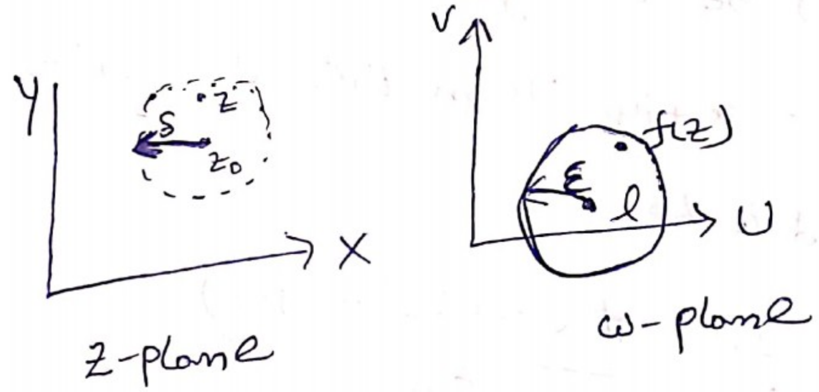
UNIT - IV

Limit of a complex function :-

A function $w = f(z)$ is said to tend to limit l as z approaches a point z_0, if for every real ε, we can find a +ve real δ such that

$$|f(z) - l| < \varepsilon \quad \text{for} \quad |z - z_0| < \delta$$

i.e for every $z \neq z_0$ in the δ-disc (dotted) of z-plane, $f(z)$ has a value lying in the ε-disc of w-plane. we denote $\lim_{z \to z_0} f(z) = l$.



Continuity of f(z) :-

A function $w = f(z)$ is said to be continu at $z = z_0$, if $\lim_{z \to z_0} f(z) = f(z_0)$.

Further $f(z)$ is said to be continuous in any regio. R of the z-plane, if it is continuous at every point of that region.

Also $f(z)$ is said to be continuous in any region R of the z-plane, if it

Also if $w = f(z) = u(x,y) + iv(x,y)$ is continuous at $z = z_0$ then $u(x,y)$ and $v(x,y)$ are also continuous at $z = z_0$. i.e at $x = x_0$ and $y = y_0$. Conversely if $u(x,y)$ and $v(x,y)$ are continuous at (x_0, y_0), then $f(z)$ will be continuous at $z = z_0$.

Derivative of f(z) :-

Let $w = f(z)$ be a single-valued function of the variable $z = x+iy$. Then the derivative of $w = f(z)$ is defined to be

$$\frac{dw}{dz} = f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}.$$

* The necessary and sufficient conditions for the derivative of the function $w = u(x,y) + iv(x,y) = f(z)$ to exist for all values of z in a region R, are

(i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous functions of x and y in R;

(ii) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

The relations (ii) are known as Cauchy - Riemann equations or briefly C-R equations.

Analytic functions :-

A fun $f(z)$ which is single-valued and possesses a unique derivative with r.t z at all points of a region R, is called an analytic function of z in that region.

* An analytic function is also called a regular function or an ~~hom~~ holomorphic function.

* A function which is analytic everywhere in the

plane, is known as an entire function. As derivati-ve of a polynomial exists at every point, a polyno-mial of any degree is an entire function.

~~ceases a derivative~~

* A point at which an analytic function ceases to possess a derivative is called a "singular point" of the function.

* The real and imaginary parts of an analytic function are called conjugate functions.

* The relation b/w two conjugate functions are C-R equations.

Cauchy - Riemann equations in polar coordinates

Let $x = r\cos\theta$
$y = r\sin\theta$

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \quad \text{and} \quad \frac{\partial v}{\partial \theta} = r.\frac{\partial u}{\partial r}$$

(or) $$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

which is the polar form of C-R equations

Prob ① Prove that the function $\sinh z$ is analytic and and find its derivative.

Sol:- Here $f(z) = u+iv = \sinh z$

$= \sinh(x+iy)$

$= \sinh x \cos y + i \cosh x \sin y$

$\therefore u = \sinh x \cos y$ and

$v = \cosh x \sin y$

$\frac{\partial u}{\partial x} = \cosh x \cos y$, $\frac{\partial u}{\partial y} = -\sinh x \sin y$

$\frac{\partial v}{\partial x} = \sinh x \sin y$, $\frac{\partial v}{\partial y} = \cosh x \cos y$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Thus C-R equations are satisfied.

Since $\sinh x$, $\cosh x$, $\sin y$ and $\cos y$ are continuous functions. $\frac{\partial u}{\partial x}$. $\frac{\partial u}{\partial y}$. $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are also continuous functions satisfying C-R eqn's.

Hence $f(z)$ is analytic everywhere

Now $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$= \cosh x \cos y + i \sinh x \sin y$

$= \cosh(x+iy) = \cosh z$.

(2) Determine the analytic function $w = u+iv$, if $v = \log(x^2+y^2) + x - 2y$.

Sol Here $v = \log(x^2+y^2) + x - 2y$

$$\therefore \quad \frac{\partial v}{\partial x} = \frac{2x}{x^2+y^2} + 1$$

$$\frac{\partial v}{\partial y} = \frac{2y}{x^2+y^2} - 2$$

Since $w = u + iv$

$$\frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

$$= \left(\frac{2y}{x^2+y^2} - 2\right) + i\left(\frac{2x}{x^2+y^2} + 1\right)$$

Replacing x by z and y by 0, we get

$$\frac{dw}{dz} = -2 + i\left(\frac{2}{z} + 1\right) = (i-2) + \frac{2i}{z}$$

Integrating w.r.t. z, we have

$$w = (i-2)z + 2i\log z + C$$

(3) S.T. the function $f(z) = \sqrt{|xy|}$ is not analytic at the origin even though C.R. equations are satisfied thereof.

Sol If $f(z) = \sqrt{|xy|} = u(x,y) + iv(x,y)$

then $u(x,y) = \sqrt{|xy|}$.

$v(x,y) = 0$.

At the origin, we have

$$\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x\to 0} \frac{0-0}{x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0} \frac{u(0,y) - u(0,0)}{y} = \lim_{x\to 0} \frac{0-0}{y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{x\to 0} \frac{v(x,0) - v(0,0)}{x} = \lim_{x\to 0} \frac{0-0}{x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{y\to 0} \frac{v(0,y) - v(0,0)}{y} = \lim_{y\to 0} \frac{0-0}{y} = 0$$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{+\partial x}$

Hence C-R equations are satisfied at the origin.

However $f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z-0}$

$$= \underset{z \to 0}{Lt} \frac{\sqrt{|xy|} - 0}{x + iy}$$

$$= \underset{x \to 0}{Lt} \frac{\sqrt{|mx^2|} - 0}{x(1 + im)} \quad [\because z \to 0 \text{ along the line } y = mx]$$

$$= \underset{x \to 0}{Lt} \frac{\sqrt{|m|}}{1 + im}$$

Now this limit is not unique since it depends on m.

$\therefore f'(0)$ does not exist.

Hence the function $f(z)$ is not analytic (regular) at the origin.

(4) P.T. the function $f(z)$ defined by

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2} \quad (z \neq 0), \ f(0) = 0$$

is continuous and the C-R eqn's are satisfied at the origin, yet $f'(0)$ does not exist.

Sol Here $f(z) = \dfrac{(x^3-y^3)+i(x^3+y^3)}{x^2+y^2}$. $z \neq 0$.

Let $f(z) = u+iv$

$$= \frac{x^3-y^3}{x^2+y^2} + i\,\frac{x^3+y^3}{x^2+y^2}$$

then $u = \dfrac{x^3-y^3}{x^2+y^2}$, $v = \dfrac{x^3+y^3}{x^2+y^2}$

Since $z \neq 0 \Rightarrow x \neq 0,\ y \neq 0$

$\therefore$ u and v are rational functions of x and y with non-zero denominators. Thus u, v and hence $f(z)$ are continuous functions when $x \neq 0$.

To test them for continuity at $z=0$. on changing u, v to polar coordinates by putting

$x = r\cos\theta$, $y = r\sin\theta$, we get

$u = r(\cos^3\theta - \sin^3\theta)$ $\quad$ $v = r(\cos^3\theta + \sin^3\theta)$

when $z \to 0$, $r \to 0$

$\therefore \lim\limits_{z\to 0} u = \lim\limits_{r\to 0} r(\cos^3\theta - \sin^3\theta) = 0$

$||^{ly}$ $\lim\limits_{z\to 0} v = 0$.

$\therefore \lim\limits_{r\to 0} f(z) = 0 = f(0)$

$\Rightarrow$ $f(z)$ is continuous at $z=0$.

Hence $f(z)$ is continuous for all values of z

At the origin $(0,0)$, we have

$$\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x} = \lim_{x\to 0} \frac{x-0}{x} = 1$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0} \frac{u(0,y)-u(0,0)}{y} = \lim_{y\to 0} \frac{-y-0}{y} = -1$$

$$\frac{\partial v}{\partial x} = \lim_{x\to 0} \frac{v(x,0)-v(0,0)}{x} = \lim_{x\to 0} \frac{x-0}{x} = 1$$

$$\frac{\partial v}{\partial y} = \lim_{y\to 0} \frac{v(0,y)-v(0,0)}{y} = \lim_{y\to 0} \frac{y-0}{y} = 1$$

$\therefore$ $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Hence C-R equ's are satisfied at the origin.

Now, $$f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z}$$

$$= \lim_{z\to 0} \frac{(x^3-y^3)+i(x^3+y^3)-0}{(x^2+y^2)(x+iy)}$$

Let $z \to 0$ along the line $y = x$, then

$$f'(0) = \lim_{x \to 0} \frac{0 + 2ix^3}{2x^3(1+i)} = \frac{i}{1+i} = \frac{i(1-i)}{2}$$

$$= \frac{1+i}{2} \rightarrow (1)$$

Also, Let $z \to 0$ along the x-axis (i.e, $y=0$).

then

$$f'(0) = \lim_{x \to 0} \frac{x^3 + ix^3}{x^3} = 1+i \rightarrow (2)$$

Since the limits (1) and (2) are different,

$f'(0)$ does not exist.

Harmonic functions :-

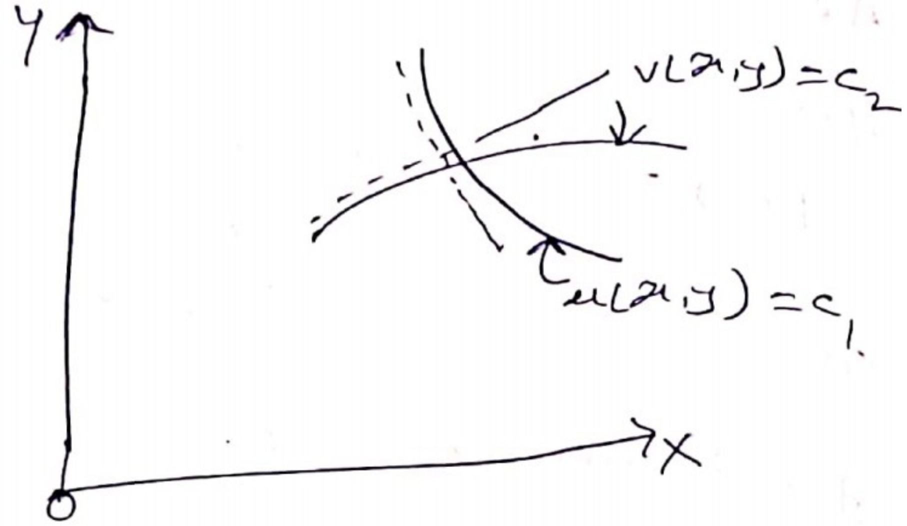
Any solution of the Laplace's equation

$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$ is called a harmonic

function.

Orthogonal system :-

Every analytic function $f(z) = u+iv$ defines two families of curves $u(x,y) = c_1$ and $v(x,y) = c_2$ which form an orthogonal system.



Consider the two families of curves

$$u(x,y) = c_1 \longrightarrow (1)$$
$$v(x,y) = c_2 \longrightarrow (2)$$

Diff (1) w.r.t. x

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-\partial u/\partial x}{\partial u/\partial y} = +m_1$$

||ly Diff (2) w.r.t x

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{\partial v/\partial x}{\partial v/\partial y} = m_2$$

$$\therefore m_1 \cdot m_2 = \frac{\frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x}}{\frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial y}} \rightarrow (3)$$

Since $f(z)$ is analytic, u & v satisfy C-R eqn

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

from (3)

$$m_1 m_2 = \frac{\frac{\partial v}{\partial y} \cdot \frac{\partial v}{\partial x}}{-\frac{\partial v}{\partial x} \cdot \frac{\partial v}{\partial y}} = -1.$$

Hence the curves intersect at right angles i.e. they form an orthogonal system.

Applications to flow problems :-

Since the real and imaginary parts of an analytic function satisfy the Laplace eqn. in two variables, these conjugate function provide soln's to a number of field and flow problems.

For example, consider the two dimension

fluid particle is along the tangent to the curve $\psi(x,y) = c'$ i.e the fluid particles move along this curve. Such curves are known as stream lines and $\psi(x,y)$ is called stream function. The curves represented by $\phi(x,y) = c$ are called equipotential lines.

* In the study of electrostatics and gravitational fields, the curves $\phi(x,y) = c$ and $\psi(x,y) = c'$ are called equipotential lines and lines of force respectively. In heat flow problems, the curves $\phi(x,y) = c$ and $\psi(x,y) = c'$ are known as isothermals and heat flow lines respectively.

Milne-Thomson's Method

This method determines the analytic function $f(z)$ when u or v is given.

Since $z = x+iy$, $\bar{z} = x-iy$

So that $x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2i}$

Let $f(z) = u(x,y) + i v(x,y) \longrightarrow (1)$

$$= u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + i v\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right)$$

Considering this as an identity in the two independent Variables $z, \bar{z}$ and putting $\bar{z} = z$, we get

$$f(z) = u(z,0) + i v(z,0)$$

which is the same as (1) if we replace x by z and y by 0.

NOW ~~$f(z) = u+iv$~~

* Thus to express any function in terms of z replace x by z and y by 0. This provides an elegant method of finding $f(z)$ when its real or imaginary part is given.

prob S.T. the function $u = e^{-2xy} \sin(x^2-y^2)$ is harmonic. Find the conjugate function V and express $u+iV$ as an analytic function of z.

Sol Here $u = e^{-2xy} \sin(x^2-y^2)$

$$\therefore \frac{\partial u}{\partial x} = -2y\, e^{-2xy} \sin(x^2-y^2) + 2x e^{-2xy} \cos(x^2-y^2)$$

$$\frac{\partial^2 u}{\partial x^2} = 4y^2 e^{-2xy} \sin(x^2-y^2) - 4xy\, e^{-2xy} \cos(x^2-y^2) + 2e^{-2xy}\cos(x^2-y^2) - 4xy\, e^{-2xy} \cos(x^2-y^2) - 4x^2 e^{-2xy} \sin(x^2-y^2) \longrightarrow (1)$$

$$\frac{\partial u}{\partial y} = -2x e^{-2xy} \sin(x^2-y^2) - 2y e^{-2xy} \cos(x^2-y^2)$$

$$\frac{\partial^2 u}{\partial y^2} = 4x^2 e^{-2xy} \sin(x^2-y^2) + 4xy\, e^{-2xy} \cos(x^2-y^2) - 2e^{-2xy}\cos(x^2-y^2) + 4xy\, e^{-2xy} \cos(x^2-y^2) - 4y^2 e^{-2xy} \sin(x^2-y^2) \longrightarrow (2)$$

(1) + (2), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$ which proves that u is harmonic.

