j|\right)\left(\sum_k q_k|k\rangle\langle k|\right)$$

$+ \dots$

Lets take the term $\left[\sum_i\sum_j\sum_k q_i q_j q_k |i\rangle\langle i||j\rangle\langle j||k\rangle\langle k|\right]$

$$= \sum_i\sum_k q_i q_k |i\rangle\langle j||k\rangle\langle k| \left(\sum_j q_j \langle i|j\rangle\right)$$

$$= \sum_i\sum_k q_i q_k |i\rangle\langle j||k\rangle\langle k| \left(\sum_j q_j \delta(i-j)\right)$$

dirac notation

$$= \sum_i\sum_k q_i q_k |i\rangle\langle i||k\rangle\langle k| \,(q_i)$$

$$= \sum_i q_i^2 |i\rangle\langle k| \left(\sum_k q_k \langle i|k\rangle\right)$$

$$= \sum_i q_i^2 |i\rangle\langle k| \left(\sum_k q_k \delta(i-k)\right)$$

dirac notation

$$= \sum_i q_i^2 |i\rangle\langle i| \,(q_i)$$

$$= \sum_i q_i^3 |i\rangle\langle i|$$

similarly $\hat{Q}\cdot\hat{Q}\cdot\hat{Q}\cdot\hat{Q}\cdot\hat{Q} = \sum_i q_i^5 |i\rangle\langle i|$

$$\therefore \sin\hat{Q} = \frac{\sum_i q_i|i\rangle\langle i|}{1!} - \frac{1}{3!}\sum_i q_i^3|i\rangle\langle i| + \frac{1}{5!}\sum_i q_i^5|i\rangle\langle i|$$

$$= \sum_i \left(q_i - \frac{q_i^3}{3!} + \frac{q_i^5}{5!} - \dots\right)|i\rangle\langle i|$$

$$= \sum_i \sin q_i \, |i\rangle\langle i|$$

Q4) Prove the following commutation identity (with full working):

$$[[A,C],[B,D]] = [[[A,B],C],D] + [[[B,C],D],A] + [[[C,D],A],B] + [[[D,A],B],C]$$

Ans) proof:

LHS:

$$[[A,C],[B,D]] = [AC - CA,\ BD - DB] = (ACBD - ACDB - CABD + CADB) - (BDAC - BDCA - DBAC + DBCA)$$

$$= ACBD + CADB + BDCA + DBAC - (ACDB + CABD + BDAC + DBCA)$$

RHS:

$$[[[A,B],C],D] = [[AB - BA, C], D] = [ABC - BAC - CAB + CBA, D]$$

$$= ABCD - BACD - CABD + CBAD - DABC + DBAC + DCAB - DCBA \rightarrow (i)$$

$$[[[B,C],D],A] = [[BC - CB, D], A] = [BCD - CBD - (DBC - DCB), A]$$

$$= BCDA - CBDA - DBCA + DCBA - (ABCD - ACBD - ADBC + ADCB)$$

$$= BCDA - CBDA - DBCA + DCBA - ABCD + ACBD + ADBC - ADCB \rightarrow (ii)$$

$$[[[C,D],A],B] = [[CD - DC, A], B] = [(CDA - DCA) - (ACD - ADC), B]$$

$$= [CDA - DCA - ACD + ADC, B] = CDAB - DCAB - ACDB + ADCB - BCDA + BDCA + BACD - BADC \rightarrow (iii)$$

$$[[[D,A],B],C] = [[DA - AD, B], C] = [DAB - ADB - BDA + BAD, C]$$

$$= DABC - ADBC - BDAC + BADC - CDAB + CADB + CBDA - CBAD \rightarrow (iv)$$

Adding (i), (ii), (iii) & (iv)

$$ABCD + CBAD + DBAC + DCAB - BACD - CABD - DABC - DCBA$$
$$BCDA + DCBA + ACBD + ADBC - CBDA - DBCA - ABCD - ADCB$$
$$CDAB + ADCB + BDCA + BACD - DCAB - ACDB - BCDA - BADC$$
$$DABC + BADC + CADB + CBDA - ADBC - BDAC - CDAB - CBAD$$

$$ACBD + CADB + BDCA + DBAC - (ACDB + CABD + BDAC + DBCA)$$

DATE

Q5) Consider a Ket space spanned by eigenkets $\{|a_i\rangle\}$ of a hermitian operator $\hat{A}$. Assume there is no degeneracy. prove that $\prod_{a_i} (\hat{A} - a_i) = \vec{0}$ → null operator.
Here $\prod_{i=1}^{n} A_i = A_1 \cdot A_2 \cdot \, \ldots \, \cdot A_n$

Ans Since $\{|a_i\rangle\}$ are eigenkets of $\hat{A}$

$$\underbrace{\hat{A}|a_1\rangle = a_1|a_1\rangle, \quad \hat{A}|a_2\rangle = a_2|a_2\rangle,}_{\text{No degeneracy.}} \ldots \quad \hat{A}|a_n\rangle = a_n|a_n\rangle$$

$(\hat{A} - a_1 \cdot I)|a_1\rangle = 0$

Either $(\hat{A} - a_1 \cdot I) = 0$ or $|a_1\rangle = 0$ → not possible

$\therefore (\hat{A} - a_1 \cdot I) = 0$

Similarly $(\hat{A} - a_2 \cdot I) = 0, (\hat{A} - a_3 \cdot I) = 0, \ldots (\hat{A} - a_n \cdot I) = 0$

$\therefore (\hat{A} - a_1) \cdot (\hat{A} - a_2)(\hat{A} - a_3) \ldots (\hat{A} - a_n) = 0$

$\prod_{a_i} (\hat{A} - a_i) = 0$

Q7) Consider 2 noncommuting operators $\hat{A}$ & $\hat{B}$, i.e., $[\hat{A}, \hat{B}] \neq 0$. Show that they cannot have a complete set of common eigen functions.

Ans) Assume that $\hat{A}$ & $\hat{B}$ are non commuting / incompatible operators and they have a complete set of common eigen kets.

$\{|a'\rangle\}$ → eigen kets of $\hat{A}$
$\{|b'\rangle\}$ → eigen kets of $\hat{B}$
Since they are same sets of eigen kets we can write them as: $|a'b'\rangle$

$[\hat{A}, \hat{B}] \neq 0$ (incompatible)

DATE

$$\hat{A}\hat{B}|a'b'\rangle = \hat{A}\, b'|a'b'\rangle = a'b'|a'b'\rangle$$

$$\hat{B}\hat{A}|a'b'\rangle = \hat{B}\, a'|a'b'\rangle = a'b'|a'b'\rangle$$

$$\hat{A}\hat{B}|a'b'\rangle - \hat{B}\hat{A}|a'b'\rangle = 0$$

$$\Rightarrow (\hat{A}\hat{B} - \hat{B}\hat{A})|a'b'\rangle = 0$$

$$\Rightarrow [\hat{A},\hat{B}] = 0 \quad \text{or} \quad |a'b'\rangle = 0$$

$\longrightarrow$ not possible

$\therefore$ $[\hat{A},\hat{B}] = 0$ is a contradiction.

They cannot have a complete set of common eigen functions.

Q6) Suppose A' and A'' are matrix representations of an operator $\hat{A}$ on a vector space V with respect to two different orthonormal basis $|v_i\rangle$ and $|w_i\rangle$. Characterize the relationship between A' and A''

Ans)

$$A' = \sum_i v_i\, |v_i\rangle\langle v_i| \qquad A'' = \sum_i w_i\, |w_i\rangle\langle w_i|$$

$$U = \sum_i |w_i\rangle\langle v_i| \qquad U^+ = \sum_i |v_i\rangle\langle w_i|$$

$$|w_i\rangle = U|v_i\rangle$$
$$\langle v_j|w_i\rangle = \langle v_j|U|v_i\rangle$$
$$\langle w_i|v_j\rangle = \langle v_i|U^+|v_j\rangle$$

$$A' = \sum_i v_i\, |v_i\rangle\langle v_i|$$

$$= \sum_i \sum_j |w_j\rangle\langle w_j|\, v_i\, |v_i\rangle\langle v_i|$$

$$= \sum_i \sum_j v_i\, |w_j\rangle\langle w_j|v_i\rangle\langle v_i|$$

$$= \sum_i \sum_j v_i\, |w_j\rangle\langle v_j|U^+|v_i\rangle\langle v_i|$$

$$A'' = \sum_i \omega_i |\omega_i\rangle\langle\omega_i|$$

$$= \sum_i \omega_i \left(\sum_j |v_j\rangle\langle v_j|\right) |\omega_i\rangle\langle\omega_i|$$

$$= \sum_i \sum_j \omega_i |v_j\rangle\langle v_j|\omega_i\rangle\langle\omega_i|$$

$$= \sum_i \sum_j \omega_i |v_j\rangle\langle v_j|U|v_i\rangle\langle\omega_i|$$

Pauli's matrices.

DATE

Q8) $\hat{S_x} = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\hat{S_y} = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\hat{S_z} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

prove:

a) $[\hat{S_i}, \hat{S_j}] = i\,\epsilon_{ijk}\hbar\,\hat{S_k}$

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if } ijk = (x,y,z) \text{ or } (y,z,x) \text{ or } (z,x,y) \\ -1 & \text{if } ijk = (z,y,x) \text{ or } (x,z,y) \text{ or } (y,z,x) \\ 0 & \text{if repetition occurs.} \end{cases}$$

Proof:

$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} = -(\hat{B}\hat{A} - \hat{A}\hat{B}) = -[\hat{B}, \hat{A}]$

The no of combinations are

$[\hat{S_x}, \hat{S_x}]$ $\quad [\hat{S_x}, \hat{S_y}]$ $\quad [\hat{S_x}, \hat{S_z}]$

$[\hat{S_y}, \hat{S_x}]$ $\quad [\hat{S_y}, \hat{S_y}]$ $\quad [\hat{S_y}, \hat{S_z}]$

$[\hat{S_z}, \hat{S_x}]$ $\quad [\hat{S_z}, \hat{S_y}]$ $\quad [\hat{S_z}, \hat{S_z}]$

$[\hat{S_x}, \hat{S_x}] = [\hat{S_y}, \hat{S_y}] = [\hat{S_z}, \hat{S_z}] = 0$

since $[\hat{A}, \hat{A}] = \hat{A}\cdot\hat{A} - \hat{A}\cdot\hat{A} = \hat{A}^2 - \hat{A}^2 = 0$ ✓

$$[\hat{S_x}, \hat{S_y}] = \hat{S_x}\cdot\hat{S_y} - \hat{S_y}\cdot\hat{S_x}$$
$$= \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} - \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$= \frac{\hbar^2}{4}\begin{pmatrix} +i & 0 \\ 0 & -i \end{pmatrix} - \frac{\hbar^2}{4}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$
$$= \frac{\hbar^2}{4}\begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix}$$
$$= i\frac{\hbar^2}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\hbar\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\hbar\,\hat{S_z}$$

$\downarrow$

$\epsilon_{ijk} = 1$

$(i\,j\,k) = (x, y, z)$

$$[\hat{S_y}, \hat{S_x}] = -[\hat{S_x}, \hat{S_y}]$$
$$= -i\hbar\,\hat{S_z} =$$

$\downarrow$

$\epsilon_{ijk} = -1$

$(i, j, k) = (y, x, z)$

DATE

$$[\hat{S}_x, \hat{S}_z] = \hat{S}_x \cdot \hat{S}_z - \hat{S}_z \cdot \hat{S}_x$$

$$= \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \frac{\hbar^2}{4}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \frac{\hbar^2}{4}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= \frac{\hbar^2}{4}\begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$$

$$= \frac{\hbar^2}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar^2}{2}\begin{pmatrix} 0 & i^2 \\ -i^2 & 0 \end{pmatrix}$$

$$= i\frac{\hbar^2}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

$$= i\hbar \cdot \left(\frac{\hbar}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\right)$$

$$= -i\hbar\,\hat{S}_y$$

$$\downarrow$$

$$\epsilon_{ijk} = -1$$

$$(i,j,k) = (x,z,y)$$

$$[\hat{S}_z, \hat{S}_x] = -[\hat{S}_x, \hat{S}_z]$$

$$= +i\hbar\,\hat{S}_y$$

$$\downarrow$$

$$\epsilon_{ijk} = +1$$

$$(i,j,k) = (z,x,y)$$

$$[\hat{S}_y, \hat{S}_z] = \hat{S}_y \cdot \hat{S}_z - \hat{S}_z \cdot \hat{S}_y$$

$$= \frac{\hbar^2}{4}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \frac{\hbar^2}{4}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} - \frac{\hbar^2}{4}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

$$= \frac{\hbar^2}{4}\begin{bmatrix} 0 & 2i \\ 2i & 0 \end{bmatrix} = i\hbar\left[\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]$$

