Probability

Introduction

Dr R Sinuvasan

Random experiment, Sample space, Sample point

Experiment: In Statistics, by the word experiment it means 'an attempt to produce a result'. It need not be a laboratory experiment.

Random Experiment: If an experiment is such that

(i) all the possible outcomes of the experiment are predictable, in advance

(ii) outcome of any trial of the experiment is not known, in advance, and

(iii) it can be repeated any number of times under identical conditions, is called a random experiment.

Sample space: The set of all possible outcomes of a random experiment is called the sample space of the experiment and is usually denoted by S (or Ω).

Sample Point: The outcome of a random experiment is called a sample point, which is an element in S.

Example

Consider the random experiment of tossing a coin once "Head" and "Tail" are the two possible outcomes. The sample space is $S = \{H, T\}$. It is a finite sample space

Example

Suppose that a study is conducted on all families with one or two children. The possible outcomes, in the order of births, are: boy only, girl only, boy and girl, girl and boy, both are boys and both are girls. Then, the sample space is S = {b,g,bg,gb,bb,gg}. It is also a finite sample space. Here, ‘b’ represents the child is a boy and ‘g’ represents the child is a girl.

Example

Consider the experiment of tossing a coin until head appears. Then, the sample space of this experiment is S = { (H), (T,H), (T,T,H), (T,T,T,H), ... }. This is a countable sample space. If head appears in the first trial itself, then the element of S is (H); if head appears in the second attempt then the element of S is (T,H); if head appears in the third attempt then the element of S is (T,T,H) and so on.

Event: A subset of the sample space is called an event.

the event that the eldest child in the families is a girl is represented as $A = \{g, gb, gg\}$

The event that the families have one boy is represented as $B = \{b, bg, gb\}$.

Mutually exclusive events: Two or more events are said to be mutually exclusive, when the occurrence of any one event excludes the occurrence of other event. Mutually exclusive events cannot occur simultaneously.

In particular, events A and B are said to be mutually exclusive if they are disjoint, that is, $A \cap B = \phi$.

Consider the case of rolling a die. Let A = {1, 2, 3} and B = {4, 5, 6} be two events. Then we find $A \cap B = \phi$. Hence A and B are said to be mutually exclusive events

Definitions of Probability

Probability is a measure of uncertainty.

Mathematical Probability:

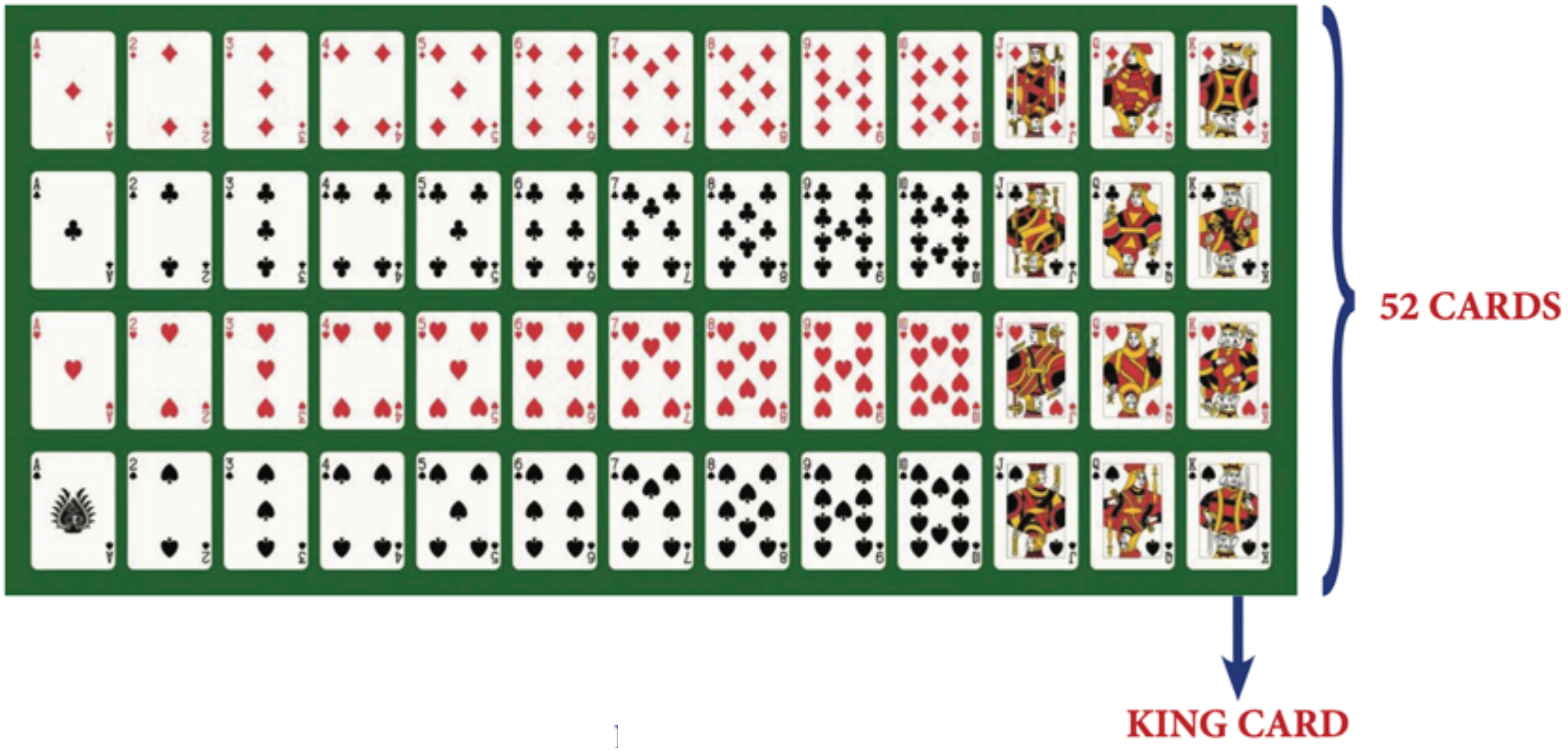
If the sample space S, of an experiment is finite with all its elements being equally likely, then the probability for the occurrence of any event, A, of the experiment is defined as

