Discrete Time Fourier Transformation (DTFT)

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1. DFT and DTFT

- DTFT is the Fourier transform of choice for analyzing infinite-length signals and systems
- Useful for conceptual, but not Matlab friendly (infinitely-long vectors)
- We will derive DTFT as the limit of the DFT as the signal length $N \to \infty$

$$\omega = \frac{2\pi}{N} k$$

The Centered DFT

- Both $x[n]$ and $X[k]$ can be interpreted as periodic with period N, so we will shift the intervals of interest in time and frequency to be centered around $n, k = 0$

$$-\frac{N}{2} \;\leq\; n, k \;\leq\; \frac{N}{2} - 1$$

- The modified forward and inverse DFT formulas are

$$X_u[k] \;=\; \sum_{n=-N/2}^{N/2-1} x[n]\, e^{-j\frac{2\pi}{N}kn}, \qquad -\frac{N}{2} \;\leq\; k \;\leq\; \frac{N}{2} - 1$$

$$x[n] \;=\; \frac{1}{N} \sum_{k=-N/2}^{N/2-1} X_u[k]\, e^{j\frac{2\pi}{N}kn} \qquad -\frac{N}{2} \;\leq\; n \;\leq\; \frac{N}{2} - 1$$

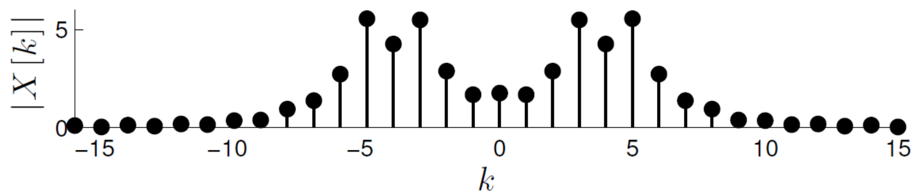
Take It To The Limit (1)

$$X_u[k] \;=\; \sum_{n=-N/2}^{N/2-1} x[n]\, e^{-j\frac{2\pi}{N}kn}, \qquad -\frac{N}{2} \;\leq\; k \;\leq\; \frac{N}{2} - 1$$

- Let the signal length N increase towards ∞ and study what happens to $X_u[k]$

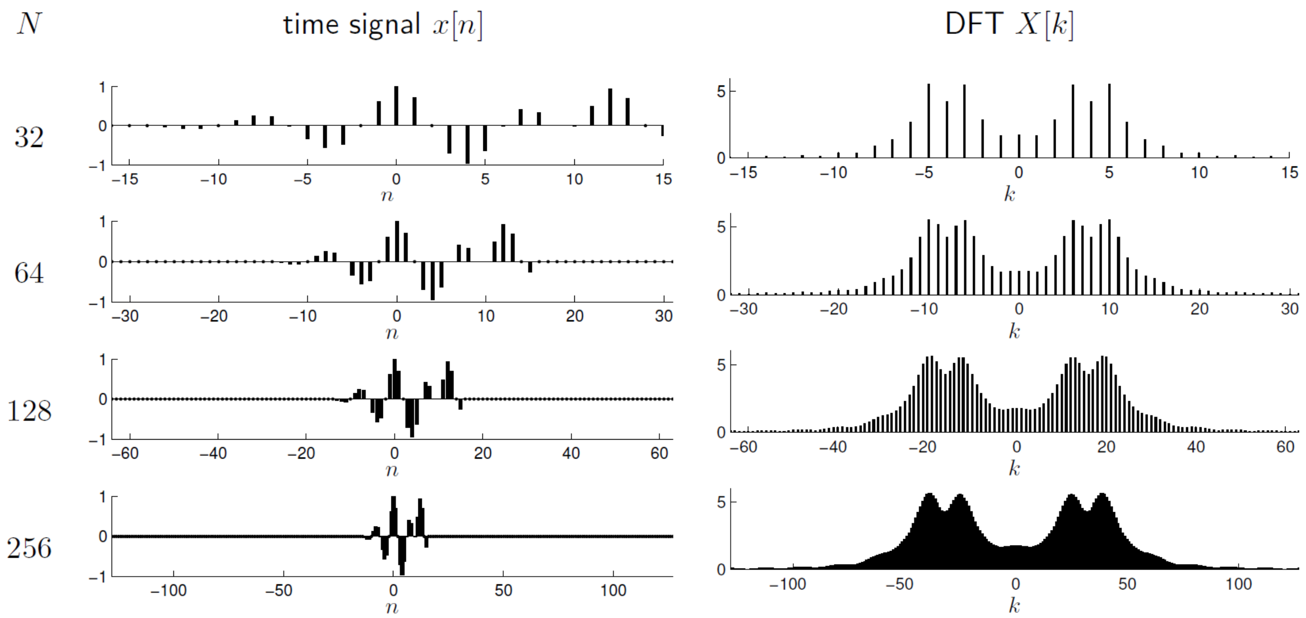
- **Key fact:** No matter how large N grows, the frequencies of the DFT sinusoids remain in the interval

$$-\pi \;\leq\; \omega_k = \frac{2\pi}{N}k \;<\; \pi$$



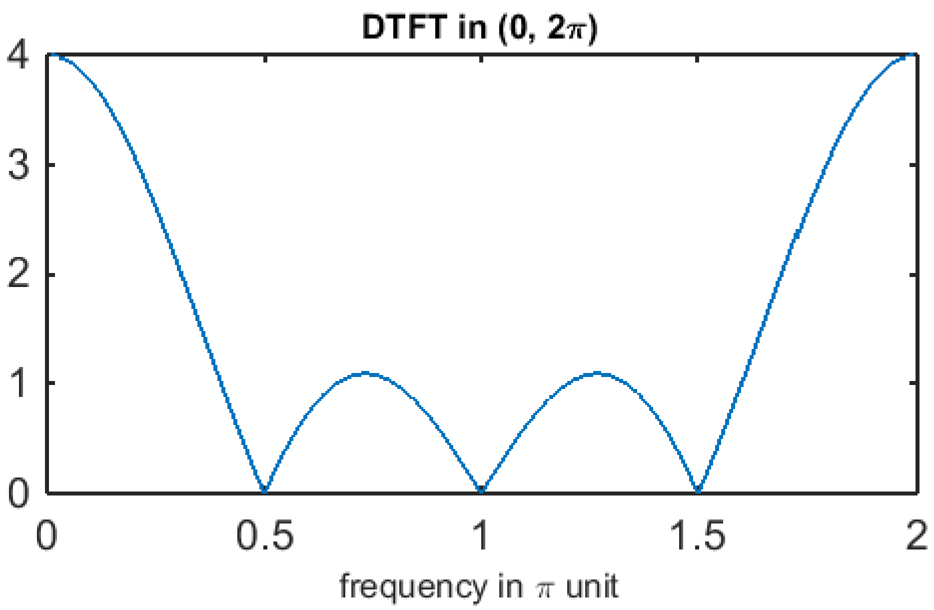
Take It To The Limit (2)

$$X_u[k] \;=\; \sum_{n=-N/2}^{N/2-1} x[n]\, e^{-j\frac{2\pi}{N}kn}$$



In [11]:

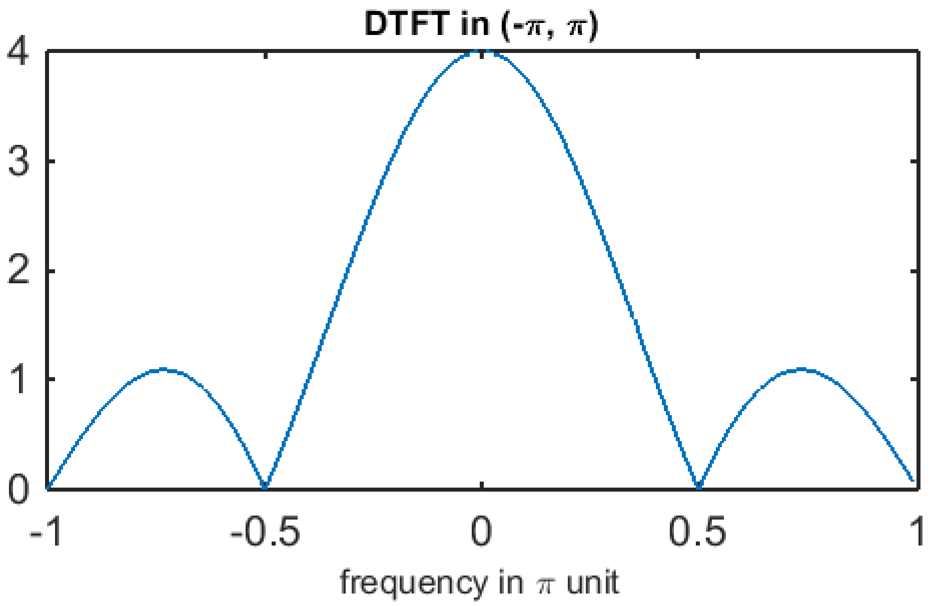
```
%plot -s 560,300

% dtft from definition
n = 0:3;
x = [1 1 1 1];

N = 200;

w2 = [0:N]*2*pi/N; w = w2(1:end-1);
Xdtft = sin(2*w)./sin(w/2).*exp(-1j*3*w/2);

plot(w/pi,abs(Xdtft))
xlabel('frequency in \pi unit','fontsize',8),
title('DTFT in (0, 2\pi)','fontsize',8)
```



