- Theorem: Let each X_i , $1 \le i \le n$ be $\{a_i , a_i + 1\}$ valued for some real a_i and X_i's be independent. Let $X = \Sigma_i X_i$ and $E[X] =$. Then for any $\delta > 0$, $\Pr[X \ge \mu + \delta n] \le \exp\{-2 \delta^2 n\}$ and $\Pr[X \le \mu - \delta n] \le \exp\{-2 \delta^2 n\}$.
  - Proof is an offline exercise for you.

Ans) $X_i \to a_i \quad Prob = P_i$

$\quad\;\; \to a_i + 1 \quad Prob = 1 - P_i$

$X_i - a_i \to 0 \quad Prob = P_i$

$\quad\quad\;\; \to 1 \quad Prob = 1 - P_i$

$$E[X_i - a_i] = P_i(0) + (1-P_i)1$$
$$= 1 - P_i$$

$$E[X] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i - a_i] + \sum_{i=1}^{n} a_i$$
$$= \sum_{i=1}^{n}(1-P_i) + \sum_{i=1}^{n} a_i = n - \sum_{i=1}^{n} P_i + \sum_{i=1}^{n} a_i$$
$$\mu = n - \sum_{i=1}^{n}(P_i - a_i)$$

$$Pr[X \geq \mu + \delta n] = Pr\left[e^{Xt} \geq e^{(\mu+\delta n)t}\right]$$

Let Y be a r.v such that $Y = e^{Xt}$

$$Y = e^{(X_1 + X_2 + \dots X_n)t} = e^{(X_1 - a_1 + X_2 - a_2 + \dots X_n - a_n + a_1 + a_2 + \dots a_n)t}$$
$$= \underbrace{e^{(X_1-a_1)t}}_{Y_1} \cdot \underbrace{e^{(X_2-a_2)t}}_{Y_2} \cdots \underbrace{e^{(X_n-a_n)t}}_{Y_n} \cdot \underbrace{e^{\sum_{i=1}^{n} a_i t}}_{const}$$

$$E[Y_i] = e^{0t} \cdot P_i + (1-P_i)e^t = P_i + (1-P_i)e^t$$

$$E[Y] = E\left[Y_1 Y_2 \dots Y_n \cdot e^{\sum_{i=1}^{n} a_i t}\right] = e^{\sum_{i=1}^{n} a_i t} E[Y_1 \cdot Y_2 \dots Y_n]$$
$$= e^{\sum_{i=1}^{n} a_i t} \cdot \prod_{i=1}^{n} E[Y_i]$$
$$= e^{\sum_{i=1}^{n} a_i t} \prod_{i=1}^{n} P_i + (1-P_i)e^t$$

$$1 + \underbrace{P_i + (1-P_i)e^t - 1}_{x} \leq \underbrace{e^{(P_i - 1) + (1-P_i)e^t}}_{ex} \quad (1 + x \leq e^x)$$

$$E[Y] = e^{\sum_{i=1}^{n} a_i t} \cdot \prod_{i=1}^{n} e^{(p_i - 1)[1-e^t]}$$

$$= e^{\sum_{i=1}^{n} a_i t} \cdot e^{(1-e^t)\left(\sum_{i=1}^{n} p_i - n\right)}$$

$$\Pr\left(e^{xt} \geq e^{(\mu+\delta n)t}\right) = \Pr\left(Y \geq e^{(\mu+\delta n)t}\right)$$

$$\leq \frac{E[Y]}{e^{(\mu+\delta n)t}}$$

$$\leq \frac{e^{\sum_{i=1}^{n} a_i t + (1-e^t)\left(\sum_{i=1}^{n} p_i - n\right)}}{e^{(\mu+\delta n)t}}$$

$$\leq e^{\left(\sum_{i=1}^{n} a_i - (\mu+\delta n)\right)t + (1-e^t)\left(\sum_{i=1}^{n} p_i - n\right)}$$

$$f(t) = \left[\sum_{i=1}^{n} a_i - (\mu+\delta n)\right]t + (1-e^{+t})\left(\sum_{i=1}^{n} p_i - n\right)$$

$$f'(t) = \sum_{i=1}^{n} a_i - (\mu+\delta n) + -e^t\left(\sum_{i=1}^{n} p_i - n\right) = 0$$

