Chi-Square distribution

The square of standard normal variable is known as a chi-square variable with 1 degree of freedom (d.f.). Thus

If $X \sim N(\mu, \sigma^2)$, then it is known that $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$. Further Z^2 is said to follow χ^2 – distribution with 1 degree of freedom (χ^2 – pronounced as chi-square)

Properties of χ^2 distribution

- It is a continuous distribution.
- The distribution has only one parameter *i.e.* n *d.f.*
- The shape of the distribution depends upon the *d.f*, n.
- The mean of the chi-square distribution is n and variance $2n$
- If U and V are independent random variables having χ^2 distributions with degree of freedom n_1 and n_2 respectively, then their sum $U + V$ has the same χ^2 distribution with *d.f* $n_1 + n_2$.

Applications of chi-square distribution

- To test the variance of the normal population
- To test the independence of attributes.
- To test the goodness of fit of a distribution.

Test of Hypotheses for population variance of the normal population (Population mean is assumed to be unknown)

Procedure

Step 1 : Let μ and σ^2 be respectively the mean and the variance of the normal population under study, where σ^2 is known and μ unknown. If $\sigma_0^{\ 2}$ is an admissible value of σ^2, then frame the

Null hypothesis as $H_0: \sigma^2 = \sigma_0^{\ 2}$

and choose the suitable alternative hypothesis from

(i) $H_1: \sigma^2 \neq \sigma_0^{\ 2}$ (ii) $H_1: \sigma^2 > \sigma_0^{\ 2}$ (iii) $H_1: \sigma^2 < \sigma_0^{\ 2}$

Step 2 : Describe the sample/data and its descriptive measures. Let $(X_1, X_2, \ldots, X_n)$ be a random sample of n observations drawn from the population, where n is small $(n < 30)$.

Step 3 : Fix the desired level of significance α.

Step 4 : Consider the test statistic $\chi^2 = \dfrac{(n-1)S^2}{\sigma_0^2}$ under H_0. The approximate sampling distribution of the test statistic under H_0 is the chi-square distribution with $(n-1)$ degrees of freedom.

Step 5 : Calculate the value of the of χ^2 for the given sample as $\chi_0{}^2 = \dfrac{(n-1)s^2}{\sigma_0^2}$

Step 6 : Choose the critical value of χ_e^2 corresponding to α and H_1 from the following table.

Alternative Hypothesis (H_1)	$\sigma^2 \neq \sigma_0^2$	$\sigma^2 > \sigma_0^2$	$\sigma^2 < \sigma_0^2$
Critical value (χ_e^2)	$\chi^2_{n-1,\frac{\alpha}{2}}$ and $\chi_0{}^2 \leq \chi^2_{n-1,1-\frac{\alpha}{2}}$	$\chi^2_{n-1,\alpha}$	$\chi^2_{n-1,1-\alpha}$

Step 7 : Decide on H_0 choosing the suitable rejection rule from the following table corresponding to H_1.

Alternative Hypothesis (H_1)	$\sigma^2 \neq \sigma_0^2$	$\sigma^2 > \sigma_0^2$	$\sigma^2 < \sigma_0^2$
Rejection Rule	$\chi^2_{n-1,\frac{\alpha}{2}}$ and $\chi_0{}^2 \leq \chi^2_{n-1,1-\frac{\alpha}{2}}$	$\chi_0{}^2 > \chi^2_{n-1,\alpha}$	$\chi_0{}^2 < \chi^2_{n-1,\,1-\alpha}$

Example

The weights (in kg.) of 8 students are 38, 42, 43, 50, 48, 45, 52 and 50. Test the hypothesis that the variance of the population is 48 kg, assuming the population is normal and μ is unknown.

Solution:

Step 1 : **Null Hypothesis** H_0: $\sigma^2 = 48$ kg.

i.e. Population variance can be regarded as 48 kg.

Alternative hypothesis H_1: $\sigma^2 \neq 48$ kg.

i.e. Population variance cannot be regarded as 48 kg.

Step 2 : The given sample information is
Sample size $(n) = 8$

Step 3 : **Level of significance**
$\alpha = 5\%$

Step 4 : Test statistic

Under null hypothesis H_0

$$\chi^2 = \frac{(n-1)S^2}{\sigma_0^2}$$

follows chi-square distribution with $(n-1)$ *d.f.*

Step 5 : Calculation of test statistic

The value of chi-square under H_0 is calculated as under:

To find $\bar{x}$ and sample variance s^2, we form the following table.

x_i	$(x_i - 46)$	$(x_i - 46)^2$
38	-8	64
42	-4	16
43	-3	9
50	4	16
48	2	4
45	-1	1
52	6	36
50	4	16
$\sum_{i=1}^{8} x_i = 368$	0	$\sum_{i=1}^{8} (x_i - \bar{x})^2 = 162$

$$\bar{x} = \frac{\sum_{i=1}^{8} x_i}{n} = \frac{368}{8} = 46$$

$$s^2 = \frac{\sum_{i=1}^{8}(x_i - \bar{x})^2}{(n-1)} = \frac{\sum_{i=1}^{8}(x_i - 46)^2}{(8-1)} = \frac{162}{7} = 23.143.$$

