$$\phi(r,\theta,z,t) = h(z,\theta,t) - r \tag{1}$$

$$\frac{D\phi}{Dt} = 0 \tag{2}$$

$$\left(\frac{\partial\phi}{\partial r}\right)\frac{dr}{dt} + \left(\frac{\partial\phi}{\partial\theta}\right)\frac{d\theta}{dt} + \left(\frac{\partial\phi}{\partial z}\right)\frac{dz}{dt} + \left(\frac{\partial\phi}{\partial t}\right) = 0 \tag{3}$$

$$\left(\frac{\partial(h-r)}{\partial r}\right)\frac{dr}{dt} + \left(\frac{\partial(h-r)}{\partial\theta}\right)\frac{d\theta}{dt} + \left(\frac{\partial(h-r)}{\partial z}\right)\frac{dz}{dt} + \left(\frac{\partial(h-r)}{\partial t}\right) = 0 \tag{4}$$

$$\left(\frac{\partial h}{\partial r}\right)u_r - u_r + \frac{u_\theta}{r}\left(\frac{\partial h}{\partial\theta}\right) + u_z\left(\frac{\partial h}{\partial z}\right) + \left(\frac{\partial h}{\partial t}\right) - u_r = 0 \tag{5}$$

$$\left(\frac{\partial h}{\partial r}\right)u_r + \frac{u_\theta}{r}\left(\frac{\partial h}{\partial\theta}\right) + u_z\left(\frac{\partial h}{\partial z}\right) + \left(\frac{\partial h}{\partial t}\right) = 2u_r \tag{6}$$

rearranging,

$$\left(\frac{\partial h}{\partial t}\right) + u_r\left(\frac{\partial h}{\partial r}\right) + \frac{u_\theta}{r}\left(\frac{\partial h}{\partial\theta}\right) + u_z\left(\frac{\partial h}{\partial z}\right) = 2u_r \tag{7}$$

we note the following ,

$$h(z,\theta,t) = \overline{h}(z) + h'(z,\theta,t) \tag{8}$$

$$u_r(h,z,\theta,t) = \overline{U}_r(r) + u_r'(h,z,\theta,t) \tag{9}$$

$$u_\theta(h,z,\theta,t) = u'(h,z,\theta,t) \tag{10}$$

(since the base flow is axisymmetric $\overline{U}_\theta = 0$)

$$u_z(h,z,\theta,t) = \overline{U}_z(r) + u_z'(h,z,\theta,t) \tag{11}$$

$$\left(\frac{\partial h(z,\theta,t)}{\partial t}\right) + u_r(h,z,\theta,t)\left(\frac{\partial h(z,\theta,t)}{\partial r}\right) +$$

$$\frac{u_\theta(h,z,\theta,t)}{r}\left(\frac{\partial h(z,\theta,t)}{\partial\theta}\right) + u_z(h,z,\theta,t)\left(\frac{\partial h(z,\theta,t)}{\partial z}\right) = 2(\overline{U_r}(r) + u_r'(h,z,\theta,t)) \tag{12}$$

$$\left(\frac{\partial\overline{h}}{\partial t}\right)+\left(\frac{\partial h'}{\partial t}\right)+\overline{U}_r\left(\frac{\partial\overline{h}(z)}{\partial r}\right)+\quad \overline{U}_r\left(\frac{\partial h'}{\partial r}\right)+u_r'\left(\frac{\partial\overline{h}(z)}{\partial r}\right)\quad +u_r'\left(\frac{\partial h'}{\partial r}\right)+$$
$$\frac{u_\theta'}{r}\left(\frac{\partial\overline{h}}{\partial\theta}\right)+\frac{u_\theta'}{r}\left(\frac{\partial h'}{\partial\theta}\right)+\quad \overline{U}_z\left(\frac{\partial\overline{h}}{\partial z}\right)+\overline{U}_z\left(\frac{\partial h'}{\partial z}\right)\quad +u_z'\left(\frac{\partial\overline{h}}{\partial z}\right)+u_z'\left(\frac{\partial h'}{\partial z}\right)$$
$$=2(\overline{U_r}(r)+u_r'(h,z,\theta,t))$$

since $\overline{h}$ is a function of z only, $\frac{\partial\overline{h}(z)}{\partial r}=\frac{\partial\overline{h}(z)}{\partial\theta}=0$ and $\frac{\partial h'(z,\theta,t)}{\partial r}=0$. We have,

$$\left(\frac{\partial\overline{h}}{\partial t}\right)+\left(\frac{\partial h'}{\partial t}\right)+\frac{u_\theta'}{r}\left(\frac{\partial h'}{\partial\theta}\right)$$

$$+\overline{U}_z\left(\frac{\partial\overline{h}}{\partial z}\right)+\overline{U}_z\left(\frac{\partial h'}{\partial z}\right)+u_z'\left(\frac{\partial\overline{h}}{\partial z}\right)+u_z'\left(\frac{\partial h'}{\partial z}\right)=2(\overline{U_r}(r)+u_r'(h,z,\theta,t)) \tag{13}$$

Equation for the mean is given by

$$\left(\frac{\partial\overline{h}}{\partial t}\right)+\overline{U}_r\left(\frac{\partial\overline{h}(z)}{\partial r}\right)+\overline{U}_z\left(\frac{d\overline{h}}{dz}\right)=2\overline{U}_r(r) \tag{14}$$

or,

$$\left(\frac{\partial\overline{h}}{\partial t}\right)+\overline{U}_z\left(\frac{d\overline{h}}{dz}\right)=2\overline{U}_r(r) \tag{15}$$

