Conditional Probability

Consider the following situations:

(i) two events occur successively or one after the other (e.g) A occurs after B has occurred and
(ii) both event A and event B occur together.

Definition of Conditional of Probability

If $P(B) > 0$, the conditional probability of A given B is defined as

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$

There are 4000 people living in a village including 1500 female. Among the people in the village, the age of 1000 people is above 25 years which includes 400 female. Suppose a person is chosen and you are told that the chosen person is a female. What is the probability that her age is above 25 years?

Here, the event of interest is selecting a female with age above 25 years. In connection with the occurrence of this event, the following two events must happen.

A: a person selected is female

B: a person chosen is above 25 years.

$$P(A) = P(\text{Selecting a female}) = \frac{1500}{4000}$$

$$P(A \cap B) = P(\text{Selecting a female with age above 25 years}) = \frac{400}{1500}$$

Hence, $$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{400}{1500} \times \frac{4000}{1500} = \frac{160}{225} = \frac{32}{45}.$$

A pair of dice is rolled and the faces are noted. Let

A: sum of the faces is odd,

B: sum of the faces exceeds 8, and

C: the faces are different then

find (i) P (A/C) (ii) P (B/C)

The outcomes favourable to the occurrence of these events are

$$A = \{ (1,2), (1,4), (1,6), (2,1), (2,3), (2,5), (3,2), (3,4), (3,6)$$
$$(4,1), (4,3), (4,5), (5,2), (5,4), (5,6), (6,1), (6,3), (6,5) \}$$

$$B = \{ (3,6), (4,5), (4,6), (5,4), (5,5), (5,6), (6,3), (6,4), (6,5), (6,6) \}$$

$$C = \{ (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,3), (2,4), (2,5), (2,6),$$
$$(3,1), (3,2), (3,4), (3,5), (3,6), (4,1), (4,2), (4,3), (4,5), (4,6),$$
$$(5,1), (5,2), (5,3), (5,4), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5) \}$$

Since A and B are proper subsets of C, $A \cap C = A$ and $B \cap C = B$.

Here, $$P(A) = \frac{18}{36} = \frac{1}{2}$$

$$P(B) = \frac{10}{36} = \frac{5}{9}$$

$$P(C) = \frac{30}{36} = \frac{5}{6}.$$

Since, $A \cap C = A$. Hence,

$$P(A \cap C) = P(A) = \frac{1}{2}$$

$$P(B \cap C) = P(B) = \frac{5}{9}.$$

Hence, the probability for the sum of the faces is an odd number given that the faces are different is

$$P(A/C) = \frac{P(A \cap C)}{P(C)} = \frac{\frac{1}{2}}{\frac{5}{6}} = \frac{3}{5}$$

Similarly, the probability for the sum of the faces exceeds 8 given that the faces are different is

$$P(B/C) = \frac{P(B \cap C)}{P(C)} = \frac{\frac{2}{9}}{\frac{5}{6}} = \frac{4}{15}$$

Definition :

Two events A and B are said to be independent of one another, if

$P(A \cap B) = P(A) \times P(B)$.

Theorem: (Multiplication Theorem of Probability)

If A and B are any two events of an experiment, then

$$P(A \cap B) = \begin{cases} P(A)P(B \mid A), & \text{if } P(A) > 0 \\ P(B)P(A \mid B), & \text{if } P(B) > 0 \end{cases}$$

In the experiment of rolling a pair of dice, the events A, B and C are defined as A: getting 2 on the first die, B: getting 2 on the second die, and C: sum of the faces of dice is an even number. Prove that the events are pair wise independent but not mutually independent?

A box contains 7 red and 3 white marbles. Three marbles are drawn from the box one after the other without replacement. Find the probability of drawing three marbles in the alternate colours with the first marble being red.

The event of interest is drawing the marbles in alternate colours with the first is red. This event can occur only when the marbles are drawn in the order (Red , White , Red)

If A and C represent the events of drawing red marbles respectively in the first and the third draws and B is the event of drawing white marble in the second draw, then the required event is $A \cap B \cap C$. The probability for the occurrence of $A \cap B \cap C$ can be calculated applying

$$P(A \cap B \cap C) = P(A)P(B/A)P(C/A \cap B)$$

Since there are 7 red and 3 white marbles in the box for the first draw,

$$P(A) = \frac{7}{10}$$

Now, there will be 6 red and 3 white marbles in the box for the second draw if the event A has occurred. Hence,

$$P(B/A) = \frac{3}{9}$$

Similarly, there will be 6 red and 2 white marbles in the box for the third draw if the events A and B have occurred. Hence,

$$P(C/A \cap B) = \frac{6}{8}.$$

$\therefore \quad P(A \cap B \cap C) = \frac{7}{10} \times \frac{3}{9} \times \frac{6}{8} = \frac{7}{40}$ is the required probability of drawing three marbles in the alternate colours with the first marble being red.

