Module 6

Chi-Square Tests and the *F*-Distribution

§ 6.1

Goodness of Fit

Multinomial Experiments

A **multinomial experiment** is a probability experiment consisting of a fixed number of trials in which there are more than two possible outcomes for each independent trial. (Unlike the **binomial** experiment in which there were only two possible outcomes.)

Example:
A researcher claims that the distribution of favorite pizza toppings among teenagers is as shown below.

Each outcome is classified into **categories**.

Topping	Frequency, f
Cheese	41%
Pepperoni	25%
Sausage	15%
Mushrooms	10%
Onions	9%

The probability for each possible outcome is fixed.

Chi-Square Goodness-of-Fit Test

A **Chi-Square Goodness-of-Fit Test** is used to test whether a frequency distribution fits an expected distribution.

To calculate the test statistic for the chi-square goodness-of-fit test, the observed frequencies and the expected frequencies are used.

The **observed frequency** O of a category is the frequency for the category observed in the sample data.

The **expected frequency** E of a category is the *calculated* frequency for the category. Expected frequencies are obtained assuming the specified (or hypothesized) distribution. The expected frequency for the ith category is

$$E_i = np_i$$

where n is the number of trials (the sample size) and p_i is the assumed probability of the ith category.

Observed and Expected Frequencies

Example:

200 teenagers are randomly selected and asked what their favorite pizza topping is. The results are shown below.

Find the observed frequencies and the expected frequencies.

Topping	Results ($n = 200$)	% of teenagers
Cheese	78	41%
Pepperoni	52	25%
Sausage	30	15%
Mushrooms	25	10%
Onions	15	9%

Observed Frequency	Expected Frequency
78	200(0.41) = 82
52	200(0.25) = 50
30	200(0.15) = 30
25	200(0.10) = 20
15	200(0.09) = 18

Chi-Square Goodness-of-Fit Test

For the chi-square goodness-of-fit test to be used, the following must be true.

1. The observed frequencies must be obtained by using a random sample.
2. Each expected frequency must be greater than or equal to 5.

The Chi-Square Goodness-of-Fit Test

If the conditions listed above are satisfied, then the sampling distribution for the goodness-of-fit test is approximated by a chi-square distribution with $k - 1$ degrees of freedom, where k is the number of categories. The test statistic for the chi-square goodness-of-fit test is

$$X^2 = \sum \frac{(O - E)^2}{E}$$

The test is always a right-tailed test.

where O represents the observed frequency of each category and E represents the expected frequency of each category.

Chi-Square Goodness-of-Fit Test

Performing a Chi-Square Goodness-of-Fit Test

In Words	*In Symbols*
1. Identify the claim. State the null and alternative hypotheses.	State H_0 and H_a.
2. Specify the level of significance.	Identify α.
3. Identify the degrees of freedom.	d.f. = $k - 1$
4. Determine the critical value.	Use Table 6 in Appendix B.
5. Determine the rejection region.	

Continued.

Chi-Square Goodness-of-Fit Test

Performing a Chi-Square Goodness-of-Fit Test

In Words	*In Symbols*
6. Calculate the test statistic.	$\chi^2 = \sum \frac{(O - E)^2}{E}$
7. Make a decision to reject or fail to reject the null hypothesis.	If χ^2 is in the rejection region, reject H_0. Otherwise, fail to reject H_0.
8. Interpret the decision in the context of the original claim.	

Chi-Square Goodness-of-Fit Test

Example:
A researcher claims that the distribution of favorite pizza toppings among teenagers is as shown below. 200 randomly selected teenagers are surveyed.

Topping	Frequency, f
Cheese	39%
Pepperoni	26%
Sausage	15%
Mushrooms	12.5%
Onions	7.5%

Using $\alpha = 0.01$, and the observed and expected values previously calculated, test the surveyor's claim using a chi-square goodness-of-fit test.

Continued.

Chi-Square Goodness-of-Fit Test

Example continued:

H_0: The distribution of pizza toppings is 39% cheese, 26% pepperoni, 15% sausage, 12.5% mushrooms, and 7.5% onions. (Claim)

H_a: The distribution of pizza toppings differs from the claimed or expected distribution.

