Dual Spaces:

As with any vector space, the dual $\mathcal{H}^*$ of a Hilbert space $\mathcal{H}$ is the space of linear maps $\mathcal{H} \to \mathbb{C}$.
That is an element $\varphi \in \mathcal{H}^*$ defines a map

$$\varphi : a\psi_1 + b\psi_2 \longrightarrow a\varphi(\psi_1) + b\varphi(\psi_2) \text{ for all}$$

$\psi_1, \psi_2 \in \mathcal{H}$ and $a, b \in \mathbb{C}$.

One way to construct such a map is to use the inner product : given some state $\phi \in \mathcal{H}$, we can define an element $(\phi, \ ) \in \mathcal{H}^*$ which acts by

$$(\phi, \ ) : \psi \longrightarrow (\phi, \psi)$$

ie, we take the inner product of $\psi \in \mathcal{H}$ with our choosen element ϕ. The linearity property of the inner product transfer to ensure that $(\phi, \ )$ is indeed a linear map. — note that since the inner product is antlinear in its first entry, it is important that our choosen element ϕ sits in the first entry.
It is to be noted that any linear map $\varphi : \mathcal{H} \to \mathbb{C}$ can be written as $(\phi, \ )$ for some fixed choice of ϕ.

This fact also holds true for infinite dimensional systems, but not trivially so. It is true via something called Riesz representation Theorem.

Dirac notation and continuum states

From now on, in this course we will use a notation for Hilbert space which was formalized by P.A.M Dirac. It is the standard notation for any Quantum mechanics course.

Any element in Hilbert space $\mathcal{H}$ is denoted by the ket vector $\rightarrow |\psi\rangle$.

Any element of the dual space $\mathcal{H}^*$ is denoted by the Bra vector $\rightarrow |\phi\rangle$

The inner product is written as $\rightarrow \langle\phi|\psi\rangle$

Given an orthogonal and normalized basis $\{|e_a\rangle\}$ of $\mathcal{H}$, we have

$$|\psi\rangle = \sum_a \psi_a |e_a\rangle$$

for any vector $|\psi\rangle$ in the Hilbert space $\mathcal{H}$.

If $|x\rangle = \sum_b x_b |e_b\rangle$, then we have the inner product

$$\langle x|\psi\rangle = \sum_{ab} \bar{x}_b \psi_a \langle e_b|e_a\rangle$$

$$= \sum_{ab} \bar{x}_b \psi_a \delta_{ab} = \sum_a \bar{x}_a \psi_a .$$

It is very useful to be able to extend this idea also to function spaces. In this case, we introduce a 'continuum basis' with element $|a\rangle$ labeled by a continuous variable a, so

$$\langle a'|a\rangle = \delta(a'-a) \rightarrow \text{Dirac Delta func}$$

Then we can write

$$|\psi\rangle = \int \psi(a)\, |a\rangle\, da$$

to expand $|\psi\rangle$ in terms of $|a\rangle$

Therefore we have

$$\langle x | \psi \rangle = \int \bar{x}(b)\, \psi(a) \langle b | a \rangle \, da\, db$$

$$= \int \psi(a)\, da \int \bar{x}(b)\, \delta(b-a)\, db.$$

We know $\int f(b)\, \delta(b-a)\, db = f(a)$.

$\therefore$ We have $\langle x | \psi \rangle = \int \bar{x}(a)\, \psi(a)\, da$

This is just the inner product extended to a continuum basis.

Expanding a general state $|\psi\rangle$ as an integral

$$|\psi\rangle = \int_{\mathbb{R}} \psi(x')\, |x'\rangle\, dx'$$

we see that the complex co-efficients are

$$\langle x | \psi \rangle = \int \psi(x') \langle x | x' \rangle \, dx'$$

$$= \int \psi(x')\, \delta(x - x')\, dx'$$

$$= \psi(x)$$

Again we could expand this same vector in any number of different basis. For exampl we could choose the ~~the~~ momentum basis $|p\rangle$ and expand

$$|\psi\rangle = \int_{\mathbb{R}} \tilde{\psi}(p)\, |p\rangle\, dp.$$

here $\tilde{\psi}(p) = \langle p | \psi \rangle$ is the momentum space wavefunction just as $\psi(x) = \langle x | \psi \rangle$ is the position space wave function.

