Question 1

Each week, Pfizer vaccines for COVID-19 arrive by air to one of two airports in Toronto: Billy Bishop Toronto City Airport (100,000 doses) or Toronto Pearson Airport (250,000 doses). They are immediately transported to 7 hospital immunization clinics and 22 city run mass vaccination sites where the city can collectively administer exactly 50,000 vaccinations per day (7 days per week).

Airport \ Vaccination Sites	**1-5**	**6-10**	**11-15**	**16-20**	**21-25**	**26-29**
Billy Bishop Toronto City Airport	\$0.05	\$0.06	\$0.07	\$0.08	\$0.09	\$0.10
Toronto Pearson Airport	\$0.08	\$0.05	\$0.09	\$0.10	\$0.07	\$0.06

Table 1: Transportation costs per dose.

The cost per dose associated with transporting vaccines from each of the airports to each of the vaccination sites is given in Table 1. Each day, the seven hospital immunization clinics can administer four times as many vaccinations as compared to the city run vaccination clinics. To ensure a feasible transportation plan, the following restrictions must be adhered to:

1. The difference between the number of doses sent from either airport to sites 1-5 combined must be within 4,800 units of each other.
2. The number of doses sent from Toronto Pearson Airport to sites 21-25 combined must be less than or equal to eight times of the doses sent from Billy Bishop airport to sites 11-15 combined.
3. The number of doses sent from Billy Bishop airport to sites 26-29 combined must be greater than or equal to 80% of the doses sent from Toronto Pearson Airport to sites 16-20 combined.

Formulate and solve a linear program to determine how many doses of vaccine should be sent from the airports to each of the 29 vaccination locations to minimize transportation costs while adhering to all constraints. Then, answer the following 10 questions below

(a) How many vaccinations can a hospital administer per week? **28000**

(b) How many doses should be sent to each city run vaccination site? **7000**

(c) How many decision variables are in the formulation? **58**

(d) What type of constraint is the first restriction? **Absolute value constraint**

(e) Write down the constraint associated with ensuring 250,000 doses are transferred from Toronto Pearson Airport to the 29 vaccination sites. $\boldsymbol{\sum_{j=1}^{29} x_{2j} = 250000}$

(f) Write down the constraint associated with the third restriction. $\boldsymbol{\sum_{j=26}^{29} x_{1j} \geq 0.8 \sum_{j=16}^{20} x_{2j}}$

(g) What is the optimal transportation cost for the week? **\$24,828**

(h) How many doses are sent from Toronto Pearson Airport to vaccination site 3? **16,400**

(i) Are both of the ratio constraints binding or not binding? **Binding**

(j) What is preventing the solution from finding a lower cost transportation plan?

The ratio and absolute value constraints. First, they are binding constraints. Second, the constraints do not necessarily have to be binding as we require the demand constraints to be binding for feasibility. Third, they represent restrictions, that if changed, would result in a lower cost without changing the transportation network.

Linear Programming Formulation: Let x_{ij} represent the number of doses sent from the $i = 2$ airports to the $j = 1, \ldots, 29$ vaccination sites (the first seven are hospitals).

$$\text{minimize} \sum_{i=1}^{2} \sum_{j=1}^{7} c_j x_{ij} \qquad \text{subject to}$$

$$\sum_{j=1}^{29} x_{1j} = 100000 \tag{1}$$

$$\sum_{j=1}^{29} x_{2j} = 250000 \tag{2}$$

$$\sum_{i=1}^{2} x_{ij} = 7000 \qquad \text{for } j = 8, ..., 29 \tag{3}$$

$$\sum_{i=1}^{2} x_{ij} = 28000 \qquad \text{for } j = 1, ..., 7 \tag{4}$$

$$\sum_{j=1}^{5} x_{1j} - \sum_{j=1}^{5} x_{2j} \leq 4800 \tag{5}$$

$$\sum_{j=1}^{5} x_{2j} - \sum_{j=1}^{5} x_{1j} \leq 4800 \tag{6}$$

$$\sum_{j=21}^{25} x_{2j} \leq 8 \sum_{j=11}^{15} x_{1j} \tag{7}$$

$$\sum_{j=26}^{29} x_{1j} \geq 0.8 \sum_{j=16}^{20} x_{2j} \tag{8}$$

$$x_{ij} \geq 0, \qquad \text{for all } i, j \tag{9}$$

The objective is to minimize the cost of storing the vaccines. Constraints (1)-(2) represent the amount that can be transferred from the airports to all the vaccination sites. Constraints (3)-(4) is the amount we must transfer directly to the city run sites and hospitals from the airport to satisfy demand. Constraints (5)-(6) represent the absolute value constraints. Constraint (7)-(8) are the two ratio constraints. Finally, (9) represents the variable bounds.