Now, let $f(z) = u + iV$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial V}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$
($\because$ C-R eqn)

$$= \left[-2y e^{-2xy} \sin(x^2-y^2) + 2x e^{-2xy} \cos(x^2-y^2)\right.$$

$$+ i\left(2x e^{-2xy} \sin(x^2-y^2) + 2y e^{-2xy} \cos(x^2-y^2)\right)$$

Replacing x by z and y by 0 we get,

$$f'(z) = 2z \cos z^2 + 2iz \sin z^2$$

$$= 2z\left[\cos z^2 + i \sin z^2\right]$$

$$= 2z\left[e^{iz^2}\right]$$

Integrating w-r-t 'z', we have

$$f(z) = -ie^{iz^2} + c \quad [\because c = A + iB \text{ complex co}$$

which expresses $u+iv$ as an analytic fun of z

Since $u + iv = -ie^{iz^2} + c$

$$= -ie^{i(x+iy)^2} + c$$

$$= -ie^{i(x^2-y^2+i2xy)} + c$$

$$= -i e^{-2xy} \cdot e^{i(x^2-y^2)} + c$$

$$= -ie^{-2xy}\left[\cos(x^2-y^2) + i \sin(x^2-y^2)\right]$$

$$= e^{-2xy} \sin(x^2-y^2) + i\left[-e^{-2xy} \cos(x^2-y^2)\right]$$

$$\therefore v = -e^{-2xy} \cos(x^2-y^2) + B.$$

(2) An electrostatic field in the xy-plane is given by the potential function $\phi = 3x^2y - y^3$. find the stream function.

Sol Let $\psi(x,y)$ be a stream function.

By C-R equations $\frac{\partial \psi}{\partial x} = -\frac{\partial \phi}{\partial y}$

$= -3x^2 + 3y^2 \rightarrow (1)$

$\frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} = 6xy \rightarrow (2)$

Integrating (1) w.r.t 'x',

$\psi = -x^3 + 3xy^2 + F(y) \longrightarrow (3)$

$\frac{\partial \psi}{\partial y} = 6xy + F'(y)$

From (2) & (3)

$6xy + F'(y) = 6xy$

$F'(y) = 0$

$\therefore F(y) = C$

Hence $\psi = -x^3 + 3xy^2 + C$.

③ If $u - v = (x-y)(x^2+4xy+y^2)$ and $f(z) = u+iv$ is an analytic function, $z = x+iy$. find $f(z)$ in terms of z

Sol Here $f(z) = u+iv$

$\therefore \quad i f(z) = iu - v$

Adding $(1+i)f(z) = (u-v) + i(u+v)$

Let $(1+i)f(z) = F(z)$

$u - v = U$

$u + v = V$. then

$F(z) = U + iV$

Now $U = u - v = (x-y)(x^2+4xy+y^2)$

$\Rightarrow \dfrac{\partial U}{\partial x} = (x^2+4xy+y^2) + (x-y)(2x+4y)$

$= 3x^2+6xy-3y^2$

and. $\dfrac{\partial U}{\partial y} = -(x^2+4xy+y^2) + (x-y)(4x+2y)$

$= 3x^2-6xy-3y^2$

$\therefore F'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$

$\because$ [C-R eqn

$= (3x^2 + 6xy - 3y^2) - i(3x^2 - 6xy - 3y^2)$

Replacing x by z and y by 0 we get

$F'(z) = 3z^2 - 3iz^2$

$= 3z^2(1-i)$

Integrating both sides

$F(z) = (1-i)(z^3) + C$

$(1+i) f(z) = (1-i) z^3 + C$ $\quad [\because F(z) = (1+i) f(z)$

$f(z) = \frac{1-i}{1+i} z^3 + \frac{C}{1+i}$

$= \frac{(1-i)^2}{1-i^2} z^3 + \frac{C}{1+i}$

Hence $f(z) = -iz^3 + A$

(4) If $f(z)$ is a regular fun of z, P.T.

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$$

(or)

$$\nabla^2 |f(z)|^2 = 4|f'(z)|^2$$

Sol Let $f(z) = u+iv$

$$|f(z)| = \sqrt{u^2+v^2}$$

$$|f(z)|^2 = u^2+v^2 = \phi(x,y)$$

$$\therefore \frac{\partial \phi}{\partial x} = 2u\frac{\partial u}{\partial x}+2v\frac{\partial v}{\partial x}$$

$$\frac{\partial^2\phi}{\partial x^2} = 2\left[u\frac{\partial^2 u}{\partial x^2}+\left(\frac{\partial u}{\partial x}\right)^2+v\frac{\partial^2 v}{\partial x^2}+\left(\frac{\partial v}{\partial x}\right)^2\right]$$

||ly $$\frac{\partial^2\phi}{\partial y^2} = 2\left[u\frac{\partial^2 u}{\partial y^2}+\left(\frac{\partial u}{\partial y}\right)^2+v\frac{\partial^2 v}{\partial y^2}+\left(\frac{\partial v}{\partial y}\right)^2\right]$$

Adding, we get

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2} = 2\left[u\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2 + v\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right] \longrightarrow (1)$$

Since $f(z) = u+iv$ is a regular function of z, u and v satisfy C-R equ's and Laplace's equ.

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ and

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}$$

From (1), we get

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 2\left[0 + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + 0 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right]$$

$$= 4\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] \rightarrow (2)$$

Now $f(z) = u+iv$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$$

From (2), we get

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = 4\,|f'(z)|^2$$

$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\,|f'(z)|^2$

(5) Find the orthogonal trajectories of the family of curves

$x^4+y^4-6x^2y^2 = \text{const.}$

Sol Take $u(x,y) = x^4+y^4-6x^2y^2$

Then the family of curves $V(x,y) = \text{Constant}$ will be the required trajectories if

$f(z) = u+iv$ is analytic

Now $\frac{\partial u}{\partial x} = 4x^3-12xy^2$

$\frac{\partial u}{\partial y} = 4y^3-12x^2y$

$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 4x^3-12xy^2$

Integrating

$V = 4x^3y-4xy^3+c(x)$

Diff partially w.r.t x

$12x^2y-4y^3+\frac{d\,c(x)}{dx} = \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

$= -4y^3 + 12x^2y$

$\therefore \frac{d\,c(x)}{dx} = 0$ or $c = \text{const}.$

Thus the required orthogonal trajectories are $V = \text{const}$ or $x^3y - xy^3 = \text{const}.$

(5)

Cauchy's ~~integral~~ theorem

If $f(z)$ is an analytic fun and $f'(z)$ is continuous at each point within and on a closed curve c, then $\int_c f(z)dz = 0$

Proof:- $f(z) = u + iv$, $dz = dx + idy$

$$\int_c f(z)\,dz = \int_c (u\,dx - v\,dy) + i\int_c (v\,dx + u\,dy)$$

Since $f'(z)$ is continuous, therefore $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are all continuous in the region D enclosed by the curve C

$$\Rightarrow \int_c f(z)dz = \int_c (u\,dx - v\,dy) + i\int_c v\,dx + u\,dy$$

W.K.T the Green's thm $\int_c \phi\,dx + \psi\,dy = \iint_E \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\,dy$

$$\int_c f(z)dz = \iint_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

Since $f(z)$ is Analytic, then u, v are satisfies CR eqns

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \ \& \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\int_c f(z)dz = \iint_D \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\right) dx\,dy$$

$$\int_c f(z)dz = 0$$

Hence proved

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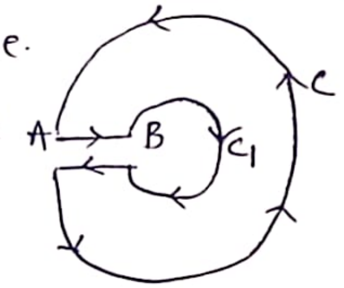
Extension of Cauchy's ~~integral~~ theorem :

If $f(z)$ is analytic in the region D between the simple closed curves C and C_1, then $\int_C f(z)\,dz = \int_{C_1} f(z)\,dz$

Proof :

Consider the region D in between the two curves C & C_1.



The paths are indicated by arrows i.e. the curve C_1 along AB in clockwise direction and the curve C along BA in Anticlockwise direction

Then by Cauchy's ~~integral~~ theorem

$$\int_{AB} f(z)\,dz + \int_{C} f(z)\,dz + \int_{BA} f(z)\,dz + \int_{C_1} f(z)\,dz = 0$$

$$\cancel{\int_{AB} f(z)\,dz} + \int_{C} f(z)\,dz - \cancel{\int_{AB} f(z)\,dz} + \int_{C_1} f(z)\,dz = 0$$

$$\int_{C} f(z)\,dz = \int_{C_1} f(z)\,dz$$

Each integration being taken in the anti clockwise direction, if $C_1, C_2, \ldots$ be any no of closed curves then $\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \int_{C_3} f(z)\,dz + \cdots$

Note:- If $f(z)$ is Analytic within and on C. $z=a$ point inside C then

$$f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\,dz$$

$$f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$

i.e. $f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$

$$f'(a) = \frac{d}{da}\left[\frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz\right]$$

$$= \frac{1}{2\pi i}\int_C \frac{\partial}{\partial a}\frac{f(z)}{(z-a)}\,dz = \frac{1}{2\pi i}\int_C \frac{-f(z)}{(z-a)^2}\,dz\,(-1)$$

$$f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\,dz$$

Singular point:-

A point at which a ~~an~~ ~~analytic~~ function $f(z)$ is not analytic is called the singular point or singularity of the function.

Cauchy's integral formula :-

If $f(z)$ is Analytic within and on a closed curve and if 'a' is any point within c, then $f(a) = \frac{1}{2\pi i} \int_c \frac{f(z)\,dz}{z-a}$

Proof :- Given $f(z)$ is analytic within c, and let $z = a$ is a point inside c.

And consider the function $\frac{f(z)}{z-a}$ is analytic inside c except at $z = a$.

Consider a small circle c_1 with centre at 'a', radius is 'r'.

Now $\frac{f(z)}{z-a}$ is analytic in the region between c and c_1

By the Extension of Cauchy integral theorem

$$\int_c \frac{f(z)}{z-a}\,dz = \int_{c_1} \frac{f(z)}{z-a}\,dz$$

Let us consider $z - a = re^{i\theta}$

$$z = re^{i\theta} + a$$

$$dz = ire^{i\theta}\,d\theta$$

$$= \int_0^{2\pi} \frac{f(a + re^{i\theta})}{\cancel{re^{i\theta}}}\, i\cancel{r}e^{i\theta}\,d\theta$$

In the limiting form as the circle c_1 shrink to the point at 'a'; i.e. $r \to 0$

$$= i\int_0^{2\pi} f(a)\,d\theta = 2\pi i\, f(a)$$

$$\Rightarrow \boxed{f(a) = \frac{1}{2\pi i} \int_c \frac{f(z)}{z-a}\,dz}$$ Hence proved

Prob(1):-

Evaluate $\int_C \frac{z^2-z+1}{z-1}\,dz$, where C is the circle

i) $|z|=1$, (ii) $|z|=\frac{1}{2}$

Sol:- (i) Since $f(z)$ is Analytic within and on circle $C:|z|=1$ an

$a=1$ lies on C

$\therefore$ By Cauchy's integral formula

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz = f(a) = f(1) = 1$$

$$\Rightarrow \int_C \frac{z^2-z+1}{z-1} = 2\pi i$$

(ii) $|z|=\frac{1}{2}$

In this case $a=1$ lies outside the circle $C:|z|=\frac{1}{2}$

so $\frac{z^2-z+1}{z-1}$ is analytic everywhere within C.

$\therefore$ By Cauchy's theorem $\int_C \frac{z^2-z+1}{z-1}\,dz = 0$

(2) Evaluate $\int_C \frac{e^{2z}}{(z-1)(z-2)}\,dz$, where 'c' is the cle $|z|=3$.

Sol:- Here $f(z)=e^{2z}$ is analytic within the circle $C:|z|=3$ and the two singular points are $a=1$, and $a=2$ are lies inside c.

$$\therefore \int_C \frac{e^{2z}}{(z-1)(z-2)}\,dz = \int e^{2z}\left(\frac{1}{z-2}-\frac{1}{z-1}\right)dz$$

$$= \int_C \frac{e^{2z}}{z-2}\,dz - \int_C \frac{e^{2z}}{z-1}\,dz$$

By Cauchy intergral formula

$$= 2\pi i\, f(2) - 2\pi i\, f(1) = 2\pi i\, e^4 - 2\pi i\, e^2$$

$$= 2\pi i\,(e^4 - e^2)$$

Prob (3): Evaluate $\int_C \frac{\sin^2 z}{(z-\pi/6)^3}\,dz$, where C is the circle $|z|=1$.

Sol: Here $f(z)=\sin^2 z$ is analytic inside the circle $C: |z|=1$ and the point $a=\pi/6$ ($=0.5$ approx) lies within C.

$\therefore$ By Cauchy integral formula

$$f''(a) = \frac{2!}{2\pi i}\int_C \frac{f(z)}{(z-a)^3}\,dz$$

$$\Rightarrow \int \frac{\sin^2 z}{(z-\pi/6)^3}\,dz = \frac{2\pi i}{2!} f''(\pi/6)$$

$$= \pi i\, f''(\pi/6)$$

$$= \pi i(1) = \pi i$$

$f(z) = \sin^2 z$

$f'(z) = 2\sin z\cos z$

$f''(z) = 2[-\sin^2 z + \cos^2 z]$

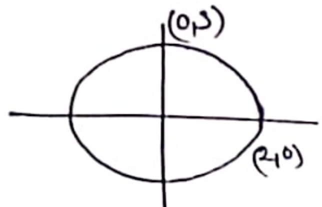
$f''(z) = 2[\cos 2z]$

$f''(\pi/6) = 2\cos 2\pi/6$

$= 2\cos\pi/3 = 2\cdot\frac{1}{2}$

$= 1$

(4) If $F(\xi) = \int_C \frac{4z^2+z+5}{z-\xi}\,dz$, where C is the ellipse $\left(\frac{x}{2}\right)^2+\left(\frac{y}{3}\right)^2=1$, find the values of $F(3.5)$, $F(i)$, $F'(-1)$, $F''(-i)$



Sol: (a) $F(3.5) = \int_C \frac{4z^2+z+5}{z-3.5}\,dz$

Here $a=3.5$, which is lies outside the given ellipse

So $\frac{4z^2+z+5}{z-3.5}$ is Analytic everywhere in ellipse

$\therefore$ By Cauchy's thm $\int_C \frac{4z^2+z+5}{z-3.5}\,dz = 0$

(b) $F(i) = \int_C \frac{4z^2+z+5}{z-i}\,dz$

Here $a=i$ i.e $(x+iy = 0+i\cdot 1) = (0,1)$ lies inside the Ellipse

$\therefore$ So $\frac{4z^2+z+5}{z-i}$ is not analytic at $(0,1)$ inside the ell

$\therefore$ By Cauchy integral formula

$$F(i) = \int_C \frac{4z^2+z+5}{z-i}\,dz = 2\pi i[4(i)^2+i+5] = 2\pi i(i+1)$$

$$= 2\pi(-1+i)$$

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ⓒ $F'(-1) = \int_C \frac{4z^2+z+5}{z+1}\,dz$

Here $a=-1$, i.e $(-1,0)$ lies inside the ellipse

$\cancel{F}$. By Cauchy integral Formula

$$F'(-1) = \frac{1}{2\pi i}\int_C \frac{4z^2+z+5}{(z+1)^2}\,dz$$

$$\Rightarrow \int_C \frac{4z^2+z+5}{(z+1)^2}\,dz = 2\pi i\, F'(-1)$$

$$= 2\pi i\,(-7)$$

$$= -14\pi i$$

$F(z) = 4z^2+z+5$
$F'(z) = 8z+1$
$F'(-1) = -8+1 = -7$

ⓓ $F''(-i) = \int_C \frac{4z^2+z+5}{z+i}\,dz$

Here $a=-i$ $(x-iy)=(0,-1)$ lies inside the ellipse

By Cauchy integral formula

$$F''(-i) = \frac{2!}{2\pi i}\int_C \frac{4z^2+z+5}{(z+i)^3}\,dz$$

$$F''(-i) \Rightarrow \int_C \frac{4z^2+z+5}{(z+i)^3}\,dz = \pi i\, F''(-i)$$

$$= (8\pi i) = 16\pi i$$

$= \frac{d}{dz}\int \frac{f(z)}{(z-\xi)^2}dz$

$F''(\xi) = 2\int_C \frac{4z^2+z+5}{(z+i)^3}dz$

$= 2 \cdot \frac{2\pi i}{2!} f''(\xi)$

$= 2\pi i(-1)^2(4) = 16$

$f(z) = 4z^2+z+5$
$f'(z) = 8z$
$F''(z) = 8$

Prob ⑤: Evaluate $f(2)$ and $f(3)$ where $f(a) = \int_C \frac{2z^2-z-2}{z-a}\,dz$ and c is the circle $|z| = 2.5$

Sol: $f(2) = \int_C \frac{2z^2-z-2}{z-2}\,dz$

Here $a=2$ is lies inside the cle $|z|=2.5$.

