GATE MA 2015 Q10

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Question

Let X be a random variable having the distribution function

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{4} & \text{if } 0 \leq x < 1 \\ \frac{1}{3} & \text{if } 1 \leq x < 2 \\ \frac{1}{2} & \text{if } 2 \leq x < \frac{11}{3} \\ 1 & \text{if } x \geq \frac{11}{3} \end{cases} \tag{1}$$

Then $E(X)$ is equal to ...

Solution 1

We will use dirac δ functions to solve the question.
Consider a unit step function $u(x)$,

$$u(x) = \begin{cases} 1 & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

We will remove the discontinuity at $x = 0$ by defining $u_\alpha(x)$ for any $\alpha > 0$,

$$u_\alpha(x) = \begin{cases} 1 & x > \frac{\alpha}{2} \\ \frac{1}{\alpha}(x + \frac{\alpha}{2}) & -\frac{\alpha}{2} \leq x \leq \frac{\alpha}{2} \\ 0 & x < -\frac{\alpha}{2} \end{cases} \tag{3}$$

Solution 1 contd...

$$\delta_\alpha(x) = \frac{du_\alpha(x)}{dx} \tag{4}$$

$$\delta_\alpha(x) = \begin{cases} \frac{1}{\alpha} & -\frac{\alpha}{2} \le x \le \frac{\alpha}{2} \\ 0 & |x| > \frac{\alpha}{2} \end{cases} \tag{5}$$

$$u(x) = \lim_{\alpha \to 0} u_\alpha(x) \tag{6}$$

$$\delta(x) = \lim_{\alpha \to 0} \delta_\alpha(x) \tag{7}$$

Lemma

$$\int_{-\infty}^{\infty} g(x)\delta(x-x_0)dx = g(x_0) \tag{8}$$

Proof:
Let I be the value of the above integral.

$$I = \lim_{\alpha\to 0}\left[\int_{-\infty}^{\infty} g(x)\delta_\alpha(x-x_0)dx\right] \tag{9}$$

$$= \lim_{\alpha\to 0}\left[\int_{x_0-\frac{\alpha}{2}}^{x_0+\frac{\alpha}{2}} \frac{g(x)}{\alpha}dx\right] \tag{10}$$

$$\int_{x_0-\frac{\alpha}{2}}^{x_0+\frac{\alpha}{2}} \frac{g(x)}{\alpha}dx = \alpha\frac{g(x_\alpha)}{\alpha} = g(x_\alpha) \tag{11}$$

Here, $x_\alpha \in \left(x_0 - \frac{\alpha}{2}, x_0 + \frac{\alpha}{2}\right)$

$$I = \lim_{\alpha\to 0} g(x_\alpha) = g(x_0) \tag{12}$$

Solution 1 contd...

$$f_X(x) = \frac{dF(x)}{dx} \tag{13}$$

$$f_X(x) = \frac{1}{4}\delta(x) + \frac{1}{12}\delta(x-1) + \frac{1}{6}\delta(x-2) + \frac{1}{2}\delta\left(x - \frac{11}{3}\right) + 0 \tag{14}$$

We will use (8) to compute $E(X)$

$$E(X) = \int_{-\infty}^{\infty} x f_X(x) dx \tag{15}$$

$$E(X) = \int_{-\infty}^{\infty} \frac{1}{4}x\delta(x) + \frac{1}{12}x\delta(x-1) + \frac{1}{6}x\delta(x-2) + \frac{1}{2}x\delta\left(x - \frac{11}{3}\right) dx \tag{16}$$

$$E(X) = \frac{1}{4} \times 0 + \frac{1}{12} \times 1 + \frac{1}{6} \times 2 + \frac{1}{2} \times \frac{11}{3} \tag{17}$$

$$E(X) = 2.25 \tag{18}$$

Solution 2

We will use Riemann Stieltjes integral.
Suppose $F(x)$ is continuously differentiable on $[a, b]$ except at a finite number of points $c_1, c_2, \ldots c_n$.

$$\int_a^b g(x)dF(x) = \int_a^{c_1} g(x)dF(x) + g(c_1)\left[F(c_1) - \lim_{\substack{x\to c_1 \\ x<c_1}} F(x)\right] + \int_{c_1}^{c_2} g(x)dF(x) + g(c_2)\left[F(c_2) - \lim_{\substack{x\to c_2 \\ x<c_2}} F(x)\right] + \cdots + \int_{c_{n-1}}^{c_n} g(x)dF(x) + g(c_n)\left[F(c_n) - \lim_{\substack{x\to c_n \\ x<c_n}} F(x)\right] + \int_{c_n}^{b} g(x)dF(x) \quad (19)$$

Solution 2 contd...

$$E(g(X)) = \int_{-\infty}^{\infty} g(x)dF(x) \tag{20}$$

$$E(X) = \int_{-\infty}^{\infty} xdF(x) \tag{21}$$

Solution 2 contd...

$$E(X) = \int_{-\infty}^{0} xdF(x) + 0\left[F(0) - \lim_{\substack{x\to 0\\ x<0}} F(x)\right] +$$
$$\int_{0}^{1} xdF(x) + 1\left[F(1) - \lim_{\substack{x\to 1\\ x<1}} F(x)\right] +$$
$$\int_{1}^{2} xdF(x) + 2\left[F(2) - \lim_{\substack{x\to 2\\ x<2}} F(x)\right] +$$
$$\int_{2}^{\frac{11}{3}} xdF(x) + \frac{11}{3}\left[F\left(\frac{11}{3}\right) - \lim_{\substack{x\to \frac{11}{3}\\ x<\frac{11}{3}}} F(x)\right] +$$
$$\int_{\frac{11}{3}}^{\infty} xdF(x) \quad (22)$$

Solution 2 contd...

$$\int_{-\infty}^{0} xdF(x) = \int_{0}^{1} xdF(x) = \int_{1}^{2} xdF(x) = \int_{2}^{\frac{11}{3}} xdF(x) = \int_{\frac{11}{3}}^{\infty} xdF(x) = 0 \quad (23)$$

$$E(X) = 2.25 \quad (24)$$