Question 2

The S&P500 is a stock market index of the 500 leading companies publicly traded in the U.S. stock market. As part of your role at a robo-advisor investment startup, your task is to design an investment portfolio based on 67 companies listed on the S&P500. The goal is to maximize the expected 1-year return (see the spreadsheet *sp500_data.xlsx*) subject to a total investment of $10 million. The portfolio must satisfy a number of restrictions to ensure it is sufficiently diversified.

- At most, $600,000 can be invested in any individual stock (for diversification purposes).
- No more than $500,000 can be invested in the Telecommunications sector.
- The amount invested in the Information Technology (IT) sector must be at least 75% the amount invested in the Telecommunications sector.
- The absolute difference between the total invested in the Consumer Discretionary sector and the Consumer Staples sector should not exceed $200,000 (complementary trends).
- At least $1 million must be invest in the Energy sector and at least $300,000 must be invested in companies headquartered in New York, New York.

Formulate and solve a linear program to design an optimal investment portfolio that maximizes the total expected return and satisfies all restrictions. Then, answer the following 10 questions below.

(a) How many companies are headquartered in New York City? **6**

(b) How many decision variables are there? **67**

(c) What does a decision variable represent? Given this definition, write down the constraint associated with the restriction that, at most, $600,000 can be invested in any individual stock. **A decision variable represents how much money is invested in company $i = 1, ..., 67$. Then, $x_i \leq 600000 \quad \forall i$**

(d) How much is invested in companies headquartered in NYC? **$600,000**

(e) What is the optimal expected return of the portfolio after 1 year? **$513,460**

(f) After seeing your report, a colleague has asked if it would be worth it to inquire whether the amount invested in the Energy sector could be reduced. If the shadow price associated with the Energy sector constraint is -1.31%, what would your answer be?

The shadow price of the Energy sector constraint is negative but it's actually -2.14% (I used slightly different numbers so you would have to actually formulate and solve questions 2a-2d). Regardless, if either -2.14% or -1.31% is used, any *decrease* in the right-hand side of that constraint would *increase* the objective function value, which represents the expected return of the portfolio. Yes, <u>reduce it!</u>

(g) It was just announced that the return for Coca-Cola Enterprises is actually 3.00%. Without re-solving the problem, if the allowable increase/decrease for this objective function coefficient is 2.4%/-Infinity, would you change your portfolio if you had this new return information?

The original return of Coca-Cola Enterprises is 2.02% while the new return is 3.00%. Thus, the objective function coefficient has increased by $3.00\% - 2.02\% = 0.98\%$. This is within the allowable increase for the objective function coefficient ($4.42\% - 2.02\% = 2.40\%$ in the context of this problem and $4.5\% - 2.02\% = 2.48\%$ using the numbers in the sensitivity report). No, the optimal portfolio would <u>not change</u>.

(h) Due to the announcement of a new version of a popular smartphone, one of your colleagues has proposed increasing the maximum amount invested in the Telecommunications sector to \$625,000. If the shadow price associated with the Telecommunications sector constraint is 9.85% and the allowable increase/decrease is $\$100,000/\$42,857.14$, is this a good decision? Without re-solving the problem, would the announcement change the composition of the optimal portfolio?

Yes, it is worth it. The fake shadow price used for this question (9.85%) and the actual shadow price of the constraint (9.78%) are both positive. Thus, increasing the right-hand-side of the constraint will increase the optimal expected return. The change is beyond the allowable increase of the Telecommunications constraint ($\$625,000 > \$100,000$); the optimal allocation of funds will change.

(i) The bank is running a special promotion and will let you borrow, at most, \$50,000. This must be paid back in exactly one one year but the 1.5% interest rate is extremely favorable. Assume that all borrowed money is used for investing and you receive the estimated returns in exactly one year. If the shadow price of the total investment constraint is 4.32% with an allowable increase/decrease of $\$75,000/\$525,000$, how much would you be willing to borrow, if anything?

The allowable increase is higher than the maximum amount we can borrow ($\$50,000$). If we borrow all of it, our increase in profit would be

$$4.36\% \times 50000 - 1.5\% \times 50000 > 0(\textbf{real})$$
$$4.32\% \times 50000 - 1.5\% \times 50000 > 0(\textbf{fake})$$

In both cases, the return on the investment is higher than the interest rate. Therefore, it makes sense to borrow from the bank and use the promotion.

(j) A company in New York City has attracted your attention by stating that it will revolutionize the Energy sector. However, with the help of a colleague based in New York City, you have estimated its yield to be 6%. Does it make sense to invest in this new opportunity?

We need to compute the reduced cost of the new variable corresponding to this novel investment option. Notice that this variable will be added to the New York City constraint (shadow price of zero), the total amount constraint (shadow price of 4.32%/4.36%), and the Energy sector constraint (shadow price of −1.31%/−2.14%):

$$\text{reduced cost} = 6 - (1 \times 0 + 1 \times 4.32\% - 1 \times 1.31\%) = 2.99\%(\textbf{real})$$
$$\text{reduced cost} = 6 - (1 \times 0 + 1 \times 4.36\% - 1 \times 2.14\%) = 3.78\%(\textbf{fake})$$

Since the reduced cost is positive, we will buy some stock in the company.

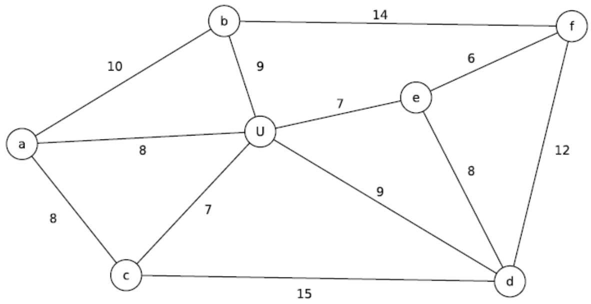
Linear Programming Formulation: Let x_i represent a decision variable that indicates how much money is invested in company i, for $i = 1, 2, \ldots, 67$. Let r_i be the return of stock i. Also, let T, I, D, S, and E be the set of indices corresponding to companies in the Telecommunications, IT, Consumer Discretionary, Consumer Staple, and Energy sectors, respectively. Finally, let NY be the indices of companies headquartered in New York city.

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{67} (r_i\, x_i)/100 && \text{subject to} \\
& \sum_{i=1}^{67} x_i = 10000000 && \text{(Budget)} \\
& \sum_{i \in T} x_i \leq 500000 && \text{(Telecommunications Restriction)} \\
& \sum_{i \in I} x_i \geq 0.75 \sum_{i \in T} x_i && \text{(IT and Telecom Restriction)} \\
& \sum_{i \in D} x_i - \sum_{i \in S} x_i \leq 200000 && \text{(Discretionary/Staple 1)} \\
& \sum_{i \in S} x_i - \sum_{i \in D} x_i \leq 200000 && \text{(Discretionary/Staple 2)} \\
& \sum_{i \in E} x_i \geq 1000000 && \text{(Energy Sector)} \\
& \sum_{i \in NY} x_i \geq 300000 && \text{(NYC)} \\
& x_i \leq 600000, \quad \text{for } i = 1, 2, \ldots, 67 && \text{(Limit per stock)} \\
& x_i \geq 0 \quad \text{for } i = 1, 2, \ldots, 67 && \text{(Non-negativity)}
\end{aligned}
$$

The objective is to maximize the expected return on the investments. The constraints represent the total budget and the various restrictions as outlined in the question.

Question 3

York University is planning to introduce a shuttle service between the Keele (U), Markham (f), and Glendon (a) campuses as well as a few other other stops. The shuttle can only travel directly from one stop to another if there is an arc between them in the network (see diagram below). All connections are two-way streets and arc labels represent the distance between stops (km).



Transportation analysts are considering 58 routes (see file *Routes.csv* for the list of routes and their maintenance cost per day) where the shuttle begins at the Keele campus (U), stops at several locations, and then returns back to the Keele campus (U). The following two factors must also be considered.

1. If the Glendon campus (a) is visited by more than one shuttle service, where the shuttles are serving different routes, an extra \$350 per day is incurred. Note that this cost is incurred for each *pair* of shuttles that visit the Glendon campus. For instance, if shuttle 1 and 2 visit the Glendon campus, an extra \$350 is incurred. If, instead, shuttles 1, 2, and 3 visit the campus, then there are three pairs (1 and 2, 1 and 3, 2 and 3). Thus, the total cost is $3 \times \$350 = \$1050$.

2. The student association will subsidize the project. That is, they will pay \$50 per day for each stop that is served by at least three routes (this does not include the Keele stop).

Formulate and solve a *linear* programming model with only binary decision variables to select a minimum-cost set of routes such that all stops are served by at least one route and the two restrictions discussed above are adhered to. Then, answer the following 10 questions:

(a) What route, among the 58 that are under consideration, has the largest cost? **35 or 51**

(b) If the cost per km is \$2.00, what is the smallest cost of a route that begins at the Keele campus (U), then visits the Glendon campus (a), the Markham campus (f) and then returns to the Keele campus (in that order)? Note that other stops may also be served in this route. **\$90**

(c) If no subsidy were to be provided by the student association, how many routes would be selected? **1 (there exists a single route that can serve all stops).**

(d) Write down the term in the objective associated with the student association subsidy.

$-\sum_{j=1}^{6} 50z_i$

(e) Write down the nonlinear term, associated with the Glendon campus (a) being visited by more than one shuttle service (i.e., the first restriction), *before* it is linearized.

$350x_ix_j$ for all $i, j \in A$ where $i > j$.

(f) Write down constraints that linearize the model.

$y_{ij} \leq x_i, y_{ij} \leq x_j,$ and $y_{ij} \geq x_i + x_j - 1, \quad \forall i, j \in A, i > j$

(g) How many decision variables are needed in the linear program?

Depending on the constraints included in the model, at least 844 (58 + 6 + $\sum_{n=1}^{40} n$), most likely, 1664 (58 + 6 + 40 × 40), and at most 3428 (58 + 6 + 58 × 58).

(h) What is minimum cost of serving all stops and satisfying the restrictions? **108**

(i) In the optimal solution, how many routes are selected? **3**

(j) In the optimal solution, how many routes stop at the Glendon campus? **1**

Linear Programming Formulation: Define A, B, C, D, E and F as sets associated with routes that serve stops a, b, c, d, e and f, respectively. Let x_i be a binary variable that equals one if route $i = 1, ..., 58$ is selected and zero otherwise. Let y_{ij} be a binary variable if route i and route j both visit the Glendon campus, i.e., i and j are both in set A for $i \neq j$, and zero otherwise. We also define z_k to be a binary variable that equals one if at least three routes serve stop $k = 1, ..., 6$ and zero otherwise. Finally, let W_i be the cost of selecting route $i = 1, ..., 58$.

$$
\begin{array}{lll}
\min & \sum_{i=1}^{58} W_i x_i + \sum_{i,j,\in A, i>j} 350 y_{ij} - \sum_{j=1}^{6} 50 z_i & \text{subject to} \\
& \sum_{i \in A} x_i \geq 1 + 2z_1 & \text{(stop a)} \\
& \sum_{i \in B} x_i \geq 1 + 2z_2 & \text{(stop b)} \\
& \sum_{i \in C} x_i \geq 1 + 2z_3 & \text{(stop c)} \\
& \sum_{i \in D} x_i \geq 1 + 2z_4 & \text{(stop d)} \\
& \sum_{i \in E} x_i \geq 1 + 2z_5 & \text{(stop e)} \\
& \sum_{i \in F} x_i \geq 1 + 2z_6 & \text{(stop f)} \\
& y_{ij} \leq x_i & \forall i, j \in A, i > j \\
& y_{ij} \leq x_j & \forall i, j \in A, i > j \\
& y_{ij} \geq x_i + x_j - 1 & \forall i, j \in A, i > j \\
& y_{ij} = 0 & \forall i, j \in A, i \leq j \\
& x_i \in \{0, 1\} & \text{for } i = 1, \dots, 58 \\
& y_{ij} \in \{0, 1\} & \text{for } i \neq j, i, j \in A \\
& z_k \in \{0, 1\} & \text{for } k = 1, \dots, 6
\end{array}
$$

Note that, in order to ensure we don't double count pairs of shuttles, we constrain $y_{ij} = 0$ for all $i, j \in A$ where $i \leq j$ (*note:* there are multiple correct ways to do this). This has two effects:

1. For all $i \leq j$, it ensures $y_{ij} = 0$, and as a result, only one of x_i and x_j can equal to one.
2. For all $i > j$, $y_{ij} \geq 0$ which means that both x_i and x_j can equal to one.

Nevertheless, even if you iterate over the set A incorrectly, you will still get the same answer although the optimal objective function value may differ depending on whether the objective function is correct.