Challenging Problem 13

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Download all python codes from

```
https://github.com/srikaran−p/AI1103/tree/main/
    ChallengingProblem13/codes
```

and latex codes from

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https://github.com/srikaran−p/AI1103/tree/main/
    ChallengingProblem13
```

Problem

If each element of an n^{th} order determinant is either 0 or 1, what is the probability that the value of the determinant is positive?
(Assume that the individual entries of the determinant are chosen independently, each value being assumed with probability $\frac{1}{2}$)

Solution

Let $X \in \{0,1\}$ be a random variable. We will compute the number of invertible matrices. Let a_{ij} represent the element in i^{th} row and j^{th} column.

$$\left(M\right) = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \\ a_{n1} & & a_{nn} \end{bmatrix} \tag{0.0.1}$$

We will first have a non-zero row in M. This can be done in $2^n - 1$ ways. Then, will have a row which is linearly independent from the first row. This can be done in $2^n - 2^1$ ways. Similarly, the third row should be linearly independent to the first 2 rows. This can be done in $2^n - 2^2$ ways. Doing so, we will get,

$$N(\det M \neq 0) = (2^n - 1)(2^n - 2)\left(2^n - 2^2\right) \dots \left(2^n - 2^{n-1}\right) \tag{0.0.2}$$

Matrices with $\det M < 0$ are the matrices resulting from swapping of first two rows of matrices with $\det M > 0$.

$$N(\det M > 0) = N(\det M < 0) \tag{0.0.3}$$

$$= \frac{1}{2} N(\det M \neq 0) \tag{0.0.4}$$

$$= \frac{1}{2} \prod_{k=0}^{n-1} \left(2^n - 2^k\right) \tag{0.0.5}$$

$$\Pr(\det M > 0) = \frac{N(\det M > 0)}{2^{n^2}} \tag{0.0.6}$$

$$= \frac{1}{2} \prod_{k=1}^{n} \left(1 - 2^{-k}\right) \tag{0.0.7}$$

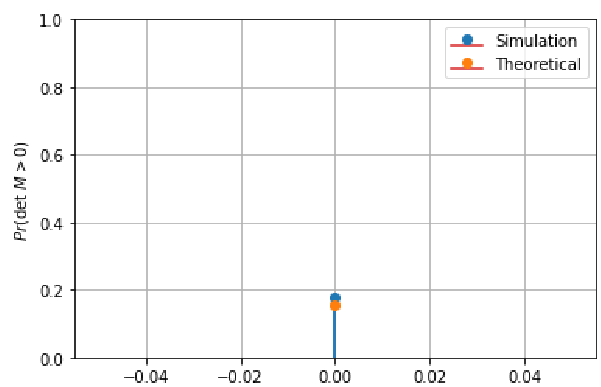

Fig. 0: Plot for Simulation v/s Theoretical (n = 4)