CS6700: Tutorial 3 - Policy Iteration

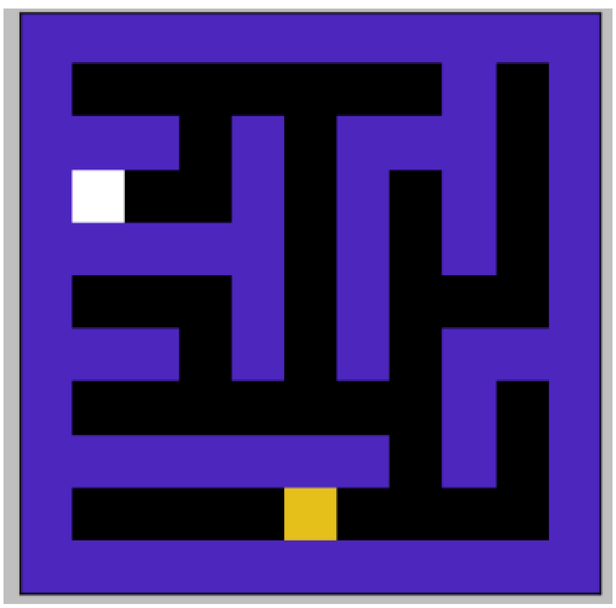
```python
import numpy as np
from enum import Enum
import copy
```

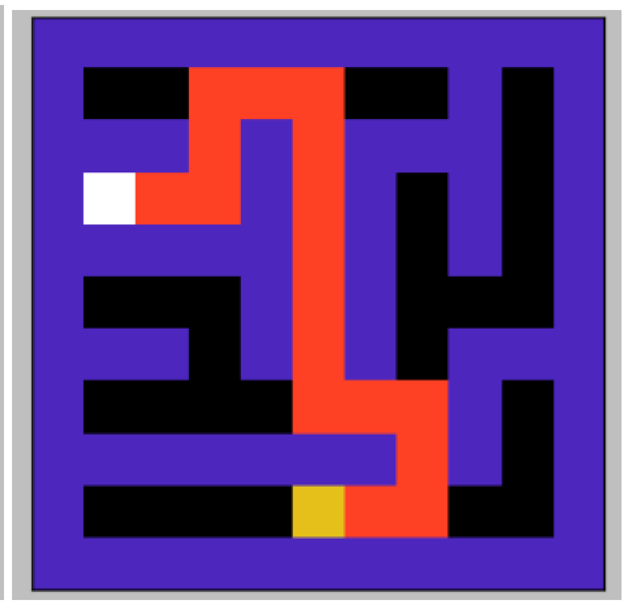
Consider a standard grid world, where only 4 (up, down, left, right) actions are allowed and the agent deterministically moves accordingly, represented as below. Here yellow is the start state and white is the goal state.

Say, we define our MDP as:

- S: 121 (11 x 11) cells
- A: 4 actions (up, down, left, right)
- P: Deterministic transition probability
- R: -1 at every step
- gamma: 0.9

Our goal is to find an optimal policy (shown in right).





```python
# Above grid is defined as below:
#   - 0 denotes an navigable tile
#   - 1 denotes an obstruction/wall
#   - 2 denotes the start state
#   - 3 denotes an goal state

# Note: Here the upper left corner is defined as (0, 0)
#       and lower right corner as (m-1, n-1)
```

```
# Optimal Path: RIGHT RIGHT UP UP LEFT LEFT UP UP UP UP UP UP LEFT
LEFT DOWN DOWN LEFT LEFT


GRID_WORLD = np.array([
    [1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1],
    [1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1],
    [1, 1, 1, 0, 1, 0, 1, 1, 1, 0, 1],
    [1, 3, 0, 0, 1, 0, 1, 0, 1, 0, 1],
    [1, 1, 1, 1, 1, 0, 1, 0, 1, 0, 1],
    [1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1],
    [1, 1, 1, 0, 1, 0, 1, 0, 1, 1, 1],
    [1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1],
    [1, 1, 1, 1, 1, 1, 1, 0, 1, 0, 1],
    [1, 0, 0, 0, 0, 2, 0, 0, 0, 0, 1],
    [1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]
])
```