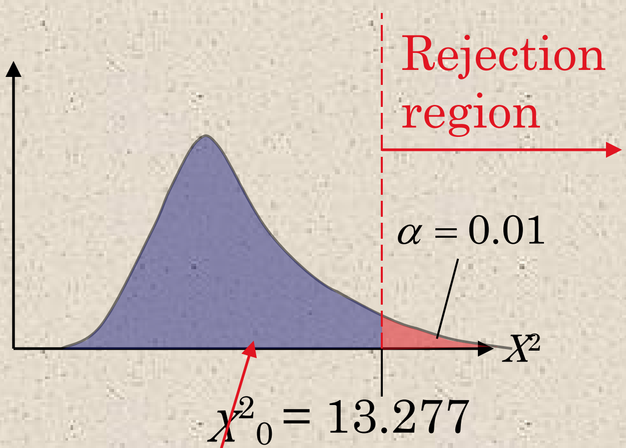
Because there are 5 categories, the chi-square distribution has $k - 1 = 5 - 1 = 4$ degrees of freedom.

With d.f. = 4 and $\alpha = 0.01$, the critical value is $\chi^2_0 = 13.277$.

Continued.

Chi-Square Goodness-of-Fit Test

Example continued:



Topping	Observed Frequency	Expected Frequency
Cheese	78	82
Pepperoni	52	50
Sausage	30	30
Mushrooms	25	20
Onions	15	18

$$X^2 = \sum \frac{(O-E)^2}{E} = \frac{(78-82)^2}{82} + \frac{(52-50)^2}{50} + \frac{(30-30)^2}{30} + \frac{(25-20)^2}{20} + \frac{(15-18)^2}{18}$$

$$\approx 2.025$$

Fail to reject H_0.

There is not enough evidence at the 1% level to reject the surveyor's claim.

§ 6.2

Independence

Contingency Tables

An $r \times c$ **contingency table** shows the observed frequencies for two variables. The observed frequencies are arranged in r rows and c columns. The intersection of a row and a column is called a cell.

The following contingency table shows a random sample of 321 fatally injured passenger vehicle drivers by age and gender. (Adapted from Insurance Institute for Highway Safety)

	Age					
Gender	16 – 20	21 – 30	31 – 40	41 – 50	51 – 60	61 and older
Male	32	51	52	43	28	10
Female	13	22	33	21	10	6

Expected Frequency

Assuming the two variables are independent, you can use the contingency table to find the expected frequency for each cell.

Finding the Expected Frequency for Contingency Table Cells

The expected frequency for a cell $E_{r,c}$ in a contingency table is

$$\text{Expected frequency } E_{r,c} = \frac{(\text{Sum of row } r) \times (\text{Sum of column } c)}{\text{Sample size}}.$$

Expected Frequency

Example:

Find the expected frequency for each “Male” cell in the contingency table for the sample of 321 fatally injured drivers. Assume that the variables, age and gender, are independent.

	Age						
Gender	16 – 20	21 – 30	31 – 40	41 – 50	51 – 60	61 and older	Total
Male	32	51	52	43	28	10	216
Female	13	22	33	21	10	6	105
Total	45	73	85	64	38	16	321

Continued.

Expected Frequency

Example continued:

	Age						
Gender	16 – 20	21 – 30	31 – 40	41 – 50	51 – 60	61 and older	Total
Male	32	51	52	43	28	10	216
Female	13	22	33	21	10	6	105
Total	45	73	85	64	38	16	321

$$\text{Expected frequency } E_{r,c} = \frac{(\text{Sum of row } r) \times (\text{Sum of column } c)}{\text{Sample size}}$$

$$E_{1,1} = \frac{216 \cdot 45}{321} \approx 30.28 \qquad E_{1,2} = \frac{216 \cdot 73}{321} \approx 49.12 \qquad E_{1,3} = \frac{216 \cdot 85}{321} \approx 57.20$$

$$E_{1,4} = \frac{216 \cdot 64}{321} \approx 43.07 \qquad E_{1,5} = \frac{216 \cdot 38}{321} \approx 25.57 \qquad E_{1,6} = \frac{216 \cdot 16}{321} \approx 10.77$$

Chi-Square Independence Test

A **chi-square independence test** is used to test the independence of two variables. Using a chi-square test, you can determine whether the occurrence of one variable affects the probability of the occurrence of the other variable.

