First Fundamental Theorem of Asset Pricing

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1 Basic terms of economics used in this topic

The fundamental theorems of asset pricing provide necessary and sufficient conditions for a market to be arbitrage-free and for a market to be complete.

1.1 Arbitrage

An arbitrage opportunity is a way of making money with no initial investment without any possibility of loss. Though arbitrage opportunities do exist briefly in real life, it has been said that any sensible market model must avoid this type of profit. It is the practice of taking advantage of a price difference between two or more markets: striking a combination of matching deals that capitalize upon the imbalance, the profit being the difference between the market prices at which the unit is traded. Usually, an arbitrage opportunity is present when there is the possibility to instantaneously buy something for a low price and sell it for a higher price. It is the possibility of a risk-free profit after transaction costs. The first theorem is important in that it ensures a fundamental property of market models.

1.2 Completeness

A complete market is one in which every contingent claim can be replicated. though this property is common in models, it is not always considered desirable or realistic. The complete market has two conditions:

1) Negligible transaction costs and therefore also perfect information.
2) There is a price for every asset in every possible state of the world.

In such a market, the complete set of possible bets on future states of the world can be constructed with existing assets without friction. here, goods are state-contingent; that is, a good includes the time and state of the world in which it is consumed. for instance, an umbrella tomorrow if it rains is a distinct good from an umbrella tomorrow if it is clear.

1.3 The binomial model

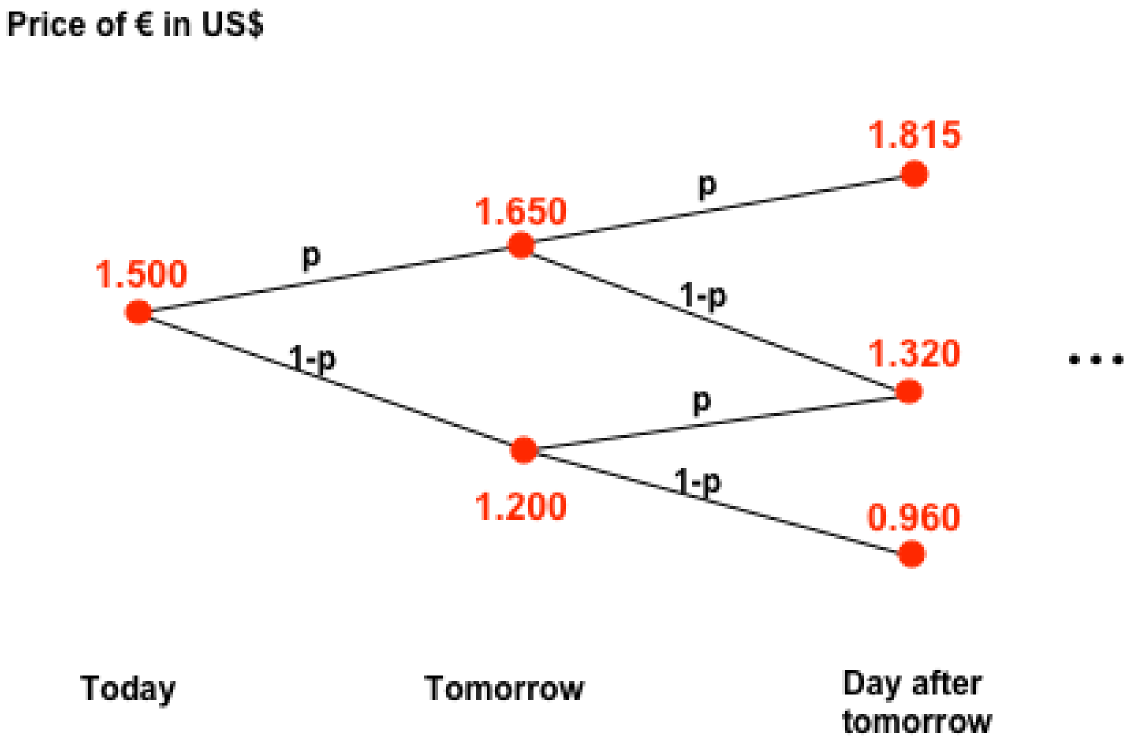
Cox, Ross and Rubinstein developed the binomial model in 1979, which is also known as the CRR model.

