Bernoulli Trails and Binomial Experiment

- **Bernoulli trials:** A Bernoulli refers to a trial that has only two possible outcomes.
- **Example of Bernoulli trials:**
  - Flipping a coin: $S = \{\text{head}, \text{tail}\}$
  - Truth of an answer: $S = \{\text{right}, \text{wrong}\}$
  - Status of a machine: $S = \{\text{working}, \text{broken}\}$
  - Quality of a product: $S = \{\text{good}, \text{defective}\}$
  - Accomplishment of a task: $S = \{\text{success}, \text{failure}\}$
- **Binomial Experiment:** A binomial experiment refers to a random experiment consisting of n repeated trials which satisfy the following conditions:
  1. The trials are independent, i.e., the outcome of a trial does not affect the outcomes of other trials,
  2. Each trial has only two outcomes, labeled as 'success' and 'failure', and
  3. The probability of a success (p) in each trial is constant.

- In other words, a **binomial experiment** consists of a series of n independent Experiment Bernoulli trials with a constant probability of success (p) in each trial.

The Binomial Probability Distribution Function

- To find, the probability of getting x success in n trails. the probability of success be p the probability of failure is $1 - p = q$,
  (or)
  The probability of obtaining 'k' successes in 'n' independent trials of a binomial experiment, where the probability of success is 'p', is given by
  - $f(k, n, p) = P(X = k) = \binom{n}{k} p^k (1-p)^{n-k} = \binom{n}{k} p^k q^{n-k}$ for $k = 0, 1, 2, \ldots, n$, where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.
  - Mean: $E(X) = \mu = np$,
  - Standard deviation: $SD(X) = \sigma = \sqrt{npq}$
  - Variance: $Var(X) = npq$

- Eight coins are tossed simultaneously. Find the probability of getting atleast six heads.
- **Solution:** Here $n = 8, \quad p = P(head) = 2, \; q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$
- Trials satisfy conditions of Binomial distribution.
- Hence

$$\begin{aligned} P(X = k) &= \binom{n}{k} p^k q^{n-k} = \binom{8}{k} p^k q^{8-k}, \quad k = 0, 1, \ldots, 8. \\ &= \binom{8}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{8-k} = \binom{8}{k} \left(\frac{1}{2}\right)^8 = \frac{\binom{8}{k}}{256} \end{aligned}$$

$$\begin{aligned} P(\text{getting atleast six heads}) &= P(X \geq 6) \\ &= P(k = 6) + P(k = 7) + P(k = 8) \\ &= \frac{\binom{8}{6}}{256} + \frac{\binom{8}{7}}{256} + \frac{\binom{8}{8}}{256} = \frac{28 + 8 + 1}{256} \\ &= \frac{37}{256} \end{aligned}$$

Example of Binomial distribution

- If 20% of the bolts produced by a machine are found by defective. Determine the probability that out of 4 bolts chosen at random (a) one (b) zero (c) at most two will be defective .
- **Solution:** Success=a bolt being defective
  $p = \frac{20}{100} = \frac{1}{5}, \quad q = 1 - p = 1 - \frac{20}{100} = \frac{80}{100} = \frac{4}{5}$
- The trails $n = 4$
- The probability $P(X = k) = \binom{n}{k} p^k q^{n-k} = \binom{4}{k} p^k q^{4-k}$
- $P(X = 1) = \binom{4}{1} (\frac{1}{5})^1 (\frac{4}{5})^{4-1} = \frac{4^4}{5^4}$
- $P(X = 0) = \binom{4}{0} (\frac{1}{5})^0 (\frac{4}{5})^{4-0} = \frac{4^4}{5^4}$
- At most two $P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$

$$P(X \leq 2) = \binom{4}{0} (\frac{1}{5})^0 (\frac{4}{5})^{4-0} + \binom{4}{1} (\frac{1}{5})^1 (\frac{4}{5})^{4-1} + \binom{4}{1} (\frac{1}{5})^2 (\frac{4}{5})^{4-2} = \frac{528}{5^4}$$

- What are the mean and standard deviation of the number of universal donors out of the 20 donors?. What is the probability that there are exactly 2 or 3 universal donors out of the 20 donors? (Here 6% of people have universal blood donors or O-negative blood donors).
- There are two outcomes:
  - success $=$ O-negative
  - failure $=$ other blood types.
- $p = 6\% = \frac{6}{100} = 0.06$
- Let X $=$ number of O-negative donors among n $=$ 20 people.
- $E(X) = np = 20 * 0.06 = 1.2$
- $SD(X) = \sqrt{npq} = \sqrt{20 * 0.06 * 0.94} \approx 1.06$

