

Mathematical formulation of linear instability problem of a viscous incompressible liquid round jet in a gaseous medium.

Srikumar Warrier

April 27, 2016

1 Governing Equations.

The continuity equation in polar cylindrical coordinates for the liquid phase is given by

$$\frac{1}{r} \frac{\partial(ru_{r1})}{\partial r} + \frac{1}{r} \frac{\partial u_{\theta 1}}{\partial \theta} + \frac{\partial u_{z1}}{\partial z} = 0 \quad (1)$$

For the gas phase,

$$\frac{1}{r} \frac{\partial(ru_{r2})}{\partial r} + \frac{1}{r} \frac{\partial u_{\theta 2}}{\partial \theta} + \frac{\partial u_{z2}}{\partial z} = 0 \quad (2)$$

The non dimensionalised momentum equation in the r direction (radial direction) is given by

$$\frac{\partial u_{r1}}{\partial t} + u_{r1} \frac{\partial u_{r1}}{\partial r} + \frac{u_{\theta 1}}{r} \frac{\partial u_{r1}}{\partial \theta} - \frac{u_{\theta 1}^2}{r} + u_{z1} \frac{\partial u_{r1}}{\partial z} = -\frac{\partial p}{\partial r} + \frac{1}{Re_1} \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_{r1}}{\partial r} \right) - \frac{u_{r1}}{r^2} + \frac{1}{r^2} \frac{\partial^2 u_{r1}}{\partial \theta^2} - \frac{2}{r^2} \frac{\partial u_{\theta 1}}{\partial \theta} + \frac{\partial^2 u_{r1}}{\partial z^2} \right] \quad (3)$$

For the gas phase,

$$\frac{\partial u_{r2}}{\partial t} + u_{r2} \frac{\partial u_{r2}}{\partial r} + \frac{u_{\theta 2}}{r} \frac{\partial u_{r2}}{\partial \theta} - \frac{u_{\theta 2}^2}{r} + u_{z2} \frac{\partial u_{r2}}{\partial z} = -\frac{\partial p}{\partial r} + \frac{1}{Re_2} \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_{r2}}{\partial r} \right) - \frac{u_{r2}}{r^2} + \frac{1}{r^2} \frac{\partial^2 u_{r2}}{\partial \theta^2} - \frac{2}{r^2} \frac{\partial u_{\theta 2}}{\partial \theta} + \frac{\partial^2 u_{r2}}{\partial z^2} \right] \quad (4)$$

In the azimuthal direction (θ direction),

$$\frac{\partial u_{\theta 1}}{\partial t} + u_{r1} \frac{\partial u_{\theta 1}}{\partial r} + \frac{u_{\theta 1}}{r} \frac{\partial u_{\theta 1}}{\partial \theta} + \frac{u_{\theta 1} u_{r1}}{r} + u_{z1} \frac{\partial u_{\theta 1}}{\partial z} = -\frac{1}{r} \frac{\partial p}{\partial \theta} + \frac{1}{Re_1} \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_{\theta 1}}{\partial r} \right) - \frac{u_{\theta 1}}{r^2} + \frac{1}{r^2} \frac{\partial^2 u_{\theta 1}}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial u_{r1}}{\partial \theta} + \frac{\partial^2 u_{\theta 1}}{\partial z^2} \right] \quad (5)$$

For the gas phase,

$$\frac{\partial u_{\theta 2}}{\partial t} + u_{r2} \frac{\partial u_{\theta 2}}{\partial r} + \frac{u_{\theta 2}}{r} \frac{\partial u_{\theta 2}}{\partial \theta} + \frac{u_{\theta 2} u_{r2}}{r} + u_{z2} \frac{\partial u_{\theta 2}}{\partial z} = -\frac{1}{r} \frac{\partial p}{\partial \theta} + \frac{1}{Re_2} \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_{\theta 2}}{\partial r} \right) - \frac{u_{\theta 2}}{r^2} + \frac{1}{r^2} \frac{\partial^2 u_{\theta 2}}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial u_{r2}}{\partial \theta} + \frac{\partial^2 u_{\theta 2}}{\partial z^2} \right] \quad (6)$$

In the axial direction (z direction) for the liquid phase

$$\frac{\partial u_{z1}}{\partial t} + u_{r1} \frac{\partial u_{z1}}{\partial r} + \frac{u_{\theta 1}}{r} \frac{\partial u_{z1}}{\partial \theta} + u_{z1} \frac{\partial u_{z1}}{\partial z} = -\frac{\partial p}{\partial z} + \frac{1}{Re_1} \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_{z1}}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u_{z1}}{\partial \theta^2} + \frac{\partial^2 u_{z1}}{\partial z^2} \right] \quad (7)$$

For the gas phase,

$$\frac{\partial u_{z2}}{\partial t} + u_{r2} \frac{\partial u_{z2}}{\partial r} + \frac{u_{\theta 2}}{r} \frac{\partial u_{z2}}{\partial \theta} + u_{z2} \frac{\partial u_{z2}}{\partial z} = -\frac{\partial p}{\partial z} + \frac{1}{Re_2} \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_{z2}}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u_{z2}}{\partial \theta^2} + \frac{\partial^2 u_{z2}}{\partial z^2} \right] \quad (8)$$

where u_r , u_θ and u_z are the velocity components in the radial, azimuthal and axial directions respectively.

2 Base flow.

The flow variables are assumed to consist of a mean part and an infinitesimally small perturbation.

$$\begin{aligned} u_r &= U_r + u'_r(r, \theta, z, t) \\ u_\theta &= U_\theta + u'_\theta(r, \theta, z, t) \\ u_z &= U_z + u'_z(r, \theta, z, t) \end{aligned} \quad (9)$$

The base flow is taken to be an incompressible viscous axisymmetric one with locally parallel flow assumption. The base flow quantities are

$$\begin{aligned} U_{r1} &= U_{r1}(r) \\ U_{r2} &= U_{r2}(r) \\ U_\theta &= 0 \\ U_{z1} &= U_{z1}(r) \\ U_{z2} &= U_{z2}(r) \\ P &= 0 \end{aligned} \quad (10)$$

2.1 Normal mode form of perturbation.

The flow is periodic in azimuthal and axial direction. We assume perturbations of the form

$$[u_r, u_\theta, u_z, p] = [\tilde{u}_r(r), \tilde{u}_\theta(r), \tilde{u}_z(r), \tilde{p}(r)] e^{i(\alpha z + m\theta - \omega t)} \quad (11)$$

