Unit 2 Homework: Characterizing Random Variables

w203: Statistics for Data Science

Applied Practice

1. Processing Pasta

A certain manufacturing process creates pieces of pasta that vary by length. Suppose that the length of a particular piece (represented as a random variable L that takes on values l), is a continuous random variable with the following probability density function.

$$f(l) = \begin{cases} 0, & l \leq 0 \\ c \cdot l, & 0 < l \leq 2 \\ 0, & 2 < l \end{cases},$$

where c is a constant that makes $f(l)$ a valid probability distribution function.

a. (1 points) Compute the constant c. What value of c makes $f(l)$ a valid pdf?
b. (1 point) What is the cumulative probability when $l \leq 0$?
c. (1 point) What is the cumulative probability function when $0 < l \leq 2$?
d. (1 point) What is the cumulative probability function when $2 \leq l$?
e. (1 points) Compute the median value of L. That is, compute l such that $P(L \leq l) = 1/2$.

Proof Practice

2. Broken Rulers

You have a ruler of length 2 and you choose a place to break it using a uniform probability distribution. Let random variable X represent the length of the left piece of the ruler. X is distributed uniformly in $[0, 2]$. You take the left piece of the ruler and once again choose a place to break it using a uniform probability distribution. Let random variable Y be the length of the left piece from the second break.

a. (3 points) Draw a picture of the region in the X-Y plane for which the joint density of X and Y is non-zero.

b. (3 points) Compute the joint density function for X and Y. (As always, make sure you write a complete expression.)

c. (3 points) Compute the marginal probability density for Y, $f_Y(y)$.

d. (3 points) Compute the conditional probability density of X, conditional on $Y = y$, $f_{X|Y}(x|y)$. (Make sure you state the values of y for which this exists.)

3. Post-Processing and Independence

What if you have two random variables, X and Y that are independent. What happens if you apply a function, f, onto X? Is this newly transformed random variable, $f(X)$ still independent of Y?

This question is posed in a way that works for any random variable, with any possible function from $\mathbb{R}$ to $\mathbb{R}$. A student with a strong math background might choose to prove this for the general case; as a hint to these students, our proof involves inverse images.

For students who are building their math fundimentals, consider a proof by cases for a particular type of random variable, a Bernoulli random variable.

That is, suppose X and Y are Bernoulli random variables that each take on values $\{0,\ 1\}$. Furthermore, suppose that f is a function that maps the input 0 either onto 0 or 1, and it maps the input 1 onto either 0 or 1. That is, $f : \{0, 1\} \to \{0, 1\}$ is a function that can be applied to X.

(3 points) Either for the general, or for the Bernoulli special case, prove that if X and Y are independent then $f(X)$ and Y are independent.

(1 points) Proving this for the general case requires more advanced math. What is the payoff for this additional work? If you prove something for the general case, does it apply to **all** special cases? If you prove something for a special case, does it apply to **all** general cases?

Note: The Homework Maximum is 100%