Mathematical formulation of linear instability problem of a viscous incompressible liquid round jet in a gaseous medium.

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1 Governing Equations.

The continuity equation in polar cylindrical coordinates for the liquid phase is given by

$$\frac{1}{r}\frac{\partial(ru_{r1})}{\partial r}+\frac{1}{r}\frac{\partial u_{\theta 1}}{\partial \theta}+\frac{\partial u_{z1}}{\partial z}=0 \tag{1}$$

For the gas phase,

$$\frac{1}{r}\frac{\partial(ru_{r2})}{\partial r}+\frac{1}{r}\frac{\partial u_{\theta 2}}{\partial \theta}+\frac{\partial u_{z2}}{\partial z}=0 \tag{2}$$

The non dimensionalised momentum equation in the r direction (radial direction) is given by

$$\frac{\partial u_{r1}}{\partial t}+u_{r1}\frac{\partial u_{r1}}{\partial r}+\frac{u_{\theta 1}}{r}\frac{\partial u_{r1}}{\partial \theta}-\frac{u_{\theta 1}^2}{r}+u_{z1}\frac{\partial u_{r1}}{\partial z}=-\frac{\partial p}{\partial r}+\frac{1}{Re_1}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_{r1}}{\partial r}\right)-\frac{u_{r1}}{r^2}+\frac{1}{r^2}\frac{\partial^2 u_{r1}}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial u_{\theta 1}}{\partial \theta}+\frac{\partial^2 u_{r1}}{\partial z^2}\right] \tag{3}$$

For the gas phase,

$$\frac{\partial u_{r2}}{\partial t}+u_{r2}\frac{\partial u_{r2}}{\partial r}+\frac{u_{\theta 2}}{r}\frac{\partial u_{r2}}{\partial \theta}-\frac{u_{\theta 2}^2}{r}+u_{z2}\frac{\partial u_{r2}}{\partial z}=-\frac{\partial p}{\partial r}+\frac{1}{Re_2}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_{r2}}{\partial r}\right)-\frac{u_{r2}}{r^2}+\frac{1}{r^2}\frac{\partial^2 u_{r2}}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial u_{\theta 2}}{\partial \theta}+\frac{\partial^2 u_{r2}}{\partial z^2}\right] \tag{4}$$

In the azimuthal direction (θ direction),

$$\frac{\partial u_{\theta 1}}{\partial t}+u_{r1}\frac{\partial u_{\theta 1}}{\partial r}+\frac{u_{\theta 1}}{r}\frac{\partial u_{\theta 1}}{\partial \theta}+\frac{u_{\theta 1}u_{r1}}{r}+u_{z1}\frac{\partial u_{\theta 1}}{\partial z}=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\frac{1}{Re_1}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_{\theta 1}}{\partial r}\right)-\frac{u_{\theta 1}}{r^2}+\frac{1}{r^2}\frac{\partial^2 u_{\theta 1}}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial u_{r1}}{\partial \theta}+\frac{\partial^2 u_{\theta 1}}{\partial z^2}\right] \tag{5}$$

For the gas phase,

$$\frac{\partial u_{\theta 2}}{\partial t}+u_{r2}\frac{\partial u_{\theta 2}}{\partial r}+\frac{u_{\theta 2}}{r}\frac{\partial u_{\theta 2}}{\partial \theta}+\frac{u_{\theta 2}u_{r2}}{r}+u_{z2}\frac{\partial u_{\theta 2}}{\partial z}=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\frac{1}{Re_2}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_{\theta 2}}{\partial r}\right)-\frac{u_{\theta 2}}{r^2}+\frac{1}{r^2}\frac{\partial^2 u_{\theta 2}}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial u_{r2}}{\partial \theta}+\frac{\partial^2 u_{\theta 2}}{\partial z^2}\right] \tag{6}$$

In the axial direction (z direction) for the liquid phase

$$\frac{\partial u_{z1}}{\partial t}+u_{r1}\frac{\partial u_{z1}}{\partial r}+\frac{u_{\theta 1}}{r}\frac{\partial u_{z1}}{\partial \theta}+u_{z1}\frac{\partial u_{z1}}{\partial z}=-\frac{\partial p}{\partial z}+\frac{1}{Re_1}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_{z1}}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_{z1}}{\partial \theta^2}+\frac{\partial^2 u_{z1}}{\partial z^2}\right] \tag{7}$$

For the gas phase,

$$\frac{\partial u_{z2}}{\partial t}+u_{r2}\frac{\partial u_{z2}}{\partial r}+\frac{u_{\theta 2}}{r}\frac{\partial u_{z2}}{\partial \theta}+u_{z2}\frac{\partial u_{z2}}{\partial z}=-\frac{\partial p}{\partial z}+\frac{1}{Re_2}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_{z2}}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_{z2}}{\partial \theta^2}+\frac{\partial^2 u_{z2}}{\partial z^2}\right] \tag{8}$$

where u_r , u_θ and u_z are the velocity components in the radial, azimuthal and axial directions respectively.

2 Base flow.

The flow variables are assumed to consist of a mean part and an infinitesimally small perturbation.

$$\begin{aligned} u_r &= U_r + u_r'(r,\theta,z,t) \\ u_\theta &= U_\theta + u_\theta'(r,\theta,z,t) \\ u_z &= U_z + u_z'(r,\theta,z,t) \end{aligned} \tag{9}$$

The base flow is taken to be an incompressible viscous axisymmetric one with locally parallel flow assumption. The base flow quantities are

$$\begin{aligned} U_{r1} &= U_{r1}(r) \\ U_{r2} &= U_{r2}(r) \\ U_\theta &= 0 \\ U_{z1} &= U_{z1}(r) \\ U_{z2} &= U_{z2}(r) \\ P &= 0 \end{aligned} \tag{10}$$

2.1 Normal mode form of perturbation.

The flow is periodic in azimuthal and axial direction. We assume perturbations of the form

$$[u_r, u_\theta, u_z, p] = [i\tilde{u}_r(r), \tilde{u}_\theta(r), \tilde{u}_z(r), \tilde{p}(r)]e^{i(\alpha z + m\theta - \omega t)} \tag{11}$$

It is to be noted that $u_r \propto i\tilde{u}_r$. This can be readily seen from the contunity equation, the phase of u_r differs by $\frac{\pi}{2}$ from that of u_θ and u_r. Substituting eq(11) into eq(3), eq(4), eq(5), eq(6), eq(7), eq(8) , eq(1) and eq(2), we have, In the radial direction,

$$-i\frac{\tilde{u}_{r1}''}{Re_1}+i\left(U_{r1}-\frac{1}{rRe_1}\right)\tilde{u}_{r1}'+\left[\omega+iU_{r1}'-\alpha U_{z1}+\frac{i}{Re_1}\left(\frac{m^2+1}{r^2}+\alpha^2\right)\right]\tilde{u}_{r1}+\frac{2im}{r^2Re_1}\tilde{u}_{\theta 1}+\tilde{p}'=0 \tag{12}$$

$$-i\frac{\tilde{u}_{r2}''}{Re_2}+i\left(U_{r2}-\frac{1}{rRe_2}\right)\tilde{u}_{r2}'+\left[\omega+iU_{r2}'-\alpha U_{z2}+\frac{i}{Re_2}\left(\frac{m^2+1}{r^2}+\alpha^2\right)\right]\tilde{u}_{r2}+\frac{2im}{r^2Re_2}\tilde{u}_{\theta 2}+\tilde{p}'=0 \tag{13}$$

