Finite Element Methods and Non-Abelian Gauge Theories

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Motivation

- Simulate the strong nuclear force using FEM
- Use non-cubic lattices and local mesh refinement

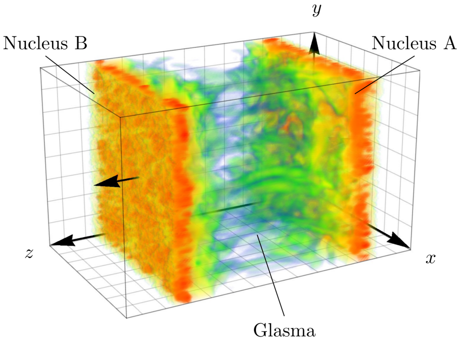


Figure: A snapshot of a 3+1D "Glasma" simulating the strong nuclear interaction of heavy ion collisions [2]

Overview

- Maxwell Theory (Electromagnetism)
- General Gauge Theories
- FEM Discretization

Maxwell Theory

Charges and Currents

- Charge density ρ
- Charge current j
- Space-time current J

$$J_\mu = \begin{pmatrix} J_0 = \rho \\ J_m = j_m \end{pmatrix}_\mu$$

- Continuity equation

$$\sum_{\mu=0}^{3} \partial^\mu J_\mu = \partial_t \rho + \sum_{k=1}^{3} \partial_k j^k = 0$$

Gauge Field and Field Strength

- Gauge field $A \in \Lambda^1(M)$ ("$\simeq$ vectorfield")
- Field strength $F = \mathrm{d}A \in \Lambda^2(M)$ ("$\simeq$ matrix")
- Maxwell equations

$$\mathrm{div} F = J$$

- Gauge transformation $A \to A + \mathrm{d}f$
- In coordinates: $A_\mu \to A_\mu + \partial_\mu f$
- Gauge invariance of F

$$A \to A + \mathrm{d}f \Rightarrow F = \mathrm{d}A \to \mathrm{d}(A + \mathrm{d}f) = \mathrm{d}A + \underbrace{\mathrm{d}\mathrm{d}}_{\equiv 0} f = \mathrm{d}A = F$$

Gauge Theories [7] [6] [3]

Gauge Theories

Physical Symmetry $\Rightarrow$ Lie group G

- $G = U(1) = \{e^{i\phi}, \phi \in \mathbb{R}\}$ ("phase symmetry") $\rightarrow$ Electromagnetism
- $G = SU(N) = \{U \in \mathbb{C}^{N\times N} | UU^\dagger = \mathbb{1}, \det U = 1\}$ $\rightarrow$ Yang-Mills Theory

Associated Lie algebra $\mathfrak{g} := T_{\mathbb{1}} G$

- $T_{\mathbb{1}} U(1) = i\mathbb{R}$
- $T_{\mathbb{1}} SU(N) = su(N) := \{\phi \in \mathbb{C}^{N\times N} | \phi^\dagger = -\phi, \mathrm{Tr}\phi = 0\}$

For matrix Lie groups/algebras:

- Lie bracket $[\cdot,\cdot] = \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g} \equiv$ matrix commutator
- Exponential map $\exp : \mathfrak{g} \rightarrow G \equiv$ matrix exponential

Gauge Theories

- Current $J : M \to \Lambda^1(M) \otimes \mathfrak{g}$
- Gauge field $A : M \to \Lambda^1(M) \otimes \mathfrak{g}$
- Field strength $F : M \to \Lambda^2(M) \otimes \mathfrak{g}$
- $F = \mathrm{d}A + \frac{1}{2}[A \wedge A]$
- Exterior covariant derivative $\mathrm{d}_A B := \mathrm{d}B + [A \wedge B]$
- Yang-Mills equations $\mathrm{div}_A F = J$
- Continuity equation $\mathrm{div}_A J = 0$

Gauge transformations $g \in C^\infty(M, G)$

- $A \to gAg^{-1} - \mathrm{d}gg^{-1}$
- $F \to gFg^{-1}$
- $J \to gJg^{-1}$

Electromagnetism as a $U(1)$-Gauge Theory

$G = U(1)$

- $U(1)$ Abelian!
- $[\cdot,\cdot] \equiv 0$
- Field strength $F = \mathrm{d}A + \frac{1}{2}[A \wedge A] = \mathrm{d}A$
- Exterior covariant derivative $\mathrm{d}_A B = \mathrm{d}B + [A \wedge B] = \mathrm{d}B$
- Yang-Mills equations $\mathrm{div}_A F = \mathrm{div} F = J$
- Continuity equation $\mathrm{div}_A J = \mathrm{div} J = 0$

