GATE EC-2018 Q39

Perambuduri Srikaran

IITH AI

October 18, 2021

Question

The input $4\,\text{sinc}\,(2t)$ is fed to a Hilbert transformer to obtain $y\,(t)$, as shown in the figure below:

$$4\,\text{sinc}\,(2t) \longrightarrow \boxed{\begin{array}{c}\text{Hilbert}\\ \text{Transform}\end{array}} \longrightarrow y\,(t)$$

Here, $\text{sinc}\,(x) = \frac{\sin(\pi x)}{\pi x}$. The value (accurate to two decimal places) of $\int_{-\infty}^{\infty} |y\,(t)\,|^2 dt$ is

Parseval's Theorem

Parseval's theorem states that there is no loss of information in Fourier transform and the amount of energy remains the same in time and frequency domains.

$$\int_{-\infty}^{\infty} |x\left(t\right)|^2 dt = \int_{-\infty}^{\infty} |X\left(f\right)|^2 df \tag{1}$$

Solution

We will define the following functions:

$$x(t) = 4\,\mathrm{sinc}(2t) \tag{2}$$

$$h(t) = \frac{1}{\pi t} \tag{3}$$

$$y(t) = x(t) * h(t) \tag{4}$$

$$x(t) \overset{\mathcal{F}}{\rightleftharpoons} X(f) \tag{5}$$

$$h(t) \overset{\mathcal{F}}{\rightleftharpoons} H(f) \tag{6}$$

$$y(t) \overset{\mathcal{F}}{\rightleftharpoons} Y(f) \tag{7}$$

Solution Contd...

Define the rectangular function,

$$\text{rect}(t) = \begin{cases} 1 & \text{if } |t| \leq \frac{1}{2} \\ 0 & \text{if } |t| > \frac{1}{2} \end{cases} \tag{8}$$

Define the signum function,

$$\text{sgn}(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t = 0 \\ -1 & \text{if } t < 0 \end{cases} \tag{9}$$

Solution Contd...

The Fourier transforms are

$$X(f) = 2\,\mathrm{rect}\left(\frac{f}{2}\right) \tag{10}$$

$$H(f) = -j\,\mathrm{sgn}(f) \tag{11}$$

Applying Convolution theorem in (4),

$$\begin{aligned} Y(f) &= X(f)H(f) && (12) \\ &= -2j\,\mathrm{rect}\left(\frac{f}{2}\right)\mathrm{sgn}(f) && (13) \\ &= 2j\,\mathrm{rect}\left(f+\frac{1}{2}\right) - 2j\,\mathrm{rect}\left(f-\frac{1}{2}\right) && (14) \end{aligned}$$

Solution Contd...

Applying Inverse Fourier Transform on $Y(f)$,

$$y(t) = 2j\,\mathrm{sinc}(t)e^{-j\pi t} - 2j\,\mathrm{sinc}(t)e^{j\pi t} \tag{15}$$

$$= -2j\,\mathrm{sinc}(t)\left(e^{j\pi t} - e^{-j\pi t}\right) \tag{16}$$

$$= 2\,\mathrm{sinc}(t)\left(\frac{e^{j\pi t} - e^{-j\pi t}}{j}\right) \tag{17}$$

$$= 4\,\mathrm{sinc}(t)\sin(\pi t) \tag{18}$$

$$= 4\pi t\,\mathrm{sinc}^2(t) \tag{19}$$

Solution Contd...

By the Parseval's theorem,

$$\int_{-\infty}^{\infty} |y(t)|^2 dt = \int_{-\infty}^{\infty} |Y(f)|^2 df \tag{20}$$

$$= \int_{-1}^{1} |2\,\mathrm{rect}(f)|^2 df \tag{21}$$

$$= 8 \tag{22}$$

Plots

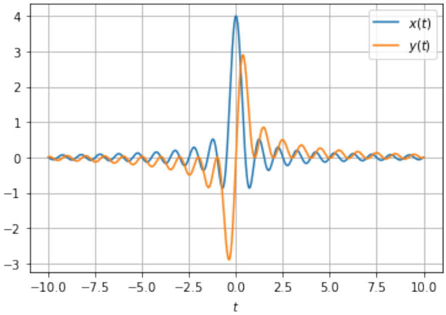

Figure: Input and Output Signals

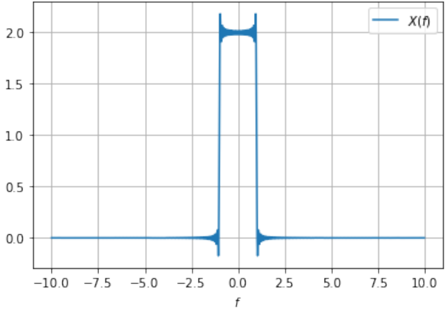

Figure: Plots of X(f) in frequency domain

Plots

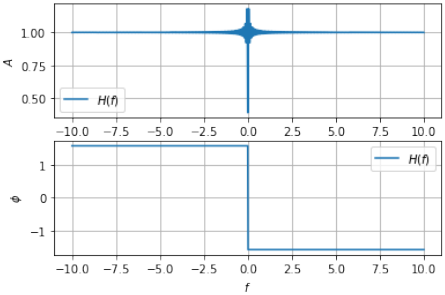

Figure: Amplitude and Phase of H(f) v/s frequency

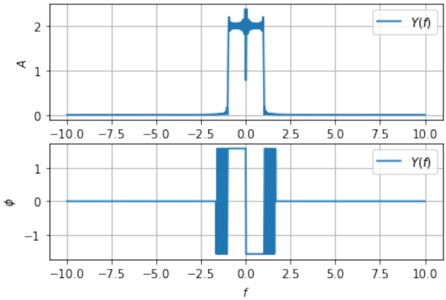

Figure: Amplitude and Phase of Y(f) v/s frequency