In the azimuthal direction,

$$-\frac{\tilde{u}_{\theta 1}''}{Re_1}+\left(U_{r1}-\frac{1}{rRe_1}\right)\tilde{u}_{\theta 1}'+\left[-i\omega+i\alpha U_{z1}+\frac{U_{r1}}{r}+\frac{1}{Re_1}\left(\frac{m^2+1}{r^2}+\alpha^2\right)\right]\tilde{u}_{\theta 1}+\left(\frac{2m}{r^2Re_1}\right)\tilde{u}_{r1}+\frac{im\tilde{p}}{r}=0 \tag{14}$$

$$-\frac{\tilde{u}''_{\theta 2}}{Re_2}+\left(U_{r2}-\frac{1}{rRe_2}\right)\tilde{u}'_{\theta 2}+\left[-i\omega+i\alpha U_{z2}+\frac{U_{r2}}{r}+\frac{1}{Re_2}\left(\frac{m^2+1}{r^2}+\alpha^2\right)\right]\tilde{u}_{\theta 2}+\left(\frac{2m}{r^2 Re_2}\right)\tilde{u}_{r2}+\frac{im\tilde{p}}{r}=0 \tag{15}$$

In the axial di

$$\phi(r,z,\theta,t)=h(z,\theta,t)-r \tag{16}$$

so that the 0 level set of ϕ describes the interface between the liquid jet and the gaseous medium. The Kinematic condition that the interface is a material surface gives ,

$$\frac{D\phi}{Dt}=0 \tag{17}$$

$$\frac{\partial h}{\partial t}+\frac{u_\theta}{r}\frac{\partial h}{\partial \theta}+u_z\frac{\partial h}{\partial z}=\left(\frac{\partial r}{\partial t}\right)_{r=R}=2u_r(h(z,\theta,t),\theta,z,t) \tag{18}$$

We assume wave-like perturbation of the interface as

$$h'(z,\theta,t)=\tilde{h}e^{i(\alpha z+m\theta-\omega t)} \tag{19}$$

$$h=\overline{h}(z)+h'(z,\theta,t) \tag{20}$$

Substituting eq(19) in eq(18) and subtracting the mean flow we get,

$$\frac{\partial h'(z,\theta,t)}{\partial t}+\frac{u'_\theta(h(z,\theta,t),z,\theta,t)}{r}\left(\frac{\partial h'(z,\theta,t)}{\partial \theta}\right)+U_z(r)\left(\frac{\partial h'(z,\theta,t)}{\partial z}\right)+u'_z(h(z,\theta,t),\theta,z,t)\frac{d\overline{h}(z)}{dz}+ \\ u'_z(h(z,\theta,t),\theta,z,t)\frac{\partial h'(z,\theta,t)}{\partial z}=2u'_r(h(z,\theta,t),\theta,z,t) \tag{21}$$

The above equation is nonlinear in h as u'_z and u'_θ are functions of h,θ,z,t.

For infintesimally small perturbation , we linearise the above equation as ,

$$\frac{\partial h'(z,\theta,t)}{\partial t}+U_z(r)\frac{\partial h'(z,\theta,t)}{\partial z}+u'_z(h(z,\theta,t),z,\theta,t)\frac{d\overline{h}(z)}{dz}=2u'_r(h(z,\theta,t),\theta,z,t) \tag{22}$$

Substituting normal mode form of perturbation, we get,

$$-i\omega\tilde{h}e^{i(\alpha z+m\theta-\omega t)}+i\alpha U_z(r)\tilde{h}e^{i(\alpha z+m\theta-\omega t)}+\tilde{u}_z e^{i(\alpha z+m\theta-\omega t)}\frac{d\overline{h}(z)}{dz}=2\tilde{u}_r(h(z,\theta,t))e^{i(\alpha z+m\theta-\omega t)} \tag{23}$$

or ,

$$-i\omega\tilde{h}+i\alpha U_z(r)\tilde{h}+\tilde{u}_z\frac{d\overline{h}(z)}{dz}=2\tilde{u}_r(h(z,\theta,t)) \tag{24}$$

At the interface, we can write ,

$$-i\omega\tilde{h}+i\alpha\tilde{h}U_{z1}(r)+\tilde{u}_{z1}\frac{d\overline{h}(z)}{dz}=\tilde{u}_{r1}(h(z,\theta,t)) \tag{25}$$

$$-i\omega\tilde{h}+i\alpha\tilde{h}U_{z2}(r)+\tilde{u}_{z2}\frac{d\overline{h}(z)}{dz}=\tilde{u}_{r2}(h(z,\theta,t)) \tag{26}$$

rection,

$$-\frac{\tilde{u}''_{z1}}{Re_1}+\left(U_{r1}-\frac{1}{rRe_1}\right)\tilde{u}'_{z1}+\left[-i\omega+i\alpha U_{z1}+\frac{1}{Re_1}\left(\frac{m^2}{r^2}+\alpha^2\right)\right]\tilde{u}_{z1}+iU'_{z1}\tilde{u}_{r1}+i\alpha\tilde{p}=0 \tag{27}$$

$$-\frac{\tilde{u}''_{z2}}{Re_2}+\left(U_{r2}-\frac{1}{rRe_2}\right)\tilde{u}'_{z2}+\left[-i\omega+i\alpha U_{z2}+\frac{1}{Re_2}\left(\frac{m^2}{r^2}+\alpha^2\right)\right]\tilde{u}_{z2}+iU'_{z2}\tilde{u}_{r2}+i\alpha\tilde{p}=0 \tag{28}$$

Continuity equation yields,

$$\tilde{u}'_{r1}+\frac{\tilde{u}_{r1}}{r}+\frac{m\tilde{u}_{\theta 1}}{r}+\alpha\tilde{u}_{z1}=0 \tag{29}$$

$$\tilde{u}'_{r2}+\frac{\tilde{u}_{r2}}{r}+\frac{m\tilde{u}_{\theta 2}}{r}+\alpha\tilde{u}_{z2}=0 \tag{30}$$

3 Boundary Conditions.

Due to the singular nature of the coordinate system on the centerline, all physical quantities must be smooth and bounded at $r = 0$ .Therefore as $r \to 0, \lim_{r\to 0}\frac{\partial \mathbf{V}}{\partial\theta}=0.$(31)

$$\lim_{r\to 0}\frac{\partial p'}{\partial\theta}=0 \tag{32}$$

where $\mathbf{V}$ is the total velocity vector. the above limits represents the boundedness and smoothness conditions on the solutions along the centerline.(Batchelor and Gill 1962, also Khorrami et al.,JCP 81,206-229 (1989)). while expanding the limits, we need to consider only the perturbation part of the velocity since the mean flow is independent of z and θ

$$\lim_{r\to 0}\frac{\partial \mathbf{V}}{\partial\theta}=\lim_{r\to 0}\frac{\partial}{\partial\theta}(u_{r1}\mathbf{e_r}+u_{\theta 1}\mathbf{e}_\theta+u_{z1}\mathbf{e_z})=0$$

But

$$\frac{\partial \mathbf{e_z}}{\partial\theta}=0;\frac{\partial \mathbf{e_r}}{\partial\theta}=\mathbf{e}_\theta;\frac{\partial \mathbf{e}_\theta}{\partial\theta}=-\mathbf{e_r} \tag{33}$$

$$-(m\tilde{u}_{r1})\mathbf{e_r}+i(\tilde{u}_{r1}+m\tilde{u}_{\theta 1})\mathbf{e}_\theta+in\tilde{u}_{z1}\mathbf{e_z}=0$$

$$imp=0$$

In order for the inequality to hold, each component of the resultant vector must be zero. This gives,

$$m\tilde{u}_{r1}=0 \tag{34}$$

$$\tilde{u}_{r1}+m\tilde{u}_{\theta 1}=0 \tag{35}$$

$$m\tilde{u}_{z1}=0 \tag{36}$$

$$mp=0 \tag{37}$$

For axisymmetric perturbation $m = 0$,

$$\tilde{u}_{r1}(0)=\tilde{u}_{\theta 1}(0)=0 \tag{38}$$

$\tilde{u}_{z1}(0)$ and $\tilde{p}(0)$ must be finite. and for $m = \pm 1$

$$\tilde{u}_{r1}(0)\pm\tilde{u}_{\theta 1}(0)=0 \tag{39}$$

$$\tilde{u}_{z1}(0)=\tilde{p}(0)=0 \tag{40}$$

if $|m| > 1$,

$$\tilde{u}_{z1}(0) = \tilde{p}(0) = 0 \tag{41}$$

$$\tilde{u}_{r1}(0) = \tilde{u}_{\theta 1}(0) = 0 \tag{42}$$

In case when $|m| = 1$, continuity equation is applied on the centerline with $\tilde{u}_z = 0$ which gives,