For the chi-square independence test to be used, the following must be true.

1. The observed frequencies must be obtained by using a random sample.
2. Each expected frequency must be greater than or equal to 5.

Chi-Square Independence Test

The Chi-Square Independence Test

If the conditions listed are satisfied, then the sampling distribution for the chi-square independence test is approximated by a chi-square distribution with

$$(r-1)(c-1)$$

degrees of freedom, where r and c are the number of rows and columns, respectively, of a contingency table. The test statistic for the chi-square independence test is

$$\chi^2 = \sum \frac{(O-E)^2}{E}$$

The test is always a right-tailed test.

where O represents the observed frequencies and E represents the expected frequencies.

Chi-Square Independence Test

Performing a Chi-Square Independence Test

In Words	*In Symbols*
1. Identify the claim. State the null and alternative hypotheses.	State H_0 and H_a.
2. Specify the level of significance.	Identify α.
3. Identify the degrees of freedom.	d.f. = $(r-1)(c-1)$
4. Determine the critical value.	Use Table 6 in Appendix B.
5. Determine the rejection region.	

Continued.

Chi-Square Independence Test

Performing a Chi-Square Independence Test

In Words	*In Symbols*
6. Calculate the test statistic.	$\chi^2 = \sum \frac{(O-E)^2}{E}$
7. Make a decision to reject or fail to reject the null hypothesis.	If χ^2 is in the rejection region, reject H_0. Otherwise, fail to reject H_0.
8. Interpret the decision in the context of the original claim.	

Chi-Square Independence Test

Example:
The following contingency table shows a random sample of 321 fatally injured passenger vehicle drivers by age and gender. The expected frequencies are displayed in parentheses. At $\alpha = 0.05$, can you conclude that the drivers' ages are related to gender in such accidents?

	Age						
Gender	16 – 20	21 – 30	31 – 40	41 – 50	51 – 60	61 and older	Total
Male	32 (30.28)	51 (49.12)	52 (57.20)	43 (43.07)	28 (25.57)	10 (10.77)	216
Female	13 (14.72)	22 (23.88)	33 (27.80)	21 (20.93)	10 (12.43)	6 (5.23)	105
	45	73	85	64	38	16	321

Chi-Square Independence Test

Example continued:

Because each expected frequency is at least 5 and the drivers were randomly selected, the chi-square independence test can be used to test whether the variables are independent.

H_0: The drivers' ages are independent of gender.

H_a: The drivers' ages are dependent on gender. (Claim)

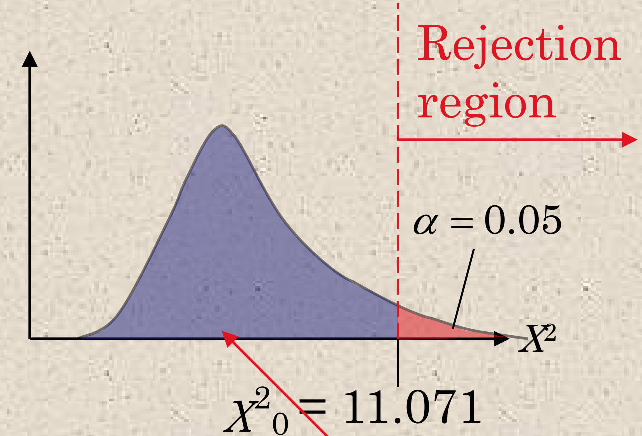
d.f. = $(r - 1)(c - 1) = (2 - 1)(6 - 1) = (1)(5) = 5$

With d.f. = 5 and $\alpha = 0.05$, the critical value is $\chi^2_0 = 11.071$.

Continued.

