Challenging Problem 15

Perambuduri Srikaran - AI20BTECH11018

Download all python codes from

https://github.com/srikaran−p/AI1103/tree/main/ChallengingProblem15/codes

and latex codes from

https://github.com/srikaran−p/AI1103/tree/main/ChallengingProblem15

Problem

(UGC-MATH 2019 Q 105) Consider a simple symmetric random walk on integers, where from every state i you move to i-1 and i+1 with the probability half each. Then which of the following are true?

1) The random walk is aperiodic
2) The random walk is irreducible
3) The random walk is null recurrent
4) The random walk is positive recurrent

Solution

The simple symmetric random walk is a Markov chain with state space S = $\{i | i \in \mathbb{Z}\}$ and with transition matrix P where,

$$p_{ij} = \begin{cases} 0, & |i-j| > 1 \\ p = \dfrac{1}{2}, & j = i+1 \\ q = 1 - p = \dfrac{1}{2}, & j = i-1 \end{cases} \tag{0.0.1}$$

and p_{ij} denotes the probability of being in state j, starting from state i after n steps or transitions.

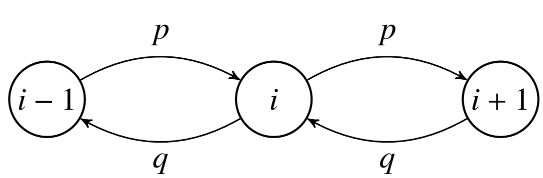

Fig. 4: Simple Symmetric random walk on integers

If the random walk is at state i, then,

$$P = \begin{array}{c c} & \begin{array}{cc} i & i+1 \end{array} \\ \begin{array}{c} i \\ i+1 \end{array} & \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix} \end{array} \tag{0.0.2}$$

Fig. 4: Transition matrix of the random walk

1) The period of a state is defined as the gcd of all $t > 0$ such that $P^t_{i,i} > 0$. The state is considered aperiodic, if its period is 1.

$$P^k_{i,i} = \begin{cases} 0, & k \bmod 2 = 1 \\ (pq)^k, & k \bmod 2 = 0 \end{cases} \tag{0.0.3}$$

Hence, the period is 2. So, the random walk is periodic.
Option **(1)** is **incorrect**

2) For a Markov chain to be **irreducible** all pairs (i, j) should communicate with each other.

$$P^{|i-j|}_{i,j} = P_{i,i+1} P_{i+1,i+2} \cdots P_{j-1,j} \tag{0.0.4}$$
$$= \prod_{k=i}^{j-1} P_{k,k+1} \tag{0.0.5}$$
$$> 0 \tag{0.0.6}$$

Similarly, $P^{|i-j|}_{j,i} > 0$.
Hence, we can say that every pair (i, j) are communicable. So, the Markov chain is irreducible.
$\therefore$Option **(2)** is **correct**

3) In a Markov Chain for state i to be **recurrent** it should satisfy ,

$$\lim_{t\to\infty} \sum_{n=1}^{t} p^n_{ii} = \infty \tag{0.0.7}$$

$$\lim_{t\to\infty}\sum_{n=1}^{t} p_{ii}^{n} = \lim_{t\to\infty}\left(\sum_{k=1}^{t} p_{ii}^{2k} + \sum_{k=1}^{t} p_{ii}^{(2k-1)}\right) \tag{0.0.8}$$

$$= \lim_{t\to\infty}\sum_{k=1}^{t} p_{ii}^{2k} \tag{0.0.9}$$

$$p_{ii}^{2k} = {}^{2k}C_k p^k q^k = \frac{2k!}{k!k!} p^k q^k \tag{0.0.10}$$

By using Stirling approximation to the (0.0.10) we get

$$p_{ii}^{2k} = \frac{\left((2k)^{2k+\frac{1}{2}}\right) \times \exp(-2k) \times (2\pi)^{\frac{1}{2}}}{\left(k^{k+\frac{1}{2}} \times \exp(-k)\right)^2 \times 2\pi} p^k q^k \tag{0.0.11}$$

$$= \frac{(4pq)^{2k}}{(k\pi)^{\frac{1}{2}}} = \frac{1}{(k\pi)^{\frac{1}{2}}} \tag{0.0.12}$$

$$\lim_{t\to\infty}\sum_{k=1}^{t} p_{ii}^{2k} = \lim_{t\to\infty}\sum_{k=1}^{t} \frac{1}{(k\pi)^{\frac{1}{2}}} \tag{0.0.13}$$

Since $\frac{1}{k^{\frac{1}{2}}}$ is divergent, Equation (0.0.7) is satisfied

$\therefore$ The random walk is **recurrent**

The first-passage time probability is

$$f_{i,j}(n) = \Pr(X_n = j, X_{n-1} \neq j, X_{n-2} \neq j, \dots X_1 \neq j | X_0 = i) \tag{0.0.14}$$

The first-passage time $T_{i,j}$ from state i to j has the PMF $f_{i,j}(n)$ and the distribution function

$$F_{i,j}(n) = \sum_{k=0}^{n} f_{i,j}(k) \tag{0.0.15}$$

For the Markov Chain to be null recurrent

$$\overline{T_{j,j}} = \infty \tag{0.0.16}$$

and for positive recurrent where $\overline{T_{j,j}}$ represents the mean time to enter j starting from j. We can calculate the mean by using the distribution function

$$\overline{T_{j,j}} = 1 + \sum_{k=0}^{n}\left(1 - F_{j,j}(k)\right) \tag{0.0.17}$$

From (0.0.15) and (0.0.17) we get (0.0.16) condition to be satisfied

$\therefore$ Option **(3)** is **correct**.

4) For a Markov chain to be positive recurrent it should satisfy the following condition

$$\overline{T_{j,j}} < \infty \tag{0.0.18}$$

Since the Markov chain satisfies the (0.0.16) but not (0.0.18) it is not positive recurrent.

$\therefore$ Option **(4)** is **incorrect**

Answer : option2, option3