1.3.1 One step model

Suppose that today €1=$1.5. Assume that you know that tomorrow the euro will be worth either $1.2 with probability $1 - p$ or $1.65 with probability p. Assume also that you can borrow or lend money in dollar currency at a fixed interest rate of 10%. Under these circumstances, the market that you are facing can be modelled by a one-step binomial model. One step because you are only given information about the euro value tomorrow, binomial because there are only two possible values of the euro tomorrow. The assumption that you can borrow or lend money at a fixed interest rate is a common assumption in short term financial models.

1.3.2 Multi step model

Suppose that today €1=1.5 and for any given day of the month the probability of the price of the euro to go down by 20% is $1-p$ while the probability of the price going up by 10% is p. As before assume that you can borrow or lend money at a fixed daily interest rate of 10%. In this case, the market can be modelled by a multi step binomial model and the best way of visualizing such a model is through a binary tree like the one shown below:



1.3.3 The general case

The CRR model with time horizon t involves a risk less bond and a risky asset (e.g. stocks, bonds, commodities such as oil or gold, currency exchange rate, etc).

The price process of the risk less bond is $S_t^0 = (1+r)^t$ for all $t = 0, 1, 2..T$ with $r > -1$.

The price process of the risky asset is denoted by $S_t = S_{t-1}(1+R_t)$ for $t = 0, 1, ..., T$ where with R_t the return in the t th trading period.

The return R_t can only take two possible values $-1 \leq a \leq b$. This implies that the price of the risky asset at any time t, either jumps to the higher value $S_t(1+b)$ or to the lower value $S_t(1+a)$.

1.4 Example of Arbitrage

The action of borrowing risky assets and selling them immediately is called selling short. Investors who believe the price of an asset is going to drop will short sell this asset.

We have in this case that the risk less bond is the dollar, the risky asset is the euro, t is the number of days remaining in the month, $r = 0.1$, $a = -0.2$ and $b = 0.1$. suppose further that you can borrow euros with no interest, that $p = 0.5$ and that there is only one day left in the month (so that you are facing a one step situation).

You could take advantage of this circumstance by using the following strategy: borrow one euro today, sell it immediately for \$1.5 and lend this money. Tomorrow you will get for sure \$1.65 since the interest rate is 10%. If the price of the euro goes up to \$1.65 you use your money to buy a euro and pay your debt obtaining a net gain of \$0. If the price of the euro goes down to \$1.2, you buy a euro to pay your debt but in this case, your ending balance is \$1.65 − \$1.2 = \$0.45. Hence by following this strategy you can make \$0.45 with a probability of 0.5 and no risk. Of course, you could do the same with an arbitrary amount of euros in the beginning, which generates even greater gains. Such a strategy is commonly known as an arbitrage opportunity.

1.5 Martingale

In probability theory, a martingale is a sequence of random variables (i.e., a stochastic process) for which, at a particular time, the conditional expectation of the next value in the sequence, regardless of all prior values, is equal to the present value.

A martingale is a process that models a fair game.

A random process $x_0, x_1, ..., x_t$, such that for every t between 0 and t, x_t can only assume finitely many values $x_1^t, x_2^t, ..., x_{nt}^t$, is said to be a martingale with respect to the probability measure p if $E^P[X^T|X_0 = x_{i_0}^0, X_1 = x_{i_1}^1, ..., X_s = x_{i_s}^s] = x_{i_s}^s$ for all s and $i_0, ...,$ is.

In other words, the expectation under p of the final outcome x_t given the outcomes up to time s is exactly the value at time s. this happens if and only if for any t $E^P[X^{t+1}|X_0 = x_{i_0}^0, X_1 = x_{i_1}^1, ..., X_t = x_{i_t}^t] = x_{i_t}^t$

1.6 Risk-neutral measure

In mathematical finance, a risk-neutral measure is a probability measure such that each share price is exactly equal to the discounted expectation of the share price under this measure.

The easiest way to remember what the risk-neutral measure is:

1) The probability measure of a transformed random variable. Typically this transformation is the utility function of the payoff. The risk-neutral measure would be the measure corresponding to an expectation of the payoff with a linear utility.

2) An implied probability measure, that is one implied from the current observable/posted/traded prices of the relevant instruments. Relevant means those instruments that are causally linked to the events in the probability space under consideration (i.e. underlying prices plus derivatives), and

3) It is the implied probability measure (solves a kind of inverse problem) that is defined using a linear (risk-neutral) utility in the payoff, assuming some known model for the payoff.