$$2\tilde{u}'_{r1}(0) + m\tilde{u}'_{\theta 1}(0) = 0 \tag{43}$$

3.1 Matching Conditions at the interface.

$$\phi(r, \theta, z, t) = h(z, \theta, t) - r \tag{44}$$

$$\frac{D\phi}{Dt} = 0 \tag{45}$$

$$\left(\frac{\partial \phi}{\partial r}\right)\frac{dr}{dt} + \left(\frac{\partial \phi}{\partial \theta}\right)\frac{d\theta}{dt} + \left(\frac{\partial \phi}{\partial z}\right)\frac{dz}{dt} + \left(\frac{\partial \phi}{\partial t}\right) = 0 \tag{46}$$

$$\left(\frac{\partial (h-r)}{\partial r}\right)\frac{dr}{dt} + \left(\frac{\partial (h-r)}{\partial \theta}\right)\frac{d\theta}{dt} + \left(\frac{\partial (h-r)}{\partial z}\right)\frac{dz}{dt} + \left(\frac{\partial (h-r)}{\partial t}\right) = 0 \tag{47}$$

$$\left(\frac{\partial h}{\partial r}\right)u_r - u_r + \frac{u_\theta}{r}\left(\frac{\partial h}{\partial \theta}\right) + u_z\left(\frac{\partial h}{\partial z}\right) + \left(\frac{\partial h}{\partial t}\right) - u_r = 0 \tag{48}$$

$$\left(\frac{\partial h}{\partial r}\right)u_r + \frac{u_\theta}{r}\left(\frac{\partial h}{\partial \theta}\right) + u_z\left(\frac{\partial h}{\partial z}\right) + \left(\frac{\partial h}{\partial t}\right) = 2u_r \tag{49}$$

rearranging,

$$\left(\frac{\partial h}{\partial t}\right) + u_r\left(\frac{\partial h}{\partial r}\right) + \frac{u_\theta}{r}\left(\frac{\partial h}{\partial \theta}\right) + u_z\left(\frac{\partial h}{\partial z}\right) = 2u_r \tag{50}$$

we note the following ,

$$h(z, \theta, t) = \overline{h}(z) + h'(z, \theta, t) \tag{51}$$

$$u_r(h, z, \theta, t) = \overline{U}_r(r) + u'_r(h, z, \theta, t) \tag{52}$$

$$u_\theta(h, z, \theta, t) = u'(h, z, \theta, t) \tag{53}$$

(since the base flow is axisymmetric $\overline{U}_\theta = 0$)

$$u_z(h, z, \theta, t) = \overline{U}_z(r) + u'_z(h, z, \theta, t) \tag{54}$$

$$\left(\frac{\partial h(z,\theta,t)}{\partial t}\right) + u_r(h,z,\theta,t)\left(\frac{\partial h(z,\theta,t)}{\partial r}\right) + \frac{u_\theta(h,z,\theta,t)}{r}\left(\frac{\partial h(z,\theta,t)}{\partial \theta}\right) + u_z(\frac{u'_\theta}{r}\left(\frac{\partial h'}{\partial \theta}\right)h,z,\theta,t)\left(\frac{\partial(\overline{h}(z) + h'(z,\theta,t))}{\partial z}\right) = 2(\overline{U_r}(r) + u'_r(h, \tag{55}$$

$$
\begin{aligned}
&\left(\frac{\partial \overline{h}}{\partial t}\right)+\left(\frac{\partial h'}{\partial t}\right)+\overline{U}_r\left(\frac{\partial \overline{h}(z)}{\partial r}\right)+\quad \overline{U}_r\left(\tfrac{\partial h'}{\partial r}\right)+u'_r\left(\tfrac{\partial \overline{h}(z)}{\partial r}\right)\quad +u'_r\left(\frac{\partial h'}{\partial r}\right)+\\
&\frac{u'_\theta}{r}\left(\frac{\partial \overline{h}}{\partial \theta}\right)+\frac{u'_\theta}{r}\left(\frac{\partial h'}{\partial \theta}\right)+\quad \overline{U}_z\left(\tfrac{\partial \overline{h}}{\partial z}\right)+\overline{U}_z\left(\tfrac{\partial h'}{\partial z}\right)\quad +u'_z\left(\frac{\partial \overline{h}}{\partial z}\right)+u'_z\left(\frac{\partial h'}{\partial z}\right)\\
&=2(\overline{U_r}(r)+u'_r(h,z,\theta,t))
\end{aligned}
$$

since $\overline{h}$ is a function of z only, $\frac{\partial \overline{h}(z)}{\partial r}=\frac{\partial \overline{h}(z)}{\partial \theta}=0$. We have,

$$
\left(\frac{\partial \overline{h}}{\partial t}\right)+\left(\frac{\partial h'}{\partial t}\right)+\overline{U}_r\left(\frac{\partial h'}{\partial r}\right)+u'_r\left(\frac{\partial h'}{\partial r}\right)+\frac{u'_\theta}{r}\left(\frac{\partial h'}{\partial \theta}\right)+\overline{U}_z\left(\frac{\partial \overline{h}}{\partial z}\right)+\overline{U}_z\left(\frac{\partial h'}{\partial z}\right)+u'_z\left(\frac{\partial \overline{h}}{\partial z}\right)+u'_z\left(\frac{\partial h'}{\partial z}\right)=2(\overline{U_r}(r)+u'_r(h,z,\theta,t)) \tag{56}
$$

Equation for the mean is given by

$$
\left(\frac{\partial \overline{h}}{\partial t}\right)+\underbrace{\overline{U}_r\left(\frac{\partial \overline{h}(z)}{\partial r}\right)}+\overline{U}_z\left(\frac{d\overline{h}}{dz}\right)=2\overline{U}_r(r) \tag{57}
$$

or,

$$
\left(\frac{\partial \overline{h}}{\partial t}\right)+\overline{U}_z\left(\frac{d\overline{h}}{dz}\right)=2\overline{U}_r(r) \tag{58}
$$

Subtracting the mean from the total flow, we get the perturbed equation of h:

$$
\left(\frac{\partial h'}{\partial t}\right)+\overline{U}_r\left(\frac{\partial h'}{\partial r}\right)+u'_r\left(\frac{\partial h'}{\partial r}\right)+\frac{u'_\theta}{r}\left(\frac{\partial h'}{\partial \theta}\right)+\overline{U}_z\left(\frac{\partial h'}{\partial z}\right)+u'_z\left(\frac{d\overline{h}}{dz}\right)+u'_z\left(\frac{\partial h'}{\partial z}\right)=2u'_r(h,z,\theta,t) \tag{59}
$$

assuming small perturbations, we neglect the product of small quantities to yield,

$$
\left(\frac{\partial h'(z,\theta,t)}{\partial t}\right)+\overline{U}_r(r)\left(\frac{\partial h'(z,\theta,t)}{\partial r}\right)+\overline{U}_z(r)\left(\frac{\partial h'(z,\theta,t)}{\partial z}\right)+\underbrace{u'_z(h,z,\theta,t)}\left(\frac{d\overline{h}(z)}{dz}\right)=2\,\underbrace{u'_r(h,z,\theta,t)} \tag{60}
$$

The above equation is nonlinear in h. We know that $h=\overline{h}(z)+h'(z,\theta,t)$
To linearize eq(17) equation, we need to expand the terms in the curly braces using taylor series about $\overline{h}$.