Later we will show that $\langle x|P\rangle = e^{ixP/\hbar}/\sqrt{2\pi\hbar}$,
So these two co-efficients are related by

$$\langle x|\psi\rangle = \int_{\mathbb{R}} \tilde{\psi}(P)\,\langle x|P\rangle\,dP = \frac{1}{\sqrt{2\pi\hbar}}\int_{\mathbb{R}} e^{ixP/\hbar}\,\tilde{\psi}(P)\,dP$$

$$\langle P|\psi\rangle = \int_{\mathbb{R}} \psi(x)\,\langle P|x\rangle\,dx = \frac{1}{\sqrt{2\pi\hbar}}\int_{\mathbb{R}} e^{-ixP/\hbar}\,\psi(x)\,dx$$

Therefore the position space wave function and the momentum space wavefunction are related via fourier transformation.

The real point is that the fundamental object is the abstract vector $|\psi\rangle \in \mathcal{H}$. All the physical information is encoded in its state vector $|\psi\rangle$.
The wavefunction $\psi(x)$ or $\tilde{\psi}(P)$ are merely the expansion co-efficients in some basis.
Like any other choice of basis, this expansion may be useful of some purposes and unhelpful for others.

Operators :

A linear operator is a map $A : \mathcal{H} \rightarrow \mathcal{H}$ that is compatible with the vector space structure in the sense that

$$A(c_1|\phi_1\rangle + c_2|\phi_2\rangle) = c_1 A|\phi_1\rangle + c_2 A|\phi_2\rangle$$

All the operators we use in quantum mechanics are linear. Operators form an algebra. Given two operator A and B, we define their sum $\alpha A + \beta B$ as

$$(\alpha A + \beta B) : |\phi\rangle = \alpha A|\phi\rangle + \beta B|\phi\rangle$$

for all $|\phi\rangle \in \mathcal{H}$, and take their product AB to be the composition

$$AB : |\phi\rangle \rightarrow A \circ B|\phi\rangle = A(B|\phi\rangle)$$

for all $|\phi\rangle \in \mathcal{H}$.

The sum and product of two linear operators is again a linear operator.

The operator ~~ab~~ algebra is associative

i.e. $A(BC) = (AB)C$, but not commutative

i.e $AB \neq BA$ in general.

The difference between the two actions is known as the commutator

$$[A, B] = AB - BA$$

The commutator obeys the following properties.

antisymmetry : $[A, B] = -[B, A]$.

linearity : $[\alpha A_1 + \beta A_2, B] = \alpha [A_1, B] + \beta [A_2, B]$

Leibniz identity : $[A, BC] = [A, B]C + B[A, C]$

Jacobi identity : $[A, [B, C]] + [B, [C, A]]$

$$+ [C, [A, B]] = 0.$$

A state $|\psi\rangle \in \mathcal{H}$ is said to be an eigenstate of an operator A if

$$A|\psi\rangle = a_\psi |\psi\rangle \quad \text{with } a_\psi \in \mathbb{C}$$

a_ψ is known as eigenvalue and $|\psi\rangle$ is the eigenvector of A.

The set of all eigenvalues of an operator A is called the spectrum of A.

While the number of linearly independent eigenstates (eigenvectors) is called the degeneracy of this eigenvalue.

The operation : $A^\dagger$ means transposition + complex conjugation

$$\langle \phi | A^\dagger | \psi \rangle = \langle \psi | A | \phi \rangle^\dagger$$

$$(A+B)^\dagger = A^\dagger + B^\dagger \qquad |\psi\rangle^\dagger = \langle\psi|$$

$$(AB)^\dagger = B^\dagger A^\dagger$$

$$(\alpha A)^\dagger = \bar{\alpha} A^\dagger$$

$$(A^\dagger)^\dagger = A$$

$$[A, B]^\dagger = -[A^\dagger, B^\dagger]$$

Note : The adjoint equation of $A|\psi\rangle = a_\psi|\psi\rangle$ is $\langle\psi|A^\dagger = \langle\psi|\bar{a}_\psi$

An operator $\mathcal{G}$ is called Hermitian if $\mathcal{G}^\dagger = \mathcal{G}$, so that $\langle\phi|\mathcal{G}|\psi\rangle = \langle\psi|\mathcal{G}|\phi\rangle^\dagger$

Harmitian operators are very special in QM and ~~have~~ have some important properties

1) The eigenvalues of harmitian operators are real.

