MID SEMESTER EXAMINATION-1
IIIT, Hyderabad, Monsoon 2019
Subject: Discrete Structures
(Code: MA01.101)

INTERNATIONAL INSTITUTE OF INFORMATION TECHNOLOGY
HYDERABAD

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8:30 - 10:00

Duration: **90 minutes** | **Instructor: Dr. P. Kumar** | Maximum Marks: **20**

Instructions

There are 3 pages and 14 questions. All questions are compulsory. Calculator is not allowed. You are not allowed to ask questions in exams. If you find that there are doubts or errors in questions, then please write it in your answer script with proper justification.

Logic Theory (Implication and equivalence table on last page)

1. Write a formula which is equivalent to the formula $P \wedge (Q \leftrightarrow R)$ and contains NAND only. [1]

2. Obtain **principle disjunctive** and **conjunctive normal forms** of the following formula. [2]

$$P \vee (\neg P \rightarrow (Q \vee (\neg Q \rightarrow R))).$$

3. Write the **negation** of the statement [1]

$$(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})((x = y^2) \vee (x < 0)),$$

where $\mathbb{R}$ is the set of real numbers.

4. Explain if the following steps in the derivation is not correct. [1]
   1. $P(a) \rightarrow Q(b)$
   2. $(\exists x)(P(x) \rightarrow Q(x)), \quad EG, (1)$

5. Derive the following using **CP rule**. [2]

$$P, \quad P \rightarrow (Q \rightarrow (R \wedge S)) \implies Q \rightarrow S.$$

6. Show the following implication. [1]

$$\neg(\forall x)(P(x) \wedge Q(x)), \quad (\forall x)P(x) \implies \neg(\forall x)Q(x).$$

7. Let $P(x, y)$ denote the sentence: $x^2 + y = 1$. What are the truth values of

$$(\forall x)(\exists y)P(x, y), \quad (\forall x)(\forall y)P(x, y), \quad (\exists x)(\forall y)P(x, y),$$

and $(\exists x)(\exists y)P(x, y)$? Here $x, y \in \mathbb{R}$, the set of all real numbers. [1]

Sets, Functions, Relations

8. Let $A = \mathbb{Z}$ (set of integers) and let $\mathcal{R} = \{(x, y) \in A \times A \mid xy + x^2 = x^2 + 1\}$. We say that $x\mathcal{R}y$ iff $(x, y) \in \mathcal{R}$. Which of the following are true? [2]

   a). $0\mathcal{R}0$.
   b). $1\mathcal{R}1$.

c). $1\mathcal{R}0$.

d). $1\mathcal{R}(-2)$.

e). $3\mathcal{R}2$.

We say that $\mathcal{R}$ is

1. **symmetric** if whenever $x\mathcal{R}y$ then $y\mathcal{R}x$.
2. **reflexive** if $x\mathcal{R}x$

Is $\mathcal{R}$ defined above symmetric and reflexive? Justify.

9. Let $\mathbb{N}_+ = \mathbb{N} \cup \{0\}$, and define the function $f : \mathbb{N}_+ \times \mathbb{N}_+ \to \mathbb{N}$ by [2]

$$f(m,n) = 2^m(2n+1).$$

Prove or disprove the following:

a). f is **injective** (one-one).

b). f is **surjective** (onto).

10. For a given set X, let $|X|$ denote the **cardinality** of X. For three sets $A, B, C \subseteq U$ (U is universal set) we have the following: [2]

$$B \subseteq A, \quad C^c \cup A = U, \quad |C^c \cap A| = 7, \quad |B \Delta C| = 5,$$
$$|A \setminus (B \setminus C)| = 6, \quad |A \setminus (C \setminus B)| = 9, \quad |(B \cup C)^c| = 5.$$

Find $|A|, |B|$, and $|C|$. If the given conditions above are inconsistent, then justify.

Here X^c denotes complement of the set X, and $\setminus$ denotes **set difference**, i.e., $X \setminus Y = \{x \in X \mid x \notin Y\}$, and Δ denotes **symmetric set difference**, i.e., $X \Delta Y = (X \setminus Y) \cup (Y \setminus X)$. Also, $|X|$ denotes number of elements in set X. You may use Venn diagram.

Proof Techniques

11. Prove the following **using contradiction**. [1]

If the mean of the four distinct numbers is $n \in \mathbb{Z}$, then **at least** one of the integers is greater than $n+1$.

12. Justify your argument using **parity**. [1]

Can a knight start at $a1$ of a chess, and go to square $h8$ visiting each of the remaining squares **exactly once** on the way?

13. Justify your argument using **Pigeon-hole principle**. [1]

Given 8 different natural numbers, none greater that 15, show that at least three pairs of them have the same positive difference.

14. For a fixed natural number m, show by using **induction** on $n \in \mathbb{N} \cup \{0\}$, $n \geq 0$ that [2]

$$\frac{m!}{0!} + \frac{(m+1)!}{1!} + \cdots + \frac{(m+n)!}{n!} = \frac{(m+n+1)!}{n!(m+1)}.$$

Choose the **base case** to be $n = 0$.

Question 14 continues on the next page...

Tables: Some Implications and Equivalences

I_1	$P \wedge Q \implies P$	(simplification)
I_2	$P \wedge Q \implies Q$	(simplification)
I_3	$P \implies P \vee Q$	(addition)
I_4	$Q \implies P \vee Q$	(addition)
I_5	$\neg P \implies P \to Q$	
I_6	$Q \implies P \to Q$	
I_7	$\neg(P \to Q) \implies P$	
I_8	$\neg(P \to Q) \implies \neg Q$	
I_9	$P,\, Q \implies P \wedge Q$	
I_{10}	$\neg P,\, P \vee Q \implies Q$	(disjunctive syllogism)
I_{11}	$P, P \to Q \implies Q$	(modus ponens)
I_{12}	$\neg Q,\, P \to Q \implies \neg P$	(modus tollens)
I_{13}	$P \to Q,\, Q \to R \implies P \to R$	(hypothetical syllogism)
I_{14}	$P \vee Q,\, P \to R,\, Q \to R \implies R$	(dilema)
E_1	$\neg\neg P \iff P$	(double negation)
E_{10}	$P \vee P \iff P$	
E_{11}	$P \wedge P \iff P$	
E_{16}	$P \to Q \iff \neg P \vee Q$	
E_{18}	$P \to Q \iff \neg Q \to \neg P$	(contrapositive)
E_{25}	$\neg(\exists x)A(x) \iff (\forall x)\neg A(x)$	
E_{26}	$\neg(\forall x)A(x) \iff (\exists x)\neg A(x)$	