Out[11]:

In [8]:

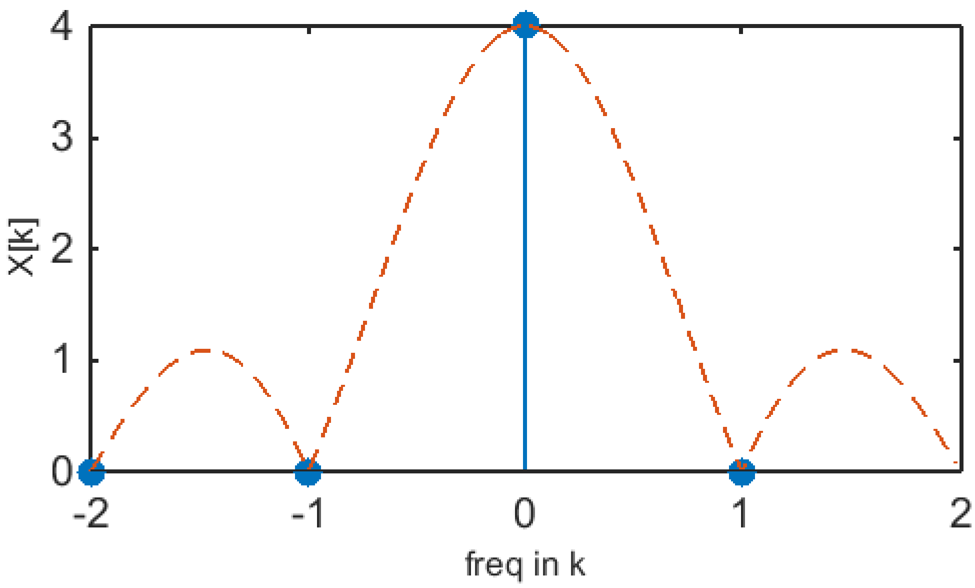
```
k = [0:N/2-1 -N/2:-1];
w = k*2*pi/N;
ws = fftshift(w);
Xdtfts = fftshift(Xdtft);

plot(ws/pi,abs(Xdtfts))
xlabel('frequency in \pi unit','fontsize',8),
title('DTFT in (-\pi, \pi)','fontsize',8)
```



Out[8]:

In [14]:

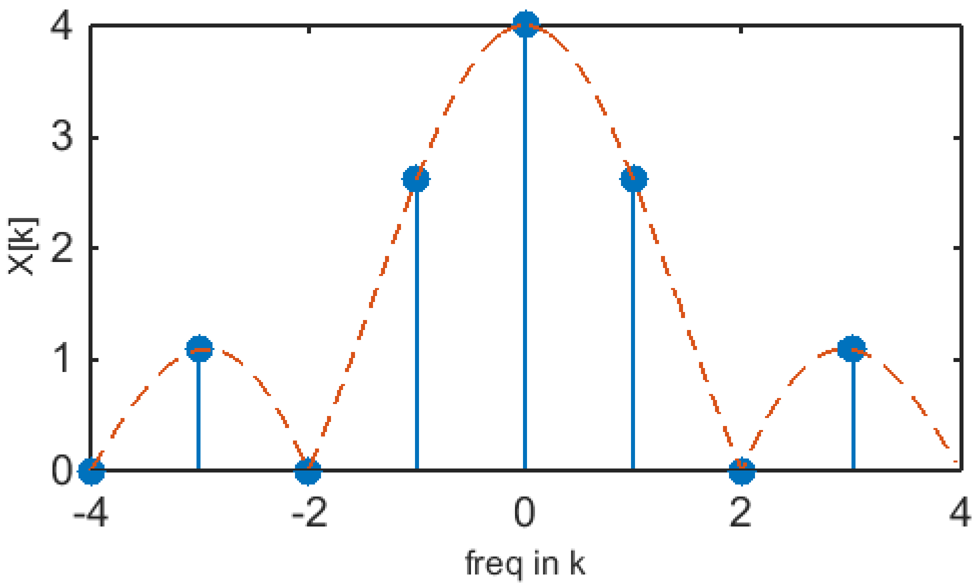
```
x = [1,1,1,1];
N = length(x);

k = [0:N/2-1 -N/2:-1];
ks = fftshift(k);

X = dft(x,N);
Xs = fftshift(X);

stem(ks,abs(Xs),'filled'),
xlabel('freq in k','fontsize',8), ylabel('X[k]','fontsize',8),
xlim([-N/2,N/2]), hold on
plot(ws*N/(2*pi),abs(Xdtfts),'--'), hold off
```



Out[14]:

In [15]:

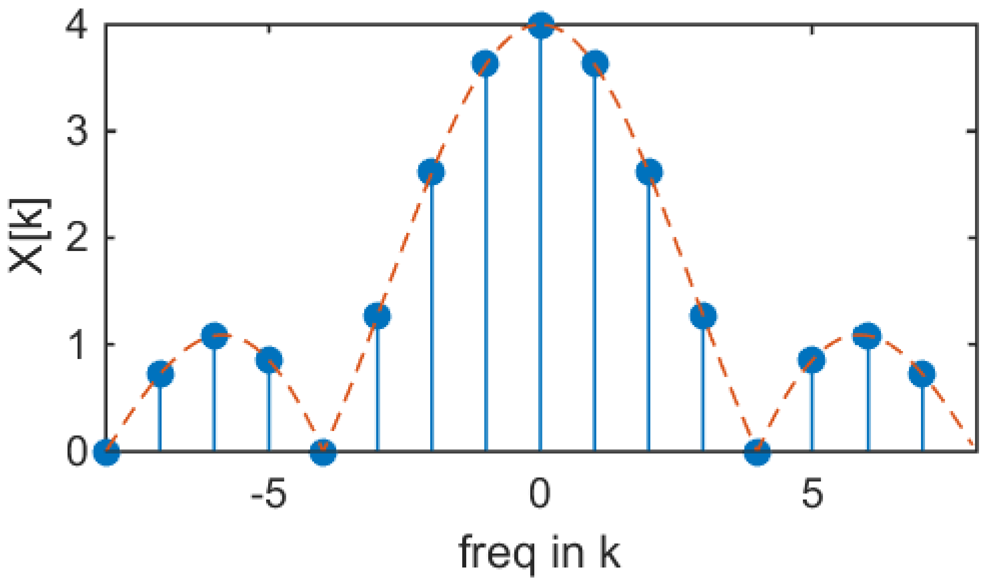
```
x = [1,1,1,1,zeros(1,4)];
N = length(x);

k = [0:N/2-1 -N/2:-1];
ks = fftshift(k);

X = dft(x,N);
Xs = fftshift(X);

stem(ks,abs(Xs),'filled'),
xlabel('freq in k','fontsize',8), ylabel('X[k]','fontsize',8),
xlim([-N/2,N/2]), hold on
plot(ws*N/(2*pi),abs(Xdtfts),'--'), hold off
```



Out[15]:

In [5]:

```
x = [1,1,1,1,zeros(1,12)];
N = length(x);

k = [0:N/2-1 -N/2:-1];
ks = fftshift(k);

X = dft(x,N);
Xs = fftshift(X);

stem(ks,abs(Xs),'filled'),
xlabel('freq in k'), ylabel('X[k]'), xlim([-N/2,N/2]), hold on
plot(ws*N/(2*pi),abs(Xdtfts),'--'), hold off
```



Out[5]:

In [2]:

```
% DTFT using the output of FFT (or DFT)

x = [1,1,1,1,zeros(1,2^6-4)];
N = length(x);

k = [0:N/2-1 -N/2:-1];
ks = fftshift(k);

X = dft(x,N);
Xs = fftshift(X);

stem(ks,abs(Xs),'filled','markersize',3)
xlabel('freq in k'), ylabel('X[k]'), xlim([-N/2,N/2]), hold on
plot(ws*N/(2*pi),abs(Xdtfts),'--'), hold off
```

Out[2]: Undefined function or variable 'ws'.

Discrete Time Fourier Transform (Forward)

- As $N \to \infty$, the forward DFT converges to a function of the **continuous frequency variable** ω that we will call the **forward discrete time Fourier transform** (DTFT)

$$\sum_{n=-N/2}^{N/2-1} x[n]\, e^{-j\frac{2\pi}{N}kn} \longrightarrow \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \;=\; X(\omega), \qquad -\pi \le \omega < \pi$$