$$P(A) = \frac{No.of\ elements\ favourable\ to\ A}{No.\ of\ elements\ in\ S}$$

$$P(A) = \frac{n(A)}{n(S)}\ .$$

Example

What is the chance of getting a king in a draw from a pack of 52 cards?



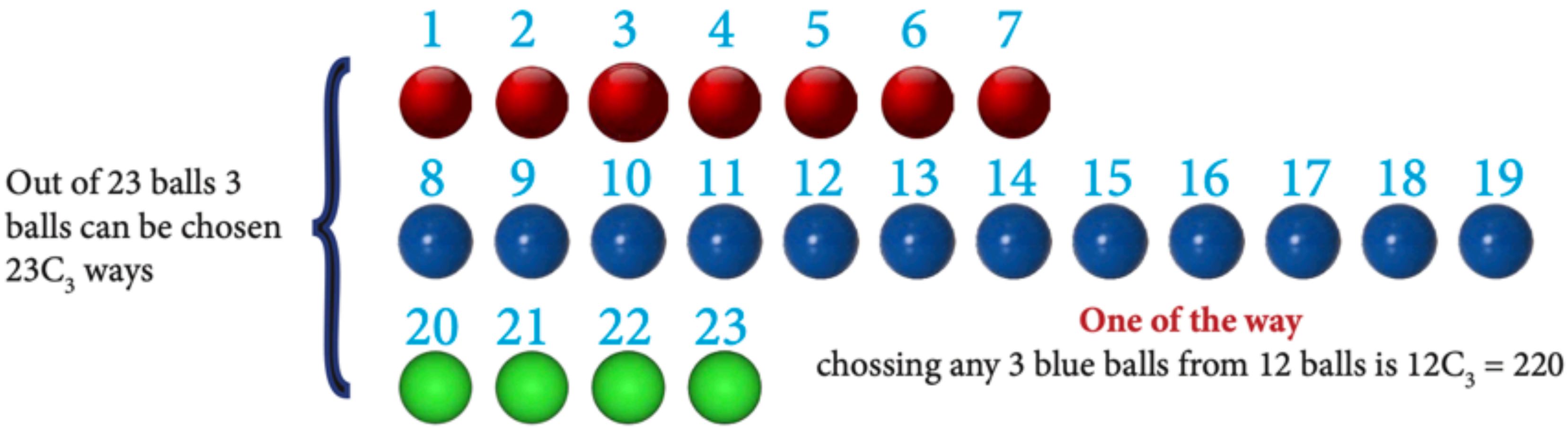
Let A be the event of choosing a card which is a king

In which, number of king cards $n(A) = 4$

Therefore probability of drawing a card which is king is = $P(A) = \frac{n(A)}{n(S)} = \frac{4}{52}$

Example

A bag contains 7 red, 12 blue and 4 green balls. What is the probability that 3 balls drawn are all blue?