It is to be noted that $u_r \propto i\tilde{u}_r$. This can be readily seen from the continuity equation, the phase of u_r differs by $\frac{\pi}{2}$ from that of u_θ and u_z . Substituting eq(11) into eq(3), eq(4), eq(5), eq(6), eq(7), eq(8), eq(1) and eq(2), we have, In the radial direction,

$$-i \frac{\tilde{u}_{r1}''}{Re_1} + i \left(U_{r1} - \frac{1}{r Re_1} \right) \tilde{u}_{r1}' + \left[\omega + i U_{r1}' - \alpha U_{z1} + \frac{i}{Re_1} \left(\frac{m^2 + 1}{r^2} + \alpha^2 \right) \right] \tilde{u}_{r1} + \frac{2im}{r^2 Re_1} \tilde{u}_{\theta 1} + \tilde{p}' = 0 \quad (12)$$

$$-i \frac{\tilde{u}_{r2}''}{Re_2} + i \left(U_{r2} - \frac{1}{r Re_2} \right) \tilde{u}_{r2}' + \left[\omega + i U_{r2}' - \alpha U_{z2} + \frac{i}{Re_2} \left(\frac{m^2 + 1}{r^2} + \alpha^2 \right) \right] \tilde{u}_{r2} + \frac{2im}{r^2 Re_2} \tilde{u}_{\theta 2} + \tilde{p}' = 0 \quad (13)$$

In the azimuthal direction,

$$-\frac{\tilde{u}_{\theta 1}''}{Re_1} + \left(U_{r1} - \frac{1}{r Re_1} \right) \tilde{u}_{\theta 1}' + \left[-i\omega + i\alpha U_{z1} + \frac{U_{r1}}{r} + \frac{1}{Re_1} \left(\frac{m^2 + 1}{r^2} + \alpha^2 \right) \right] \tilde{u}_{\theta 1} + \left(\frac{2m}{r^2 Re_1} \right) \tilde{u}_{r1} + \frac{im\tilde{p}}{r} = 0 \quad (14)$$

$$-\frac{\tilde{u}_{\theta 2}''}{Re_2} + \left(U_{r2} - \frac{1}{rRe_2}\right)\tilde{u}_{\theta 2}' + \left[-i\omega + i\alpha U_{z2} + \frac{U_{r2}}{r} + \frac{1}{Re_2}\left(\frac{m^2+1}{r^2} + \alpha^2\right)\right]\tilde{u}_{\theta 2} + \left(\frac{2m}{r^2 Re_2}\right)\tilde{u}_{r2} + \frac{im\tilde{p}}{r} = 0 \quad (15)$$

In the axial direction,

$$-\frac{\tilde{u}_{z1}''}{Re_1} + \left(U_{r1} - \frac{1}{rRe_1}\right)\tilde{u}_{z1}' + \left[-i\omega + i\alpha U_{z1} + \frac{1}{Re_1}\left(\frac{m^2}{r^2} + \alpha^2\right)\right]\tilde{u}_{z1} + iU_{z1}'\tilde{u}_{r1} + i\alpha\tilde{p} = 0 \quad (16)$$

$$-\frac{\tilde{u}_{z2}''}{Re_2} + \left(U_{r2} - \frac{1}{rRe_2}\right)\tilde{u}_{z2}' + \left[-i\omega + i\alpha U_{z2} + \frac{1}{Re_2}\left(\frac{m^2}{r^2} + \alpha^2\right)\right]\tilde{u}_{z2} + iU_{z2}'\tilde{u}_{r2} + i\alpha\tilde{p} = 0 \quad (17)$$

Continuity equation yields,

$$\tilde{u}_{r1}' + \frac{\tilde{u}_{r1}}{r} + \frac{m\tilde{u}_{\theta 1}}{r} + \alpha\tilde{u}_{z1} = 0 \quad (18)$$

$$\tilde{u}_{r2}' + \frac{\tilde{u}_{r2}}{r} + \frac{m\tilde{u}_{\theta 2}}{r} + \alpha\tilde{u}_{z2} = 0 \quad (19)$$

3 Boundary Conditions.

Due to the singular nature of the coordinate system on the centerline, all physical quantities must be smooth and bounded at $r = 0$. Therefore as $r \rightarrow 0$, $\lim_{r \rightarrow 0} \frac{\partial \mathbf{V}}{\partial \theta} = 0$. (20)

$$\lim_{r \rightarrow 0} \frac{\partial p'}{\partial \theta} = 0 \quad (21)$$

where \mathbf{V} is the total velocity vector. the above limits represents the boundedness and smoothness conditions on the solutions along the centerline. (Batchelor and Gill 1962, also Khorrami et al., JCP 81,206-229 (1989)). while expanding the limits, we need to consider only the perturbation part of the velocity since the mean flow is independent of z and θ

$$\lim_{r \rightarrow 0} \frac{\partial \mathbf{V}}{\partial \theta} = \lim_{r \rightarrow 0} \frac{\partial}{\partial \theta} (u_{r1}\mathbf{e}_r + u_{\theta 1}\mathbf{e}_\theta + u_{z1}\mathbf{e}_z) = 0$$

But

$$\begin{aligned} \frac{\partial \mathbf{e}_z}{\partial \theta} &= 0; \quad \frac{\partial \mathbf{e}_r}{\partial \theta} = \mathbf{e}_\theta; \quad \frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\mathbf{e}_r \\ -(m\tilde{u}_{r1})\mathbf{e}_r + i(\tilde{u}_{r1} + m\tilde{u}_{\theta 1})\mathbf{e}_\theta + in\tilde{u}_{z1}\mathbf{e}_z &= 0 \\ imp &= 0 \end{aligned} \quad (22)$$

In order for the inequality to hold, each component of the resultant vector must be zero. This gives,

$$m\tilde{u}_{r1} = 0 \quad (23)$$

$$\tilde{u}_{r1} + m\tilde{u}_{\theta 1} = 0 \quad (24)$$

$$m\tilde{u}_{z1} = 0 \quad (25)$$

$$mp = 0 \quad (26)$$

For axisymmetric perturbation $m = 0$,

$$\tilde{u}_{r1}(0) = \tilde{u}_{\theta 1}(0) = 0 \quad (27)$$

$\tilde{u}_{z1}(0)$ and $\tilde{p}(0)$ must be finite. and for $m = \pm 1$

$$\tilde{u}_{r1}(0) \pm \tilde{u}_{\theta 1}(0) = 0 \quad (28)$$

$$\tilde{u}_{z1}(0) = \tilde{p}(0) = 0 \quad (29)$$

if $|m| > 1$,

$$\tilde{u}_{z1}(0) = \tilde{p}(0) = 0 \quad (30)$$

$$\tilde{u}_{r1}(0) = \tilde{u}_{\theta 1}(0) = 0 \quad (31)$$

In case when $|m| = 1$, continuity equation is applied on the centerline with $\tilde{u}_z = 0$ which gives,

$$2\tilde{u}'_{r1}(0) + m\tilde{u}'_{\theta 1}(0) = 0 \quad (32)$$

3.1 Matching Conditions at the interface.

Let

$$\phi(r, z, \theta, t) = h(z, \theta, t) - r \quad (33)$$

so that the 0 level set of ϕ describes the interface between the liquid jet and the gaseous medium. The Kinematic condition that the interface is a material surface gives ,

$$\frac{D\phi}{Dt} = 0 \quad (34)$$

$$\frac{\partial h}{\partial t} + \frac{u_\theta}{r} \frac{\partial h}{\partial \theta} + u_z \frac{\partial h}{\partial z} = \left(\frac{\partial r}{\partial t} \right)_{r=R} = u_r(h(z, \theta, t), \theta, z, t) \quad (35)$$

