Linear Partial Differential Equation and Variational Calculus

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1 Question 6

The equation of state of a system is given by $pV = \alpha U(T,V)$, where α is a constant and *U(T, V)* is the specific internal energy. Show that the specific internal energy and specific entropy can be expressed in the form $U = V^{-\alpha}\phi(TV^{\alpha}); S = \psi(TV^{\alpha})$, where it is given that $\phi'(x) = x\phi'(x)$

Hint : use $\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_V - p$ and then finally you have to use Lagrange's method of solving first order PDE

2 Solution

We know,

$$\frac{PV}{\alpha} = U(T,V)$$

$$dU = \left(\frac{\partial U}{\partial T}\right)_V dT + \left(\frac{\partial U}{\partial V}\right)_T dV$$

$$\left(\frac{\partial U}{\partial T}\right)_V = \frac{V}{\alpha}\left(\frac{\partial P}{\partial T}\right)_V$$

$$\left(\frac{\partial U}{\partial V}\right)_T = \frac{P}{\alpha}\left(\frac{\partial V}{\partial V}\right) + \frac{V}{\alpha}\left(\frac{\partial P}{\partial V}\right)_T = \frac{P}{\alpha} + \frac{V}{\alpha}\left(\frac{\partial P}{\partial V}\right)_T$$

$$dU = \frac{V}{\alpha}\left(\frac{\partial P}{\partial T}\right)_V + \left(\frac{P}{\alpha} + \frac{V}{\alpha}\left(\frac{\partial P}{\partial V}\right)_T\right) dV$$

From the hint,

$$\frac{P}{\alpha} + \frac{V}{\alpha}\left(\frac{\partial P}{\partial V}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_V - P$$

$$P + \frac{P}{\alpha} = T\left(\frac{\partial P}{\partial T}\right)_V - \frac{V}{\alpha}\left(\frac{\partial P}{\partial V}\right)_T$$

So, the Lagrange's equation:

$$-\frac{\partial V}{\frac{V}{\alpha}} = \frac{dT}{T} = \frac{\partial P}{P\left(\frac{1+\alpha}{\alpha}\right)}$$

Solving the first equality,

$$-\log V^d = \log T + C_1$$

$$C_1 = V^\alpha T$$

Solving the second equality,

$$\log T = \frac{\alpha}{1+\alpha} \log P + C_2$$

$$C_2 = \frac{T}{P^{\frac{\alpha}{1+\alpha}}}$$

Solving the remaining equality,

$$-\alpha \log V = \frac{\alpha}{(1+\alpha)} \log P + C_3$$

$$C_3 = VP^{\frac{1}{1+\alpha}}$$

Writing P in terms of T,

$$P^{\frac{\alpha}{(1+\alpha)}} = \frac{T}{C_2}$$

$$P = \frac{T^{\frac{(\alpha+1)}{\alpha}}}{C_2'}$$

Now,

$$U(T,V) = \frac{PV}{\alpha} = \frac{V}{\alpha}\frac{T^{\frac{(\alpha+1)}{\alpha}}}{C_2'} = \frac{V.T.T^{\frac{1}{\alpha}}}{\alpha C_2'}$$

If,

$$U(T,V) = V^{-\alpha}\phi(TV^\alpha)$$

then,

$$U(T,V) = V^{-\alpha}.\frac{V^{1+\alpha}.T.T^{\frac{1}{\alpha}}}{\alpha C_2'}$$

$$= V^{-\alpha}.\frac{(V^\alpha T)(VT^{\frac{1}{\alpha}}}{\alpha C_2'}$$

Therefore,

$$\phi(TV^{\alpha}) = \frac{TV^{\alpha}T^{\frac{1}{\alpha}V}}{\alpha C_2^{'}}$$

$$\phi(x) = \frac{x.x^{\frac{1}{\alpha}}}{\alpha C_2^{'}} = \frac{x^{1+\frac{1}{\alpha}}}{\alpha C_2^{'}}$$

$$\phi^{'}(x) = \left(1 + \frac{1}{\alpha}\right)\frac{x^{\frac{1}{\alpha}}}{\alpha C_2^{'}} = x\psi^{'}(x)$$

$$\psi^{'}(x) = \left(1 + \frac{1}{\alpha}\right)\frac{x^{\frac{1}{\alpha}-1}}{\alpha C_2^{'}}$$

$$\psi(x) = \left(1 + \frac{1}{\alpha}\right)\frac{1}{\alpha C_2^{'}}\frac{x^{\frac{1}{\alpha}}}{\frac{1}{\alpha}}$$

$$= \frac{\alpha + 1}{\alpha C_2^{'}}x^{\frac{1}{\alpha}}$$

Therefore,

$$\psi(TV^{\alpha}) = \frac{\alpha + 1}{\alpha C_2^{'}}T^{\frac{1}{\alpha}}V = S$$