The calculated value of chi-square is $\chi_0^{\,2} = \dfrac{(n-1)s^2}{\sigma^2} = \dfrac{\sum_{i=1}^{8}(x_i - \bar{x})^2}{\sigma_0^2} = \dfrac{162}{48} = 3.375$

Step 6 : Critical values

Since H_1 is a two sided alternative, the critical values at α =0.05 are $\chi^2_{7,\,0.025} = 16.01$ and $\chi^2_{7,0.975} = 1.69$.

Step 7 : Decision

Since it is a two-tailed test, the elements of the critical region are determined by the rejection rule $\chi_0^2 \geq \chi^2_{n-1,\frac{\alpha}{2}}$ or $\chi_0^2 \leq \chi^2_{n-1,1-\frac{\alpha}{2}}$.

For the given sample information, the rejection rule does not hold, since

$$1.69 = \chi^2_{7,\,0.975} < \chi_0^2\ (=3.375) < \chi^2_{7,0.025} = 16.01.$$

Hence, H_0 is not rejected in favour of H_1. Thus, Population variance can be regarded as 48 kg.

Example

A normal population has mean μ (unknown) and variance 9. A sample of size 9 observations has been taken and its variance is found to be 5.4. Test the null hypothesis H_0: $\sigma^2 = 9$ against H_1: $\sigma^2 > 9$ at 5% level of significance.

Solution:

Step 1 : **Null Hypothesis** H_0: $\sigma^2 = 9$.

i.e., Population variance regarded as 9.

Alternative hypothesis H_1: $\sigma^2 > 9$.

i.e. Population variance is regarded as greater than 9.

Step 2 : **Data**

Sample size $(n) = 9$

Sample variance $(s^2) = 5.4$

Step 3 : **Level of significance**

$\alpha = 5\%$

Step 4 : Test statistic

Under null hypothesis H_0

$$\chi^2 = \frac{(n-1)S^2}{\sigma_0^2}$$

follows chi-square distribution with (n-1) degrees of freedom.

Step 5 : Calculation of test statistic

The value of chi-square under H_0 is calculated as

$$\chi_0^2 = \frac{(n-1)s^2}{\sigma_0^2} = \frac{8 \times 5.4}{9} = 4.8$$

Step 6 : Critical value

Since H_1 is a one-sided alternative, the critical values at α =0.05 is $\chi_e^2 = \chi^2_{8,\,0.05} = 15.507$.

Step 7 : Decision

Since it is a one-tailed test, the elements of the critical region are determined by the rejection rule $\chi_0^2 > \chi_e^2$.

For the given sample information, the rejection rule does not hold , since $\chi_0^2 = 4.8 < \chi^2_{8,\,0.05} = 15.507$. Hence, H_0 is not rejected in favour of H_1. Thus, the population variance can be regarded as 9.

Example

A normal population has mean μ (unknown) and variance 0.018. A random sample of size 20 observations has been taken and its variance is found to be 0.024. Test the null hypothesis $H_0: \sigma^2 = 0.018$ against $H_1: \sigma^2 < 0.018$ at 5% level of significance.

Solution:

Step 1 : Null Hypothesis $H_0: \sigma^2 = 0.018$.

i.e. Population variance regarded as 0.018.

Alternative hypothesis $H_1: \sigma^2 < 0.018$.

i.e. Population variance is regarded as lessthan 0.018.

Step 2 : Data

Sample size $(n) = 20$

Sample variance $(s^2) = 0.024$

Step 3 : Level of significance

$\alpha = 5\%$

Step 4 : Test statistic

Under null hypothesis H_0

$$\chi^2 = \frac{(n-1)S^2}{\sigma_0^2}$$

follows chi-square distribution with $(n-1)$ degrees of freedom.

Step 5 : Calculation of test statistic

The value of chi-square under H_0 is calculated as

$$\chi_0^2 = \frac{(n-1)s^2}{\sigma_0^2} = \frac{19 \times 0.024}{0.018} = 25.3$$

Step 6 : Critical value

Since H_1 is a one-sided alternative, the critical values at α =0.05 is $\chi_e^2 = \chi^2_{19,\,0.95} = 10.117$.

Step 7 : Decision

Since it is a one-tailed test, the elements of the critical region are determined by the rejection rule $\chi_0^2 < \chi_e^2$

For the given sample information, the rejection rule does not hold, since

$\chi_0^2 = 25.3 > \chi_e^2 = \chi^2_{19,\,0.95} = 10.117.$

Hence, H_0 is not rejected in favour of H_1. Thus, the population variance can be regarded as 0.018.