We assume wave-like perturbation of the interface as

$$h'(z, \theta, t) = \tilde{h}e^{i(\alpha z + m\theta - \omega t)} \quad (36)$$

$$h = \bar{h}(z) + h'(z, \theta, t) \quad (37)$$

Substituting eq(36) in eq(35) and subtracting the mean flow we get,

$$\begin{aligned} \frac{\partial h'(z, \theta, t)}{\partial t} + \frac{u'_\theta(h(z, \theta, t), z, \theta, t)}{r} \left(\frac{\partial h'(z, \theta, t)}{\partial \theta} \right) + U_z(r) \left(\frac{\partial h'(z, \theta, t)}{\partial z} \right) + u'_z(h(z, \theta, t), \theta, z, t) \frac{d\bar{h}(z)}{dz} + \\ u'_z(h(z, \theta, t), \theta, z, t) \frac{\partial h'(z, \theta, t)}{\partial z} = u'_r(h(z, \theta, t), \theta, z, t) \end{aligned} \quad (38)$$

The above equation is nonlinear in h as u'_z and u'_θ are functions of h, θ, z, t .

For infinitesimally small perturbation , we linearise the above equation as ,

$$\frac{\partial h'(z, \theta, t)}{\partial t} + U_z(r) \frac{\partial h'(z, \theta, t)}{\partial z} + u'_z(h(z, \theta, t), z, \theta, t) \frac{d\bar{h}(z)}{dz} = u'_r(h(z, \theta, t), \theta, z, t) \quad (39)$$

Substituting normal mode form of perturbation, we get,

$$-i\omega \tilde{h}e^{i(\alpha z + m\theta - \omega t)} + i\alpha U_z(r) \tilde{h}e^{i(\alpha z + m\theta - \omega t)} + \tilde{u}_z e^{i(\alpha z + m\theta - \omega t)} \frac{d\bar{h}(z)}{dz} = \tilde{u}_r(h(z, \theta, t))e^{i(\alpha z + m\theta - \omega t)} \quad (40)$$

or ,

$$-i\omega\tilde{h} + i\alpha U_z(r)\tilde{h} + \tilde{u}_z \frac{d\bar{h}(z)}{dz} = \tilde{u}_r(h(z, \theta, t)) \quad (41)$$

At the interface, we can write ,

$$-i\omega\tilde{h} + i\alpha\tilde{h}U_{z1}(r) + \tilde{u}_{z1} \frac{d\bar{h}(z)}{dz} = \tilde{u}_{r1}(h(z, \theta, t)) \quad (42)$$

$$-i\omega\tilde{h} + i\alpha\tilde{h}U_{z2}(r) + \tilde{u}_{z2} \frac{d\bar{h}(z)}{dz} = \tilde{u}_{r2}(h(z, \theta, t)) \quad (43)$$

4 Stress boundary conditions

The normal stress in terms of the local pressure and velocity field is given by

$$\tau = -pI + \mu[(\nabla u) + (\nabla u)^T] \quad (44)$$

where,

$$\nabla u = \left(\mathbf{e}_r \frac{\partial}{\partial r} + \frac{\mathbf{e}_\theta}{r} \frac{\partial}{\partial \theta} + \mathbf{e}_z \frac{\partial}{\partial z} \right) (u_r \mathbf{e}_r + u_\theta \mathbf{e}_\theta + u_z \mathbf{e}_z) \quad (45)$$

$$\nabla u = \mathbf{e}_r \frac{\partial}{\partial r} (u_r \mathbf{e}_r + u_\theta \mathbf{e}_\theta + u_z \mathbf{e}_z) + \frac{\mathbf{e}_\theta}{r} \frac{\partial}{\partial \theta} (u_r \mathbf{e}_r + u_\theta \mathbf{e}_\theta + u_z \mathbf{e}_z) + \mathbf{e}_z \frac{\partial}{\partial z} (u_r \mathbf{e}_r + u_\theta \mathbf{e}_\theta + u_z \mathbf{e}_z) \quad (46)$$

The following identities are to be noted,

$$\begin{aligned} \frac{\partial \mathbf{e}_r}{\partial r} &= 0; \frac{\partial \mathbf{e}_\theta}{\partial r} = 0; \frac{\partial \mathbf{e}_z}{\partial r} = 0; \\ \frac{\partial \mathbf{e}_r}{\partial \theta} &= \mathbf{e}_\theta; \frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\mathbf{e}_r; \frac{\partial \mathbf{e}_z}{\partial \theta} = 0; \\ \frac{\partial \mathbf{e}_r}{\partial z} &= 0; \frac{\partial \mathbf{e}_\theta}{\partial z} = 0; \frac{\partial \mathbf{e}_z}{\partial z} = 0; \end{aligned}$$

We evaluate eq(46) term by term by using the above identities

$$\mathbf{e}_r \frac{\partial}{\partial r} (u_r \mathbf{e}_r + u_\theta \mathbf{e}_\theta + u_z \mathbf{e}_z) = \mathbf{e}_r \mathbf{e}_r \frac{\partial u_r}{\partial r} + \mathbf{e}_r \mathbf{e}_\theta \frac{\partial u_\theta}{\partial r} + \mathbf{e}_r \mathbf{e}_z \frac{\partial u_z}{\partial r} \quad (47)$$

$$\frac{\mathbf{e}_\theta}{r} \frac{\partial}{\partial \theta} (u_r \mathbf{e}_r + u_\theta \mathbf{e}_\theta + u_z \mathbf{e}_z) = \mathbf{e}_\theta \mathbf{e}_r \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) + \mathbf{e}_\theta \mathbf{e}_\theta \left(\frac{u_r}{r} \right) + \mathbf{e}_\theta \mathbf{e}_\theta \left(\frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right) - \mathbf{e}_\theta \mathbf{e}_r \left(\frac{u_\theta}{r} \right) + \mathbf{e}_\theta \mathbf{e}_z \left(\frac{1}{r} \frac{\partial u_z}{\partial \theta} \right) \quad (48)$$