$$\Rightarrow e^t = \frac{\sum_{i=1}^{n} a_i - \mu - \delta n}{\sum_{i=1}^{n} p_i - n} = \frac{\mu - \sum_{i=1}^{n} a_i + \delta n}{\mu - \sum_{i=1}^{n} a_i}$$

$$t = \log\left(\frac{\mu - \sum_{i=1}^{n} a_i + \delta n}{\mu - \sum_{i=1}^{n} a_i}\right)$$

$$\Pr[X \le \mu - \delta\mu] = \Pr[-X \ge \delta\mu - \mu]$$

$$= \Pr\left[e^{-Xt} \ge e^{(\delta\mu - \mu)t}\right]$$

$$Y = e^{-Xt} = e^{-(X_1 + X_2 + \cdots X_n)t}$$

R.V such that

$$= e^{-(X_1 - a_1 + X_2 - a_2 + \cdots X_n - a_n)t - \sum_{i=1}^{n} a_i t}$$

$$= \underbrace{e^{-(X_1-a_1)t}}_{Y_1} \cdot \underbrace{e^{-(X_2-a_2)t}}_{Y_2} \cdots \underbrace{e^{-(X_n-a_n)t}}_{Y_n} \cdot e^{\sum_{i=1}^{n} a_i t}$$

$$E[Y_i] = P_i e^{-t(0)} + (1-P_i)e^{-t(1)}$$

$$= P_i + (1-P_i)e^{-t}$$

$$E[Y] = E\left[Y_1 Y_2 \cdots Y_n \cdot e^{\sum_{i=1}^{n} a_i t}\right] = e^{\sum_{i=1}^{n} a_i t} \cdot E[Y_1 \cdot Y_2 \cdots Y_n]$$

$$= e^{\sum_{i=1}^{n} a_i t} \cdot \prod_{i=1}^{n} P_i + (1-P_i)e^{-t}$$

$$1 + \underbrace{P_i + (1-P_i)e^{-t} - 1}_{x} \le e^{P_i - 1 + (1-P_i)e^{-t}} \qquad (1+x \le e^x)$$

$$\le e^{(P_i - 1)(1-e^{-t})}$$

$$\therefore E[Y] = e^{\sum_{i=1}^{n} a_i t} \cdot e^{(1-e^{-t})\left(\sum_{i=1}^{n} P_i - n\right)}$$

$$= e^{\sum_{i=1}^{n} a_i t + (1-e^{-t})\left(\sum_{i=1}^{n} P_i - n\right)}$$

$$E\left[e^{-Xt} \ge e^{(\delta\mu - \mu)t}\right] = E\left[Y \ge e^{(\delta\mu - \mu)t}\right] \le \frac{E[Y]}{e^{(\delta\mu - \mu)t}}$$

$$\Pr\left[Y \geq e^{(\delta n - \mu)t}\right] \leq e^{\left(\sum_{i=1}^{n} a_i + \mu - \delta n\right)t + (1-e^{-t})\left(\sum_{i=1}^{n} P_i - n\right)}$$

$$f(t) = t\left(\sum_{i=1}^{n} a_i + \mu - \delta n\right) + (1-e^{-t})\left(\sum_{i=1}^{n} P_i - n\right)$$

$$f'(t) = \sum_{i=1}^{n} a_i + \mu - \delta n + e^{-t}\left(\sum_{i=1}^{n} P_i - n\right) = 0$$

$$\Rightarrow e^{-t}\left(\sum_{i=1}^{n} P_i - n\right) = \delta n - \mu - \sum_{i=1}^{n} a_i$$

$$e^{-t} = \frac{\delta n - \mu - \sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} P_i - n}$$

$$-t = \log_e\left(\frac{\delta n - \mu - \sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} P_i - n}\right)$$

$$t = \log_e\left(\frac{\sum_{i=1}^{n} P_i - n}{\delta n - \mu - \sum_{i=1}^{n} a_i}\right)$$

$$= \log_e\left(\frac{\sum_{i=1}^{n} a_i - \mu}{\delta n - \left(\mu + \sum_{i=1}^{n} a_i\right)}\right)$$