Approximation Algorithms: Load Balancing

Approximations

- The word *approximation* is used for several flavors of algorithms
- Approximating optimization problems
  - An algorithm is a c-approximation if its cost is $c \cdot$ OPT, where OPT is the optimum cost; ($c < 1$ or a maximization problem, $c > 1$ for min)
- Approximations in an online setting
  - An online algorithm is a c-approximation/c-competitive, if it has cost $c \cdot$ OPT, where OPT is the cost of an offline algorithm (that knows the entire input ahead of time); ($c < 1$ or a max problem, $c > 1$ for min)
- The word approximate is also used to indicate that the algorithm is permitted to make one-sided errors (such false positive)
  - Bloom filter
  - Approximate hitter algorithm (Problem 5(b) on assignment)

Challenges: Approximation Algorithms

- Approximating problems that are NP hard
  - Main challenge is showing that the algorithm performs close to optimal when the optimal solution is not known/NP hard
  - Usually done by lower (upper) bounding the cost of the optimal solution for minimization (maximization) problems

- Approximation for online algorithms
  - High benchmark. Comparison against an optimal that knows the entire future, while the algorithm does not even know the next element
  - Sometimes dealt with using “resource augmentation”—allowing the algorithm some flexibility compared to the optimal

Online: Ski Rental Problem

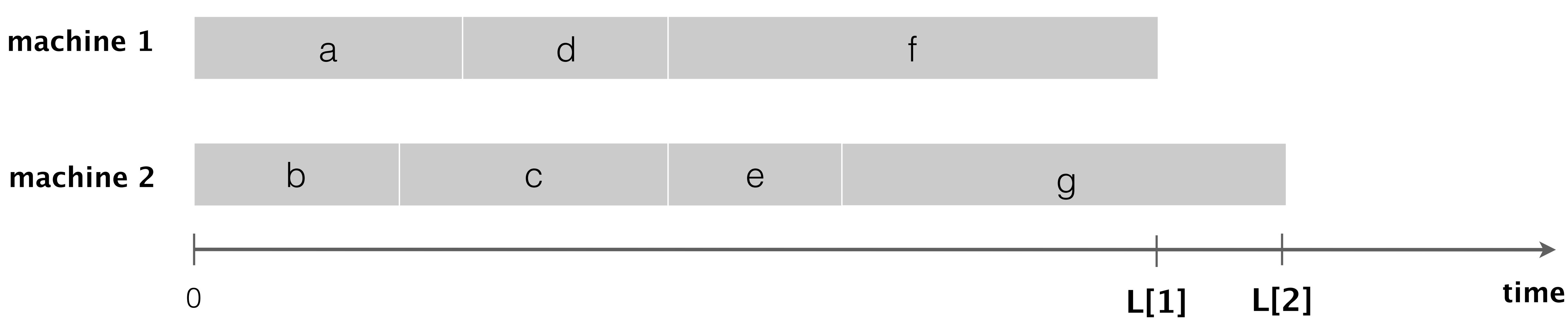
- Assume that you are taking ski lessons
- After each lesson you decide (depending on how much you liked it and how cold you are) whether to continue to ski or to quit entirely
- Question: rent or buy?
- Cost of renting $\$1$ (say)
- Cost of buying $\$B$
- **Offline strategy.** If you knew in advance how many times you would ski, say t times, what is the best strategy?
  - If $t \geq B$ times, then buy, else rent
  - In other words, optimal offline cost is $\min\{t, B\}$

Online: Ski Rental Problem

- **Online strategy.** We need to figure out a decision point, a number k such you buy skis on the kth visit (renting before then)
- **Claim.** If we set $k = B$ (the cost of buying skis), we are gauranteed to never pay more than twice of the best offline optimal strategy
- That is, buying on the Bth ski visit is $2-$competitive
- Even if you quit right after the Bth visit, $t \geq B$
- Offline cost is $\min\{t, B\} = B$
- Online strategy's cost?
    - $(k-1) \cdot 1 + B = (B-1) + B = 2B - 1$
- Competitive/Approximation ratio?
    - $2B - 1/B = 2 - 1/B \leq 2$

Load Balancing

- **Input.** m identical machines; n jobs, and processing times $t_1, \ldots, t_m$, where job j has processing time t_j (on any machine)
- Job j must run contiguously on one machine
- A machine can process at most one job at a time.
- Let $S[i]$ be the subset of jobs assigned to machine i.
- The **load of machine** i is $L[i] = \sum_{j \in S[i]} t_j$ (total processing time).



