The norm of an operator $\|T\|$

Does this norm so defined satisfy the four Prop of a norm ?

$\|Tx\| \leq C\|x\|$ C is a real number.

$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|}$ $C = \|T\|$

$\|Tx\| \leq \|T\|\|x\|$

$\|T\| = \sup_{\|x\|=1} \|Tx\|$

$\|T\| = 0 \Rightarrow T = 0$ operator.

① $\|\cdot\| \geq 0$ ✓

$\|x\| = 0 \Rightarrow x = 0$

$\|\alpha T\| = \sup_{\|x\|=1} \|\alpha Tx\| = \sup_{\|x\|=1} |\alpha|\|Tx\| = |\alpha| \sup_{\|x\| \geq 1} \|Tx\| = |\alpha|\|T\|$

$\|T\|$ → Norm of an operator

Not the same as $\|Tx\|$

Triangle inequality.

$\|T_1 + T_2\| \leq \|T_1\| + \|T_2\|$

$\sup_{\|x\|=1} \|(T_1+T_2)x\| = \sup_{\|x\|=1} \|T_1x + T_2x\| \leq \sup_{\|x\|=1} \|T_1x\| + \sup_{\|x\|=1} \|T_2x\| = \|T_1\| + \|T_2\|$

Norms of various operators.

Identity operator $I : X \to X$ $X \neq \{0\}$

$\|I\| = 1$

$I(x) = x$

$\|I\| = \sup_{\|x\|=1} \{\|I(x)\|\} = \sup_{\|x\|=1} \{\|x\|\} = 1$

Zero operator $0 : X \to Y$ $\|0\| = 0$

Differentiation Operator. Let X be the normed space of all Polynomials on $J = [0,1]$, $\|x\| = \max_{t \in J} |x(t)|$, $t \in J$

$\|x\|$ → x is a Polynomial

A differentiation operator $T : T(x(t)) = x'(t)$ diff w.R.To t.

Is this Linear?

$T(\alpha x + \beta y) = \alpha x' + \beta y' = \alpha Tx + \beta Ty$ ✓ Linear.

Is this bounded?

Let $x_n(t) = t^n$ $n \in \mathbb{N}$, $\|x_n\| = \max_{t \in J} |x(t)| = 1$

$J = [0,1]$

$Tx_n(t) = x_n'(t) = n t^{n-1}$

$\|Tx_n\| = n$ $t \to 1$

$\|T\| = \sup_{\|x_n\|=1} \|Tx_n\| = \sup n = n$

$n \in \mathbb{N}$ is arbitrary, for $t^n \to \|T\| = n$

for $t^m \to \|T\| = m$

There is no fixed number such that $\frac{\|Tx_n\|}{\|x_n\|} \leq C$

This operator is linear but not bounded