- Recall: Inner product for infinite-length signals

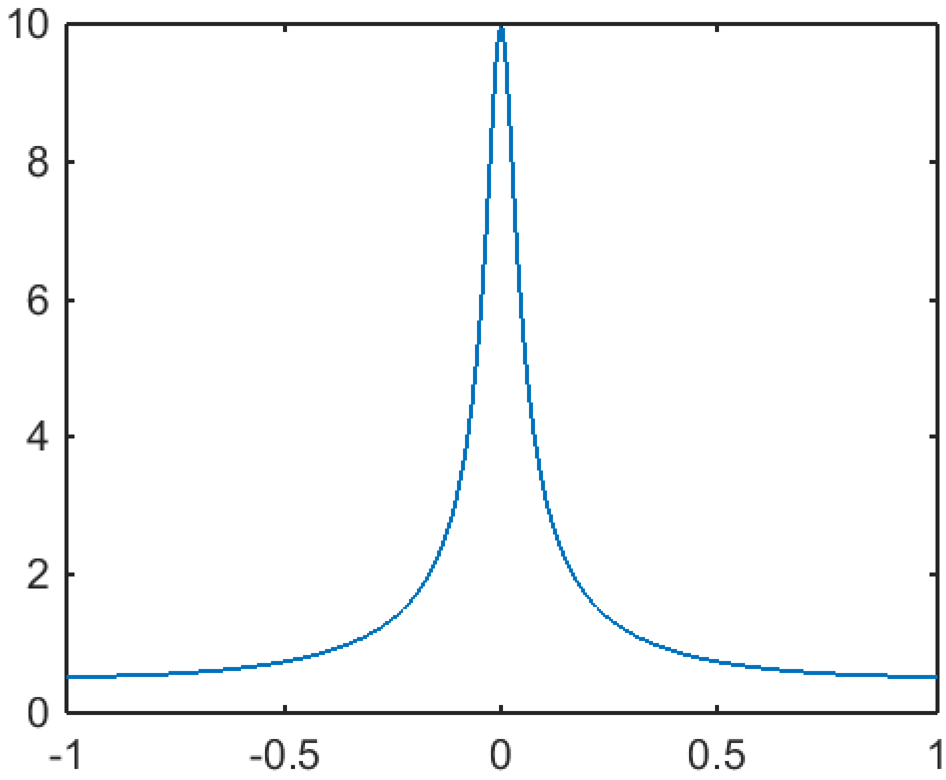
$$\langle x, y\rangle \;=\; \sum_{n=-\infty}^{\infty} x[n]\, y[n]^*$$

- **Analysis interpretation:** The value of the DTFT $X(\omega)$ at frequency ω measures the similarity of the infinite-length signal $x[n]$ to the infinite-length sinusoid $e^{j\omega n}$

1.1. Analytic form of DTFT (exact transformation)

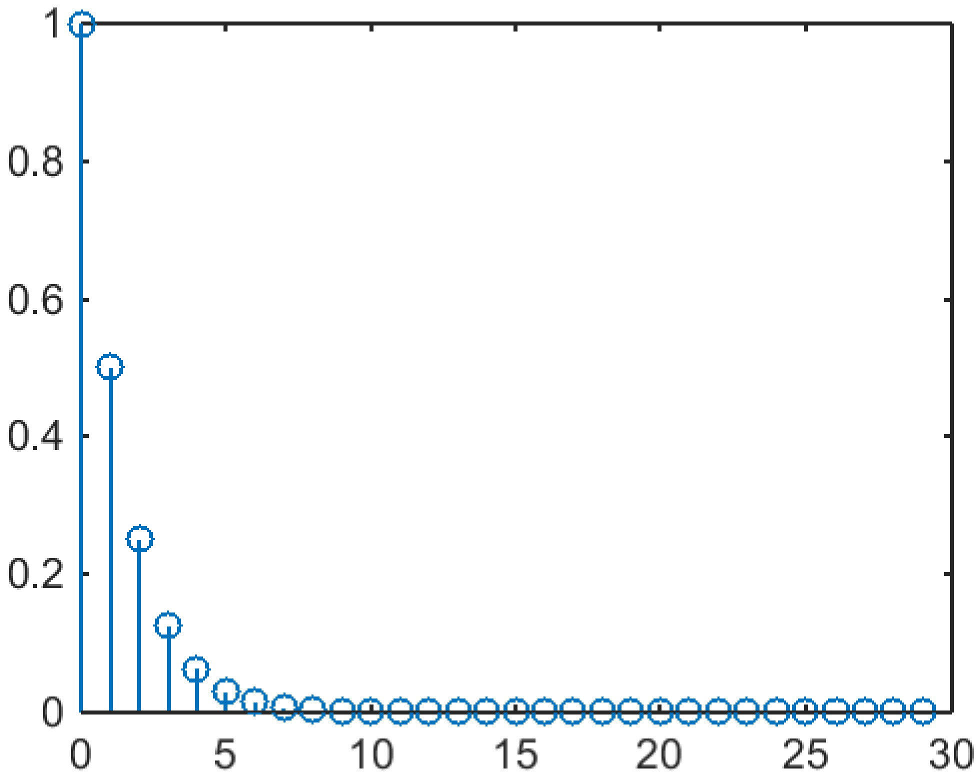
$$x[n] = (0.5)^n u[n] \qquad \leftrightarrow \qquad X(e^{j\omega}) = \frac{e^{j\omega}}{e^{j\omega} - 0.5}$$

In [4]:
```
w = linspace(-pi,pi,2^8);
X = exp(1j*w)./(exp(1j*w) - 0.9);
plot(w/pi,abs(X))
```



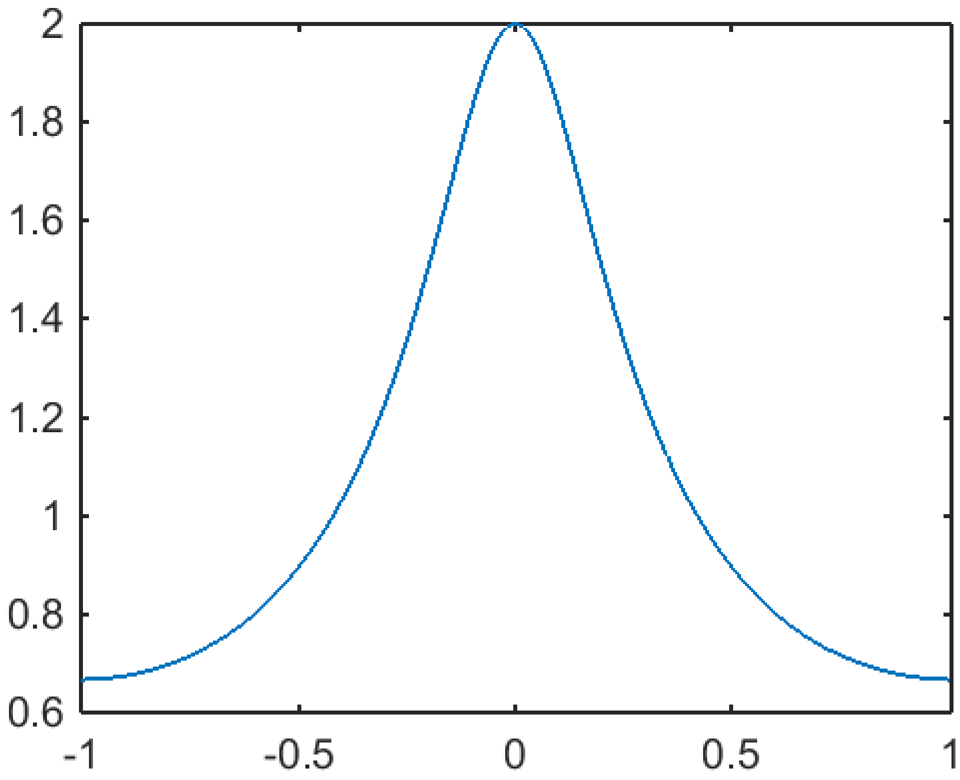
Out[4]:

In [6]:
```
N = 30;
n = 0:N-1;
x = zeros(1,N);
for i = 1:N
    x(i) = 0.5^(i-1);
end
stem(n,x)
```



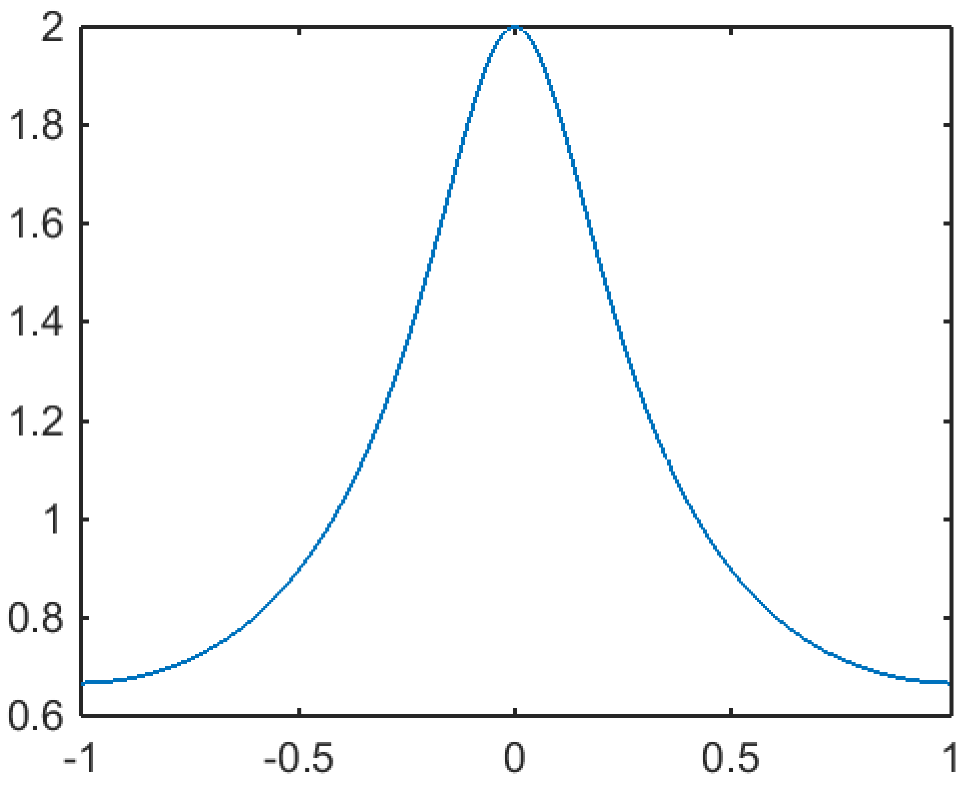
Out[6]:

In [9]:
```
w = linspace(-1,1,2^8)*pi;
X = exp(-1j*(w'*n))*x';

plot(w/pi,abs(X))
```



Out[9]:

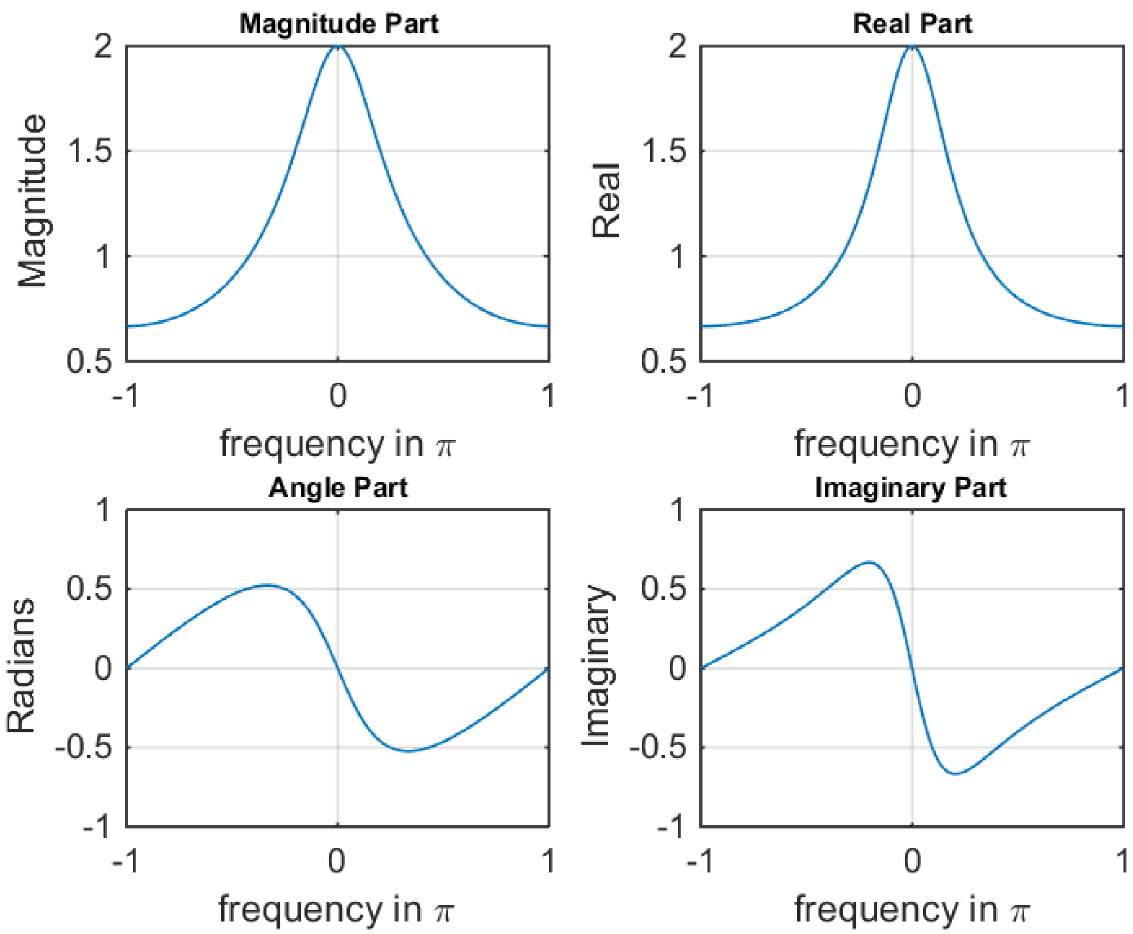
In [11]:
```
X = dtft(x,n,w);
plot(w/pi,abs(X))
```



Out[11]:

In [7]:

```
%plot -s 800,600

N = 500;
w2 = [-1:1/N:1]*pi;
w = w2(1:end-1);

% closed form of DTFT
X = exp(1j*w)./(exp(1j*w) - 0.5*ones(size(w)));

magX = abs(X);  angX = angle(X);
realX = real(X);    imagX = imag(X);

subplot(2,2,1); plot(w/pi,magX); grid
xlabel('frequency in \pi'); title('Magnitude Part','fontsize',8); ylabel('Magnitude')

subplot(2,2,3); plot(w/pi,angX); grid
xlabel('frequency in \pi'); title('Angle Part','fontsize',8); ylabel('Radians')

subplot(2,2,2); plot(w/pi,realX); grid
xlabel('frequency in \pi'); title('Real Part','fontsize',8); ylabel('Real')

subplot(2,2,4); plot(w/pi,imagX); grid
xlabel('frequency in \pi'); title('Imaginary Part','fontsize',8); ylabel('Imaginary')
```

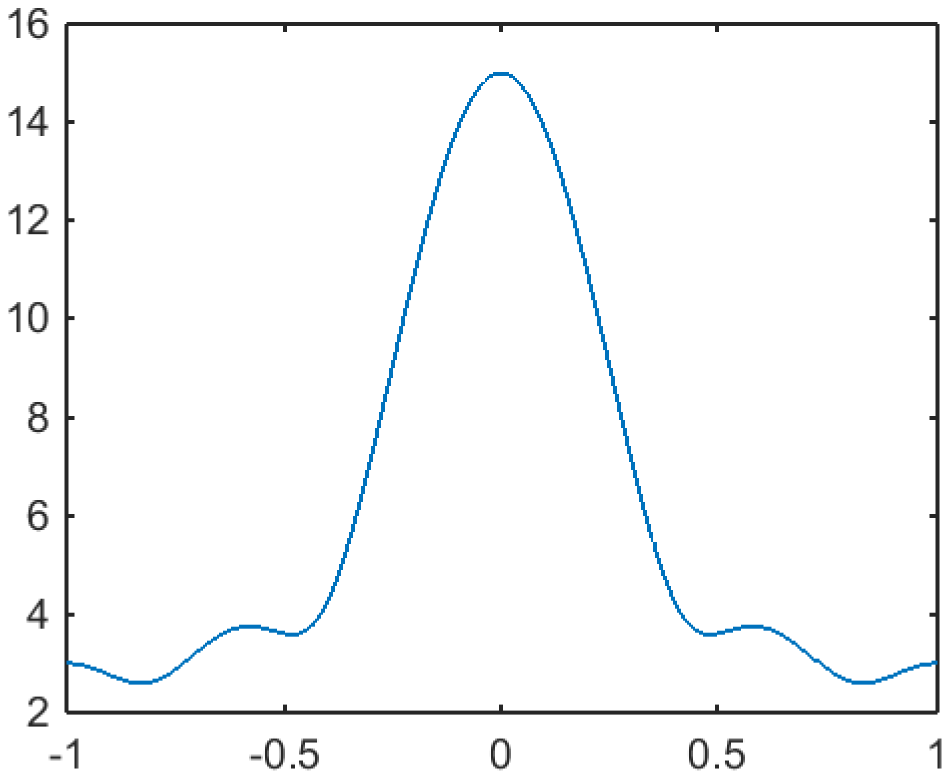


Out[7]:

1.2. DTFT of a numerical computation (using a definition)

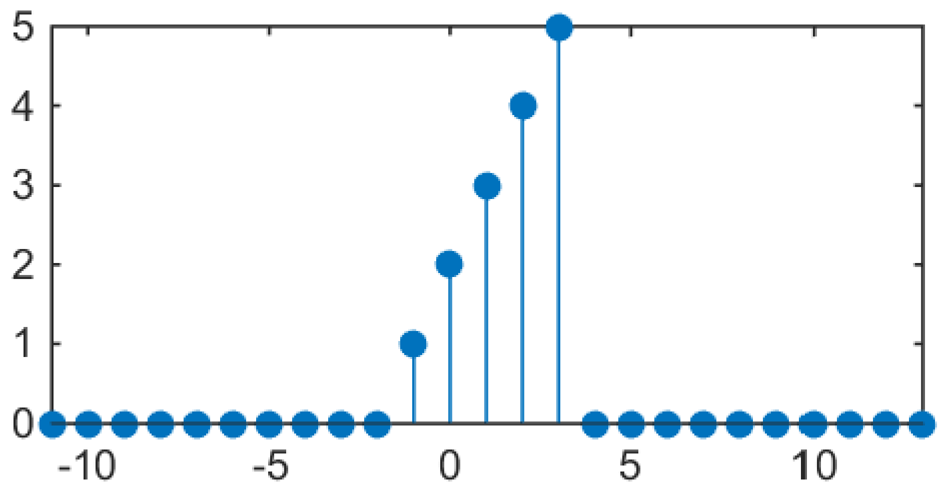
Example 1

In [13]:
```
x = [0,1,2,3,4,5];
n = -2:3;
X = dtft(x,n,w);
plot(w/pi,abs(X))
```



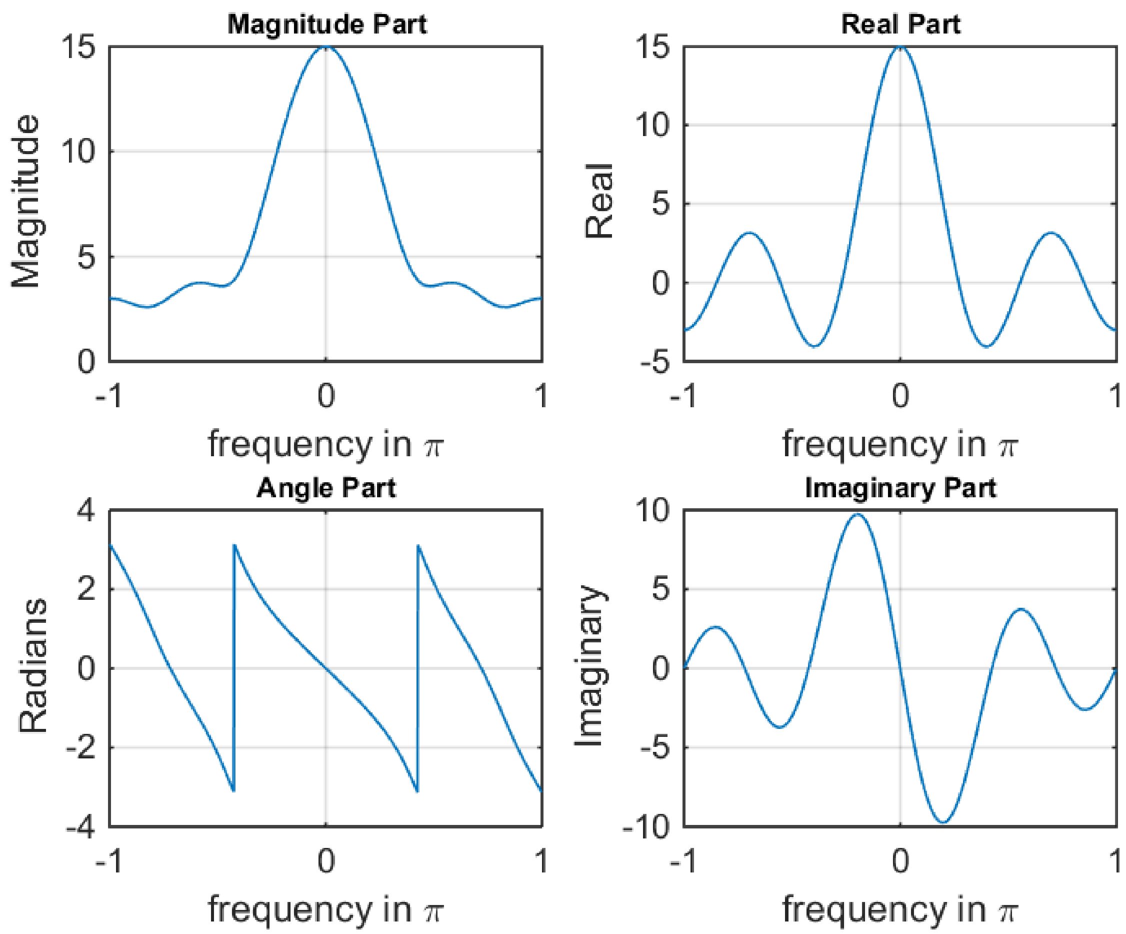
Out[13]:

In [8]:
```
%plot -s 560,250
% in MATLAB, we represent sequences and indices as row vectors

n = -1:3;
x = 1:5;                                        % sequence x(n)

% just for a nice plot
m = 10;
stem([n(1)-m:n(end)+m],[zeros(1,m),x,zeros(1,m)],'filled'), axis tight
```



Out[8]:

In [9]:
```
%plot -s 800,600

%% DTFT code
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
N = 500;
k = -N:N-1; w = (pi/N)*k;
X = x * (exp(-1j*pi/500)).^(n'*k);       % DTFT using matrix-vector product
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% plots
magX = abs(X);  angX = angle(X);
realX = real(X);    imagX = imag(X);

subplot(2,2,1); plot(w/pi,magX); grid
xlabel('frequency in \pi'); title('Magnitude Part','fontsize',8); ylabel('Magnitude')

subplot(2,2,3); plot(w/pi,angX); grid
xlabel('frequency in \pi'); title('Angle Part','fontsize',8); ylabel('Radians')

subplot(2,2,2); plot(w/pi,realX); grid
xlabel('frequency in \pi'); title('Real Part','fontsize',8); ylabel('Real')

subplot(2,2,4); plot(w/pi,imagX); grid
xlabel('frequency in \pi'); title('Imaginary Part','fontsize',8); ylabel('Imaginary')
```



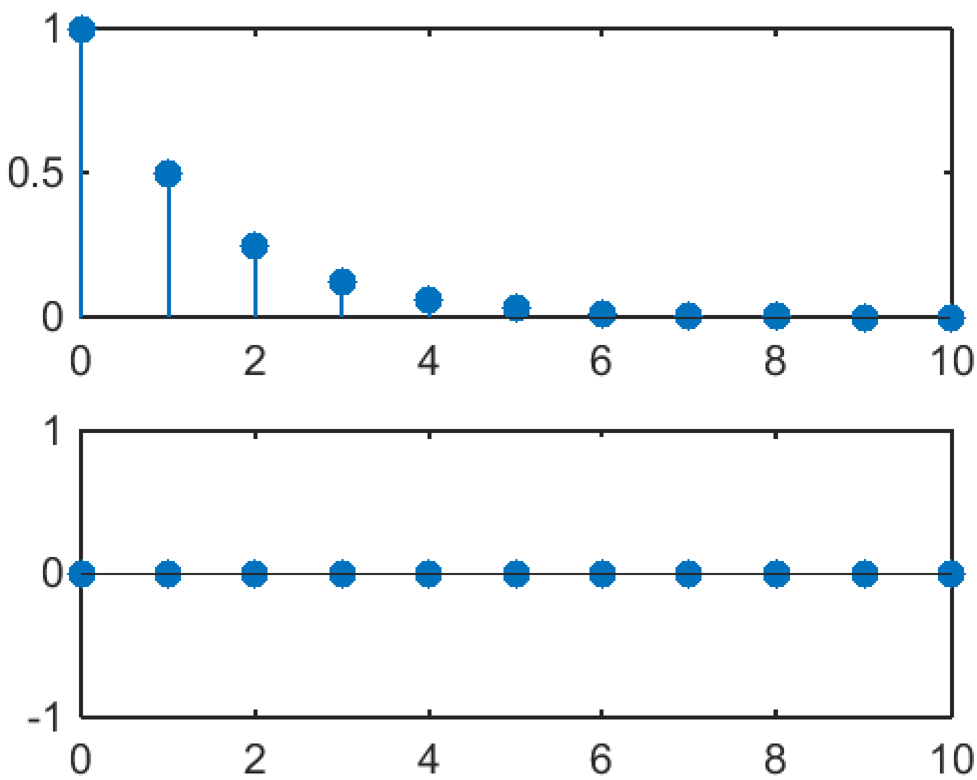
Out[9]:

Example 2

$$x[n] = \left(0.5e^{(j\pi/3)}\right)^n$$

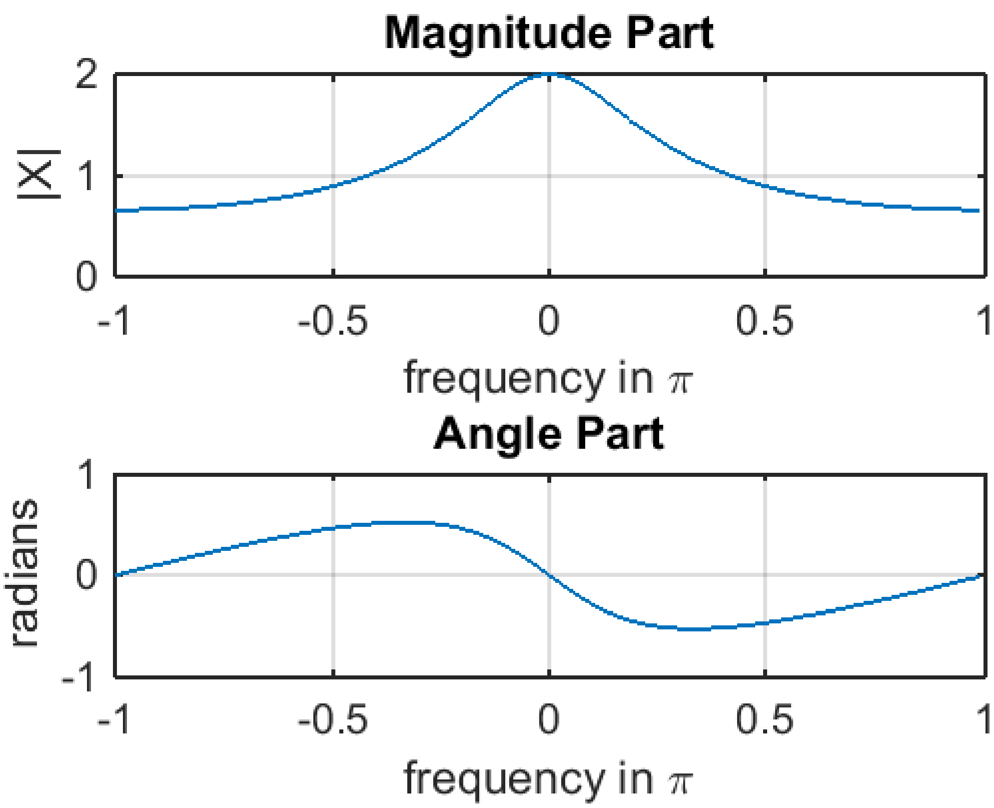
In [20]:

```
%plot -s 560,420
n = 0:10;
x = (0.5).^n;

subplot(2,1,1),  stem(n,real(x),'filled')
subplot(2,1,2),  stem(n,imag(x),'filled')
```