$$\mathbf{e}_z \frac{\partial}{\partial z} (u_r \mathbf{e}_r + u_\theta \mathbf{e}_\theta + u_z \mathbf{e}_z) = \mathbf{e}_z \mathbf{e}_r \frac{\partial u_r}{\partial z} + \mathbf{e}_z \mathbf{e}_\theta \frac{\partial u_\theta}{\partial z} + \mathbf{e}_z \mathbf{e}_z \frac{\partial u_z}{\partial z} \quad (49)$$

$$\nabla u = \begin{bmatrix} \frac{\partial u_r}{\partial r} & \frac{\partial u_\theta}{\partial r} & \frac{\partial u_z}{\partial r} \\ \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} \right) & \left(\frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right) & \frac{1}{r} \frac{\partial u_z}{\partial \theta} \\ \frac{\partial u_r}{\partial z} & \frac{\partial u_\theta}{\partial z} & \frac{\partial u_z}{\partial z} \end{bmatrix} \quad (50)$$

$$(\nabla u)^T = \begin{bmatrix} \frac{\partial u_r}{\partial r} & \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} \right) & \frac{\partial u_r}{\partial z} \\ \frac{\partial u_\theta}{\partial r} & \left(\frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right) & \frac{\partial u_\theta}{\partial z} \\ \frac{\partial u_z}{\partial r} & \frac{1}{r} \frac{\partial u_z}{\partial \theta} & \frac{\partial u_z}{\partial z} \end{bmatrix} \quad (51)$$

$$[\nabla u + (\nabla u)^T] = \begin{bmatrix} 2 \frac{\partial u_r}{\partial r} & \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r} \right) & \left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} \right) \\ \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r} \right) & 2 \left(\frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right) & \left(\frac{1}{r} \frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z} \right) \\ \left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} \right) & \left(\frac{1}{r} \frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z} \right) & 2 \frac{\partial u_z}{\partial z} \end{bmatrix} \quad (52)$$

eq(44) becomes ,

$$\tau = \begin{bmatrix} -p + 2\mu \frac{\partial u_r}{\partial r} & \mu \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r} \right) & \mu \left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} \right) \\ \mu \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r} \right) & -p + 2\mu \left(\frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right) & \mu \left(\frac{1}{r} \frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z} \right) \\ \mu \left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} \right) & \mu \left(\frac{1}{r} \frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z} \right) & -p + 2\mu \frac{\partial u_z}{\partial z} \end{bmatrix} \quad (53)$$

4.1 Stress balance conditions at the liquid-gas interface.

We now derive the normal and tangential stress conditions at the liquid-gas interface. Consider an interfacial surface S bounded by a closed curve C. Surface tension σ is defined as the force per unit length in the s-direction at every point along C that acts to flatten the surface S. Performing a force balance on a volume element V enclosing the interfacial surface S defined by the contour C :

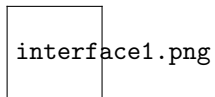
$$\int_V \rho \frac{D\mathbf{u}}{Dt} dV = \int_V f dV + \int_S [(\mathbf{t}_1(\mathbf{n})) + (\mathbf{t}_2(\mathbf{n}))] dS + \int_C \sigma s dl \quad (54)$$

dl is elemental length segment along the curve C. $\tau = -pI + \mu[(\nabla u) + (\nabla u)^T]$ is the total normal stress tensor. $\mathbf{t}_1(\mathbf{n}) = -\mathbf{n} \cdot \tau_1$ is the total normal stress vector exerted by the liquid phase on the interface while $\mathbf{t}_2(\mathbf{n}) = \mathbf{n} \cdot \tau_2$ is the total normal stress vector exerted by the gas phase on the interface. Now if ϵ is the typical length scale of the element V, then the acceleration and body forces will scale as ϵ^3 , while the surface forces scale as ϵ^2 . Hence in the limit of $\epsilon \rightarrow 0$, only the surface forces must balance,

$$\int_S [(\mathbf{t}_1(\mathbf{n})) + (\mathbf{t}_2(\mathbf{n}))] dS + \int_C \sigma s dl = 0 \quad (55)$$

consider $\int_C \sigma s dl$. using stokes theorem we can write ,

$$\int_C \sigma s dl = \int_S \mathbf{n} \sigma (\nabla \cdot \mathbf{n}) ds \quad (56)$$



proof

From stokes theorem we know that,

$$\int_C \mathbf{F} \cdot d\mathbf{l} = \int_S \mathbf{n} \cdot (\nabla \times \mathbf{F}) ds$$

Along the curve C, $d\mathbf{l} = \mathbf{m} dl$

$$\int_C \mathbf{F} \cdot \mathbf{m} dl = \int_S \mathbf{n} \cdot (\nabla \times \mathbf{F}) ds$$

choose $\mathbf{F} = \mathbf{f} \times \mathbf{b}$. Where \mathbf{b} is an arbitrary constant vector.

$$\int_C (\mathbf{f} \times \mathbf{b}) \cdot \mathbf{m} dl = \int_S \mathbf{n} \cdot (\nabla \times (\mathbf{f} \times \mathbf{b})) ds$$

using the vector identities ,

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = -\mathbf{B} \cdot (\mathbf{A} \times \mathbf{C}) \quad (57)$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} \quad (58)$$

the LHS can be written $\int_C (\mathbf{f} \times \mathbf{b}) \cdot \mathbf{m} dl = \int_C -\mathbf{b} \cdot (\mathbf{f} \times \mathbf{m}) dl$, the RHS can be written as

$$\int_S \mathbf{n} \cdot [\mathbf{f}(\nabla \cdot \mathbf{b}) - \mathbf{b}(\nabla \cdot \mathbf{f}) + (\mathbf{b} \cdot \nabla)\mathbf{f} - (\mathbf{f} \cdot \nabla)\mathbf{b}] ds$$

since \mathbf{b} is an constant vector $(\nabla \cdot \mathbf{b}) = 0$ and $\nabla \mathbf{b} = 0$

$$\int_C -\mathbf{b} \cdot (\mathbf{f} \times \mathbf{m}) dl = \int_S \mathbf{n} \cdot (-\mathbf{b}(\nabla \cdot \mathbf{f}) + (\mathbf{b} \cdot \nabla)\mathbf{f}) ds \quad (59)$$

since \mathbf{b} is an arbitrary vector

$$b \cdot \int_C (\mathbf{f} \times \mathbf{m}) dl = b \cdot \int_S (\mathbf{n}(\nabla \cdot \mathbf{f}) - (\mathbf{n} \cdot \nabla)\mathbf{f}) ds \quad (60)$$

choose $\mathbf{f} = \sigma \mathbf{n}$.