Load Balancing

- The makespan of an algorithm is the maximum load on any machine $L = \max_i L[i]$
- Load balancing Problem. Assign jobs to machines so as to minimize makespan.
- **Claim.** Load balancing is NP hard even with $m = 2$ machines
- **Proof.** Reduction from PARTITION problem.
- We will design an approximation algorithm for this problem
- [Greedy returns!] Consider the following greedy strategy:
    - Fix some order on the jobs
    - Assign job j to machine i whose load is smallest so far

Load Balancing: Greedy

LIST-SCHEDULING $(m, n, t_1, t_2, \ldots, t_n)$

FOR $i = 1$ TO m

  $L[i] \leftarrow 0.$ ⟵ load on machine i

  $S[i] \leftarrow \varnothing.$ ⟵ jobs assigned to machine i

FOR $j = 1$ TO n

  $i \leftarrow \operatorname{argmin}_k L[k].$ ⟵ machine i has smallest load

  $S[i] \leftarrow S[i] \cup \{j\}.$ ⟵ assign job j to machine i

  $L[i] \leftarrow L[i] + t_j.$ ⟵ update load of machine i

RETURN $S[1], S[2], \ldots, S[m].$

- Running time?
  - $O(n \log m)$ using a priority queue for loads $L[k]$

Load Balancing: Greedy Analysis

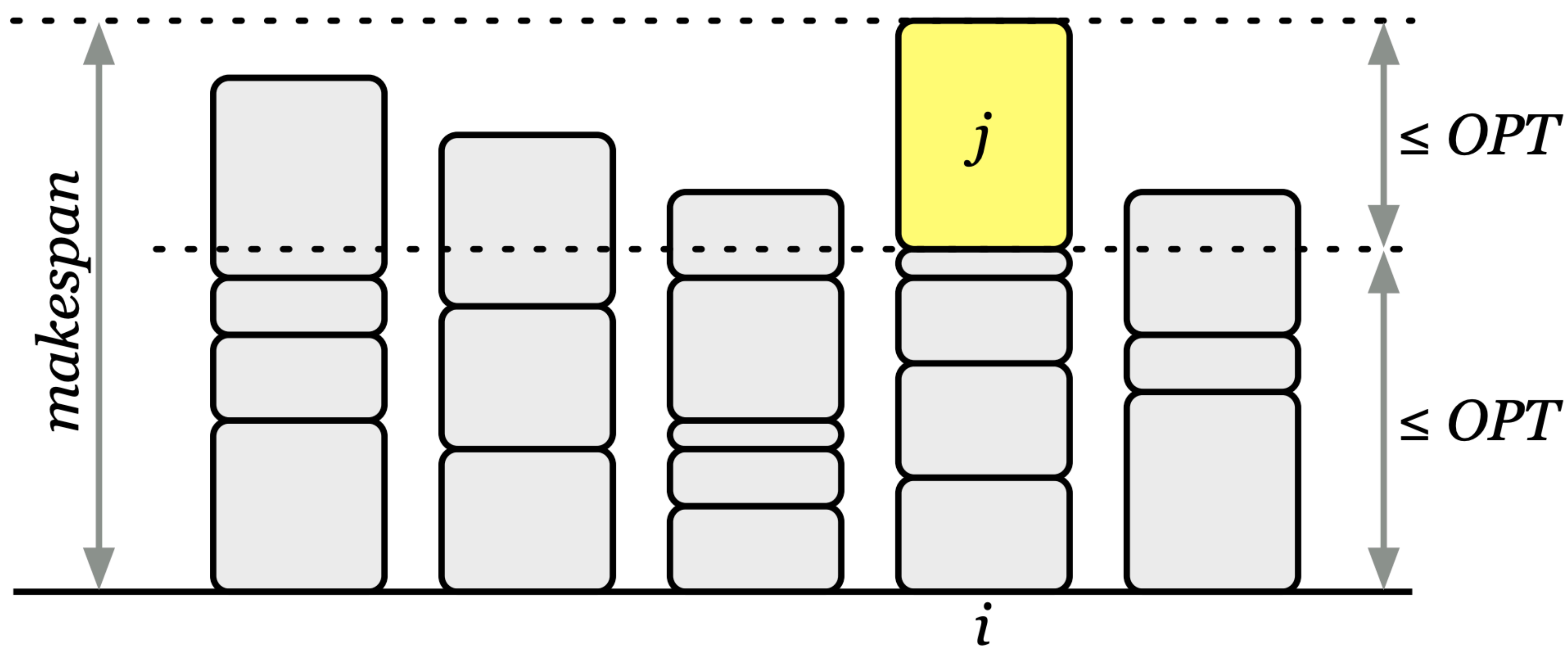
- **Claim.** Greedy algorithm is a 2-approximation.
- To show this, we need to show greedy solution never more than a factor two worse than the optimal
- **Challenge.** We don't know the optimal solution. In fact, finding the optimal is NP hard.
- Technique used in approximation algorithm (minimization problem)
    - Lower bound the cost of optimal solution
    - A good enough lower bound can help show that our algorithm cannot be too much worse than the optimal
- In our problem, what are some lower bounds on the makespan of even an optimal algorithm?

