REGGE CALCULUS AS A LOCAL THEORY OF THE POINCARÉ GROUP

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We reformulate Regge calculus in terms of dynamical variables belonging to the Poincaré group. Our formulation uses the dual lattice, is naturally related in the continuum limit to the Einstein action written in terms of differential forms, and retains the possibility of choosing between the first order and the second order formalism.

1. Introduction

One of the most intriguing features of general relativity is its close resemblance (and yet not equivalence) with an ordinary gauge theory. This is particularly evident in the first order (vierbein) formalism, where the dynamical variables are exactly those appropriate to a gauge theory of the Poincaré group.

This similarity led to various attempts to write down regularized versions of GR on a hypercubic lattice, with the hope of applying to gravity the successful techniques of lattice gauge theories [1,2]. In particular, in two previous papers [2], we have presented a formulation of lattice gravity as a spontaneously broken gauge theory of the Poincaré group, and we have found that the coupling to matter of such a theory leads to the phenomenon of doubling not just for fermions, but for all particles. On the other hand, it is well known since the seminal work of Regge [3] that there is a natural and geometrically appealing way of discretizing gravity on a simplicial lattice. Regge calculus has been a steadily active field of research during its three decades history (see, for example, refs. [4,5]), and has recently attracted a growing interest in connection with two-dimensional quantum gravity, and therefore with conformal field theory and string theory [6].

One of the drawbacks of the simplicial lattice is that it is naturally related to the second order (metric) formalism of GR, and therefore its connection with a gauge theory is rather obscure. The aim of this letter is to show that it is possible to reformulate Regge calculus in terms of dynamical variables belonging to the Poincaré group, in such a way to make an explicit connection with the corresponding gauge theory, and also with the formulation on the hypercubic lattice. This is done at first in the second order formalism, by assigning the rotational degrees of freedom (Lorentz connections) to the links of a lattice dual to the original Regge skeleton, so that they are functions of the translational degrees of freedom (vierbein). The transition to a first order formulation, with all the variables defined on the links of the dual lattice, is then natural, and is also discussed in detail.

2. A group action for simplicial gravity

A simplicial lattice in d dimensions is a collection of flat d-simplices, together with an incidence matrix containing the information about how the simplices

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are connected to each other. Each d-simplex S can be characterized by its $d+1$ vertices

$$S \equiv \{P_1, ..., P_{d+1}\} ,$$

and its geometry is completely determined by the $d(d+1)/2$ lengths of the edges (P_iP_j), which are the dynamical variables of Regge calculus. In general, a d-simplex S contains N^d_k k-dimensional simplices, with $N^d_k = \binom{d+1}{k+1}$. In particular, S has $d+1$ faces

$$f_i \equiv \{P_1, ..., \hat{P}_i, ..., P_{d+1}\} ,$$

where $\hat{P}_i$ denotes the omission of the point P_i. Two faces f_i and f_j have in common a hinge

$$h_{ij} \equiv \{P_1, ..., \hat{P}_i, ..., \hat{P}_j, ..., P_{d=1}\} ,$$

and so on.

In a lattice of flat simplices the curvature is concentrated on the hinges and is measured by the deficit angle $\epsilon(h)$, defined as the rotation undergone by a vector when parallel transported along a closed curve surrounding the hinge h. The Regge–Einstein action on the lattice is then

$$I_R = \sum_{\text{hinges}} \epsilon(h) V(h) , \qquad (1)$$

where $V(h)$ is the volume of the hinge h.

Given a simplicial lattice there is a general procedure [4,5] for the construction of a dual lattice, whose cells are polyhedra (Voronoi polyhedra). The Voronoi polyhedron dual to a vertex P is the set of all the points in the lattice that are closer to P than to any other vertex. In this way a k-simplex in the lattice is dual to a $(d-k)$-polyhedron in the dual lattice, and the k-dimensional linear space identified by the k-simplex is orthogonal to the $(d-k)$-dimensional space spanned by the corresponding polyhedron. In particular, the point D dual to d-simplex S is the centre of the $(d-1)$-dimensional sphere passing through all the vertices of S.

We will now show how to associate uniquely with each link of the dual lattice a Poincaré transformation, and we will build with these new variables an action that reduces to I_R in the small curvature limit.