Short proof. Let Q be a harmitian operator with eigenvector $|q\rangle$ such that

$$Q|q\rangle = q|q\rangle \quad \text{and} \quad \langle q|Q = \langle q|\bar{q}$$

Now we know $\langle q|Q|q\rangle = \langle q|Q|q\rangle^{+}$.

Since $Q^{+} = Q$.

Therefore $q\langle q|q\rangle = \bar{q}\langle q|q\rangle$

which gives $q = \bar{q}$

Secondly suppose $|a_1\rangle$ and $|a_2\rangle$ are both eigenstates of Q, having distinct eigenvalues a_1 and a_2.

Therefore $\langle a_1|Q^{+}|a_2\rangle = \langle a_2|Q|a_1\rangle^{+} = \langle a_1|Q|a_2\rangle$

$$\Rightarrow \quad a_1\langle a_1|a_2\rangle - a_2\langle a_1|a_2\rangle = 0$$

$$\Rightarrow \quad (a_1 - a_2)\langle a_1|a_2\rangle = 0.$$

Now $a_1 \neq a_2$, therefore $\langle a_1|a_2\rangle = 0$

So eigenstates of distinct eigenvalues of a Harmitian operator, are orthogonal.

In finite dimension, the set of eigenvectors of a given Hermitian operator form a basis of $\mathcal{H}$. i.e. they form a complete orthonormal set. This property allows us to express Hermitian operators in a very useful form.

If, $\{|n\rangle\}$ is the set of orthonormal basis of a Hermitian operator $\mathcal{Q}$ with eigenvalues $\{q_n\}$, then we can write

$$\mathcal{Q} = \sum_n q_n |n\rangle\langle n|$$

$\hookrightarrow$ Spectral dicomposition

If $|\psi\rangle$ be a vector with $|\psi\rangle \in \mathcal{H}$,

$$\mathcal{Q}|\psi\rangle = \sum_n q_n |n\rangle\langle n|\psi\rangle$$

Note that $\mathcal{Q}|\psi\rangle \in \mathcal{H}$, since $\langle n|\psi\rangle$ is just a complex number, whereas each of the terms involves a $|n\rangle \in \mathcal{H}$.

In particular, expressing $|\psi\rangle$ in this basis as $|\psi\rangle = \sum_m c_m |m\rangle$ gives

$$\mathcal{Q}|\psi\rangle = \sum_{n,m} q_n c_m |n\rangle\langle n|m\rangle$$

$$= \sum_{n,m} q_n c_m \delta_{mn}$$

$$= \sum_n q_n c_n .$$

One reason this representation of Q is useful since it allows us to define functions of operators.

We set $$f(Q) = \sum_n f(q_n)\, |n\rangle\langle n|$$

Example: $$f(Q) = Q^{-1} = \sum_n \frac{1}{q_n} |n\rangle\langle n| .$$

which makes sense, provided $q_n \neq 0 \;\forall n$.

whereas $$f(Q) = \ln(Q) = \sum_n \ln q_n \,|n\rangle\langle n| .$$

which is meaningful when $q_n > 0 \;\forall n$.

Representation of Identity operator:

The identity operator $I_{\mathcal{H}}$ on $\mathcal{H}$ is
$$I_{\mathcal{H}} = \sum_n |n\rangle\langle n| \quad \text{or} \quad I_{\mathcal{H}} = \int |q\rangle\langle q|\, dq$$

Example:

Let $\{|p\rangle\}$ denotes a complete set of eigenstates of the momentum operator P, with $P|p\rangle = p|p\rangle$

Then we have,
$$P|\psi\rangle = \int P|p\rangle\langle p|\psi\rangle\, dp \quad \left[\int |p\rangle\langle p|\, dp = I_{\mathcal{H}} \right]$$

