Theorem (Law of Total Probability)

If $B_1, B_2, \dots, B_n$ are mutually exclusive events such that $\bigcup_{j=1}^{n} B_j = S$ and $P(B_j) > 0$ for $j = 1, 2, \dots, n$, Then for any event A

$$P(A) = P(A/B_1)P(B_1) + P(A/B_2)P(B_2) + \dots + P(A/B_n)P(B_n).$$

Theorem (Bayes' Theorem)

Let $B_1, \dots, B_n$ be n mutually exclusive events such that where S is the sample space of the random experiment. If $P(B_j) > 0$ for $j = 1, 2, \dots, n$, then for any event A of the same experiment with $P(A) > 0$,

$$P(B_j/A) = \frac{P(A/B_j)P(B_j)}{P(A/B_1)P(B_1) + P(A/B_2)P(B_2) + \dots + P(A/B_n)P(B_n)}, \quad j = 1, 2, \dots, n.$$

In a certain assembly plant, three machines, B1,B2 and B3, made 30%, 45% and 25%, respectively, of the products. It is known from past experience that 2%, 3% and 2% of the products made by each machine, respectively, are defective. Now, suppose that a finished product is randomly selected. What is the probability that it if defective?

***Solution*:** Consider the following events:

A: the product is defective,

B_1: the product is made by machine B_1,

B_2: the product is made by machine B_2,

B_3: the product is made by machine B_3.

Applying the rule of elimination, we can write

$$P(A) = P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + P(B_3)P(A|B_3).$$

Referring to the tree diagram of Figure 2.15, we find that the three branches give the probabilities

$$P(B_1)P(A|B_1) = (0.3)(0.02) = 0.006,$$
$$P(B_2)P(A|B_2) = (0.45)(0.03) = 0.0135,$$
$$P(B_3)P(A|B_3) = (0.25)(0.02) = 0.005,$$

and hence

$$P(A) = 0.006 + 0.0135 + 0.005 = 0.0245.$$

At a certain university, 4% of men are over 6 feet tall and 1% of women are over 6 feet tall. The total student population is divided in the ratio 3:2 in favour of women. If a student is selected at random from among all those over six feet tall, what is the probability that the student is a woman?

Let M={Student is Male}, F={Student is Female}.
Note that M and F partition the sample space of students.
Let T={Student is over 6 feet tall}.

We know that $\mathsf{P}(M) = 2/5$, $\mathsf{P}(F) = 3/5$, $\mathsf{P}(T|M) = 4/100$ and $\mathsf{P}(T|F) = 1/100$.

We require $\mathsf{P}(F|T)$. Using Bayes' theorem we have:

$$\begin{aligned}
\mathsf{P}(F|T) &= \frac{\mathsf{P}(T|F)\mathsf{P}(F)}{\mathsf{P}(T|F)\mathsf{P}(F) + \mathsf{P}(T|M)\mathsf{P}(M)} \\
&= \frac{\frac{1}{100} \times \frac{3}{5}}{\frac{1}{100} \times \frac{3}{5} + \frac{4}{100} \times \frac{2}{5}} \\
&= \frac{3}{11}
\end{aligned}$$

A factory production line is manufacturing bolts using three machines, A, B and C. Of the total output, machine A is responsible for 25%, machine B for 35% and machine C for the rest. It is known from previous experience with the machines that 5% of the output from machine A is defective, 4% from machine B and 2% from machine C. A bolt is chosen at random from the production line and found to be defective. What is the probability that it came from

(a) machine A (b) machine B (c) machine C?

Let

D={bolt is defective},

A={bolt is from machine A},

B={bolt is from machine B},

C={bolt is from machine C}.

We know that $\mathsf{P}(A) = 0.25$, $\mathsf{P}(B) = 0.35$ and $\mathsf{P}(C) = 0.4$.

Also

$$\mathsf{P}(D|A) = 0.05, \mathsf{P}(D|B) = 0.04, \mathsf{P}(D|C) = 0.02.$$

A statement of Bayes' theorem for three events A, B and C is

$$\begin{aligned}
\mathsf{P}(A|D) &= \frac{\mathsf{P}(D|A)\mathsf{P}(A)}{\mathsf{P}(D|A)\mathsf{P}(A) + \mathsf{P}(D|B)\mathsf{P}(B) + \mathsf{P}(D|C)\mathsf{P}(C)} \\
&= \frac{0.05 \times 0.25}{0.05 \times 0.25 + 0.04 \times 0.35 + 0.02 \times 0.4} \\
&= 0.362
\end{aligned}$$

Similarly

$$\begin{aligned}
\mathsf{P}(B|D) &= \frac{0.04 \times 0.35}{0.05 \times 0.25 + 0.04 \times 0.35 + 0.02 \times 0.4} \\
&= 0.406 \\
\mathsf{P}(C|D) &= \frac{0.02 \times 0.4}{0.05 \times 0.25 + 0.04 \times 0.35 + 0.02 \times 0.4} \\
&= 0.232
\end{aligned}$$