Let us concentrate on a particular hinge $h \equiv \{P_1, ..., P_{d-1}\}$, shared by the set of simplices $\{S_1, ..., S_N\}$, characterized by

$$S_\alpha \equiv \{P_1, ..., P_{d-1}, Q_{\alpha-1,\alpha}, Q_{\alpha,\alpha+1}\} \quad (\alpha = 1, ..., N) .$$

A two dimensional example, with $N=5$, is shown in fig. 1. Let us further choose in each simplex S_α an origin and a lorentzian frame, and let us denote the coordinates of the vertices of S_α in this frame by

$$P_i \equiv \{y^a_i(\alpha)\} ,$$

$$Q_{\alpha-1,\alpha} \equiv \{z^a_{\alpha-1,\alpha}(\alpha)\} ,$$

$$Q_{\alpha,\alpha+1} \equiv \{z^a_{\alpha,\alpha+1}(\alpha)\} .$$

It is now natural to assign to each vertex D_α of the dual lattice its coordinates in the frame α, $x^a(\alpha)$. Furthermore, there is a unique and natural way to associate to each link of the dual lattice a Poincaré transformation $U(\alpha, \alpha+1) \equiv \{U^a_b\alpha, \alpha+1), U^a(\alpha, \alpha+1\}$. We define $U(\alpha, \alpha+1)$ by demanding that

$$U^a_b(\alpha, \alpha+1) y^b_i(\alpha+1) + U^a(\alpha, \alpha+1) = y^a_i(\alpha) ,$$

$$U^a_b(\alpha, \alpha+1) z^b_{\alpha,\alpha+1}(\alpha+1) + U^a(\alpha, \alpha+1)$$
$$= z^a_{\alpha,\alpha+1}(\alpha) .$$

In other words $U(\alpha, \alpha+1)$ is the Poincaré transformation that, acting on the coordinates in $S_{\alpha+1}$, makes the ones belonging to the interface $S_\alpha \cap S_{\alpha+1}$ coincide with those measured in S_α.

The arbitrariness of the choice of the reference frame in each simplex S_α becomes in the dual lattice the gauge invariance under local Poincaré transformations $\Lambda(\alpha)$:

$$U(\alpha, \alpha+1) \to \Lambda(\alpha) U(\alpha, \alpha+1) \Lambda^{-1}(\alpha+1) ,$$

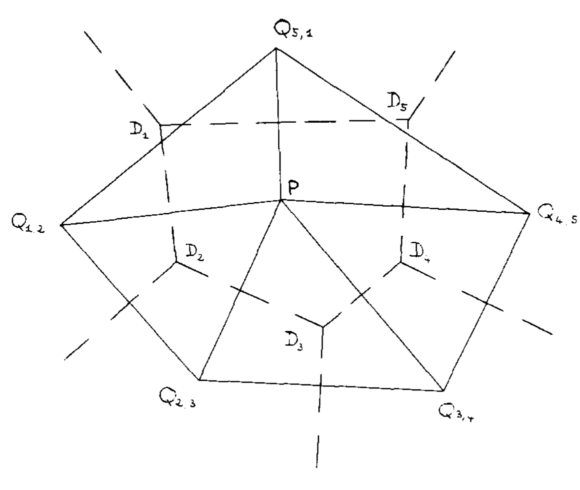

Fig. 1. A two-dimensional example of Regge skeleton.

$$x^a(\alpha)\to\Lambda^a_b(\alpha)x^b(\alpha)+\Lambda^a(\alpha)\,.$$

Given a Regge lattice, it is always possible, by suitably fixing the translational part of the gauge, to set $x^a(\alpha)=0$ for all α. The coordinates $x^a(\alpha)$ are therefore gauge phantoms, similar to the local coordinates introduced in ref. [2] on the hypercubic lattice.

Consider now the plaquette variable on the dual lattice around the hinge h,

$$W_\alpha(h)=U(\alpha,\alpha+1)U(\alpha+1,\alpha+2)...U(\alpha-1,\alpha)\,.$$

By definition, $W_\alpha(h)$ leaves the coordinates of all the vertices P_i of the hinge h unchanged and its translational part is zero. Therefore, $W_\alpha(h)$ is a rotation in a plane perpendicular to the hinge h, by an angle, say, $\theta(h)$. If we denote by θ_α the dihedral angle in the simplex S_α between the faces $\{P_i, Q_{\alpha-1,\alpha}\}$ and $\{P_i, Q_{\alpha,\alpha+1}\}$, then it is easy to see that

$$\sum_{\alpha=1}^{N}\theta_\alpha+\theta(h)=2\pi\,,$$

so that $\theta(h)$ is the deficit angle around the hinge h. This result leads naturally to the following proposal for a gravitational action on the lattice, in terms of the plaquette variables $W_\alpha(h)$

$$\begin{aligned}I=&\sum_{\text{hinges}}\Bigg(W^{a_1a_2}_\alpha(h)\\&\times\prod_{i=1}^{d-2}\,[y_i(\alpha)-y_{d-1}(\alpha)]^{a_{i+2}}\epsilon_{a_1...a_d}\Bigg)\\=&\sum_{\text{hinges}}\sin\theta(h)V(h)\,,\end{aligned}\tag{2}$$

where some care must be taken to insure that the orientation of the hinge h is chosen consistently with the direction of the rotation $W_\alpha(h)$ for all h.

