Assignment 3

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Download all python codes from

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https://github.com/srikaran−p/AI1103/tree/main/
    Assignment3/codes
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and latex codes from

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https://github.com/srikaran−p/AI1103/tree/main/
    Assignment3
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Problem

(GATE-MA 2015 Q17) Let X be a random variable having the distribution function

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{4} & \text{if } 0 \le x < 1 \\ \frac{1}{3} & \text{if } 1 \le x < 2 \\ \frac{1}{2} & \text{if } 2 \le x < \frac{11}{3} \\ 1 & \text{if } x \ge \frac{11}{3} \end{cases} \tag{0.0.1}$$

Then $E(X)$ is equal to ...

Solution

Consider a unit step function $u(x)$,

$$u(x) = \begin{cases} 1 & x \ge 0 \\ 0 & \text{otherwise} \end{cases} \tag{0.0.2}$$

We will remove the discontinuity at $x = 0$ by defining $u_\alpha(x)$ for any $\alpha > 0$,

$$u_\alpha(x) = \begin{cases} 1 & x > \frac{\alpha}{2} \\ \frac{1}{\alpha}(x + \frac{\alpha}{2}) & -\frac{\alpha}{2} \le x \le \frac{\alpha}{2} \\ 0 & x < -\frac{\alpha}{2} \end{cases} \tag{0.0.3}$$

$$\delta_\alpha(x) = \frac{du_\alpha(x)}{dx} \tag{0.0.4}$$

$$\delta_\alpha(x) = \begin{cases} \frac{1}{\alpha} & -\frac{\alpha}{2} \le x \le \frac{\alpha}{2} \\ 0 & |x| > \frac{\alpha}{2} \end{cases} \tag{0.0.5}$$

$$u(x) = \lim_{\alpha \to 0} u_\alpha(x) \tag{0.0.6}$$

$$\delta(x) = \lim_{\alpha \to 0} \delta_\alpha(x) \tag{0.0.7}$$

$$\int_{-\infty}^{\infty} g(x)\delta(x - x_0)dx = g(x_0) \tag{0.0.8}$$

We will add dirac δ functions to remove discontinuities in $F(x)$.

$$f_X(x) = \frac{dF(x)}{dx} \tag{0.0.9}$$

$$f_X(x) = \frac{1}{4}\delta(x) + \frac{1}{12}\delta(x-1) + \frac{1}{6}\delta(x-2) + \frac{1}{2}\delta\left(x - \frac{11}{3}\right) + 0 \tag{0.0.10}$$

We will use (0.0.8) to compute E(X)

$$E(X) = \int_{-\infty}^{\infty} x f_X(x) dx \tag{0.0.11}$$

$$E(X) = \int_{-\infty}^{\infty} \frac{1}{4}x\delta(x) + \frac{1}{12}x\delta(x-1) + \frac{1}{6}x\delta(x-2) + \frac{1}{2}x\delta\left(x - \frac{11}{3}\right) dx \tag{0.0.12}$$

$$E(X) = \frac{1}{4} \times 0 + \frac{1}{12} \times 1 + \frac{1}{6} \times 2 + \frac{1}{2} \times \frac{11}{3} \tag{0.0.13}$$

$$E(X) = 2.25 \tag{0.0.14}$$

Alternative Solution

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) dF(x) \tag{0.0.15}$$

Suppose $F(x)$ is continuously differentiable on $[a, b]$ except at a finite number of points $c_1, c_2, \ldots c_n$.

$$\int_a^b g(x)dF(x) = \int_a^{c_1} g(x)dF(x) + g(c_1)\left[F(c_1) - \lim_{\substack{x\to c_1\\ x<c_1}} F(x)\right] + \int_{c_1}^{c_2} g(x)dF(x) + g(c_2)\left[F(c_2) - \lim_{\substack{x\to c_2\\ x<c_2}} F(x)\right] + \cdots + \int_{c_{n-1}}^{c_n} g(x)dF(x) + g(c_n)\left[F(c_n) - \lim_{\substack{x\to c_n\\ x<c_n}} F(x)\right] + \int_{c_n}^{b} g(x)dF(x) \quad (0.0.16)$$

As per the question, $g(X) = X$

$$E(X) = \int_{-\infty}^{\infty} xdF(x) \quad (0.0.17)$$

$$E(X) = \int_{-\infty}^{0} xdF(x) + 0\left[F(0) - \lim_{\substack{x\to 0\\ x<0}} F(x)\right] + \int_{0}^{1} xdF(x) + 1\left[F(1) - \lim_{\substack{x\to 1\\ x<1}} F(x)\right] + \int_{1}^{2} xdF(x) + 2\left[F(2) - \lim_{\substack{x\to 2\\ x<2}} F(x)\right] + \int_{2}^{\frac{11}{3}} xdF(x) + \frac{11}{3}\left[F\left(\frac{11}{3}\right) - \lim_{\substack{x\to \frac{11}{3}\\ x<\frac{11}{3}}} F(x)\right] + \int_{\frac{11}{3}}^{\infty} xdF(x) \quad (0.0.18)$$

$$\int_{-\infty}^{0} xdF(x) = \int_{0}^{1} xdF(x) = \int_{1}^{2} xdF(x) = \int_{2}^{\frac{11}{3}} xdF(x) = \int_{\frac{11}{3}}^{\infty} xdF(x) = 0 \quad (0.0.19)$$

$$E(X) = 2.25 \quad (0.0.20)$$

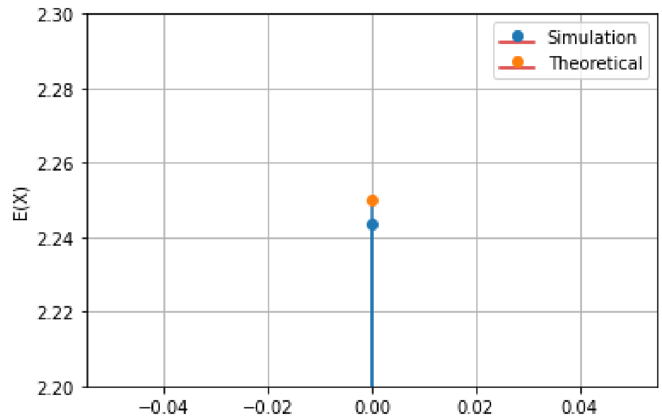

Fig. 0: Plot for Simulation v/s Theoretical