Challenging Problem 13

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Question

If each element of an n^{th} order determinant is either 0 or 1, what is the probability that the value of the determinant is positive?
(Assume that the individual entries of the determinant are chosen independently, each value being assumed with probability $\frac{1}{2}$)

Solution

We will compute the number of invertible matrices. Let a_{ij} represent the element in i^{th} row and j^{th} column.

$$(M) = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \\ a_{n1} & & a_{nn} \end{bmatrix} \tag{1}$$

$$(v_i) = \begin{bmatrix} a_{i1} & \dots & a_{in} \end{bmatrix} \tag{2}$$

Solution (contd...)

Let $0_{1\times n}$ represent row zero vector.

$$\Pr\left(v_1 \neq 0_{1\times n}\right) = \frac{2^n - 1}{2^n} \tag{3}$$

$$\Pr\left(k_1 v_1 + k_2 v_2 \neq 0 | v_1 \neq 0_{1\times n}\right) = \frac{2^n - 2^1}{2^n} \tag{4}$$

$$\Pr\left(l_1 v_1 + l_2 v_2 + l_3 v_3 \neq 0 | k_1 v_1 + k_2 v_2 \neq 0\right) = \frac{2^n - 2^2}{2^n} \tag{5}$$

Similarly, we can find the probability of representing the row vectors.

$$\Pr\left(\det M \neq 0\right) = \left(1 - 2^{-1}\right)\left(1 - 2^{-2}\right)\left(1 - 2^{-3}\right)\ldots\left(1 - 2^{-n}\right) \tag{6}$$

Solution (contd...)

Matrices with $\det M < 0$ are the matrices resulting from swapping of first 2 rows of matrices with $\det M > 0$.

$$\Pr(\det M > 0) = \Pr(\det M < 0) \tag{7}$$
$$= \frac{1}{2}\Pr(\det M \neq 0) \tag{8}$$
$$= \frac{1}{2}\prod_{k=1}^{n}\left(1 - 2^{-k}\right) \tag{9}$$

Simulation

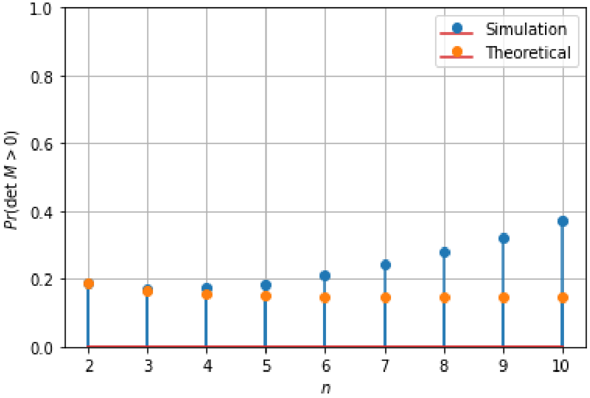

Figure: Plot for Simulation v/s Theoretical