Assignment 1

AI1110: Probability and Random Variables
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10.13.3.36: Question. Two dice are thrown at the same time. Determine the probability that the difference of the numbers on the two dice is 2
Answer: $\frac{8}{36}$.

Random Variable	Definition
$p_X(k)$	Probability that the outcome of die is k
$p_E(n)$	Probability that the difference between outcome of dice is n

Assuming the die to be fair, the probability mass function (pmf) is expressed as

$$p_X(k) = \begin{cases} \frac{1}{6} & 1 \le k \le 6 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

When 2 dice are rolled, each die will have 6 outcomes and the events are independent

For the random variable $p_E(n)$
$n \in \{0, 1, 2, 3, 4, 5\}$

Let us assume X_1 be the outcome of first die and X_2 be the outcome of second die

$$\begin{aligned} p_E(n) &= Pr(|X_1 - X_2| = n) \\ &= Pr((X_1 - X_2 = n) + (X_1 - X_2 = -n)) \end{aligned}$$

Let E_1 be the event satisfing $X_1 - X_2 = n$ and E_2 be the event satisfing $X_1 - X_2 = -n$

$$p_E(n) = Pr(E_1 + E_2)$$

for n=0

Since events E_1 and E_2 are identical

$$Pr(E_1 + E_2) = Pr(E_1)$$

$$\begin{aligned} p_E(0) &= Pr(X_1 - X_2 = 0) = Pr(X_1 = X_2) \\ &= \sum_k Pr(X_1 = k|X_2 = k)p_{X_2}(k) \end{aligned} \tag{2}$$

after unconditioning. X_1 and X_2 are independent,

$$\begin{aligned} Pr(X_1 = k|X_2 = k) &= Pr(X_1 = k) \\ &= p_{X_1}(k) \end{aligned} \tag{3}$$

From (2) and (3),

$$p_E(0) = \sum_k p_{X_1}(k)p_{X_2}(k) \quad = p_{X_1}(n) * p_{X_2}(n) \tag{4}$$

where * denotes the convolution operation. Substituting from (1) in (7)

$$p_E(0) = \frac{1}{6}\sum_{k=1}^{6} p_{X_1}(k) \tag{5}$$

Substituting from (1) in (5)

$$\begin{aligned} p_E(0) &= \frac{1}{6}\sum_{1}^{6} p_{X_1}(k) \\ &= \frac{1}{6}(\sum_{1}^{6} p_{X_1}(k) \\ &= \frac{1}{6}(6\frac{1}{6}) \\ &= \frac{6}{36} \end{aligned}$$

for 1 ≤n≤ 6

since events E_1 and E_2 are disjoint

$$\begin{aligned} p_E(n) &= Pr(E_1 + E_2) \\ &= Pr(E_1) + Pr(E_2) \end{aligned}$$

Because of symmetry between E_1 and E_2

$$\begin{aligned} Pr(E_1) &= Pr(E_2) \\ p_E(n) &= 2 * Pr(E_1) \end{aligned}$$

$$\begin{aligned} Pr(E_1) = Pr(X_1 - X_2 = n) &= Pr(X_1 = X_2 + n) \\ &= \sum_k Pr(X_1 = k + n | X_2 = k) p_{X_2}(k) \quad (6) \end{aligned}$$

after unconditioning. X_1 and X_2 are independent,

$$\begin{aligned} Pr(X_1 = k + n | X_2 = k) &= Pr(X_1 = k + n) \\ &= p_{X_1}(k + n) \qquad (7) \end{aligned}$$

From (6) and (7),

$$Pr(E_1) = \sum_k p_{X_1}(k + n) p_{X_2}(k) \quad = p_{X_1}(n) * p_{X_2}(n) \qquad (8)$$

where * denotes the convolution operation. Substituting from (1) in (7)

$$Pr(E_1) = \frac{1}{6} \sum_{k=1}^{6} p_{X_1}(k + n) = \frac{1}{6} \sum_{1+n}^{6+n} p_{X_1}(k) \quad (9)$$

Substituting from (1) in (9)

$$\begin{aligned} Pr(E_1) &= \frac{1}{6} \sum_{1+n}^{6+n} p_{X_1}(k) \\ &= \frac{1}{6}(\sum_{1+n}^{6} p_{X_1}(k) + \sum_{7}^{1+6} p_{X_1}(k)) \\ &= \frac{1}{6}(6 - n\frac{1}{6} + 0) \\ &= \frac{6 - n}{36} \end{aligned}$$

Therefore,the probability of $p_E(n)$ is

$$\begin{aligned} p_E(n) &= 2 \times Pr(E_1) \\ &= 2 \times \frac{6 - n}{36} \\ &= \frac{12 - 2n}{36} \end{aligned}$$

$$p_E(n) = \begin{cases} 0 & n < 0 \\ \frac{6}{36} & n = 0 \\ \frac{12-2n}{36} & 1 \leq n \leq 5 \\ 0 & n > 5 \end{cases} \qquad (10)$$

From (10) probability that the difference of the numbers on the two dice is 2 is

$$\begin{aligned} p_E(2) &= \frac{12 - 2 \times 2}{36} \\ &= \frac{12 - 4}{36} \\ &= \frac{8}{36} \end{aligned}$$