Chi-Square Independence Test

Example continued:



$$X^2 = \sum \frac{(O-E)^2}{E} = 2.84$$

Fail to reject H_0.

O	E	$O - E$	$(O - E)^2$	$\frac{(O-E)^2}{E}$
32	30.28	1.72	2.9584	0.0977
51	49.12	1.88	3.5344	0.072
52	57.20	–5.2	27.04	0.4727
43	43.07	–0.07	0.0049	0.0001
28	25.57	2.43	5.9049	0.2309
10	10.77	–0.77	0.5929	0.0551
13	14.72	–1.72	2.9584	0.201
22	23.88	–1.88	3.5344	0.148
33	27.80	5.2	27.04	0.9727
21	20.93	0.07	0.0049	0.0002
10	12.43	–2.43	5.9049	0.4751
6	5.23	0.77	0.5929	0.1134

There is not enough evidence at the 5% level to conclude that age is dependent on gender in such accidents.

§ 6.3

Comparing Two Variances

F-Distribution

Let s_1^2 and s_2^2 represent the sample variances of two different populations. If both populations are normal and the population variances σ_1^2 and σ_2^2 are equal, then the sampling distribution of

$$F = \frac{s_1^2}{s_2^2}$$

is called an ***F*-distribution**.

There are several properties of this distribution.

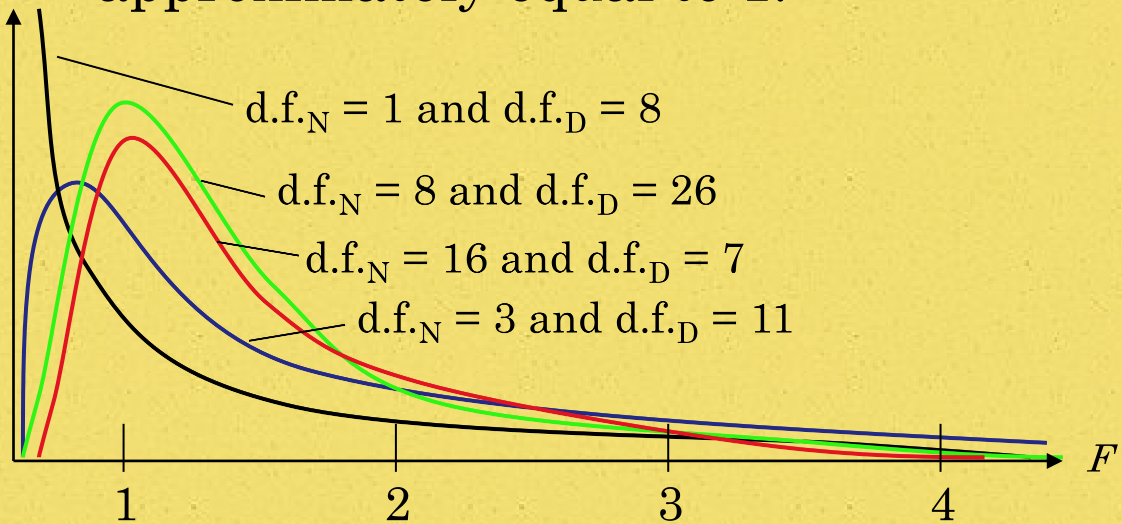
1. The F-distribution is a family of curves each of which is determined by two types of degrees of freedom: the degrees of freedom corresponding to the variance in the numerator, denoted **d.f.$_N$**, and the degrees of freedom corresponding to the variance in the denominator, denoted **d.f.$_D$**.

Continued.

F-Distribution

Properties of the F-distribution continued:

2. F-distributions are positively skewed.
3. The total area under each curve of an F-distribution is equal to 1.
4. F-values are always greater than or equal to 0.
5. For all F-distributions, the mean value of F is approximately equal to 1.



Critical Values for the F-Distribution

Finding Critical Values for the F-Distribution

1. Specify the level of significance α.
2. Determine the degrees of freedom for the numerator, d.f._N.
3. Determine the degrees of freedom for the denominator, d.f._D.
4. Use Table 7 in Appendix B to find the critical value. If the hypothesis test is

   a. one-tailed, use the α F-table.

   b. two-tailed, use the $\frac{1}{2}\alpha$ F-table.