Out[20]:

In [21]:
```
N = 100;
k = -N:N-1; w = (pi/N)*k;

X = x*(exp(-1j*pi/N)).^(n'*k);     % DTFT

magX = abs(X); angX = angle(X);

subplot(2,1,1); plot(w/pi,magX);grid
xlabel('frequency in \pi'); ylabel('|X|')
title('Magnitude Part')

subplot(2,1,2); plot(w/pi,angX);grid
xlabel('frequency in \pi'); ylabel('radians')
title('Angle Part')
```



Out[21]:

2. Numerical DTFT Computation

```
function X = dtft(x,n,w)
% X = dtft(x, n, w)
%
% X = DTFT values computed at w frequencies
% x = finite duration sequence over n
% n = sample position vector
% w = frequency location vector

X = exp(-1j*(w'*n))*x';

end
```

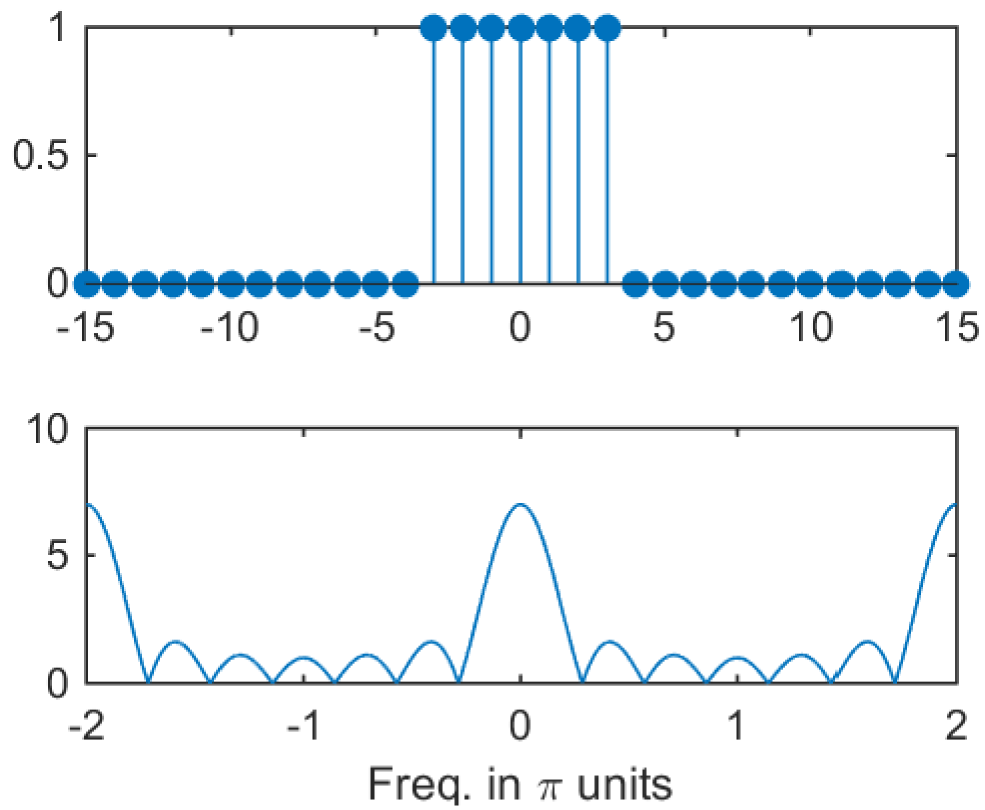
DTFT of the unit pulse

$$p[n] = \begin{cases} 1 & -M \le n \le M \\ 0 & \text{otherwise} \end{cases}$$

$$P(\omega) = \sum_{n=-\infty}^{\infty} p[n]\, e^{-j\omega n} = \sum_{n=-M}^{M} e^{-j\omega n} = \frac{e^{j\omega M} - e^{-j\omega(M+1)}}{1 - e^{-j\omega}} =$$

$$\frac{e^{-j\omega/2}\left(e^{j\omega\frac{2M+1}{2}} - e^{-j\omega\frac{2M+1}{2}}\right)}{e^{-j\omega/2}\left(e^{j\omega/2} - e^{-j\omega/2}\right)} = \frac{2j\,\sin\left(\omega\frac{2M+1}{2}\right)}{2j\,\sin\left(\frac{\omega}{2}\right)}$$

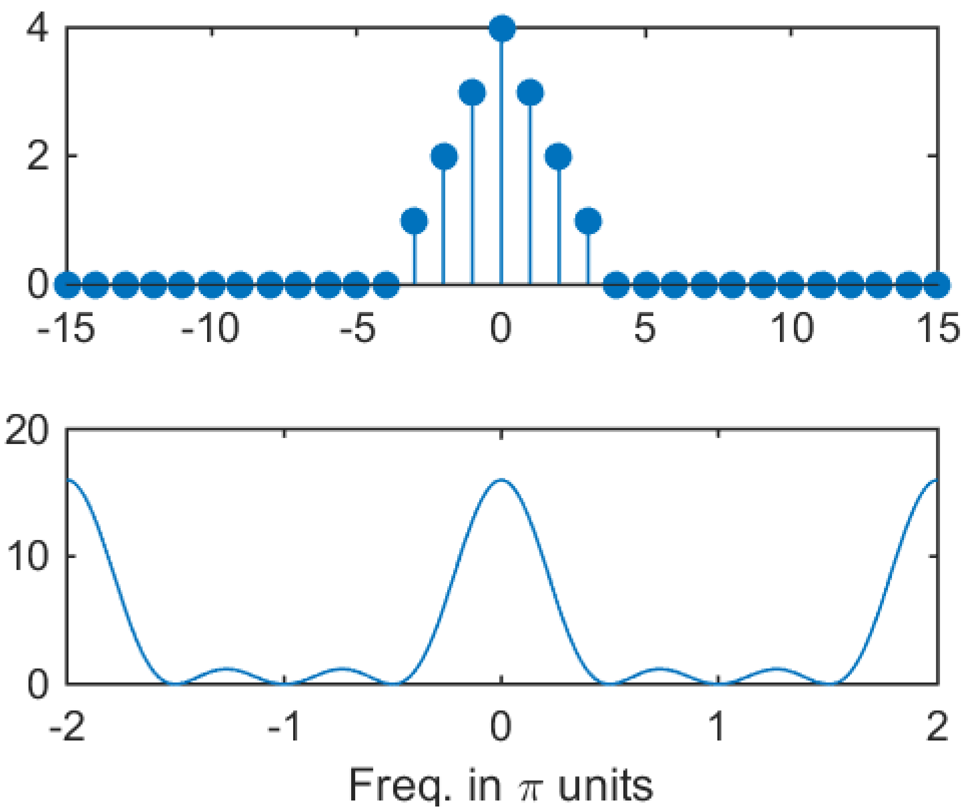
In [12]:
```
%plot -s 560,420

x = [1 1 1 1 1 1 1];
n = -3:3;
w = linspace(-2,2,2^10)*pi;

X = dtft(x,n,w);

xd = zeros(1,31);
nd = -15:15;
[y,ny] = sigadd(xd,nd,x,n);

subplot(2,1,1), stem(ny,y,'filled')
subplot(2,1,2), plot(w/pi,abs(X)), xlabel('Freq. in \pi units')
```



Out[12]:

DTFT of triangle

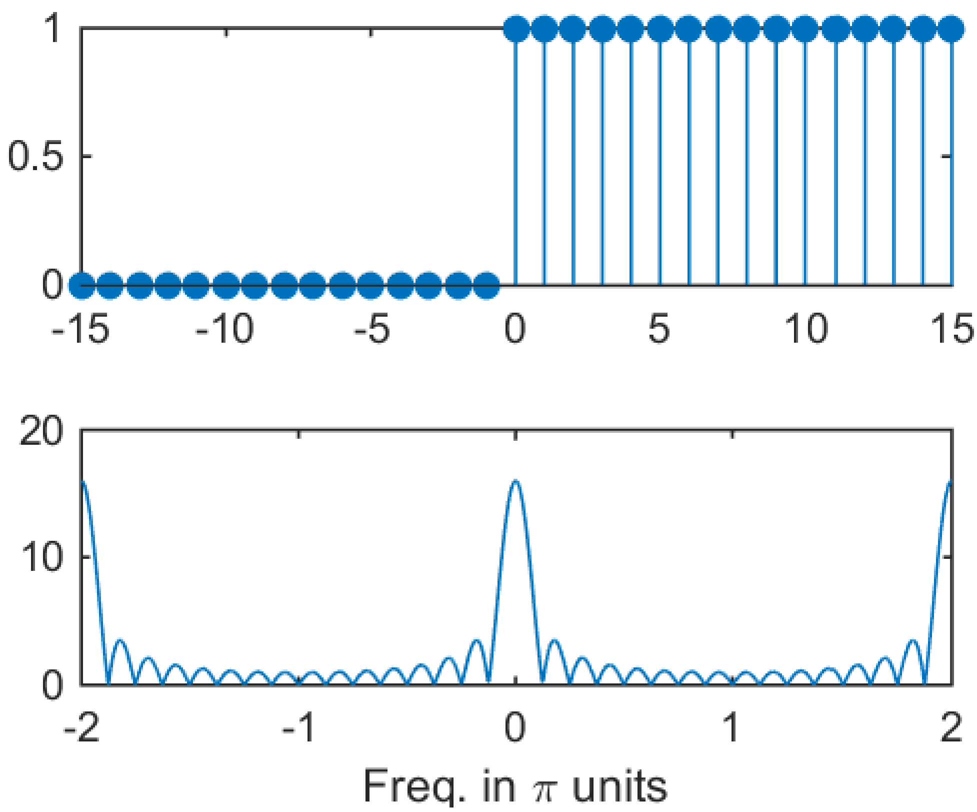
In [13]:
```
%plot -s 560,420

x = [1 2 3 4 3 2 1];
n = -3:3;
w = linspace(-2,2,2^10)*pi;

X = dtft(x,n,w);

xd = zeros(1,31);
nd = -15:15;
[y,ny] = sigadd(xd,nd,x,n);

subplot(2,1,1), stem(ny,y,'filled')
subplot(2,1,2), plot(w/pi,abs(X)), xlabel('Freq. in \pi units')
```