$$
u'_r(h,z,\theta,t)=u'_r(\overline{h}(z),z,\theta,t)+(h-\overline{h})\left(\frac{\partial u'_r}{\partial h}\right)_{(\overline{h},z,\theta,t)}+(z-z)\left(\frac{\partial u'_z}{\partial h}\right)_{(\overline{h},z,\theta,t)}+(\theta-\theta)\left(\frac{\partial u'_\theta}{\partial h}\right)_{(\overline{h},z,\theta,t)}+\left\{O(h-\overline{h})^2\right\}+.... \tag{61}
$$

Neglecting higher order terms,

$$
u'_r(h,z,\theta,t)=u'_r(\overline{h}(z),z,\theta,t)+(h-\overline{h})\left(\frac{\partial u'_r}{\partial h}\right)_{(\overline{h},z,\theta,t)} \tag{62}
$$

but $h-\overline{h}=h'$ which is a function of z,θ,t .
i.e.,

$$u'_r(h,z,\theta,t) = u'_r(\overline{h}(z),z,\theta,t) + h'(z,\theta,t)\left(\frac{\partial u'_r}{\partial h}\right)_{(\overline{h},z,\theta,t)} \tag{63}$$

Similarly,

$$u'_z(h,z,\theta,t) = u'_z(\overline{h}(z),z,\theta,t) + h'(z,\theta,t)\left(\frac{\partial u'_z}{\partial h}\right)_{(\overline{h},z,\theta,t)} \tag{64}$$

substituting eq (20) and eq(21) into eq(17)

$$\left(\frac{\partial h'(z,\theta,t)}{\partial t}\right) + \overline{U}_r(r)\left(\frac{\partial h'(z,\theta,t)}{\partial r}\right) + \overline{U}_z(r)\left(\frac{\partial h'(z,\theta,t)}{\partial z}\right) \quad + \left\{u'_z(\overline{h}(z),z,\theta,t) + h'(z,\theta,t)\left(\frac{\partial u'_z}{\partial h}\right)_{(\overline{h},z,\theta,t)}\right\}\left(\frac{d\overline{h}(z)}{dz}\right) \quad = 2\left\{u'_r(\overline{h}(z),z\right.$$

4 Stress boundary conditions

The normal stress in terms of the local pressure and velocity field is given by

$$\tau = -pI + \mu[(\nabla u) + (\nabla u)^T] \tag{65}$$

where,

$$\nabla u = \left(\mathbf{e_r}\frac{\partial}{\partial \mathbf{r}} + \frac{\mathbf{e}_\theta}{\mathbf{r}}\frac{\partial}{\partial \theta} + \mathbf{e_z}\frac{\partial}{\partial z}\right)(u_r\mathbf{e_r} + u_\theta\mathbf{e}_\theta + u_z\mathbf{e_z}) \tag{66}$$

$$\nabla u = \mathbf{e_r}\frac{\partial}{\partial r}(u_r\mathbf{e_r} + u_\theta\mathbf{e}_\theta + u_z\mathbf{e_z}) + \frac{\mathbf{e}_\theta}{r}\frac{\partial}{\partial \theta}(u_r\mathbf{e_r} + u_\theta\mathbf{e}_\theta + u_z\mathbf{e_z}) + \mathbf{e_z}\frac{\partial}{\partial z}(u_r\mathbf{e_r} + u_\theta\mathbf{e}_\theta + u_z\mathbf{e_z}) \tag{67}$$

The following identities are to be noted,

$$\frac{\partial \mathbf{e_r}}{\partial r} = 0; \frac{\partial \mathbf{e}_\theta}{\partial r} = 0; \frac{\partial \mathbf{e_z}}{\partial r} = 0;$$
$$\frac{\partial \mathbf{e_r}}{\partial \theta} = \mathbf{e}_\theta; \frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\mathbf{e_r}; \frac{\partial \mathbf{e_z}}{\partial \theta} = 0;$$
$$\frac{\partial \mathbf{e_r}}{\partial z} = 0; \frac{\partial \mathbf{e}_\theta}{\partial z} = 0; \frac{\partial \mathbf{e_z}}{\partial z} = 0;$$

We evaluate eq(67) term by term by using the above identities

$$\mathbf{e_r}\frac{\partial}{\partial r}(u_r\mathbf{e_r} + u_\theta\mathbf{e}_\theta + u_z\mathbf{e_z}) = \mathbf{e_r}\mathbf{e_r}\frac{\partial u_r}{\partial r} + \mathbf{e_r}\mathbf{e}_\theta\frac{\partial u_\theta}{\partial r} + \mathbf{e_r}\mathbf{e_z}\frac{\partial u_z}{\partial r} \tag{68}$$

$$\frac{\mathbf{e}_\theta}{r}\frac{\partial}{\partial \theta}(u_r\mathbf{e_r} + u_\theta\mathbf{e}_\theta + u_z\mathbf{e_z}) = \mathbf{e}_\theta\mathbf{e_r}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta}\right) + \mathbf{e}_\theta\mathbf{e}_\theta\left(\frac{u_r}{r}\right) + \mathbf{e}_\theta\mathbf{e}_\theta\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\right) - \mathbf{e}_\theta\mathbf{e_r}\left(\frac{u_\theta}{r}\right) + \mathbf{e}_\theta\mathbf{e_z}\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta}\right) \tag{69}$$

$$\mathbf{e_z}\frac{\partial}{\partial z}(u_r\mathbf{e_r} + u_\theta\mathbf{e}_\theta + u_z\mathbf{e_z}) = \mathbf{e_z}\mathbf{e_r}\frac{\partial u_r}{\partial z} + \mathbf{e_z}\mathbf{e}_\theta\frac{\partial u_\theta}{\partial z} + \mathbf{e_z}\mathbf{e_z}\frac{\partial u_z}{\partial z} \tag{70}$$

$$\nabla u = \begin{bmatrix} \frac{\partial u_r}{\partial r} & \frac{\partial u_\theta}{\partial r} & \frac{\partial u_z}{\partial r} \\ \left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}\right) & \left(\frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\right) & \frac{1}{r}\frac{\partial u_z}{\partial \theta} \\ \frac{\partial u_r}{\partial z} & \frac{\partial u_\theta}{\partial z} & \frac{\partial u_z}{\partial z} \end{bmatrix} \tag{71}$$

$$(\nabla u)^T = \begin{bmatrix} \frac{\partial u_r}{\partial r} & \left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}\right) & \frac{\partial u_r}{\partial z} \\ \frac{\partial u_\theta}{\partial r} & \left(\frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\right) & \frac{\partial u_\theta}{\partial z} \\ \frac{\partial u_z}{\partial r} & \frac{1}{r}\frac{\partial u_z}{\partial \theta} & \frac{\partial u_z}{\partial z} \end{bmatrix} \tag{72}$$

$$[\nabla u + (\nabla u)^T] = \begin{bmatrix} 2\frac{\partial u_r}{\partial r} & \left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r}\right) & \left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right) \\ \left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r}\right) & 2\left(\frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\right) & \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}\right) \\ \left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right) & \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}\right) & 2\frac{\partial u_z}{\partial z} \end{bmatrix} \tag{73}$$

eq(65) becomes ,

$$\tau = \begin{bmatrix} -p + 2\mu\frac{\partial u_r}{\partial r} & \mu\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r}\right) & \mu\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right) \\ \mu\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r}\right) & -p + 2\mu\left(\frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\right) & \mu\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}\right) \\ \mu\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right) & \mu\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}\right) & -p + 2\mu\frac{\partial u_z}{\partial z} \end{bmatrix} \tag{74}$$