Total number of balls = 7+12+14=23 balls

Out of 23 balls 3 balls can be selected in = $n(s)= 23C_3$ ways

Let A be the event of choosing 3 balls which is blue

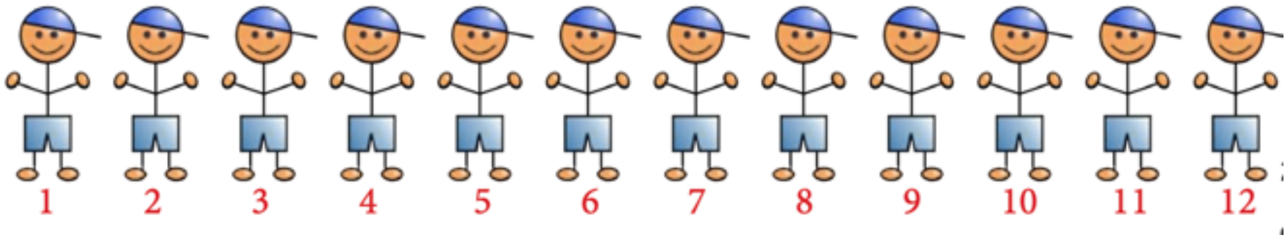
Number of possible ways of drawing 3 out of 12 blue balls is = $n(A)=12C_3$ ways

Therefore, $P(A) = \dfrac{n(A)}{n(S)} = \dfrac{12C_3}{23C_3} = \dfrac{220}{1771}$

$= 0.1242$

Example

A class has 12 boys and 4 girls. Suppose 3 students are selected at random from the class. Find the probability that all are boys.



Total number of students = 12+4=16

Three students can be selected out of 16 students in $16C_3$ ways

i.e. $n(s) = 16C_3 = \dfrac{16 \times 15 \times 14}{1 \times 2 \times 3} = 560$

Three boys can be selected from 12 boys in $12C_3$ ways

i.e. $n(A) = 12C3 = \dfrac{12 \times 11 \times 10}{1 \times 2 \times 3} = 220$

The required probability $P(A) = \dfrac{n(A)}{n(S)} = \dfrac{220}{560} = \dfrac{11}{28}$

$= 0.392$

Axiomatic approach to probability:

Let S be the sample space of a random experiment. If a number $P(A)$ assigned to each event $A \in S$ satisfies the following axioms, then $P(A)$ is called the probability of A.

Axiom-1 : $P(A) \geq 0$
Axiom-2 : $P(S) = 1$
Axiom-3 : If $\{A1, A2, \ldots\}$ is a sequence of mutually exclusive events i.e., $A_i \cap A_j = \phi$ when $i \neq j$, then

$$P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$$

Theorem: The probability of impossible event is 0 i.e., $P(\phi) = 0$.

Theorem: If S is the sample space and A is any event of the experiment, then $P(\bar{A}) = 1 - P(A)$.

Theorem: If A and B are two events in an experiment such that $A \subset B$, then $P(B\text{-}A) = P(B) - P(A)$.

Example

In the experiment of tossing an unbiased coin (or synonymously balanced or fair coin), the sample space is $S = \{H, T\}$. What is the probability of getting head or tail?

Since $n(A_1) = 1 = n(A_2)$ and $n(S) = 2$,

$$P(A_1) = \frac{n(A_1)}{n(S)} = \frac{1}{2}.$$

Similarly, $P(A_2) = \frac{1}{2}$.

If a biased coin is tossed and the outcome of head is thrice as likely as tail,

If a biased coin is tossed and the outcome of head is thrice as likely as tail, then $P(A_1) = 3P(A_2)$.

Substituting this in $P(S) = P(A_1)+P(A_2)$, it follows that

$$1 = P(A_1)+P(A_2) = 4P(A_2)$$

It gives that $P(A_2) = \frac{1}{4}$ and hence $P(A_1) = \frac{3}{4}$.

It should be noted that the probabilities for getting head and tail differ for a biased coin.

Example

A box contains 3 red and 4 blue socks. Find the probability of choosing two socks of same colour.

