Summer Road trip

At a gas station on Route 66, 40% of customers use regular gas (event R), 35% use mid-grade gas (event M), and 25% use premium gas (event P). Of the customers that use regular gas, 30% fill their tanks (event F). Of the customers that use mid-grade gas, 60% fill their tanks, while of those that use premium, 50% fill their tanks. Assume that each customer is drawn independently from the entire pool of customers.

$P(R) = 40/100$

$P(M) = 35/100$

$P(C) = 25/100$

$P(F|R) = 30/100$

$P(F|M) = 60/100$

$P(F|C) = 50/100$

1. (1 point) What is the probability that the next customer will request regular gas and fill the tank?

   $P(R \cap F) = P(F|R) * P(R)$
   $= 30/100 * 40/100$
   $= 3/25$

2. (1 point) What is the probability that the next customer will fill the tank with any kind of gas?

   $P(F) = P(R \cap F) + P(M \cap F) + P(P \cap F)$
   $= P(F|R) * P(R) + P(F|M) * P(M) + P(F|P) * P(P)$
   $= 40/100 * 30/100 + 35/100 * 60/100 + 25/100 * 50/100$
   $= 0.455$

3. (1 point) Given that the next customer fills the tank, what is the conditional probability that they use regular gas?

   $P(R|F) = (P(F|R) * P(R)) / P(F)$
   $= (30/100 * 40/100) / .455$
   $= 0.26373$

The Claaaaaw

Suppose that in a claw game at an arcade, there is a collection of toys that have the following characteristics:

- 2/5 are red;
- 3/5 are waterproof;
- 1/2 are cool. (When we write that 2/5 are red, this means that 3/5 are not red. But we're not saying anything about whether those toys are waterproof or whether they are cool.)

Furthermore:

- 1/5 are both red and waterproof;
- 1/5 are both red and cool;
- 3/10 are both waterproof and cool.

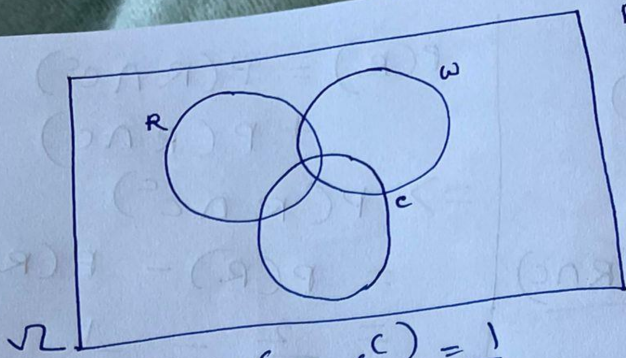
(When we write that 1/5 are both red and waterproof, this contains no information about how cool they are.) Finally:

- 1/10 are neither red, waterproof, nor cool. (These are pretty lame toys.)

Since working those claws is so hard, suppose that any toy in the game has an equal chance of being selected.

$P(R) = 2/5$
$P(W) = 3/5$
$P(C) = ½$
$P(R \cap C) = 1/5$
$P(R \cap W) = 1/5$
$P(C \cap W) = 3/10$
$P(R^c \cap W^c \cap C^c) = 1/10$

1. (3 points) Draw an area diagram to represent these events. For as many of the events that you can, compute the probability (for example, $P(R \cap C \cap W)$)).



$P(R) = \frac{2}{5}$

$P(W) = \frac{3}{5}$

$P(C) = \frac{1}{2}$

$P(R \cap W) = \frac{1}{5}$

$P(R \cap C) = \frac{1}{5}$

$P(W \cap C) = \frac{3}{10}$

$P(R^c \cap W^c \cap C^c) = \frac{1}{10}$

$$P(R) + P(W) + P(C) - P(R\cap C) - P(R \cap W) - P(W \cap C) + P(R \cap W \cap C) + P(R^c \cap W^c \cap C^c) = 1$$

$$\Rightarrow \frac{\cancel{2}}{5} + \frac{3}{5} + \frac{1}{2} - \frac{\cancel{1}}{5} - \frac{\cancel{1}}{5} - \frac{3}{10} + P(R \cap W \cap C) + P(R^c \cap W^c \cap C^c) = 1$$

$$P(R \cap W \cap C) + \frac{1}{10} = 1$$

$$\Rightarrow \frac{6}{10} + \frac{5}{10} - \frac{3}{10} + P(R \cap W \cap C) = 1$$

$$\Rightarrow \frac{11}{10} - \frac{3}{10} + \frac{1}{10} + P(R \cap W \cap C)$$

$$\Rightarrow P(R \cap W \cap C) = 1 - \frac{9}{10} = \frac{1}{10}$$

2. (1 points) What is the probability of drawing a toy that is red and waterproof and cool?

