Solutions for Modern Complexity Theory Homework Zero

By writing my name here I affirm that I am aware of all policies and abided by them while working on this problem set:

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Collaborators:

Problem 0 (5 points): Read fully the **Mathematical Background Chapter** from the textbook at `https://introtcs.org/public/lec_00_math_background.pdf`. This is probably the most important exercise in this problem set!!

Solution 0: I certify that I fully read the mathematical background chapter

Question 1 (3 points): Write a logical expression $\varphi(x)$ involving the variables x_0, x_1, x_2 and the operators $\wedge$ (AND), $\vee$ (OR), and $\neg$ (NOT), such that $\varphi(x)$ is true if and only if the majority of the inputs are *False*.

Solution 1:
Let us assume that the x_0, x_1, x_2 take the values TRUE or FALSE only. they will be boolean variables. Im denoting TRUE as 1 and FALSE as 0.

Table 1: Truth table

x_0	x_1	x_2	$\varphi(x)$
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	0

By solving this truth table we get $\varphi(x) = (\neg x_0 \wedge \neg x_1) \vee (\neg x_0 \wedge \neg x_2) \vee (\neg x_1 \wedge \neg x_2)$.

Question 2: Use the logical quantifiers $\forall$ (for all), $\exists$ (exists), as well as $\wedge, \vee, \neg$ and the arithmetic operations $+, \times, =, >, <$ to write the following:

Question 2.1 (3 points): An expression $\psi(n, k)$ such that for every natural numbers n, k, $\psi(n, k)$ is true if and only if k divides n.

Solution 2.1:$\psi(n, k) = \{n, k \in \mathbb{N} \ : \ \exists_{y \in \mathbb{N}} n = k * y\}$

Question 2.2 (3 points bonus): An expression $\varphi(n)$ such that for every natural number n, $\varphi(n)$ is true if and only if n is a power of three.

Solution 2.2:$\varphi(n) = \{n \in \mathbb{N} \ : \ \exists_{y \in \mathbb{Z}} n = 3^y\}$

Question 3: In this question, you need to describe in words sets that are defined using a formula with quantifiers. For example, the set $S = \{x \in \mathbb{N} \ : \ \exists_{y \in \mathbb{N}} x = 2y\}$ is the set of even numbers.

Question 3.1 (3 points): Describe in words the following set S:

$$S = \{x \in \{0,1\}^{100} : \forall_{i \in \{0,\ldots,98\}} x_i = x_{i+1}\}$$

(Recall that, as written in the mathematical background chapter, we use zero-based indexing in this course, and so a string $x \in \{0,1\}^{100}$ is indexed as $x_0 x_1 \cdots x_{99}$.

Solution 3.1:
S is a set which contains strings of length 100, where each character is either 0 or 1. S contains only those strings where each character of the string is same.

$$S = \{00......(length = 100), 111....(length = 100)\}$$

Question 3.2 (3 points): Describe in words the following set T:

$$T = \{x \in \{0,1\}^* : |x| > 1 \text{ and } \forall_{i \in \{2,\ldots,|x|-1\}} \forall_{j \in \{2,\ldots,|x|-1\}} i \cdot j \neq |x|\}$$

Solution 3.2:
T is a set which contains strings of length greater than 1 and the length of each string in the set is a prime number.

Question 4: This question deals with sets, their cardinalities, and one to one and onto functions. You can cite results connecting these notions from the course's textbook, MIT's "Mathematics for Computer Science" or any other discrete mathematics textbook.

Question 4.1 (4 points): Define $S = \{0,1\}^6$ and T as the set $\{n \in [100] \mid n \text{ is prime } \}$. *Prove or disprove:* There is a one to one function from S to T.

Solution 4.1:
S is a set which contains strings of length 6, where each character of the string is either 0 or 1. Cardinality of set S is 64 (2^6).
T is a set of all prime numbers between 0-99. So T = { 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97.}.
Cardinality of T is 25.
For there to be a one to one relation between set S and T, the codomain set T must atleast have as many elements as the Domain set S for there to be a one to one correspondence.
Cardinality of S > Cardinality of T. Atleast 2 element in S will map to the same element in T.
Therefore we cannot have a one to one function from S to T.

Question 4.2 (4 points): Let $n = 100$, $S = [n] \times [n] \times [n]$ and $T = \{0,1\}^n$. *Prove or disprove:* There is an onto function from T to S.

