Complexity and Advanced Algorithms Spring 2021

Approximation Algorithms – A Brief Introduction

- Suppose a problem is known to be NP-Complete.
- No hope to solve in polynomial time unless P =
 NP.
- But, several practical problems fall in this category.
- Need some solution to this issue.
- This is where approximation algorithms help.

- For a problem P, let A be an approximation algorithm.
- Suppose that the problem P is a minimization problem.
- Then, the performance of Algorithm A for P is measured as its approximation ratio defined as follows.
- For an instance I of P, let OPT(I) denote the best possible solution.
- Let A(I) denote the solution produced by the algorithm A.
- The approximation ratio of algorithm A is $\max_{I} |A(I)|/|OPT(I)|$.
- Notice that the ratio is always at least 1.

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- We have seen an example of this definition earlier.
- In the context of MAXSAT.
- Today, we will study two more problems and approximation algorithms for them.

m Ax f() fest possible
$$I: f(q(q)=20)$$

for any alq. $f(A(x)) \le 20$
for any input, if $|ap_1(x)| \le 20$
 $|A(x)| > (2) |ap_1(x)|$ $|A(x)|$

- •Consider m machines M₁, M₂, ..., M_m, that are identical in all respects.
 - Like the m cores of your multicore computer.
- •We have n jobs, J₁, J₂,..., Jn to be processed and any job can be processed by any machine.
- Our goal is to minimize the time spent by any machine.

- We define the following quantities.
- Let A(i) be the jobs assigned to machine M_i.
- Job j has a time requirement t_i , for j = 1 to n.
- The makespan of machine M_i is $T_i := \sum_{j \in A(i)} t_j$.

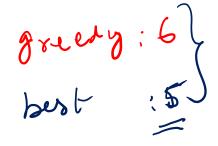
- The makespan of an assignment $T = \max_i T_i$.
- The goal of the problem is to find an assignment that minimizes the makespan.

- One of the popular algorithms is to use a greedy technique.
- We can assume that all the jobs are given apriori.
 - A bit unlike the real world setting where user jobs are fired any time.
- Assign the next job to the machine that is presently least loaded.
 - Note that all jobs are given at the beginning.
 - Assignment is done before any machine starts its processing.

- GreedyAssign(Jobs J, m)
- begin
 - Set A(i) = empty for all i between 1 to m.
 - Set T_i = 0 for all i between 1 to m
 - For j = 1 to n do
 - Find an index i such that machine M_i has minimum T_i
 - $A(i) = A(i) \cup \{j\}$
 - $\bullet T_i = T + t_j$
 - Endfor
- End

• Illustrate how this algorithm works on the following set of jobs and four machines.

- Find the best possible makespan and the makespan produced by the greedy algorithm.
 - For j = 1 to n do
 - Find an index i such that machine M_i has minimum T_i
 - $A(i) = A(i) \cup \{j\}$
 - $\bullet \ \mathsf{T_i} = \mathsf{T} \ + \ \mathsf{t_j}$
 - Endfor



- We start with two observations that help us prove the approximation ratio of the greedy algorithm.
- Let T* denote the best possible makespan.
- Observation 1: $T^* \ge (1/m) \sum_{j=1}^n t_j$.
- Comes from the fact that there is a total work of $\Sigma_{j=1}^m$ t, and some machine will have to work for at least a 1/m fraction of the total.
- Observation 2: T* ≥ max_j t_j.
 - The longest job will be on some machine which will work for at least that much time.

- Let us now turn our attention to the greedy algorithm.
- Consider the machine that has the largest makespan according to the assignment produced by the greedy algorithm.
- Let this be the machine M_i, and its makespan be T_i.
- Let t_i be the last job assigned to M_i.
- Why did we assign t_i to M_i?
 - As M_i has the least load just before assigning t_j to M_i.

- The load of M_i before assigning T_i is $T_i t_i$.
- At this time, every other machine has a load at least T_i - t_i.
 - Plus any other jobs assigned to them later on.
- So, the sum of the time for all jobs is lower bounded as follows.

•
$$\sum_{j=1}^{n} t_j \ge m(T_i - t_j)$$
, or

•
$$T_i - t_j \leq (1/m) \sum_{j=1}^n t_j \leq T^*.$$

- Further, notice that $t_i \le T^*$.
- Use Observation 2.

- Therefore, $T_i = T_i t_j + t_j = (T_i t_j) + (t_j) \le T^* + T^*$ = 2T*.
- Hence, $T_i \le 2T^*$.

$$\frac{|A(T)|}{|PT(T)|} = \frac{T_i}{T^*} \leq \frac{2T^*}{T^*}$$

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- What would be one way to improve the solution produced by the greedy algorithm?
- Sort the jobs in descending order according to their runtime and assign them just as earlier.

- Let us call this as SortedGreedyAssignment.
- We will now analyze this approach.

• Convince yourself that also SortedGreedyAssignment does not produce the best possible makespan.

- The proof is not much different from the earlier proof.
- We will obtain a better bound for t_j, the last job assigned to the machine M_i that eventually has the largest makespan, T_i.
- Consider the case where there are fewer than m jobs.
- Then, the best possible assignment is to give each job to a different machine.
 - Our algorithm also does the same.
 - So, we indeed produce the best possible makespan.

- Let us consider the case that n is more than m.
- More jobs than machines.
- In any assignment, the job t_{m+1} in sorted order, is given to some machine that already has one additional job that is at least as long as t_{m+1} .
- So, $T^* \ge 2t_{m+1}$.

- Further, note that for machine M that has the largest makespan in our algorithm, if t_j is the last job assigned to M_i , then $t_j \leq t_{m+1}$.
- From the previous, $t_{m+1} \le 1/2 T^*$.
- Now, $T_i = T_i t_j + t_j \le T^* + 1/2 T^* = 3/2 T^*$.