Subtracting the mean from the total flow, we get the perturbed equation of h:

$$\left(\frac{\partial h'}{\partial t}\right)+\frac{u_\theta'}{r}\left(\frac{\partial h'}{\partial\theta}\right)+\overline{U}_z\left(\frac{\partial h'}{\partial z}\right)+u_z'\left(\frac{d\overline{h}}{dz}\right)+u_z'\left(\frac{\partial h'}{\partial z}\right)=2u_r'(h,z,\theta,t) \tag{16}$$

assuming small perturbations, we neglect the product of small quantities to yield,

$$\left(\frac{\partial h'(z,\theta,t)}{\partial t}\right)+\overline{U}_z(r)\left(\frac{\partial h'(z,\theta,t)}{\partial z}\right)+\underbrace{u_z'(h,z,\theta,t)}\left(\frac{d\overline{h}(z)}{dz}\right)=2\,\underbrace{u_r'(h,z,\theta,t)} \tag{17}$$

The above equation is nonlinear in h. We know that $h=\overline{h}(z)+h'(z,\theta,t)$

To linearize eq(17) equation, we need to expand the terms in the curly braces using taylor series about $\overline{h}$.

$$u_r'(h,z,\theta,t)=u_r'(\overline{h}(z),z,\theta,t)+(h-\overline{h})\left(\frac{\partial u_r'}{\partial h}\right)_{(\overline{h},z,\theta,t)}$$

$$+(z-z)\left(\frac{\partial u'_z}{\partial h}\right)_{(\overline{h},z,\theta,t)}+(\theta-\theta)\left(\frac{\partial u'_\theta}{\partial h}\right)_{(\overline{h},z,\theta,t)}+\left\{O(h-\overline{h})^2\right\}+.... \tag{18}$$

Neglecting higher order terms,

$$u'_r(h,z,\theta,t)=u'_r(\overline{h}(z),z,\theta,t)+(h-\overline{h})\left(\frac{\partial u'_r}{\partial h}\right)_{(\overline{h},z,\theta,t)} \tag{19}$$

but $h-\overline{h}=h'$ which is a function of z,θ,t .
i.e.,

$$u'_r(h,z,\theta,t)=u'_r(\overline{h}(z),z,\theta,t)+h'(z,\theta,t)\left(\frac{\partial u'_r}{\partial h}\right)_{(\overline{h},z,\theta,t)} \tag{20}$$

Similarly,

$$u'_z(h,z,\theta,t)=u'_z(\overline{h}(z),z,\theta,t)+h'(z,\theta,t)\left(\frac{\partial u'_z}{\partial h}\right)_{(\overline{h},z,\theta,t)} \tag{21}$$

substituting eq (20) and eq(21) into eq(17)

$$\left(\frac{\partial h'(z,\theta,t)}{\partial t}\right) \qquad +\overline{U}_z(r)\left(\frac{\partial h'(z,\theta,t)}{\partial z}\right)+$$

$$\left\{u'_z(\overline{h}(z),z,\theta,t)+h'(z,\theta,t)\left(\frac{\partial u'_z}{\partial h}\right)_{(\overline{h},z,\theta,t)}\right\} \quad \left(\frac{d\overline{h}(z)}{dz}\right) \quad =2\left\{u'_r(\overline{h}(z),z,\theta,t)+h'(z,\theta,t)\left(\frac{\partial u'_r}{\partial h}\right)_{(\overline{h},z,\theta,t)}\right\}$$

$$\frac{\partial h'}{\partial t}+\overline{U}_z(r)\left(\frac{\partial h'}{\partial z}\right)+u'_z\frac{d\overline{h}(z)}{dz}+h'\left(\frac{\partial u'_z}{\partial h}\right)_{(\overline{h},z,\theta,t)}\frac{d\overline{h}(z)}{dz}=2u'_r+2h'\left(\frac{\partial u'_r}{\partial h}\right)_{(\overline{h},z,\theta,t)} \tag{22}$$

$$\frac{\partial h'}{\partial t}+\overline{U}_z(r)\left(\frac{\partial h'}{\partial z}\right)+u'_z\frac{d\overline{h}(z)}{dz}+\underbrace{h'\left(\frac{\partial u'_z}{\partial h}\right)_{(\overline{h},z,\theta,t)}}\left(\frac{d\overline{h}(z)}{dz}\right)=2u'_r+2\underbrace{h'\left(\frac{\partial u'_r}{\partial h}\right)_{(\overline{h},z,\theta,t)}} \tag{23}$$

We can further linearize the above equation assuming small perturbation, so that the terms in the curly brackets are ignored.

Note: consider the term $\underbrace{h'\left(\frac{\partial u'_z}{\partial h}\right)_{(\overline{h},z,\theta,t)}}\left(\frac{d\overline{h}(z)}{dz}\right)$. *Even though the term in the curly bracket is small ,* $\frac{d\overline{h}(z)}{dz}$ *need not be,* *so that their product is finite.But we assume that* $\frac{d\overline{h}(z)}{dz}$ *is bounded, hence the entire term can be neglected.*

$$\frac{\partial h'(z,\theta,t)}{\partial t} + \overline{U}_z(r)\left(\frac{\partial h'(z,\theta,t)}{\partial z}\right) + u'_z(\overline{h}(z),z,\theta,t)\left(\frac{d\overline{h}(z)}{dz}\right) = 2u'_r(\overline{h}(z),z,\theta,t) \tag{24}$$

The above equation is linear .