Q1) $X = X_1 + X_2 + \cdots + X_n$

$$X_i \begin{cases} \to 1 & \text{prob } 1/2 \\ \to -1 & \text{prob } 1/2 \end{cases}$$

$\Pr(X \geq a) = ?$ $\quad a > 0$

$$E[X_i] = 1 \Pr(X_i = 1) - 1 \Pr(X_i = -1)$$

$$= 1 \cdot \tfrac{1}{2} - 1 \cdot \tfrac{1}{2} = 0$$

Linearity of Expectations

$$E[X] = E[X_1 + X_2 + \cdots X_n] = E[X_1] + E[X_2] + \cdots E[X_n]$$

$$= 0. \qquad (\therefore \mu = 0)$$

$$\Pr(X \geq a) = \Pr(e^{Xt} \geq e^{at})$$

$t \to$ const (arbitrary)

Let Y be a r.v such that $Y = e^{Xt}$

$$Y = e^{(X_1 + X_2 + \cdots X_n)t} = \underbrace{e^{X_1 t}}_{Y_1} \cdot \underbrace{e^{X_2 t}}_{Y_2} \cdot e^{X_3 t} \cdots \underbrace{e^{X_n t}}_{Y_n}$$

$$Y = Y_1 \cdot Y_2 \cdot Y_3 \cdots Y_n$$

Y_i's are pairwise independent.

$$E[Y_i] = \Pr(X_i = 1)\, e^{t(1)} + \Pr(X_i = -1)\, e^{t(-1)}$$

$$= \frac{1}{2} e^t + \frac{1}{2} e^{-t} = \left(\frac{e^t + e^{-t}}{2}\right)$$

$$E[Y] = E[Y_1 \cdot Y_2 \cdots Y_n]$$

pair wise independent Y_i's

$$= E[Y_1] \cdot E[Y_2] \cdots E[Y_n]$$

$$= \prod_{i=1}^{n} \left(\frac{e^t + e^{-t}}{2}\right)$$

Using the inequality $1 + x \leq e^x$

$$1 + \left(\frac{e^t + e^{-t}}{2} - 1\right) \leq e^{\left(\frac{e^t + e^{-t}}{2} - 1\right)}$$

$$E[Y] = \prod_{i=1}^{n} e^{\left(\frac{e^t+e^{-t}}{2}-1\right)} = e^{n\left(\frac{e^t+e^{-t}}{2}-1\right)}$$

$$Pr(e^{xt} \geq e^{at}) = Pr\left(Y \geq \underbrace{\frac{e^{at}}{E[Y]}}_{\to c} E[Y]\right)$$

$$\leq \frac{E[Y]}{e^{at}} \quad \left(\leq 1/c\right)$$

$$\leq e^{n\left(\frac{e^t+e^{-t}}{2}-1\right)-at} \longrightarrow (i)$$

$$f(t) = n\left(\frac{e^t+e^{-t}}{2}-1\right) - at$$

$$f'(t) = n\left(\frac{e^t+e^{-t}(-1)}{2}\right) - a = 0$$

$$\Rightarrow \frac{n}{2}(e^t - e^{-t}) = a \Rightarrow e^t - e^{-t} = \frac{2a}{n}$$

Let $e^t = x$, $x - 1/x = 2a/n$

$$x^2 - 1 = \frac{2a}{n}x \Rightarrow x^2 - \frac{2a}{n}x - 1 = 0$$

$$e^t = x = \frac{\frac{2a}{n} \pm \sqrt{\left(\frac{2a}{n}\right)^2+4}}{2} = \frac{\frac{2a}{n} + \sqrt{\left(\frac{2a}{n}\right)^2+4}}{2}$$

'−' value of the root of x cant chosen as $x > 0$

$$\Rightarrow t = \log\left(\frac{\frac{2a}{n} + \sqrt{\left(\frac{2a}{n}\right)^2+4}}{2}\right)$$

$$f''(t) = \frac{n}{2}(e^t + e^{-t}) > 0 \checkmark \qquad \therefore t \to \text{min value}$$

Substituting the value of t in eq (i)

$$Pr(X \geq a) = Pr(Y \geq e^{at}) \leq e^{n\left(\frac{e^t+e^{-t}}{2}-1\right)-at}$$

where $t = \log\left(\frac{\frac{2a}{n} + \sqrt{\left(\frac{2a}{n}\right)^2+4}}{2}\right)$

