Tail Inequalities

- Given the earlier conditions, it holds that for any $\delta > 0$,

$$\Pr(X \geq \mu(1+\delta)) \leq \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{\mu}$$

- So, $\Pr\left(Y \geq e^{t\mu(1+\delta)}\right) = e^{-\mu(1-e^{t}) - t\mu(1+\delta)}$
- Since t is a free parameter in the above, we can find a t that minimizes the right hand side.
- To simplify, let $f(t) = \ln e^{-\mu(1-e^{t}) - t\mu(1+\delta)}$

$$= -\mu(1-e^{t}) - t\mu(1+\delta)$$

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- Differentiating f(t) wrt t, we get $f'(t) = \mu e^t - \mu(1+\delta)$

- So, f'(t) = 0 at $t = \ln(1+\delta)$

- Verify that the above t corresponds to a minima. OFFLINE.

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- With $t = \ln(1+\delta)$, we get that

$$\Pr(X \geq \mu(1+\delta)) \leq \frac{e^{-\mu(1-(1+\delta))}}{(1+\delta)^{\mu(1+\delta)}} = \frac{e^{\mu\delta}}{(1+\delta)^{1+\delta}}$$

$$= \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{\mu}$$

- completing the proof.

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- A simplification of the RHS gives

$$\Pr(X \geq \mu(1+\delta)) \leq \begin{cases} e^{-\mu\delta^2/4} & \text{if } \delta \leq 1 \\ e^{-\mu\delta\ln\delta} & \text{if } \delta > 1 \end{cases}$$

A Simple Example

- Here is one more practical application of the Chernoff bounds.
- Consider once again counting the number of heads out of tossing n fair coins independently.
  - So, $p = ½$.
- Let X_i denote the random variable that takes 1 if the ith coin toss results in a head, and 0 otherwise.
  - $E[X_i] = ½$.
- Let $X = \Sigma_i X_i$
- X counts the total number of heads over the n tosses.
  - $E[X] = n/2$.
  - With n = 100, say, we expect 50 heads over 100 coin tosses.

A Simple Example

- Markov inequality tells that the probability that X takes a value beyond 70 is $\Pr(X \geq 70) = \Pr(X \geq 70/50 \times 50) <= 5/7 = 0.7$ (approx.)
- To apply Chbychev's inequality, we need to do some extra work.
- $Var(X_i)$ for any i is computed as $E[X_i^2] - E[X_i]^2$.
- $E[X_i^2] = 0x(1/2) + 1x(1/2) = ½$.
- $Var(X_i) = ½ - (½)^2 = ¼$.
- $Var(X) = 100x\ Var(X_1)$ due to independence.
- So, $Var(X) = 100/4 = 25$, and $\sigma_X = 5$.
- Now, $\Pr(X \geq 70)$ can be rewritten as $\Pr(|X - 50| \geq 20)$ and further as $\Pr(|X - 50| \geq (20x\ 1/ \sigma_X) \sigma_X))$ which is now at most $25/400 = 1/16 = 0.0625$.