

Challenging Problem 15

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<https://github.com/srikan-p/AI1103/tree/main/ChallengingProblem15/codes>

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<https://github.com/srikan-p/AI1103/tree/main/ChallengingProblem15>

PROBLEM

(UGC-MATH 2019 Q 105) Consider a simple symmetric random walk on integers, where from every state i you move to $i-1$ and $i+1$ with the probability half each. Then which of the following are true?

- 1) The random walk is aperiodic
- 2) The random walk is irreducible
- 3) The random walk is null recurrent
- 4) The random walk is positive recurrent

SOLUTION

The simple symmetric random walk is a Markov chain with state space $S = \{i | i \in \mathbb{Z}\}$ and with transition matrix P where,

$$p_{ij} = \begin{cases} 0, & |i - j| > 1 \\ p = \frac{1}{2}, & j = i + 1 \\ q = 1 - p = \frac{1}{2}, & j = i - 1 \end{cases} \quad (0.0.1)$$

and p_{ij} denotes the probability of being in state j , starting from state i after n steps or transitions.

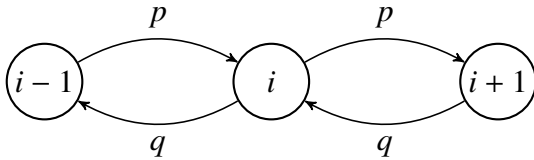


Fig. 4: Simple Symmetric random walk on integers

If the random walk is at state i , then,

$$P = \begin{matrix} & \begin{matrix} i & i+1 \end{matrix} \\ \begin{matrix} i \\ i+1 \end{matrix} & \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix} \end{matrix} \quad (0.0.2)$$

Fig. 4: Transition matrix of the random walk

- 1) The period of a state is defined as the gcd of all $t > 0$ such that $P^t_{i,i} > 0$. The state is considered aperiodic, if its period is 1.

$$P^k_{ij} = \begin{cases} 0, & k \bmod 2 = 1 \\ (2)^{-k}, & k \bmod 2 = 0 \end{cases} \quad (0.0.3)$$

Hence, the period is 2. So, the random walk is periodic.

Option (1) is **incorrect**

- 2) For a Markov chain to be **irreducible** all pairs (i, j) should communicate with each other. Let us assume that the chain starts from state i and it requires m forward and $(n - m)$ backward steps to reach j . Let $i < j$ wlog

$$j - i = m - (n - m) = 2m - n \quad (0.0.4)$$

$$m = \frac{(j - i) + n}{2} \quad (0.0.5)$$

$$p_{ij} = {}^nC_m p^m q^{n-m} \quad (0.0.6)$$

$$p_{ij} = {}^nC_{\left(\frac{(j-i)+n}{2}\right)} p^m q^{n-m} > 0$$

$$n = (j - i) + 2k \quad \forall k \in \mathbb{W} \quad (0.0.7)$$

Here i and j are general, hence all pairs i and j communicate with each other.

\therefore Option (2) is **correct**

- 3) In a Markov Chain for state i to be **recurrent** it should satisfy,

$$\lim_{t \rightarrow \infty} \sum_{n=1}^t p^n_{ii} = \infty \quad (0.0.8)$$

$$\lim_{t \rightarrow \infty} \sum_{n=1}^t p_{ii}^n = \lim_{t \rightarrow \infty} \left(\sum_{k=1}^t p_{ii}^{2k} + \sum_{k=1}^t p_{ii}^{(2k-1)} \right) \quad (0.0.9)$$

$$= \lim_{t \rightarrow \infty} \sum_{k=1}^t p_{ii}^{2k} \quad (0.0.10)$$

$$p_{ii}^{2k} = {}^{2k}C_k p^k q^k = \frac{2k!}{k!k!} p^k q^k \quad (0.0.11)$$

By using Stirling approximation to the (0.0.11) we get

$$p_{ii}^{2k} = \frac{\left((2k)^{2k+\frac{1}{2}} \right) \times \exp(-2k) \times (2\pi)^{\frac{1}{2}}}{\left(k^{k+\frac{1}{2}} \times \exp(-k) \right)^2 \times 2\pi} p^k q^k \quad (0.0.12)$$

$$= \frac{(4pq)^{2k}}{(k\pi)^{\frac{1}{2}}} = \frac{1}{(k\pi)^{\frac{1}{2}}} \quad (0.0.13)$$

$$\lim_{t \rightarrow \infty} \sum_{k=1}^t p_{ii}^{2k} = \lim_{t \rightarrow \infty} \sum_{k=1}^t \frac{1}{(k\pi)^{\frac{1}{2}}} \quad (0.0.14)$$

Since $\frac{1}{k^{\frac{1}{2}}}$ is divergent, Equation (0.0.8) is satisfied

\therefore The random walk is **recurrent**

The first-passage time probability is

$$f_{i,j}(n) = \Pr(X_n = j, X_{n-1} \neq j, X_{n-2} \neq j, \dots, X_1 \neq j | X_0 = i) \quad (0.0.15)$$

The first-passage time $T_{i,j}$ from state i to j has the PMF $f_{i,j}(n)$ and the distribution function

$$F_{i,j}(n) = \sum_{k=0}^n f_{i,j}(k) \quad (0.0.16)$$

For the Markov Chain to be null recurrent

$$\overline{T_{j,j}} = \infty \quad (0.0.17)$$

and for positive recurrent where $\overline{T_{j,j}}$ represents the mean time to enter j starting from j . We can calculate the mean by using the distribution function

$$\overline{T_{j,j}} = 1 + \sum_{k=0}^n (1 - F_{j,j}(k)) \quad (0.0.18)$$

From (0.0.16) and (0.0.18) we get (0.0.17) condition to be satisfied

\therefore Option (3) is **correct**.

4) For a Markov chain to be positive recurrent it

should satisfy the following condition

$$\overline{T_{j,j}} < \infty \quad (0.0.19)$$

Since the Markov chain satisfies the (0.0.17) but not (0.0.19) it is not positive recurrent.

\therefore Option (4) is **incorrect**

Answer : option2, option3