$$\int_C (\sigma \mathbf{n} \times \mathbf{m}) dl = \int_S [\mathbf{n}(\nabla \cdot \sigma \mathbf{n}) - (\mathbf{n} \cdot \nabla)\sigma \mathbf{n}] ds \quad (61)$$

we note the following,

- 1) $\nabla \cdot \sigma \mathbf{n} = \nabla \sigma \cdot \mathbf{n} + \sigma \nabla \cdot \mathbf{n}$.
- 2) $\nabla(\sigma \mathbf{n}) = \sigma \nabla \mathbf{n} + \mathbf{n} \nabla \sigma$.
- 3) $\mathbf{n} \times \mathbf{m} = -\mathbf{s}$
- 4) $\nabla \sigma \cdot \mathbf{n} = 0$
- 5) $\nabla \mathbf{n} \cdot \mathbf{n} = \frac{1}{2} \nabla(\mathbf{n} \cdot \mathbf{n}) = 0$
- 6) We assume that $\nabla \sigma = 0$

This yields,

$$-\int_C \sigma \mathbf{s} dl = \int_S \mathbf{n} \sigma (\nabla \cdot \mathbf{n}) ds$$

end of proof.

Rewriting eq(55),

$$\begin{aligned} \int_S [(\mathbf{t}_1(\mathbf{n})) + (\mathbf{t}_2(\mathbf{n}))] dS + \int_C \sigma s dl &= 0 \\ \int_S [\mathbf{n} \cdot \boldsymbol{\tau}_2 - \mathbf{n} \cdot \boldsymbol{\tau}_1] ds - \int_S \mathbf{n} \sigma (\nabla \cdot \mathbf{n}) ds &= 0 \end{aligned} \quad (62)$$

Since the surface element is arbitrary , the integral must vanish identically.

$$\mathbf{n} \cdot \boldsymbol{\tau}_2 - \mathbf{n} \cdot \boldsymbol{\tau}_1 = \mathbf{n} \sigma (\nabla \cdot \mathbf{n}) \quad (63)$$

4.2 Normal stress balance

Taking $\mathbf{n} \cdot (eq63)$ yields the normal stress balance at the interface :

$$\mathbf{n} \cdot \boldsymbol{\tau}_2 \cdot \mathbf{n} - \mathbf{n} \cdot \boldsymbol{\tau}_1 \cdot \mathbf{n} = \sigma (\nabla \cdot \mathbf{n}) \quad (64)$$

4.3 Tangential stress balance

Taking $\mathbf{t} \cdot (eq63)$ yields the tangential stress balance at the interface :

$$\mathbf{n} \cdot \boldsymbol{\tau}_2 \cdot \mathbf{t} - \mathbf{n} \cdot \boldsymbol{\tau}_1 \cdot \mathbf{t} = 0 \quad (65)$$

where ,

$$\tau_1 = \begin{bmatrix} -p + 2\mu_1 \frac{\partial u_{r1}}{\partial r} & \mu_1 \left(\frac{1}{r} \frac{\partial u_{r1}}{\partial \theta} - \frac{u_{\theta 1}}{r} + \frac{\partial u_{\theta 1}}{\partial r} \right) & \mu_1 \left(\frac{\partial u_{z1}}{\partial r} + \frac{\partial u_{r1}}{\partial z} \right) \\ \mu_1 \left(\frac{1}{r} \frac{\partial u_{r1}}{\partial \theta} - \frac{u_{\theta 1}}{r} + \frac{\partial u_{\theta 1}}{\partial r} \right) & -p + 2\mu_1 \left(\frac{u_{r1}}{r} + \frac{1}{r} \frac{\partial u_{\theta 1}}{\partial \theta} \right) & \mu_1 \left(\frac{1}{r} \frac{\partial u_{z1}}{\partial \theta} + \frac{\partial u_{\theta 1}}{\partial z} \right) \\ \mu_1 \left(\frac{\partial u_{z1}}{\partial r} + \frac{\partial u_{r1}}{\partial z} \right) & \mu_1 \left(\frac{1}{r} \frac{\partial u_{z1}}{\partial \theta} + \frac{\partial u_{\theta 1}}{\partial z} \right) & -p + 2\mu_1 \frac{\partial u_{z1}}{\partial z} \end{bmatrix} \quad (66)$$

$$\tau_2 = \begin{bmatrix} -p + 2\mu_2 \frac{\partial u_{r2}}{\partial r} & \mu_2 \left(\frac{1}{r} \frac{\partial u_{r2}}{\partial \theta} - \frac{u_{\theta 2}}{r} + \frac{\partial u_{\theta 2}}{\partial r} \right) & \mu_2 \left(\frac{\partial u_{z2}}{\partial r} + \frac{\partial u_{r2}}{\partial z} \right) \\ \mu_2 \left(\frac{1}{r} \frac{\partial u_{r2}}{\partial \theta} - \frac{u_{\theta 2}}{r} + \frac{\partial u_{\theta 2}}{\partial r} \right) & -p + 2\mu_2 \left(\frac{u_{r2}}{r} + \frac{1}{r} \frac{\partial u_{\theta 2}}{\partial \theta} \right) & \mu_2 \left(\frac{1}{r} \frac{\partial u_{z2}}{\partial \theta} + \frac{\partial u_{\theta 2}}{\partial z} \right) \\ \mu_2 \left(\frac{\partial u_{z2}}{\partial r} + \frac{\partial u_{r2}}{\partial z} \right) & \mu_2 \left(\frac{1}{r} \frac{\partial u_{z2}}{\partial \theta} + \frac{\partial u_{\theta 2}}{\partial z} \right) & -p + 2\mu_2 \frac{\partial u_{z2}}{\partial z} \end{bmatrix} \quad (67)$$

4.4 Finding unit normal vector and unit tangent vector to the interface.

To find the tangent vector , we write the parametric form of the surface for which two independent parameters and three dependent variables are required.

Let the equation of surface in parametrized form be $\mathbf{X}(u, v) = [\mathbf{z}(u, v), \theta(u, v), \mathbf{r} = \mathbf{h}(u, v)]$, for some u and v intervals, define a surface S in the u-v plane. \mathbf{X}_u is the tangent in the u direction , Since v is held constant. similarly \mathbf{X}_v is the tangent in the v direction , Since u is held constant.

$$\mathbf{X}_u = \frac{\partial \mathbf{X}}{\partial u} = [1, 0, \frac{\partial \mathbf{h}}{\partial u}]$$

Let the position of the interface at any instant of time t be $r = \mathbf{h}(\mathbf{z}, \theta)$.