$$= i\hbar\,\hat{S}_x \rightarrow \epsilon_{ijk} = 1$$

$$\downarrow$$

$$(i,j,k) = (y,z,x)$$

$$[\hat{S}_z, \hat{S}_y] = -[\hat{S}_y, \hat{S}_z]$$

$$= -i\hbar\,\hat{S}_x \rightarrow \epsilon_{ijk} = -1$$

$$(i,j,k) = (z,y,x)$$

ii) $\{\hat{S}_i, \hat{S}_j\} = \frac{\hbar^2}{2} \delta_{ij} I$

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Proof:

$$\{\hat{A}, \hat{B}\} = \hat{A}\cdot\hat{B} + \hat{B}\cdot\hat{A} = \{\hat{B}, \hat{A}\}$$

$$\{\hat{A}, \hat{A}\} = \hat{A}\cdot\hat{A} + \hat{A}\cdot\hat{A} = \hat{A}^2 + \hat{A}^2 = 2\hat{A}^2$$

$$\therefore \{S_x, S_x\} = 2\cdot\hat{S}_x^2 = 2\cdot\frac{\hbar^2}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \frac{\hbar^2}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{\hbar^2}{2} I \rightarrow \delta_{ij} = 1 \quad (i,k) = (x,x)$$

$$\{S_y, S_y\} = 2\cdot\hat{S}_y^2 = 2\cdot\frac{\hbar^2}{4}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \frac{\hbar^2}{2}\begin{pmatrix} -i^2 & 0 \\ 0 & -i^2 \end{pmatrix} = \frac{\hbar^2}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{\hbar^2}{2} I$$

$\delta_{ij} = 1$ $(i,j) = (y,y)$

Similarly

$$\{S_z, S_z\} = \frac{\hbar^2}{2} I \rightarrow \delta_{ij} = 1 \quad (i,j) = (z,z)$$

$$\{\hat{S}_x, \hat{S}_y\} = \hat{S}_x\cdot\hat{S}_y + \hat{S}_y\cdot\hat{S}_x$$

$$= \frac{\hbar^2}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{\hbar^2}{4}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \frac{\hbar^2}{4}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + \frac{\hbar^2}{4}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = \underline{\underline{0}} = \{\hat{S}_y, \hat{S}_x\}$$

$\rightarrow \delta_{ij} = 0 \quad (i \neq j)$

$$\{\hat{S}_x, \hat{S}_z\} = \hat{S}_x\cdot\hat{S}_z + \hat{S}_z\cdot\hat{S}_x$$

$$= \frac{\hbar^2}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \frac{\hbar^2}{4}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \frac{\hbar^2}{4}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 0 = \{\hat{S}_z, \hat{S}_x\}$$

$\rightarrow \delta_{ij} = 0 \quad (i \neq j)$

$$\{\hat{S}_y, \hat{S}_z\} = \hat{S}_y\cdot\hat{S}_z + \hat{S}_z\cdot\hat{S}_y$$

$$= \frac{\hbar^2}{4}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \frac{\hbar^2}{4}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + \frac{\hbar^2}{4}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = 0 = \{\hat{S}_z, \hat{S}_y\}$$

$\rightarrow \delta_{ij} = 0 \quad (i \neq j)$

c) $[\hat{S}^2, \hat{S}_i] = 0$ Here $\hat{S}^2 = \hat{S}\cdot\hat{S} = \hat{S}_x^2 + \hat{S}_y + \hat{S}_z$

Proof:

$$[\text{const}, \hat{A}] = \text{const}\cdot\hat{A} - \hat{A}\cdot\text{const}$$
$$= \text{const}\cdot\hat{A} - \text{const}\cdot\hat{A}$$
$$= \underline{\underline{0}}$$

From prev result,

$$\{S_x, S_x\} = 2\hat{S}_x^2 = \frac{\hbar^2}{2} I, \Rightarrow \hat{S}_x^2 = \frac{\hbar^2}{4} I$$

Similarly, $\hat{S}_y^2 = \frac{\hbar^2}{4} I$ & $\hat{S}_z^2 = \frac{\hbar^2}{4} I$

$$\hat{S}^2 = \frac{\hbar^2}{4} I + \frac{\hbar^2}{4} I + \frac{\hbar^2}{4} I = \frac{3\hbar^2}{4} I = \text{const}.$$

$$[\hat{S}^2, \hat{S}_i] = \hat{S}^2\cdot\hat{S}_i - \hat{S}_i\cdot\hat{S}^2$$
$$= \hat{S}^2\cdot\hat{S}_i - \hat{S}^2\cdot\hat{S}_i$$
$$= 0 \quad \text{for all } \underline{\underline{i}}$$