Load Balancing: Greedy Analysis

- Let OPT be the optimal makespan.
- **Lemma.** OPT $\geq \max_j t_j$.
- **Proof.** Some machine must process the most time-consuming job.
- Any other lower bounds?
- **Lemma.** OPT $\geq \frac{1}{m} \sum_j t_j$
- **Proof.**
  - The total processing time is $\sum_j t_j$
  - Some machine must do a $1/m$ fraction of the total work.

Greedy is a 2-Approximation

- **Proof.** Consider load $L[i]$ of bottleneck machine i ← machine that ends up with highest load
  - Let j be the last scheduled job on machine i
  - When job j was assigned to machine i, i must have had the smallest load
  - That is, $L[i] - t_j \leq L[k] \quad \forall 1 \leq k \leq m$



Greedy is a 2-Approximation

- **Proof.** Consider load $L(i)$ of bottleneck machine i
  - Let j be the last scheduled job on machine i
  - When job j was assigned to machine i, i must have had the smallest load
  - That is, $L[i] - t_j \leq L[k] \quad \forall 1 \leq k \leq m$
  - Summing over all k and diving by m

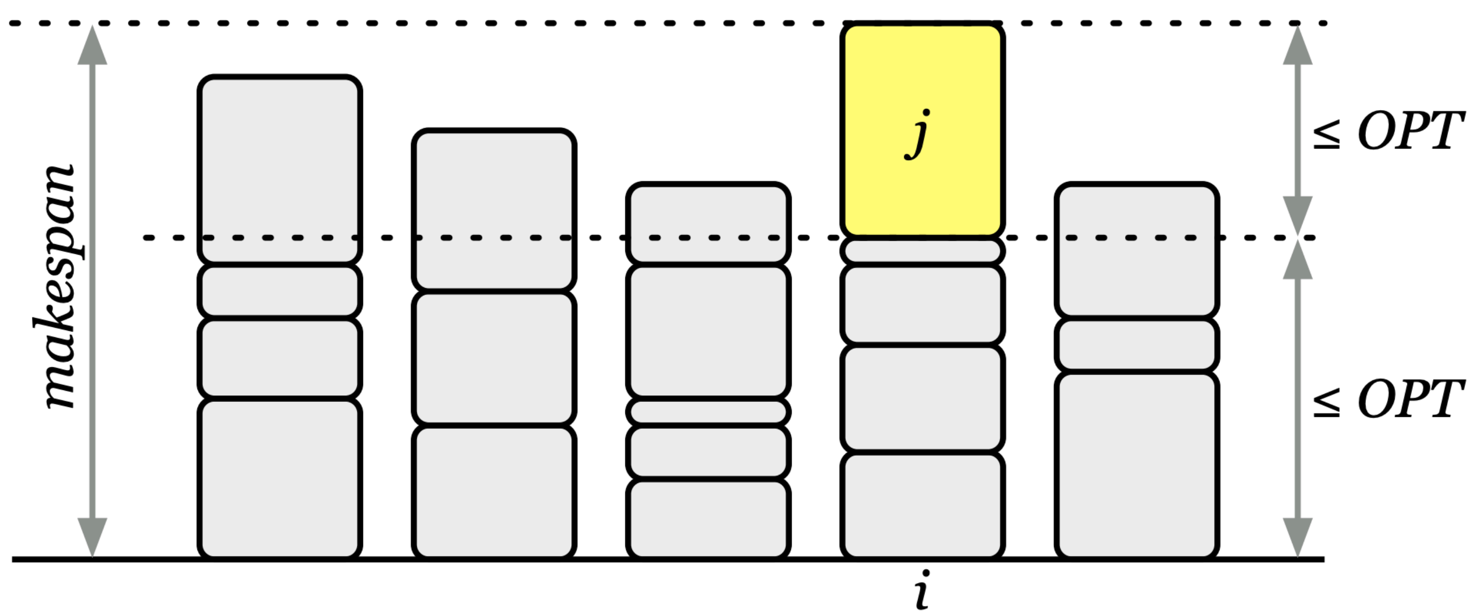
$$L[i] - t_j \leq \frac{1}{m} \sum_k L[k]$$