Gauge transformations with $g = e^{\phi}$

- $A \to gAg^{-1} - \mathrm{d}gg^{-1} = A - \mathrm{d}\phi$
- $F \to gFg^{-1} = F$
- $J \to gJg^{-1} = J$

Gauge Theories - Differential Geometry [7] [6]

In a differential geometric context [7]:

- Gauge field $A : M \to \Lambda^1(M) \otimes \mathfrak{g}$ (connection 1-form)
- Field strength $F : M \to \Lambda^2(M) \otimes \mathfrak{g}$ (curvature 2-form)
- $F = \mathrm{d}A + \frac{1}{2}[A \wedge A]$ (Cartan structure equation)

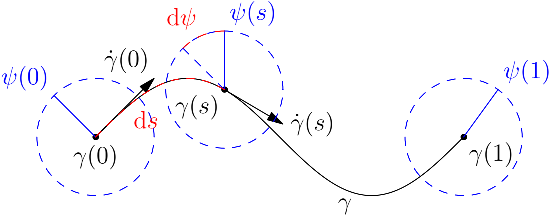
Parallel Transport

- Idea: relate objects belonging to different spatial points.
- Connection $\Rightarrow$ finite parallel transport
- Gauge field $A \in \mathfrak{g} \Rightarrow$ Gauge group element $U \in G$ [6]

$$\psi(s) = U(s)\psi_0$$

$$\nabla_{\dot{\gamma}}\psi := \partial_s \psi(s) - A(\gamma(s))\psi(s) \overset{!}{=} 0$$

$$\partial_s U(s) = A(\gamma(s))U(s)$$



FEM Discretization

FEM Discretization - Goals

Goals

- Gauge symmetry
- Charge conservation
- Local mesh refinement

Challenges

- Non-commutativity
- Non-linearities

Lie Group valued FE functions

∃ Geodesic Finite Elements [5]
⇒ compatible with gauge structure?
Gauge transformations ≡ functions $g : M \to G$
Consecutive transformations ⇒ Single transformation

$$\psi(\cdot) \xrightarrow{g_1(\cdot)} g_1(\cdot)\psi(\cdot) \xrightarrow{g_2(\cdot)} \underbrace{g_2(\cdot)g_1(\cdot)}_{g_{12}(\cdot)}\psi(\cdot) = g_{12}(\cdot)\psi(\cdot)$$

Goal: Lie group finite element spaces G_h such that

$$g_1, g_2 \in G_h \Rightarrow g_1 g_2 \in G_h$$

SU(2) discretization

parametrization of $SU(2)$-elements

- $a \in SU(2)$
- $\alpha \in \mathbb{R}$
- $\hat{\alpha} \in \mathbb{R}^3, \ \|\hat{\alpha}\| = 1$
- Pauli matrices $\sigma_k \in \mathbb{C}^{2\times 2}$ [1]

$$a = \exp\{\alpha \sum_k \hat{\alpha}_k \sigma_k\} = \cos\alpha \mathbb{1} + i \sin\alpha \sum_k \hat{\alpha}_k \sigma_k$$

Multiply two elements $(\alpha, \hat{\alpha}), (\beta, \hat{\beta})$

$$ab = \ldots = \underbrace{\left(\cos\alpha \cos\beta - \sin\alpha \sin\beta \hat{\alpha} \cdot \hat{\beta}\right)}_{=\cos\gamma} \mathbb{1} + \tag{1}$$

$$+ i \sum_k \underbrace{\left(\sin\alpha \cos\beta \hat{\alpha}_k + \cos\alpha \sin\beta \hat{\beta}_k\right)}_{\sin\gamma\hat{\gamma}} \sigma_k \tag{2}$$