Critical Values for the F-Distribution

Example:

Find the critical F-value for a right-tailed test when $\alpha = 0.05$, $\text{d.f.}_N = 5$ and $\text{d.f.}_D = 28$.

Appendix B: Table 7: F-Distribution

d.f._D: Degrees of freedom, denominator	$\alpha = 0.05$					
	d.f._N: Degrees of freedom, numerator					
	1	2	3	4	5	6
1	161.4	199.5	215.7	224.6	230.2	234.0
2	18.51	19.00	19.16	19.25	19.30	19.33
27	4.21	3.35	2.96	2.73	2.57	2.46
28	4.20	3.34	2.95	2.71	2.56	2.45
29	4.18	3.33	2.93	2.70	2.55	2.43

The critical value is $F_0 = 2.56$.

Critical Values for the *F*-Distribution

Example:

Find the critical *F*-value for a two-tailed test when $\alpha = 0.10$, $d.f._N = 4$ and $d.f._D = 6$.

$$\frac{1}{2}\alpha = \frac{1}{2}(0.10) = 0.05$$

Appendix B: Table 7: *F*-Distribution

d.f.$_D$: Degrees of freedom, denominator	$\alpha = 0.05$					
	d.f.$_N$: Degrees of freedom, numerator					
	1	**2**	**3**	**4**	**5**	**6**
1	161.4	199.5	215.7	224.6	230.2	234.0
2	18.51	19.00	19.16	19.25	19.30	19.33
3	10.13	9.55	9.28	9.12	9.01	8.94
4	7.71	6.94	6.59	6.39	6.26	6.16
5	6.61	5.79	5.41	5.19	5.05	4.95
6	5.99	5.14	4.76	4.53	4.39	4.28
7	5.59	4.74	4.35	4.12	3.97	3.87

The critical value is $F_0 = 4.53$.

Two-Sample *F*-Test for Variances

Two-Sample *F*-Test for Variances

A two-sample F-test is used to compare two population variances σ_1^2 and σ_2^2 when a sample is randomly selected from each population. The populations must be independent and normally distributed. The test statistic is

$$F = \frac{s_1^2}{s_2^2}$$

where s_1^2 and s_2^2 represent the sample variances with $s_1^2 \geq s_2^2$. The degrees of freedom for the numerator is d.f.$_N = n_1 - 1$ and the degrees of freedom for the denominator is d.f.$_D = n_2 - 1$, where n_1 is the size of the sample having variance s_1^2 and n_2 is the size of the sample having variance s_2^2.

Two-Sample *F*-Test for Variances

Using a Two-Sample F-Test to Compare σ_1^2 and σ_2^2

In Words	*In Symbols*
1. Identify the claim. State the null and alternative hypotheses.	State H_0 and H_a.
2. Specify the level of significance.	Identify α.
3. Identify the degrees of freedom.	$\text{d.f.}_N = n_1 - 1$ $\text{d.f.}_D = n_2 - 1$
4. Determine the critical value.	Use Table 7 in Appendix B.

Continued.

Two-Sample *F*-Test for Variances

Using a Two-Sample F-Test to Compare σ_1^2 and σ_2^2

In Words	*In Symbols*
5. Determine the rejection region.	
6. Calculate the test statistic.	$F = \dfrac{s_1^2}{s_2^2}$
7. Make a decision to reject or fail to reject the null hypothesis.	If F is in the rejection region, reject H_0. Otherwise, fail to reject H_0.
8. Interpret the decision in the context of the original claim.	

Two-Sample *F*-Test

Example:

A travel agency's marketing brochure indicates that the standard deviations of hotel room rates for two cities are the same. A random sample of 13 hotel room rates in one city has a standard deviation of $27.50 and a random sample of 16 hotel room rates in the other city has a standard deviation of $29.75. Can you reject the agency's claim at $\alpha = 0.01$?

Because 29.75 > 27.50, $s_1^2 = 885.06$ and $s_2^2 = 756.25$.