$$\frac{1}{m} \sum_k t_k$$

$$\leq \text{OPT}$$

Greedy is a 2-Approximation

- **Proof.**
- Consider load $L(i)$ of bottleneck machine i
- $$L[i] - t_j \leq \frac{1}{m}\sum_k L[k]$$
  $$\frac{1}{m}\sum_k t_k$$
  $$\leq \text{OPT}$$



- We know that $t_j \leq \text{OPT}$
- Thus, $L = L[i] \leq \text{OPT} + t_j$
  $$\leq 2\text{OPT} \quad \blacksquare$$

Greedy is a 2-Approximation

- Is our analysis tight?
- Close to it.
- Consider $m(m-1)$ jobs of length 1 + 1 job of length m
- How would greedy schedule these jobs?
    - Greedy will evenly divide the first $m(m-1)$ jobs among m machines, will place the final long job on any one machine
    - Makespan: $m-1+m=2m-1$
- How would optimal schedule it?
    - Give the long job to one machine, the rest split the other small jobs with a makespan m
- Ratio: $(2m-1)/m \approx 2$

Greedy is Online

- Notice that our greedy algorithm is an online algorithm
- Assigns jobs to machines in the order they arrive
    - Does not depend on future jobs
- Online approximation algorithms are very useful as often the entire input is not known ahead of time
- In online settings, it may be impossible to compute an optimum solution in polynomial time, even when the offline problem is polynomial time solvable
- Can we do better, if we assume all jobs are available at start time?
- **Offline.** Slight modification of greedy gets better approximation!