Choosing $\mathbf{z} = u$; , $\theta = v$. then $\mathbf{r} = \mathbf{h}(u, v)$, then the unit tangent vector in the r direction is given by,

$$\mathbf{t}_r = [1, 0, \frac{\partial \mathbf{h}}{\partial z}] \left(\frac{1}{\sqrt{1 + (\frac{\partial \mathbf{h}}{\partial z})^2}} \right) \quad (68)$$

Similarly the unit tangent vector in the θ direction is given by,

$$\mathbf{t}_\theta = [0, 1, \frac{1}{r} \frac{\partial \mathbf{h}}{\partial \theta}] \left(\frac{1}{\sqrt{1 + (\frac{1}{r} \frac{\partial \mathbf{h}}{\partial \theta})^2}} \right) \quad (69)$$

Since both \mathbf{t}_r and \mathbf{t}_θ are orthogonal (\mathbf{u}, \mathbf{v} are orthogonal), their cross product yields a vector which is orthogonal to both \mathbf{t}_r and \mathbf{t}_θ which is the normal vector to the interface.

$$\mathbf{n} = \frac{\mathbf{t}_r \times \mathbf{t}_\theta}{\|\mathbf{t}_r \times \mathbf{t}_\theta\|} \quad (70)$$

$$\mathbf{t}_r \times \mathbf{t}_\theta = \left[-\frac{\partial h}{\partial z}, -\frac{1}{r} \frac{\partial h}{\partial \theta}, 1 \right] \quad (71)$$

$$\|\mathbf{t}_r \times \mathbf{t}_\theta\| = \sqrt{\left(\frac{\partial h}{\partial z}\right)^2 + \left(\frac{1}{r} \frac{\partial h}{\partial \theta}\right)^2 + 1} \quad (72)$$

$$\mathbf{n} = \left[-\frac{\partial h}{\partial z}, -\frac{1}{r} \frac{\partial h}{\partial \theta}, 1 \right] \frac{1}{\sqrt{\left(\frac{\partial h}{\partial z}\right)^2 + \left(\frac{1}{r} \frac{\partial h}{\partial \theta}\right)^2 + 1}} \quad (73)$$

4.5 Evaluating normal stress and shear stress at the interface.

rewriting eq(64) ,

$$\mathbf{n} \cdot \boldsymbol{\tau}_2 \cdot \mathbf{n} - \mathbf{n} \cdot \boldsymbol{\tau}_1 \cdot \mathbf{n} = \sigma(\nabla \cdot \mathbf{n})$$

evaluating ,

$$\mathbf{n} \cdot \boldsymbol{\tau} \cdot \mathbf{n} = \sum_{l=1}^3 (n_l \mathbf{e}_l) \cdot \left(\sum_{i=1}^3 \sum_{j=1}^3 (\tau_{ij} \mathbf{e}_i \mathbf{e}_j) \right) \cdot \left(\sum_{l=1}^3 (n_l \mathbf{e}_l) \right) \quad (74)$$

$$\mathbf{n} \cdot \boldsymbol{\tau} \cdot \mathbf{n} = \sum_{l=1}^3 (n_l \mathbf{e}_l) \cdot \left(\sum_{i=1}^3 \sum_{j=1}^3 \sum_{l=1}^3 (\tau_{ij} n_l (\mathbf{e}_i \mathbf{e}_j) \cdot \mathbf{e}_l) \right)$$

$$\mathbf{n} \cdot \boldsymbol{\tau} \cdot \mathbf{n} = \sum_{l=1}^3 (n_l \mathbf{e}_l) \cdot \left(\sum_{i=1}^3 \sum_{j=1}^3 \sum_{l=1}^3 (\tau_{ij} n_l (\mathbf{e}_i \delta_{jl})) \right)$$

$$\mathbf{n} \cdot \boldsymbol{\tau} \cdot \mathbf{n} = \sum_{l=1}^3 (n_l \mathbf{e}_l) \cdot \left(\sum_{i=1}^3 \sum_{j=1}^3 \tau_{ij} n_j \mathbf{e}_i \right)$$

$$\begin{aligned}
\mathbf{n} \cdot \boldsymbol{\tau} \cdot \mathbf{n} &= \sum_{l=1}^3 \sum_{i=1}^3 \sum_{j=1}^3 \tau_{ij} n_j n_l (\mathbf{e}_i \cdot \mathbf{e}_l) \\
\mathbf{n} \cdot \boldsymbol{\tau} \cdot \mathbf{n} &= \sum_{l=1}^3 \sum_{j=1}^3 \tau_{lj} n_j n_l = \sum_{l=1}^3 n_l \sum_{j=1}^3 \tau_{lj} n_j \\
\mathbf{n} \cdot \boldsymbol{\tau} \cdot \mathbf{n} &= n_1(\tau_{11}n_1 + \tau_{12}n_2 + \tau_{13}n_3) + n_2(\tau_{21}n_1 + \tau_{22}n_2 + \tau_{23}n_3) + n_3(\tau_{31}n_1 + \tau_{32}n_2 + \tau_{33}n_3)
\end{aligned} \tag{75}$$

$$n_1(\tau_{11}n_1 + \tau_{12}n_2 + \tau_{13}n_3) = \frac{1}{\left(\left(\frac{\partial h}{\partial z}\right)^2 + \left(\frac{1}{r} \frac{\partial h}{\partial \theta}\right)^2 + 1\right)} \left[(-p + 2\mu \frac{\partial u_r}{\partial r}) + \mu \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r} \right) \left(\frac{-1}{r} \frac{\partial h}{\partial \theta} \right) + \mu \left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} \right) \left(\frac{-\partial h}{\partial z} \right) \right] \tag{76}$$

$$n_2(\tau_{21}n_1 + \tau_{22}n_2 + \tau_{23}n_3) = \frac{\frac{-1}{r} \frac{\partial h}{\partial \theta}}{\left(\left(\frac{\partial h}{\partial z}\right)^2 + \left(\frac{1}{r} \frac{\partial h}{\partial \theta}\right)^2 + 1\right)} \left[\mu \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r} \right) + \left(-p + 2\mu \left(\frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right) \left(\frac{-1}{r} \frac{\partial h}{\partial \theta} \right) \right) + \mu \left(\frac{1}{r} \frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z} \right) \left(\frac{-\partial h}{\partial z} \right) \right] \tag{77}$$

$$n_3(\tau_{31}n_1 + \tau_{32}n_2 + \tau_{33}n_3) = \frac{\frac{-\partial h}{\partial z}}{\left(\left(\frac{\partial h}{\partial z}\right)^2 + \left(\frac{1}{r} \frac{\partial h}{\partial \theta}\right)^2 + 1\right)} \left[\mu \left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} \right) + \mu \left(\frac{1}{r} \frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z} \right) \left(\frac{-1}{r} \frac{\partial h}{\partial \theta} \right) + (-p + 2\mu \frac{\partial u_z}{\partial z}) \left(\frac{-\partial h}{\partial z} \right) \right] \tag{78}$$