H_0: $\sigma_1^2 = \sigma_2^2$ (Claim)

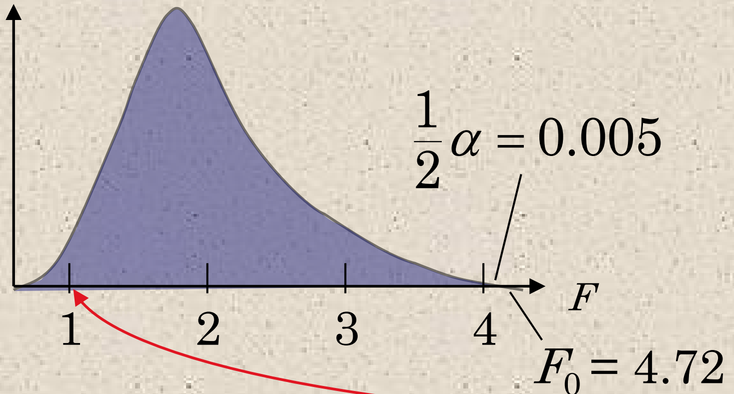
H_a: $\sigma_1^2 \neq \sigma_2^2$

Continued.

Two-Sample F-Test

Example continued:

This is a two-tailed test with $\frac{1}{2}\alpha = \frac{1}{2}(0.01) = 0.005$, d.f.$_N = 15$ and d.f.$_D = 12$.



The critical value is $F_0 = 4.72$.

The test statistic is

$$F = \frac{s_1^2}{s_2^2} = \frac{885.06}{756.25} \approx 1.17.$$

Fail to reject H_0.

There is not enough evidence at the 1% level to reject the claim that the standard deviation of the hotel room rates for the two cities are the same.

§ 6.4

Analysis of Variance

One-Way ANOVA

One-way analysis of variance is a hypothesis-testing technique that is used to compare means from three or more populations. Analysis of variance is usually abbreviated **ANOVA**.

In a one-way ANOVA test, the following must be true.

1. Each sample must be randomly selected from a normal, or approximately normal, population.
2. The samples must be independent of each other.
3. Each population must have the same variance.

One-Way ANOVA

$$\text{Test statistic} = \frac{\text{Variance between samples}}{\text{Variance within samples}}$$

1. The variance between samples MS_B measures the differences related to the treatment given to each sample and is sometimes called the **mean square between**.
2. The variance within samples MS_W measures the differences related to entries within the same sample. This variance, sometimes called the **mean square within**, is usually due to sampling error.

One-Way ANOVA

One-Way Analysis of Variance Test

If the conditions listed are satisfied, then the sampling distribution for the test is approximated by the F-distribution. The test statistic is

$$F = \frac{MS_B}{MS_W}.$$

The degrees of freedom for the F-test are

$$\text{d.f.}_{\text{N}} = k - 1$$

and

$$\text{d.f.}_{\text{D}} = N - k$$

where k is the number of samples and N is the sum of the sample sizes.

Test Statistic for a One-Way ANOVA

Finding the Test Statistic for a One-Way ANOVA Test

In Words	*In Symbols*
1. Find the mean and variance of each sample.	$\bar{x} = \frac{\sum x}{n} \quad s^2 = \frac{\sum (x - \bar{x})^2}{n-1}$
2. Find the mean of all entries in all samples (the grand mean).	$\bar{\bar{x}} = \frac{\sum x}{N}$
3. Find the sum of squares between the samples.	$SS_B = \sum n_i(\bar{x}_i - \bar{\bar{x}})^2$
4. Find the sum of squares within the samples.	$SS_W = \sum (n_i - 1)s_i^2$

Continued.

