1. Using basic calculations, prove that the maximum ionisation occurs at optical depth unity. Assume an isothermal atmosphere (i.e. n(z)=n_0 exp(-z/H)) where H can be assumed constant) and use the definition of optical depth, tau= sigma X integral (n *dz). Note that the formula for n(z) allows for a very covenient representation of the integral or derivative of n(z) with respect to z. You’ll need that

$$I(s_\lambda) = solar\ flux$$

$$\sigma^a = absorption\ crosssection$$

$$The\ decrease\ intensity\ after\ it\ travels\ an\ incremental\ distance:$$

$$dI(s_\lambda) = -I(s_\lambda) n(z) \sigma^a ds_\lambda$$

$$\chi \rightarrow Solar\ Zenith\ angle$$

$$ds_\lambda = -dz \sec \chi$$

$$dI(s_\lambda) = -I(z) n(z) \sigma^a \sec \chi dz$$

$$\int_{I_\infty}^{I(z)} \frac{dI(z)}{I(z)} = \int_z^\infty \sigma^a \sec \chi n(z) dz$$

$$\ln\left(\frac{I_\infty}{I(z)}\right) = \sigma^a \sec \chi \int_z^\infty n(z) dz = \sec \chi \tau$$

$$I(z) = I_\infty \exp(-\tau \sec \chi)$$

$$Optical\ Depth: \tau = \sigma^a \int_z^\infty n(z) dz$$

$$Isothermal\ atmosphere: n(z) = n_0 \exp\left(\frac{-(z-z_0)}{H}\right)$$

$$\ln\left(\frac{I_\infty}{I(z)}\right) = \sigma^a \sec \chi \int_z^\infty n_0 \exp\left(\frac{-(z-z_0)}{H}\right) dz$$

$$= \sigma^a \sec \chi H n_0 \exp\left(\frac{-(z-z_0)}{H}\right)$$

$$= \sigma^a \sec \chi H n(z)$$

$$I(z) = I_\infty \exp(-\sigma^a \sec \chi H n(z))$$

$$\tau(z) = \sigma^a H n(z)$$

$$Photoionisation: P_c(z,\chi) = I(z,\chi)\beta\sigma^a n(z)$$
$$\beta = probabiltiy\ of\ ionisation$$
$$P_c(z,\chi) = I_\infty \exp(-\tau(z)\sec\chi)\beta\sigma^a n(z)$$
$$Maximum\ ionisation: \frac{dP_c}{dz} = 0$$
$$\frac{dP_c}{dz} = I_\infty \beta\sigma^a[\exp(-\tau(z)\sec\chi)n'(z) - n(z)\exp(-\tau(z)\sec\chi)\sec\chi\,\tau'(z)] = 0$$
$$\Rightarrow n'(z) - n(z)\sec\chi\,\tau'(z) = 0$$
$$[n'(z) - n(z)\sec\chi\,\tau'(z)] = 0$$
$$\tau(z) = \sigma^a H\, n(z) \Rightarrow \tau'(z) = \sigma^a H\, n'(z) (assuming\ H\ constant)$$
$$\frac{\tau'(z)}{\sigma^a H} - n(z)\sec\chi\,\tau'(z) = 0$$
$$\tau'(z) - \sigma^a H\, n(z)\sec\chi\,\tau'(z) = 0$$
$$\tau'(z)(1 - \sigma^a H\, n(z)\sec\chi) = 0$$
$$\tau'(z)(1 - \tau(z)\sec\chi) = 0$$
$$\tau(z)\sec\chi = 1$$
$$For\ \sec\chi = 1: overhead\ sun\ \ \tau(z) = 1\ for\ maximum\ ionisation$$

2. Having shown that tau=1, substitute the atmospheric density variation according to the above formula and derive a value for the appropriate cross-section for maximum absorption as a function of altitude (cover the range from 50- 120 km)

$$Assuming\ all\ the\ energy\ absorbed\ goes\ into\ ionisation: \tau(z)\sec\chi = 1$$
$$\sigma^a H\, n(z)\sec\chi = 1$$
$$n_0 \exp\left(\frac{-(z - z_0)}{H}\right) = \frac{1}{\sigma^a H \sec\chi}$$
$$-\frac{(z - z_0)}{H} = \ln\left(\frac{1}{H\, n_0\, \sigma^a \sec\chi}\right)$$
$$z = z_0 + H\ln\left(H\, n_0\, \sigma^a \sec\chi\right)$$
$$Also:$$
$$n_0 \exp\left(\frac{-(z - z_0)}{H}\right) = \frac{1}{\sigma^a H \sec\chi}$$
$$\sigma^a = \frac{1}{H\sec\chi\, n_0 \exp\left(\frac{-(z - z_0)}{H}\right)}$$
$$\sigma^a = \frac{1}{H\, n_0 \sec\chi}\exp\left(\frac{z - z_0}{H}\right)$$

3. Assume pure N_2 absorption (for mars you would use CO_2), use the Henke curves, convert the cross-sections profile to energy, and hence wavelength of incident photon and present again as a function of altitude. This should give a profile of photon wavelength vs altitude where maximum absorption occurs. Then answer- what altitude corresponds to maximum absorption of the Fe XV line at 1.9 A?