$P(R) + P(W) + P(C) – P(R \cap C) - P(R \cap W) - P(C \cap W) + P(R \cap C \cap W) + P(R^c \cap W^c \cap C^c) = 1$

- ⇨ **$2/5 + 3/5 + ½ - 1/5 – 1/5 – 3/10 + P(R \cap C \cap W) + 1/10 =1$**
- ⇨ **$P(R \cap C \cap W) = 1/10$**

3. (1 points) Suppose that you pull out a toy at random, and you observe only the color, noting that it is red. Conditional on just this information, what is the probability that the toy is not cool?

P(!C | R) = P(!C ∩ R) / P(R)

Now, P(R) = P(R ∩ C) U P(R ∩ !C)
=> P(R ∩ !C) = P(R) - P(R ∩ C)
=> P(R ∩ !C) = 2/5 – 1/5 = 1/5

P(!C | R) = (1/5) / (2/5)
= 1/2

4. (1 points) Given that a randomly selected toy is either red or waterproof, what is the probability that it is cool?

P(C | (R U W)) = P(C ∩ (R U W)) / P(R U W)
= P(C ∩ R) U P(C ∩ W) / P(R U W)
= P(R ∩ W ∩ C) / P(R U W)
= 1/10 / (P(R) + P(W) – P(R ∩ W))
= 1/10 / (2/5 + 3/5 – 1/5)
= 1/10 / 4/5
= 1/8
= 0.125

In the question that follows, to argue that a number is a maximum, you actually have to argue two separate things:
(a) That the number is possible; but, importantly,
(b) That no greater number is possible.
Without part (b) you wouldn't have complete proof.
The same goes for proving a minimum, just replace greater with smaller.

On the Overlap of Two Events

Suppose for events A and B, $P(A) = 1/2$, $P(B) = 3/4$, but we have no more information about the events.

1. (3 points) What are the maximum and minimum possible values for $P(A \cap B)$?

By addition rule, we have the following: -

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Probability bounds from Kolmogorov's Axioms: $\forall A \in S$ (S is the sample space), $0 \le P(A) \le 1$.

Hence, $0 \le P(A \cup B) \le 1$

and

$0 \le P(A \cap B) \le 1$

$\frac{1}{2} + \frac{3}{4} - P(A \cap B) \le 1$

- ⇨ **$2/4 + \frac{3}{4} - P(A \cap B) \le 1$**
- ⇨ **$5/4 - P(A \cap B) \le 1$**
- ⇨ **$5/4 - 1 \le P(A \cap B)$**
- ⇨ **$\frac{1}{4} \le P(A \cap B)$**

So, the minimum value for $P(A \cap B) = \frac{1}{4}$.

Now, $A \cap B \subseteq A$ and $A \cap B \subseteq B$. From the rule of monotonicity, we know that:-

$P(A \cap B) <= \min\{P(A), P(B)\}$

$<= \min(1/2, \frac{3}{4})$

$= \frac{1}{2}$

So, combining both values, we get: -

$\frac{1}{4} \le P(A \cap B) \le \frac{1}{2}$.

2. (3 points) What are the maximum and minimum possible values for $P(A|B)$?

By the Conditional Probability Rule, for $A, B \in S$ with $P(B) > 0$, the conditional probability of A given B is $P(A|B) = P(A \cap B) / P(B)$ (1) and $P(B) > 0$.

Now, from the prior step, we derived: -
$¼ \le P(A \cap B) \le ½$ (2)

Now, $P(B)$ is already given and fixed whereas **$P(A \cap B)$ ranges between ¼ and ½.**

If we take the minimum value of $P(A \cap B)$ from equation (2) and substitute it in the above equation(1), then, we get:-

$$
\begin{aligned}
P(A|B) &= P(A \cap B) / P(B) \\
&= (¼) / (3/4) \\
&= 1/3
\end{aligned}
$$

So, the minimum value for $P(A|B)$ is 1/3.

If we take the maximum value of $P(A \cap B)$ from equation (2) and substitute it in the above equation(1), then, we get:-

$$
\begin{aligned}
P(A|B) &= P(A \cap B) / P(B) \\
&= (1/2) / (3/4) \\
&= (2/4) / (3/4) \\
&= 2/3
\end{aligned}
$$

So, the maximum value for $P(A|B)$ is 2/3.

So, combining both values, we get: -

$1/3 \le P(A \mid B) \le 2/3$