By Cauchy integral Formula

$$f(2) = \frac{1}{2\pi i}\int_C \frac{2z^2-z-2}{z-2}\,dz$$

$$\Rightarrow \int_C \frac{2z^2-z-2}{z-2}\,dz = 2\pi i\, f(2) = 2\pi i\,[8-4] = 8\pi i$$

Now $f(3) = \int \frac{2z^2-z-2}{z-3}\,dz$

Here $a=3$ lies outside the cle $|z|=2.5$.

So $\frac{2z^2-z-2}{z-3}$ is analytic every where in C

By Cauchy's thm $\int_C \frac{2z^2-z-2}{z-3}\,dz = 0$

(6) Evaluate $\int_C \frac{z^3-2z+1}{(z-i)^2}\,dz$ where C is the cle $|z|=2$

Sol: Here $f(z) = z^3-2z+1$

$a=i$ i.e. $x+iy = 0+i\cdot 1 = (0,1)$ is lies inside the cle $|z|=2$

By Cauchy's integral formula

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz = f(a)$$

$$\frac{1!}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\,dz = f'(a)$$

$$\frac{1!}{2\pi i}\int_C \frac{z^3-2z+1}{(z-i)^2}\,dz = f'(i)$$

$f(z) = z^3-2z+1$

$f'(z) = 3z^2-2$

$f'(i) = -3-2 = -5$

$$\Rightarrow \int_C \frac{z^3-2z+1}{(z-i)^2}\,dz = 2\pi i\, f'(i)$$

$$= 2\pi i\,(-5)$$

$$= -10\pi i$$

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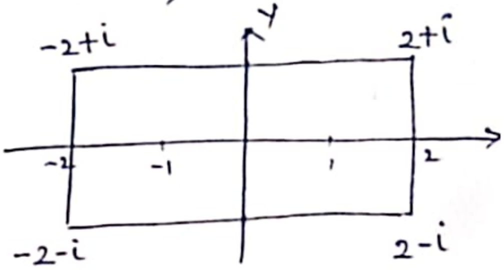
Prob(9):-

Evaluate using Cauchy's integral formula, $\int_C \frac{\cos\pi z}{z^2-1}\,dz$ around a rectangle with vertices $2\pm i, -2\pm i$

Sol:- Given rectangle with vertices is $2\pm i, -2\pm i$



Here ~~the~~ Singular points are $1, -1$

i.e. $z^2-1=0 \Rightarrow z^2=1 \Rightarrow z=\pm 1$

$\therefore$ $f(z)=\cos\pi z$ is analytic in the region bounded by the given rectangle and the two singular points $a=1, a=-1$ are lies inside this rectangle.

$\therefore$ By using Cauchy's integral formula

$$\int_C \frac{\cos\pi z}{z^2-1}\,dz = \frac{1}{2}\int_C \left(\frac{1}{z-1}-\frac{1}{z+1}\right)\cos\pi z\,dz$$

$$=\frac{1}{2}\left[\int_C \frac{\cos\pi z}{z-1}\,dz - \int_C \frac{\cos\pi z}{z+1}\,dz\right]$$

$$=\frac{1}{2}\left[2\pi i\cos\pi(1) - 2\pi i\cdot\cos\pi(-1)\right]$$

$$\int_C \frac{\cos\pi z}{z^2-1}\,dz = 0$$

* Prob(10): Evaluate $\int_C \frac{e^{2z}}{(z+1)^4}\,dz$, where c is the cle $|z|=2$

Sol:- Here $a=-1$ is the singular point lies inside the given cle $|z|=2$.

And the function $f(z)=e^{2z}$ is Analytic within the cle $|z|=2$.

By Cauchy's integral formula

$$\boxed{f(z)=e^{2z},\ f'(z)=2e^{2z},\ f'''(z)=8e^{2z}}$$

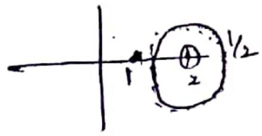
$$f'''(a)=\frac{6}{2\pi i}\int_C \frac{e^{2z}}{(z+1)^4}\,dz$$

$$\rightarrow \int_C \frac{e^{2z}}{(z+1)^4}\,dz = \frac{2\pi i}{6}f'''(-1) = \frac{\cancel{2}\pi i}{\cancel{6}}\cdot 8e^{-2} = \frac{8\pi i e^{-2}}{3}$$

Prob (11): Evaluate $\int_C \frac{z}{z^2-3z+2}\,dz$, where C is $|z-2| = \frac{1}{2}$

Sol:

$$\int_C \frac{z}{z^2-3z+2}\,dz = \int_C \frac{z}{(z-1)(z-2)}\,dz = \int_C \frac{z}{z-2}\,dz - \int_C \frac{z}{z-1}\,dz \rightarrow (1)$$



Here the cle is $|z-2| = \frac{1}{2}$

i.e. the cle with centre at $z=2$ and its radius is $\frac{1}{2}$

Here the poles are $z = 2, 1$

Here the only one singular point is $z=2$ lies inside the cle $|z-2| = \frac{1}{2}$

From (1)

$$\int_C \frac{z}{z^2-3z+2}\,dz = 2\pi i f(2) - 0 = 4\pi i$$

Prob (12): Evaluate $\int_C \frac{\log z}{(z-1)^3}\,dz$, where $C : |z-1| = 1/2$ by using Cauchy integral formula.

Sol: $f(z) = \frac{\log z}{(z-1)^3}$

$f(z) = \log z$

$f'(z) = \frac{1}{z}$

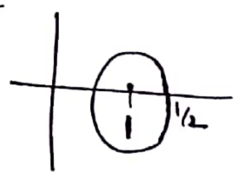
$f''(z) = -\frac{1}{z^2}$

Here $a = 1$

Given circle $|z-1| = 1/2$

i.e. centre at 1 and radius is $1/2$

∴ Here $a=1$ lies inside the given cle $|z-1| = 1/2$



∴ By Cauchy integral formula

$$\int_C \frac{\log z}{(z-1)^3}\,dz = \frac{2\pi i}{2} f''(a)$$

$$= \pi i \left(-1/z^2\right) = -\pi i$$

Series of Complex terms:

consider $(a_1+ib_1)+(a_2+ib_2)+(a_3+ib_3)+\cdots+\infty$ be an infinite series of complex terms, ~~and~~ $a_1, a_2, \ldots a_\infty$ and $b_1, b_2, \ldots \infty$ are real numbers and is denoted by $\sum_{n=1}^{\infty} a_n+ib_n$

i) If $\sum a_n$, $\sum b_n$ are both convergent then $\sum a_n+ib_n$ is convergent

ii) If $|a_1+ib_1|+|a_2+ib_2|+\cdots+\infty$ is convergent then $\sum(a_n+ib_n)$ is said to be absolutely convergent

iii) If a power series $\sum a_n z^n$ is convergent at $z=z_1$ then the series converges absolutely for all z with $|z|<|z_1|$

Circle of Convergence:

The circle $|z|=R$ in which the power series $\sum a_n z^n$ converges is said to be the circle of convergence and radius of the circle is said to be Radius of convergence of the series.

Taylor's Series:

If $f(z)$ is analytic inside a circle 'c' containing with centre 'a' then for all z in c

$$f(z)=f(a)+(z-a)f'(a)+\frac{(z-a)^2}{2!}f''(a)+\cdots$$

$$f(z)=a_0+a_1(z-a)+a_2(z-a)^2+a_3(z-a)^3+\cdots$$

Where $a_n=\dfrac{1}{2\pi i}\displaystyle\int_c \frac{f(z)}{(z-a)^{n+1}}dz.$

Laurent's Series:

If $f(z)$ is analytic inside the ring shaped region R' bounded between the circles 'c' and c_1 with radii r and r_1 ($r_1 < r$) and with centre 'a' then for all z in region R

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots$$
$$+ a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \cdots$$

Where $a_n = \dfrac{1}{2\pi i}\displaystyle\int_{\Gamma} \dfrac{f(z)}{(z-a)^{n+1}}\,dz$

Here Γ is a closed curve in a region R.

Prob:-

1) Expand $f(z) = \dfrac{1}{(z-1)(z-2)}$ in the region

a) $|z|<1$, (b) $1<|z|<2$ (c) $|z|>2$ (d) $0<|z-1|<1$

Sol: (a) $|z|<1$

Given $f(z) = \dfrac{1}{(z-1)(z-2)} = \dfrac{1}{z-2} - \dfrac{1}{z-1} \longrightarrow (1)$

if $|z|<1 \Rightarrow |z/2|<1$

$$\frac{1}{z-1} = -\frac{1}{1-z} = -[1-z]^{-1} = -[1+z+z^2+z^3+\cdots]$$

$$\frac{1}{z-2} = -\frac{1}{2-z} = -\frac{1}{2}\frac{1}{(1-z/2)} = -\frac{1}{2}(1-z/2)^{-1} = -\frac{1}{2}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\cdots\right]$$

Now $f(z) = -\dfrac{1}{2}\left[1+\dfrac{z}{2}+\left(\dfrac{z}{2}\right)^2+\left(\dfrac{z}{2}\right)^3+\cdots\right] + [1+z+z^2+z^3+\cdots]$

$$= \left(1-\frac{1}{2}\right) + \left(1-\frac{1}{2^2}\right)z + \left(1-\frac{1}{2^3}\right)z^2 + \left(1-\frac{1}{2^4}\right)z^3 + \cdots$$

$$= \frac{1}{2} + \frac{3}{4}z + \frac{7}{8}z^2 + \frac{15}{16}z^3 + \cdots$$

(b). For $1<|z|<2$

$$\left|\frac{1}{z}\right|<1 \;,\; \left|\frac{z}{2}\right|<1$$

$$\frac{1}{z-1} = \frac{1}{z\left(1-\frac{1}{z}\right)} = \frac{1}{z}\left[1-\frac{1}{z}\right]^{-1} = \frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right]$$

$$\frac{1}{z-2} = \frac{1}{-2\left[1-z/2\right]} = -\frac{1}{2}\left[1-z/2\right]^{-1} = -\frac{1}{2}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\left(z/2\right)^3+\cdots\right]$$

$$① \Rightarrow f(z) = \left[-\frac{1}{2}-\frac{z}{2^2}-\frac{z^2}{2^3}-\frac{z^3}{2^4}-\cdots\cdots\right]-\left[\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\frac{1}{z^4}+\cdots\right]$$

(c) For $|z|>2$

$$|2/z|<1 \text{ and also } |1/z|<1$$

$$\frac{1}{z-1} = \frac{1}{z\left(1-\frac{1}{z}\right)} = \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} = \frac{1}{z}\left[1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\left(\frac{1}{z}\right)^3+\cdots\cdots\right]$$

$$\frac{1}{z-2} = \frac{1}{z\left(1-2/z\right)} = \frac{1}{z}\left(1-2/z\right]^{-1} = \frac{1}{z}\left[1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\left(2/z\right)^3+\cdots\right]$$

$$① \Rightarrow f(z) = \cancel{\frac{1}{z}}+\frac{2}{z^2}+\frac{4}{z^3}+\frac{8}{z^4}+\cdots\cdots - \cancel{\frac{1}{z}}-\frac{1}{z^2}-\frac{1}{z^3}-\cdots$$

$$= (2-1)\frac{1}{z^2}+(4-1)\frac{1}{z^3}+(8-1)\frac{1}{z^4}+\cdots\cdots$$

$$= \frac{1}{z^2}+\frac{3}{z^3}+\frac{7}{z^4}+\cdots\cdots$$

(d) $0<|z-1|<1$

$$f(z) = \frac{1}{z-2}-\frac{1}{z-1}$$

$$= \frac{1}{z-1-1}-\frac{1}{z-1} = \frac{1}{(z-1)(1-\ldots)} = -\frac{1}{(1-(z-1))}-\frac{1}{z-1}$$

$$= -\frac{1}{z-1}-\left(1-(z-1)\right)^{-1} = -\frac{1}{z-1}-\left[1+(z-1)+(z-1)^2+\cdots\cdots\right]$$

$$f(z) = -\frac{1}{z-1}-1-(z-1)-(z-1)^2-(z-1)^3-\cdots\cdots$$

(2) Find Taylors Expansion of $f(z)=\frac{2z^3+1}{z^2+z}$ about the point $z=i$

Sol:- To expand $\frac{2z^3+1}{z^2+z}$ in powers $z=i$, put $z-i=t$

$$f(z)=\frac{2z^3+1}{z(z+1)}=2z-2+\frac{2z+1}{z(z+1)}$$

$$z^2+z)\ 2z^3+1\ (2z-2$$
$$2z^3+2z^2$$
$$1-2z^2$$
$$-2z-2z^2$$
$$1+2z$$

$$=2i+2z-2-2i+\frac{1}{z}+\frac{1}{z+1}$$

$$=(2i-2)+2(z-i)+\frac{1}{z}+\frac{1}{z+1}\rightarrow (1)$$

$$\frac{1}{z}=\frac{1}{t+i}=\frac{1}{i}\frac{1}{(1+t/i)}=\frac{1}{i}(1+t/i)^{-1}$$ (Binomial expansion

$$=\frac{1}{i}\left[1-\frac{t}{i}+\frac{t^2}{i^2}-\frac{t^3}{i^3}+\dots\infty\right]$$

$$=\frac{1}{i}+\frac{t}{1}+\frac{t^2}{i^3}-\frac{t^3}{i^4}+\frac{t^4}{i^5}-\dots\infty$$

$$=\frac{i}{i^2}+\frac{t}{1}+\frac{t^2}{i^3}-\frac{t^3}{i^4}+\frac{t^4}{i^5}-\dots$$

$$=-i+(z-i)+\sum_{n=2}^{\infty}(-1)^n\frac{(z-i)^n}{i^{n+1}}\rightarrow (2)$$

$$\frac{1}{z+1}=\frac{1}{t+i+1}=\frac{1}{1+i\left(1+\frac{t}{1+i}\right)}=\frac{1}{1+i}\left(1+\frac{t}{1+i}\right)^{-1}$$

$$=\frac{1}{1+i}\left[1-\frac{t}{1+i}+\frac{t^2}{(1+i)^2}-\frac{t^3}{(1+i)^3}+\dots\infty\right]$$

$$=\frac{1+i}{(1-i)(1-i)}\left[1-\frac{t}{1+i}+\frac{t^2}{(1+i)^2}-\frac{t^3}{(1+i)^3}+\dots\infty\right]$$

$$=\frac{1-i}{1-i^2}\left[1-\frac{t}{1+i}+\frac{t^2}{(1+i)^2}-\frac{t^3}{(1+i)^3}+\dots\infty\right]$$

$$=\frac{1-i}{2}-$$

$$=\frac{1}{1+i}-\frac{t}{2i}+\frac{t^2}{(1+i)^3}-\frac{t^3}{(1+i)^4}+\dots$$

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$$= \frac{1-i}{(1-i)(1+i)} - \frac{t}{2i} + \frac{t^2}{(1+i)^3} - \frac{t^3}{(1+i)^4} + \cdots$$

$$= \frac{1-i}{2} - \frac{t}{2i} + \frac{t^2}{(1+i)^3} - \frac{t^3}{(1+i)^4} + \cdots$$

$$= \frac{1}{2} - \frac{i}{2} - \frac{z-i}{2i} + \sum_{n=2}^{\infty} \frac{(-1)^n (z-i)^n}{(1+i)^{n+1}} \longrightarrow (3)$$

$\therefore$ from eq^n (1) (2) & (3)

$$f(z) = 2i-2+2(z-i) - i + (z-i) + \sum_{n=2}^{\infty} (-1)^n \frac{(z-i)^n}{i^{n+1}} +$$

$$+ \frac{1}{2} - \frac{i}{2} - \frac{z-i}{2i} + \sum_{n=2}^{\infty} (-1)^n \frac{(z-i)^n}{(1+i)^{n+1}}$$

$$= \left(2i-2-i+\frac{1}{2}-\frac{i}{2}\right) + \left(2+1-\frac{1}{2i}\right)(z-i) +$$

$$\sum_{n=2}^{\infty} (-1)^n \left(\frac{1}{i^{n+1}} + \frac{1}{(1+i)^{n+1}}\right)(z-i)^n$$

$$= \left(\frac{i}{2} - \frac{3}{2}\right) + \left(3 - \frac{i}{2i^2}\right)(z-i) + \sum_{n=2}^{\infty} (-1)^n \left(\frac{1}{i^{n+1}} + \frac{1}{(1+i)^{n+1}}\right)(z-i)^n$$

$$f(z) = \left(\frac{i}{2} - \frac{3}{2}\right) + \left(3 + \frac{i}{2}\right)(z-i) + \sum_{n=2}^{\infty} (-1)^n \left(\frac{1}{i^{n+1}} + \frac{1}{(1+i)^{n+1}}\right)(z-i)^n$$

(3) Find the Laurents expansion of $f(z) = \dfrac{7z-2}{(z+1)z(z-2)}$ the region $1 < z+1 < 3$

Sol: let $z+1 = t \Rightarrow z = t-1$

$$f(z) = \frac{7(t-1)-2}{t(t-1)(t-3)} = \frac{7t-9}{t(t-1)(t-3)}$$

$$= -\frac{3}{t} + \frac{1}{t(1-1/t)} - \frac{2}{3(1-t/3)}$$

$$= -\frac{3}{t} + \frac{1}{t}(1-1/t)^{-1} - \frac{2}{3}(1-t/3)^{-1} \quad \lceil \text{Binomial exp}$$

$$= -\frac{3}{t} + \frac{1}{t}\left(1+\frac{1}{t}+\frac{1}{t^2}+\frac{1}{t^3}+\cdots\infty\right) - \frac{2}{3}\left(1+\frac{t}{3}+\frac{t^2}{3^2}+\frac{t^3}{3^3}+\cdots\right)$$

$$= -\frac{2}{t} + \frac{1}{t^2} + \frac{1}{t^3} + \frac{1}{t^4} + \cdots\infty - \frac{2}{3}\left(1+\frac{t}{3}+\frac{t^2}{3^2}+\frac{t^3}{3^3}+\cdots\right)$$

$$f(z) = \frac{-2}{z+1} + \frac{1}{(z+1)^2} + \frac{1}{(z+1)^3} + \frac{1}{(z+1)^4} + \cdots\infty - \frac{2}{3}\left(1+\frac{z+1}{3}+\frac{(z+1)^2}{3^2}+\cdots\right)$$

(4) Find the Laurents Series expansion of $f(z) = \dfrac{z^2-6z-1}{(z-1)(z-3)(z+2)}$ in the region $3 < |z+2| < 5$

Sol: Given region $3 < |z+2| < 5$

$$\Rightarrow \left|\frac{3}{z+2}\right| < 1, \quad \left|\frac{z+2}{5}\right| < 1$$

$$\frac{z^2-6z-1}{(z-1)(z-3)(z+2)} = \frac{A}{z-1} + \frac{B}{z-3} + \frac{C}{z-2}$$

$$z^2-6z-1 = A(z-3)(z-2) + B(z-1)(z-2) + C(z-1)(z-3)$$

put $z = 3$

$$9-18-1 = B(2)(1)$$

$$-10/2 = B$$

$$\boxed{B = -5}$$

put $z = 1$

$1 - 6 - 1 = 2A \Rightarrow \boxed{A = -3}$

put $z = 2$

$4 - 12 - 1 = -C \Rightarrow \boxed{C = 9}$

$$f(z) = \frac{z^2 - 6z - 1}{(z-1)(z-3)(z+2)} =$$

let $z + 2 = u$

$z = u - 2 \Rightarrow z - 1 = u - 3$

$z - 3 = u - 5$

Now $$f(z) = \frac{(u-2)^2 - 6(u-2) - 1}{(u-3)(u-5)u} = \frac{A}{u} + \frac{B}{u-3} + \frac{C}{u-5}$$

$A = 1, \ B = 1, \ C = -1$

$$f(z) = \frac{1}{u} + \frac{1}{u-3} + \frac{-1}{u-5}$$ Since $3 < |u| < 5$

$\left|\frac{3}{u}\right| < 1, \ \left|\frac{u}{5}\right| < 1$

$$f(z) = \frac{1}{u} + \frac{1}{u-3} - \frac{1}{u-5}$$

~~Since $3 < |u| < 5 \Rightarrow \frac{3}{u}$~~

$$= \frac{1}{u} + \frac{1}{u(1 - 3/u)} + \frac{1}{5(1 - u/5)} = \frac{1}{u} + \frac{1}{u}(1 - 3/u)^{-1} + \frac{1}{5}(1 - u/5)^{-1}$$

$$= \frac{1}{u} + \frac{1}{u}\left[1 + \frac{3}{u} + \frac{3^2}{u^2} + \frac{3^3}{u^3} + \cdots\right] + \frac{1}{5}\left[1 + \frac{u}{5} + \frac{u^2}{5^2} + \cdots\right]$$

$$= \frac{1}{u} + \frac{1}{u} + \frac{3}{u^2} + \frac{3^2}{u^3} + \frac{3^3}{u^4} + \cdots + \frac{1}{5}\left[1 + \frac{z+2}{5} + \frac{(z+2)^2}{5^2} + \cdots\right]$$

$$= \frac{2}{z+2} + \frac{3}{(z+2)^2} + \frac{3^2}{(z+2)^3} + \cdots + \frac{1}{5}\left[1 + \frac{z+2}{5} + \frac{(z+2)^2}{5^2} + \cdots\right]$$

(5) Expand the function $\frac{1}{(z-1)(z+3)}$ for $1<|z|<3$ in Laurent's series

Sol: $f(z) = \frac{1}{(z-1)(z+3)} = \frac{A}{z-1} + \frac{B}{z+3}$ $\begin{cases} A = 1/4 \\ B = -1/4 \end{cases}$

Given region $1<|z|<3 \Rightarrow \left|\frac{1}{z}\right|<1, \left|\frac{z}{3}\right|<1$

$$= \frac{1}{4}\frac{1}{z(1-\frac{1}{z})} + \frac{1}{4}\frac{1}{3(1+z/3)}$$

$$= \frac{1}{4}\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} + \frac{1}{4}\frac{1}{3}\left(1+z/3\right)^{-1}$$

$$= \frac{1}{4}\frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\infty\right] + \frac{1}{4}\frac{1}{3}\left(1-\frac{z}{3}+\frac{z^2}{3^2}-\frac{z^3}{3^3}+\cdots\right.$$

$$= \frac{1}{4}\left(\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\frac{1}{z^4}+\cdots\infty\right] + \frac{1}{4}\left(\frac{1}{3}-\frac{z}{9}\right.$$

$$f(z) = \frac{1}{4z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\infty\right] + \frac{1}{12}\left[1-\frac{z}{3}+\frac{z^2}{9}-\frac{z^3}{27}+\cdots\right.$$

Isolated Singularity:-

If $z=a$ is a singular point of the function $f(z)$ that there exist a circle with centre 'a' which has no other singular point of $f(z)$. Then $z=a$ is called an isolated singular point. ~~Of~~

If $z=a$ is an isolated singularity of $f(z)$ then $f(z)$ has Laurent's series expansion about $z=a$ is given by

$$\left.\begin{aligned} f(z) &= a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots \\ &\quad + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \cdots \end{aligned}\right\} \rightarrow (1)$$

pole of order n:-

If all the negetive powers of $(z-a)$ in (1) after the term n^{th} are missing then ~~the~~ the singular point $z=a$ is called pole of order n.

If the pole of order is '1' then the pole is said to be simple pole.

Removable Singularity :-

An isolated singular point of $f(z)$ is said to be removable singularity of $f(z)$, if all the negetive power of $(z-a)$ in the Laurent's series expansion of $f(z)$ is missing.

Essential Singularity:-

If in the Laurent's series expansion of $f(z)$ about $z=a$ contains infinite no of negetive powers of $z-a$ then the isolated singularity $z=a$ of $f(z)$ is called the essential singularity.

(1) Find the nature and location of Singularity of

$$f(z) = \frac{e^{2z}}{(z-1)^4}$$

Sol: Singularity point of $f(z) = \frac{e^{2z}}{(z-1)^4}$ is $z=1$

clearly $z=1$ is an isolated singularity.

Let $z-1=t$

$$f(z) = \frac{e^{2z}}{(z-1)^4} = \frac{e^{2(t+1)}}{t^4} = \frac{e^2}{t^4} e^{2t}$$

$$= \frac{e^2}{t^4}\left[1+\frac{2t}{1!}+\frac{(2t)^2}{2!}+\frac{(2t)^3}{3!}+\frac{(2t)^4}{4!}+\frac{(2t)^5}{5!}+\cdots\right]$$

$$= e^2\left(\frac{1}{t^4}+\frac{2}{t^3}+\frac{2}{t^2}+\frac{4}{3t}+\frac{2}{3}+\frac{4t}{15}\cdots\right)$$

$$= e^2\left(\frac{1}{(z-1)^4}+\frac{2}{(z-1)^3}+\frac{2}{(z-1)^2}+\frac{4}{3(z-1)}+\frac{2}{3}+\frac{4}{15}(z-1)+\cdots\right)$$

Here the finite (4) number of terms containing -ve powers of $(z$

$\therefore z=1$ is a pole of order 4.

1) Find the nature and location of singularities of

$f(z) = \frac{e^z}{(z-1)^4}$ $\quad [\because e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots]$

Sol:- Given $f(z) = \frac{e^z}{(z-1)^4}$

$$= \frac{e^{z+1-1}}{(z-1)^4} = \frac{e^{z-1} \cdot e}{(z-1)^4}$$

$$= \frac{e}{(z-1)^4}\left[1 + \frac{z-1}{1!} + \frac{(z-1)^2}{2!} + \frac{(z-1)^3}{3!} + \cdots\right]$$

$$f(z) = \frac{e}{(z-1)^4} + \frac{e}{(z-1)^3} + \frac{e}{2!(z-1)^2} + \frac{e}{3!(z-1)} + \frac{e}{4!} + \cdots$$

Here the finite number (4) of terms containing -ve powers of $(z-1)$

$z=1$ is a pole of order 4.

2) Find the nature and location of singularities of

$f(z) = \sin\left(\frac{1}{1-z}\right)$

Sol:- $f(z) = \sin\left(\frac{1}{1-z}\right) = -\sin\left(\frac{1}{z-1}\right)$ $\quad \left[\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right]$

$$= -\left[\frac{(z-1)^{-1}}{1!} - \frac{(z-1)^{-3}}{3!} + \frac{(z-1)^{-5}}{5!} - \cdots\right]$$

Here $z=1$ is an isolated singularity

And the no of -ve powers of $(z-1)$ in the Laurent series expansion of $f(z)$ about $z=1$ is infinite

So, the isolated singularity $z=1$ of $f(z)$ is an essential singularity

3) $f(z) = z e^{1/z^2}$

$$= z\left(1 + \frac{1}{1! z^2} + \frac{1}{2! z^4} + \frac{1}{3! z^6} + \cdots\right) = z + z^{-1} + \frac{z^{-3}}{2} + \frac{z^{-5}}{6} + \cdots \infty$$

Here $z=0$ is an simple pole. And here infinite no of -ve powers of z in the series. Then $z=0$ is an essential singularity of $f(z)$.

4) Find the nature and location of Singularities of the function $\frac{z - \sin z}{z^2}$

Sol:- $f(z) = \frac{z - \sin z}{z^2}$

$\sin z = \frac{z^1}{1!} - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$

$= \frac{1}{z^2}\left[\cancel{z} - \frac{\cancel{z}}{1!} + \frac{z^3}{3!} - \frac{z^5}{5!} + \cdots\right]$

$= \frac{z}{6} - \frac{z^3}{120} + \cdots$

Here there is the no of -ve powers of z are missing.

$\therefore z=0$ is a Removable Singularity.

5) $f(z) = (z+1)\sin\frac{1}{z-2}$

Sol:

Let $z-2 = t \Rightarrow z = t+2$

$z+1 = t+3$

$f(z) = (z+1)\sin\frac{1}{z-2} = (t+3)\sin\frac{1}{t} = (t+3)\left(\frac{1}{1!t} - \frac{1}{3!t^3} + \frac{1}{5!t^5} \cdots\right)$

$= \left(1 - \frac{1}{3!t^2} + \frac{1}{5!t^4} - \cdots\right) + \left(\frac{3}{t} - \frac{3}{3!t^3} + \frac{3}{5!t^5} - \cdots\right)$

$= \left(1 - \frac{1}{6t^2} + \frac{1}{120t^4} - \cdots\right) + \frac{3}{t} - \frac{1}{2t^3} + \frac{1}{40t^5} - \cdots$

$= 1 + \frac{3}{t} - \frac{1}{6t^2} - \frac{1}{2t^3} + \frac{1}{120t^4} + \frac{1}{40t^5} - \cdots$

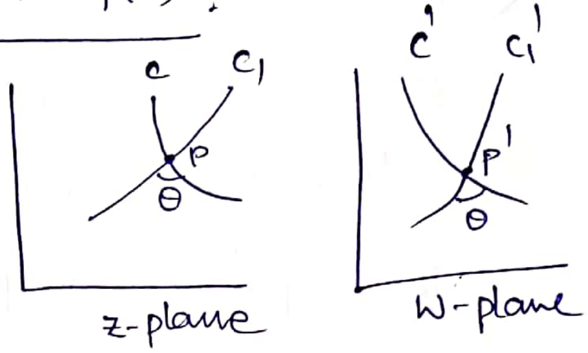
$= 1 + \frac{3}{z-2} - \frac{1}{6(z-2)^2} - \frac{1}{2(z-2)^3} + \frac{1}{120(z-2)^4} + \frac{1}{40(z-2)^5} - \cdots$

Here $z=2$ is the isolated Singularity

And in the above series the no of -ve powers of $z=2$ are infinite.

$\therefore z=2$ is an essential Singularity.

Geometrical Representation of $w=f(z)$:



Transformation (mapping):

A curve 'c' in z-plane is mapped into the corresponding curve c' in the w-plane by the function $w=f(z)$ is said to be mapping (or) Transformation of the z-plane into w-plane.

Conformal Transformation:

Suppose two curves c and c_1 in the z-plane intersect at a point P and corresponding curves c' and c_1' in the w-plane intersect at P' if the angles of intersection of the curves at P. is the same as the angle of intersection of the curves P' then the transformation is said to be Conformal transformation.

Note:- 1) The transformation effected by an analytic function $w=f(z)$ is Conformal at every point of the z-plane where $f'(z)\neq 0$

2) A point at which $f'(z)=0$ is called critical point of the transformation.

Standard Properties :-

1) Translation:

If $w=z+c$, where c is the complex constant if $z=x+iy$, $c=c_1+ic_2$ and $w=u+iv$ then the transformation becomes $w=z+c$

$$u+iv = x+iy + c_1 + ic_2$$
$$= (x+c_1) + i(y+c_2)$$

Here $u = x+c_1$, $v = y+c_2$, i.e. the point $P(x,y)$ in the z-plane is mapped into the point $p'(x+c_1, y+c_2)$ in the w-plane.

lly the other points of z-plane are mapped into w-p

Thus if W-plane is superposed on the z-plane

This transformation maps a figure in the z-plane in a figure in the w-plane of same shape and size

2) Magnification and Rotation :

If $W = cz$, where c is the complex constant, her $c = \rho e^{i\alpha}$, $z = re^{i\theta}$, $W = Re^{i\phi}$

$$W = cz$$
$$Re^{i\phi} = \rho e^{i\alpha} . re^{i\theta}$$
$$Re^{i\phi} = \rho r \, e^{i(\alpha+\theta)}$$
$$\Rightarrow R = \rho r \qquad \phi = \alpha + \theta$$

The point $P(r,\theta)$ in the z-plane is mapped onto the point $p'(\rho r, \theta+\alpha)$ in the W-plane.

Hence any figure in the z-plane is transformed into a geometrically similar figure in the W-plane.

Inversion and Reflection

Suppose w-plane is ~~Supposed~~ Superposed on z-plane if

$$z = re^{i\theta}, \quad w = Re^{i\phi}$$

Then $w = \frac{1}{z}$

$$Re^{i\phi} = \frac{1}{re^{i\theta}} \Rightarrow Re^{i\phi} = \frac{1}{r} e^{-i\theta}$$

$$\Rightarrow \text{Here } R = \frac{1}{r}, \quad \phi = -\theta$$

Thus if P be (r, θ) and P_1 be $(1/r, -\theta)$

i.e. P_1 is the inverse of P w.r.to the unit circle with centre, then the reflection P' of P_1 in the real axis represents $w = 1/z$.

Hence this transformation is an inversion of z w.r.to the unit circle $|z| = 1$ followed by reflection of the inverse into the real axis.

Bilinear transformation:

The transformation $w = \frac{az+b}{cz+d}$ is said to be Bilinear transformation. Where a, b, c, d are complex constants and $ad - bc \neq 0$ ensure that $\frac{dw}{dz} \neq 0$ i.e. the transformation is conformal. If $ad - bc = 0$ then every point on the z-plane is critical point.

Invariant points of bilinear Transformation:

If z-maps into itself in the w-plane (i.e. $w = z$) then

$$w = \frac{az+b}{cz+d} \Rightarrow z = \frac{az+b}{cz+d}$$

$$z(cz+d) = az+b$$

$$cz^2 + z(d-a) - b = 0$$

The roots of this equation are defined as the invariant (or) the fixed points of the bilinear transformation.

Properties of Bilinear transformation:

1) A bilinear Transformation maps circles into circles

By actual division of $w = \frac{az+b}{cz+d}$ is

$$w = \frac{a}{c} + \frac{bc-ad}{c^2} \cdot \frac{1}{z + d/c}$$

$$w_1 = z + d/c \;,\; w_2 = \frac{1}{w_1} \;,\; w_3 = \frac{bc-ad}{c^2} w_2$$

$$\Rightarrow w = \frac{a}{c} + w_3$$

By this transformation we successfully pass from

z-plane $\rightarrow$ w_1 plane $\rightarrow$ w_2 plane $\rightarrow$ w_3 plane $\rightarrow$ w plane

Hence the transformation maps circles into circles

2) A bilinear transformation preserves cross Ratio of four points.

Consider the four points z_1, z_2, z_3, z_4 of z-plane mapps into the points w_1, w_2, w_3, w_4 of w-plane by bilinear Transformation., if these points are infinite then

$$w_j - w_k = \frac{az_j+b}{cz_j+d} - \frac{az_k+b}{cz_k+d} = \frac{(ad-bc)(z_j-z_k)}{(cz_j+d)(cz_k+d)}$$

by using this relation for $j = 1,2,3,4 \ldots\ldots ; k = 1,2,3,4 \ldots\ldots$

$$\frac{(w_1-w_2)(w_3-w_4)}{(w_1-w_4)(w_3-w_2)} = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}$$

Thus the cross ratio of four points are invariant under bilinear transformation.

Problems:-

(1) Find the bilinear transformation which maps the points $z = 1, i, -1$ onto the points $w = i, 0, -i$, and hence find

a) the image of $|z|<1$

b) the invariant points of this transformation

Sol:- Let the points are $z_1 = 1$, $z_2 = i$, $z_3 = -1$ and $z_4 = z$ map onto the points $w_1 = i$, $w_2 = 0$, $w_3 = -i$. and $w_4 = w$

By using cross ratio under bilinear transformation

$$\frac{(1-i)(-1-z)}{(1-z)(-1-i)} = \frac{(i-0)(-i-w)}{(i-w)(-i-0)}$$

$$\Rightarrow \frac{w+i}{w-i} = \frac{(z+1)(1-i)}{(z-1)(1+i)}$$

By Componendo and dividendo

$$\frac{w+\cancel{i}+w-\cancel{i}}{w+i-\cancel{w}+i} = \frac{(z+1)(1-i)+(z-1)(1+i)}{(z+1)(1-i)-(z-1)(1+i)} = \frac{z-\cancel{iz}+\cancel{1}-\cancel{i}+z+\cancel{iz}-\cancel{1}-i}{\cancel{z}-iz+1-\cancel{i}-\cancel{z}-iz+1+\cancel{i}}$$

$$\frac{2w}{2i} = \frac{2z-2i}{2-2iz} \Rightarrow \frac{w}{i} = \frac{z-i}{1-iz}$$

$$\Rightarrow w = \frac{1+iz}{1-iz} \longrightarrow (1)$$

Which is the required bilinear transformation

a) $W = \dfrac{1+iz}{1-iz} \Rightarrow W - Wiz = 1+iz$

$\Rightarrow W-1 = z(i+Wi)$

$\Rightarrow z = \dfrac{W-1}{i(1+W)}$

$|z|<1 \Rightarrow \left|\dfrac{W-1}{i(1+W)}\right| < 1$

$\Rightarrow |W-1| < |i||1+W|$

$\Rightarrow |u+iv-1| < |1+u+iv|$

$\Rightarrow (u-1)^2+v^2 < (u+1)^2+v^2$

$\Rightarrow \cancel{1}+\cancel{u^2}-2u+\cancel{v^2} < \cancel{u^2}+\cancel{1}+2u+\cancel{v^2}$

$\Rightarrow 4u>0 \Rightarrow u>0$

Hence the interior of the circle $x^2+y^2=1$ in the z-plane mapped onto the entire half of the w-plane. to the right of imaginary a

b) To find the invariant points

$z = \dfrac{az+b}{cz+d} \Rightarrow cz^2+z(d-a)-b=0$

from ① by comp. $a=i,\ b=1,\ c=-i,\ d=1$

$\Rightarrow -iz^2+z(1-i)-1=0 \Rightarrow -iz^2-z(i-1)-1=0$
$iz^2+z(i-1)+1=0$

$\Rightarrow \cancel{\dfrac{-(1-i)\pm\sqrt{(1-i)^2-4i}}{-2i}} = \dfrac{-(i-1)\pm\sqrt{(i-1)^2-4i}}{2i}$

$= \dfrac{(-i+1)\pm i\sqrt{6i}}{2i} = \dfrac{1-i\pm i\sqrt{6i}}{2i}$ (By multo by i

$= \cancel{\dfrac{-i^3+i\pm i^2\sqrt{6i}}{2i^2}} = \dfrac{i-i^2\pm i^2\sqrt{6i}}{2i^2}$

$= \dfrac{1+i\pm\sqrt{6i}}{-2} = -\dfrac{1}{2}\left(1+i\mp\sqrt{6i}\right)$

Which are the required invariant points

(2) Find the bilinear transformation which maps $1, i, -1$ to $2, i, -2$ respectively. Find the fixed and critical points of the transformation.

Sol:- Let the given points are $z_1 = 1,\ z_2 = i,\ z_3 = -1,\ z_4 = z$

$w_1 = 2,\ w_2 = i,\ w_3 = -2,\ w_4 = w$

By using cross ratio

$$\frac{(w_1 - w_2)(w_3 - w_4)}{(w_1 - w_4)(w_3 - w_2)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$$

$$\frac{(2-i)(-2-w)}{(2-w)(-2-i)} = \frac{(1-i)(-1-z)}{(1-z)(-1-i)} \Rightarrow \frac{(2-i)(2+w)}{(2-w)(2+i)} = \frac{(1-i)(1+z)}{(1-z)(1+i)}$$

By componendo and dividendo

$$\frac{(2-i)(2+w)+(2-w)(2+i)}{(2-i)(2+w)-(2-w)(2+i)} = \frac{(1-i)(1+z)+(1-z)(1+i)}{(1-i)(1+z)-(1-z)(1+i)}$$

$$\frac{2w+4-2i-iw+4+2i-2w-wi}{4+2w-2i-iw-4-2i+2w+iw} = \frac{1+z-i-iz+1+i-z-zi}{1+z-i-iz-1-i+z+iz}$$

$$\frac{8-2iw}{4w-4i} = \frac{2-2iz}{2z-2i} \Rightarrow \frac{4-iw}{2w-2i} = \frac{1-iz}{z-i}$$

$$\Rightarrow 4z-4i-iwz-w = 2w-2iwz-2i-2z$$

$$\Rightarrow 6z-2i+iwz-3w=0$$

$$w(iz-3) = 2i-6z$$

$$\Rightarrow w = \frac{2i-6z}{iz-3}$$ By comparing with $\frac{az+b}{cz+d}$

$a=-6;\ b=2i,\ c=i,\ d=-3$

$ad-bc = (-6)(-3)-(2i)(i) = 18+2 = 20 \neq 0$

No critical points

Now $cz^2 + z(d-a) - b = 0$

$iz^2 + z(-3+6) - 2i = 0 \Rightarrow iz^2 + 3z - 2i = 0$

$a = i,\ b = 3,\ c = -2i$

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{9 - 4i(-2i)}}{2i}$$

$$z = \frac{-3 \pm \sqrt{9-8}}{2i} = \frac{-3 \pm 1}{2i}$$

$z = \frac{-3+1}{2i}$

$= -\frac{2}{2i} =$

$= -\frac{1}{i} = \frac{i^2}{i}$

$\boxed{z = i}$

$z = \frac{-3-1}{2i}$

$= -\frac{4}{2i} = -\frac{2}{i} = \frac{2i^2}{i}$

$\boxed{z = 2i}$

$\therefore$ The fixed points are $i, 2i$

(3) Find the Bilinear transformation which maps the points $z = 1, i, -1$ into the points $w = 0, 1, \infty$

Sol: Let the given points are $z_1 = 1,\ z_2 = i,\ z_3 = -1\ \ z_4 = z$

$w_1 = 0,\ w_2 = 1,\ w_3 = \infty,\ w_4 = w$

By using cross Ratio $\frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)} = \frac{(w_1 - w_2)(w_3 - w_4)}{(w_1 - w_4)(w_3 - w_2)}$

$$\frac{(1-i)(-1-z)}{(1-z)(-1-i)} = \frac{(0-1)(1-\infty)}{(0-w)(\infty-1)} \Rightarrow \frac{1}{w} = \frac{(1-i)(z+1)}{(1+i)(1-z)}$$

$$\Rightarrow w = \frac{(1-z)(1+i)}{(1-i)(z+1)} = \frac{i(1-z)(1+i)}{(i-i^2)(1+z)} = \frac{i\cancel{(1+i)}(1-z)}{\cancel{(1+i)}(1+z)}$$

$$\Rightarrow w = \frac{i(1-z)}{(1+z)}$$

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(4) S.T. the transformation $w = \frac{i(1-z)}{(1+z)}$ maps the circle $|z|=1$ into the real axis of the w-plane and the interior of the circle $|z|<1$ into the upper half of the w plane.

Sol:- Given transformation $w = \frac{i(1-z)}{(1+z)}$

$$\Rightarrow w + wz = i - iz$$

$$\Rightarrow z(w+i) = i - w$$

$$\Rightarrow z = \frac{i-w}{w+i}$$

$$|z| = \left|\frac{i-w}{i+w}\right| = 1$$

$$\Rightarrow |i-w| = |i+w|$$

$$\Rightarrow |i-u-iv| = |i+u+iv|$$

$$\Rightarrow |i(1-v)-u| = |u+i(1+v)|$$

$$\Rightarrow u^2 + (1-v)^2 = u^2 + (1+v)^2$$

$$\Rightarrow \cancel{u^2} + \cancel{1} + \cancel{v^2} - 2v = \cancel{u^2} + \cancel{1} + \cancel{v^2} + 2v$$

$$4v = 0 \Rightarrow v = 0$$

$\therefore$ If $v=0$ it maps the real axis of the w-plane

$|z|<1$

$$\Rightarrow |z| = \left|\frac{i-w}{i+w}\right| < 1 \Rightarrow |i-w| < |i+w|$$

$$\Rightarrow |i-u-iv| < |i+u+iv| \Rightarrow |i(1-v)-u| < |u+i(1+v)|$$

$$\Rightarrow \cancel{u^2} + \cancel{1} + \cancel{v^2} - 2v < \cancel{u^2} + \cancel{1} + \cancel{v^2} + 2v$$

$$\Rightarrow 4v > 0 \Rightarrow v > 0$$

for $|z| < 1$, $v > 0$ it maps into upper half of the w-plane.

5) S.T. the bilinear transformation $w = \frac{2z+3}{z-4}$ maps the circle $x^2+y^2-4x=0$ into the line $4u+3=0$

Sol:- Given Bilinear transformation $w = \frac{2z+3}{z-4} \rightarrow (1)$

Given circle is $x^2+y^2-4x=0$

$$(x-2)^2+y^2=4$$

$$\Rightarrow |(x-2)+iy|^2 = 2^2$$

$$\text{i.e } |z-2|=2 \longrightarrow (2)$$

Given $w = \frac{2z+3}{z-4}$

$$w(z-4) = 2z+3 \Rightarrow z = \frac{4w+3}{w-2}$$

$$(2) \Rightarrow |z-2|=2 \Rightarrow \left|\frac{4w+3}{w-2}-2\right| = 2$$

$$\Rightarrow \left|\frac{4w+3-2w+4}{w-2}\right| = 2$$

$$\Rightarrow |2w+7| = 2|w-2|$$

$$\Rightarrow |7+2u+i2v| = 2|u+iv-2|$$

$$\Rightarrow (7+2u)^2+4v^2 = 4((u-2)^2+v^2)$$

$$\Rightarrow 49+28u+\cancel{4u^2}+\cancel{4v^2} = \cancel{4u^2}+16-16u+\cancel{4v^2}$$

$$\Rightarrow 49+28u+16u-16=0$$

$$\Rightarrow 44u+33=0$$

$$\Rightarrow 4u+3=0$$

(6) S.T. the Condition for transformation $w = \frac{az+b}{cz+d}$ to make the circle $|w|=1$ correspond to a straight line in the z-plane $|a|=|c|$

Sol:- Given circle $|w|=1$

$$\Rightarrow \left|\frac{az+b}{cz+d}\right| = 1$$

$$|az+b| = |cz+d|$$

$$\Rightarrow |a(x+iy)+b| = |c(x+iy)+d|$$

$$\Rightarrow |ax+aiy+b| = |cx+ciy+d|$$

$$\Rightarrow |a| \left|x+iy+\frac{b}{a}\right| = |c| \left|x+iy+d/c\right|$$

$$\Rightarrow \left|\frac{a}{c}\right| \left|\left(x+\frac{b}{a}\right)+iy\right| = \left|\left(x+\frac{d}{c}\right)+iy\right|$$

$$\Rightarrow \left|\frac{a}{c}\right|^2 \left(x^2+\left(\frac{b}{a}\right)^2+\frac{2bx}{a}+y^2\right) = x^2+\left(\frac{d}{c}\right)^2+\frac{2xd}{c}+y^2$$

$$\Rightarrow \left(\frac{a}{c}\right)^2 \left(x^2+y^2+\frac{2bx}{a}+\left(\frac{b}{a}\right)^2\right) = x^2+y^2+\frac{2xd}{c}+\frac{d^2}{c^2}$$

$$\Rightarrow x^2+y^2\left(\left(\frac{a}{c}\right)^2-1\right)+\left(\frac{2ba}{c^2}-\frac{2d}{c}\right)x+\left(\frac{b^2}{c^2}-\frac{d^2}{c^2}\right)=0$$

This equation reduces to the equation of the form $c_1x+c_2=0$ when provided $|a|=|c|$

Hence $|w|=1$ is transform to a Stright line if $|a|=|c|$

Residues

In the Laurent series expansion the coefficients of -ve powers of $(z-a)$ of $f(z)$ is said to be Residues

$$\text{Res } f(a) = \frac{1}{2\pi i}\int_C f(z)\,dz$$

$$\int_C f(z)\,dz = 2\pi i \ \text{Res} f(a)$$

Residue Theorem:

Statement: If $f(z)$ is analytic in a closed curve c except at a finite no of singular points within c then

$\int_C f(z)\,dz = 2\pi i$ (Sum of the residues at the singular pt within c)

Proof:

Let us consider the singular points $a_1, a_2, a_3, \ldots a_n$ by a small circles $c_1, c_2, c_3, \ldots c_n$.

These circles together with 'c' form an multiply connected region in $f(z)$ is analytic.

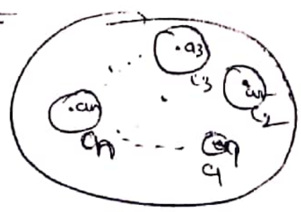
By extension of Cauchy's theorem

$$\int_C f(z)dz = \int_{c_1} f(z)dz + \int_{c_2} f(z)dz + \cdots + \int_{c_n} f(z)\,dz$$

$$= 2\pi i \text{ Res} f(a_1) + 2\pi i \text{ Res} f(a_2) + \cdots + 2\pi i \text{ Res} f(a_n)$$

$$\Rightarrow \int_C f(z)dz = 2\pi i\left(\text{Res} f(a_1) + \text{Res} f(a_2) + \cdots + \text{Res} f(a_n)\right)$$

$\int_C f(z)\,dz = 2\pi i$ (Sum of the residues at the singular pt within c)



Calculation of Residues:

i) If $f(z)$ has a simple pole at $z=a$ then

$$\text{Res } f(a) = \underset{z\to a}{Lt}\ (z-a)f(z)$$

ii) If $f(z)$ has a pole of order n at $z=a$ then

$$\text{Res } f(a) = \underset{z\to a}{Lt}\ \frac{1}{(n-1)!}\ \frac{d^{n-1}}{dz^{n-1}}\ (z-a)^n f(z)$$

Problems

* 1) Determine the poles of the function $f(z) = \dfrac{z^2}{(z-1)^2(z+2)}$ and residue at each pole.

Hence Evaluate $\oint_c f(z)dz$, where c is the cle $|z|=2.5$

Sol: Given $f(z) = \dfrac{z^2}{(z-1)^2(z+2)}$

The function has simple pole $z=-2$

$$\text{Res } f(-2) = \underset{z\to -2}{Lt}\ \cancel{(z+2)}\ \frac{z^2}{(z-1)^2\cancel{(z+2)}} = \frac{4}{9}$$

And the fun has a pole of order 2 at $z=1$

$$\text{Res } f(1) = \underset{z\to 1}{Lt}\ \frac{1}{(2-1)!}\ \frac{d^{2-1}}{dz^{2-1}}\ \cancel{(z-1)^2}\ \frac{z^2}{\cancel{(z-1)^2}(z+2)}$$

$$= \underset{z\to 1}{Lt}\ \frac{d}{dz}\left(\frac{z^2}{z+2}\right) = \underset{z\to 1}{Lt}\ \frac{z^2+4z}{(z+2)^2} = \frac{5}{9}$$

Clearly $f(z)$ is analytic on $|z| = 2.5$ at all points inside except at $z=-2$ and $z=1$

By Residue theorem.

$$\oint_c f(z)\,dz = 2\pi i\left[\text{Res } f(-2) + \text{Res } f(1)\right]$$

$$= 2\pi i\left(\frac{4}{9} + \frac{5}{9}\right) = 2\pi i$$

2) Evaluate the integral $\int_C \frac{3z^2+z+1}{(z^2-1)(z+3)} dz$, $C: |z|=2$

Sol: $f(z) = \frac{3z^2+z+1}{(z^2-1)(z+3)}$

$= \frac{3z^2+z+1}{(z+1)(z-1)(z+3)}$

Here the poles are $-1, 1, -3$

The poles $-1, 1$ are inside the circle $|z|=2$

The pole -3 is outside the circle $|z|=2$

$$\int_C \frac{3z^2+z+1}{(z^2-1)(z+3)} dz = 2\pi i \left[\operatorname{Res} f(1) + \operatorname{Res} f(-1) \right]$$

$$\operatorname{Res} f(1) = \lim_{z\to 1} (z-1) \frac{3z^2+z+1}{(z+1)(z-1)(z+3)} = \frac{3+1+1}{2\times 4} = 5/8$$

$$\operatorname{Res} f(-1) = \lim_{z\to -1} (z+1) \frac{3z^2+z+1}{(z+1)(z-1)(z+3)} = \frac{3-1+1}{-2\times 2} = -3/4$$

$$\Rightarrow \int_C \frac{3z^2+z+1}{(z^2-1)(z+3)} dz = 2\pi i \left[5/8 - 3/4 \right] = 2\pi i \left[-1/8 \right]$$

$$= -\pi i/4$$

3) Evaluate $\int_C \frac{2z+1}{(2z-1)^2} dz$, where c is $|z|=1$

Sol: Here $f(z) = \frac{2z+1}{(2z-1)^2} = \frac{2z+1}{4(z-1/2)^2}$

Here the pole is $1/2$ of order 2 inside the circle $|z|=1$

$$\int_C \frac{2z+1}{(2z-1)^2} dz = 2\pi i \operatorname{Res} f(1/2)$$

$$\operatorname{Res} f(1/2) = \lim_{z\to 1/2} \frac{1}{(2-1)!} \frac{d^{2-1}}{dz^{2-1}} (z-1/2)^2 \cdot \frac{2z+1}{4(z-1/2)^2} = \frac{2}{4} = 1/2$$

$$\Rightarrow \int_C \frac{2z+1}{(2z-1)^2} dz = 2\pi i (1/2)$$

$$= \pi i$$

(4) Evaluate $\int_C \frac{z+4}{z^2+2z+5} dz$ where c is $|z+1-i|=2$

Sol: $f(z) = \int \frac{z+4}{z^2+2z+5} dz$

Consider z^2+2z+5

$$z = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

Here the poles are $-1+2i, -1-2i$

The pole $-1+2i$ inside the cle $|z+1-i|=2$

The pole $-1-2i$ outside the cle $|z+1-i|=2$

$$\int_C \frac{z+4}{z^2+2z+5} dz = 2\pi i \; Res(-1+2i)$$

$$Res(-1+2i) = \lim_{z \to -1+2i} (z+1-2i) \frac{z+4}{(z+1-2i)(z+1+2i)}$$

$$= \frac{-1+2i+4}{-1+2i+1+2i} = \frac{2i+3}{4i}$$

$$\Rightarrow \int_C \frac{z+4}{z^2+2z+5} dz = 2\pi i \left(\frac{2i+3}{4i}\right) = \frac{4\pi i + 6\pi}{4} = \pi i + \frac{3\pi}{2}$$

*5) Evaluate $\int_C \frac{z\,dz}{(z-1)(z-2)^2}$; $c: |z-2| = 1/2$

Sol: Here the poles are 1, 2

The pole 1 is a simple pole outside the cle $|z-2| = 1/2$

The pole 2 is of order 2 inside the cle. $|z-2| = 1/2$

$$\therefore \int_C \frac{z\,dz}{(z-1)(z-2)^2} = 2\pi i \; Res\, f(2)$$

$$Res\, f(2) = \lim_{z \to 2} \frac{d}{dz} (z-2)^2 \frac{z}{(z-1)(z-2)^2}$$

$$= \lim_{z \to 2} \left(\frac{z-1-z}{(z-1)^2}\right) = -1$$

$$\Rightarrow \int_C \frac{z}{(z-1)(z-2)^2} dz = 2\pi i \times -1 = -2\pi i$$

(6.) Evaluate $\int_C \frac{3z^2+2}{(z-1)(z^2+9)}\,dz$, $C: |z-2|=2$ *

Sol: Given $f(z) = \frac{3z^2+2}{(z-1)(z^2+9)} = \frac{3z^2+2}{(z-1)(z+3i)(z-3i)}$

Here the poles are $1, 3i, -3i$

The pole 1 lies inside the cle $|z-2|=2$

The poles $3i, -3i$ are lies outside the circle $|z-2|=2$

$$\int_C \frac{3z^2+2}{(z-1)(z^2+9)}\,dz = 2\pi i \ \text{Res}\, f(1)$$

$$\text{Res}\, f(1) = \underset{z\to 1}{\text{Lt}}\ \cancel{(z-1)}\ \frac{3z^2+2}{\cancel{(z-1)}(z+3i)(z-3i)}$$

$$= \frac{3+2}{1+9} = \frac{5}{10} = \frac{1}{2}$$

$$\Rightarrow \int_C \frac{3z^2+2}{(z-1)(z^2+9)}\,dz = 2\pi i\left(\frac{1}{2}\right) = \pi i$$

*(7) Evaluate $\int_C \frac{dz}{(z^2+4)^2}$; $C: |z-i|=2$

Sol: Given $f(z) = \frac{1}{(z^2+4)^2} = \frac{1}{(z+2i)^2(z-2i)^2}$

Here the poles are $2i, -2i$ are of order 2

The pole $2i$ lies inside the cle $|z-i|=2$

and the pole $-2i$ lies outside of the cle $|z-i|=2$

$$\int_C \frac{dz}{(z^2+4)^2} = 2\pi i \ \text{Res}\, f(2i)$$

$$\text{Res}\, f(2i) = \underset{z\to 2i}{\text{Lt}}\ \frac{d}{dz}\cancel{(z-2i)^2}\ \frac{1}{(z+2i)^2\cancel{(z-2i)^2}} = \frac{-2}{(4i)^3} = \frac{-2}{-64i} = \frac{1}{32i} = \frac{-i}{32}$$

$$\Rightarrow \int_C \frac{dz}{(z^2+4)^2} = 2\pi i\left(-\frac{i}{32}\right) = \frac{2\pi}{32} = \frac{\pi}{16}$$

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(8) Evaluate $\int_C \frac{\sin^2 z}{(z-\pi/6)^2} dz$, $C: |z|=2$

Sol: Here the pole is $\pi/6$ of order 2

The pole lies inside the circle $|z|=2$

$$\int_C \frac{\sin^2 z}{(z-\pi/6)^2} dz = 2\pi i \; \text{Res} f(\pi/6)$$

$$\text{Res} f(\pi/6) = \lim_{z\to\pi/6} \frac{d}{dz} \cancel{(z-\pi/6)^2} \frac{\sin^2 z}{\cancel{(z-\pi/6)^2}} = \lim_{z\to\pi/6} 2\sin z\cos z$$

$$= [\sin 2z]_{z=\pi/6} = \sin \pi/3 = \sqrt{3}/2$$

$$\Rightarrow \int_C \frac{\sin^2 z}{(z-\pi/6)^2} dz = \cancel{2}\pi i \; \sqrt{3}/\cancel{2} = \sqrt{3}\pi i$$

(9) $\int_C \frac{z \sec z}{(1-z)^2} dz$, $C: |z|=3$

Sol: Here the pole is 1 of order 2

The pole lies inside the circle $|z|=3$

$$\int_C \frac{z\sec z}{(1-z)^2} dz = 2\pi i \; \text{Res} f(1)$$

$$\text{Res} f(1) = \lim_{z\to 1} \frac{d}{dz} \cancel{(z-1)^2} \frac{z\sec z}{\cancel{(z-1)^2}}$$

$$= [z \tan z \sec z + \sec z]_{\text{at } z=1} = \tan 1 \sec 1 + \sec 1$$

$$= \sec 1 \, [1 + \tan 1]$$

$$\Rightarrow \int_C \frac{z\sec z}{(1-z)^2} dz = 2\pi i \, [\sec 1 (1+\tan 1)]$$

(10) Evaluate $\int_C \frac{1-2z}{z(z-1)(z-2)} dz$, $C: |z| = 1.5$

Sol:

$$f(z) = \frac{1-2z}{z(z-1)(z-2)}$$

Here the poles are $0, 1, 2$

$z = 0, 1$ are lies inside the o^{le} $|z| = 1.5$

$z = 2$ is lies outside the o^{le} $|z| = 1.5$

$$\int_C \frac{1-2z}{z(z-1)(z-2)} dz = 2\pi i \left[\operatorname{Res} f(0) + \operatorname{Res} f(1)\right]$$

$$\operatorname{Res} f(0) = \underset{z \to 0}{Lt} \cancel{(z-0)} \frac{1-2z}{\cancel{z}(z-1)(z-2)} = \frac{1}{2}$$

$$\operatorname{Res} f(1) = \underset{z \to 1}{Lt} \cancel{(z-1)} \frac{1-2z}{z\cancel{(z-1)}(z-2)} = 1$$

$$\Rightarrow \int_C \frac{1-2z}{z(z-1)(z-2)} dz = 2\pi i \left(\frac{1}{2} + 1\right) = 3\pi i$$

*

(11) Evaluate $\int_C \frac{4z^2 - 4z + 1}{(z-2)(z^2+4)} dz$, $C: |z| = 1$

Sol: $f(z) = \frac{4z^2 - 4z + 1}{(z-2)(z+2i)(z-2i)}$

Here the poles are $2, -2i, 2i$

All the poles are outside the o^{le} $|z| = 1$

By Cauchys theorem

$$\int_C \frac{4z^2 - 4z + 1}{(z-2)(z^2+4)} dz = 0.$$

(12) Evaluate $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2 (z-2)} dz$, where C is the Circle $|z| = 3$
*

Sol:- $f(z) = \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2 (z-2)}$

Here the poles are 1, 2

Here the pole 1 is of order 2 lies inside the cle $|z| = 3$

And the pole 2 also lies inside the cle $|z| = 3$

$$\text{Res} f(1) = \underset{z \to 1}{Lt} \frac{d}{dz} \cancel{(z-1)^2} \frac{\sin \pi z^2 + \cos \pi z^2}{\cancel{(z-1)^2} (z-2)}$$

$$= \left[\frac{(z-2)\left[2\pi z \cos \pi z^2 - 2\pi z \sin \pi z^2\right] - \left[\sin \pi z^2 + \cos \pi z^2\right]}{(z-2)} \right]_{z=1}$$

$$= (-1)(-2\pi) - (-1) = 2\pi + 1$$

$$\text{Res} f(2) = \underset{z \to 2}{Lt} \cancel{(z-2)} \frac{\sin \pi z^2 + \cos \pi z^2}{\cancel{(z-2)}(z-1)^2} = 1$$

$$\Rightarrow \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2 (z-2)} dz = 2\pi i \left[\text{Res} f(1) + \text{Res} f(2)\right]$$

$$= 2\pi i \left[(2\pi + 1) + 1\right]$$

$$= 2\pi i \left[2\pi + 2\right]$$

$$= 4\pi^2 i + 4\pi i$$

$$= 4\pi i(\pi + 1)$$

13) Evaluate $\int_C \frac{dz}{(z^2+1)(z^2-4)}$, $C: |z| = 1.5$

Sol: Consider $f(z) = \frac{1}{(z+i)(z-i)(z+2)(z-2)}$

Here the poles are $-i, i, -2, 2$

The poles $-i, i$ are lies inside the circle $|z| = 1.5$

The poles $-2, 2$ " " outside the " "

$$\int_C \frac{dz}{(z^2+1)(z^2-4)} = 2\pi i \left[\text{Res} f(-i) + \text{Res} f(i)\right]$$

$$\text{Res } f(i) = \underset{z\to i}{Lt} \; \cancel{(z-i)} \frac{1}{(z+i)\cancel{(z-i)}(z+2)(z-2)} = \frac{1}{-5(2i)} = -\frac{1}{10i}$$

$$\text{Res } f(-i) = \underset{z\to -i}{Lt} \; \cancel{(z+i)} \frac{1}{\cancel{(z+i)}(z-i)(z+2)(z-2)} = \frac{1}{-5(-2i)} = \frac{1}{10i}$$

$$\Rightarrow \int_C \frac{dz}{(z^2+1)(z^2-4)} = \frac{1}{10i} - \frac{1}{10i} = 0$$

14) Evaluate $\int_C \frac{z \cos z}{(z-\pi/2)^3} dz$, $C: |z-1| = 1$

Sol: Consider $f(z) = \frac{z \cos z}{(z-\pi/2)^3}$

Here the pole is $\pi/2$ is of order 3

$$\int_C \frac{z \cos z}{(z-\pi/2)^3} dz = 2\pi i \; \text{Res} f(\pi/2)$$

$$\text{Res} f(\pi/2) = \underset{z\to \pi/2}{Lt} \frac{1}{(3-1)!} \frac{d^2}{dz^2} \cancel{(z-\pi/2)^3} \frac{z \cos z}{\cancel{(z-\pi/2)^3}}$$

$$= \frac{1}{2} \frac{d}{dz}\left[-z \sin z + \cos z\right]_{z=\pi/2} = \frac{1}{2}\left[-z \cos z - \sin z - \sin z\right]_{z=\pi/2}$$

$$= \frac{1}{2}\left[-z \cos z - 2 \sin z\right]_{z=\pi/2} = \frac{1}{2}(-2) = -1$$

$$\Rightarrow \int_C \frac{z \cos z}{(z-\pi/2)^3} dz = -2\pi i$$

Evaluation of Real Definite integrals:

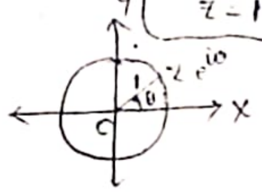
I) Integration around the unit circle

An integral of the type $\int_0^{2\pi} f(\sin\theta, \cos\theta)\, d\theta$, where the integral is a rational function of $\sin\theta$ and $\cos\theta$ can be evaluated by writing $e^{i\theta} = z$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1}{2}\left[z + \frac{1}{z}\right]$$

$\because z = ~~re^{i\theta}~~\ re^{i\theta}$
$z = 1 \cdot e^{i\theta} \cdot e^{i\theta}$
$|z| = 1$



Then the integral takes the form $\int_C f(z)\, dz$ where $f(z)$ is a rational function of z and c is a unit circle $|z| = 1$

Prob (1):-

Show that $\displaystyle\int_0^{2\pi} \frac{\cos 2\theta\, d\theta}{1 - 2a\cos\theta + a^2} = \frac{2\pi a^2}{1 - a^2}$, $(a^2 < 1)$

Sol:- Let us consider $z = e^{i\theta}$

Applying 'log' on b.s.

$\log z = \log e^{i\theta} \Rightarrow \log z = i\theta$

By diff.

$$\frac{1}{z}\, dz = i\, d\theta$$

$$\Rightarrow d\theta = \frac{1}{zi}\, dz$$

$\because \cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

$\Rightarrow \cos\theta = \frac{1}{2}\left[z + \frac{1}{z}\right]$

$\cos 2\theta = \frac{1}{2}\left[z^2 + \frac{1}{z^2}\right]$

Now $$\int_C \frac{\cos 2\theta}{1 - 2a\cos\theta + a^2}\, d\theta = \int_C \frac{\frac{1}{2}\left[z^2 + \frac{1}{z^2}\right]}{1 - 2a\, \frac{1}{2}\left(z + \frac{1}{z}\right) + a^2}\, \frac{1}{zi}\, dz$$

$$= \frac{1}{2} \int_C \frac{z^2 + \frac{1}{z^2}}{1 - az - \frac{a}{z} + a^2}\, \frac{dz}{iz}$$

$$= \frac{1}{2}\int_C \frac{\frac{z^4+1}{z^2}}{\frac{z - az^2 - a + a^2 z}{z}} \frac{dz}{iz}$$

$$= \frac{1}{2i}\int_C \frac{z^4+1}{z^2(z-az^2-a+a^2z)} dz = \frac{1}{2i}\int_C \frac{z^4+1}{z^2(z(1-az)-a(1-az))} dz$$

$$= \frac{1}{2i}\int_C \frac{z^4+1}{z^2(1-az)(z-a)} dz$$

$$= \frac{1}{2i}\int_C \frac{-(z^4+1)}{az^2(z-1/a)(z-a)} dz$$

Here the poles are $z = 0, \frac{1}{a}, a$

The poles $z = 1/a$ and 'a' are Simple poles

And the pole $z=0$ is of order 2

Hence the poles $z=0$ and 'a' are lies inside the circle $|z|=1$

$$\text{Res } f(a) = \underset{z\to a}{Lt} \cancel{(z-a)} \frac{-(z^4+1)}{2iaz^2(z-1/a)\cancel{(z-a)}}$$

$$= -\frac{1}{2ia} \frac{a^4+1}{a^2\left(\frac{a^2-1}{a}\right)} = -\frac{1}{2ia^2}\frac{a^4+1}{(a^2-1)} = \frac{-(a^4+1)}{2ia^2(a^2-1)}$$

$$= \frac{a^4+1}{2ia^2(1-a^2)}$$

$$\text{Res } f(0) = \underset{z\to 0}{Lt} (z-0)^2 \frac{d}{dz} \frac{-(z^4+1)}{2ia\cancel{z^2}\cdot(z-1/a)(z-a)}$$

$$= \left[\frac{-1}{2ia}\left[\frac{d}{dz} \frac{z^4+1}{z^2-az-\frac{z}{a}+1}\right]\right]_{\text{at } z=0}$$

$$= \left[\frac{-1}{2ia} \frac{(z^2-az-\frac{z}{a}+1)4z^3 - (z^4+1)(2z-a-\frac{1}{a})}{(z^2-az-\frac{z}{a}+1)^2}\right]_{z=0}$$

$$= -\frac{1}{2ia}\left(\frac{0-\frac{[-a^2-1]}{a}}{1}\right) = -\frac{1}{2ia}\left[\frac{a^2+1}{a}\right]$$

$$\text{Res}\, f(0) = -\frac{(a^2+1)}{2ia^2}$$

By using Residue theorem

$$\int_0^{2\pi} \frac{\cos 2\theta}{1-2a\cos\theta+a^2}\, d\theta = 2\pi i\left[\text{Res}\, f(a) + \text{Res}\, f(0)\right]$$

$$= 2\pi i\left[\frac{(a^4+1)}{2ia^2(1-a^2)} - \frac{(a^2+1)}{2ia^2}\right]$$

$$= \frac{\cancel{2}\pi i}{\cancel{2}ia^2}\left[\frac{a^4+1}{(1-a^2)} - (a^2+1)\right]$$

$$= \frac{\pi \cancel{i}}{\cancel{i}a^2}\left[\frac{a^4+1-(1-a^2)(1+a^2)}{(1-a^2)}\right]$$

$$= \frac{\pi}{a^2}\left[\frac{a^4+\cancel{1}-\cancel{1}+a^4}{(1-a^2)}\right]$$

$$= \frac{2\pi a^4}{a^2(1-a^2)} = \frac{2\pi a^2}{1-a^2}$$

$$\Rightarrow \int_0^{2\pi} \frac{\cos 2\theta}{1-2a\cos\theta+a^2}\, d\theta = \frac{2\pi a^2}{1-a^2}$$

(2) By integrating around a unit circle, evaluate

$$\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\, d\theta .$$

Sol:- Consider $z=e^{i\theta}$,

$\log z = i\theta$

$\frac{1}{z}\,dz = i\,d\theta \Rightarrow d\theta = \frac{1}{iz}\,dz$

$\cos\theta = \frac{1}{2}\left[z+\frac{1}{z}\right]$

$\cos 3\theta = \frac{1}{2}\left(z^3+\frac{1}{z^3}\right)$

$$\int_C \frac{\cos 3\theta}{5-4\cos\theta}\,d\theta = \int \frac{\frac{1}{2}\left(z^3+\frac{1}{z^3}\right)}{5-4\cdot\frac{1}{2}\left(z+\frac{1}{z}\right)}\,\frac{1}{iz}\,dz$$

$$= \frac{1}{2i}\int_C \frac{\frac{z^6+1}{z^3}}{\frac{5z-2z^2-2}{z}}\,\frac{1}{z}\,dz = \frac{1}{2i}\int_C \frac{z^6+1}{z^3(5z-2z^2-2)}\,dz$$

$$= \frac{1}{2i}\int_C \frac{z^6+1}{-[z^3(z-2)(2z-1)]}\,dz = -\frac{1}{2i}\int_C \frac{z^6+1}{2z^3\cdot(z-2)(z-1/2)}\,dz$$

$$= -\frac{1}{4i}\int_C \frac{z^6+1}{z^3(z-2)(z-1/2)}\,dz$$

Here the poles are $0, 2, 1/2$

Here $z=0$ is the pole of older 3 ($z=1/2$) lies inside the c^le $|z|=1$

and $z=2$. is the pole lies outside the circle $|z|=1$

$$\text{Res} f(0) = \lim_{z\to 0} \frac{1}{2!}\frac{d^2}{dz^2}(z-0)^3\left(-\frac{1}{4i}\cdot\frac{z^6+1}{z^3(z-2)(z-1/2)}\right)$$

$$= \left[-\frac{1}{8i}\frac{d^2}{dz^2}\frac{(z^6+1)}{\frac{(2z^2-5z+2)}{2}}\right]_{\text{at } z=0} = \left[-\frac{1}{4i}\frac{d^2}{dz^2}\frac{z^6+1}{(2z^2-5z+2)}\right]_{z=0}$$

$$= \left[-\frac{1}{4i}\frac{d}{dz}\frac{(2z^2-5z+2)(6z^5)-(z^6+1)(4z-5)}{(2z^2-5z+2)^2}\right]_{\text{at } z=0}$$

$$= \left[-\frac{1}{4i}\frac{d}{dz}\frac{12z^7-30z^6+12z^5-4z^7+5z^6-4z+5}{(2z^2-5z+2)^2}\right]_{z=0}$$

$$= \left[-\frac{1}{4i}\frac{d}{dz}\frac{8z^7-25z^6+12z^5-4z+5}{(2z^2-5z+2)^2}\right]_{z=0}$$

$$= -\frac{1}{4i}\left[\frac{(2z^2-5z+2)^2(56z^6-150z^5-60z^4-4)-(8z^7-25z^6+12z^5-4z+5)\,2(2z^2-5z+2)(4z-5)}{(2z^2-5z+2)^4}\right]_{z=0}$$

$$= -\frac{1}{4i}\left[\frac{4(-4)-(10)(2)(-5)}{16}\right] = \frac{-1}{4i}\left(\frac{-16+100}{16}\right)$$

$$\text{Res} f(0) = -\frac{1}{4i}\left(\frac{\cancel{84}^{21}}{\cancel{16}_4}\right) = -\frac{21}{16i}$$

$$\text{Res} f(1/2) = \lim_{z\to 1/2}\cancel{(z-1/2)}\;-\frac{1}{4i}\frac{z^6+1}{z^3(z-2)\cancel{(z-1/2)}}$$

$$= \left[-\frac{1}{4i}\frac{(1/2)^6+1}{(1/2)^3(1/2-2)}\right] = -\frac{1}{4i}\frac{\frac{1}{64}+1}{\frac{1}{8}(-3/2)}$$

$$= -\frac{1}{4i}\frac{65/\cancel{64}_4}{-3/\cancel{16}} = -\frac{1}{4i}\frac{65}{4\times -3} = \frac{65}{48i}$$

$$\Rightarrow \int_0^{2\pi}\frac{\cos 3\theta}{5-4\cos\theta}\,d\theta = 2\pi i\left[\text{Res} f(0)+\text{Res} f(1/2)\right]$$

$$= 2\pi \cancel{i}\left(-\frac{21}{16\cancel{i}}+\frac{65}{48\cancel{i}}\right]$$

$$= 2\pi\left[\frac{-63+65}{48}\right] = \underline{\underline{\pi/12}}$$

(3) Apply the Calculus of residues, to P.T

(i) $\int_0^{2\pi} \frac{d\theta}{1-2p\sin\theta+p^2} = \frac{2\pi}{1-p^2}$ $(0<p<1)$

(ii) $\int_0^{2\pi} \frac{d\theta}{1-2r\cos\theta+r^2} = \frac{2\pi}{1-r^2}$ $(0<r<1)$

Sol: (ii) let $e^{i\theta}=z$

$dz = ie^{i\theta}d\theta \Rightarrow d\theta = \frac{dz}{ie^{i\theta}} = \frac{dz}{iz}$

$\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$

$$\int_0^{2\pi}\frac{d\theta}{1-2r\cos\theta+r^2} = \int_C \frac{1}{1-2r\frac{1}{2}\left(z+\frac{1}{z}\right)+r^2}\frac{dz}{iz} = \int_C \frac{1}{1-r\left(z+\frac{1}{z}\right)+r^2}\frac{dz}{iz}$$

$$= \int_C \frac{z}{z-r(z^2+1)+r^2z}\frac{dz}{iz} = \frac{1}{i}\int_C \frac{dz}{(z-r)(1-rz)} \rightarrow (1)$$

let $f(z) = \frac{1}{(z-r)(1-rz)} = \frac{1}{(z-r)(-r)\left(z-\frac{1}{r}\right)}$

Here the poles are $z=r, 1/r$

$C: |z|=1$

$z=r$ lies inside the ck $C:|z|=1$

$z=1/r$ lies outside the ck $C:|z|=1$, $\left(\begin{array}{l}0<r<1\\ |r|<1\end{array}\right.$

$$\operatorname{Res} f(r) = \lim_{z\to r}(z-r)\frac{1}{(z-r)(-r)(z-1/r)} = \frac{1}{(-r)\left(\frac{r^2-1}{r}\right)} = \frac{1}{1-r^2}$$

By Residue theorem

$$\int_0^{2\pi}\frac{d\theta}{1-2r\cos\theta+r^2} = 2\pi i\left[\operatorname{Res} f(r)\right] = \frac{2\pi i}{1-r^2}$$

from (1) $\frac{1}{i}\int_0^{2\pi}\frac{d\theta}{1-2r\cos\theta+r^2} = \frac{1}{i}\left(\frac{2\pi i}{1-r^2}\right) = \frac{2\pi}{1-r^2}$

~~(4)~~ (i) ~~Apply~~ P.T. $\int_0^{2\pi} \frac{d\theta}{1-2p\sin\theta+p^2} = \frac{2\pi}{1-p^2}$ $(0<p<1)$

Let $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}d\theta \Rightarrow d\theta = \frac{dz}{iz}$

$$\int_0^{2\pi} \frac{d\theta}{1-2p\sin\theta+p^2} = \int_C \frac{1}{1-2p\frac{1}{2i}(z-\frac{1}{z})+p^2}\frac{dz}{iz} = \int_C \frac{1}{\frac{iz-pz^2+p+p^2iz}{iz}}\frac{dz}{iz}$$

$$= \int_C \frac{1}{-pz^2+ip^2z+zi+p}dz = \int_C \frac{1}{(iz+p)(ipz+1)}dz \rightarrow (1)$$

Let $f(z) = \frac{1}{(iz+p)(ipz+1)}$

Here the poles are $z = \frac{-p}{i} = \frac{i^2p}{i} = ip \Rightarrow |ip| < 1$ $[\because 0<p<1$, $p^2<1^2$, $p^2<1$, $\frac{1}{p}>1$

$z = \frac{-1}{ip} = \frac{i^2}{ip} = \frac{i}{p} \Rightarrow |\frac{i}{p}| > 1$

$\therefore$ The pole pi is lies inside the c $|z|=1$

$$\text{Res } f(pi) = \lim_{z\to pi}(z-pi)\frac{1}{(iz+p)(ipz+1)} = \lim_{z\to pi}\frac{(iz+p)}{i(iz+p)(ipz+1)}$$

$$= \frac{1}{i(1-p^2)}$$

By Residue theorem

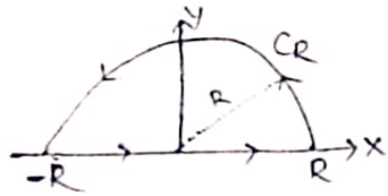
$$\int_C \frac{dz}{(iz+p)(ipz+1)} = 2\pi i\,[\text{Res } f(pi)] = 2\pi i\frac{1}{i(1-p^2)} = \frac{2\pi}{1-p^2}$$

$$\Rightarrow (1) \Rightarrow \int_0^{2\pi} \frac{d\theta}{1-2p\sin\theta+p^2} = \frac{2\pi}{1-p^2}$$

II) Integration around a Small Semi Circle:-

To evaluate $\int_{-\infty}^{\infty} f(x)dx$, we consider $\int_C f(z)dz$,

Where C is the contour consisting of the semicircle $C_R : |z| = R$, together with the diameter. That closes it



If $f(z)$ has no singular point on the real axis, By Residue Thm

$$\int_C f(z)dz = 2\pi i \; [\text{ Sum of the Residues }]$$

$$\Rightarrow \int_{C_R} f(z)dz + \int_{-R}^{R} f(x)dx = 2\pi i \left[\sum \text{Res}(f(a))\right]$$

If $\int_{C_R} f(z)dz \longrightarrow 0$ as $R \to \infty$

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \left[\sum \text{Res}\, f(ai)\right]$$

1) Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)(x^2+4)}$

Sol:- Consider $\displaystyle\int_C f(z)dz = \int_C \frac{z^2 dz}{(z^2+1)(z^2+4)} = \int_C \frac{z^2 dz}{(z+i)(z-i)(z+2i)(z-2i)}$

Where C is the small Semi circle (Contour) C_R of radius R, together with the part of the real axis from -R to R

Here the poles are $i, -i, 2i, -2i$

The poles $i, 2i$ are lies inside the circle $C : |z| = R$

By Residue theorem

$$\int_C f(z)dz = 2\pi i[\text{Res} f(i) + \text{Res} f(2i)]$$

$$\text{Res} f(i) = \lim_{z\to i} \cancel{(z-i)} \frac{z^2}{(z+i)\cancel{(z-i)}(z+2i)(z-2i)} = \frac{i^2}{(2i)(3i)(-i)}$$

$$= \frac{\cancel{i^2}}{6\cancel{i^2}} = \frac{i}{6}$$

$$\text{Res} f(2i) = \lim_{z\to 2i} \cancel{(z-2i)} \frac{z^2}{(z+i)(z-i)\cancel{(z-2i)}(z+2i)}$$

$$= \frac{-\cancel{4}}{3i(i)(\cancel{4}i)} = \frac{-1}{-3i} = \frac{i\cancel{{}^2}}{-3\cancel{i^2}} = -\frac{i}{3}$$

$$\Rightarrow \int_C f(z)dz = 2\pi i\left[\frac{i}{6} - \frac{i}{3}\right] = 2\pi i\left[\frac{i-2i}{6}\right] = \cancel{2}\pi i\left(\frac{-i}{\cancel{6}}\right) = \pi/3$$

Now $\int_C f(z)dz = \int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz \longrightarrow (1)$

Now let $R\to\infty$ as $|z|\to 0$

$$\frac{\pi}{3} = \int_{-\infty}^{\infty} f(x)dx + \int_{C_R} f(z)dz \longrightarrow (2)$$

$$f(z) = \frac{z^2}{(z^2+1)(z^2+4)} = \frac{\cancel{z^2}}{z^{\cancel{4}}(1+\frac{1}{z^2})(1+\frac{4}{z^2})} = \frac{1}{z^2(1+\frac{1}{z^2})(1+\frac{4}{z^2})}$$

$$\lim_{z\to\infty}\int f(z)\,dz = \lim_{z\to\infty}\int \frac{1}{z^2(1+\frac{1}{z^2})(1+\frac{4}{z^2})} dz = 0 \longrightarrow (3)$$

from (2) & (3)

$$\frac{\pi}{3} = \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)} dx$$

(2) Apply the calculus of Residues to P.T

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)}\,dx \qquad (a,b>0)$$

Sol: Given $f(x) = \dfrac{x^2}{(x^2+a^2)(x^2+b^2)}$

$$\Rightarrow f(z) = \frac{z^2}{(z^2+a^2)(z^2+b^2)} = \frac{z^2}{(z+ai)(z-ai)(z+bi)(z-bi)}$$

Here the poles are $ai, -ai, bi, -bi$

Here only the poles ai, bi are inside

Now $$\int_{C_R} f(z)\,dz + \int_{-R}^{R} f(x)\,dx = 2\pi i\left(\sum \text{Res}(a_i)\right) \longrightarrow (1)$$

$$\Rightarrow \int_{C_R} f(z)\,dz = \int_{C_R} \frac{z^2}{z^4\left(1+\frac{a^2}{z^2}\right)\left(1+\frac{b^2}{z^2}\right)}\,dz$$

$$= \int_{C_R} \frac{1}{z^2\left(1+\frac{a^2}{z^2}\right)\left(1+\frac{b^2}{z^2}\right)}\,dz$$

$$\Rightarrow \lim_{R\to\infty} \int_{C_R} f(z)\,dz = \lim_{R\to\infty} \int_{C_R} \frac{1}{z^2\left(1+\frac{a^2}{z^2}\right)\left(1+\frac{b^2}{z^2}\right)} = 0 \longrightarrow (2)$$

$$\text{Res } f(ai) = \lim_{z\to ai} \cancel{(z-ai)} \frac{z^2}{(z+ai)\cancel{(z-ai)}(z^2+b^2)}$$

$$= \frac{a^2 i^2}{2ai(-a^2+b^2)} = \frac{ai}{2(b^2-a^2)}$$

$$\text{Res } f(bi) = \lim_{z\to bi} \cancel{(z-bi)} \frac{z^2}{(z^2+a^2)(z+bi)\cancel{(z-bi)}} = \frac{b^2 i^2}{2bi(-b^2+a^2)}$$

$$= \frac{bi}{2(a^2-b^2)}$$

By Residue theorem

$$\int_C f(z)\,dz = 2\pi i\left(\operatorname{Res} f(ai) + \operatorname{Res} f(bi)\right)$$

$$= 2\pi i\left(\frac{ai}{2(b^2-a^2)} + \frac{bi}{2(a^2-b^2)}\right) = \frac{\cancel{2}\pi i}{\cancel{2}}\left[\frac{ai-bi}{b^2-a^2}\right]$$

$$= -\pi\left[\frac{a-b}{b^2-a^2}\right] = \pi\left[\frac{\cancel{a-b}}{(b+a)\cancel{(a-b)}}\right] = \frac{\pi}{a+b} \longrightarrow (3)$$

From equations (1), (2), (3)

$$0 + \int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{\pi}{a+b}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{\pi}{a+b}$$

3) S.T. $\int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9}\,dx = 5\pi/12$

Sol: Let $f(z) = \frac{z^2-z+2}{z^4+10z^2+9}$

Consider $z^4+10z^2+9=0$

$(z^2+1)(z^2+9)=0 \Rightarrow z = \pm i, \pm 3i$

The singular points above the real axis are $z = i, 3i$

Consider the contour 'C' consists of semi circle $C_R: |z|=R$

~~above the line~~

By Residue theorem

$$\int_C f(z)dz = 2\pi i\,[\text{Res } f(i) + \text{Res } f(3i)] \longrightarrow (1)$$

Now $$\int_{C_R} f(z)dz + \int_{-R}^{R} f(x)dx = 2\pi i\,[\text{Res } f(i) + \text{Res } f(3i)] \longrightarrow (2)$$

$$\text{Res } f(i) = \lim_{z\to i} \cancel{(z-i)} \frac{z^2-z+2}{(z+i)\cancel{(z-i)}(z+3i)(z-3i)} = \lim_{z\to i} \frac{z^2-z+2}{(z+i)(z^2+9)}$$

$$= \frac{-1-i+2}{2i(-1+9)} = \frac{1-i}{16i}$$

$$\text{Res } f(3i) = \lim_{z\to 3i} \cancel{(z-3i)} \frac{z^2-z+2}{(z^2+1)\cancel{(z-3i)}(z+3i)}$$

$$= \frac{-9-3i+2}{(-9+1)(6i)} = \frac{-7-3i}{-48i} = \frac{7+3i}{48i}$$

$$(1) \Rightarrow \int_C f(z)dz = 2\pi i\left[\frac{1-i}{16i} + \frac{7+3i}{48i}\right] = \frac{2\pi i}{i}\left[\frac{3-\cancel{3i}+7+\cancel{3i}}{48}\right]$$

$$= 2\pi\left[\frac{10}{48}\right] = \frac{20\pi}{48} = \frac{5\pi}{12} \longrightarrow (3)$$

$$\int_{C_R} f(z)dz = \int \frac{z^2\left[(1-1/z)+2/z^2\right]}{z^4(1+10/z^2+9/z^4)}\,dz$$

$$\int_{C_R} f(z)\,dz = \int_{C_R} \frac{1}{z^2} \frac{\left(1-\frac{1}{z}+2/z^2\right)}{\left(1+\frac{10}{z^2}+9/z^4\right)}\,dz \to 0 \quad \text{as } R\to\infty \longrightarrow (4)$$

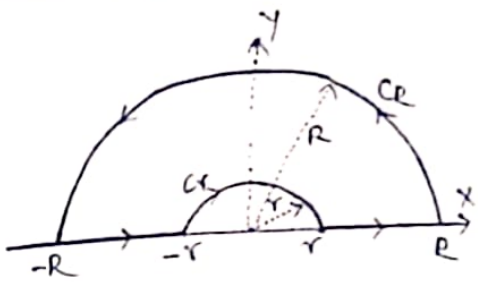
from (1) (2) (3) (4)

$$0+\int_{-\infty}^{\infty} f(x)\,dx = 5\pi/12$$

$$\Rightarrow \boxed{\int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9}\,dx = 5\pi/12}$$

Indenting the Contours having poles on the real axis

To evaluate the function $\int_{-\infty}^{\infty} f(z)dz$



Where $f(z)$ has poles on the real axis, when the integral has a simple pole on the real axis. We delete it from the region by indenting the contour i.e. by drawing a small circle having the pole for the centre

Prob:- (1)

Evaluate $\int_{0}^{\infty} \frac{\sin mx}{x} dx$, when $m>0$

Sol:- $f(x) = \frac{\sin mx}{x} \Rightarrow f(z) = \frac{\sin mz}{z}$

$$\int_{C_R} f(z)dz + \int_{r}^{R} f(x)dx + \int_{C_r} f(z)dz + \int_{-R}^{-r} f(x)dx = 0$$

→ In the real axis the limits from R to -R is 0 to π. In the main region C_R from $|z|=R$

→ In the real axis the limits from -r to r is π to 0. In the semi circle C_r from $|z|=r$

We have to P.T.

$$\int_{C_R} f(z)dz + \int_{r}^{R} f(x)dx + \int_{C_r} f(z)dz + \int_{-R}^{-r} f(x)dx = 0 \longrightarrow (1)$$

$$\int_{C_R} f(z)dz = \int_{C_R} \frac{e^{imz}}{z} dz$$

$z = Re^{i\theta}$

$dz = iRe^{i\theta}d\theta$

$$\left|\int_{C_R} f(z)\,dz\right| = \left| i\int_{C_R} e^{imR(\cos\theta + i\sin\theta)}\,d\theta \right|$$

$$= i\int_{C_R} |e^{imR\cos\theta - mR\sin\theta}|\,d\theta$$

$$\Rightarrow \int_{C_R} f(z)\,dz = 0 \xrightarrow{\text{as } R\to\infty} (2)$$

$$\int_{C_r} f(z)\,dz = \int_{C_r} \frac{e^{imz}}{z}\,dz$$

$$= \int_{C_r} \frac{e^{imre^{i\theta}}}{re^{i\theta}}\, ire^{i\theta}\,d\theta = i\int_{C_r} e^{imr(\cos\theta + i\sin\theta)}\,d\theta$$

$$= i\int_{\pi}^{0} e^{0}\,d\theta = -\pi i \xrightarrow{\text{as } r\to 0} (3)$$

from (1)

As $R\to\infty$, & $r\to 0$

$$0 + \int_{0}^{\infty} f(x)\,dx - \pi i + \int_{-\infty}^{0} f(x)\,dx = 0$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \pi i$$

$$\int_{-\infty}^{\infty} \frac{e^{imx}}{x}\,dx = \pi i$$

$$\int_{-\infty}^{\infty} \frac{\cos mx + i\sin mx}{x}\,dx = \pi i$$

$$\int_{-\infty}^{\infty} \frac{\cos mx}{x}\,dx + i\int_{-\infty}^{\infty} \frac{\sin mx}{x}\,dx = \pi i$$

By Comparing Real & imaginary parts

$$\int_{-\infty}^{\infty} \frac{\cos mx}{x}\,dx = 0, \quad \int_{0}^{\infty} \frac{\sin mx}{x}\,dx = \pi/2$$