Now $$\langle x|P|\psi\rangle = \int \langle x|P|p\rangle \underbrace{\langle p|\psi\rangle}_{\hat{\psi}(p)} dp .$$

$$= \int p \langle x|p\rangle\, \hat{\psi}(p)\, dp$$

$$= \int p\, e^{ixp/\hbar}\, \hat{\psi}(p)\, dp$$

$$= -i\hbar \left[\frac{\partial}{\partial x} \int \frac{e^{ixp/\hbar}}{\sqrt{2\pi\hbar}}\, \hat{\psi}(p)\, dp \right]$$

Here we first use the property $\int |p\rangle\langle p|\, dp = I_H$ and then use a standard property of Fourier transformation. (Don't worry about it if you donot remember Fourier trans.)

Furthermore we have,

$$\langle x|P|\psi\rangle = -i\hbar \frac{\partial}{\partial x}\langle x|\psi\rangle$$

Commutator identity:

i) Antisymmetry: $[A; B] = -[B, A]$

Proof: $[A, B] = AB - BA = -(BA - AB) = -[B, A]$

(Proved)

ii) linearity: $[\alpha A_1 + \beta A_2, B] = \alpha [A_1, B] + \beta [A_2, B]$

Proof: $[\alpha A_1 + \beta A_2, B]$

$= (\alpha A_1 + \beta A_2) B - B(\alpha A_1 + \beta A_2)$

$= \alpha A_1 B + \beta A_2 B - \alpha B A_1 - \beta B A_2$

$= \alpha (A_1 B - B A_1) + \beta (A_2 B - B A_2)$

$= \alpha [A_1, B] + \beta [A_2, B]$

(Proved).

iii) Leibniz identity: $[A, BC] = [A, B] C + B [A, C]$

Proof: $[A, BC] = ABC - BCA$

$= ABC - BAC + BAC - BCA$

$= (AB - BA) C + B (AC - CA)$

$= [A, B] C + B [A, C]$.

(Proved).

Jacobi identity : $[A,[B,C]] + [B,[C,A]]$
$+ [C,[A,B]] = 0$

Proof : $[A,[B,C]] + [B,[C,A]] + [C,[A,B]]$

$$= [A,(BC-CB)] + [B,(CA-AC)] + [C,(AB-BA)]$$

$$= A(BC-CB) - (BC-CB)A + B(CA-AC) - (CA-AC)B$$
$$+ C(AB-BA) - (AB-BA)C$$

$$= A\cancel{B}C - A\cancel{C}B - B\cancel{C}A + C\cancel{B}A + B\cancel{C}A - B\cancel{A}C - C\cancel{A}B + \cancel{A}CB$$
$$+ CA\cancel{B} - C\cancel{B}A - A\cancel{B}C + B\cancel{A}C$$

$$= 0 \qquad \text{(Proved)}.$$

Function of Hermitian operators.

i) Q be a Hermitian operator:
with spectral representation $Q = \sum_i a_i |i\rangle\langle i|$.

Therefore $Q^{-1} = \sum_i a_i^{-1} |i\rangle\langle i|$.

Consequence: Non-full rank matrices do not have inverse.

ii) $\exp(Q) = \sum_i \exp(a_i) |i\rangle\langle i|$.

Proof: $\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$= \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

$$\therefore \exp(Q) = \mathbb{I} + Q + \frac{Q \cdot Q}{2!} + \frac{QQQ}{3!} + \cdots$$

$$= \mathbb{I} + \sum_i a_i |i\rangle\langle i| + \frac{1}{2!}\sum_i a_i |i\rangle\langle i| \left(\sum_j a_j |j\rangle\langle j|\right)$$

$$+ \cdots$$

$$= \sum_i |i\rangle\langle i| + \sum_i a_i |i\rangle\langle i| + \sum_i \frac{a_i^2}{2!} |i\rangle\langle i|$$

$$+ \cdots$$

$$= \sum_i \left(1 + a_i + \frac{a_i^2}{2!} + \frac{a_i^3}{3!} + \cdots\right) |i\rangle\langle i|$$

$$= \sum_i \exp(a_i) |i\rangle\langle i|$$

(Proved).