This means that you try to find the risk-neutral measure by solving the equation where current prices are the expected present value of the future pay-offs under the risk-neutral measure. The concept of a unique risk-neutral measure is most useful when one imagines making prices across a number of derivatives that would make a unique risk-neutral measure since it implies a kind of consistency in ones hypothetical un traded prices and, theoretically points to arbitrage opportunities in markets where bid/ask prices are visible.

2 Hahn–Banach separation theorems

The key element of the Hahn–Banach theorem is fundamentally a result about the separation of two convex sets: $-p(-x-n) - f(n) : n \in M$, and $p(m+x) - f(m) : m \in M$.

Let be a real locally convex topological vector space and let A and B be non-empty convex subsets. If $IntA \neq \phi$ and $B \cap IntA = \phi$ then there exists a continuous linear functional f on X such that $supf(A) \leq inff(B)$ and $f(a) < intf(B)$ for all $a \in IntA$(such is f is necessarily non-zero).

Let X be a real topological vector space and choose A, B convex non-empty disjoint subsets of X.

1) If A is open then A and B are separated by a (closed) hyper-plane. Explicitly, this means that there exists a continuous linear map f : X → K and s ∈ R such that $f(a) < s \leq f(b)$ for all $a \in A, b \in B$. If both A and B are open then the right-hand side may be taken strict as well.

2) If X is locally convex, A is compact, and B closed, then A and B are strictly separated: there exists a continuous linear map f : X → K and s, t ∈ R such that $f(a) < t < s < f(b)$ for all $a \in A, b \in B$.

2.1 Other Applications of Hahn-Banach Theorem

As the Hahn-Banach separation theorem is a general theorem, it has applications in other fields of mathematics as well.

For instance, it is used in the proof of Farkas Lemma and other Farkas like lemmas, which deal with finite systems of linear inequalities.

It states that:

Let $A \in \mathbb{R}^{mxn}$ and $b \in \mathbb{R}^m$. Then exactly one of the two following assertions is true: 1. There exists an $x \in \mathbb{R}^n$ such that $Ax = b$ and $x \geq 0$. or 2. There exists a $y \in \mathbb{R}^m$ such that $A^T y \geq 0$ and $b^T y < 0$.

3 First Fundamental Theorem of Asset Pricing (FFTAP)

Proving that a market is arbitrage-free may be very tedious, even under very simple circumstances. The first fundamental theorem of asset pricing is a result that provides an alternative way to test the existence of arbitrage opportunities in a given market. When applied to binomial markets, this theorem gives a very precise condition that is extremely easy to verify.

The first version of this theorem was proven by M.Harrison and D. Kreps in 1979. More general versions of the theorem were proven in 1981 by M. Harrison and S.Pliska and in 1994 by F. Delbaen and W. Schachermayer.

The Theorem states that:

A discrete market, on a discrete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, is arbitrage-free if, and only if, there exists at least one risk neutral probability measure that is equivalent to the original probability measure,P.

A financial market with time horizon t and price processes of the risky asset and risk less bond given by $S_1, S_2 ..., S_t$ and $S_1^0, ..., S_t^0$, respectively, is arbitrage-free under the probability P if and only if there exists another probability measure Q such that

1) For any event A, $P(A) = 0$ if and only if $Q(A) = 0$. we say in this case that P and Q are equivalent probability measures.

2) The discounted price process, $x_0 := S_0/S_0^0, ..., x_t := S_t/S_t^0$is a martingale under Q.

A measure Q that satisfies 1) and 2) is known as a risk-neutral measure.

The first of the conditions, namely that the two probability measures have to be equivalent, is explained by the fact that the concept of arbitrage depends only on events that have or do not have measure 0. More specifically, an arbitrage opportunity is a self-financing trading strategy such that the probability that the value of the final portfolio is negative is zero and the probability that it is positive is not 0, and we are not really concerned about the exact probability of this last event.

In the second condition we are not requiring the price process of the risky asset to be a martingale (i.e a fair game) but the discounted price process.

