# The Hilbert Action in Regge Calculus

Warner A. Miller
Theoretical Astrophysics Group (T-6, MS B288)
Theoretical Division
Los Alamos National Laboratory
Los Alamos, NM 87545
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### Abstract

The Hilbert action is derived for a simplicial geometry. I recover the usual Regge calculus action by way of a decomposition of the simplicial geometry into 4-dimensional cells defined by the simplicial (Delaunay) lattice as well as its dual (Voronoi) lattice. Within the simplicial geometry, the Riemann scalar curvature, the proper 4-volume, and hence, the Regge action is shown to be exact, in the sense that the definition of the action does not require one to introduce an averaging procedure, or a sequence of continuum metrics which were common in all previous derivations. It appears that the unity of these two dual lattice geometries is a salient feature of Regge calculus.

# I. Gravity as Simplicial Geometry.

Einstein gave us a description of gravitation as geometry. A spacetime geometry can be represented arbitrarily closely by a lattice of flat 4-dimensional triangles (4-simplices). Such a simplicial geometry is a straight-forward generalization to four dimensions of the familiar architectural geodesic dome. Here however, each building block is not a flat triangle but a flat 4-simplex, and its intrinsic geometry is Lorentzian rather than Euclidean. It was Tullio Regge who showed us, for the first time, how to encode the principles of General Relativity onto a simplicial geometry — a mathematics popularly known today as Regge calculus. 1–3

In his original paper, Regge introduced the notion of curvature on a simplicial geometry and gave us the simplicial analogue of (1) the Hilbert action, (2) Einstein equations, and (3) the Bianchi identities.<sup>1</sup> The objective of this paper will be to reinvestigate the Regge calculus form of the Hilbert action and to provide a simple geometric derivation of it. The derivation presented here will shed new light on the foundations of Regge calculus.

Although I am primarily interested in a 4-dimensional spacetime realization of Regge calculus, the derivation of the action presented here is valid for an arbitrary *D*-dimensional simplicial lattice. In accord with this dimensional generality, I will derive the action here in *D* dimensions. Throughout this paper I will refer to the simplicial lattice as a Delaunay lattice and to its circumcentric dual lattice as a Voronoi lattice. I have introduced these lattices and terminology for two reasons. First, the Voronoi lattice appears to be a fundamental feature of Regge calculus. Second, by adopting this standard terminology Regge calculus might become more accessible to a broader community and might aid in cross fertilization of different scientific fields. Precise definitions of the Voronoi and

Delaunay lattices can be found in the literature.<sup>4</sup> Finally, throughout this paper I use geometric units where G = c = 1.

There are currently two derivations of the Hilbert action in Regge calculus. first was introduced by Regge and involved an averaging procedure. The second was by R. Friedberg and T. D. Lee which employed an embedding of the simplicial lattice into a higher dimensional Euclidean geometry and a sequence of continuum metrics which converged to the simplicial spacetime. While both derivations are consistent, and the latter provides a useful mathematical linkage between Regge calculus and the continuum, they lead one to believe that the Regge-Hilbert action is an approximate, truncated expression. I will demonstrate in this paper that the notions of parallel transport, curvature and proper volumes are precisely defined in Regge calculus; and this leads to an internally exact geometric construction of the Regge-Hilbert action. Furthermore, the emergence of the Voronoi lattice as a natural facet of Regge calculus is (1) consistent with an earlier and independent derivation of the simplicial form of the Einstein equations (Regge equations) as a sum of moment-of-rotation trivectors, <sup>6</sup> and is (2) in conformity with a previous derivation of the  $R^2$  action terms in Regge calculus. What I wish to emphasize in this paper is that the principles of general relativity can be applied directly to the simplicial spacetime geometry to yield an exact expression for the Regge calculus form of the Hilbert action.