Test Statistic for a One-Way ANOVA

Finding the Test Statistic for a One-Way ANOVA Test

In Words	*In Symbols*
5. Find the variance between the samples.	$MS_B = \dfrac{SS_B}{k-1} = \dfrac{SS_B}{\text{d.f.}_\text{N}}$
6. Find the variance within the samples	$MS_W = \dfrac{SS_W}{N-k} = \dfrac{SS_W}{\text{d.f.}_\text{D}}$
7. Find the test statistic.	$F = \dfrac{MS_B}{MS_W}$

Performing a One-Way ANOVA Test

Performing a One-Way Analysis of Variance Test

In Words	*In Symbols*
1. Identify the claim. State the null and alternative hypotheses.	State H_0 and H_a.
2. Specify the level of significance.	Identify α.
3. Identify the degrees of freedom.	$\text{d.f.}_N = k - 1$ $\text{d.f.}_D = N - k$
4. Determine the critical value.	Use Table 7 in Appendix B.

Continued.

Performing a One-Way ANOVA Test

Performing a One-Way Analysis of Variance Test

In Words	*In Symbols*
5. Determine the rejection region.	
6. Calculate the test statistic.	$F = \dfrac{MS_B}{MS_W}$
7. Make a decision to reject or fail to reject the null hypothesis.	If F is in the rejection region, reject H_0. Otherwise, fail to reject H_0.
8. Interpret the decision in the context of the original claim.	

ANOVA Summary Table

A table is a convenient way to summarize the results in a one-way ANOVA test.

Variation	Sum of squares	Degrees of freedom	Mean squares	F
Between	SS_B	d.f._N	$MS_B = \dfrac{SS_B}{\text{d.f.}_\text{N}}$	$MS_B \div MS_W$
Within	SS_W	d.f._D	$MS_W = \dfrac{SS_W}{\text{d.f.}_\text{D}}$	

Performing a One-Way ANOVA Test

Example:

The following table shows the salaries of randomly selected individuals from four large metropolitan areas. At $\alpha = 0.05$, can you conclude that the mean salary is different in at least one of the areas? (Adapted from US Bureau of Economic Analysis)

Pittsburgh	Dallas	Chicago	Minneapolis
27,800	30,000	32,000	30,000
28,000	33,900	35,800	40,000
25,500	29,750	28,000	35,000
29,150	25,000	38,900	33,000
30,295	34,055	27,245	29,805

Continued.

Performing a One-Way ANOVA Test

Example continued:

H_0: $\mu_1 = \mu_2 = \mu_3 = \mu_4$

H_a: At least one mean is different from the others. (Claim)

Because there are k = 4 samples, $\text{d.f.}_N = k - 1 = 4 - 1 = 3$.

The sum of the sample sizes is
$N = n_1 + n_2 + n_3 + n_4 = 5 + 5 + 5 + 5 = 20$.

$\text{d.f.}_D = N - k = 20 - 4 = 16$

Using $\alpha = 0.05$, $\text{d.f.}_N = 3$, and $\text{d.f.}_D = 16$,
the critical value is $F_0 = 3.24$.

Continued.

Performing a One-Way ANOVA Test

Example continued:

To find the test statistic, the following must be calculated.

$$\bar{\bar{x}} = \frac{\sum x}{N} = \frac{140745 + 152705 + 161945 + 167805}{20} = 31160$$

$$MS_B = \frac{SS_B}{\text{d.f.}_N} = \frac{\sum n_i(\bar{x}_i - \bar{\bar{x}})^2}{k-1}$$

$$= \frac{5(28149 - 31160)^2 + 5(30541 - 31160)^2}{4-1} +$$

$$\frac{5(32389 - 31160)^2 + 5(33561 - 31160)^2}{4-1}$$

$$\approx 27874206.67$$

Continued.

Performing a One-Way ANOVA Test

Example continued:

$$MS_W = \frac{SS_W}{\text{d.f.}_D} = \frac{\sum(n_i - 1)s_i^2}{N - k}$$

$$\approx \frac{(5-1)(3192128.94) + (5-1)(13813030.08)}{20-4} +$$

$$\frac{(5-1)(24975855.83) + (5-1)(17658605.02)}{20-4}$$

$$= 14909904.97$$

Test statistic

Critical value

$$F = \frac{MS_B}{MS_W} = \frac{27874206.67}{14909904.34} \approx 1.870$$

$$1.870 < 3.24.$$

Fail to reject H_0.

There is not enough evidence at the 5% level to conclude that the mean salary is different in at least one of the areas.