Improving on Online Greedy

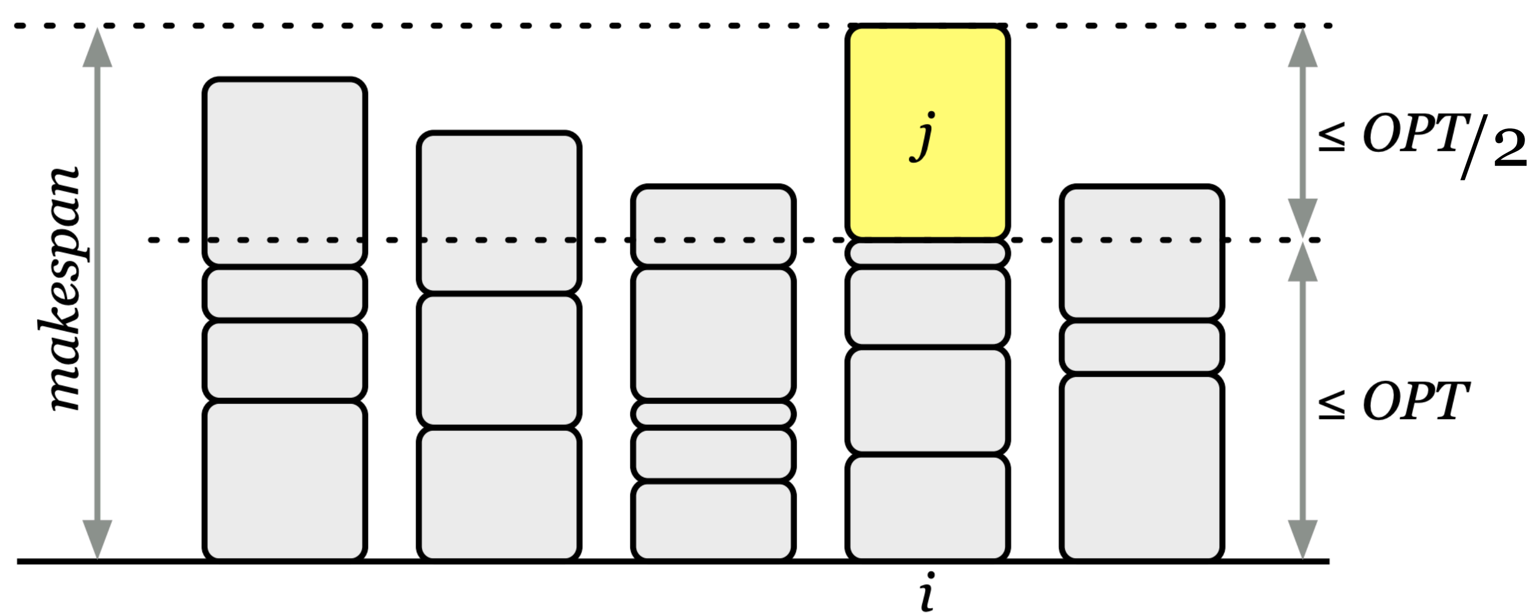
- Worst case of our greedy algorithm: spreading jobs out evenly when a giant job at the end screwed things up
- What can we do to avoid this?
  - Idea: deal with larger jobs first
  - Small jobs can only hurt so much
- Turns out this improves our approximation factor
- **Longest-processing-time (LPT) first**. Sort n jobs in decreasing order of processing times; then run the greedy algorithm on them
- **Claim.** LPT has a makespan at most $1.5 \cdot$ OPT
- **Observation.** If we have fewer than m jobs, then the greedy solution is clearly optimal (as it puts each job on its own machine)

LPT-first is a 1.5-Approximation

- **Lemma.** LPT-first has a makespan at most $1.5 \cdot$ OPT
- **Observation.**
  - If we have fewer than m jobs, then the greedy solution is clearly optimal (as it puts each job on its own machine)
- **Claim.** If more than m jobs then, OPT $\geq 2 \cdot t_{m+1}$
- **Proof.** Consider the first $m + 1$ jobs in sorted order.
  - They each take at least t_{m+1} time
  - $m + 1$ jobs and m machines, there must be a machine with at least two jobs
  - Thus the optimal makespan OPT $\geq 2 \cdot t_{m+1}$

LPT-first is a 1.5-Approximation

- **Lemma.** LPT-first has a makespan at most $1.5 \cdot$ OPT
- **Proof.** Similar to our original proof. Consider the machine M_i that has the maximum load
- If M_i has a single job, then our algorithm is optimal
- Suppose M_i has at least two jobs and let t_j be the last job assigned to the machine, note that $j \geq m + 1$ (why?)
- Thus, $t_j \leq t_{m+1} \leq \frac{1}{2}$OPT