# II. Dual Voronoi-Delaunay Derivation of the Hilbert Action in Regge Calculus.

In a D-dimensional Delaunay lattice geometry the curvature is concentrated on the (D-2)-dimensional simplices. We refer to these co-dimension two simplices as hinges. In order to transcribe the Hilbert action into Regge calculus, we need to define the scalar curvature tensor,  $^{(D)}R$ , and the D-dimensional volume elements,  $d^{(D)}V_{\text{proper}}$ .

$$I_{\text{Hilbert}} = \frac{1}{16\pi} \int {}^{\scriptscriptstyle{(D)}}\!R \; d^{\scriptscriptstyle{(D)}}\!V_{\text{proper}}. \tag{1}$$

Curvature can be revealed in a number of ways ranging from geodesic deviation to surface area deficit. I know of no more appropriate vehicle toward the understanding of curvature on a simplicial geometry than the parallel transportation of a vector around a closed loop.

$$K = \begin{pmatrix} \text{Gaussian} \\ \text{Curvature} \end{pmatrix} = \frac{\text{Angle that Vector is Rotated}}{\text{Area Circumnavigated}}$$
(2)

However, in our D-dimensional simplicial geometry the curvature at each hinge takes the form of a conical singularity. Consequently, the deflection of a gyroscope when carried one complete circuit around a closed loop encircling a given hinge (h) is independent of the area of the loop. This circumstance presents us with our first dilemma. How do we define the curvature of a hinge? Equivalently, we may ask, what area, content or measure do we assign to a given hinge, h? A natural measure for h emerges if we consider the collection of points in the Delaunay lattice geometry closer to h than to any other hinge. If there are  $N_h$  D-dimensional simplices sharing hinge h, then the set of points thus assignable to h naturally form  $N_h$  simplices. These simplices are formed by taking the Voronoi polygon  $h^*$  dual to hinge h and connecting its  $N_h$  vertices to the (D-1) vertices of h. Each corner on  $h^*$  lies at the circumcenter of one of the D-simplices in the Delaunay geometry which share

hinge h. By construction, the Voronoi polygon  $h^*$  is orthogonal to hinge h. We use this orthogonality (Appendix) to show that the D-volume of these  $N_h$  simplices, determined by both the Delaunay lattice (hinge, h) and its dual Voronoi lattice (polygon,  $h^*$ ), is proportional to the simple product of (1)  $A_h$ , the (D-2)-dimensional volume of hinge h, and (2)  $A_h^*$ , the area of the Voronoi polygon  $h^*$ .

$$\Delta^{(D)}V = \frac{2}{D(D-1)} A_h A_h^*.$$
 (3)

The proportionality constant is related to the dimension, D, of our lattice geometry.

These dual Voronoi-Delaunay volume elements provide us with a natural decomposition for our Hilbert action (Eq. 1) as well as a natural structure to define the scalar curvature of any hinge, h.

$$\int d^{(D)}V_{\text{proper}} \Longrightarrow \sum_{\substack{hinges, \\ h}} \frac{2}{D(D-1)} A_h A_h^*. \tag{4}$$

To define the curvature concentrated at hinge h it seems natural to assign the area  $A_h^*$  to the hinge h. If we parallel transport a vector around the perimeter of  $h^*$  it will have traversed the flat geometry of the interior of each of the  $N_h$  simplices sharing h. Furthermore, it will return rotated in a plane perpendicular to h (hence parallel to  $h^*$ ) by an angle  $\varepsilon_h$ . This deficit angle  $\varepsilon_h$  is defined by the hyperdihedral angles  $(\theta_i)$  of each of the  $N_h$  simplices of the Delaunay lattice sharing hinge h.

$$\varepsilon_h = 2\pi - \sum_{i=1}^{N_h} \theta_i \tag{5}$$

In particular,  $\theta_i$  is the angle between the (D-1)-dimensional faces of simplex i hinging on h. The Gaussian curvature for h is thus,

$$K_h = \frac{\varepsilon_h}{A_h^*}. (6)$$

Since the plane of rotation and the rotation bivector are both orthogonal to hinge h, and since there are no other curvature sources associated with  $h^*$  then the Riemann scalar curvature is simply proportional to  $K_h$ .

$$\begin{pmatrix}
\text{Riemann Scalar Curvature} \\
\text{Associated with Hinge, } h
\end{pmatrix} = {}^{(D)}R_h = \frac{D(D-1)\varepsilon_h}{A_h^*}.$$
(7)

Therefore, one can translate the Hilbert action into its Regge calculus form by using the dual Voronoi-Delaunay construction presented here through equations (4) and (7).

