Challenging Problem 13

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Download all python codes from

```
https://github.com/srikaran−p/AI1103/tree/main/
    ChallengingProblem13/codes
```

and latex codes from

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https://github.com/srikaran−p/AI1103/tree/main/
    ChallengingProblem13
```

Problem

If each element of an n^{th} order determinant is either 0 or 1, what is the probability that the value of the determinant is positive?
(Assume that the individual entries of the determinant are chosen independently, each value being assumed with probability $\frac{1}{2}$)

Solution

Let $X \in \{0, 1\}$ be a random variable. We will compute the number of invertible matrices. Let a_{ij} represent the element in i^{th} row and j^{th} column.

$$\left(M\right) = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \\ a_{n1} & & a_{nn} \end{bmatrix} \tag{0.0.1}$$

$$\left(v_i\right) = \begin{bmatrix} a_{i1} & \dots & a_{in} \end{bmatrix} \tag{0.0.2}$$

Let $0_{1\times n}$ represent row zero vector.

$$\Pr\left(v_1 \neq 0_{1\times n}\right) = \frac{2^n - 1}{2^n} \tag{0.0.3}$$

$$\Pr\left(k_1 v_1 + k_2 v_2 \neq 0 | v_1 \neq 0_{1\times n}\right) = \frac{2^n - 2^1}{2^n} \tag{0.0.4}$$

$$\Pr\left(l_1 v_1 + l_2 v_2 + l_3 v_3 \neq 0 | k_1 v_1 + k_2 v_2 \neq 0\right) = \frac{2^n - 2^2}{2^n} \tag{0.0.5}$$

Similarly, we can find the probability of representing the other row vectors.

$$\Pr\left(\det M \neq 0\right) = \left(1 - 2^{-1}\right)\left(1 - 2^{-2}\right)\left(1 - 2^{-3}\right) \dots \left(1 - 2^{-n}\right) \tag{0.0.6}$$

Matrices with $\det M < 0$ are the matrices resulting from swapping of first two rows of matrices with $\det M > 0$.

$$\Pr\left(\det M > 0\right) = \Pr\left(\det M < 0\right) \tag{0.0.7}$$

$$= \frac{1}{2}\Pr\left(\det M \neq 0\right) \tag{0.0.8}$$

$$= \frac{1}{2}\prod_{k=1}^{n}\left(1 - 2^{-k}\right) \tag{0.0.9}$$

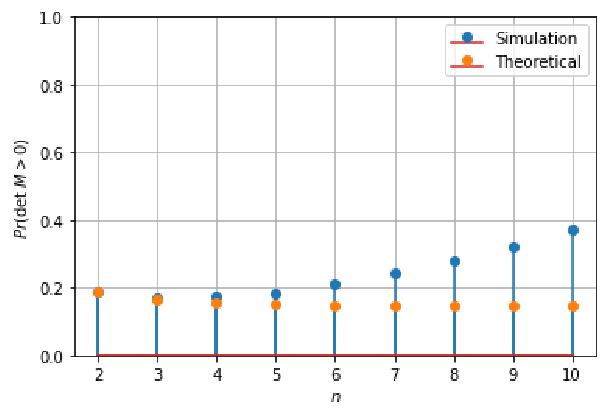

Fig. 0: Plot for Simulation v/s Theoretical