This can be explained by the following reasoning: suppose that the price of the risky asset is measured in dollars. If at time t the price of the risky asset is \$x, the fair price the seller should charge at time 0 for the asset should be $\$x/(1+r)^t$, with r the interest rate. If he charges $\$y \leq \$x/(1+r)^t$ then the buyer could take advantage of the situation by borrowing \$y at time 0 to buy the asset and then selling at time t to repay his debt of $\$y(1+r)^t$, obtaining a positive profit of $\$(x - y(1+r)^t)$. If he charges $\$y > \$x/(1+r)^t$then he could take advantage of the situation by selling the asset at time 0 and lending \$y dollars so that at time t he would receive $\$y(1+r)^t$and after buying back the asset he would make a positive profit of $\$(y(1+r)^t - x)$. Basically, when one considers the discounted price process every price is being measured in the unity of the risk less asset S^0 which is also commonly known in the literature as the Numéraire.

If p is not 0 or 1 the market is arbitrage-free if and only if $a < r < b$.

How the Hahn-Banach theorem comes into play

The Hahn-Banach theorem guarantees the existence of a linear functional which splits two disjoint sets. This concept is very relevant in mathematical finance and is related to martingale measures, i.e. risk-neutral pricing.

In this context, we have two economically interesting sets. The first one is the set of all terminal wealth which are attainable through trading in a financial market, starting from zero wealth. The second set consists of all Non negative terminal wealth. these are the so-called arbitrage opportunities.

Up to some technical conditions, no-arbitrage is formally defined as the non-intersection of these two sets.

As a consequence, the Hahn-Banach guarantees a linear functional which is positive on the positive wealth and non-positive on the wealth attained by trading. The output of this functional can be thought of as the risk-neutral price of some terminal payoff. Associated with this functional (by duality theory) is a probability measure so that the original random process modelling asset prices is a martingale under this new measure.

The relationship between no-arbitrage and existence of pricing rules is called the first fundamental theorem of asset pricing.

4 Mathematical Proof Behind Asset Pricing

There are two models proposed with related theorem given below

4.1 One Period Model

This model is depicted by a vector, $x \in R^m$, speaking to the costs of m instruments toward the start of the period, a set of all potential results over the period, and a limited capacity $X : \Omega \longrightarrow R^m$, speaking to the costs of the m instruments toward the finish of the period relying upon the result, $\omega \in \Omega$.These are the few definitions associated with proofing the theorem.

Arbitrage exists if there is a vector $\eta \in R^m$ such that $\eta.x < 0$ and $\eta.X(\omega) \leq 0$ for all $\omega \in \Omega$.

The expense of setting up the position will be $\eta x = \eta_1 x_1 + \cdots + \eta_m x_m$. This being negative methods cash is made by putting on the position. At the point when the position is exchanged toward the finish of the period, the returns are ηX. This being non-negative methods no cash is lost.an arbitrary probability measure on Ω and use the conditions $\eta \cdot \mathrm{x} = 0$ and $\eta \cdot \mathrm{X} \geq 0$ with $\mathrm{E}[\eta \cdot \mathrm{X}] > 0$ to define an arbitrage opportunity.Making nothing when setting up a position and having a nonzero likelihood of making a positive measure of cash with no gauge of either the likelihood or measure of cash to be made is anything but a sensible definition of an exchange opportunity.This is strategy of putting money with no risk involved making it is strong arbitrage.Define the realized return for a position, η, by $R_\eta = \eta \cdot \mathrm{X}/\eta \cdot \mathrm{x}$, whenever $\eta \cdot \mathrm{x} \neq 0$.If there exists $\gamma \in R^m$ with $\gamma \cdot \mathrm{X}(\omega) = 1$ for $\omega \in \Omega$ (a zero coupon bond) then the price is $\gamma \cdot \mathrm{x} = 1/R_\gamma$. Zero interest rates correspond to a realized return of 1.It can be observed that arbitrage is equivalent to the condition $R_\eta < 0$ on Ω for some $\eta \in R_m$ .Also negative interest rates do not necessarily imply arbitrage.

Theorem 1: Arbitrage exists if and only if x does not belong to the smallest closed cone containing the range of X. If x_ϵ is the nearest point in the cone to x, then $\eta = x_\epsilon - x$ is an arbitrage.