A_1 = Selection of black socks,

$$n(A_1) = 3C_2 \qquad = 3$$

$$P(A_1) = \frac{n(A_1)}{n(S)} \qquad = \frac{3}{21}$$

A_2 = Selection of blue socks,

$$n(A_2) = 4C_2 \qquad = 6$$

$$P(A_2) = \frac{n(A2)}{n(S)} \qquad = \frac{6}{21}$$

then $A_1 \cup A_2$ represents the event of selecting 2 socks of same colour. Since the occurrence of one event excludes the occurrence of the other, these two events are mutually exclusive. Then, by Axiom-3,

$$P(A_1 \cup A_2) = P(A_1) + P(A_2)$$

Therefore, $P(A_1 \cup A_2) = \frac{3}{21} + \frac{6}{21} = \frac{9}{21} = \frac{3}{7}$

Thus, the probability of selecting two socks of same colour is 3/7.

Example

Angel selects three cards at random from a pack of 52 cards. Find the probability of drawing: (i) 3 spade cards. (ii) one spade and two diamonds cards (iii) one spade, one diamonds and one heart cards.

Total no. of ways of drawing 3 cards = n(S) = $52\ C_3 = 22100$

(a) Let A_1 = drawing 3 spade cards.

Since there are 13 Spades cards in a pack of cards,

No. of ways of drawing 3 spade cards = $n(A_1) = 13\ C_3 = 286$

Therefore, $P(A_1) = \frac{n(A_1)}{n(S)} = \frac{286}{22100}$

(b) Let A_2 = drawing one spade and two knave cards

No. of ways of drawing one spade card = $13C_1 = 13$

No. of ways of drawing two knave cards = $13C_2 = 78$

Since drawing a spade and 2 knaves should occur together,

No. of ways drawing one spade and two knave cards = $n(A_2) = 13 \times 78 = 1014$

$$\text{Therefore, } P(A_2) = \frac{n(A_2)}{n(S)} = \frac{13 \times 78}{22100}$$

$$\text{Hence, } P(A_2) = \frac{1014}{22100} = \frac{507}{11050}$$

(c) Let A_3 = drawing one spade, one knave and one heart cards

No. of ways of drawing one spade, one knave and one heart cards is

$$n(A_3) = 13C_1 \times 13C_1 \times 13C_1 = 13 \times 13 \times 13$$

$$\text{Therefore, } P(A_3) = \frac{n(A_3)}{n(S)} = \frac{13 \times 13 \times 13}{22100}$$

$$\text{Hence, } P(A_3) = \frac{2197}{22100}.$$

Theorem (Addition Theorem of Probability for Two Events)

If A and B are any two events in a random experiment, then

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Example

In the Annual sports meet, among the 260 students, 90 participated in Badminton, 120 participated in Hockey, and 50 participated in Badminton and Hockey. A Student is selected at random. Find the probability that the student participated in (i) Either Badminton or Hockey, (ii) Neither of the two tournaments, (iii) Hockey only, (iv) Badminton only, (v) Exactly one of the tournaments.

$n(s)=260$

Let A : the event that the student participated in Kabadi

B : the event that the student participated in Hockey.

$n(A) = 90;\quad n(B) = 120;\quad n(A \cap B) = 50$

$$P(A) = \frac{n(A)}{n(S)} = \frac{90}{260}$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{120}{260}$$

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{50}{260}$$

(i) The probability that the student participated in either Kabadi or Hockey is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\frac{90}{260} + \frac{120}{260} - \frac{50}{260} = \frac{160}{260} = \frac{8}{13}$$

(ii) The probability that the student participated in neither of the two tournaments in

$$P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B}) \text{ (By De Morgan's law } \overline{A \cup B} = \overline{A} \cap \overline{B}\text{)}$$

$$= 1 - P(A \cup B)$$

$$= 1 - \frac{8}{13} = \frac{5}{13}$$

(iii) The probability that the student participated in Hockey only is

$$P(\overline{A} \cap B) = P(B) - P(A \cap B)$$

$$= \frac{120}{260} - \frac{50}{260} = \frac{70}{260} = \frac{7}{26}$$

(iv) The probability that the student participated in Kabadi only

$$P(A \cap \overline{B}) = P(A) - P(A \cap B)$$

$$= \frac{90}{260} - \frac{50}{260} = \frac{40}{260} = \frac{2}{13}$$

(v) The probability that the student participated in exactly one of the tournaments is

$P[(A \cap \overline{B}) \cup (\overline{A} \cap B)] = P(A \cap \overline{B}) + P(\overline{A} \cap B)$ [$\because A \cap \overline{B}, \overline{A} \cap B$ are mutually exclusive events]

$$= \frac{70}{260} + \frac{40}{260} = \frac{110}{260} = \frac{11}{26}$$