Here I used this formula obtained from the previous part:

$$z = z_0 + H \ln\left(H n_0 \sigma^a \sec\chi\right)$$

Fe XV line at 1.9 A, the altitude comes out to be 65 km. From Henke, I got the cross-section to be 8.46E-22 cm^2

4. Then you can answer the following: what sampling for the cross-section corresponds to a grid vertical resolution of:
a. 10 km
b. 1 per 7 km scale height
c. 2 per 7 km scale height

$$\text{Let the vertical resolution be } \Delta V$$

$$z_2 - z_1 = H\left(\ln\left(H n_0 \sigma_2^a(\lambda_2) \sec\chi\right) - \ln\left(H n_0 \sigma_1^a(\lambda_1) \sec\chi\right)\right)$$

$$z_2 - z_1 = H \ln\left(\frac{\sigma_2^a(\lambda_2)}{\sigma_1^a(\lambda_1)}\right)$$

$$\left|\frac{\sigma_2^a(\lambda_2)}{\sigma_1^a(\lambda_1)}\right| = \exp\left(\frac{z_2 - z_1}{H}\right)$$

$$\left|\frac{\sigma_2^a(\lambda_2)}{\sigma_1^a(\lambda_1)}\right| = \exp\left(\frac{\Delta V}{H}\right)$$

$$\left|\frac{\sigma_2^a(\lambda_2)}{\sigma_1^a(\lambda_1)}\right| = \exp\left(\frac{\Delta V}{H}\right)$$

$$\text{Assuming scale height } H = 7\ km$$

$$a.\ \Delta V = 10\ km$$

$$\left|\frac{\sigma_2^a(\lambda_2)}{\sigma_1^a(\lambda_1)}\right| = \exp\left(\frac{10}{7}\right) = 4.17$$

$$b.\ 1\ cross-section\ per\ 7\ km\ scale\ height: \Delta V = 7\ km$$

$$\left|\frac{\sigma_2^a(\lambda_2)}{\sigma_1^a(\lambda_1)}\right| = \exp\left(\frac{7}{7}\right) = e^1 = 2.7$$

$$c.\ 2\ cross-sections\ per\ 7\ km\ scale\ height: \Delta V = 3.5\ km$$

$$\left|\frac{\sigma_2^a(\lambda_2)}{\sigma_1^a(\lambda_1)}\right| = \exp\left(\frac{3.5}{7}\right) = e^{0.5} = 1.6$$

Here I have added all the sampling of the cross-sections data for the O_2 and N_2 for a vertical resolution of 5km, calculated using the above derived relationship.

O2 data for Vertical resolution of 5km calculated cross-section (cm2)	H+F cross-section	H+F wavelength (A)	H+F energy (eV)	Altitude(Tau=1) (km)
8.50E-24	8.5024E-024	0.4	30992.5	31
1.74E-23	1.44541E-023	0.5	24794	36
3.55E-23	2.56666E-023	0.6	20661.7	41
7.25E-23	7.91786E-023	0.8	15496.2	46
1.48E-22	7.91786E-023	0.8	15496.2	51
3.02E-22	3.04492E-022	1.3	9536.15	56
6.18E-22	5.8454E-022	1.5	8264.67	61
1.26E-21	1.53575E-021	2.1	5903.33	66
2.58E-21	2.62512E-021	2.5	4958.8	71
5.26E-21	6.3768E-021	3.4	3646.18	76
1.08E-20	1.19034E-020	4.2	2951.67	81
2.20E-20	2.52946E-020	5.4	2295.74	86
4.49E-20	3.51255E-020	6.1	2032.3	91
9.17E-20	8.5024E-020	8.3	1493.61	96
1.87E-19	1.9E-019	33.74	367.43	101
3.82E-19	3.84E-019	47.7	259.9	106

N2 data for Vertical resolution of 5km calculated cross-section (cm2)	H+F cross-section	H+F wavelength (A)	H+F energy (eV)	Altitude(Tau=1) (km)
4.54E-24	4.5357E-24	0.4	30992.5	31
9.26E-24	7.76884E-24	0.5	24794	36
1.89E-23	1.3863E-23	0.6	20661.7	41
3.86E-23	1.3863E-23	0.8	15496.2	46
7.90E-23	4.32171E-23	0.8	15496.2	51
1.61E-22	1.65611E-22	1.3	9536.15	56
3.29E-22	3.18662E-22	1.5	8264.67	61
6.73E-22	5.1172E-22	1.8	6887.22	66
1.37E-21	1.45608E-21	2.5	4958.8	71
2.81E-21	3.57739E-21	3.4	3646.18	76
5.74E-21	6.79192E-21	4.2	2951.67	81
1.17E-20	9.95528E-21	4.7	2637.66	86
2.39E-20	2.07479E-20	6.1	2032.3	91
4.89E-20	2.07479E-20	8.3	1493.61	96
9.99E-20	9.62964E-20	10.4	1192.02	101
2.04E-19	2.09E-19	47.7	259.9	106