# MIP Relaxations of a Nonconvex MINLP Arising in Production Planning Problems with Increasing Byproducts

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We study a nonconvex optimization problem arising in production planning. In this application, the created products $p \in \mathcal{P}$ are either useful ($p \in \mathcal{P}^+$) or byproducts ($p \in \mathcal{P}^-$). The fraction of useful products created decreases monotonically during production. Conversely, the fraction of byproducts increases monotonically as a function of the total production. The production rate is limited by a concave function $f(\cdot)$ of the total production. Figure 1 gives an example of how the maximum production rate $f(\cdot)$, the useful product fraction $g_{p^+}(\cdot)$, and the byproduct fraction $g_{p^-}(\cdot)$ evolve as a function of the total production $v(t)$. In addition, there are fixed costs associated with starting production, requiring the use of {0,1}-decision variables. Production planning problems with these characteristics arise in chemical engineering applications [1, 2, 3].
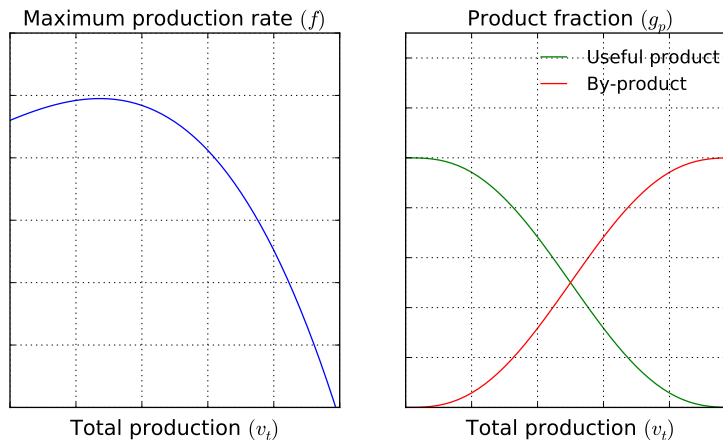


Figure 1: An illustration of the nonlinear production functions used in model (1)

The production set we study can be expressed using the following continuous-time model (1):

$$v(t) = \int_0^t x(s)\mathrm{d}s \qquad \forall t \in [0, T], \tag{1a}$$

$$x(t) \le f(v(t)) \qquad \forall t \in [0, T], \tag{1b}$$

$$y_p(t) = x(t)\, g_p(v(t)) \qquad \forall t \in [0, T], \tag{1c}$$

$$v(t) \le \mathrm{M}^v z(t) \qquad \forall t \in [0, T], \tag{1d}$$

where $x(t)$ represents the production rate at time $t \in [0, T]$, $v(t)$ is the cumulative total production up to time $t$, and $y_p(t)$ is the rate of production of product $p \in \mathcal{P}$ at time $t \in [0, T]$. To facilitate modeling of a fixed cost for beginning production at time $t \in [0, T]$, our model contains a binary step function $z : [0, T] \to \{0, 1\}$ to indicate if production can occur at time $t \in [0, T]$.

Figure 2: MIP approximations and relaxations for the discrete time MINLP formulation of (1)
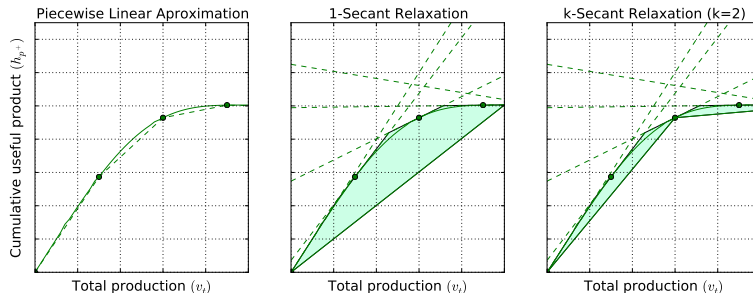


Our work makes the following three main contributions:

1. We propose a novel discrete-time Mixed Integer Nonlinear Programming (MINLP) formulation of (1). Our model uses the univariate functions $h_p(v) \overset{\text{def}}{=} \int_0^v g_p(s) \mathrm{d}s$ to define the cumulative production of product $p$ as $w_p(v(t)) = h_p(v(t))$. Since $g_p(\cdot)$ is nonincreasing for useful products $p \in \mathcal{P}^+$, and $g_p(\cdot)$ is nondecreasing for byproducts $p \in \mathcal{P}^-$, it follows that $h_p(\cdot)$ is concave for $p \in P^+$ and convex for $p \in P^-$. Our formulation has two major advantages in comparison with previous work [2]: it is a more accurate discrete time formulation of model (1), and it is more computationally tractable, since it relies on the over and under-estimation of *univariate* functions $h_p(\cdot)$ instead of bilinear functions $y_p(t) = x(t)\ g_p(v(t))$.

2. We formulate three tractable, piecewise-linear, mixed-integer programming (MIP) approximations and relaxations of our discrete time formulations (see Figure 2).

3. We introduce a technique to strengthen our new MIP models using the on-off nature of the production functions. This technique is applicable in any setting where the argument of a piecewise linear function has an associated variable upper bound constraint.

We performed numerical experiments on realistic, but randomly generated test instances to demonstrate the effectiveness of each of our contributions. First, we showed that our proposed formulation was up to 50% more accurate than past models [2]. Second, we observed that our MIP-based approximations and relaxations can be used to obtain tight solution bounds as well as near feasible solutions for the underlying MINLP. Finally, we demonstrated that our formulation strengthening technique can produce a 2-3 fold improvement in performance of the MIP solver.

# References

[1] R. R. Iyer, I. E. Grossmann, S. Vasantharajan, and A. S. Cullick, "Optimal planning and scheduling of offshore oil field infrastructure investment and operations," *Industrial & Engineering Chemistry Research*, vol. 37, no. 4, pp. 1380–1397, 1998.

[2] B. Tarhan, I. E. Grossmann, and V. Goel, "Stochastic programming approach for the planning of offshore oil or gas field infrastructure under decision-dependent uncertainty," *Industrial & Engineering Chemistry Research*, vol. 48, no. 6, pp. 3078–3097, 2009.

[3] S. A. Van den Heever and I. E. Grossmann, "An iterative aggregation/disaggregation approach for the solution of a mixed-integer nonlinear oilfield infrastructure planning model," *Industrial & Engineering Chemistry Research*, vol. 39, no. 6, pp. 1955–1971, 2000.