Actions

```
class Actions(Enum):
  UP    = (0, (-1, 0))  # index = 0, (xaxis_move = -1 and yaxis_move =
0)
  DOWN  = (1, (1, 0))   # index = 1, (xaxis_move = 1 and yaxis_move =
0)
  LEFT  = (2, (0, -1))  # index = 2, (xaxis_move = 0 and yaxis_move =
-1)
  RIGHT = (3, (0, 1))   # index = 3, (xaxis_move = 0 and yaxis_move =
-1)

  def get_action_dir(self):
    _, direction = self.value
    return direction

  @property
  def index(self):
    indx, _ = self.value
    return indx

  @classmethod
  def from_index(cls, index):
    action_index_map = {a.index: a for a in cls}
    return action_index_map[index]

# How to use Action enum
for a in Actions:
  print(f"name: {a.name}, action_id: {a.index}, direction_to_move:
{a.get_action_dir()}")
```

```python
print("\n-----------------------------------\n")

# find action enum from index 0
a = Actions.from_index(0)
print(f"0 index action is: {a.name}")
```

```
name: UP, action_id: 0, direction_to_move: (-1, 0)
name: DOWN, action_id: 1, direction_to_move: (1, 0)
name: LEFT, action_id: 2, direction_to_move: (0, -1)
name: RIGHT, action_id: 3, direction_to_move: (0, 1)

-----------------------------------

0 index action is: UP
```

Policy

```python
class BasePolicy:
  def update(self, *args):
    pass

  def select_action(self, state_id: int) -> int:
    raise NotImplemented


class DeterministicPolicy(BasePolicy):
  def __init__(self, actions: np.ndarray):
    # actions: its a 1d array (|S| size) which contains action for
each state
    self.actions = actions

  def update(self, state_id, action_id):
    assert state_id < len(self.actions), f"Invalid state_id
{state_id}"
    assert action_id < len(Actions), f"Invalid action_id {action_id}"
    self.actions[state_id] = action_id

  def select_action(self, state_id: int) -> int:
    assert state_id < len(self.actions), f"Invalid state_id
{state_id}"
    return self.actions[state_id]
```

Environment

```python
class Environment:
  def __init__(self, grid):
    self.grid = grid
    m, n = grid.shape
    self.num_states = m*n
```

```
  def xy_to_posid(self, x: int, y: int):
    _, n = self.grid.shape
    return x*n + y

  def posid_to_xy(self, posid: int):
    _, n = self.grid.shape
    return (posid // n, posid % n)

  def isvalid_move(self, x: int, y: int):
    m, n = self.grid.shape
    return (x >= 0) and (y >= 0) and (x < m) and (y < n) and
(self.grid[x, y] != 1)

  def find_start_xy(self) -> int:
    m, n = self.grid.shape
    for x in range(m):
      for y in range(n):
        if self.grid[x, y] == 2:
          return (x, y)
    raise Exception("Start position not found.")

  def find_path(self, policy: BasePolicy) -> str:
    max_steps = 50
    steps = 0

    P, R = self.get_transition_prob_and_expected_reward()
    num_actions, num_states = R.shape
    all_possible_state_posids = np.arange(num_states)

    path = ""
    curr_x, curr_y = self.find_start_xy()
    while (self.grid[curr_x, curr_y] != 3) and (steps < max_steps):
      curr_posid = self.xy_to_posid(curr_x, curr_y)
      action_id = policy.select_action(curr_posid)
      next_posid = np.random.choice(
          all_possible_state_posids, p=P[action_id, curr_posid])
      action = Actions.from_index(action_id)
      path += f" {action.name}"
      curr_x, curr_y = self.posid_to_xy(next_posid)
      steps += 1
    return path

  def get_transition_prob_and_expected_reward(self):  # P(s_next | s,
a), R(s, a)
    m, n = self.grid.shape
    num_states = m*n
    num_actions = len(Actions)
    P = np.zeros((num_actions, num_states, num_states))
    R = np.zeros((num_actions, num_states))
    for a in Actions:
```

```
      for x in range(m):
        for y in range(n):
          xmove_dir, ymove_dir = a.get_action_dir()
          xnew, ynew = x + xmove_dir, y + ymove_dir  # find the new
co-ordinate after the action a

          posid = self.xy_to_posid(x, y)
          new_posid = self.xy_to_posid(xnew, ynew)


          if self.grid[x, y] == 3:
            # the current state is a goal state
            P[a.index, posid, posid] = 1
            R[a.index, posid] = 0
          elif (self.grid[x, y] == 1) or (not self.isvalid_move(xnew,
ynew)):
            # the current state is a block state or the next state is
invalid
            P[a.index, posid, posid] = 1
            R[a.index, posid] = -1
          else:
            # action a is valid and goes to a new position
            P[a.index, posid, new_posid] = 1
            R[a.index, posid] = -1
    return P, R
```

