Mathematical formulation of linear instability problem of a viscous incompressible liquid round jet in a gaseous medium.

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March 29, 2016

1 Governing Equations.

The continuity equation in polar cylindrical coordinates for the liquid phase is given by

$$\frac{1}{r}\frac{\partial(ru_{r1})}{\partial r}+\frac{1}{r}\frac{\partial u_{\theta 1}}{\partial \theta}+\frac{\partial u_{z1}}{\partial z}=0 \tag{1}$$

For the gas phase,

$$\frac{1}{r}\frac{\partial(ru_{r2})}{\partial r}+\frac{1}{r}\frac{\partial u_{\theta 2}}{\partial \theta}+\frac{\partial u_{z2}}{\partial z}=0 \tag{2}$$

The non dimensionalised momentum equation in the r direction (radial direction) is given by

$$\frac{\partial u_{r1}}{\partial t}+u_{r1}\frac{\partial u_{r1}}{\partial r}+\frac{u_{\theta 1}}{r}\frac{\partial u_{r1}}{\partial \theta}-\frac{u_{\theta 1}^2}{r}+u_{z1}\frac{\partial u_{r1}}{\partial z}=-\frac{\partial p}{\partial r}+\frac{1}{Re_1}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_{r1}}{\partial r}\right)-\frac{u_{r1}}{r^2}+\frac{1}{r^2}\frac{\partial^2 u_{r1}}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial u_{\theta 1}}{\partial \theta}+\frac{\partial^2 u_{r1}}{\partial z^2}\right] \tag{3}$$

For the gas phase,

$$\frac{\partial u_{r2}}{\partial t}+u_{r2}\frac{\partial u_{r2}}{\partial r}+\frac{u_{\theta 2}}{r}\frac{\partial u_{r2}}{\partial \theta}-\frac{u_{\theta 2}^2}{r}+u_{z2}\frac{\partial u_{r2}}{\partial z}=-\frac{\partial p}{\partial r}+\frac{1}{Re_2}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_{r2}}{\partial r}\right)-\frac{u_{r2}}{r^2}+\frac{1}{r^2}\frac{\partial^2 u_{r2}}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial u_{\theta 2}}{\partial \theta}+\frac{\partial^2 u_{r2}}{\partial z^2}\right] \tag{4}$$

In the azimuthal direction (θ direction),

$$\frac{\partial u_{\theta 1}}{\partial t}+u_{r1}\frac{\partial u_{\theta 1}}{\partial r}+\frac{u_{\theta 1}}{r}\frac{\partial u_{\theta 1}}{\partial \theta}+\frac{u_{\theta 1}u_{r1}}{r}+u_{z1}\frac{\partial u_{\theta 1}}{\partial z}=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\frac{1}{Re_1}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_{\theta 1}}{\partial r}\right)-\frac{u_{\theta 1}}{r^2}+\frac{1}{r^2}\frac{\partial^2 u_{\theta 1}}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial u_{r1}}{\partial \theta}+\frac{\partial^2 u_{\theta 1}}{\partial z^2}\right] \tag{5}$$

For the gas phase,

$$\frac{\partial u_{\theta 2}}{\partial t}+u_{r2}\frac{\partial u_{\theta 2}}{\partial r}+\frac{u_{\theta 2}}{r}\frac{\partial u_{\theta 2}}{\partial \theta}+\frac{u_{\theta 2}u_{r2}}{r}+u_{z2}\frac{\partial u_{\theta 2}}{\partial z}=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\frac{1}{Re_2}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_{\theta 2}}{\partial r}\right)-\frac{u_{\theta 2}}{r^2}+\frac{1}{r^2}\frac{\partial^2 u_{\theta 2}}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial u_{r2}}{\partial \theta}+\frac{\partial^2 u_{\theta 2}}{\partial z^2}\right] \tag{6}$$

In the axial direction (z direction) for the liquid phase

$$\frac{\partial u_{z1}}{\partial t}+u_{r1}\frac{\partial u_{z1}}{\partial r}+\frac{u_{\theta 1}}{r}\frac{\partial u_{z1}}{\partial \theta}+u_{z1}\frac{\partial u_{z1}}{\partial z}=-\frac{\partial p}{\partial z}+\frac{1}{Re_1}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_{z1}}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_{z1}}{\partial \theta^2}+\frac{\partial^2 u_{z1}}{\partial z^2}\right] \tag{7}$$

For the gas phase,

$$\frac{\partial u_{z2}}{\partial t}+u_{r2}\frac{\partial u_{z2}}{\partial r}+\frac{u_{\theta 2}}{r}\frac{\partial u_{z2}}{\partial \theta}+u_{z2}\frac{\partial u_{z2}}{\partial z}=-\frac{\partial p}{\partial z}+\frac{1}{Re_2}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_{z2}}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_{z2}}{\partial \theta^2}+\frac{\partial^2 u_{z2}}{\partial z^2}\right] \tag{8}$$

where u_r , u_θ and u_z are the velocity components in the radial, azimuthal and axial directions respectively.