- **Calculate the probability of 2 or 3 successes.**

$$\begin{aligned}
P(X = 2 \text{ or } 3) &= P(X=2) + P(X=3) \\
&= \binom{20}{2}(0.06)^2(0.94)^{20-2} + \binom{20}{3}(0.06)^2(0.94)^{20-3} \\
&= \binom{20}{2}(0.06)^2(0.94)^{18} + \binom{20}{3}(0.06)^2(0.94)^{17} \\
&\approx 0.2246 + 0.08860 \\
&= 0.3106
\end{aligned}$$

Practise problems of Binomial distribution

1. Out of 800 families with 5 children each. How many would you except to have (a) 3 boys (b) 5 girls. Assume equal probability for boys and girls.
2. If the probability of success is $\frac{1}{20}$, how many trails are necessary in order that the probability of at least one success is just greater than $\frac{1}{2}$
3. The incidence of an occupational disease in an industry is such that the workers have a 20% chance of suffering from in it, what is the probability that out of 6 workers at random, four or more will suffer from the disease?

- The Poisson distribution can be obtained as a limiting case of binomial distribution under the following conditions:
    - The number of trails 'n' is indefinitely large i.e., $n \to \infty$
    - The probability of a success 'p' for each trial is very small i.e., $p \to 0$
    - $np = \lambda$ is finite.
- The probability distribution function is

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

- Mean of the poison distribution

$$\text{Mean} = E(x) = \sum_{x=0}^{\infty} xP(x) = \lambda$$

- Variance of the poisson distribution

$$\text{Variance} = \sum (x - \mu)^2 P(x) = \lambda$$

- If 2% of electric bulbs manufactured by a certain company are defective find the probability that in a sample of 200 bulbs (i) less than 2 bulbs are defective (ii) more than 3 bulbs are defective. $[e^{-4} = 0.0183]$
- **Solution:** $E(x) = \sum_{x=0}^{\infty} xP(x)$
- Let X denote the number of defective bulbs
- $P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!},\ x = 0, 1, 2, \ldots, \infty$
- Given $\quad p = P(\text{a defective bulb}) = 2\% = \frac{2}{100} = 0.02$
- n=200
- $\lambda = np = 200 * 0.02 = 4$
- $P(X = x) = \frac{e^{-4}4^x}{x!},\ x = 0, 1, 2, \ldots, \infty$

- (i) P(less than 2 bulbs are defective)

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\begin{aligned}
&= P(X < 2) \\
&= P(X = 0) + P(X = 1) \\
&= \frac{e^{-4}4^0}{0!} + \frac{e^{-4}4^1}{1!} \\
&= e^{-4}(1 + 4) = 0.0183 * 5 \\
&= 0.0915
\end{aligned}
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- (ii) P(more than 3 defectives)

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\begin{aligned}
&= P(X > 3) = 1 - P(X \leq 3) \\
&= P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) \\
&= \frac{e^{-4}4^0}{0!} + \frac{e^{-4}4^1}{1!} + \frac{e^{-4}4^2}{2!} + \frac{e^{-4}4^3}{3!} \\
&= 1 - e^{-4}\{1 + 4 + 8 + 10.667\} \\
&= 1 - 0.0183 * 23.667 = 0.567
\end{aligned}
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- Suppose that the average number of accidents occurring weekly on a particular stretch of a highway equals 3. Calculate the probability that there is at least one accident this week.
- **Solution:** Let X denote the number of accidents occurring on the stretch of highway in question during this week. Because it is reasonable to suppose that there are a large number of cars passing along that stretch, each having a small probability of being involved in an accident, the number of such accidents should be approximately Poisson distributed. Hence,

$$\begin{aligned} P(X \geq 1) &= 1 - P(X = 0) \\ &= 1 - \frac{e^{-3}3^0}{0!} \\ &= 1 - e^{-3} \approx 0.9502 \end{aligned}$$

Practise Problems of poission Distribution

- A car-hire firm has two cars which it hires out day by day. The number of demands for a car on each day is distributed as a Poisson distribution with mean 1.5. Calculate the proportion of days (i) on which there is no demand (ii) on which demand is refused.
  Answer:(i) 0.2231 (ii) 0.1913
- A manufacture of cotter clips know that 5% of his product is defective. If he sells clips in boxes of 100 and guarantees that not more than 10 clips will be defective. What is the probability that a box will fail to meet the guarantee quality?
  Answer: 0.0137

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