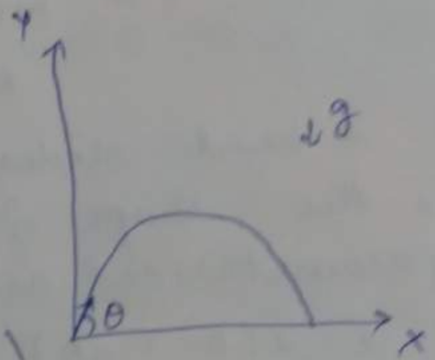
5.) A.) Equation of motion of projectile from principle of least action.



We define a lagrangian $L(t, \theta, \dot{\theta})$ and use Euler Lagrangian equation to get the equation of motion.

$$L \equiv T - V = L(t, q_1, q_2, \cdots q_n, \dot{q}_1, \dot{q}_2, \cdots \dot{q}_n)$$

The action S,

$$S \equiv \int L(t, q_1, q_2, \cdots q_n, \dot{q}_1, \dot{q}_2, \cdots \dot{q}_n)\, dt$$

Euler Lagrangian eq. is

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0 \quad , i = 1, 2, \cdots n$$

For a projectile motion, $x = R\cos\theta$, $y = R\sin\theta$

Kinetic Energy, $T = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) = \frac{1}{2} m R^2 \dot{\theta}^2$

Potential Energy, $V = mgy = mgR\sin\theta$.

$$L = T - V = \frac{1}{2} m R^2 \dot{\theta}^2 - mgR\sin\theta$$

Applying Euler-Lagrangian eq.

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0$$

$$\frac{\partial L}{\partial \theta} = -mgR\cos\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = mR^2\dot{\theta} \quad \Rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = mR^2\ddot{\theta}$$

$$\therefore \; -mgR\cos\theta - mR^2\ddot{\theta} = 0$$

$$\Rightarrow mR^2\ddot{\theta} = -mgR\cos\theta$$

$$R\ddot{\theta} = -g\cos\theta$$

B) A bead slides without friction on a wire in the shape of a cycloid which can be parameterized as

$$x = a(\theta - \sin\theta)$$

$$y = a(1+\cos\theta)$$

where a is a constant.

$$\dot{x} = a(\dot{\theta} - \dot{\theta}\cos\theta)$$

$$\dot{y} = -a\sin\theta\,\dot{\theta}$$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$= \frac{1}{2}m\left[a^2(\dot{\theta}^2 + \dot{\theta}^2\cos^2\theta - 2\dot{\theta}^2\cos\theta) + a^2\dot{\theta}^2\sin^2\theta\right]$$

$$= \frac{1}{2}ma^2(2\dot{\theta}^2 - 2\dot{\theta}^2\cos\theta) = ma^2(1-\cos\theta)\dot{\theta}^2$$

$$V = mgy = mga(1+\cos\theta)$$

$$L = T - V = ma^2(1-\cos\theta)\dot{\theta}^2 - mga(1+\cos\theta)$$

$$\frac{\partial L}{\partial \dot{\theta}} = 2ma^2\dot{\theta}(1-\cos\theta)$$

$$\frac{\partial L}{\partial \theta} = ma^2\dot{\theta}^2\sin\theta + mga\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 2ma^2\ddot{\theta}(1-\cos\theta) + 2ma^2\dot{\theta}^2\sin\theta.$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0$$

$$\Rightarrow 2ma^2(1-\cos\theta)\ddot{\theta} + 2ma^2\dot{\theta}^2\sin\theta - ma^2\dot{\theta}^2\sin\theta - mga\sin\theta = 0$$

$$\Rightarrow 2ma^2(1-\cos\theta)\ddot{\theta} + ma^2\dot{\theta}^2\sin\theta - mga\sin\theta = 0$$

$$(1-\cos\theta)\ddot{\theta} + \frac{1}{2}\dot{\theta}^2\sin\theta - \frac{g\sin\theta}{2a} = 0$$

Lagrange Equation for relativistic particle.

We know that Action is a scalar and we can make it Lorrentz invariant.

$$S = \int_{t_1}^{t_2} L\,dt$$ But here S depends on t.

$dt = \gamma\, d\tau$ where $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$

$$\Rightarrow S = \int_{t_1}^{t_2} L\gamma\, d\tau$$

So here we have to make $L\gamma$ constant

$$L\gamma = \text{const}$$

$$\underset{\text{energy unit}}{L} = \frac{\text{const}}{\underset{\text{unitless}}{\gamma}}$$ $\Rightarrow$ const has to be unit of energy.

Let, $L = \dfrac{\alpha mc^2}{\gamma}$ (for free particle)

$$P = \frac{\partial L}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}}\left(\alpha\sqrt{1-\frac{\dot{x}^2}{c^2}}\, mc^2\right)$$

$$= \alpha m c^2 \cdot \frac{-2\,\dot{x}/c^2}{2\sqrt{1-\dot{x}^2/c^2}} = -\alpha m \gamma \dot{x}$$

for $v << c$, $\gamma = 1 \Rightarrow p = m\dot{x}$ for $\alpha = -1$

$$\therefore \quad L = -\frac{mc^2}{\gamma}, \quad p = \gamma m \dot{x}$$

For particle under potential V

$$L = -\frac{mc^2}{\gamma} - V$$