4.6 Evaluation of tangential stress.

We saw from the stress balance equations that the tangential stress balance across the interface gives,

$$\mathbf{n} \cdot \boldsymbol{\tau}_2 \cdot \mathbf{t} - \mathbf{n} \cdot \boldsymbol{\tau}_1 \cdot \mathbf{t} = 0 \tag{79}$$

where \mathbf{t} is a unit tangent vector to the interface surface. This has got 2 components \mathbf{t}_r and \mathbf{t}_θ ,

$$\mathbf{n} \cdot \boldsymbol{\tau}_2 \cdot \mathbf{t}_r - \mathbf{n} \cdot \boldsymbol{\tau}_1 \cdot \mathbf{t}_r = 0 \tag{80}$$

and ,

$$\mathbf{n} \cdot \boldsymbol{\tau}_2 \cdot \mathbf{t}_\theta - \mathbf{n} \cdot \boldsymbol{\tau}_1 \cdot \mathbf{t}_\theta = 0 \tag{81}$$

evaluating ,

$$\begin{aligned}
\mathbf{n} \cdot \boldsymbol{\tau} \cdot \mathbf{t}_r &= \sum_{l=1}^3 (n_l \mathbf{e}_l) \cdot \left(\sum_{i=1}^3 \sum_{j=1}^3 (\tau_{ij} \mathbf{e}_i \mathbf{e}_j) \right) \cdot \left(\sum_{k=1}^3 (t_k \mathbf{e}_k) \right) \\
\mathbf{n} \cdot \boldsymbol{\tau} \cdot \mathbf{t}_r &= \sum_{l=1}^3 (n_l \mathbf{e}_l) \cdot \left(\sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 (\tau_{ij} t_k (\mathbf{e}_i \delta_{jk})) \right) \\
\mathbf{n} \cdot \boldsymbol{\tau} \cdot \mathbf{t}_r &= \sum_{l=1}^3 (n_l \mathbf{e}_l) \cdot \left(\sum_{i=1}^3 \sum_{j=1}^3 \tau_{ij} t_j \mathbf{e}_i \right)
\end{aligned} \tag{82}$$

$$\begin{aligned}
\mathbf{n} \cdot \boldsymbol{\tau} \cdot \mathbf{t}_r &= \sum_{l=1}^3 \sum_{i=1}^3 \sum_{j=1}^3 \tau_{ij} t_j n_l (\mathbf{e}_i \cdot \mathbf{e}_l) \\
\mathbf{n} \cdot \boldsymbol{\tau} \cdot \mathbf{t}_r &= \sum_{l=1}^3 \sum_{j=1}^3 \tau_{lj} t_j n_l = \sum_{l=1}^3 n_l \sum_{j=1}^3 \tau_{lj} t_j \\
\mathbf{n} \cdot \boldsymbol{\tau} \cdot \mathbf{t}_r &= n_1(\tau_{11}t_1 + \tau_{12}t_2 + \tau_{13}t_3) + n_2(\tau_{21}t_1 + \tau_{22}t_2 + \tau_{23}t_3) + n_3(\tau_{31}t_1 + \tau_{32}t_2 + \tau_{33}t_3)
\end{aligned} \tag{83}$$

$$\begin{aligned}
\mathbf{t}_r &= [t_r, t_\theta, t_z] = [t_1, t_2, t_3] = \left[\frac{\partial h}{\partial z}, 0, 1 \right] \left(\frac{1}{\sqrt{1 + \left(\frac{\partial \mathbf{h}}{\partial z} \right)^2}} \right) \\
n_1(\tau_{11}t_1 + \tau_{12}t_2 + \tau_{13}t_3) &= \frac{1}{\left(\sqrt{\left(\frac{\partial h}{\partial z} \right)^2 + 1} \right)} \frac{1}{\sqrt{\left(\frac{\partial h}{\partial z} \right)^2 + \left(\frac{1}{r} \frac{\partial h}{\partial \theta} \right)^2 + 1}} \left[(-p + 2\mu \frac{\partial u_r}{\partial r}) \left(\frac{\partial h}{\partial z} \right) + \mu \left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} \right) \right]
\end{aligned} \tag{84}$$

$$n_2(\tau_{21}t_1 + \tau_{22}t_2 + \tau_{23}t_3) = \frac{1}{\left(\sqrt{\left(\frac{\partial h}{\partial z} \right)^2 + 1} \right)} \frac{\frac{-1}{r} \frac{\partial h}{\partial \theta}}{\sqrt{\left(\frac{\partial h}{\partial z} \right)^2 + \left(\frac{1}{r} \frac{\partial h}{\partial \theta} \right)^2 + 1}} \left[\mu \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r} \right) \left(\frac{\partial h}{\partial z} \right) + \mu \left(\frac{1}{r} \frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z} \right) \right] \tag{85}$$

$$n_3(\tau_{31}t_1 + \tau_{32}t_2 + \tau_{33}t_3) = \frac{1}{\left(\sqrt{\left(\frac{\partial h}{\partial z} \right)^2 + 1} \right)} \frac{\frac{-\partial h}{\partial z}}{\sqrt{\left(\frac{\partial h}{\partial z} \right)^2 + \left(\frac{1}{r} \frac{\partial h}{\partial \theta} \right)^2 + 1}} \left[\mu \left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} \right) \left(\frac{\partial h}{\partial z} \right) + \left(-p + 2\mu \frac{\partial u_z}{\partial z} \right) \right] \tag{86}$$

evaluating ,

$$\mathbf{n} \cdot \boldsymbol{\tau} \cdot \mathbf{t}_\theta = \sum_{l=1}^3 (n_l \mathbf{e}_l) \cdot \left(\sum_{i=1}^3 \sum_{j=1}^3 (\tau_{ij} \mathbf{e}_i \mathbf{e}_j) \right) \cdot \left(\sum_{k=1}^3 (t_k \mathbf{e}_k) \right) \tag{87}$$

$$\mathbf{n} \cdot \boldsymbol{\tau} \cdot \mathbf{t}_\theta = \sum_{l=1}^3 (n_l \mathbf{e}_l) \cdot \left(\sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 (\tau_{ij} t_k (\mathbf{e}_i \delta_{jk})) \right)$$

$$\mathbf{n} \cdot \boldsymbol{\tau} \cdot \mathbf{t}_\theta = \sum_{l=1}^3 (n_l \mathbf{e}_l) \cdot \left(\sum_{i=1}^3 \sum_{j=1}^3 \tau_{ij} t_j \mathbf{e}_i \right)$$