Proof: In the event that x has a place with the cone, it is subjectively near a limited entirety $\sum_j X(_j)\pi_j$, where $_j \in \Omega$ and $\pi_j > 0$ for all j. On the off chance that $\eta \cdot \mathrm{X}(\omega) \geq 0$ for all $\omega \in \Omega$ at that point $\eta \cdot \sum_j X(\omega_j)\pi_j \geq 0$, consequently ηx can't be negative. The other heading is an outcome of the accompanying with C being the littlest shut cone containing $X(\Omega)$.

Associated with this there is one lemma on defining closed clone.It is stated as follows,

Lemma 1: If $C \subset R^m$ is a closed cone and $x \notin C$, then there exists $\eta \in R^m$ such that $\eta x < 0$ and $\eta y \geq 0$ for all $y \in C$.

proof: This result is well known, but here is an elementary self-contained proof.Since C is closed and convex, there exists $x_\epsilon \in C$ such that $\|x_\epsilon - x\| \leq \| y - x \|$ for all $y \in C$. We have $\|x_\epsilon - x\| \leq \| tx_\epsilon$ - x $\|$ for $t \geq 0$, so $0 \leq (t^2 - 1) \| x_\epsilon \|^2 - 2(t-1)x_\epsilon.x = f(t)$.

Because $f(t)$ is quadratic in t and vanishes at $t = 1$, we have $0 = f\ '(1) = 2\| x_\epsilon \|^2 \ 2x_\epsilon.x$, hence $\eta.x_\epsilon = 0$. Now $0 < \| \eta \|^2 = \eta.x_\epsilon - \eta.x$, so $\eta.x < 0$

Since $\| x_\epsilon - x \| \leq \| ty + x_\epsilon - x \|$ for $t \geq 0$ and $y \in C$, we have $0 \leq t^2 \| y \|^2 + 2ty.(x_\epsilon x)$. Dividing by t and setting $t = 0$ shows $y.\eta \geq 0$.

4.2 Multi-Period Model

The multi-period model is indicated by an expanding grouping of times $(t_j)_{0\leq j\leq n}$ at which exchanges can happen, a succession of algebras $(A_j)_{0\leq j\leq n}$ on the arrangement of potential results Ω where A_j speaks to the data accessible at time t_j , an arrangement of limited R^m esteemed capacities $(X_j)_{0\leq j\leq n}$ with X_j being A_j quantifiable that speak to the costs of m instruments, and an arrangement of limited R^m esteemed. capacities $(C_j)_{1\leq j\leq n}$ with C_j being A_j quantifiable that speak to the incomes related with holding one portion of each instrument throughout the former time span. We further accept the cardinality of A_0 is limited, and the A_j are expanding.

An exchanging technique is the grouping of limited R m esteemed capacities $(\Gamma_j)_{0\leq j\leq n}$ with $_j$ being A_j quantifiable that speak to the sum in every security bought at time t_j . Your position is $\Sigma_j = \Gamma_0 + \cdots + \Gamma_j$, the amassing of exchanges over time.An exchanging methodology is called finished off at time t_j if $\Sigma_j = 0$. Note in the one-time frame case finished off exchanging systems have the structure $\Gamma_0 = \gamma$, $\Gamma_1 = \gamma$.

The sum your record makes at time t_j is $A_j = \Sigma_{j-1}$. C_j - Σ_j . X_j , $0 \leq j \leq n$,where we utilize the show $C_0 = 0$. The budgetary understanding is that at time t_j you get incomes dependent on the position held from t_{j1} to t_j and are charged for exchanging Γ_j shares at costs X_j.

Some Mathematical foundations requires to prove fundamental theorem asset pricing.

Let $B(\Omega, A, R^m)$ mean the Banach polynomial math of limited A quantifiable capacities on Ω taking qualities in R^m . We compose this as $B(\Omega, A)$ when $m = 1$. Review that if B is a Banach variable based math we can characterize the item $yy^* \in B^*$ for $y \in B$ and $y^* \in B^*$ by $\langle$ x, yy^* $\rangle$ = $\langle$ xy, y^* $\rangle$ for $x \in B$, a reality we will use beneath.

The standard assertion of the FTAP utilizes restrictive desire. This form utilizes limitation of measures, a lot more straightforward idea. The restrictive desire for integral of an arbitrary A variable is characterized by $Y = E[X|A]$ if and just Y is A quantifiable and $\int_{A'} Y dP = \int_{A'} X dP$ for each of the $A' \in A$. Utilizing the double blending this says $\langle 1_A Y, P\rangle = \langle 1_A X, P\rangle$ for each of the $A' \in A$. Utilizing the item characterized we can compose this as $\langle 1_A, Y P\rangle = \langle 1_A, XP\rangle$ so $Y P(A) = XP(A)$ for each of the $A' \in A$. On the off chance that P has area A this says $Y P = XP\|A$.

