Assignment 4

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Download latex codes from

https://github.com/srikaran−p/AI1103/tree/main/Assignment4

Problem

(GATE-MA 2015 Q17) Let τ_1 be the usual topology on $\mathbb{R}$. Let τ_2 be the topology on $\mathbb{R}$ generated by $\mathcal{B} = \{[a\ ,b) \subset \mathbb{R} : -\infty < a < b < \infty\}$. Then the set $\left\{x \in \mathbb{R} : 4sin^2x \leq 1\right\} \cup \left\{\frac{\pi}{2}\right\}$ is

(A) closed in $(\mathbb{R}, \tau_1)$ but NOT in $(\mathbb{R}, \tau_2)$

(B) closed in $(\mathbb{R}, \tau_2)$ but NOT in $(\mathbb{R}, \tau_1)$

(C) closed in both $(\mathbb{R}, \tau_1)$ and $(\mathbb{R}, \tau_2)$

(D) neither closed in $(\mathbb{R}, \tau_1)$ nor closed in $(\mathbb{R}, \tau_2)$

Solution

Let A be the set of all the solutions of the given inequality,

$$A = \bigcup_{n\in\mathbb{Z}} \left[2n\pi - \frac{\pi}{6}, 2n\pi + \frac{\pi}{6}\right] + \bigcup_{n\in\mathbb{Z}} \left[2n\pi + \frac{5\pi}{6}, 2n\pi + \frac{7\pi}{6}\right] + \left\{\frac{\pi}{2}\right\} \quad (0.0.1)$$

$$A' = \left(\frac{-5\pi}{6}, \frac{-\pi}{6}\right) + \left(\frac{\pi}{6}, \frac{\pi}{2}\right) + \left(\frac{\pi}{2}, \frac{5\pi}{6}\right) + \bigcup_{n\in\mathbb{Z}-\{0\}} \left(2n\pi - \frac{5\pi}{6}, 2n\pi - \frac{\pi}{6}\right) + \bigcup_{n\in\mathbb{Z}-\{0\}} \left(2n\pi + \frac{\pi}{6}, 2n\pi + \frac{5\pi}{6}\right) \quad (0.0.2)$$

0.1 Definition

A set U of real numbers is said to be open if for all $x \in U$, there exists $\delta(x) > 0$ such that $(x - \delta(x), x + \delta(x)) \subset U$.

The intervals in A' are open sets by 0.1.

0.2 Theorem

If $\{U_\alpha\}$ is any collection (finite, infinite, countable or uncountable) of open sets, then $\bigcup_\alpha U_\alpha$ is an open set.

A' is an open set by 0.2.

$$A' \in \tau_1 \quad (0.2.1)$$

A' is not closed in $(\mathbb{R}, \tau_1)$.

$\implies$ A is closed in $(\mathbb{R}, \tau_1)$.

$$(a, b) = \bigcup_{n=1}^{\infty} \left[a + \frac{1}{n}, b\right) \quad (0.2.2)$$

The intervals in A' can be written as (0.2.2).

$$A' \in \tau_2 \quad (0.2.3)$$

A' is not closed in $(\mathbb{R}, \tau_2)$.

$\implies$ A is closed in $(\mathbb{R}, \tau_2)$.

Hence, option (C) is correct.