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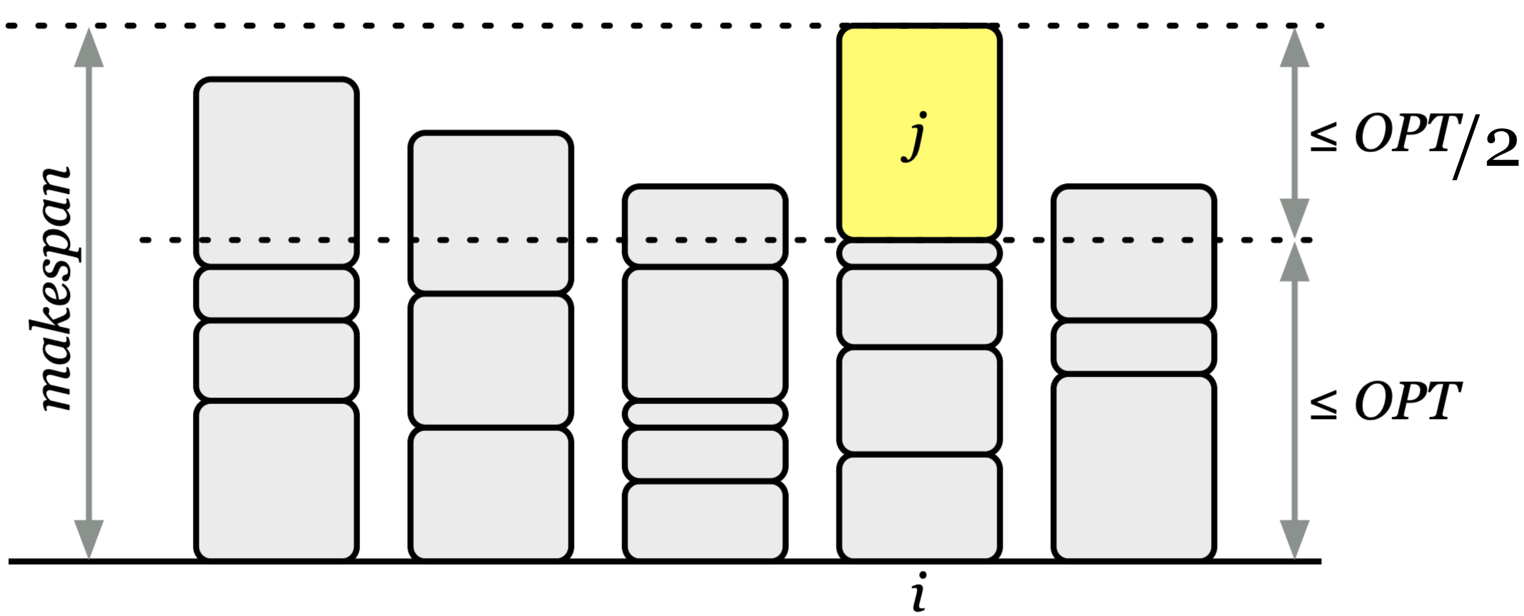
- $T_i - t_j \leq$ OPT
- $T_i \leq \frac{3}{2}$OPT ∎



Is our 1.5-Approximation tight?

- **Question.** Is out $3/2$-approximation analysis tight?
    - Turns out, no
- **Theorem [Graham 1969].** LPT-first is a $4/3$-approximation.
    - Proof via a more sophisticated analysis of the same algorithm
- **Question.** Is the $4/3$-approximation analysis tight?
    - Pretty much.
- Example
    - m machines, $n = 2m + 1$ jobs
    - 2 jobs each of length $m, m + 1, \ldots, 2m - 1$ + one job of length m
    - Approximation ratio $= (4m - 1)/3m \approx 4/3$

Formal Definition

- Consider an arbitrary optimization problem
- Let OPT(X): the cost of the optimal solution on a given input X
- Let $A(X)$: the cost of algorithm A on the same input X
- A is a $\alpha(n)$-approximation iff

$$\frac{\text{OPT(X)}}{A(X)} \leq \alpha(n) \quad \text{and} \quad \frac{\text{A(X)}}{\text{OPT}(X)} \leq \alpha(n)$$

  for all input X of size n
- Maximization problem: second inequality is trivial, first matters
- Minimization problem: first inequality is trivial, second matters
- Goal. Find a useful function of the input to upper and lower on the cost of OPT and A, e.g. OPT$(X) \geq f(X)/2$ and $A(X) \leq 4f(X)$ means A is a 8-approximation

Acknowledgments

- Some of the material in these slides are taken from
  - Kleinberg Tardos Slides by Kevin Wayne (https://www.cs.princeton.edu/~wayne/kleinberg-tardos/pdf/04GreedyAlgorithmsI.pdf)
  - Jeff Erickson's Algorithms Book (http://jeffe.cs.illinois.edu/teaching/algorithms/book/Algorithms-JeffE.pdf)
  - Lecture slides: https://web.stanford.edu/class/archive/cs/cs161/cs161.1138/