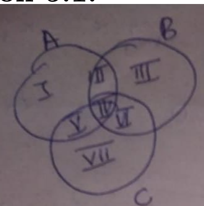
Solution 4.2:
Given n =100
Set S is $\{0, 1, .., 99\}^3$. It contains all the strings of length 3. where each character of the string can take any value between 0 to 99.Cardinality of S is 100^3.
Set T is $\{0, 1\}^{100}$. It contains all strings of length 100. where each character of the string can take either 0 or 1. Cardinality of T is 2^{100}.
For a function from T to S to be onto, every element in S must have a preimage in T. That is, the range must be equal to its co domain.
For there to be an onto relation between set T and S, the domain set T must atleast have as many elements as the Codomain set S for there to be an onto correspondence. Otherwise a function itself cannot be defined.
Cardinailty of T is more than cardinality of S.
Therefore we can have a onto function from T to S.

Question 4.3 (4 points): Let $n = 100$, let $S = \{0,1\}^{n^3}$ and T be the set of all functions mapping $\{0,1\}^n$ to $\{0,1\}$. *Prove or disprove:* There is a one to one function from S to T.

Solution 4.3:
Given n =100
In the function T, we have the domain d_T which is the set of all strings of length 100, whose characters can be either 0 or 1. Therefore the cardinality of d_T 2^{100}. We have the codomain cd_T which is either 0 or 1. For T to be a function, each string from d_T can be mapped with either 0 or 1. Total no of fucntions possible from d_T to cd_T $2^{2^{100}}$.
The Set S is a set which contain all strings of length 10^6 whose characters are either 0 or 1. Therefore the cardinality of S is $2^{1000000}$.
For there to be a one to one relation between set S and T, the codomain set T must atleast have as many elements as the Domain set S for there to be a one to one correspondence.
Cardinality of S < Cardinality of T. Therefore we can have a one to one function from S to T.

Question 5.1 (5 points): Prove that for every finite sets A, B, C, $|A \cup B \cup C| \leq |A| + |B| + |C|$.

Solution 5.1:



$|A \cup B \cup C| = \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} + \mathrm{V} + \mathrm{VI} + \mathrm{VII} \rightarrow (i)$

$|A| = \mathrm{I} + \mathrm{II} + \mathrm{IV} + \mathrm{V}$

$|B| = \mathrm{II} + \mathrm{III} + \mathrm{IV} + \mathrm{VI}$

$|C| = \mathrm{V} + \mathrm{IV} + \mathrm{VI} + \mathrm{VII}$

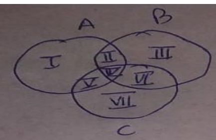
$|A| + |B| + |C| = \mathrm{I} + 2\cdot\mathrm{II} + 2\cdot\mathrm{V} + 2\cdot\mathrm{VI} + 3\cdot\mathrm{IV} + \mathrm{VII} + \mathrm{III} \rightarrow (ii)$

Comparing (i) & (ii)

$\therefore |A \cup B \cup C| \leq |A| + |B| + |C|$

Question 5.2 (5 points bonus): Prove that for every finite sets A, B, C, $|A \cup B \cup C| \geq |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C|$.

Solution 5.2:



$|A| = \mathrm{I} + \mathrm{II} + \mathrm{IV} + \mathrm{V}$

$|B| = \mathrm{III} + \mathrm{II} + \mathrm{IV} + \mathrm{VI}$

$|C| = \mathrm{VII} + \mathrm{VI} + \mathrm{V} + \mathrm{IV}$

$|A \cap B| = \mathrm{II} + \mathrm{IV}$

$|B \cap C| = \mathrm{VI} + \mathrm{IV}$

$|A \cap C| = \mathrm{V} + \mathrm{IV}$

$|A \cup B \cup C| = \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} + \mathrm{V} + \mathrm{VI} + \mathrm{VII} \longrightarrow (i)$

$|A| + |B| + |C| - (|A \cap B| + |B \cap C| + |C \cap A|)$

$= \mathrm{I} + 2 \cdot \mathrm{II} + 2 \cdot \mathrm{V} + 2 \cdot \mathrm{VI} + 3 \cdot \mathrm{IV} + \mathrm{VII} + \mathrm{III}$