2 Base flow.

The flow variables are assumed to consist of a mean part and an infinitesimally small perturbation.

$$\begin{aligned} u_r &= U_r + u_r'(r,\theta,z,t) \\ u_\theta &= U_\theta + u_\theta'(r,\theta,z,t) \\ u_z &= U_z + u_z'(r,\theta,z,t) \end{aligned} \tag{9}$$

The base flow is taken to be an incompressible viscous axisymmetric one with locally parallel flow assumption. The base flow quantities are

$$\begin{aligned} U_{r1} &= U_{r1}(r) \\ U_{r2} &= U_{r2}(r) \\ U_\theta &= 0 \\ U_{z1} &= U_{z1}(r) \\ U_{z2} &= U_{z2}(r) \\ P &= 0 \end{aligned} \tag{10}$$

2.1 Normal mode form of perturbation.

The flow is periodic in azimuthal and axial direction. We assume perturbations of the form

$$[u_r, u_\theta, u_z, p] = [i\tilde{u}_r(r), \tilde{u}_\theta(r), \tilde{u}_z(r), \tilde{p}(r)]e^{i(\alpha z + m\theta - \omega t)} \tag{11}$$

It is to be noted that $u_r \propto i\tilde{u}_r$. This can be readily seen from the contunity equation, the phase of u_r differs by $\frac{\pi}{2}$ from that of u_θ and u_r. Substituting eq(11) into eq(3), eq(4), eq(5), eq(6), eq(7), eq(8) , eq(1) and eq(2), we have, In the radial direction,

$$-i\frac{\tilde{u}_{r1}''}{Re_1}+i\left(U_{r1}-\frac{1}{rRe_1}\right)\tilde{u}_{r1}'+\left[\omega+iU_{r1}'-\alpha U_{z1}+\frac{i}{Re_1}\left(\frac{m^2+1}{r^2}+\alpha^2\right)\right]\tilde{u}_{r1}+\frac{2im}{r^2Re_1}\tilde{u}_{\theta 1}+\tilde{p}'=0 \tag{12}$$

$$-i\frac{\tilde{u}_{r2}''}{Re_2}+i\left(U_{r2}-\frac{1}{rRe_2}\right)\tilde{u}_{r2}'+\left[\omega+iU_{r2}'-\alpha U_{z2}+\frac{i}{Re_2}\left(\frac{m^2+1}{r^2}+\alpha^2\right)\right]\tilde{u}_{r2}+\frac{2im}{r^2Re_2}\tilde{u}_{\theta 2}+\tilde{p}'=0 \tag{13}$$

In the azimuthal direction,

$$-\frac{\tilde{u}_{\theta 1}''}{Re_1}+\left(U_{r1}-\frac{1}{rRe_1}\right)\tilde{u}_{\theta 1}'+\left[-i\omega+i\alpha U_{z1}+\frac{U_{r1}}{r}+\frac{1}{Re_1}\left(\frac{m^2+1}{r^2}+\alpha^2\right)\right]\tilde{u}_{\theta 1}+\left(\frac{2m}{r^2Re_1}\right)\tilde{u}_{r1}+\frac{im\tilde{p}}{r}=0 \tag{14}$$

$$-\frac{\tilde{u}''_{\theta 2}}{Re_2} + \left(U_{r2} - \frac{1}{rRe_2}\right)\tilde{u}'_{\theta 2} + \left[-i\omega + i\alpha U_{z2} + \frac{U_{r2}}{r} + \frac{1}{Re_2}\left(\frac{m^2+1}{r^2} + \alpha^2\right)\right]\tilde{u}_{\theta 2} + \left(\frac{2m}{r^2 Re_2}\right)\tilde{u}_{r2} + \frac{im\tilde{p}}{r} = 0 \qquad (15)$$

In the axial direction,

$$-\frac{\tilde{u}''_{z1}}{Re_1} + \left(U_{r1} - \frac{1}{rRe_1}\right)\tilde{u}'_{z1} + \left[-i\omega + i\alpha U_{z1} + \frac{1}{Re_1}\left(\frac{m^2}{r^2} + \alpha^2\right)\right]\tilde{u}_{z1} + iU'_{z1}\tilde{u}_{r1} + i\alpha\tilde{p} = 0 \qquad (16)$$

$$-\frac{\tilde{u}''_{z2}}{Re_2} + \left(U_{r2} - \frac{1}{rRe_2}\right)\tilde{u}'_{z2} + \left[-i\omega + i\alpha U_{z2} + \frac{1}{Re_2}\left(\frac{m^2}{r^2} + \alpha^2\right)\right]\tilde{u}_{z2} + iU'_{z2}\tilde{u}_{r2} + i\alpha\tilde{p} = 0 \qquad (17)$$

Continuity equation yields,

$$\tilde{u}'_{r1} + \frac{\tilde{u}_{r1}}{r} + \frac{m\tilde{u}_{\theta 1}}{r} + \alpha\tilde{u}_{z1} = 0 \qquad (18)$$

$$\tilde{u}'_{r2} + \frac{\tilde{u}_{r2}}{r} + \frac{m\tilde{u}_{\theta 2}}{r} + \alpha\tilde{u}_{z2} = 0 \qquad (19)$$

3 Boundary Conditions.

Due to the singular nature of the coordinate system on the centerline, all physical quantities must be smooth and bounded at r = 0 .Therefore as $r \to 0, \lim_{r\to 0} \frac{\partial \mathbf{V}}{\partial \theta} = 0$.(20)

$$\lim_{r\to 0} \frac{\partial p'}{\partial \theta} = 0 \qquad (21)$$

where $\mathbf{V}$ is the total velocity vector. the above limits represents the boundedness and smoothness conditions on the solutions along the centerline.(Batchelor and Gill 1962, also Khorrami et al.,JCP 81,206-229 (1989)). while expanding the limits, we need to consider only the perturbation part of the velocity since the mean flow is independent of z and θ

$$\lim_{r\to 0} \frac{\partial \mathbf{V}}{\partial \theta} = \lim_{r\to 0} \frac{\partial}{\partial \theta}(u_{r1}\mathbf{e_r} + u_{\theta 1}\mathbf{e}_\theta + u_{z1}\mathbf{e_z}) = 0$$

But

$$\frac{\partial \mathbf{e_z}}{\partial \theta} = 0; \frac{\partial \mathbf{e_r}}{\partial \theta} = \mathbf{e}_\theta; \frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\mathbf{e_r} \qquad (22)$$

$$-(m\tilde{u}_{r1})\mathbf{e_r} + i(\tilde{u}_{r1} + m\tilde{u}_{\theta 1})\mathbf{e}_\theta + in\tilde{u}_{z1}\mathbf{e_z} = 0$$

$$imp = 0$$

In order for the inequality to hold, each component of the resultant vector must be zero. This gives,

$$m\tilde{u}_{r1} = 0 \qquad (23)$$

$$\tilde{u}_{r1} + m\tilde{u}_{\theta 1} = 0 \qquad (24)$$

$$m\tilde{u}_{z1} = 0 \qquad (25)$$

$$mp = 0 \qquad (26)$$

For axisymmetric perturbation $m = 0$,

$$\tilde{u}_{r1}(0) = \tilde{u}_{\theta 1}(0) = 0 \qquad (27)$$

$\tilde{u}_{z1}(0)$ and $\tilde{p}(0)$ must be finite. and for $m = \pm 1$

$$\tilde{u}_{r1}(0) \pm \tilde{u}_{\theta 1}(0) = 0 \tag{28}$$

$$\tilde{u}_{z1}(0) = \tilde{p}(0) = 0 \tag{29}$$

if $|m| > 1$,

$$\tilde{u}_{z1}(0) = \tilde{p}(0) = 0 \tag{30}$$

$$\tilde{u}_{r1}(0) = \tilde{u}_{\theta 1}(0) = 0 \tag{31}$$

In case when $|m| = 1$, continuity equation is applied on the centerline with $\tilde{u}_z = 0$ which gives,

$$2\tilde{u}'_{r1}(0) + m\tilde{u}'_{\theta 1}(0) = 0 \tag{32}$$

3.1 Matching Conditions at the interface.

Let

$$\phi = h(z, \theta, t) - r \tag{33}$$

so that the 0 level set of ϕ describes the interface between the liquid jet and the gaseous medium. The Kinematic condition that the interface is a material surface gives , $\frac{D\phi}{Dt} = 0$

$$\frac{\partial h}{\partial t} + \frac{u_\theta}{r}\frac{\partial h}{\partial \theta} + u_z \frac{\partial h}{\partial z} = \left(\frac{\partial r}{\partial t}\right)_{r=R} \tag{34}$$