There are 13 boys and 6 girls in a class. Four students are selected randomly one after another from that class. Find the probability that: (i) all are girls, (ii) first two are boys and next are girls

B: all the randomly selected students are girls

There will be 6 girls among 19 students, in total, while selecting the first student; there will be 5 girls among 18 students, in total, while selecting the second student; 4 girls among 17 students, in total, while selecting the third student; and 3 girls among the remaining 16 students, in total, while selecting the fourth student.

$$P(B) = \frac{6}{19} \times \frac{5}{18} \times \frac{4}{17} \times \frac{3}{16}$$

$$P(B) = \frac{5}{1292}.$$

(ii) Suppose that

C: In the randomly selected students the first two are boys and the next are girls

There will be 13 boys among the 19 students, in total, while selecting the first student; there will be 12 boys among 18 students, in total, while selecting the second student; 6 girls among 17 students, in total, while selecting the third student; and 5 girls among the remaining 16 students, in total, while selecting the fourth student.

Then, by applying the Theorem 8.5 for simultaneous occurrence of these four events, it follows that

$$P(C) = \frac{13}{19} \times \frac{12}{18} \times \frac{6}{17} \times \frac{5}{16} = \frac{65}{1292}$$

A box contains six 10 Ω resistors and ten 30 Ω resistors. The resistors are all unmarked and are of the same physical size.

(a) One resistor is picked at random from the box; find the probability that:

(i) It is a 10 Ω resistor.

(ii) It is a 30 Ω resistor.

(b) At the start, two resistors are selected from the box. Find the probability that:

(i) Both are 10 Ω resistors.
(ii) The first is a 10 Ω resistor and the second is a 30 Ω resistor.

(iii) Both are 30 Ω resistors.

(a) (i) As there are six 10 Ω resistors in the box that contains a total of $6 + 10 = 16$ resistors, and there is an *equally likely chance* of any resistor being selected, then

$$\text{P}(10\ \Omega) = \frac{6}{16} = \frac{3}{8}$$

(ii) As there are ten 30 Ω resistors in the box that contains a total of $6 + 10 = 16$ resistors, and there is an *equally likely chance* of any resistor being selected, then

$$\text{P}(30\ \Omega) = \frac{10}{16} = \frac{5}{8}$$

(b) (i) As there are six 10 Ω resistors in the box that contains a total of $6 + 10 = 16$ resistors, and there is an *equally likely chance* of any resistor being selected, then

$$\text{P(first selected is a 10 } \Omega \text{ resistor)} = \frac{6}{16} = \frac{3}{8}$$

If the first resistor selected was a 10 Ω one, then when the second resistor is selected, there are only five 10 Ω resistors left in the box which now contains $5 + 10 = 15$ resistors.

Hence, $\quad \text{P(second selected is also a 10 } \Omega \text{ resistor)} = \dfrac{5}{15} = \dfrac{1}{3}$

And, $\quad \text{P(both are 10 } \Omega \text{ resistors)} = \dfrac{3}{8} \times \dfrac{1}{3} = \dfrac{1}{8}$

(b) (ii) As before, P(first selected is a 10 Ω resistor) $= \dfrac{6}{18} = \dfrac{3}{8}$

If the first resistor selected was a 10 Ω one, then when the second resistor is selected, there are still ten 30 Ω resistors left in the box which now contains $5 + 10 = 15$ resistors. Hence, P(second selected is a 30 Ω resistor) $= \dfrac{10}{15} = \dfrac{2}{3}$

And, P(first was a 10 Ω resistor and second was a 30 Ω resistor) $= \dfrac{3}{8} \times \dfrac{2}{3} = \dfrac{1}{4}$

(b) (iii) As there are ten 30 Ω resistors in the box that contains a total of $6 + 10 = 16$ resistors, and there is an *equally likely chance* of any resistor being selected, then

P(first selected is a 30 Ω resistor) $= \dfrac{10}{16} \times \dfrac{5}{8}$

If the first resistor selected was a 30 Ω one, then when the second resistor is selected, there are only nine 30 Ω resistors left in the box which now contains $5 + 10 = 15$ resistors.

Hence, P(second selected is also a 30 Ω resistor) $= \dfrac{9}{15} = \dfrac{3}{5}$

And, P(both are 30 Ω resistors) $= \dfrac{5}{8} \times \dfrac{3}{5} = \dfrac{3}{8}$

A box contains 4 bad tubes and 6 good tubes. Two are drawn out together. One of them is tested and found to be good. What is the probability that the other one is also good?

Let $G_i = \{i^{th} \text{ tube is good}\}$ $\quad B_i = \{i^{th} \text{ tube is bad}\}$

$\mathsf{P}(G_2|G_1) = \frac{5}{9}$ (only 5 good tubes left out of 9).

A man owns a house in town and a cottage in the country. In any one year the probability of the town house being burgled is 0.01 and the probability of the country cottage being burgled is 0.05. In any one year what is the probability that: (a) both will be burgled? (b) one or the other (but not both) will be burgled ?

(a) $H = \{\text{house is burgled}\}$ $\;C = \{\text{cottage is burgled}\}$

(b) $\mathsf{P}(H \cap C) = \mathsf{P}(H)\mathsf{P}(C) = (0.01)(0.05) = 0.0005$ since events independent

$$\begin{aligned}\mathsf{P}(\text{one or the other (but not both)}) &= \mathsf{P}((H \cap C') \cup (H' \cap C)) = \mathsf{P}(H \cap C') + \mathsf{P}(H' \cap C) \\ &= \mathsf{P}(H)\mathsf{P}(C') + \mathsf{P}(H')\mathsf{P}(C) \\ &= (0.01)(0.95) + (0.99)(0.05) = 0.059.\end{aligned}$$