4.1 Stress balance conditions at the liquid-gas interface.

We now derive the normal and tangential stress conditions at the liquid-gas interface. Consider an interfacial surface S bounded by a closed curve C. Surface tension σ is defined as the force per unit length in the s-direction at every point along C that acts to flatten the surface S. Performing a force balance on a volume element V enclosing the interfacial surface S defined by the contour C :

$$\int_V \rho \frac{D\mathbf{u}}{Dt} dV = \int_V f dV + \int_S [(\mathbf{t_1}(\mathbf{n})) + (\mathbf{t_2}(\mathbf{n}))] dS + \int_C \sigma \mathbf{s} dl \tag{75}$$

dl is elemental length segment along the curve C. $\tau = -pI + \mu[(\nabla u) + (\nabla u)^T]$ is the total normal stress tensor. $\mathbf{t_1}(\mathbf{n}) = -\mathbf{n} \cdot \tau_\mathbf{1}$ is the total normal stress vector exerted by the liquid phase on the interface while $\mathbf{t_2}(\mathbf{n}) = \mathbf{n} \cdot \tau_\mathbf{2}$ is the total normal stress vector exerted by the gas phase on the interface. Now if ϵ is the typical length scale of the element V ,then the acceleration and body forces will scale as ϵ^3, while the surface forces scale as ϵ^2. Hence in the limit of $\epsilon \to 0$, only the surface forces must balance,

$$\int_S [(\mathbf{t_1}(\mathbf{n})) + (\mathbf{t_2}(\mathbf{n}))] dS + \int_C \sigma \mathbf{s} dl = 0 \tag{76}$$

consider $\int_C \sigma \mathbf{s} dl$. using stokes theorem we can write ,

$$\int_C \sigma \mathbf{s} dl = \int_S \mathbf{n}\sigma(\nabla \cdot \mathbf{n}) ds \tag{77}$$

proof

From stokes theorem we know that,

$$\int_C \mathbf{F}\cdot\mathbf{dl} = \int_S \mathbf{n}\cdot(\nabla\times\mathbf{F})ds$$

Along the curve C, $\mathbf{dl} = \mathbf{m}dl$

$$\int_C \mathbf{F}\cdot\mathbf{m}dl = \int_s \mathbf{n}\cdot(\nabla\times\mathbf{F})ds$$

choose $\mathbf{F} = \mathbf{f}\times\mathbf{b}$. Where $\mathbf{b}$ is an arbitrary constant vector.

$$\int_C (\mathbf{f}\times\mathbf{b}).\mathbf{m}dl = \int_S \mathbf{n}\cdot(\nabla\times(\mathbf{f}\times\mathbf{b}))ds$$

using the vector identities ,

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = -\mathbf{B}\cdot(\mathbf{A}\times\mathbf{C}) \tag{78}$$

$$\nabla\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A}) + (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} \tag{79}$$

the LHS can be written $\int_C (\mathbf{f}\times\mathbf{b}).\mathbf{m}dl = \int_C -\mathbf{b}\cdot(\mathbf{f}\times\mathbf{m})dl$, the RHS can be written as

$\int_S \mathbf{n}\cdot[\mathbf{f}(\nabla\cdot\mathbf{b}) - \mathbf{b}(\nabla\cdot\mathbf{f}) + (\mathbf{b}\cdot\nabla\mathbf{f}) - (\mathbf{f}\cdot\nabla\mathbf{b})ds$

since $\mathbf{b}$ is an constant vector $(\nabla\cdot b) = 0$ and $\nabla b = 0$

$$\int_C -\mathbf{b}\cdot(\mathbf{f}\times\mathbf{m})dl = \int_S \mathbf{n}\cdot(-\mathbf{b}(\nabla\cdot\mathbf{f}) + (\mathbf{b}\cdot\nabla\mathbf{f}))ds \tag{80}$$

since $\mathbf{b}$ is an arbitrary vector

$$b\cdot\int_C (\mathbf{f}\times\mathbf{m})dl = b\cdot\int_S (\mathbf{n}(\nabla\cdot\mathbf{f}) - (\mathbf{n}\cdot\nabla\mathbf{f}))ds \tag{81}$$

choose $\mathbf{f} = \sigma\mathbf{n}$.

$$\int_C (\sigma\mathbf{n}\times\mathbf{m})dl = \int_S [\mathbf{n}(\nabla\cdot\sigma\mathbf{n}) - (\mathbf{n}\cdot\nabla\sigma\mathbf{n})]ds \tag{82}$$

we note the following,

1) $\nabla\cdot\sigma\mathbf{n} = \nabla\sigma\cdot\mathbf{n} + \sigma\nabla\cdot\mathbf{n}$.

2) $\nabla(\sigma\mathbf{n}) = \sigma\nabla\mathbf{n} + \mathbf{n}\nabla\sigma$.

3) $\mathbf{n}\times\mathbf{m} = -\mathbf{s}$

4) $\nabla\sigma\cdot\mathbf{n} = 0$

5) $\nabla\mathbf{n}\cdot\mathbf{n} = \frac{\mathbf{1}}{\mathbf{2}}\nabla(\mathbf{n}\cdot\mathbf{n}) = 0$

6) We assume that $\nabla\sigma = 0$

This yields,

$$-\int_C \sigma \mathbf{s} dl = \int_S \mathbf{n}\sigma(\nabla \cdot \mathbf{n}) ds$$

end of proof.

Rewriting eq(76),

$$\int_S [(\mathbf{t_1}(\mathbf{n})) + (\mathbf{t_2}(\mathbf{n}))] dS + \int_C \sigma \mathbf{s} dl = 0$$

$$\int_S [\mathbf{n} \cdot \tau_{\mathbf{2}} - \mathbf{n} \cdot \tau_{\mathbf{1}}] ds - \int_S \mathbf{n} \sigma (\nabla \cdot \mathbf{n}) ds = 0 \tag{83}$$

Since the surface element is arbitrary , the integral must vanish identically.

$$\mathbf{n} \cdot \tau_{\mathbf{2}} - \mathbf{n} \cdot \tau_{\mathbf{1}} = \mathbf{n} \sigma (\nabla \cdot \mathbf{n}) \tag{84}$$

4.2 Normal stress balance

Taking $\mathbf{n} \cdot (eq84)$ yields the normal stress balance at the interface :

$$\mathbf{n} \cdot \tau_{\mathbf{2}} \cdot \mathbf{n} - \mathbf{n} \cdot \tau_{\mathbf{1}} \cdot \mathbf{n} = \sigma (\nabla \cdot \mathbf{n}) \tag{85}$$

4.3 Tangential stress balance

Taking $\mathbf{t} \cdot (eq84)$ yields the tangential stress balance at the interface :

$$\mathbf{n} \cdot \tau_{\mathbf{2}} \cdot \mathbf{t} - \mathbf{n} \cdot \tau_{\mathbf{1}} \cdot \mathbf{t} = 0 \tag{86}$$

where ,

$$\tau_1 = \begin{bmatrix} -p + 2\mu_1 \frac{\partial u_{r1}}{\partial r} & \mu_1 \left(\frac{1}{r} \frac{\partial u_{r1}}{\partial \theta} - \frac{u_{\theta 1}}{r} + \frac{\partial u_{\theta 1}}{\partial r} \right) & \mu_1 \left(\frac{\partial u_{z1}}{\partial r} + \frac{\partial u_{r1}}{\partial z} \right) \\ \mu_1 \left(\frac{1}{r} \frac{\partial u_{r1}}{\partial \theta} - \frac{u_{\theta 1}}{r} + \frac{\partial u_{\theta 1}}{\partial r} \right) & -p + 2\mu_1 \left(\frac{u_{r1}}{r} + \frac{1}{r} \frac{\partial u_{\theta 1}}{\partial \theta} \right) & \mu_1 \left(\frac{1}{r} \frac{\partial u_{z1}}{\partial \theta} + \frac{\partial u_{\theta 1}}{\partial z} \right) \\ \mu_1 \left(\frac{\partial u_{z1}}{\partial r} + \frac{\partial u_{r1}}{\partial z} \right) & \mu_1 \left(\frac{1}{r} \frac{\partial u_{z1}}{\partial \theta} + \frac{\partial u_{\theta 1}}{\partial z} \right) & -p + 2\mu_1 \frac{\partial u_{z1}}{\partial z} \end{bmatrix} \tag{87}$$