Policy Iteration

Policy Iteration (using iterative policy evaluation) for estimating $\pi \approx \pi_*$

1. Initialization
   $V(s) \in \mathbb{R}$ and $\pi(s) \in \mathcal{A}(s)$ arbitrarily for all $s \in \mathcal{S}$; $V(terminal) \doteq 0$

2. Policy Evaluation
   Loop:
      $\Delta \leftarrow 0$
      Loop for each $s \in \mathcal{S}$:
         $v \leftarrow V(s)$
         $V(s) \leftarrow \sum_{s',r} p(s', r \,|\, s, \pi(s)) \big[r + \gamma V(s')\big]$
         $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
   until $\Delta < \theta$ (a small positive number determining the accuracy of estimation)

3. Policy Improvement
   policy-stable $\leftarrow$ *true*
   For each $s \in \mathcal{S}$:
      old-action $\leftarrow \pi(s)$
      $\pi(s) \leftarrow \operatorname{argmax}_a \sum_{s',r} p(s', r \,|\, s, a) \big[r + \gamma V(s')\big]$
      If *old-action* $\neq \pi(s)$, then *policy-stable* $\leftarrow$ *false*
   If *policy-stable*, then stop and return $V \approx v_*$ and $\pi \approx \pi_*$; else go to 2

```
def policy_evaluation(P: np.ndarray, R: np.ndarray, gamma: float,
                      policy: BasePolicy, theta: float,
                      init_V: np.ndarray=None):
  _, num_states = R.shape

  # Please try different starting point for V you will find it will
always
  # converge to the same V_pi value.

  if init_V is None:
    init_V = np.zeros(num_states)

  V = copy.deepcopy(init_V)

  delta = 100.0

  while delta > theta:
    delta = 0.0

    for state_id in range(num_states):

      action_id = policy.select_action(state_id)
```

```
      # Following equation is a different way of writing the same
equation given in the slide.
      # Note here R is an expected reward term.
      # v to keep trach previous value of V[state_id]
      v = V[state_id]
      V[state_id] = R[action_id, state_id] + gamma *
np.dot(P[action_id, state_id], V)

      # YOUR CODE HERE
      # delta being updated
      delta = max(delta, abs(v - V[state_id])) # Calculate delta which
determines when to terminate the evaluation step

  return V


def policy_improvement(P: np.ndarray, R: np.ndarray, gamma: float,
                      policy: BasePolicy, V: np.ndarray):
  _, num_states = R.shape
  policy_stable = True

  for state_id in range(num_states):

    old_action_id = policy.select_action(state_id)

    # YOUR CODE HERE
    # Q is the value function for each action in the given state
    Q = [R[action_id, state_id] + gamma * np.dot(P[action_id,
state_id], V) for action_id in range(len(Actions))]
    # the index of the maximum value of this list will be the new
action_id
    new_action_id = Q.index(max(Q)) # update new_action_id based on
the value function.

    policy.update(state_id, new_action_id)
    if old_action_id != new_action_id:
      policy_stable = False

  return policy_stable


def policy_iteration(P: np.ndarray, R: np.ndarray, gamma: float,
                    theta: float=1e-3, init_policy: BasePolicy =
None):
  _, num_states = R.shape

  # Please try exploring different policies you will find it will
always
  # converge to the same optimal policy for valid states.
```

```
  if init_policy is None:
    # Say initial policy = all up actions.
    init_policy = DeterministicPolicy(actions=np.zeros(num_states,
dtype=int))

  # creating a copy of a initial policy
  policy = copy.deepcopy(init_policy)
  policy_stable = False

  while not policy_stable:
    V = policy_evaluation(P, R, gamma, policy, theta)
    policy_stable = policy_improvement(P, R, gamma, policy, V)

  return policy, V
```