$$\mathbf{n} \cdot \boldsymbol{\tau} \cdot \mathbf{t}_\theta = \sum_{l=1}^3 \sum_{i=1}^3 \sum_{j=1}^3 \tau_{ij} t_j n_l (\mathbf{e}_i \cdot \mathbf{e}_l)$$

$$\mathbf{n} \cdot \boldsymbol{\tau} \cdot \mathbf{t}_\theta = \sum_{l=1}^3 \sum_{j=1}^3 \tau_{lj} t_j n_l = \sum_{l=1}^3 n_l \sum_{j=1}^3 \tau_{lj} t_j$$

$$\mathbf{n} \cdot \boldsymbol{\tau} \cdot \mathbf{t}_\theta = n_1(\tau_{11}t_1 + \tau_{12}t_2 + \tau_{13}t_3) + n_2(\tau_{21}t_1 + \tau_{22}t_2 + \tau_{23}t_3) + n_3(\tau_{31}t_1 + \tau_{32}t_2 + \tau_{33}t_3) \tag{88}$$

$$\mathbf{t}_\theta = [0, 1, \frac{1}{r} \frac{\partial h}{\partial \theta}] \left(\frac{1}{\sqrt{1 + (\frac{1}{r} \frac{\partial h}{\partial \theta})^2}} \right)$$

$$[t_1, t_2, t_3] = [\frac{1}{r} \frac{\partial h}{\partial \theta}, 1, 0] \left(\frac{1}{\sqrt{1 + (\frac{1}{r} \frac{\partial h}{\partial \theta})^2}} \right)$$

$$n_1(\tau_{11}t_1 + \tau_{12}t_2 + \tau_{13}t_3) = \frac{1}{\left(\sqrt{\left(\frac{1}{r} \frac{\partial h}{\partial \theta}\right)^2 + 1}\right)} \frac{1}{\sqrt{\left(\frac{\partial h}{\partial z}\right)^2 + \left(\frac{1}{r} \frac{\partial h}{\partial \theta}\right)^2 + 1}} \left[\left(-p + 2\mu \frac{\partial u_r}{\partial r} \right) \left(\frac{1}{r} \frac{\partial h}{\partial \theta} \right) + \mu \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r} \right) \right] \quad (89)$$

$$n_2(\tau_{21}t_1 + \tau_{22}t_2 + \tau_{23}t_3) = \frac{1}{\left(\sqrt{\left(\frac{1}{r} \frac{\partial h}{\partial \theta}\right)^2 + 1}\right)} \frac{\frac{-1}{r} \frac{\partial h}{\partial \theta}}{\sqrt{\left(\frac{\partial h}{\partial z}\right)^2 + \left(\frac{1}{r} \frac{\partial h}{\partial \theta}\right)^2 + 1}} \left[\mu \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r} \right) \left(\frac{1}{r} \frac{\partial h}{\partial \theta} \right) + \left(-p + 2\mu \left(\frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right) \right) \right] \quad (90)$$

$$n_3(\tau_{31}t_1 + \tau_{32}t_2 + \tau_{33}t_3) = \frac{1}{\left(\sqrt{\left(\frac{1}{r} \frac{\partial h}{\partial \theta}\right)^2 + 1}\right)} \frac{\frac{-\partial h}{\partial z}}{\sqrt{\left(\frac{\partial h}{\partial z}\right)^2 + \left(\frac{1}{r} \frac{\partial h}{\partial \theta}\right)^2 + 1}} \left[\mu \left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} \right) \left(\frac{1}{r} \frac{\partial h}{\partial \theta} \right) + \mu \left(\frac{1}{r} \frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z} \right) \right] \quad (91)$$

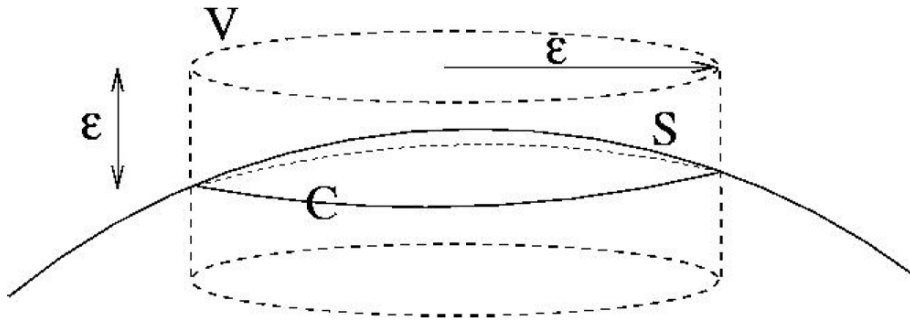


Figure 2: A Gaussian fluid pillbox of height and radius ϵ spanning the interface evolves under the combined influence of volume and surface forces.

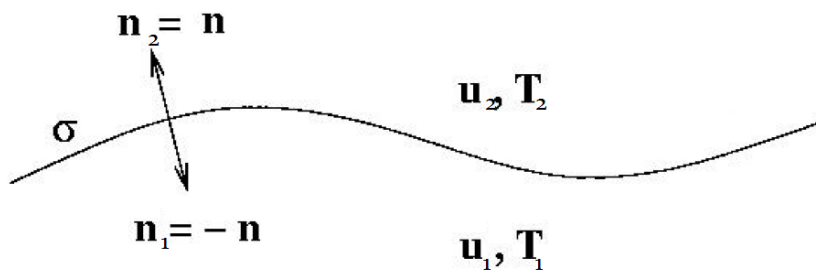


Figure 3: A definitional sketch of a fluid-fluid interface.

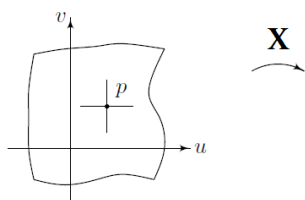


Figure 4.1: Point p on the parametric surface.

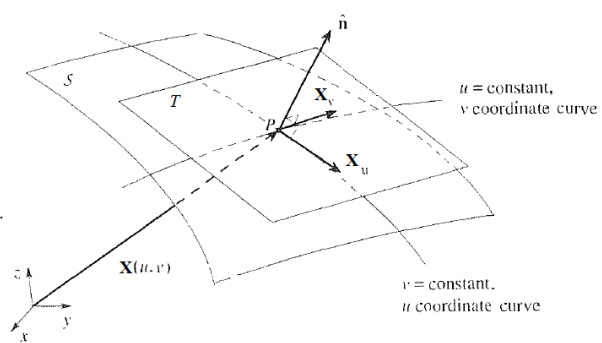


Figure 4. Tangent plane and normal.