This action has the following properties:

(i) It is invariant under Poincaré transformations in each simplex (gauge invariance).

(ii) It does not depend upon the particular simplex S_α chosen as a starting point to define the plaquette variable $W_\alpha(h)$.

(iii) It reduces to the Regge action (1) in the limit of small deficit angles.

(iv) It bears a striking resemblance with the continuum action written in terms of differential forms,

$$I_c=\int R^{a_1a_2}\wedge\mathscr{V}^{a_3}\wedge...\wedge\mathscr{V}^{a_d}\epsilon_{a_1...a_d}\,,\tag{3}$$

suggesting the natural identification in the continuum limit

$$y^a_i(\alpha)-y^a_j(\alpha)\equiv\mathscr{V}^a_{ij}(\alpha)\to\mathscr{V}^a_\mu\,.$$

As it has been written, the action (2) is still in the second order formalism, since the $W_\alpha(h)$ have been explicitly calculated from the $\mathscr{V}^a_{ij}(\alpha)$. Furthermore, it depends on variables defined both on the original and on the dual lattice. An alternative formulation, suitable for a better understanding of these two problems, is given in the next section.

3. The action on the dual lattice

At this stage, in the action (2) the only independent dynamical variables are the vectors $\mathscr{V}^a_{ij}(\alpha)$. The Lorentz connections $U^a_b(\alpha,\alpha+1)$ are complicated functions of the $\mathscr{V}^a_{ij}(\alpha)$, just as in ordinary Regge calculus the deficit angles are functions of the edge lengths. At this point one might build a first order formalism by considering the $U^a_b(\alpha,\alpha+1)$ as independent variables, and then recover their expression in terms of the $\mathscr{V}^a_{ij}(\alpha)$ as a solution to the equation of motion. This first order formulation, though strictly connected to the usual one associated with the continuum action (3), is slightly awkward from the lattice point of view, since half the variables belong to the links of the dual lattice, and the other half to the links of the simplicial lattice. We seek therefore an alternative formulation, in which all the variables are defined on the dual lattice. For this purpose we notice that, given a face $f_{\alpha,\beta}\equiv\{P_1,...,P_d\}$, and the coordinates $x^a_i(\alpha)$ of the points P_i in the reference frame α of one of the two simplexes to which $f_{\alpha,\beta}$ belongs, one can naturally associate to $f_{\alpha,\beta}$ the vector

$$\begin{aligned}b^a_{\alpha,\beta}(\alpha)&=\epsilon^{ab_1...b_{d-1}}\prod_{i=1}^{d-1}\,[x_i(\alpha)-x_d(\alpha)]_{b_i}\\&=\epsilon^{ab_1...b_{d-1}}\prod_{i=1}^{d-1}\mathscr{V}_{id,b_i}(\alpha)\,.\end{aligned}\tag{4}$$

The analogous vector $b^a_{\alpha,\beta}(\beta)$ calculated in the reference frame of the other simplex S_β is related to the above by

$$U^a_b(\alpha,\beta)b^b_{\alpha,\beta}(\beta)=b^a_{\alpha,\beta}(\alpha)\,.$$

If the edges $\mathscr{V}^a_{ij}(\alpha)$ form a closed simplex, the vectors $b^a_{\alpha,\beta}(\alpha)$ clearly do not depend on the choice of the $(d-1)$ edges used for their definition, except for the orientation, which we choose conventionally to be pointing toward the outside of S_α. Under these assumptions, the $b^a_{\alpha,\beta}(\alpha)$ have the following properties:

(i) They are orthogonal to the $(d-1)$-simplex $\{P_i, Q_{\alpha,\beta}\}$, and their length is proportional to the volume of $\{P_i, Q_{\alpha,\beta}\}$ itself.