Q2) $X = X_1 + X_2 + \dots X_n$

X_i: $\rightarrow c \rightarrow$ prob p

$\rightarrow c+1 \rightarrow$ prob p

$\rightarrow c+2$

$\vdots$

$d \rightarrow$ prob p

Assume the set has 'k' elements

$\therefore d = c + k - 1$

$k = d + 1 - c$

prob $p = \frac{1}{k}$

$E[X_i] = \frac{1}{k} \cdot c + \frac{1}{k}(c+1) + \dots \frac{1}{k}(c+k-1)$

$= \frac{1}{k}(\underbrace{c + c+1 + \dots c+k-1}_{\text{sum of A.P}}) = \frac{1}{k} \cdot \frac{k}{2}(c + c+k-1)$

$= \frac{1}{2}(c+d)$

$E[X] = E[X_1 + X_2 + \dots X_n]$ linearity of expectation.

$= E[X_1] + E[X_2] + \dots + E[X_n] = \frac{n}{2}(c+d) \; (= \mu)$

$Pr[X \geq \mu(1+\delta)] = Pr[e^{Xt} \geq e^{\mu t(1+\delta)}]$

Let Y be a r.v such that $Y = e^{Xt}$

$Y = e^{(X_1 + X_2 + \dots X_n)t} = \underbrace{e^{X_1 t}}_{Y_1} \cdot \underbrace{e^{X_2 t}}_{Y_2} \dots \underbrace{e^{X_n t}}_{Y_n}$

Y_i's are pairwise independent.

$E[Y] = E[Y_1 \cdot Y_2 \dots Y_n] = \prod_{i=1}^{n} E[Y_i]$ (Independent events)

$E[Y_i] = \frac{1}{k} e^{t \cdot c} + \frac{1}{k} e^{t(c+1)} + \frac{1}{k} e^{t(c+2)} \dots \frac{1}{k} e^{t(c+k-1)}$

$= \frac{1}{k} e^{ct} [1 + \underbrace{e^{t} + e^{2t} + \dots e^{(k-1)t}}_{\text{sum of gp}}]$

$$E[Y_i] = \frac{1}{k} e^{ct} \left(\frac{e^{kt} - 1}{e^t - 1} \right) \qquad k = ~~c~~ \; d+1-c$$

$$= \frac{1}{d+1-c} e^{ct} \frac{\left(e^{(d+1-c)t} - 1 \right)}{e^t - 1}$$

$$= \frac{1}{d+1-c} \left[\frac{e^{(d+1)t} - e^{ct}}{e^t - 1} \right]$$

$$E[Y] = \prod_{i=1}^{n} \frac{1}{(d+1-c)} \left(\frac{e^{(d+1)t} - e^{ct}}{e^t - 1} \right)$$

Using the inequality $1+x \le e^x$

$$1 + \left[\left(\frac{1}{d+1-c}\right) \left(\frac{e^{(d+1)t} - e^{ct}}{e^t - 1} \right) - 1 \right] \le e^{\left(\frac{1}{d+1-c}\right)\left[\frac{e^{(d+1)t} - e^{ct}}{e^t - 1}\right] - 1}$$

$$\therefore E[Y] = \prod_{i=1}^{n} e^{\frac{1}{d+1-c} \left(\frac{e^{(d+1)t} - e^{ct}}{e^t - 1} \right) - 1}$$

$$= e^{n \left(\frac{1}{d+1-c} \left(\frac{e^{(d+1)t} - e^{ct}}{e^t - 1} \right) - 1 \right)}$$

$$\Pr\left(e^{xt} \ge e^{\mu t (1+\delta)}\right) = \Pr\left(Y \ge \frac{e^{\mu t (1+\delta)}}{E[Y]} E[Y]\right)$$

$$\le \frac{E[Y]}{e^{\mu t (1+\delta)}}$$

$$\le e^{n \left(\frac{1}{d+1-c} \left(\frac{e^{(d+1)t} - e^{ct}}{e^t - 1} \right) - 1 \right) - \mu t (1+\delta)}$$

$\longrightarrow (i)$

Let $f(t) = n\left(\frac{1}{d+1-c}\left(\frac{e^{(d+1)t} - e^{ct}}{e^{t} - 1}\right) - 1\right) - \mu t(1+\delta)$

$$f'(t) = \frac{n}{d+1-c}\left[\frac{(e^{t}-1)\left((d+1)e^{(d+1)t} - c\,e^{ct}\right) - \left(e^{(d+1)t} - e^{ct}\right)(e^{t})}{(e^{t}-1)^{2}}\right]$$

$$- \mu(1+\delta) = 0 \longrightarrow (ii)$$

Solve for 't' in eq (ii)