$$\tau_2 = \begin{bmatrix} -p + 2\mu_2 \frac{\partial u_{r2}}{\partial r} & \mu_2 \left(\frac{1}{r} \frac{\partial u_{r2}}{\partial \theta} - \frac{u_{\theta 2}}{r} + \frac{\partial u_{\theta 2}}{\partial r} \right) & \mu_2 \left(\frac{\partial u_{z2}}{\partial r} + \frac{\partial u_{r2}}{\partial z} \right) \\ \mu_2 \left(\frac{1}{r} \frac{\partial u_{r2}}{\partial \theta} - \frac{u_{\theta 2}}{r} + \frac{\partial u_{\theta 2}}{\partial r} \right) & -p + 2\mu_2 \left(\frac{u_{r2}}{r} + \frac{1}{r} \frac{\partial u_{\theta 2}}{\partial \theta} \right) & \mu_2 \left(\frac{1}{r} \frac{\partial u_{z2}}{\partial \theta} + \frac{\partial u_{\theta 2}}{\partial z} \right) \\ \mu_2 \left(\frac{\partial u_{z2}}{\partial r} + \frac{\partial u_{r2}}{\partial z} \right) & \mu_2 \left(\frac{1}{r} \frac{\partial u_{z2}}{\partial \theta} + \frac{\partial u_{\theta 2}}{\partial z} \right) & -p + 2\mu_2 \frac{\partial u_{z2}}{\partial z} \end{bmatrix} \tag{88}$$

4.4 Finding unit normal vector and unit tangent vector to the interface.

To find the tangent vector , we write the parametric form of the surface for which two independent parameters and three dependent variables are required.
Let the equation of surface in parametrized form be $\mathbf{X}(u,v) = [\mathbf{z}(u,v), \theta(u,v), \mathbf{r} = \mathbf{h}(u,v)]$, for some u and v intervals, define a surface S in the u-v plane. $\mathbf{X}_u$ is the tangent in the u direction , Since v is held constant. similarly $\mathbf{X}_v$ is the tangent in the v direction , Since u is held constant.

$$\mathbf{X}_u = \frac{\partial X}{\partial u} = [1, 0, \frac{\partial h}{\partial u}]$$

Let the position of the interface at any instant of time t be $r = \mathbf{h}(\mathbf{z}, \theta)$.
Choosing $\mathbf{z} = u$; , $\theta = v$. then $\mathbf{r} = \mathbf{h}(u, v)$, then the unit tangent vector in the r direction is given by,

$$\mathbf{t_r} = [1, 0, \frac{\partial \mathbf{h}}{\partial z}]\left(\frac{1}{\sqrt{1 + (\frac{\partial \mathbf{h}}{\partial z})^2}}\right) \tag{89}$$

Similarly the unit tangent vector in the θ direction is given by,

$$\mathbf{t}_\theta = [0, 1, \frac{1}{r}\frac{\partial \mathbf{h}}{\partial \theta}]\left(\frac{1}{\sqrt{1 + (\frac{1}{r}\frac{\partial \mathbf{h}}{\partial \theta})^2}}\right) \tag{90}$$

Since both $\mathbf{t_r}$ and $\mathbf{t}_\theta$ are orthogonal ($\mathbf{u}$,$\mathbf{v}$ are orthogonal),their cross product yiels a vector which is orthogonal to both $\mathbf{t_r}$ and $\mathbf{t}_\theta$ which is the normal vector to the interface.

$$\mathbf{n} = \frac{\mathbf{t_r} \times \mathbf{t}_\theta}{||\mathbf{t_r} \times \mathbf{t}_\theta||} \tag{91}$$

$$\mathbf{t_r} \times \mathbf{t}_\theta = \left[-\frac{\partial h}{\partial z}, -\frac{1}{r}\frac{\partial h}{\partial \theta}, 1\right] \tag{92}$$

$$||\mathbf{t_r} \times \mathbf{t}_\theta|| = \sqrt{\left(\frac{\partial h}{\partial z}\right)^2 + \left(\frac{1}{r}\frac{\partial h}{\partial \theta}\right)^2 + 1} \tag{93}$$

$$\mathbf{n} = \left[-\frac{\partial h}{\partial z}, -\frac{1}{r}\frac{\partial h}{\partial \theta}, 1\right]\frac{1}{\sqrt{\left(\frac{\partial h}{\partial z}\right)^2 + \left(\frac{1}{r}\frac{\partial h}{\partial \theta}\right)^2 + 1}} \tag{94}$$

4.5 Evaluating normal stress and shear stress at the interface.

rewriting eq(85) ,

$$\mathbf{n} \cdot \tau_\mathbf{2} \cdot \mathbf{n} - \mathbf{n} \cdot \tau_\mathbf{1} \cdot \mathbf{n} = \sigma(\nabla \cdot \mathbf{n})$$

evaluating ,

$$\mathbf{n} \cdot \tau \cdot \mathbf{n} = \sum_{l=1}^{3}(n_l \mathbf{e_l}) \cdot \left(\sum_{i=1}^{3}\sum_{j=1}^{3}(\tau_{ij}\mathbf{e_i}\mathbf{e_j})\right) \cdot \left(\sum_{l=1}^{3}(n_l \mathbf{e_l})\right) \tag{95}$$

$$\mathbf{n} \cdot \tau \cdot \mathbf{n} = \sum_{l=1}^{3}(n_l \mathbf{e_l}) \cdot \left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{l=1}^{3}(\tau_{ij} n_l(\mathbf{e_i}\mathbf{e_j}) \cdot \mathbf{e_l})\right)$$

$$\mathbf{n} \cdot \tau \cdot \mathbf{n} = \sum_{l=1}^{3}(n_l \mathbf{e_l}) \cdot \left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{l=1}^{3}(\tau_{ij} n_l(\mathbf{e_i}\delta_{\mathbf{jl}})\right)$$

$$\mathbf{n} \cdot \tau \cdot \mathbf{n} = \sum_{l=1}^{3}(n_l \mathbf{e_l}) \cdot \left(\sum_{i=1}^{3}\sum_{j=1}^{3}\tau_{ij} n_j \mathbf{e_i}\right)$$

$$\mathbf{n}\cdot\tau\cdot\mathbf{n} = \sum_{l=1}^{3}\sum_{i=1}^{3}\sum_{j=1}^{3}\tau_{ij}n_j n_l(\mathbf{e_i}\cdot\mathbf{e_l})$$

$$\mathbf{n}\cdot\tau\cdot\mathbf{n} = \sum_{l=1}^{3}\sum_{j=1}^{3}\tau_{lj}n_j n_l = \sum_{l=1}^{3}n_l\sum_{j=1}^{3}\tau_{lj}n_j$$

$$\mathbf{n}\cdot\tau\cdot\mathbf{n} = n_1(\tau_{11}n_1+\tau_{12}n_2+\tau_{13}n_3)+n_2(\tau_{21}n_1+\tau_{22}n_2+\tau_{23}n_3)+n_3(\tau_{31}n_1+\tau_{32}n_2+\tau_{33}n_3) \tag{96}$$

$$\mathrm{n}_1(\tau_{11}n_1+\tau_{12}n_2+\tau_{13}n_3) = \frac{1}{\left(\left(\frac{\partial h}{\partial z}\right)^2+\left(\frac{1}{r}\frac{\partial h}{\partial\theta}\right)^2+1\right)}\left[(-p+2\mu\frac{\partial u_r}{\partial r})+\mu\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta}{r}+\frac{\partial u_\theta}{\partial r}\right)(\frac{-1}{r}\frac{\partial h}{\partial\theta})+\mu\left(\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}\right)(\frac{-\partial h}{\partial z})\right] \tag{97}$$