(ii) If the simplex S_β is closed, and $b^a_i(\beta)$ are the vectors associated to its faces, then

$$\sum_{i=1}^{d+1} b^a_i(\beta)=0 .$$

(iii) With reference to fig. 1, the volume of the hinge surrounded by the simplices S_α's can be written as

$$\frac{1}{(d-2)!}\epsilon^{a_1\dots a_{d-2}c_1c_2}\prod_{i=1}^{d-2}[\mathscr{V}_{i,d-1}(\alpha)]_{a_i}$$
$$=\frac{1}{V(\alpha)}[b^{c_1}_{\alpha-1,\alpha}(\alpha)b^{c_2}_{\alpha,\alpha+1}(\alpha)$$
$$-b^{c_2}_{\alpha-1,\alpha}(\alpha)b^{c_1}_{\alpha,\alpha+1}(\alpha)] ,$$

where $V(\alpha)$ is the volume of the simplex S_α, which can in turn be written in terms of the b's as

$$[V(\alpha)]^{d-1}=\frac{1}{d!}\epsilon_{a_1\dots a_d}\epsilon^{i_1\dots i_d j}b^{a_1}_{i_1}(\alpha)\dots b^{a_d}_{i_d}(\alpha) . \quad (5)$$

This expression does not depend on the choice of the index j in the ϵ-symbol that contracts the labels of the faces. In other words it does not depend on which of the b's is left out.

(iv) In a given simplex S_β, let us denote by $z^a_i(\beta)$ $(i=1, \dots, d+1)$ the coordinates of the vertices. If the reference frame in S_β is chosen in such a way that

$$\sum_{i=1}^{d+1} z^a_i(\beta)=0$$

(baricentric coordinates, [7]), then the variables $b^a_i(\beta)$ are given by

$$b^a_i(\beta)=\frac{1}{(d-1)!^2}\sum_{k\neq i}\epsilon^a_{a_1\dots a_{d-1}}\epsilon^{kj_1\dots j_{d-1}}_i$$
$$\times z^{a_1}_{j_1}(\beta)\dots z^{a_{d-1}}_{j_{d-1}}(\beta) . \quad (6a)$$

This expression can be inverted to give

$$z^a_i(\beta)=\frac{1}{(d-1)!}\frac{1}{V(\beta)^{d-2}}\sum_{k\neq i}\epsilon^a_{a_1\dots a_{d-1}}\epsilon^{kj_1\dots j_{d-1}}_i$$
$$\times b^{a_1}_{j_1}(\beta)\dots b^{a_{d-1}}_{j_{d-1}}(\beta) . \quad (6b)$$

In terms of the new variables $b^a_i(\alpha)$, the action (3) can be rewritten as

$$I=\sum_h W^\alpha_{a_1a_2}(h)\frac{1}{V(\alpha)}$$
$$\times[b^{a_1}_\alpha(\alpha)b^{a_2}_{\alpha+1}(\alpha)-b^{a_2}_\alpha(\alpha)b^{a_1}_{\alpha+1}(\alpha)] , \quad (7)$$

where the sum is over all the plaquettes h of the dual lattice, and α corresponds to an arbitrarily chosen simplex belonging to the plaquette h.

We are now free to interpret the action (7) either as a second order action, with all the variables functions of the vectors b, or as a first order action, treating the b's and the Lorentz transformations $U^a_b(\alpha, \alpha+1)$ as independent variables. The transition to the first order formalism implies relaxing the constraint that the b vectors describe a well defined lattice of closed simplices: one allows for the appearance of "defects" representing the presence of torsion, and recovers the Regge lattice only as a solution (not necessarily the only one) of the equations of motion for the Lorentz connections. The configurations with non-zero torsion, even though not realized classically, are then bound to be important at the quantum level, where the above mentioned defects may appear as quantum fluctuations of the space–time manifold.

The action (7) also has a natural interpretation in the continuum limit: the normalized variables $\hat{b}^a_\alpha \equiv b^a_\alpha(\alpha)/V(\alpha)$ have the correct dimension to be identified with the inverse vierbein e^μ_a, so that in the limit $V(\alpha)\to 0$ we get

$$\sum_h W^{a_1a_2}_\alpha(h)[\hat{b}^\alpha_{a_1}(\alpha)\hat{b}^{\alpha+1}_{a_2}(\alpha)$$
$$-\hat{b}^\alpha_{a_2}(\alpha)\hat{b}^{\alpha+1}_{a_1}(\alpha)]V(\alpha)$$
$$\to\int d^dx\sqrt{g}\,R^{ab}_{\mu\nu}e^\mu_a e^\nu_b ,$$

where $V(\alpha)$ is naturally identified with the volume element $d^dx\sqrt{g}$.