$$I_{\text{Hilbert}} = \frac{1}{16\pi} \int_{(D)}^{(D)} R \, d^{(D)} V_{\text{proper}} \Longrightarrow \frac{1}{16\pi} \sum_{\substack{\text{hinges,} \\ h}} \underbrace{\left(\frac{D(D-1)\varepsilon_h}{A_h^*}\right)}_{(D)} \underbrace{\left(\frac{2}{D(D-1)} A_h A_h^*\right)}_{\Delta^{(D)} V}$$
(8)

We see that the Voronoi area  $A_h^*$  cancels as well as the dimensionality constants D(D-1). We are left with the usual Regge calculus expression for the Hilbert action.

$$I_{\text{Regge}} = \frac{1}{8\pi} \sum_{\substack{\text{hinges,} \\ h}} A_h \varepsilon_h.$$
 (9)

# III. Voronoi Lattice as Equally as Important as the Simplicial Delaunay Lattice in Regge Calculus.

Two new insights into the inner workings of Regge calculus are obtained in this paper. First, we showed that the principles of General Relativity can be applied directly to the lattice geometry to yield an internally exact expression for the Riemann curvature, proper volume; and hence, the Hilbert action. Second, we showed that Regge calculus required a unity of the simplicial geometry (Delaunay lattice) and its dual geometry (Voronoi lattice). The complex of simplices we formed by connecting a given hinge in the Delaunay lattice to its corresponding Voronoi polygon appears to be a natural geometric structure to support the Hilbert action. The orthogonality between these two lattices provided a compact expression for the volume of this complex of simplices which then led to an interesting cancellation of terms in the action. In the final expression for the action (Eq. 9) we see nowhere any trace of the Voronoi lattice — the areas of the Voronoi polygons all cancelled. This apparently deep union of these two lattices can be observed in an action—independent derivation of the simplicial version of the Einstein equations.<sup>6</sup> In particular, the author needed, in this earlier work, to introduce the Voronoi lattice geometry in order to construct the Cartan—derived Einstein equations as well as the corresponding Bianchi identities.

Regge calculus provides us with a geometric, discrete and finite rendering of one of the most beautiful theories in physics — Einstein's geometric theory of gravitation. One wonders what facets of this theory one can distill out to gain an even deeper understanding of nature? Perhaps the geometric duality highlighted in this article is one such indicator?

#### IV. Acknowledgements

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# Appendix: Definition of $A_h^*$ and Proof of Eq. 3.

To derive Eq. 3, we consider a (D-2)-dimensional hinge, h, in a D-dimensional simplicial geometry.

$$h = \{H_0, H_1, \dots, H_{(D-2)}\}. \tag{A1}$$

Here, the H's label the vertices of the hinge. We assume that there are  $N_h$  D-simplices sharing hinge h. This collection of  $N_h$  simplices  $\{S_0, S_1, \ldots, S_{N_h-1}\}$  is formed by connecting the H's to a chain of  $N_h$  vertices,  $\{S_0, S_1, \ldots, S_{N_h-1}\}$ , where

$$S_j = \{H_0, H_1, \dots, H_{(D-2)}, S_j, S_k\}, \quad \text{and} \quad k = j + 1 \mod N_h.$$
 (A2)

Let  $O_h$  be the circumcenter of hinge h. In other words,  $O_h$  is the point in the (D-2)dimensional plane formed by h that is equidistant from all the H's. Similarly, let  $C_j$  be
the circumcenter of the corresponding simplex  $S_j$ , for each j. Then the Voronoi polygon  $h^*$  dual to hinge h is simply the the collection of these  $N_h$  circumcenters,

$$h^* = \{C_0, C_1, \dots, C_{N_h - 1}\}. \tag{A3}$$

The area of  $h^*$ , which we have referred to as  $A_h^*$  is formed by connecting  $O_h$  to the vertices of  $h^*$ . This breaks the Voronoi polygon  $(h^*)$  into  $N_h$  triangles,  $h^* = \{h_i^*, i = 0, 1, \ldots, N_h - 1\}$ , where  $h_i^* = \{O_h, C_i, C_j\}$  with  $j = i + 1 \mod N_h$ . Here

$$A_h^* = \sum_{i=0}^{N_h - 1} A_{h_i}^*,\tag{A4}$$

where  $A_{h_i}^*$  is the area of triangle  $h_i^*$ . Each of these triangles  $A_{h_i}^*$  is orthogonal to h.