$$n_2(\tau_{21}n_1+\tau_{22}n_2+\tau_{23}n_3) = \frac{\frac{-1}{r}\frac{\partial h}{\partial\theta}}{\left(\left(\frac{\partial h}{\partial z}\right)^2+\left(\frac{1}{r}\frac{\partial h}{\partial\theta}\right)^2+1\right)}\left[\mu\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta}{r}+\frac{\partial u_\theta}{\partial r}\right)+\left(-p+2\mu\left(\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right)(\frac{-1}{r}\frac{\partial h}{\partial\theta})\right)+\mu\left(\frac{1}{r}\frac{\partial u_z}{\partial\theta}+\frac{\partial u_\theta}{\partial z}\right)(\frac{-\partial h}{\partial z} \tag{98}$$

$$n_3(\tau_{31}n_1+\tau_{32}n_2+\tau_{33}n_3) = \frac{\frac{-\partial h}{\partial z}}{\left(\left(\frac{\partial h}{\partial z}\right)^2+\left(\frac{1}{r}\frac{\partial h}{\partial\theta}\right)^2+1\right)}\left[\mu\left(\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}\right)+\mu\left(\frac{1}{r}\frac{\partial u_z}{\partial\theta}+\frac{\partial u_\theta}{\partial z}\right)(\frac{-1}{r}\frac{\partial h}{\partial\theta})+(-p+2\mu\frac{\partial u_z}{\partial z})(\frac{-\partial h}{\partial z})\right] \tag{99}$$

4.6 Evaluation of tangential stress.

We saw from the stress balance equations that the tangential stress balance across the interface gives,

$$\mathbf{n}\cdot\tau_\mathbf{2}\cdot\mathbf{t}-\mathbf{n}\cdot\tau_\mathbf{1}\cdot\mathbf{t} = 0 \tag{100}$$

where $\mathbf{t}$ is a unit tangent vector to the interface surface. This has got 2 components $\mathbf{t_r}$ and $\mathbf{t}_\theta$,

$$\mathbf{n}\cdot\tau_\mathbf{2}\cdot\mathbf{t_r}-\mathbf{n}\cdot\tau_\mathbf{1}\cdot\mathbf{t_r} = 0 \tag{101}$$

and ,

$$\mathbf{n}\cdot\tau_\mathbf{2}\cdot\mathbf{t}_\theta-\mathbf{n}\cdot\tau_\mathbf{1}\cdot\mathbf{t}_\theta = 0 \tag{102}$$

evaulating ,

$$\mathbf{n}\cdot\tau\cdot\mathbf{t_r} = \sum_{l=1}^{3}(n_l\mathbf{e_l})\cdot\left(\sum_{i=1}^{3}\sum_{j=1}^{3}(\tau_{ij}\mathbf{e_i}\mathbf{e_j})\right)\cdot\left(\sum_{k=1}^{3}(t_k\mathbf{e_k})\right) \tag{103}$$

$$\mathbf{n}\cdot\tau\cdot\mathbf{t_r} = \sum_{l=1}^{3}(n_l\mathbf{e_l})\cdot\left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}(\tau_{ij}t_k(\mathbf{e_i}\delta_\mathbf{jk})\right)$$

$$\mathbf{n}\cdot\tau\cdot\mathbf{t_r} = \sum_{l=1}^{3}(n_l\mathbf{e_l})\cdot\left(\sum_{i=1}^{3}\sum_{j=1}^{3}\tau_{ij}t_j\mathbf{e_i}\right)$$

$$\mathbf{n}\cdot\tau\cdot\mathbf{t_r} = \sum_{l=1}^{3}\sum_{i=1}^{3}\sum_{j=1}^{3}\tau_{ij}t_j n_l(\mathbf{e_i}\cdot\mathbf{e_l})$$

$$\mathbf{n}\cdot\tau\cdot\mathbf{t_r} = \sum_{l=1}^{3}\sum_{j=1}^{3}\tau_{lj}t_j n_l = \sum_{l=1}^{3} n_l \sum_{j=1}^{3}\tau_{lj}t_j$$

$$\mathbf{n}\cdot\tau\cdot\mathbf{t_r} = n_1(\tau_{11}t_1+\tau_{12}t_2+\tau_{13}t_3)+n_2(\tau_{21}t_1+\tau_{22}t_2+\tau_{23}t_3)+n_3(\tau_{31}t_1+\tau_{32}t_2+\tau_{33}t_3) \tag{104}$$

$$\mathbf{t_r} = [t_r, t_\theta, t_z] = [t_1, t_2, t_3] = [\frac{\partial h}{\partial z}, 0, 1]\left(\frac{1}{\sqrt{1+(\frac{\partial \mathbf{h}}{\partial z})^2}}\right)$$

$$n_1(\tau_{11}t_1+\tau_{12}t_2+\tau_{13}t_3) = \frac{1}{\left(\sqrt{\left(\frac{\partial h}{\partial z}\right)^2+1}\right)}\frac{1}{\sqrt{\left(\frac{\partial h}{\partial z}\right)^2+\left(\frac{1}{r}\frac{\partial h}{\partial \theta}\right)^2+1}}\left[(-p+2\mu\frac{\partial u_r}{\partial r})(\frac{\partial h}{\partial z})+\mu\left(\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}\right)\right] \tag{105}$$

$$n_2(\tau_{21}t_1+\tau_{22}t_2+\tau_{23}t_3) = \frac{1}{\left(\sqrt{\left(\frac{\partial h}{\partial z}\right)^2+1}\right)}\frac{\frac{-1}{r}\frac{\partial h}{\partial \theta}}{\sqrt{\left(\frac{\partial h}{\partial z}\right)^2+\left(\frac{1}{r}\frac{\partial h}{\partial \theta}\right)^2+1}}\left[\mu\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta}{r}+\frac{\partial u_\theta}{\partial r}\right)(\frac{\partial h}{\partial z})+\mu\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta}+\frac{\partial u_\theta}{\partial z}\right)\right] \tag{106}$$

$$n_3(\tau_{31}t_1+\tau_{32}t_2+\tau_{33}t_3) = \frac{1}{\left(\sqrt{\left(\frac{\partial h}{\partial z}\right)^2+1}\right)}\frac{\frac{-\partial h}{\partial z}}{\sqrt{\left(\frac{\partial h}{\partial z}\right)^2+\left(\frac{1}{r}\frac{\partial h}{\partial \theta}\right)^2+1}}\left[\mu\left(\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}\right)(\frac{\partial h}{\partial z})+\left(-p+2\mu\frac{\partial u_z}{\partial z}\right)\right] \tag{107}$$

evaluating ,

$$\mathbf{n}\cdot\tau\cdot\mathbf{t_\theta} = \sum_{l=1}^{3}(n_l\mathbf{e_l})\cdot\left(\sum_{i=1}^{3}\sum_{j=1}^{3}(\tau_{ij}\mathbf{e_i}\mathbf{e_j})\right)\cdot\left(\sum_{k=1}^{3}(t_k\mathbf{e_k})\right) \tag{108}$$

$$\mathbf{n}\cdot\tau\cdot\mathbf{t_\theta} = \sum_{l=1}^{3}(n_l\mathbf{e_l})\cdot\left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}(\tau_{ij}t_k(\mathbf{e_i}\delta_{\mathbf{jk}})\right)$$

$$\mathbf{n}\cdot\tau\cdot\mathbf{t_\theta} = \sum_{l=1}^{3}(n_l\mathbf{e_l})\cdot\left(\sum_{i=1}^{3}\sum_{j=1}^{3}\tau_{ij}t_j\mathbf{e_i}\right)$$

$$\mathbf{n}\cdot\tau\cdot\mathbf{t_\theta} = \sum_{l=1}^{3}\sum_{i=1}^{3}\sum_{j=1}^{3}\tau_{ij}t_j n_l(\mathbf{e_i}\cdot\mathbf{e_l})$$