& Substitute the 't' value in eq (i) for finding the bound

$$\therefore \Pr\left(x \geq \mu(1+\delta)\right) = \Pr\left(y \geq e^{\mu t(1+\delta)}\right) \leq e^{n\left(\frac{1}{d+1-c}\left(\frac{e^{(d+1)t} - e^{ct}}{e^{t} - 1}\right) - 1\right) - \mu t(1+\delta)}$$

Q3) $X = X_1 + X_2 + \dots X_n$

$$X_i \begin{cases} \to 0 & \text{prob} = p = 1/2 \\ \to 1 & \text{prob} = p = 1/2 \end{cases}$$

$$\Pr[X \le E[X](1-\delta)] = ?$$

Ans) $E[X_i] = 0 \cdot \Pr(X_i = 0) + 1 \cdot \Pr(X_i = 1)$

$= 1/2$

$$E[X] = E[X_1 + X_2 + \dots + X_n]$$

Linearity of Expectation

$$= E[X_1] + E[X_2] + \dots + E[X_n]$$

$$= 1/2 + 1/2 + \dots = n/2 = \mu$$

$$\Pr[X \le \mu(1-\delta)] = \Pr[-X \ge \mu(\delta - 1)]$$

$$= \Pr[e^{-Xt} \ge e^{\mu(\delta-1)t}]$$

Let Y be a r.v such that $Y = e^{-Xt}$

$$Y = e^{-(X_1 + X_2 + \dots X_n)t} = \underbrace{e^{-X_1 t}}_{Y_1} \cdot \underbrace{e^{-X_2 t}}_{Y_2} \cdots \underbrace{e^{-X_n t}}_{Y_n}$$

$$Y = Y_1 \cdot Y_2 \cdot \dots Y_n$$

Y_i's are pairwise independent.

$$E[Y_i] = \text{Prob}(X_i = 0) \cdot e^{-t \cdot 0} + \text{Prob}(X_i = 1) e^{-t}$$

$$= \frac{1}{2} + \frac{1}{2} e^{-t} = \frac{e^{-t} + 1}{2}$$

$$E[Y] = E[Y_1 \cdot Y_2 \cdots Y_n] = E[Y_1] \cdot E[Y_2] \cdots E[Y_n]$$
$$= \prod_{i=1}^{n} \left(\frac{e^{-t}+1}{2} \right)$$

Using the inequality $1+x \leq e^x$,

$$1 + \left(\frac{e^{-t}+1}{2} - 1 \right) \leq e^{\left(\frac{e^{-t}-1}{2}\right)}$$

$$\therefore E[Y] = \prod_{i=1}^{n} e^{\left(\frac{e^{-t}-1}{2}\right)} = e^{\frac{n}{2}(e^{-t}-1)}$$

$$\Pr\left(e^{-Xt} \geq e^{\mu(\delta-1)t}\right) = \Pr\left[Y \geq \underbrace{\frac{e^{\mu(\delta-1)t}}{E[Y]}}_{\to c} E[Y]\right]$$

$$\leq \frac{E[Y]}{e^{\mu(\delta-1)t}} \qquad \left(\leq \frac{1}{c}\right)$$

$$\leq e^{n/2(e^{-t}-1) - \mu t(\delta-1)} \longrightarrow (i)$$

Let $f(t) = \frac{n}{2}(e^{-t}-1) - \frac{n}{2}t(\delta-1)$

$$f'(t) = \frac{n}{2} e^{-t}(-1) - 0 - \frac{n}{2}(\delta-1) = 0$$

$$\Rightarrow -e^{-t} = \delta - 1$$

$$\Rightarrow t = -\log(1-\delta) = \log\left(\frac{1}{1-\delta}\right)$$

Sub 't' in (i)

$$\Pr(Y \geq c \cdot E[Y]) \leq e^{n/2(-\delta) - \frac{n}{2}\log(1-\delta)(1-\delta)}$$

$$\leq \left(\frac{e^{-\delta}}{(1-\delta)^{(1-\delta)}}\right)^{n/2}$$

$\mu = n/2$