α, β polynomials $\Rightarrow \gamma$ not a polynomial in general

Lie Group discretization

$$ab = \underbrace{\left(\cos\alpha\cos\beta - \sin\alpha\sin\beta\hat{\alpha}\cdot\hat{\beta}\right)}_{=\cos\gamma}\mathbb{1} + \\ + i\sum_k \underbrace{\left(\sin\alpha\cos\beta\hat{\alpha}_k + \cos\alpha\sin\beta\hat{\beta}_k\right)}_{\sin\gamma\hat{\gamma}}\sigma_k$$

But constants work!
$\Rightarrow$ Piecewise constant gauge transformations
$\Rightarrow$ Discontinuous function spaces

Distributional Connections

Tullio Regge: "General Relativity without coordinates"[4]
Similar idea, but with modifications:

- Objects to transport: particle colors, ...
- Parallel transporter $U_{kl} \in SU(N)$ between cells k, l
- Parallel transport along loop of cells $\simeq$ curvature
- Gauge transformations $U_{kl} \to g_l U_{kl} g_k^{-1}$

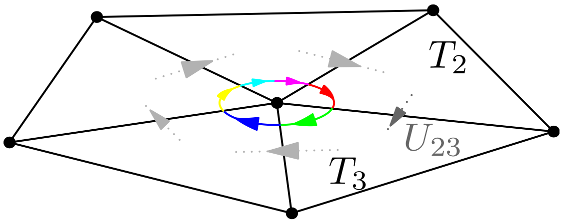


Figure: Parallel transporting a color along a loop using a distributional connection in form of Gauge Links U.

Resemblance with Lattice Gauge Theory (LGT) [1]

FEM+LGT Formulation - Work In Progress ...

- Gauge links $(U_E)_{E \in \mathcal{E}} \in G$
  as parallel transports across elements
- Charges $\rho \in H^1_{dc} \otimes \mathfrak{g}$
- Currents $j \in H(\mathrm{div})_{dc} \otimes \mathfrak{g}$
- Enforce continuity of ρ, j at element borders
  $[\rho]_E := U_{kl}^{-1} \rho_k U_{kl} - \rho_l \Big|_E \overset{!}{=} 0$

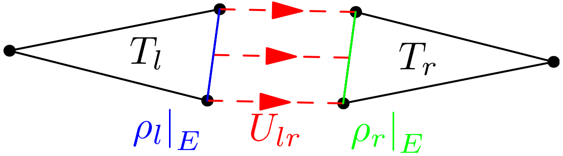


Figure: Discontinuous elements with "glueing" via gauge links U

FEM+LGT Formulation - Work In Progress ...

- Per-element gauge transformations $(g_T)_{T \in \mathcal{T}}$
- $U_{kl} \to \tilde{U}_{kl} = g_l U_{kl} g_k^{-1}$
- $\rho_k \to \tilde{\rho}_k = g_k \rho_k g_k^{-1}$
- $j_k \to \tilde{j}_k = g_k j_k g_k^{-1}$

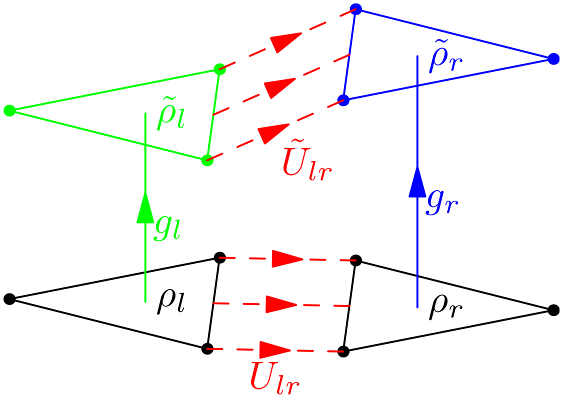


Figure: Discontinuous elements with "glueing" via gauge links U and per-element gauging.

Summary and Outlook

Summary

- Yang-Mills Theories - generalization of Maxwell Theory
- Differential geometric nature
- Challenges: non-commutativity, non-linearity, gauge transformations

Outlook

- Suitable FE-spaces: gauge transformations, gauge and field strength, charges and currents
- Discrete weak formulation for the Yang-Mills equations
- Discrete weak formulation for the evolution of charge currents
- Explore applicable FEM techniques
- Combine the above and simulate the “Glasma”

Thank you for your attention!

Literature

[1] Christof Gattringer and Christian Lang. *Quantum chromodynamics on the lattice: an introductory presentation*, volume 788. Springer Science & Business Media, 2009.

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