$$\mathbf{n}\cdot\tau\cdot\mathbf{t_\theta} = \sum_{l=1}^{3}\sum_{j=1}^{3}\tau_{lj}t_j n_l = \sum_{l=1}^{3} n_l \sum_{j=1}^{3}\tau_{lj}t_j$$

$$\mathbf{n}\cdot\tau\cdot\mathbf{t}_\theta = n_1(\tau_{11}t_1+\tau_{12}t_2+\tau_{13}t_3)+n_2(\tau_{21}t_1+\tau_{22}t_2+\tau_{23}t_3)+n_3(\tau_{31}t_1+\tau_{32}t_2+\tau_{33}t_3) \tag{109}$$

$$\mathbf{t}_\theta = [0,1,\frac{1}{r}\frac{\partial h}{\partial\theta}]\left(\frac{1}{\sqrt{1+(\frac{1}{r}\frac{\partial \mathbf{h}}{\partial\theta})^2}}\right)$$

$$[t_1,t_2,t_3] = [\frac{1}{r}\frac{\partial h}{\partial\theta},1,0]\left(\frac{1}{\sqrt{1+(\frac{1}{r}\frac{\partial \mathbf{h}}{\partial\theta})^2}}\right)$$

$$n_1(\tau_{11}t_1+\tau_{12}t_2+\tau_{13}t_3) = \frac{1}{\left(\sqrt{\left(\frac{1}{r}\frac{\partial h}{\partial\theta}\right)^2+1}\right)}\frac{1}{\sqrt{\left(\frac{\partial h}{\partial z}\right)^2+\left(\frac{1}{r}\frac{\partial h}{\partial\theta}\right)^2+1}}\left[\left(-p+2\mu\frac{\partial u_r}{\partial r}\right)(\frac{1}{r}\frac{\partial h}{\partial\theta})+\mu\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta}{r}+\frac{\partial u_\theta}{\partial r}\right)\right] \tag{110}$$

$$n_2(\tau_{21}t_1+\tau_{22}t_2+\tau_{23}t_3) = \frac{1}{\left(\sqrt{\left(\frac{1}{r}\frac{\partial h}{\partial\theta}\right)^2+1}\right)}\frac{\frac{-1}{r}\frac{\partial h}{\partial\theta}}{\sqrt{\left(\frac{\partial h}{\partial z}\right)^2+\left(\frac{1}{r}\frac{\partial h}{\partial\theta}\right)^2+1}}\left[\mu\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta}{r}+\frac{\partial u_\theta}{\partial r}\right)(\frac{1}{r}\frac{\partial h}{\partial\theta})+\left(-p+2\mu\left(\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right)\right)\right] \tag{111}$$

$$n_3(\tau_{31}t_1+\tau_{32}t_2+\tau_{33}t_3) = \frac{1}{\left(\sqrt{\left(\frac{1}{r}\frac{\partial h}{\partial\theta}\right)^2+1}\right)}\frac{\frac{-\partial h}{\partial z}}{\sqrt{\left(\frac{\partial h}{\partial z}\right)^2+\left(\frac{1}{r}\frac{\partial h}{\partial\theta}\right)^2+1}}\left[\mu\left(\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}\right)(\frac{1}{r}\frac{\partial h}{\partial\theta})+\mu\left(\frac{1}{r}\frac{\partial u_z}{\partial\theta}+\frac{\partial u_\theta}{\partial z}\right)\right] \tag{112}$$

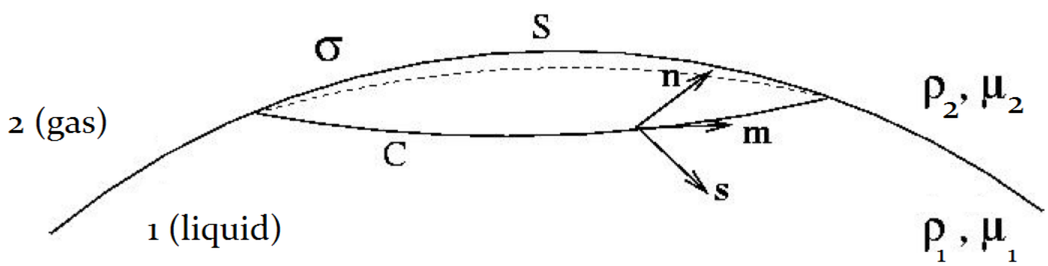

Figure 1: A surface S and bounding contour C on an interface between two fluids. The upper fluid (**2**) has density ρ and viscosity μ; the lower fluid (**1**), ρ and μ. $\mathbf{n}$ represents the unit outward normal to the surface, and $\mathbf{n} = -\mathbf{n}$ the unit inward normal. $\mathbf{m}$ the unit tangent to the contour C and $\mathbf{s}$ the unit vector normal to C but tangent to S.

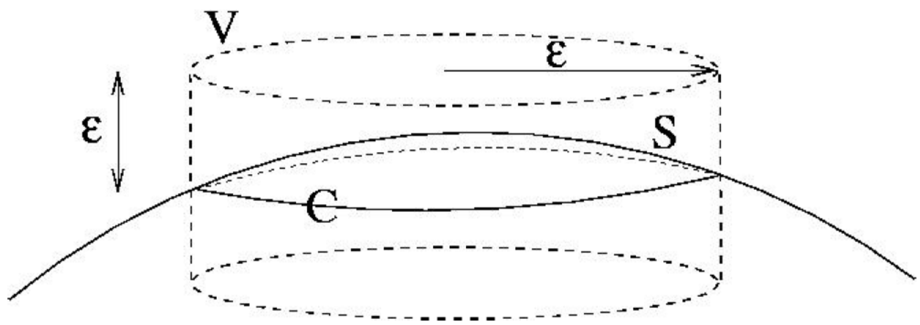

Figure 2: A Gaussian fluid pillbox of height and radius ϵ spanning the interface evolves under the combined influence of volume and surface forces.

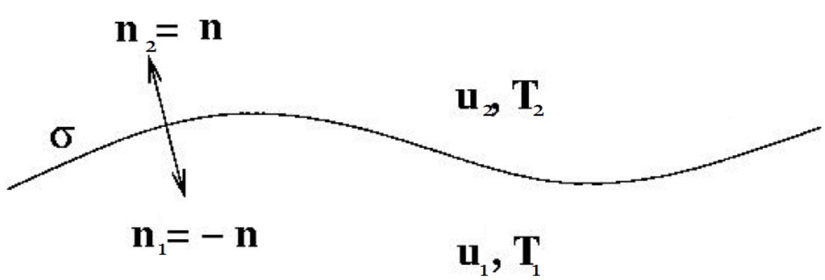

Figure 3: A definitional sketch of a fluid-fluid interface.

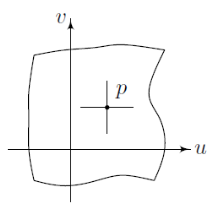

Figure 4.1: Point p on the parametric surface.

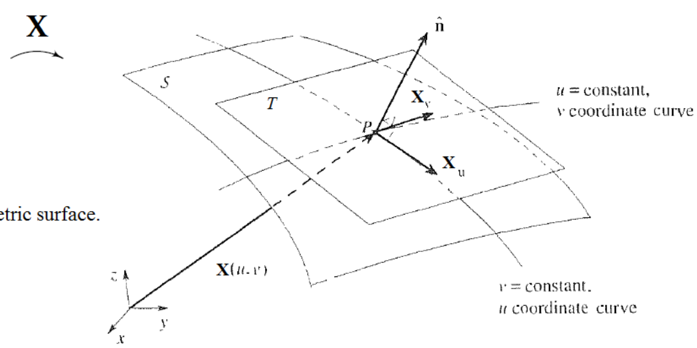

Figure 4. Tangent plane and normal.