We need a slight speculation. On the off chance that Y is A quantifiable, P has area A, and $\langle 1_A Y, P\rangle = \langle 1_A X, Q\rangle$ for every one of the $A' \in A$, at that point $Y P = XQ|_A$. There is no necessity that P and $\bigoplus Q$ be likelihood measures.

Let $P \subset \bigoplus_{j=0..n} B(\Omega, A_j)$ be the cone of all $\bigoplus_j P_j$ with the end goal that $P_0 > 0$ and $P_j \geq 0$, $1 \leq j \leq n$. The double cone, P^+ is characterized to be the arrangement of all $\bigoplus_j \pi_j$ in $\bigoplus_{j=0..n} ba(\Omega, A_j)$ with the end goal that $\langle P, \pi\rangle = \langle \bigoplus_j P_j, \bigoplus_j \pi_j\rangle = \sum_j \langle P_j, \pi_j\rangle > 0$.

Lemma: It is required to prove 2^{nd} fundamental theorem of asset pricing and it states that The dual cone P^+ consists of $\bigoplus_j \pi_j$ such that $\pi_0 > 0$, and $\pi_j \geq 0$ for $1 \leq j \leq n$.

Theorem 2: There is no arbitrage if and only if there exists $\bigoplus_i \pi_i \pi P^+$ such that $X_i\pi_i = (C_{i+1} + X_{i+1})\pi_{i+1}|_{Ai}$, $0 \leq i <$ n.
In this equation is a vector-valued measure and recall $\pi|_A$ denotes the measure π restricted to the algebra A.

Proof: Characterize A : $\bigoplus_{i=0}^n B(\Omega, A_i, R^m) \longrightarrow \bigoplus_{i=0}^n B(\Omega, A_i)$ by $A = \bigoplus_{0\leq i\leq n} A_i$. Define C to be the subspace of techniques that are finished off by time t_n .With P as over, no exchange is equal to $AC \cap P = \phi$. Once more, the standard geography guarantees that P has an inside point so the Hahn-Banach hypothesis infers there exists a hyper-plane $H = \{X \in \bigoplus_{i=0..n} B(\Omega, A_i) : \langle X, \pi\rangle = 0\}$ for a few $\pi = \bigoplus_{i=0..n} \pi_i$ containing AC that doesn't meet P. It is preposterous that P, π takes on various signs. In any case the convexity of P would suggest $0 = \langle$ P', $\pi\rangle$ for some $P' \in P$ so we may accept $\pi \in P^+$.

Note $0 = \langle A(\bigoplus_i \gamma_i), \bigoplus_i \pi_i\rangle = \sum_{i=0}^n \langle \eta_{i1}.C_i - \gamma_i.X_i, \pi_i\rangle$ for all $\bigoplus_i \gamma_i \in C$. Taking finished off procedures of the structure $\gamma_i = \Gamma$, $\gamma_{i+1} = -\gamma$ having all different terms zero yields, where γ is A_i quantifiable, gives 0 $= \langle \eta_{i-1}.C_i - \gamma_i.X_i, \pi_i\rangle + \langle \eta_i.C_{i+1} - \gamma_{i+1}.X_{i+1}, \pi_{i+1}\rangle = \langle -\gamma.X_i, \pi_i\rangle + \langle \gamma.C_{i+1} + \gamma.X_{i+1}, \pi_{i+1}\rangle$, consequently $\langle \gamma, X_i, \pi_i\rangle = \langle \gamma, (C_{i+1} + X_{i+1})\pi_{i+1}\rangle$ for each of the A_i quantifiable γ. Taking γ to be a trademark work demonstrates $X_i\pi_i = (C_{i+1} + X_{i+1})\pi_{i+1}|_{Ai}$ for $0 \leq i < n$.

5 Conclusion

To summarize the first fundamental theorem of asset pricing, in a complete market a financial derivative's price is the discounted expected value of the future payoff under the unique risk-neutral measure. Such a measure exists if and only if the market is arbitrage-free.