$\quad - (\mathrm{II} + \mathrm{VI} + \mathrm{V} + 3 \cdot \mathrm{IV})$

$= \mathrm{I} + \mathrm{II} + \mathrm{V} + \mathrm{VI} + \mathrm{VII} + \mathrm{III} \longrightarrow (ii)$

comparing (i) & (ii)

$\therefore |A \cup B \cup C| \geq |A| + |B| + |C| - (|A \cap B| + |B \cap C| + |C \cap A|)$

Question 6.1 (5 points): Prove that if G is a directed acyclic graph (DAG) on n vertices, if u and v are two vertices of G such that there is a directed path of length $n-1$ from u to v then u has no in-neighbors.[1]

Solution 6.1:
Given:
G is a DAG containing n vertices.
There exists a path between vertex u and v such that the length of the path is n-1.
Proof:
If G is a DAG, then the topological sort on G is possible.
The topological sort starts with that vertex whose indegree is 0 and finds the dfs of the graph from that vertex.
Hence if there exists a directed path from u to v with path length n-1 then u must be the vertex of in degree 0.
If a topological sort doesnt exist, then G is not a DAG.
If the minimum indegree of a each vertex in the graph > 0 then the graph contains a cycle, it cannot be a DAG.

Question 6.2 (5 points): Prove that for every undirected graph G of 1000 vertices, if every vertex has degree at most 4, then there exists a subset S of at least 200 vertices such that no two vertices in S are neighbors of one another.

Solution 6.2:
Given:

[1] *Hint:* You can use the topological sorting theorem shown in the mathematical background chapter.

G is an undirected graph
n=1000
degree of each vertex less than or equal to 4
Proof:
Start with an empty set S.
Take any vertex v in the graph, and add it to S.
Remove v and its neighbours from the graph, and repeat the above step with the new graph.
End when there are no vertices left.
At each step, at most five vertices removed at each step, since you remove the chosen vertex v and its neighbours (degree of atleast 4).
n =1000, which means that the steps given above repeat at least 200 times. For each step, some new vertex is added to S.
S has at least 200 elements
S is a Maximal Independent Set.

Question 7: For each pair of functions f, g below, state whether or not $f = O(g)$ and whether or not $g = O(f)$.

Question 7.1 (3 points): $f(n) = n(\log n)^3$ and $g(n) = n^2$.

Solution 7.1:
To find which of the 2 functions grows asymptotically faster we can use $\lim_{x\to\infty} f(x)/g(x)$. If this limit is 0 then we can say f(n) = o(g(n)), that is g(n) grows faster than f(n). If the limit tends to ∞ then we can say g(n) = o(f(n)), that is f(n) rows faster than g(n).
$\lim_{x\to\infty} f(x)/g(x)$
$=\lim_{x\to\infty}(n(\log n)^3)/n^2)$
$=\lim_{x\to\infty}((\log n)^3)/n)$
$=\infty/\infty$
Use L'Hopital's rule
Differentiating numerator and denominator seperately
$=\lim_{x\to\infty}(3(\log n)^2 * 1/n)/1)$
$=\lim_{x\to\infty}(3(\log n)^2/n)$
$=\infty/\infty$
Use L'Hopital's rule again
Differentiating numerator and denominator seperately
$=\lim_{x\to\infty}(6(\log n) * 1/n)/1)$
$=\lim_{x\to\infty}(6(\log n)/n)$
$=\infty/\infty$
Use L'Hopital's rule again
Differentiating numerator and denominator seperately
$=\lim_{x\to\infty}(6 * 1/n)/1)$
$=\lim_{x\to\infty}(6/n)$
$=\infty$
Therefore $g(n) = n^2$ grows asymptotically faster than $f(n) = n(\log n)^3$.
f(n)=o(g(n)).
But this is not a tight bound.

We can abuse this notation further and say that f(n)=O(g(n)).

Question 7.2 (3 points): $f(n) = n^{\log n}$ and $g(n) = n^2$.
Solution 7.2:
Let us equate f(n)=g(n) for n_0 where logn_0=2. From this we can calculate n_0 to be 100.
For n > 100 (to ∞), f(n) grows asypmtotically faster than g(n).
Therefore g(n)=O(f(n)).

Question 7.3 (3 points bonus):$f(n) = \binom{n}{\lceil 0.2n \rceil}$ (where $\binom{n}{k}$ is the number of k-sized subsets of a set of size n) and $g(n) = 2^{0.1n}$

Solution 7.3: