MULTIPLE INTEGRALS

Definition: A double or triple integral is know as multiple integral is an extension of a definite integral of a function of single variable to a function of two or three variables.

Double Integrals:

Consider a Region 'R' in the xy plane bounded by one or more curves. Let $f(x,y)$ be a function defined at all points of R. Let this region capital be divided into small sub regions each of area $\delta R_1, \delta R_2, \delta R_3, ------ \delta R_n$ which are pair vise non-over lapping. Let (x_i, y_i) be a orbitary point within the subregion δR_i

Consider the sum $f(x_1, y_1)\delta R_1 + f(x_2, y_2)\delta R_2 ---$
$---- + f(x_n, y_n)\delta R_n$

If the sum tends to a finite limit as $n \to \infty$ such that the maximum $\delta R_i \to 0$ respective the point x_i, y_i, the limit is called double integral of $f(x,y)$ over the region R and it is denoted by $\iint_R f(x,y)\,dR$ (or)

$$\iint_R f(x,y)\,dx\,dy$$

→ Evaluation of Double integrals

Double integrals over a region R may be evaluated by two successive integrations as follows.

Suppose that R can be described by the inequalities of the form $a \le x \le b$, $y_1(x) \le y_2(x)$ where $y = y_1(x)$ and $y = y_2(x)$ represent the boundary of R then

$$\int_{x=a}^{b}\int_{y=y_1(x)}^{y_2(x)} f(x,y)\,dx\,dy = \int_{x=a}^{b}\left(\int_{y=y_1(x)}^{y_2(x)} f(x,y)\,dy\right)dx$$

① Evaluate $\int_0^3\int_1^2 xy(1+x+y)\,dy\,dx$

$$= \int_{x=0}^{3}\int_1^2 (xy + x^2y + xy^2)\,dy\,dx$$

$$= \int_0^3 \left[x\left(\frac{y^2}{2}\right)_1^2 + x^2\left(\frac{y^2}{2}\right)_1^2 + x\left(\frac{y^3}{3}\right)_1^2\right]dx$$

$$= \int_0^3 \left[x(2-1/2) + x^2(2-1/2) + x(8/3 - 1/3)\right]dx$$

$$= \int_0^3 \left(\frac{3x}{2} + \frac{3x^2}{2} + \frac{7x}{3}\right)dx$$

$$= \frac{3}{2}\left(\frac{x^2}{2}\right)_0^3 + \frac{3}{2}\left(\frac{x^3}{3}\right)_0^3 + \frac{7}{3}\left(\frac{x^2}{2}\right)_0^3$$

$$= \frac{3}{2}\cdot\frac{9}{2} + \frac{3}{2}\,9 + \frac{7}{3}\cdot\frac{9}{2}$$

$$= \frac{123}{4}$$

② Evaluate $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\,dy\,dx$

$$= \int_0^1 \int_{y=x}^{\sqrt{x}} (x^2+y^2)\,dy\,dx$$

$$= \int_0^1 \left(x^2 (y)_x^{\sqrt{x}} + \left(\frac{y^3}{3}\right)_x^{\sqrt{x}} \right) dx$$

$$= \int_0^1 \left(x^2(\sqrt{x}-x) + \frac{(x^{3/2}-x^3)}{3} \right) dx$$

$$= \int_0^1 \left((x^{5/2}-x^3) + \frac{1}{3}(x^{3/2}-x^3) \right) dx$$

$$= \left(\frac{x^{7/2}}{7/2}\right)_0^1 - \left(\frac{x^4}{4}\right)_0^1 + \frac{1}{3}\left(\frac{x^{5/2}}{5/2} - \frac{x^4}{4}\right)_0^1$$

$$= \frac{2}{7} - \frac{1}{4} + \frac{1}{3}\left(\frac{2}{5} - \frac{1}{4}\right)$$

$$= \frac{3}{35}$$

③ Evaluate $\int_{y=0}^{a} \int_{x=0}^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\,dx\,dy$

$$= \int_0^a \int_{x=0}^{\sqrt{a^2-y^2}} \sqrt{(a^2-y^2)-x^2}\,dx\,dy$$

$$= \int_0^a \int_{x=0}^{\sqrt{a^2-y^2}} \sqrt{(\sqrt{a^2-y^2})^2-x^2}\,dx\,dy$$

$$= \int_0^a \left(\frac{x}{2}\sqrt{a^2-y^2-x^2} + \frac{(\sqrt{a^2-y^2})^2}{2} \sin^{-1}\frac{x}{\sqrt{a^2-y^2}} \right)_0^{\sqrt{a^2-y^2}} dy$$

$$= \int_0^a \left(\frac{(\sqrt{a^2-y^2})}{2} \left(\sqrt{a^2-y^2-a^2+y^2}\right) + \frac{a^2-y^2}{2} \sin^{-1}(1) \right) dy$$

$$= \frac{\pi}{2} \int_0^a \frac{a^2-y^2}{2}\,dy$$

$$= \frac{\pi}{2}\frac{a^2}{2}(y)_0^a - \frac{\pi}{2}\frac{1}{2}\frac{(y^3)_0^a}{3}$$

$$= \frac{\pi a^3}{6}$$

(4) $$\int_{x=0}^{1}\int_{y=0}^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2}\,dy\,dx$$

$$= \int_0^1\left(\int_{y=0}^{\sqrt{1+x^2}} \frac{1}{(\sqrt{1+x^2})^2+y^2}\,dy\right)dx$$

$$= \int_0^1\int_0^{\sqrt{1+x^2}} \frac{1}{p^2+y^2}\,dy\,dx$$

$$= \int_0^1\left(\frac{1}{p}\tan^{-1}\frac{y}{p}\right)_0^{\sqrt{1+x^2}} dx$$

$$= \int_0^1\left(\frac{1}{\sqrt{1+x^2}}\tan^{-1}\frac{\sqrt{1+x^2}}{\sqrt{1+x^2}}\right)dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}}\tan^{-1}(1)\,dx$$

$$= \frac{\pi}{4}(\sinh^{-1} x)_0^1$$

$$= \frac{\pi}{4}\sinh^{-1}(1) - \frac{\pi}{4}\sinh^{-1}(0)$$

$$= \frac{\pi}{4}\sinh^{-1}(1)$$

(5) Evaluate $$\int_0^1\int_0^1 \frac{dx\,dy}{\sqrt{(1+x^2)(1-y^2)}}$$

$$\int_{y=0}^{1}\left(\int_{x=0}^{1}\frac{dx}{\sqrt{1-x^2}}\right)\frac{dy}{\sqrt{1-y^2}}$$

$$= \int_0^1 (\sin^{-1} x)_0^1 \frac{dy}{\sqrt{1-y^2}}$$

$$= \frac{\pi}{2}\int_0^1 \frac{dy}{\sqrt{1-y^2}}$$

$$= \frac{\pi^2}{4}$$

⑥ $\int_0^2\int_0^x e^{x+y}\,dy\,dx$

$= \int_0^2\int_0^x e^x e^y\,dy\,dx$

$= \int_0^2 e^x(e^x-1)\,dx$

$= \int_0^2 (e^{2x}-e^x)\,dx$

$= \left(\frac{e^{2x}}{2}\right)_0^2 - (e^x)_0^2$

$= \left[\frac{e^{2(2)}-e^{2(0)}}{2}\right] - \left[e^2-e^0\right]$

$= \left(\frac{e^4-1}{2}\right) - (e^2-1)$

$= \frac{e^4-1}{2} - e^2+1$

$= \frac{e^4-2e^2+1}{2}$

⑦ $\int_0^4\int_{y^2/4}^{y} \frac{y}{x^2+y^2}\,dx\,dy$

$= \int_0^4\left(\int_{\frac{y^2}{4}}^{y} y\,\frac{1}{x^2+y^2}\,dx\right)dy$

$= \int_0^4 y\left(\frac{1}{y}\tan^{-1}\frac{x}{y}\right)_{\frac{y^2}{4}}^{y} dy$

$= \int_0^4 y\left(\frac{1}{y}\tan^{-1}(1) - \frac{1}{y}\tan^{-1}\frac{y^2}{4y}\right)dy$

$= \int_{y=0}^4 y\left(\frac{1}{y}\frac{\pi}{4} - \frac{1}{y}\tan^{-1}\frac{y}{4}\right)dy$

$= \frac{\pi}{4}\int_0^4 dy - \int_0^4 \tan^{-1}\frac{y}{4}\,dy$

$= \pi - \left[\tan^{-1}\frac{y}{4}\int dy - \int\left(\frac{d}{dy}\left(\tan^{-1}\frac{y}{4}\right)\int dy\right)dy\right]_0^4$

$= \pi - \left[y\tan^{-1}\frac{y}{4} - \frac{1}{4}\int\frac{1}{1+\frac{y^2}{16}}\,y\,dy\right]_0^4$

$= \pi - \left[y\tan^{-1}\frac{y}{4} - \frac{16}{8}\int\frac{2}{y^2+4^2}\,y\,dy\right]_0^4$

Scanned with CamScanner

$$= \pi - \left[y\tan^{-1}\frac{y}{4} - \frac{8}{4}\log(y^2+4^2) \right]_0^4$$

$$= \pi - \left[\left[4\tan^{-1}\left(\frac{4}{4}\right) - 0 \right] - 2\left[\log(4^2+4^2) - \log(4^2) \right] \right]$$

$$= \pi - \left[4\tan^{-1}(1) - 2\left[\log(32) - \log(16) \right] \right]$$

$$= \pi - \left[4\frac{\pi}{4} - 2\left[\log 2^5 - \log 2^4 \right] \right]$$

$$= \pi - \pi + 2\left[5\log 2 - 4\log 2 \right]$$

$$= 2\log 2$$

(9) Evaluate $\iint_R y\,dx\,dy$, $\iint y^2\,dx\,dy$, where R is the region bounded by the parabolas $y^2 = 4x$ and $x^2 = 4y$

Sol:

$y^2 = 4x$

$x^2 = 4y$

$x^4 = 16y^2$

$= 16(4x)$

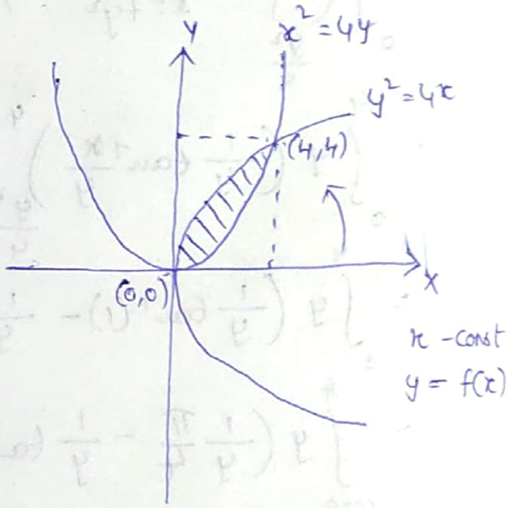
$x^4 - 64x = 0$

$x(x^3 - 64) = 0$

$x = 0 \quad x = 4$

$x = 0 \Rightarrow y = 0$

$x = 4 \Rightarrow y = 4$



$x \to 0$ to 4

$y \to x^2/4$ to $2\sqrt{x}$

The intersecting points of the given parabolas are $(0,0)$ and $(4,4)$

$\therefore$ The limits of x are 0 to 4 &

The limits of y are from $\frac{x^2}{4}$ to $2\sqrt{x}$

$$\int_{x=0}^{4}\int_{y=x^2/4}^{2\sqrt{x}} y\,dx\,dy = \int_0^4 \left(\frac{y^2}{2}\right)_{x^2/4}^{2\sqrt{x}} dx$$

$$= \int_0^4 \frac{1}{2}\left(4x - \frac{x^4}{16}\right)dx$$

$$= (x^2)_0^4 - \frac{1}{32}\left(\frac{x^5}{5}\right)_0^4$$

$$= 16 - \frac{1}{32}\,\frac{1024}{5}$$

$$= 16 - \frac{32}{5}$$

$$= 48/5 \text{ sq. units}$$

If clock wise

$y \to 0$ to 4

$x \to \frac{y^2}{4}$ to $2\sqrt{y}$

$$\Rightarrow \int_{y=0}^{4}\int_{x=y^2/4}^{2\sqrt{y}} y\,dx\,dy$$

$$= \int_0^4 y(x)_{y^2/4}^{2\sqrt{y}}\,dy$$

$$\Rightarrow \int_0^4\int_{x^2/4}^{2\sqrt{x}} y^2\,dx\,dy = \int_0^4 \left(\frac{y^3}{3}\right)_{x^2/4}^{2\sqrt{x}} dx$$

$$= \frac{1}{3}\int_0^4 \left(8x^{3/2} - \frac{x^6}{4}\right)dx$$

$$= \frac{1}{3}\left(8\,\frac{2}{5}\left(x^2\sqrt{x}\right)_0^4 - \frac{1}{4}\left(\frac{x^7}{7}\right)_0^4\right)$$

$$= \frac{1}{3}\left(\frac{16}{5}\left(16(2)\right) - \frac{1}{28}\left(4^7 - 4^0\right)\right)$$

$$= \frac{768}{35}$$

(10) Evaluate $\iint_R xy(x+y)\,dx\,dy$ over the region R bounded by $y = x^2$ and $y = x$.

Sol: The given curves are $x^2 = y$, $y = x$

$x^2 = y$

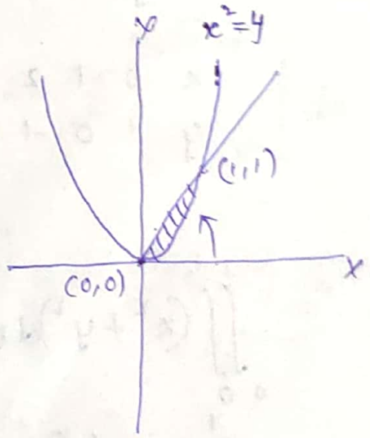
$= x$

$x^2 - x = 0$

$x(x-1) = 0$

$x = 0 \quad x = 1$

$y = 0 \quad y = 1$



$x \to 0$ to 1

$y \to x^2$ to x

$$\int_0^1\int_{x^2}^{x} xy(x+y)\,dx\,dy$$

$$= \int_0^1\int_{x^2}^{x} (x^2y + xy^2)\,dx\,dy$$

$$= \int_0^1 x^2\left(\frac{y^2}{2}\right)_{x^2}^{x} + x\left(\frac{y^3}{3}\right)_{x^2}^{x} dx$$

$$= \int_0^1 \frac{1}{2}\,x^2(x^2-x^4) + \frac{1}{3}\,x(x^3-x^6)\,dx$$

$$= \int_0^1 \frac{1}{2}(x^4-x^6) + \frac{1}{3}(x^4-x^7)\,dx$$

$$= \frac{1}{2}\left(\frac{x^5}{5}\right)_0^1 - \frac{1}{2}\left(\frac{x^7}{7}\right)_0^1 + \frac{1}{3}\left(\frac{x^5}{5}\right)_0^1 - \frac{1}{3}\left(\frac{x^8}{8}\right)_0^1$$

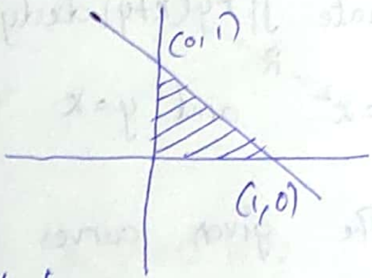
$$= \frac{3}{56}$$

(ii) Evaluate the Double integral $(x^2+y^2)\,dx\,dy$ in the positive quadrant for which $x+y=1$

Sol: $x+y=1$

$y=1-x$

x	0	1	2	3	4
y	1	0	-1	-2	-3



$x \to 0$ to 1

$y \to 0$ to $1-x$

$$\int_0^1\int_0^{1-x} (x^2+y^2)\,dx\,dy$$

$$\int_0^1 x^2(y)_0^{1-x} + \left(\frac{y^3}{3}\right)_0^{1-x} dx$$

$$\int_0^1 x^2(1-x) + \frac{1}{3}(1-x)^3\,dx$$

$$= \int_0^1 \left[\left(x^2 - x^3\right) + \frac{1}{3}(1-x)^3 \right] dx$$

$$= \frac{1}{3} - \frac{1}{4} + \frac{1}{3}\left[\frac{(1-x)^4}{-4}\right]_0^1$$

$$= \frac{1}{3} - \frac{1}{4} - \frac{1}{12}(-1) = \frac{1}{6}$$

(12) Find double integral $(x+y)^2\,dx\,dy$ over the region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Sol^n

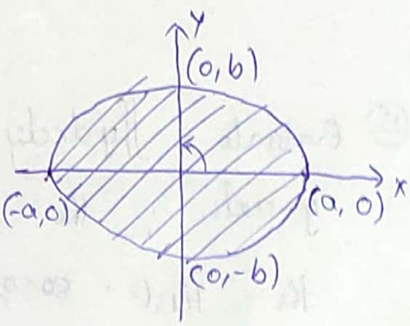
$x \longrightarrow -a$ to a

$y \longrightarrow -\frac{b}{a}\sqrt{a^2-x^2}$ to $\frac{b}{a}\sqrt{a^2-x^2}$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$y^2 = b^2\left(\frac{a^2-x^2}{a^2}\right)$$

$$y = \pm \frac{b}{a}\sqrt{a^2-x^2}$$



$$\int_{x=-a}^{a}\int_{y=-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} (x+y)^2\,dx\,dy = \int_{-a}^{a}\int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{b/a\sqrt{a^2-x^2}} (x^2+y^2+2xy)\,dx\,dy$$

$$= \int_{-a}^{a}\left[x^2(y)_{-b/a\sqrt{a^2-x^2}}^{b/a\sqrt{a^2-x^2}} + \left(\frac{y^3}{3}\right)_{-\frac{b}{a}\sqrt{a^2-x^2}}^{b/a\sqrt{a^2-x^2}} + 2x\left(\frac{y^2}{2}\right)_{-\frac{b}{a}\sqrt{a^2-x^2}}^{b/a\sqrt{a^2-x^2}} \right] dx$$

$$= \int_{-a}^{a} x^2\left(\frac{2b}{a}\sqrt{a^2-x^2}\right) + \frac{1}{3}\left(\frac{2b^3}{a^3}(a^2-x^2)^{3/2}\right) + x\underbrace{\left(\frac{b^2}{a^2}(a^2-x^2) - \frac{b^2}{a^2}(a^2-x^2)\right)}_{0} dx$$

$$= \frac{4b}{a}\int_0^a x^2\sqrt{a^2-x^2}\,dx + \frac{4b^3}{3a^3}\int_0^a (a^2-x^2)^{3/2}\,dx$$

take $x = a\cos\theta$

$dx = -a\sin\theta\,d\theta$

$x = 0 \Rightarrow \theta = \pi/2$, $x = a \Rightarrow \theta = 0$

$$= \frac{4b}{a}\int_{\pi/2}^{0} a^3\cos^2\theta\sin\theta(-a\sin\theta)\,d\theta + \frac{4b^3}{3a^3}\int_{\pi/2}^{0}(a\sin^2\theta)^{3/2}(-a\sin\theta)\,d\theta$$

$$= \frac{4b}{a}a^4\int_0^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta + \frac{4b^3}{3a^3}a^2\int_0^{\pi/2}\sin^4\theta\,d\theta$$

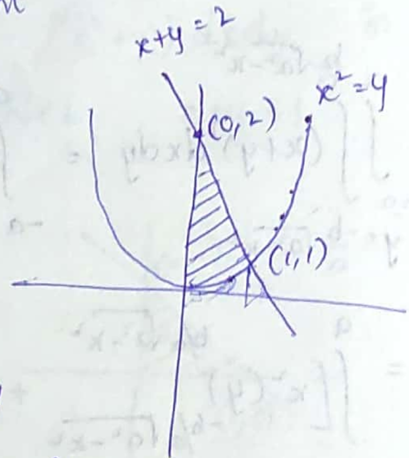
$$= 4ba^3\,\frac{3(1)(1)}{4(2)}\,\frac{\pi}{2} + \frac{4b^3}{3a}\,\frac{3}{4}\,\frac{1}{2}\,\frac{\pi}{2}$$

$$= \frac{3}{2}ba^3\frac{\pi}{2} + \frac{b^3}{a}\frac{\pi}{4}$$

$=$

(13) Evaluate $\iint y\,dx\,dy$ where R is the domain bounded by y-axis ; the curve $y = x^2$ and the line $x+y=2$ in the first ~~co-ordinate~~ quadrant

Sol:



$y = x^2$

$x+y = 2$

$y = 2-x$

x	0	1	2	3
y	2	1	0	-1

$x \to 0$ to 1

$y \to x^2$ to $2-x$

$$\int_0^1\int_{x^2}^{2-x} y\,dx\,dy$$

$$\int_0^1\left(\frac{y^2}{2}\right)_{x^2}^{2-x}dx = \int_0^1 \frac{1}{2}\left((2-x)^2-(x^2)^2\right)dx$$

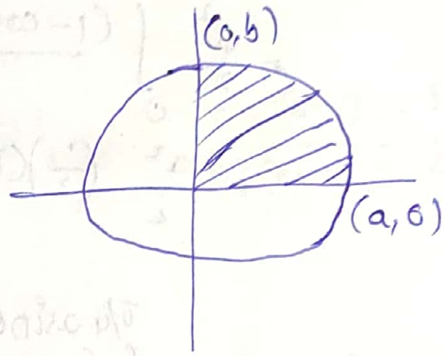
$$= \frac{1}{2}\left[\cancel{2(x)_0^1 - \left(\frac{x^2}{2}\right)_0^1} - \left(\frac{x^3}{3}\right)_0^1\right] = \frac{1}{2}\int_0^1 4+x^2-4x-x^4$$

$$= \cancel{\frac{1}{2}\left[2 - \frac{1}{2} - \frac{1}{3}\right]} = \frac{1}{2}\left(4x + \frac{x^3}{3} - 2x^2 - \frac{x^5}{5}\right)$$

$$= \frac{1}{2}\left(4 + \frac{1}{3} - 2 - \frac{1}{5}\right)$$

$$= \frac{16}{15}$$

(14) Find the value of $\iint xy \, dx\,dy$ taken over the +ve quadrant of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



Sol

$x \rightarrow 0$ to a

$y \rightarrow 0$ to $\frac{b}{a}\sqrt{a^2 - x^2}$

$$\int_0^a \int_0^{b/a\sqrt{a^2-x^2}} xy \, dx\,dy$$

$$\int_0^a x \left(\frac{y^2}{2}\right)_0^{b/a\sqrt{a^2-x^2}} dx$$

$$\int_0^a \frac{x}{2}\left(\frac{b^2}{a^2}(a^2 - x^2)\right) dx$$

$$= \frac{b^2}{2a^2} \int_0^a x(a^2 - x^2)\,dx$$

$$= \frac{b^2}{2a^2} a^2 \frac{(a^2)}{2} - \frac{b^2}{2a^2} \frac{(a^4)}{4}$$

$$= \frac{b^2}{4a^2} a^4 - \frac{b^2 a^4}{8a^2}$$

$$= \frac{b^2 a^4}{8a^2}$$

$$= \frac{b^2 a^2}{8}$$

Double Integrals in polar coordinates

① Evaluate $\int_0^{\pi}\int_0^{a\sin\theta} r\,dr\,d\theta$

Sol:

$$= \int_0^{\pi} \left(\frac{r^2}{2}\right)_0^{a\sin\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\pi} a^2\sin^2\theta \, d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi} \frac{(1-\cos 2\theta)}{2} d\theta$$

$$= \frac{a^2}{2}\left(\frac{1}{2}\right)(\pi) - \frac{a^2}{4}\left(\frac{\sin 2\theta}{2}\right)_0^{\pi} = \frac{\pi a^2}{4}$$

② Evaluate $\int_0^{\pi/4}\int_0^{a\sin\theta} \frac{r\,dr\,d\theta}{\sqrt{a^2-r^2}}$

Sol^n:

$$= \frac{1}{-2}\int_0^{\pi/4}\int_0^{a\sin\theta} \frac{-2r\,dr\,d\theta}{\sqrt{a^2-r^2}}$$

$a^2-r^2=t$

$-2r\,dr = dt$

$r=0 \Rightarrow t=a^2$

$r=a\sin\theta \Rightarrow t=a^2\cos^2\theta$

$$= \frac{-1}{2}\int_0^{\pi/4}\int_{a^2}^{a^2\cos^2\theta} \frac{dt}{\sqrt{t}}\,d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/4} (2t^{1/2})_{a^2}^{a^2\cos^2\theta}\, d\theta$$

$$= -\int_0^{\pi/4} (a\cos\theta - a)\,d\theta = -a\left(\frac{1}{\sqrt{2}}\right) + a\frac{\pi}{4}$$

③ Evaluate $\iint r^3\,dr\,d\theta$ over the area included between the circles $r=2\sin\theta$ and $r=4\sin\theta$

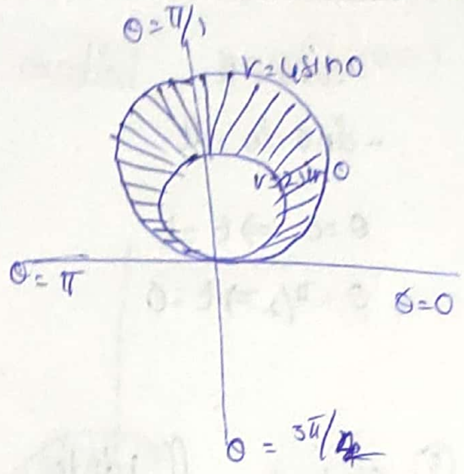
Sol

$r=2\sin\theta$

$\theta =$	0	$\pi/4$	$\pi/2$	$3\pi/4$	π
$r =$	0	$\sqrt{2}$	2		

$\theta \longrightarrow 0$ to π

$r \longrightarrow 2\sin\theta$ to $4\sin\theta$



$$\int_{\theta=0}^{\pi}\int_{r=2\sin\theta}^{4\sin\theta} r^3\,dr\,d\theta = \int_{\theta=0}^{\pi}\left(\frac{r^4}{4}\right)_{2\sin\theta}^{4\sin\theta} d\theta$$

$$= \frac{1}{4}\int_0^{\pi}(256\sin^4\theta - 16\sin^4\theta)\,d\theta$$

$$= \frac{240}{4}\int_0^{\pi}\sin^4\theta\,d\theta$$

$$= 120\int_0^{\pi/2}\sin^4\theta\,d\theta \qquad \left(\because \int_0^{\pi/2}\sin^n\theta\,d\theta = \frac{(n-1)(n-3)\cdots}{(n)(n-2)\cdots}\frac{\pi}{2}\right)$$

$$= 45\pi/2$$

(4) S.T $\iint r^2\sin\theta\,dr\,d\theta$ equals to $\frac{2a^3}{3}$ where r is the semi circle, $r = 2a\cos\theta$ above the intial line

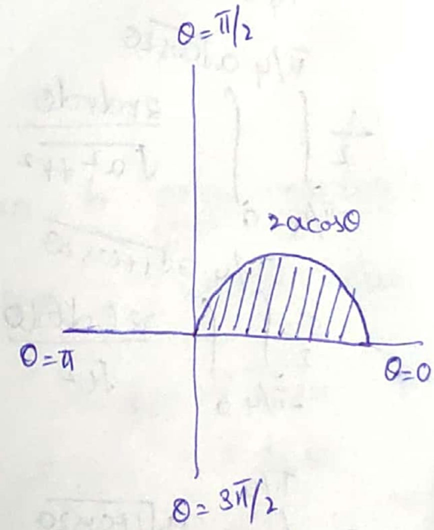
Sol: $r = 2a\cos\theta$

$\theta = 0,\ \pi/2\ \ \pi\ \ 3\pi/2\ \ 2\pi$

$r = 2a\ \ 0\ \ -2a\ \ 0$

$\theta = 0\ \ \pi/4\ \ \pi/2\ \ 3\pi/4\ \ \pi$

$r = 2a\ \ a\sqrt{2}\ \ 0\ \ -a\sqrt{2}\ \ -2a$



$r \longrightarrow 0$ to $2a\cos\theta$

$\theta \longrightarrow 0$ to $\pi/2$

$$\int_0^{\pi/2}\int_0^{2a\cos\theta} r^2\sin\theta\,dr\,d\theta = \int_0^{\pi/2}\left(\frac{r^3}{3}\right)_0^{2a\cos\theta}\sin\theta\,dr\,d\theta$$

$$= \frac{8a^3}{3}\int_0^{\pi/2}\cos^3\theta\sin\theta\,d\theta$$

$\cos\theta = t$

$-\sin\theta\, d\theta = dt$

$\theta = 0 \Rightarrow t = 1$

$\theta = \pi/2 \Rightarrow t = 0$

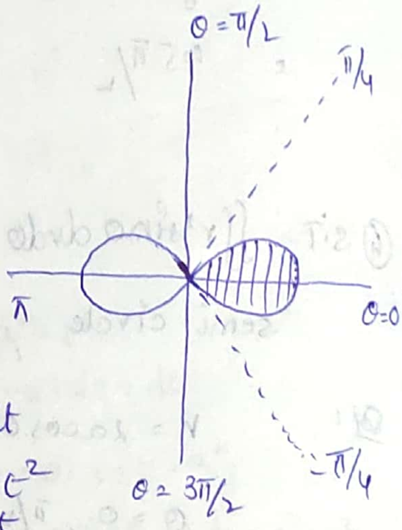
$$= \frac{-8a^3}{3}\int_1^0 t^3\, dt$$

$$= \frac{-8a^3}{3}\left(\frac{-1}{4}\right) = \frac{2a^3}{3}$$

⑤ Evaluate $\iint \frac{r\,dr\,d\theta}{\sqrt{a^2+r^2}}$ over one loop of the lemniscate $r^2 = a^2\cos 2\theta$

Solⁿ $\theta = 0 \quad \pi/4 \quad \pi/2 \quad 3\pi/4 \quad \pi \quad 5\pi/4 \quad 3\pi/2 \quad 2\pi$

$r = \pm a \quad 0 \quad ai$



$\theta \longrightarrow -\pi/4$ to $\pi/4$

$r \longrightarrow 0$ to $a\sqrt{\cos 2\theta}$

$$\int_{-\pi/4}^{\pi/4}\int_0^{a\sqrt{\cos 2\theta}} \frac{r\,dr\,d\theta}{\sqrt{a^2+r^2}}$$

$$\frac{1}{2}\int_{-\pi/4}^{\pi/4}\int_0^{a\sqrt{\cos 2\theta}} \frac{2r\,dr\,d\theta}{\sqrt{a^2+r^2}}$$

$$\frac{1}{2}\int_{-\pi/4}^{\pi/4}\int_0^{a\sqrt{1+\cos 2\theta}} \frac{2t\,dt\,d\theta}{\sqrt{t^2}}$$

$$= \int_{-\pi/4}^{\pi/4} (t)_a^{a\sqrt{1+\cos 2\theta}}\, d\theta$$

$$= \int_{-\pi/4}^{\pi/4} (a\sqrt{1+\cos 2\theta} - a)\, d\theta$$

$a^2 + r^2 = t^2$

$0 + 2r\,dr = 2t\,dt$

$r = 0 \Rightarrow a^2 = t^2$

$a = t$

$r = a\sqrt{\cos 2\theta} \Rightarrow a^2 + a^2(\cos 2\theta) = t^2$

$a^2(1+\cos 2\theta) = t^2$

$a\sqrt{1+\cos 2\theta} = t$

$$a\int_{-\pi/4}^{\pi/4} (\sqrt{1+\cos 2\theta} - 1)\, d\theta$$

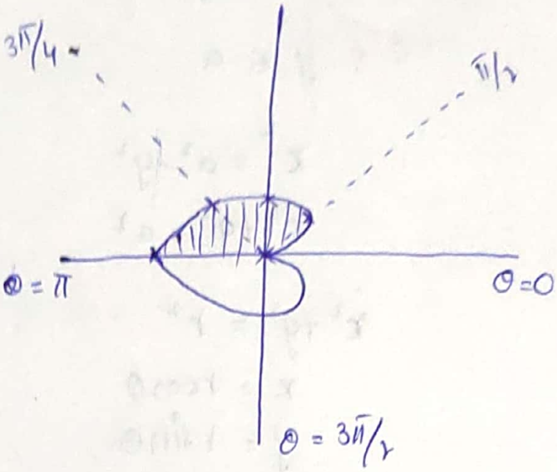
$$a\int_{-\pi/4}^{\pi/4} (\sqrt{2}\cos\theta - 1)\, d\theta$$

$$= a\sqrt{2}(\sin\theta)_{-\pi/4}^{\pi/4} - a(\theta)_{-\pi/4}^{\pi/4}$$

$$= a\sqrt{2}\,\frac{2}{\sqrt{2}} - a\frac{\pi}{2}$$

$$= 2a - a\pi/2$$

⑥ Evaluate $\iint r\sin\theta \, dr\, d\theta$ over the cardiod $r = a(1-\cos\theta)$ above the initial line.



Sol

$\theta \longrightarrow 0$ to π

$r \longrightarrow 0$ to $a(1-\cos\theta)$

$$\int_0^{\pi}\int_0^{a(1-\cos\theta)} r\sin\theta \, dr\, d\theta$$

$$= \int_0^{\pi} \left(\frac{r^2}{2}\right)_0^{a(1-\cos\theta)} \sin\theta \, d\theta$$

$$= \frac{1}{2}\int_0^{\pi} (a^2(1-\cos\theta)^2)\sin\theta$$

$$= \frac{1}{2}\int_0^{\pi} \sin\theta\,(a^2(1+\cos^2\theta - 2\cos\theta))d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi} \sin\theta(1+\cos^2\theta - 2\cos\theta)d\theta$$

$\cos\theta = t$

$-\sin\theta \, d\theta = dt$

$\theta = 0 \Rightarrow t = 1$

$\theta = \pi \Rightarrow t = -1$

$$= \frac{a^2}{2}\int_{-1}^{+1} (1+t^2-2t)dt$$

$$= \frac{a^2}{2}\left(t + \frac{t^3}{3} - t^2\right]_{-1}^{1}$$

$$= \frac{a^2}{2}\left(2+\frac{2}{3} - (1-1)\right) = \frac{4a^2}{3}$$

Change of Variables from cartesian to polar coordinates

The polar coordinates are $x = r\cos\theta$, $y = r\sin\theta$. Now the Jacobian $J = \dfrac{\partial(x,y)}{\partial(r,\theta)}$

$dx\,dy = J\,dr\,d\theta$

$= r\,dr\,d\theta$

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta$$

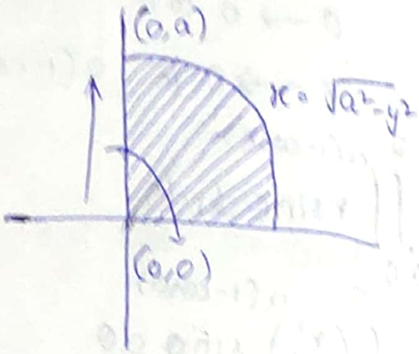
$$= r$$

① Evaluate the double integral $\int_{y=0}^{a}\int_{x=0}^{\sqrt{a^2-y^2}} (x^2+y^2)\,dy\,dx$

Sol

$0 \le x \le \sqrt{a^2-y^2}$

$0 \le y \le a$



$x^2 = a^2-y^2$

$x^2+y^2 = a^2$

$x^2+y^2 = r^2$

$x = r\cos\theta$

$y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

$\theta \longrightarrow 0$ to $\pi/2$

$r \longrightarrow 0$ to a

$$\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,dy\,dx$$

$$= \int_0^{\pi/2}\int_{r=0}^{a} r^2 r\,dr\,d\theta \quad = \int_0^{\pi/2} \left(\frac{r^4}{4}\right)_0^a d\theta$$

$$= \left(\frac{1}{4}\right) a^4 \left(\pi/2\right)$$

$$= \frac{\pi a^4}{8}$$

② S.T $\int_0^{4a}\int_{y^2/4a}^{y} \frac{x^2-y^2}{x^2+y^2}\,dx\,dy = 8a^2\left(\frac{\pi}{2} - \frac{5}{3}\right)$ by changing polar coordinates

Sol:

$x = \dfrac{y^2}{4a}$

$y^2 = 4ax$

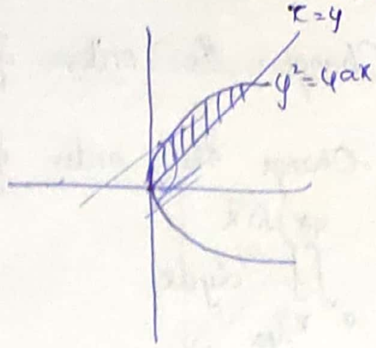
$x = y$

Take the polar coordinates $x = r\cos\theta$

$y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

$\theta \to \pi/4$ to $\pi/2$



$y^2 = 4ax$

$r^2\sin^2\theta = 4a\, r\cos\theta$

$r = \dfrac{4a\cos\theta}{\sin^2\theta}$

$r=0$

$x = y$

$r\cos\theta = r\sin\theta$

$\theta = \pi/4$

③ Evaluate $\displaystyle\int_0^2\int_0^{\sqrt{2x-x^2}} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}}$ by changing polar coordinates

Sol: $y = \sqrt{2x-x^2} \Rightarrow y^2 = 2x - x^2 \Rightarrow x^2+y^2 = 2x$

$x = r\cos\theta$
$y = r\sin\theta$ $\Rightarrow r^2 = 2r\cos\theta \Rightarrow r = 2\cos\theta$

$r \to 0$ to $2\cos\theta$

$\theta \to 0$ to $\pi/2$

$\displaystyle\int_0^{\pi/2}\int_0^{2\cos\theta} \frac{r\cos\theta\, r\,dr\,d\theta}{\sqrt{r^2}}$

$= \displaystyle\int_0^{\pi/2}\int_0^{2\cos\theta} r\cos\theta\, dr\, d\theta$

$= \displaystyle\int_0^{\pi/2} \left(\frac{r^2}{2}\right)_0^{2\cos\theta} \cos\theta$

$= \displaystyle\int_0^{\pi/2} 2\cos\theta\cos^2\theta\, d\theta$

$= 2\left(\dfrac{2}{3\times 1}\right)$

$= 4/3$

→ Change the order of Integration

① Change the order of integration and evaluate

$$\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\,dx$$

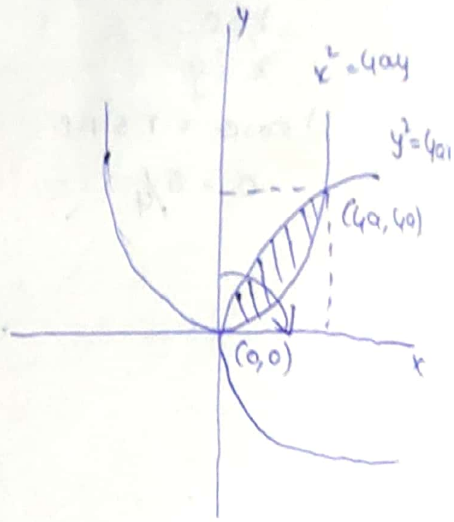
Sol:

$$y = \frac{x^2}{4a} \Rightarrow x^2 = 4ay$$

This is parabola eq^n

$$y = 2\sqrt{ax}$$

$$y^2 = 4ax$$



$$y^4 = 16a^2(4ay)$$

$$y^3 = 16a^3$$

$$y = 4a$$

$$(4a)^2 = 4ax$$

$$x = 4a$$

$y \to 0$ to $4a$

$x \to y^2/4a$ to $2\sqrt{ay}$

By change the order of integration we have

$$\int_{y=0}^{4a}\int_{x=y^2/4a}^{2\sqrt{ay}} dx\,dy = \int_0^{4a} (x)_{y^2/4a}^{2\sqrt{ay}}\, dy$$

$$= \int_0^{4a}\left(2\sqrt{ay} - \frac{y^2}{4a}\right) dy$$

$$= 2\sqrt{a}\left(\frac{y^{3/2}}{3/2}\right)_0^{4a} - \frac{1}{4a}\left(\frac{y^3}{3}\right)_0^{4a}$$

$$= \frac{4\sqrt{a}}{3}\, 8a^{3/2} - \frac{1}{12a}\, 64a^3$$

$$= \frac{32}{3}a^2 - \frac{16a^2}{3}$$

$$= \frac{16a^2}{3}$$

Scanned with CamScanner

(2) Change the order of Integration and Evaluate

$$\int_{0}^{a}\int_{x/a}^{\sqrt{x/a}} (x^2+y^2)\,dx\,dy$$

x^2 ... y^2

$y = \frac{x}{a}$ $\quad y = \sqrt{\frac{x}{a}}$

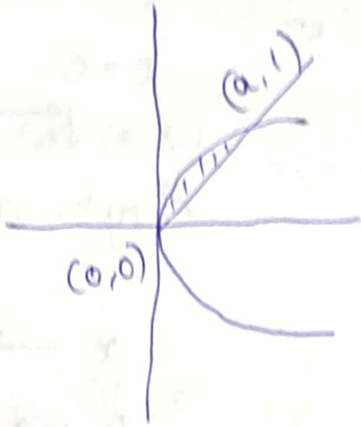
$y^2 = \frac{x}{a}$

$y^2 = y$ $\quad x = 0$, $x = a$

$y^2 - y = 0$

$y(y-1) = 0$

$y = 0 \;,\; y = 1$



$y = 0$ to 1

$x = ay^2$ to ay

$$\int_{0}^{1}\int_{ay^2}^{ay} (x^2+y^2)\,dx\,dy$$

$$= \int_0^1 \left[\left(\frac{x^3}{3}\right)_{ay^2}^{ay} dy + y^2 (x)_{ay^2}^{ay}\, dy\right]$$

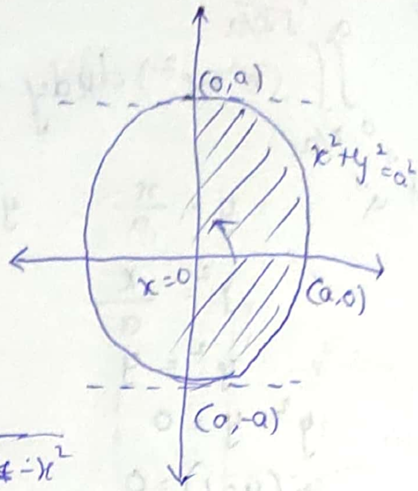
$$= \int_0^1 \frac{a^3y^3}{3} - \frac{a^3y^6}{3} + ay^3 - ay^4$$

$$= \frac{a^3}{3}\left(\frac{y^4}{4}\right)_0^1 - \frac{a^3}{3}\left(\frac{y^7}{7}\right)_0^1 + a\left(\frac{y^4}{4}\right)_0^1 - a\left(\frac{y^5}{5}\right)_0^1$$

$$= \frac{a^3}{3}\left(\frac{1}{4}\right) - \frac{a^3}{3}\left(\frac{1}{7}\right) + a\left(\frac{1}{4}\right) - a\left(\frac{1}{5}\right)$$

$$= \frac{a^3}{28} + \frac{a}{20}$$

(5) $$\int_{y=-a}^{a}\int_{x=0}^{\sqrt{a^2-y^2}} f(x,y)\,dx\,dy$$



$x=0$

$x=\sqrt{a^2-y^2}$

$x^2+y^2=a^2$

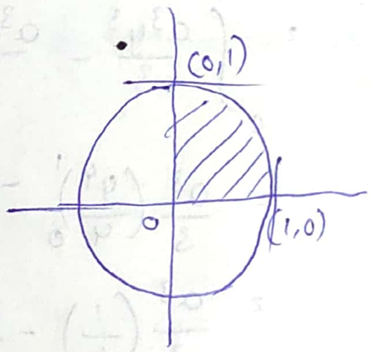
$x \to 0$ to a

$y \to -\sqrt{a^2-x^2}$ to $\sqrt{a^2-x^2}$

$$= \int_{0}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} f(x,y)\,dx\,dy$$

(4) By Changing the order of integration. Evaluate

$$\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} y^2\,dy\,dx$$



$y=\sqrt{1-x^2}$

$x^2+y^2=1$

$y \to 0$ to 1

$x \to 0$ to $\sqrt{1-y^2}$

$$\int_{y=0}^{1}\int_{x=0}^{\sqrt{1-y^2}} y^2\,dy\,dx = \int_0^1 y^2 (x)_0^{\sqrt{1-y^2}}\,dy$$

$$= \int_0^1 y^2\left(\sqrt{1-y^2}-0\right) dy$$

$y = \sin\theta$

$dy = \cos\theta\,d\theta$

$y=0 \Rightarrow \theta=0$, $y=1 \Rightarrow \theta=\pi/2$

$$\int_0^{\pi/2} \sin^2\theta \cos^2\theta \, d\theta$$

$$2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \beta(n,m)$$

$$= \frac{1}{2}\beta\left(\tfrac{3}{2}, \tfrac{3}{2}\right)$$

$$\beta(m,n) = \frac{\gamma(m)\gamma(n)}{\gamma(m+n)}$$

$$= \frac{1}{2}\,\frac{\gamma(3/2)\,\gamma(3/2)}{\gamma(3/2+3/2)}$$

$\gamma(n+1) = n\gamma(n)$

$\gamma(n+1) = n!$

$$= \frac{\frac{1}{2} \times \frac{\sqrt{\pi}}{2}\,\frac{\sqrt{\pi}}{2}}{2} = \pi/6$$

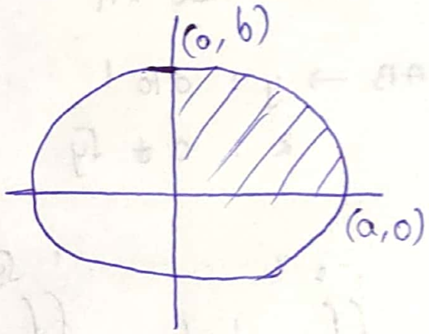
$\gamma(3/2) = \gamma(1/2 + 1)$

$= 1/2\,\gamma(1/2)$

$= \sqrt{\pi}/2$

⑤ Change the order of integration and Evaluate

$$\int_0^b \int_0^{\frac{a}{b}\sqrt{b^2-y^2}} xy \, dx\, dy$$



$x = 0$

$x = \frac{a}{b}\sqrt{b^2 - y^2}$

$x^2 = \frac{a^2}{b^2}(b^2 - y^2)$

$x^2 b^2 = a^2 b^2 - a^2 y^2$

$x^2 b^2 + y^2 a^2 = a^2 b^2$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$x \longrightarrow 0$ to a

$y \longrightarrow 0$ to $\frac{b}{a}\sqrt{a^2 - x^2}$

$$\int_{x=0}^{a} \int_{y=0}^{b/a\sqrt{a^2-x^2}} xy \, dx\, dy = \int_0^a x\left(\frac{y^2}{2}\right)_0^{b/a\sqrt{a^2-x^2}}$$

$$= \int_0^a \frac{x}{2}\left(\frac{b^2}{a^2}(a^2 - x^2)\right) dx$$

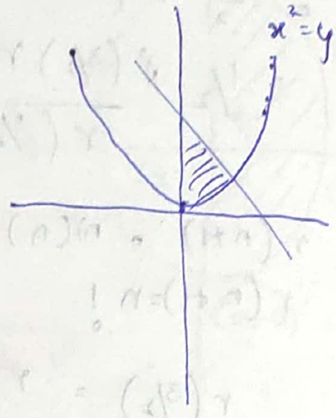
$$= \frac{b^2}{2a^2}\int_0^a (a^2 x - x^3)\, dx$$

$$= \frac{b^2}{2a^2}\left[a^2\left(\frac{x^2}{2}\right)_0^a - \left(\frac{x^4}{4}\right)_0^a\right]$$

$$= \frac{a^2 b^2}{8}$$

⑥ Change the order of integration $\int_{x=0}^{1}\int_{y=x^2}^{2-x} xy\,dx\,dy$

Sol



$y=2-x$

$x+y=2$

$\frac{x}{2}+\frac{y}{2}=1$

$y=x^2$

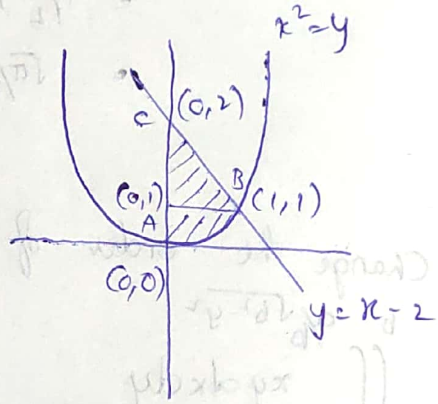
$y=2-x$

$2-x=x^2$

$(x+2)(x-1)=0$

$x=-2$

$x=1 \Rightarrow y=1$



$OAB + ABC = R$

$OAB \to y : 0$ to 1

$x : 0$ to $\sqrt{y}$

$ABC \to y : \to 1$ to 2

$x \to 0$ to $2-y$

$$\int_0^1\int_{x^2}^{2-x} xy\,dx\,dy = \int_{y=0}^{1}\int_0^{\sqrt{y}} xy\,dx\,dy + \int_{y=1}^{2}\int_0^{2-y} xy\,dx\,dy$$

$$= \int_0^1 y\left(\frac{x^2}{2}\right)_0^{\sqrt{y}} dy + \frac{1}{2}\int_1^2 y\,(x^2)_0^{2-y}\,dy$$

$$= \frac{1}{2}\int_0^1 y(y-0)\,dy + \frac{1}{2}\int_1^2 y(2-y)^2\,dy$$

$$= \frac{1}{2}\int_0^1 y^2\,dy + \frac{1}{2}\int_1^2 y(4-y^2-4y)\,dy$$

$$= \frac{1}{6} + \frac{1}{2}\left[4\left(\frac{y^2}{2}\right)_1^2 - \left(\frac{y^4}{4}\right)_1^2 - 4\left(\frac{y^3}{3}\right)_1^2\right]$$

$$= \left[\frac{1}{6} + \frac{1}{2}\left[4\left(\frac{3}{2}\right) - \left(\frac{15}{4}\right) - 4\left(\frac{7}{3}\right)\right]\right]$$

Unit - 2 TRIPLE INTEGRALS

① Evaluate triple integral $\int_{z=0}^{1}\int_{y=1}^{2}\int_{x=2}^{3} xyz \, dx\,dy\,dz$

Sol: $\int_0^1\int_1^2\int_2^3 xyz \, dx\,dy\,dz$

$\int_0^1\int_1^2 yz\left(\frac{x^2}{2}\right)_2^3 dy\,dz$

$\frac{5}{2}\int_0^1 z\left(\frac{y^2}{2}\right)_1^2 dz$

$\frac{5}{2}\times\frac{3}{2}\left(\frac{z^2}{2}\right)_0^1 = \frac{5}{2}\times\frac{3}{2}\times\frac{1}{2} = \frac{15}{8}$

② $\int_{x=0}^{a}\int_{y=0}^{x}\int_{z=0}^{x+y} e^{x+y+z} dx\,dy\,dz$

$\int_0^a\int_0^x\int_0^{x+y} (e^x)(e^y)(e^z) dx\,dy\,dz$

$= \int_0^a\int_0^x e^x e^y (e^z)_0^{x+y} dx\,dy$

$= \int_0^a\int_0^x e^x e^y e^x e^y \, dy\,dx$

$= \int_0^a\int_0^x e^{2x} e^{2y} dy\,dx$

$= \int_0^a e^{2x}\left(\frac{e^{2y}}{2}\right)_0^x dx = \int_0^a \frac{e^{2x}}{2}(e^{2x} - e^0) dx$

$= \frac{1}{2}\int_0^a (e^{4x} - e^{2x}) dx$

② Evaluate $\iiint xy^2z \, dx\,dy\,dz$ taken through the positive octanct of the sphere $x^2+y^2+z^2=a^2$

Sol:

$z \to 0$ to $\sqrt{a^2-x^2-y^2}$

$z=0$, $y \to 0$ to $\sqrt{a^2-x^2}$

$y=0, z=0$ $x \to 0$ to a

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} xy^2 \left(\frac{z^2}{2}\right)_0^{\sqrt{a^2-x^2-y^2}} dx\,dy$$

$$= \frac{1}{2} \int_0^a \int_0^{\sqrt{a^2-x^2}} xy^2(a^2-x^2-y^2) dx\,dy$$

$$= \frac{1}{2} \int_0^a \int_0^{\sqrt{a^2-x^2}} \left(x(a^2-x^2)y^2 - xy^4\right) dx\,dy$$

$$= \frac{1}{2} \int_0^a \left[x(a^2-x^2)\left(\frac{y^3}{3}\right)_0^{\sqrt{a^2-x^2}} - x\left(\frac{y^5}{5}\right)_0^{\sqrt{a^2-x^2}}\right] dx$$

$$= \frac{1}{6} \int_0^a x(a^2-x^2)(a^2-x^2)^{3/2} dx - \frac{1}{10} \int_0^a x(a^2-x^2)^{5/2} dx$$

Scanned with CamScanner

(4) Evaluate $\iiint z^2 \, dx\,dy\,dz$ taken over the volume bounded by surfaces $x^2+y^2=a^2$, $x^2+y^2=z$, $z=0$

Sol:

$z \rightarrow 0$ to x^2+y^2

$y \rightarrow -\sqrt{a^2-x^2}$ to $\sqrt{a^2-x^2}$

$x \rightarrow -a$ to a

Scanned with CamScanner

(6) Evaluate $\iiint \frac{dx\,dy\,dz}{(x+y+z+1)^3}$ taken over the volume bounded by the planes $x=0$, $y=0$, $z=0$ and the plane $x+y+z=1$

Sol:

$$z = 1-x-y$$
$$y = 1-x$$
$$x = 1$$

$$\int_0^1\int_0^{1-x}\int_0^{1-x-y} \frac{dx\,dy\,dz}{(x+y+z+1)^3} = \int_0^1\int_0^{1-x}\left[\frac{-1}{2(x+y+z+1)^2}\right]_0^{1-x-y}$$

$$= -\frac{1}{2}\int_0^1\int_0^{1-x}\left[\frac{1}{(x+y+1-x-y+1)^2} - \frac{1}{(x+y+1)^2}\right]$$

$$= -\frac{1}{2}\int_0^1\int_0^{1-x}\left[\frac{1}{4} - \frac{1}{(x+y+1)^2}\right]dx\,dy$$

$$= -\frac{1}{2}\int_0^1\left[\frac{y}{4} + \frac{1}{(x+y+1)}\right]_0^{1-x}$$

$$= -\frac{1}{2}\int_0^1\left[\frac{1-x}{4} + \frac{1}{x+1-x+1} - \frac{1}{x+1}\right]$$

$$= -\frac{1}{2}\int_0^1\left[\frac{1-x}{4} + \frac{1}{2} - \frac{1}{x+1}\right]dx$$

$$= -\frac{1}{2}\left[-\frac{(1-x)^2}{8} + \frac{x}{2} - \log(x+1)\right]_0^1$$

$$= -\frac{1}{2}\left[\frac{1}{2} - \log 2 + \frac{1}{\cdot}\right.$$

(5) Evaluate $\iiint_V dx\,dy\,dz$ where V is the final region of space formed by the planes $x=0, y=0, z=0$ and $2x+3y+4z=12$

$z \to 0$ to $\frac{12-2x-3y}{4}$

$y \to 0$ to $12-2x$

$x \to 0$ to 6

$$\int_0^6 \int_0^{12-2x} \int_0^{\frac{12-2x-3y}{4}} dx\,dy\,dz$$

$$\int_0^6 \int_0^{12-2x} \left(\frac{12-2x-3y}{4}\right) dx\,dy$$

$$2\int_0^6 \int_0^{12-2x} \left(3 - \frac{x}{2} - \frac{3}{4}y\right) dx\,dy$$

$$= \int_0^6 3(y)_0^{12-2x} - \left(\frac{xy}{4}\right)^{12-2x} - \frac{3}{4}\frac{1}{2}(12-2x)^2\,dx$$

$$= \int_0^6 \left(36 - 12x + x^2 - \frac{3}{8}(144+4x^2-48x)\right) dx$$

$$= 36(x)_0^6 - 12\left(\frac{x^2}{2}\right)_0^6 + \left(\frac{x^3}{3}\right)_0^6 - 54(6) - \frac{1}{2}(216) + \frac{18}{2}(36)$$

$= 36$ sq units.

Change of variables in triple integrals

Let the function $x = \phi_1(u,v,w)$, $y = \phi_2(u,v,w)$
$z = \phi_3(u,v,w)$ with the transformations from Cartesian coordinates to curvylinear coordinates by u,v,w

The Jacobian for this transformation is given by

$J = \dfrac{\partial(x,y,z)}{\partial(u,v,w)}$ then $\iiint\limits_V f(x,y,z)\,dx\,dy\,dz =$

$$\iiint\limits_{V'} f(\phi_1,\phi_2,\phi_3)\,|J|\,du\,dv\,dw$$

where V' is the corresponding domain in curvy linear coordinates u,v,w

Change of variables from cartezian to spherical polar coordinate system

Let $u = r$, $v = \theta$, $w = \phi$ now the spherical polar coordinates are

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\sin\theta \text{ (or } r\cos\theta)$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cancel{\sin\theta}\ \cos\theta & \cancel{r\cos\theta}\ -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta$$

$$\iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_V f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

Change of variables from Cartezian to cylindical polar coordinate system

Let $u = r,\quad v = \theta,\quad w = z$

Now the cylindrical polar coordinata are

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$$\iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_V f(r\cos\theta, r\sin\theta, z)\,r\,dr\,d\theta\,dz$$

① Evaluate $\iiint(x^2+y^2+z^2)\,dx\,dy\,dz$ taken over the volume enclosed by sphere $x^2+y^2+z^2=1$ by transforming into spherical polar coordinates.

Sol:

$$\int_{r=0}^{1}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}(x^2+y^2+z^2)\,dx\,dy\,dz$$

$$dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= \int_{r=0}^{1}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} (r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi + r^2\cos^2\theta)\, r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$= \int_0^1\int_0^{\pi}\int_0^{2\pi} (r^2)\, r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$= \int_0^1\int_0^{\pi}\int_0^{2\pi} r^4\sin\theta\, dr\, d\theta\, d\phi = \int_0^1\int_0^{\pi} r^4\sin\theta\,(2\pi)\, dr\, d\theta$$

$$= 2\pi \int_0^1\int_0^{\pi} r^4\sin\theta\, d\theta\, dr$$

$$= 2\pi \int_0^1 r^4 (-\cos\theta)_0^{\pi} = 2\pi \int_0^1 r^4 - (-1-1) =$$

$$= 4\pi \int_0^1 r^4 = \frac{4\pi}{5}$$

② Find the volume of the portion of the sphere $x^2+y^2+z^2=a^2$ lying inside the cylinder $x^2+y^2=ax$

Sol: cylindrical polar coordinates are

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$x^2+y^2+z^2=a^2$

$r^2\cos^2\theta + r^2\sin^2\theta + z^2 = a^2$

$r^2+z^2=a^2$

$z = \sqrt{a^2-r^2}$

$x^2+y^2=ax$

$r^2 = ar\cos\theta$

$r = a\cos\theta$

$z \to 0$ to $\sqrt{a^2-r^2}$

$r \to 0$ to $a\cos\theta$

$\theta \to 0$ to 2π

The volume of the sphere is

$$\iiint dx\,dy\,dz = \int_{\theta=0}^{2\pi}\int_{r=0}^{a\cos\theta}\int_{z=0}^{\sqrt{a^2-r^2}} r\,dr\,d\theta\,dz$$

$$= \int_0^{2\pi}\int_0^{a\cos\theta} \left(z\right)_0^{\sqrt{a^2-r^2}} \, d\theta\, dr$$

$$= \int_0^{2\pi}\int_0^{a\cos\theta} r\left(\sqrt{a^2-r^2}\right) d\theta\, dr$$

$$= -\int_0^{2\pi}\int_a^{a\sin\theta} t^2 \, d\theta\, dt$$

$a^2 - r^2 = t^2$

$-2r\,dr = 2t\,dt$

$r\,dr = -t\,dt$

$r = 0 \rightarrow t = a$

$r = a\cos\theta$

$\rightarrow t = a\sin\theta$

③ Evaluate $\iiint z(x^2+y^2+z^2)\,dx\,dy\,dz$ to the volume of the cylinder $x^2+y^2=a^2$ intercepted by the plane $z=0$, $z=h$

→ Special functions : Beta, Gama functions

Consider the integral $\int_a^b f(x)dx$. such an integral for which

1. Either the interval of integration is not finite i.e $a = -\infty$ or $b = \infty$ ~~and~~ or both

2. Or the function $f(x)$ is unbounded at one or more points in $[a,b]$ is called improper integral

Beta function

The integral $\int_0^1 x^{m-1}(1-x)^{n-1}dx$ is called the Beta function. and is denoted by $\beta(m,n)$ i.e

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx \qquad m,n > 0$$

Properties of Beta function

1. $\beta(m,n) = \beta(n,m)$

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx \qquad \begin{array}{l} 1-x=t \\ dx = -dt \end{array}$$

$$= \int_1^0 (1-t)^{m-1} t^{n-1}(-dt) \qquad \begin{array}{l} x=0 \Rightarrow t=1 \\ x=1 \Rightarrow t=0 \end{array}$$

$$= \int_0^1 t^{n-1}(1-t)^{m-1}dt$$

$$= \int_0^1 x^{n-1}(1-x)^{m-1}dx$$

$$= \beta(n,m)$$

2. $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

By def of Beta we have $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$

$$= \int_0^{\pi/2} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1} 2\sin\theta\cos\theta d\theta \qquad \text{put } x = \sin^2\theta$$

$$dx = 2\sin\theta\cos\theta d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-2}\theta \cos^{2n-2}\theta \sin\theta\cos\theta d\theta \qquad \begin{array}{l} x=0 \Rightarrow \theta=0 \\ x=1 \Rightarrow \theta=\pi/2 \end{array}$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

3. $\beta(m,n) = \beta(m+1,n) + \beta(m,n+1)$

$$= \int_0^1 x^{(m+1)-1}(1-x)^{n-1}dx + \int_0^1 x^{m-1}(1-x)^{(n+1)-1}dx$$

$$= \int_0^1 x^m(1-x)^{n-1}dx + \int_0^1 x^{m-1}(1-x)^n dx$$

$$= \int_0^1 \left[x^m(1-x)^{n-1} + x^{m-1}(1-x)^n\right]dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}(x+1-x)\,dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$= \beta(m,n)$$

① To show that $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}$

Sol:

By def of Beta we have $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

$$x = \frac{1}{1+y}$$

$$dx = \frac{-1}{(1+y)^2}dy$$

$x = 0 \Rightarrow y = \infty$

$x = 1 \Rightarrow y = 0$

$$\beta(m,n) = \int_\infty^0 \left(\frac{1}{1+y}\right)^{m-1}\left(1-\frac{1}{1+y}\right)^{n-1}\left(\frac{-1}{(1+y)^2}\right)dy$$

$$= \int_0^\infty \frac{1}{(1+y)^{m-1}} \frac{y^{n-1}}{(1+y)^{n-1}} \frac{1}{(1+y)^2}\,dy$$

$$= \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\,dy$$

$$= \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx \quad \text{(or)} \quad \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}$$

② To show that $\beta(m,n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx$

Sol: By def of Beta we have $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$

$$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx \quad \text{—— ①}$$

Consider $\int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$

$$x = \frac{1}{y}$$

$$dx = \frac{-1}{y^2}\,dy$$

$x = 1 \Rightarrow y = 1$

$x = \infty \Rightarrow y = 0$

$$\int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_1^0 \frac{\left(\frac{1}{y}\right)^{m-1}}{\left(1+\frac{1}{y}\right)^{m+n}}\left(\frac{-1}{y^2}\right)dy$$

$$= \int_0^1 \frac{1}{y^{m-1}}\,\frac{y^{m+n}}{(1+y)^{m+n}}\,\frac{1}{y^2}\,dy$$

$$= \int_0^1 \frac{y^{m+n}}{y^{m+1}(1+y)^{m+n}}\,dy$$

$$= \int_0^1 \frac{y^{n-1}}{(1+y)^{m+n}}\,dy$$

$$= \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx \quad \text{—— ②}$$

Substitute ② in ① we get

$$\beta(m,n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

$$= \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx$$

(3) $$\int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

we have $$2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \beta(m,n)$$

$2m-1 = p$ $\qquad$ $2n-1 = q$

$m = \frac{p+1}{2}$ $\qquad$ $n = \frac{q+1}{2}$

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

(or)

$$\int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \int_0^{\pi/2} \sin^{m-1}\theta \cos^{n-1}\theta \sin\theta\cos\theta \, d\theta$$

$$= \int_0^{\pi/2} (\sin^2\theta)^{\frac{m-1}{2}} (\cos^2\theta)^{\frac{n-1}{2}} \sin\theta\cos\theta \, d\theta$$

put $x = \sin^2\theta$

$dx = 2\sin\theta\cos\theta \, d\theta$

$\theta = 0 \Rightarrow x = 0$

$\theta = \pi/2 \Rightarrow x = 1$

$$= \frac{1}{2}\int_0^1 x^{\frac{m-1}{2}} (1-x)^{\frac{n-1}{2}} dx$$

$$= \frac{1}{2}\int_0^1 x^{\frac{m+1}{2}-1} (1-x)^{\frac{n+1}{2}-1} dx = \frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

④ Express the following integrals in Beta functions

(i) $\int_0^1 \frac{x}{\sqrt{1-x^2}} dx$

put $x^2 = t$

$2x\,dx = dt$

$x\,dx = \frac{1}{2} dt$

$x = 0 \Rightarrow t = 0$

$x = 1 \Rightarrow t = 1$

$= \frac{1}{2} \int_0^1 t^{1-1} (1-t)^{1/2 - 1} dt$

$= \frac{1}{2} \beta(1, 1/2)$

(ii) $\int_0^3 \frac{dx}{\sqrt{9-x^2}}$

put $x^2 = 9t$

$2x\,dx = 9\,dt$

$dx = \frac{9}{2x} dt$

$= \frac{9}{2 \times 3\sqrt{t}} dt$

$= \frac{3}{2\sqrt{t}} dt$

$x = 0 \Rightarrow t = 0$

$x = 3 \Rightarrow t = 1$

$= \frac{3}{2} \int_0^1 \frac{1}{\sqrt{t}} \frac{1}{\sqrt{9-9t}} dt$

$= \frac{3}{2} \frac{1}{3} \int_0^1 t^{-1/2} (1-t)^{-1/2} dt$

$= \frac{1}{2} \int_0^1 t^{1/2 - 1} (1-t)^{1/2 - 1} dt$

$= \frac{1}{2} \beta(1/2, 1/2)$

⑤ P.T $\int_0^1 \frac{x\,dx}{\sqrt{1-x^5}} = \frac{1}{5} \beta\left(\frac{2}{5}, \frac{1}{2}\right)$

$x^5 = t$

$5x^4 dx = dt$

$x\,dx = \frac{1\,dt}{5x^3}$

$x = (t)^{1/5}$

$\frac{1}{5} \int_0^1 \frac{1}{x^3} \frac{1}{\sqrt{1-t}} dt$

$= \frac{1}{5} \int_0^1 (t)^{-3/5} (1-t)^{-1/2} dt$

$= \frac{1}{5} \int_0^1 (t)^{\frac{2}{5}-1} (1-t)^{1/2 - 1} dt$

$= \frac{1}{5} \beta\left(\frac{2}{5}, \frac{1}{2}\right)$

(10) If m and n are positive integers then

$$\beta(m,n) = \frac{(m-1)(n-1)!}{(m+n-1)!}$$

Sol:

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$= \left[x^{m-1}\frac{(1-x)^n}{-n}\right]_0^1 - \int_0^1 (m-1)x^{m-2}\frac{(1-x)^n}{-n}\,dx$$

$$= -\frac{1}{n}[0] + \frac{(m-1)}{n}\int_0^1 x^{m-2}(1-x)^n\,dx$$

$$= \frac{m-1}{n}\int_0^1 x^{(m-1)-1}(1-x)^{n+1-1}\,dx$$

$$= \frac{m-1}{n}\beta(m-1,n+1) \quad \text{——— (1)}$$

from (1) $\beta(m,n) = \frac{m-1}{n}\beta(m-1,n+1)$

$$\beta(m-1,n+1) = \frac{m-1-1}{n+1}\beta(m-1-1,n+1+1)$$

$$= \frac{m-2}{n+1}\beta(m-2,n+2)$$

from (1)

$$\beta(m,n) = \frac{m-1}{n}\frac{m-2}{n+1}\beta(m-2,n+2) \quad \text{——— (2)}$$

$$\beta(m-2,n+2) = \frac{m-2-1}{n+2}\beta(m-2-1,n+2+1)$$

$$= \frac{m-3}{n+2}\beta(m-3,n+3)$$

from (2)

$$\beta(m,n) = \frac{m-1}{n}\frac{m-2}{n+1}\frac{m-3}{n+2}\beta(m-3,n+3)$$

$$\beta(m,n) = \frac{(m-1)(m-2)(m-3)\cdots(m-(m-1))}{n(n+1)(n+2)\cdots(n+(m-2))}\beta(m-(m-1),n+(m-1))$$

Scanned with CamScanner

$$= \frac{(m-1)!}{n(n+1)(n+2)\cdots(n+m-2)} \beta(1, n+m-1) \quad \text{--- (4)}$$

$$\beta(1, n+m-1) = \int_0^1 x^{1-1} (1-x)^{n+m-1-1} dx$$

$$= \int_0^1 (1-x)^{n+m-2} dx$$

$$= \int_0^1 t^{n+m-2} dx$$

$$= \left(\frac{t^{n+m-1}}{n+m-1} \right)_0^1$$

$$= \frac{1}{n+m-1}$$

from (4)

$$\beta(m,n) = \frac{(m-1)!}{n(n+1)(n+2)\cdots(n+m-2)} \frac{1}{(n+m-1)} \frac{(n-1)!}{(n-1)!}$$

$$= \frac{(m-1)(n-1)!}{(m+n-1)!}$$

(11) P.T $\frac{\beta(p,q+1)}{q} = \frac{\beta(p+1,q)}{p} = \frac{\beta(p,q)}{p+q}$ where $p>0$ and $q>0$

Sol

$$\frac{\beta(p,q+1)}{q} = \frac{1}{q} \int_0^1 x^{p-1} (1-x)^q dx$$

$$= \frac{1}{q}\left[\left(-x^{p-1} \frac{(1-x)^{q+1}}{q+1}\right)_0^1 - \frac{1}{q}\int_0^1 (p-1)x^{p-2} \left(-\frac{(1-x)^{q+1}}{q+1}\right) dx\right.$$

$$= \frac{-1}{q(q+1)}(0) + \frac{p-1}{q(q+1)} \int_0^1 x^{p-2} (1-x)^{q+1} dx$$

Scanned with CamScanner

$$= \frac{p-1}{q(q+1)} \int_0^1 x^{p-1-1} (1-x)^{q+2-1} dx$$

$$\frac{\beta(p, q+1)}{q} = \frac{1}{q} \int_0^1 x^{p-1} (1-x)^q dx$$

$$= \frac{1}{q} \left[(1-x)^q \frac{x^p}{p} \right]_0^1 - \frac{1}{q} \int_0^1 -q (1-x)^{q-1} \frac{x^p}{p} dx$$

$$= 0 + \frac{1}{p} \int_0^1 x^p (1-x)^{q-1}$$

$$= \frac{1}{p} \int_0^1 x^{p+1-1} (1-x)^{q-1} dx$$

$$= \frac{\beta(p+1, q)}{p}$$

* $\sqrt{n} = \int_0^\infty e^{-x} x^{n-1} dx$

Gamma Function,

The definite integral

Properties of gamma functions

① $\gamma(1) = 1$

$$r(n) = \int_0^\infty e^{-x} x^{n-1} dx$$

$$r(1) = \int_0^\infty e^{-x} x^{1-1} dx$$

$$= -(e^{-x})_0^\infty = -\left(\frac{1}{\infty} - 1\right) = 1$$

② $r(n) = (n-1)\, r(n-1)$

$$r(n) = \int_0^\infty e^{-x} x^{n-1} dx$$

$$= -(x^{n-1} e^{-x})_0^\infty + \int_0^\infty (n-1) x^{n-2} e^{-x} dx$$

$$= (n-1) \int_0^\infty e^{-x} x^{(n-1)-1} dx$$

$$= (n-1) \sqrt{(n-1)}$$

Note:

① $r(n+1) = n\, r(n)$

② If n is a positive fraction then

$$r(n) = (n-1)(n-2) - - - (n-r)\, r(n-r)$$ where $n-r>0$

(5) $\Gamma(n+1) = n!$

Relation b/w Beta & Gamma

(i) $\beta(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$, $m,n>0$

$$\Gamma(m) = \int_0^\infty e^{-x} x^{m-1} dx = \int_0^\infty e^{-yt} y^{m-1} t^{m-1} y\,dt$$

take $x = yt$
$dx = y\,dt$

$$= y^m \int_0^\infty e^{-yt} t^{m-1} dt$$

$x=0 \Rightarrow t=0$
$x=\infty \Rightarrow t=\infty$

$$\Rightarrow \frac{\Gamma(m)}{y^m} = \int_0^\infty e^{-yx} x^{m-1} dx \quad \text{—— (1)}$$

multiplying with $e^{-y} y^{m+n-1}$ on both sides and integrating from 0 to ∞ w.r.t y both sides.

$$\int_0^\infty \frac{\Gamma(m)}{y^m} e^{-y} y^{m+n-1} dy = \int_0^\infty\int_0^\infty e^{-yx} x^{m-1} e^{-y} y^{m+n-1} dx\,dy$$

$$\Gamma(m)\int_0^\infty e^{-y} y^{n-1} dy = \int_0^\infty\left(\int_0^\infty e^{-y(x+1)} y^{m+n-1}\right) dy\; x^{m-1} dx$$

$$\Gamma(m)\,\Gamma(n) = \int_0^\infty \frac{\Gamma(m+n)}{(1+x)^{m+n}} x^{m-1} dx$$

$$= \Gamma(m+n)\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

$$= \Gamma(m+n)\,\beta(m,n)$$

(ii) $\Gamma(1/2) = \sqrt{\pi}$

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$m = n = 1/2$

$$\beta(1/2, 1/2) = \frac{[\Gamma(1/2)]^2}{\Gamma(1)}$$

$$(\Gamma(1/2))^2 = \beta(1/2, 1/2)$$

① $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$$

$x^2 = t \Rightarrow x = \sqrt{t}$

$2x\,dx = dt$

$dx = \frac{dt}{2\sqrt{t}}$

$$\int_0^\infty e^{-x^2} dx = \int_{t=0}^\infty e^{-t} \frac{dt}{2\sqrt{t}}$$

$$= \frac{1}{2}\int_0^\infty e^{-t} t^{-1/2} dt$$

$$= \frac{1}{2}\int_0^\infty e^{-t} t^{1/2-1} dt$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{\sqrt{\pi}}{2}$$

(or) $n = 1/2$

$$\Gamma(1/2) = \int_0^\infty e^{-x} x^{1/2-1} dx$$

$$= \int_0^\infty e^{-x} x^{-1/2} dx$$

$x = t^2$

$dx = 2t\,dt$

$$= \int_{t=0}^\infty e^{-t^2} (t^2)^{-1/2} dt\,(2t)$$

$$= 2\int_0^\infty e^{-t^2} t^0 dt$$

$$= 2\int_0^\infty e^{-t^2} dt$$

$$\frac{1}{2}\Gamma(1/2) = \int_0^\infty e^{-x^2} dx$$

$$\frac{\sqrt{\pi}}{2} = \int_0^\infty e^{-x^2} dx$$

$\rightarrow$ $$\int_{y=0}^\infty \int_{x=0}^\infty e^{-(x^2+y^2)} dx\,dy = \int_0^\infty e^{-x^2} dx \cdot \int_0^\infty e^{-y^2} dy$$

$$= \frac{\sqrt{\pi}}{2} \frac{\sqrt{\pi}}{2} = \frac{\pi}{4}$$

Scanned with CamScanner

$x = r\cos\theta$

$y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

$$\iint e^{-(x^2+y^2)}\,dx = \int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2}\, r\,dr\,d\theta$$

② Evaluate the value of $\Gamma(9/2)\,\Gamma(-1/2)\,\Gamma(-7/2)$

Sol

$* \; \boxed{\Gamma(n) = (n-1)\Gamma(n-1)}$

$\Gamma(9/2) = (9/2 - 1)\,\Gamma(9/2 - 1)$

$= 7/2\;\Gamma(7/2)$

$= \frac{7}{2}\left(\frac{7}{2}-1\right)\Gamma(5/2)$

$= 7/2 \; 5/2 \; \Gamma(5/2)$

$= \frac{7}{2}\,\frac{5}{2}\,\frac{3}{2}\,\frac{1}{2}\,\Gamma(1/2)$

$= \frac{7\cdot5\cdot3\cdot1\cdot\sqrt{\pi}}{2^5}$

$\Gamma(n+1) = \cancel{n!}\, n\,\Gamma(n)$

$* \; \boxed{\Gamma(n) = \frac{\Gamma(n+1)}{n}}$

$\Gamma(-1/2) = \frac{\Gamma(-1/2+1)}{-1/2}$

$= -2\,\Gamma(1/2)$

$= -2\sqrt{\pi}$

$\Gamma(-7/2) = \frac{\Gamma(-7/2+1)}{-7/2}$

$= \frac{2^4}{7\cdot5\cdot3\cdot1}\sqrt{\pi}$

$* \; \boxed{\Gamma(n+1) = n!}$

$\Gamma(4) = \Gamma(3+1) = 3!$

$= 6$

③ S.T

i) $\beta(m+1, n) = \frac{m}{m+n}\,\beta(m,n) \quad (m, n > 0)$

$\beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$

$\beta(m+1, n) = \frac{\Gamma(m+1)\,\Gamma(n)}{\Gamma(m+n+1)}$

$= \frac{m\,\Gamma(m)\,\Gamma(n)}{(m+n)\,\Gamma(m+n)}$

$= \frac{m}{m+n}\,\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \frac{m}{m+n}\,\beta(m,n)$

Scanned with CamScanner

(ii) $\beta(m, n+1) = \frac{n}{m+n} \beta(m,n)$

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$\beta(m,n+1) = \frac{\Gamma(m)\Gamma(n+1)}{\Gamma(m+n+1)}$$

$$= \frac{n\Gamma(n)\Gamma(m)}{m+n\ \Gamma(m+n)}$$

$$= \frac{n}{m+n} \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$= \frac{n}{m+n} \beta(m,n)$$

(iii) $\beta(m,n) = \beta(m+1,n) + \beta(m,n+1)$

$$= \frac{m}{m+n}\beta(m,n) + \frac{n}{m+n}\beta(m,n)$$

$$= \beta(m,n)\left(\frac{m+n}{m+n}\right)$$

$$= \beta(m,n)$$

(4) S.T $\Gamma(n) = \int_0^1 \left(\log\frac{1}{x}\right)^{n-1} dx, \quad n>0$

Sol: $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx$

$$x = \log\frac{1}{y} \Rightarrow \frac{1}{y} = e^{x}$$

$$y = e^{-x}$$

$x = 0 \Rightarrow y = 1$

$x = \infty \Rightarrow y = 0$

$$dx = \frac{1}{1/y}\left(\frac{-1}{y^2}\right) = \frac{-1}{y}dy$$

$$\Gamma(n) = \int_{y=1}^{0} y\left(\log\frac{1}{y}\right)^{n-1}\left(\frac{-1}{y}\right)dy$$

$$= \int_0^1 \left(\log\frac{1}{y}\right)^{n-1} dy = \int_0^1 \left(\log\frac{1}{x}\right)^{n-1} dx$$

⑥ S.T $\int_0^\infty x^{n-1} e^{-kx} dx = \frac{\Gamma(n)}{k^n}$

$x = ky$
$dx = k dy$

$x = 0 \Rightarrow y = 0$
$x = \infty \Rightarrow y = \infty$

$$\int_0^\infty (y/k)^{n-1} e^{-y} \frac{dy}{k} = \int_0^\infty \frac{y^{n-1}}{k^n} e^{-y} dy$$

$$= \frac{\Gamma(n)}{k^n}$$

⑥ $\int_0^1 x^5 (1-x)^3 dx = ?$

$$\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

$$\int_0^1 x^5 (1-x)^3 = \beta(6,4)$$

$$= \frac{\Gamma(6)\Gamma(4)}{\Gamma(10)}$$

$$= \frac{5!\,3!}{9!}$$

$\Gamma(n+1) = n!$

$\beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$

⑨ $\int_0^1 x^{5/2} (1-x^2)^{3/2} dx = ?$

$$\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

$$= \int_0^1 x^{7/2-1} (1-x^2)^{5/2-1} dx$$

$$= \beta(7/2, 5/2)$$

(8) Evaluate $\int_0^1 \frac{dx}{(1-x^3)^{1/3}}$

$x^3 = y$ $\quad 3x^2 dx = dy$

$x = y^{1/3}$ $\quad x = 0 \Rightarrow y = 0$ $\quad dx = \frac{dy}{3x^2} = \frac{1}{3}\frac{dy}{(y)^{2/3}}$

$x = 1 \Rightarrow y = 1$

$$\int_0^1 \frac{1}{(1-y)^{1/3}} \frac{1}{3(y)^{2/3}} dy$$

$$\frac{1}{3}\int_0^1 (1-y)^{-1/3} (y)^{-2/3} dy$$

$$\frac{1}{3}\int_0^1 (1-y)^{2/3 - 1} (y)^{\frac{1}{3}-1} dy$$

$$= \frac{1}{3} \frac{\Gamma(1/3)\Gamma(2/3)}{\Gamma(\frac{1}{3}+\frac{2}{3})} = \frac{1}{3}\Gamma(1/3)\Gamma(2/3)$$

(9) Evaluate $\int_0^{\pi/2} \sin^5\theta \cos^{7/2}\theta \, d\theta$

Sol: $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

$2m-1 = 5$ $\quad 2n-1 = 7/2$

$m = 3$ $\quad n = 9/4$

$$\int_0^{\pi/2} \sin^5\theta \cos^{7/2}\theta \, d\theta = \frac{1}{2}\beta(3, 9/4)$$

(10) $\int_0^{\pi/2} \sin^2\theta \, d\theta$ $\qquad$ (11) $\int_0^{\pi/2} \sqrt{\cot\theta} \, d\theta$

$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

$2m-1 = 2$ $\quad 2n-1 = 0$

$m = 3/2$ $\quad n = 1/2$

$$\int_0^{\pi/2} \sin^2\theta \, d\theta = \frac{1}{2}\beta(3/2, 1/2)$$

(11) $\int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta = \int_0^{\pi/2} \cot^{1/2}\theta\, d\theta = \int_0^{\pi/2} \sin^{-1/2}\theta \cos^{-1/2}\theta\, d\theta$

$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$

$2m-1 = 0 \; -1/2 \qquad 2n-1 = 1/2$

$\cancel{m=1/2}\; m = 1/4 \qquad n = 3/4$

$\int_0^{\pi/2} \cot^{1/2}\theta\, d\theta = \frac{1}{2}\beta(1/4, 3/4)$

(12) P.T $\int_0^1 \frac{x^2\,dx}{\sqrt{1-x^4}} \times \int_0^1 \frac{dx}{\sqrt{1+x^4}} = \frac{\pi}{4\sqrt{2}}$

$I_1 = \int_{x=0}^1 \frac{x^2\,dx}{\sqrt{1-x^4}} = \int_0^{\pi/2} \frac{\sin\theta \frac{\cos\theta}{2\sqrt{\sin\theta}}}{\sqrt{1-\sin^2\theta}} = \frac{1}{2}\int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta$

$x^2 = \sin\theta$

$2x\,dx = \cos\theta\, d\theta$

$x = 0 \Rightarrow \theta = 0$

$x = 1 \Rightarrow \theta = \pi/2$

$= \frac{1}{2}\int_0^{\pi/2} \sin^{1/2}\theta \cos^0\theta\, d\theta$

$= \frac{1}{\cancel{2}\,4} \beta(3/4, 1/2)$

$= \frac{1}{4} \frac{\Gamma(3/4)\Gamma(1/2)}{\Gamma(5/4)}$

$= \frac{\sqrt{\pi}}{4} \frac{\Gamma(3/4)}{\Gamma(5/4)}$

$I_2 = \int_0^1 \frac{dx}{\sqrt{1+x^4}}$

$x^2 = \tan\theta$

$2x\,dx = \sec^2\theta\, d\theta$

$dx = \frac{\sec^2\theta\, d\theta}{2\sqrt{\tan\theta}}$

$x = 0 \Rightarrow \theta = 0$

$x = 1 \Rightarrow \theta = \pi/4$

$= \frac{1}{2}\int_0^{\pi/4} \frac{\frac{\sec^2\theta\, d\theta}{\sqrt{\tan\theta}}}{\sqrt{1+\tan^2\theta}}$

$= \frac{1}{2}\int_0^{\pi/4} \frac{\sec^2\theta}{\sqrt{\tan\theta}} \frac{1}{\sec\theta}\, d\theta$

$= \frac{1}{2}\int_0^{\pi/4} \frac{\sec\theta}{\sqrt{\tan\theta}}\, d\theta$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{1}{\cos\theta}\,\frac{\sqrt{\cos\theta}}{\sqrt{\sin\theta}}$$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{1}{\sqrt{\frac{\sin 2\theta}{2}}}$$

$$= \frac{1}{\sqrt{2}}\int_0^{\pi/4} \frac{1}{\sqrt{\sin 2\theta}}\,d\theta \qquad = \frac{1}{\sqrt{2}}\int_0^{\pi/4} \frac{1}{\sqrt{\sin 2\theta}}\,d\theta$$

$2\theta = t$

$2\,d\theta = dt$

$\theta = 0 \Rightarrow t = 0$

$\theta = \pi/4 \Rightarrow t = \pi/2$

$$= \frac{1}{2\sqrt{2}}\int_{t=0}^{\pi/2} \frac{1}{\sqrt{\sin t}}\,dt$$

$$= \frac{1}{2\sqrt{2}}\int_0^{\pi/2} \sin^{-1/2} t\,dt$$

$$= \frac{1}{2\sqrt{2}}\,\beta(1/4, 1/2)$$

$$= \frac{1}{2\sqrt{2}}\,\cancel{\beta}\,\frac{\Gamma(1/4)\,\Gamma(1/2)}{\Gamma(3/4)}$$

$$\boxed{\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}}$$

$$= \frac{\sqrt{\pi}}{2\sqrt{2}}$$

(18) P.T $\Gamma(n)\Gamma(1-n) = \dfrac{\pi}{\sin n\pi}$

Provided $\displaystyle\int_0^\infty \frac{x^{2m}}{1+x^{2n}}\,dx = \frac{\pi}{2n}\operatorname{cosec}\left(\frac{2m+1}{2n}\right)\pi$ where $m>0$, $n>0$

Sol:
$$\beta(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$m+n = 1$$

$$\beta(1-n,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)}\,dx$$

$$\frac{\Gamma(1-n)\Gamma(n)}{\Gamma(1)} = \int_0^\infty \frac{x^{n-1}}{1+x}\,dx$$

$$\Gamma(n)\Gamma(1-n) = \int_0^\infty \frac{x^{n-1}}{1+x}\,dx \quad \text{——— (1)}$$

we have $\displaystyle\int_0^\infty \frac{x^{2m}}{1+x^{2n}}\,dx = \frac{\pi}{2n}\operatorname{cosec}\left(\frac{2m+1}{2n}\right)\pi$

$$x^{2n} = t \qquad \frac{2m+1}{2n} = s$$

$$x = t^{1/2n}$$

$$dx = \frac{1}{2n}\,t^{1/2n-1}\,dt$$

$$x = 0 \Rightarrow t = 0$$
$$x = \infty \Rightarrow t = \infty$$

$$\int_0^\infty \frac{(t^{1/2n})^{2m}}{1+t}\,\frac{1}{2n}\,t^{1/2n-1}\,dt = \frac{\pi}{2n}\operatorname{cosec} s\pi$$

$$\frac{1}{2n}\int_0^\infty \frac{t^{\frac{2m+1}{2n}-1}}{1+t}\,dt = \frac{\pi}{2n}\operatorname{cosec} s\pi$$

$$\int_0^\infty \frac{t^{s-1}}{1+t}\,dt = \pi\operatorname{cosec} s\pi = \frac{\pi}{\sin s\pi}$$

$$\int_0^\infty \frac{x^{n-1}}{1+x}\,dx = \frac{\pi}{\sin s\pi} = \Gamma(n)\Gamma(1-n)$$

(14) when n is a positive integer

P.T $2^n \Gamma(n+\frac{1}{2}) = 1 \cdot 3 \cdot 5 \cdots (2n-1)\sqrt{\pi}$

Sol:
$$\Gamma(n+\tfrac{1}{2}) = \Gamma(n-\tfrac{1}{2}+1)$$
$$= (n-\tfrac{1}{2})\Gamma(n-\tfrac{1}{2})$$
$$= (n-\tfrac{1}{2})\Gamma(n-\tfrac{3}{2}+1)$$
$$= (n-\tfrac{1}{2})(n-\tfrac{3}{2})\Gamma(n-\tfrac{3}{2})$$
$$= (n-\tfrac{1}{2})(n-\tfrac{3}{2})\Gamma(n-\tfrac{5}{2}+1)$$
$$= (n-\tfrac{1}{2})(n-\tfrac{3}{2})(n-\tfrac{5}{2})\Gamma(n-\tfrac{5}{2})$$
$$= \frac{(2n-1)(2n-3)(2n-5)\cdots 5 \cdot 3 \cdot 1}{2^n}$$
$$2^n\Gamma(n+\tfrac{1}{2}) = 1 \cdot 3 \cdot 5 \cdots (2n-1)\sqrt{\pi}$$