When we connect hinge h with the Voronoi polygon  $h^*$  we form a complex of  $N_h$  simplices,  $\{V_i, i=0,1,\ldots,N_h-1\}$ . Each of these simplices  $(V_i)$  is formed by connecting the i'th edge of  $h^*$  to the hinge h,

$$\mathcal{V}_i = \{ H_0, H_1, \dots, H_{(D-2)}, C_i, C_{i+1 \bmod N_b} \}. \tag{A5}$$

To derive Eq. 3 we need only show that the D-volume of simplex  $\mathcal{V}_i$  is given by,

$$V_i = \frac{2}{D(D-1)} A_h A_{h_i}^*. (A6)$$

When we insert the circumcenter  $(O_h)$  of hinge h into simplex  $\mathcal{V}_i$  and connect it to its D+1 vertices, we form D-1 new D-simplices. Now,

$$V_i = \sum_{j=0}^{D-2} V_{i_j},\tag{A7}$$

and  $V_{i_j}$  is formed by replacing the vertex  $H_j$  of simplex  $V_i$  with the circumcenter  $(O_h)$  of the hinge h. Now we are in a position to verify Eq. A6 and thus Eq. 3.

Using the volume formula for a D-dimensional simplex we notice that,

$$V_{i_j} = \frac{1}{D!} \left( \overrightarrow{O_h H_0} \wedge \overrightarrow{O_h H_1} \wedge \dots \overrightarrow{O_h H_{j-1}} \wedge \overrightarrow{O_h H_{j+1}} \wedge \dots \overrightarrow{O_h H_{D-2}} \wedge \overrightarrow{O_h C_i} \wedge \overrightarrow{O_h C_{i+1 \bmod N_h}} \right), \tag{A8}$$

for all  $i = 0, 1, ..., N_h - 1$ . However, triangle  $h_i^*$  is orthogonal to hinge h. Therefore we can factor the last wedge product out of Eq. A8.

$$V_{i_{j}} = \frac{1}{D!} \underbrace{\left(\overrightarrow{O_{h}H_{0}} \wedge \overrightarrow{O_{h}H_{1}} \wedge \dots \overrightarrow{O_{h}H_{j-1}} \wedge \overrightarrow{O_{h}H_{j+1}} \wedge \dots \overrightarrow{O_{h}H_{D-2}}\right)}_{(D-2)!A_{h_{j}}} \underbrace{\left(\overrightarrow{O_{h}C_{i}} \wedge \overrightarrow{O_{h}C_{i+1} \bmod N_{h}}\right)}_{2A_{i}^{*}}.$$

$$(A9)$$

However, this last underbraced term is just twice the area of triangle  $h_i^*$ , and the first term is just (D-2)! times that fraction of the (D-2)-volume  $(A_{h_j})$  of the hinge h formed by replacing vertex  $H_j$  with its circumcenter  $O_h$ . Therefore,

$$V_{i_j} = \frac{2}{D(D-1)} A_{h_j} A_i^*, \tag{A10}$$

and by using this equation and Eq. A7 we have verified Eq. A6. This together with Eq. A4 completes the derivation of Eq. 3 and the definition of the area of  $h^*$ .

### References.

- <sup>1</sup> T. Regge, "General Relativity Without Coordinates," Nuovo Cimento 19 (1961) 558-571.
- <sup>2</sup> C. W. Misner, K.S. Thorne and J. A. Wheeler, *Gravitation* (Freeman, New York, 1973), Ch. 42.
- <sup>3</sup> R. M. Williams and P. A. Tuckey, "Regge Calculus: A Brief Review and Bibliography," Class. Quantum Grav. 9 (1992) 1409-1422.
- <sup>4</sup> A. Okabe, B. Boots and K. Sugihara, Spatial Tesselations, Concepts and Applications of Voronoi Diagrams (John Wiley & Sons, New York, 1992), Ch. 2.
- <sup>5</sup> R. Friedberg and T. D. Lee, "Derivation of Regge's Action from Einstein's Theory of General Relativity," *Nucl. Phys.* **B242** (1984) 145-166.
- <sup>6</sup> W. A. Miller, "The Geometrodynamic Content of the Regge Equations as Illuminated by the Boundary of a Boundary Principle," Found. Phys. **16** (1986) 143-169.
- <sup>7</sup> H. W. Hamber and R. M. Williams, "Higher Derivative Quantum Gravity on a Simplicial Lattice," *Nucl. Phys.* **B248** (1984) 392-414.