Experiments

```
def is_same_optimal_value(V1, V2, diff_theta=1e-3):
  diff = np.abs(V1 - V2)
  return np.all(diff < diff_theta)

seed = 0
np.random.seed(seed)

gamma = 0.9
theta = 1e-5

env = Environment(GRID_WORLD)
P, R = env.get_transition_prob_and_expected_reward()
```

Exercise 1: Using Policy iteration algorithm find the optimal path from start to goal position

```
# # Start with random choice of init_policy.
# One such choice could be: init_policy = np.ones(env.num_states,
dtype=int)
init_policy = DeterministicPolicy(actions=np.ones(env.num_states,
dtype=int))

pitr_policy, pitr_V_star = policy_iteration(P, R, gamma, theta=theta,
init_policy=init_policy)
pitr_path = env.find_path(pitr_policy)
print(pitr_path)

 RIGHT RIGHT UP UP LEFT LEFT UP UP UP UP UP UP LEFT LEFT DOWN DOWN
LEFT LEFT
```

We observe that the policy converges to the optimal path.

Exercise 2: Using initial guess for V as random values, find the optimal value function using policy evaluation and compare it with the optimal value function

```
# Start with random choice of init_V.
# One such choice could be: init_V = np.random.randn(env.num_states)
# Another choice could be: init_V = 10*np.ones(env.num_states)
init_V = 10*np.ones(env.num_states)

V_star = policy_evaluation(P, R, gamma, pitr_policy, theta, init_V)
is_same_optimal_value(pitr_V_star, V_star)
```

```
True
```

```
# V initialized to random values
init_V = np.random.randn(env.num_states)

V_star = policy_evaluation(P, R, gamma, pitr_policy, theta, init_V)
is_same_optimal_value(pitr_V_star, V_star)
```

```
True
```

We see that even for different initialization of V, we converge to the same optimal path and optimal values of V.

To-do: Repeat Exercise 1 with a random Deterministic policy

```
# Here the initial policy is chosen to be random actions. This leads
to a random Deterministic policy
init_policy = DeterministicPolicy(actions=np.random.randint(0,
len(Actions), env.num_states))

pitr_policy, pitr_V_star = policy_iteration(P, R, gamma, theta=theta,
init_policy=init_policy)
pitr_path = env.find_path(pitr_policy)
print(pitr_path)
```

```
 RIGHT RIGHT UP UP LEFT LEFT UP UP UP UP UP UP LEFT LEFT DOWN DOWN
LEFT LEFT
```

```
V_star = policy_evaluation(P, R, gamma, pitr_policy, theta) # Here the
initial V values are initialized to 0
is_same_optimal_value(pitr_V_star, V_star)
```

```
True
```

```
init_V = 10*np.ones(env.num_states)

V_star = policy_evaluation(P, R, gamma, pitr_policy, theta,
init_V=init_V) # Here the initial V values are initialized to 10
is_same_optimal_value(pitr_V_star, V_star)
```

```
True
```

```
init_V = np.random.randn(env.num_states)

V_star = policy_evaluation(P, R, gamma, pitr_policy, theta,
init_V=init_V) # Here the initial V values are initialized to random
values
is_same_optimal_value(pitr_V_star, V_star)
```

```
True
```

We see that when a random initial policy is used, we converge to the same optimal path and optimal values of V, irrespective of the initial values of V for all states.

We can infer that irrespective of the initial policy or values of V we choose we converge to the same optimal values of V for all states and the policy is greedy with respect to these optimal values fo V. The converged optimal policy is one of possibly many optimal policies (in this case they all converge to the same optimal policy).