Finally, the presence of the explicit factor $V(\alpha)$ in the action (7) for any $d>2$ suggests a natural way of introducing in the theory a lattice scale: one can impose that all the simplices should have the same volume, $V(\alpha)=V_0$. Introducing this constraint in the action by means of a Lagrange multiplier and measuring all the lengths in units of the Planck length, one gets

$$\frac{I}{\hbar}=\frac{1}{V_0}\sum_{h}\left[W_{\alpha}^{a_1a_2}(h)\,b^{\alpha}_{[a_1}(\alpha)\,b^{\alpha+1}_{a_2]}(\alpha)\right.$$
$$\left.+\lambda(\alpha)(V(\alpha)-V_0)\right], \qquad (8)$$

where $V(\alpha)$ is given by (5). We remark that the choice of the scale V_0 is not a gauge fixing of any of the continuum gauge degrees of freedom. Rather it is the choice of a particular slice in the space of all discretizations of a given surface. The continuum limit in the action (8) is then obtained by letting $V_0\to 0$. Alternatively [4], one can take a more radical point of view, and advocate the existence of a fundamental length providing a cut-off for ultraviolet divergencies in the quantum theory. The most natural choice is then $V_0=V_{\mathrm{Planck}}=1$.

4. Concluding remarks

We have given two equivalent formulations (eqs. (2) and (7)) of GR on a simplicial lattice, which reduce to Regge calculus in the small curvature limit, and clearly display the gauge structure of the theory. They are closely connected in the continuum limit to the Einstein action written in terms of differential forms, and therefore can be treated as first order or second order formulations. The most striking difference with Regge calculus is the appearance in the action of $\sin\theta(h)$ instead of $\theta(h)$. This might simply be considered as a lattice artifact, but one should notice that in Regge calculus the continuum limit is obtained by requiring that the local curvatures be small on the scale of the local lattice spacing: a condition which must be imposed by hand. Consider however the action (7), in the first order formalism. The crucial question is: do the equations of motion obtained by varying the Lorentz connections U lead to the correct second order expression for U as a function of b? The answer is yes in the limit of small $\theta(h)$, that is if $\sin\theta\sim\theta$. This can be seen by noticing that in Regge calculus the variation of $\theta(h)$ does not contribute to the equations of motion [3]. This suggests that the appearance of $\sin\theta$ might provide a dynamical mechanism to force the curvature to be small at the lattice scale near the classical ("perturbative") continuum limit. To establish the plausibility of such an interpretation, it will be necessary to study in detail the solutions of the equations of motion (possibly "non-perturbative" solutions with $\theta(h)\gg 0$), and the behaviour of the path integral in the neighborhood of such solutions. We remark that for this purpose, and in general for the purpose of numerical simulations, the action (8) seems particularly well suited. We also hope that this formulations, being closer to the group-theoretical structure of gravity, will be amenable to various generalizations, including coupling to matter fields and extensions to other groups.

References

[1] P. Menotti and A. Pelissetto, Ann. Phys. (NY) 170 (1986) 287;
L. Smolin, Nucl. Phys. B 148 (1979) 333;
C. Mannion and J.G. Taylor, Phys. Lett. B 100 (1981) 261;
K. Kondo, Progr. Theor. Phys. 72 (1984) 841;
E.T. Tomboulis, Phys. Rev. Lett. 52 (1984) 1173;
A. Das, M. Kaku and P.K. Townsend, Phys. Lett. B 81 (1979) 11;
M. Kaku, Phys. Rev. D27 (1983) 2819.

[2] M. Caselle, A. D'Adda and L. Magnea, Phys. Lett. B 192 (1987) 406; Phys. Lett. B 192 (1987) 411.

[3] T. Regge, Nuovo Cimento 19 (1961) 558;
G. Ponzano and T. Regge, in: Spectroscopic and group theoretical methods in physics, eds. F. Bloch et al. (Wiley, New York, 1968).

[4] R. Friedberg and T.D. Lee, Nucl. Phys. B 242 (1984) 145;
G. Feinberg, R. Friedberg, T.D. Lee and H.C. Ren, Nucl. Phys. B 245 (1984) 343;
N.H. Christ, R. Friedberg and T.D. Lee, Nucl. Phys. B 202 (1982) 89; Nucl. Phys. B 210 (1982) 337;
T.D. Lee, Discrete mechanics, in: Proc. Intern. School of subnuclear physics (Erice, 1983) p. 15.

[5] H. Hamber and R.M. Williams, Nucl. Phys. B 267 (1986) 482; Nucl. Phys. B 269 (1986) 712; Nucl. Phys. B 248 (1986) 392.

[6] V.A. Kazakov and A.A. Migdal, Nucl. Phys. B 311 (1988) 171.

[7] R. Sorkin, Phys. Rev. D 12 (1975) 385.