Out[13]:

DTFT of pulse

In [14]:
```
n = 0:15;
x = ones(1,length(n));
w = linspace(-2,2,2^10)*pi;

X = dtft(x,n,w);

xd = zeros(1,31);
nd = -15:15;
[y,ny] = sigadd(xd,nd,x,n);

subplot(2,1,1), stem(ny,y,'filled')
subplot(2,1,2), plot(w/pi,abs(X)), xlabel('Freq. in \pi units')
```

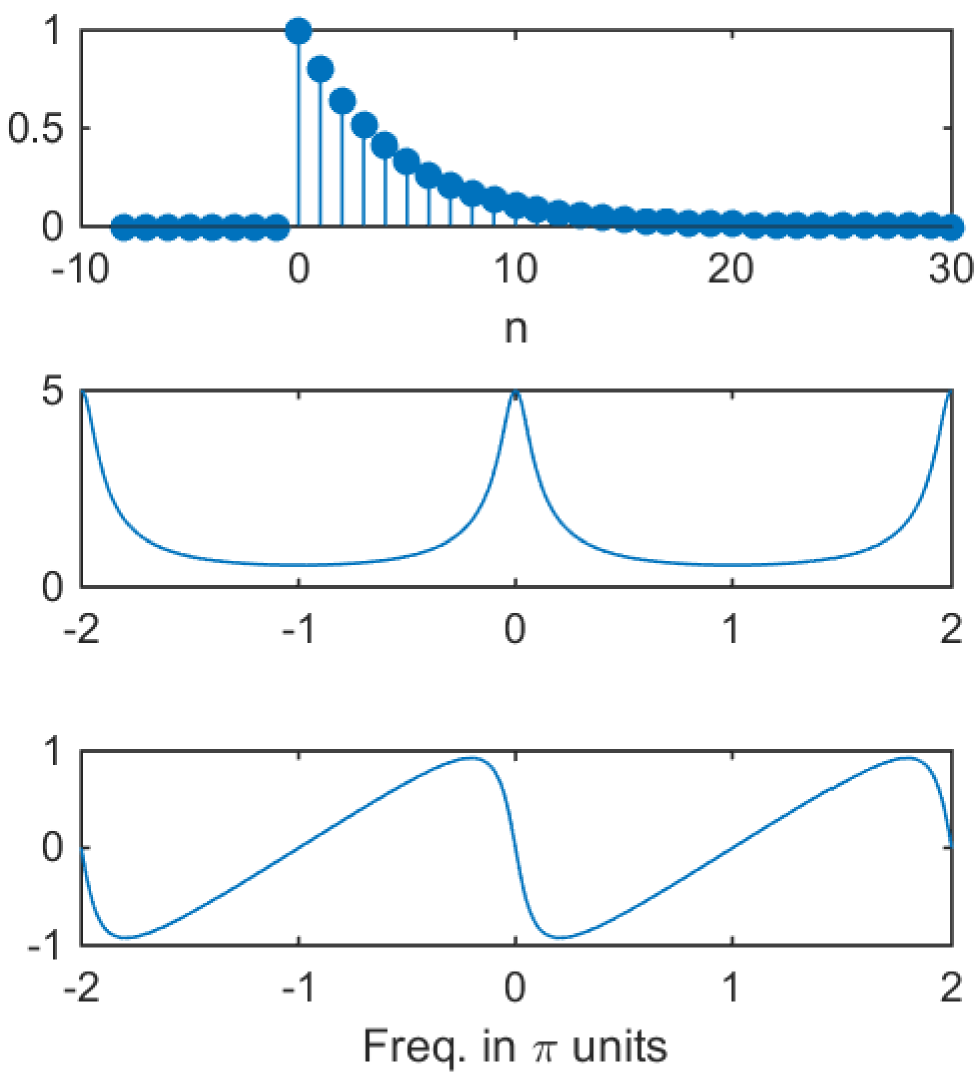


Out[14]:

DTFT of a one-sided exponential

$$h[n] = \alpha^n u[n] \qquad \longleftrightarrow \qquad H(\omega) = \frac{1}{1 - \alpha e^{-j\omega}}$$

In [15]:
```
%plot -s 560,600

N = 30;

x = zeros(1,N);
for i = 1:N
    x(i) = 0.8^(i-1);
end

n = 0:N-1;
w = linspace(-2,2,2^10)*pi;

X = dtft(x,n,w);

nd = -8:N;
xd = zeros(size(nd));
[y,ny] = sigadd(xd,nd,x,n);

subplot(3,1,1), stem(ny,y,'filled'), xlabel('n')
subplot(3,1,2), plot(w/pi,abs(X)),
subplot(3,1,3), plot(w/pi,phase(X)), xlabel('Freq. in \pi units')
```



Out[15]:

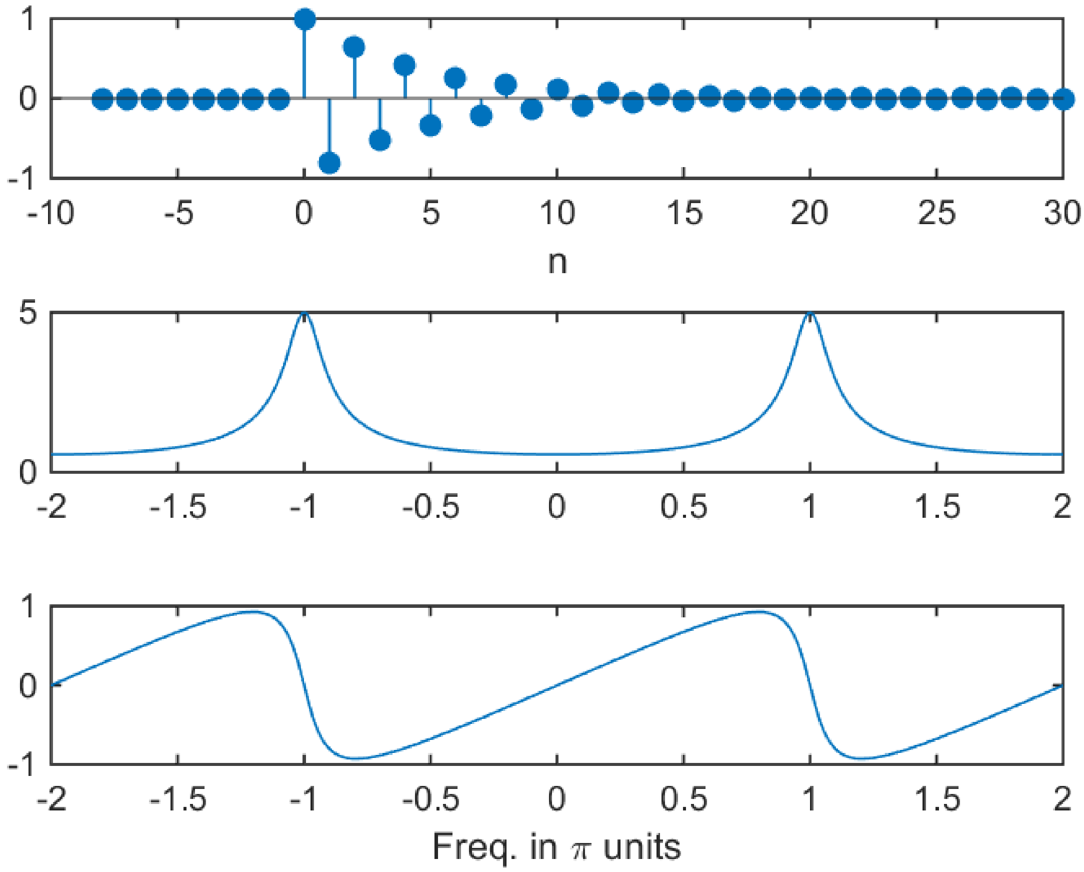
3. Property:

DTFT and Modulation

$$e^{-j\omega_0 n}x[n] \quad \longleftrightarrow \quad X(\omega-\omega_0)$$
$$e^{-j\omega_0 n}x[n] = (-1)^n x[n] \qquad \text{when} \quad \omega_0 = \frac{2\pi}{N}\frac{N}{2} = \pi$$

In [16]:

```
%plot -s 800,600

N = 30;
nd = -8:N;
xd = zeros(size(nd));

x = zeros(1,N);
for i = 1:N
    x(i) = (-0.8)^(i-1);
end
n = 0:N-1;
%w = linspace(-1,1,2^10)*pi;
w = linspace(-2,2,2^10)*pi;

X = dtft(x,n,w);

subplot(3,1,1), stem(nd,sigadd(xd,nd,x,n),'filled'), xlabel('n')
subplot(3,1,2), plot(w/pi,abs(X)),
subplot(3,1,3), plot(w/pi,phase(X)), xlabel('Freq. in \pi units')
```



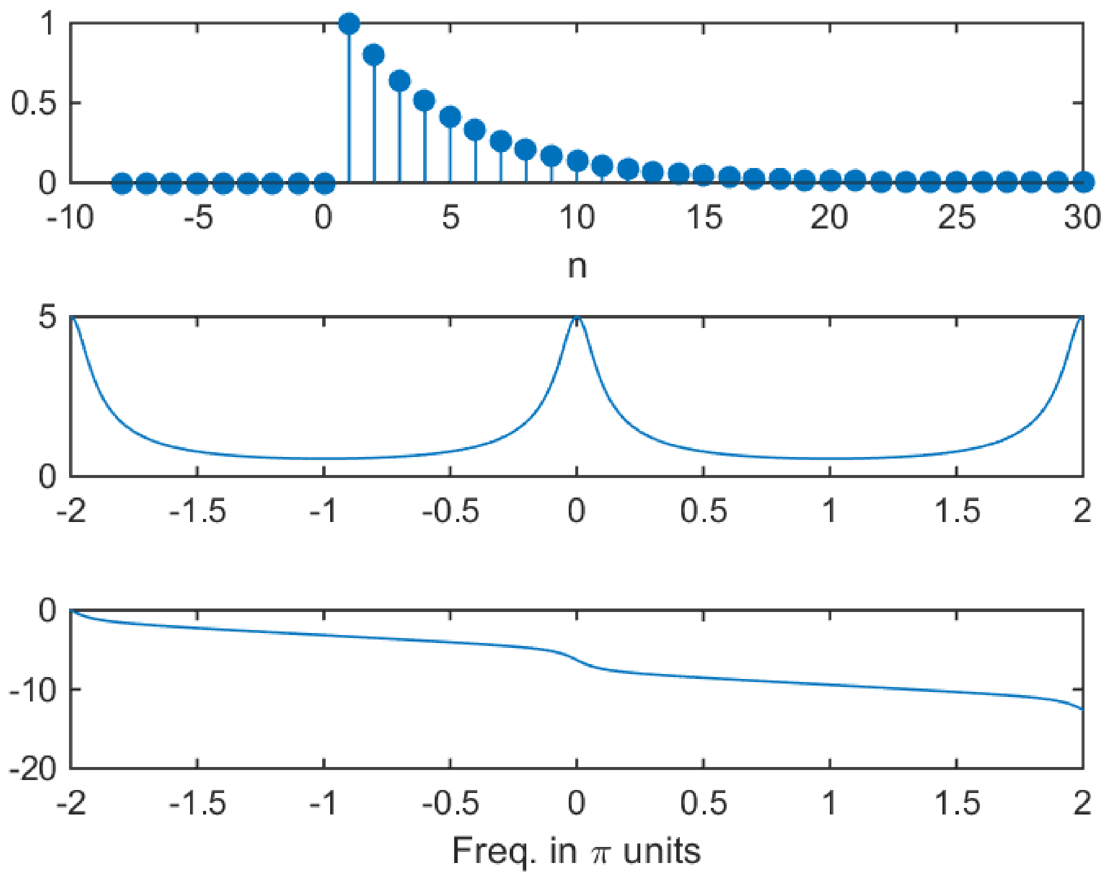
Out[16]:

DTFT and Time Shift

$$x[n-m] \quad \longleftrightarrow \quad e^{-j\omega m}X(\omega)$$

- same amplitude
- phase changed (linearly $-\angle\omega m$)

In [17]:
```
%plot -s 800,600
N = 30;
nd = -8:N;
xd = zeros(size(nd));

x = zeros(1,N);
for i = 1:N
    x(i) = 0.8^(i-1);
end

m = 1;          % m = 1 => one sample delay or shift
n = 0+m:N-1+m;
[y,ny] = sigadd(xd,nd,x,n);

% w = linspace(-1,1,2^10)*pi;
w = linspace(-2,2,2^10)*pi;

X = dtft(x,n,w);

subplot(3,1,1), stem(ny,y,'filled'), xlabel('n')
subplot(3,1,2), plot(w/pi,abs(X)),
subplot(3,1,3), plot(w/pi,phase(X)), xlabel('Freq. in \pi units')
```



Out[17]:

4. Filters

Ideal lowpass filter and discrete-time sinc function

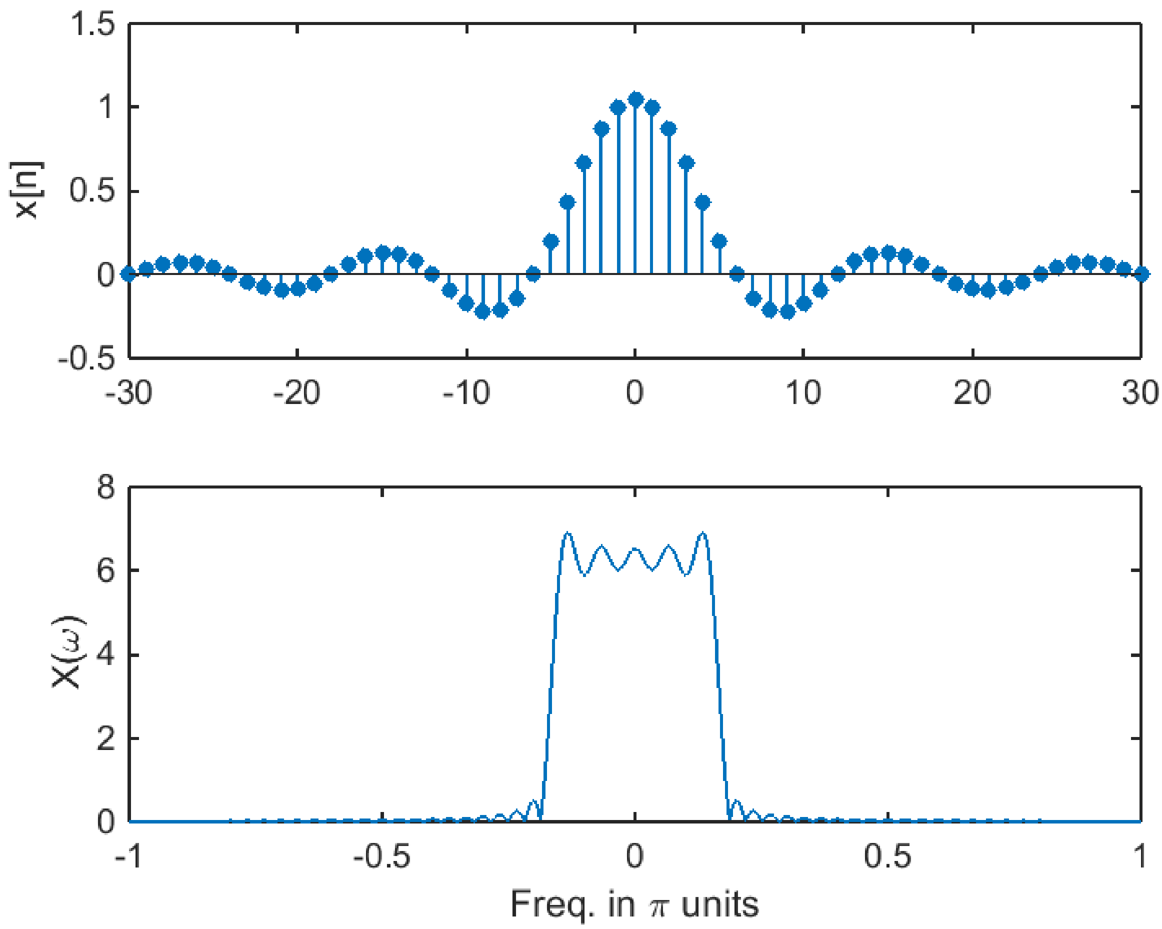
Impulse Response of the Ideal Lowpass Filter

$$h[n] = 2\omega_c \frac{sin(\omega_c n)}{\omega_c n} \qquad \longleftrightarrow \qquad H(\omega) = \begin{cases} 1 & -\omega_c \leq \omega \leq \omega_c \\ 0 & \text{otherwise} \end{cases}$$

$$h[n] \; = \; \int_{-\pi}^{\pi} H(\omega)\, e^{j\omega n} \frac{d\omega}{2\pi} \; = \; \int_{-\omega_c}^{\omega_c} e^{j\omega n} \frac{d\omega}{2\pi} \; = \; \frac{e^{j\omega_c n} \; - \; e^{-j\omega_c n}}{jn} \; = 2\omega_c \frac{\sin(\omega_c n)}{\omega_c n}$$

$$\operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

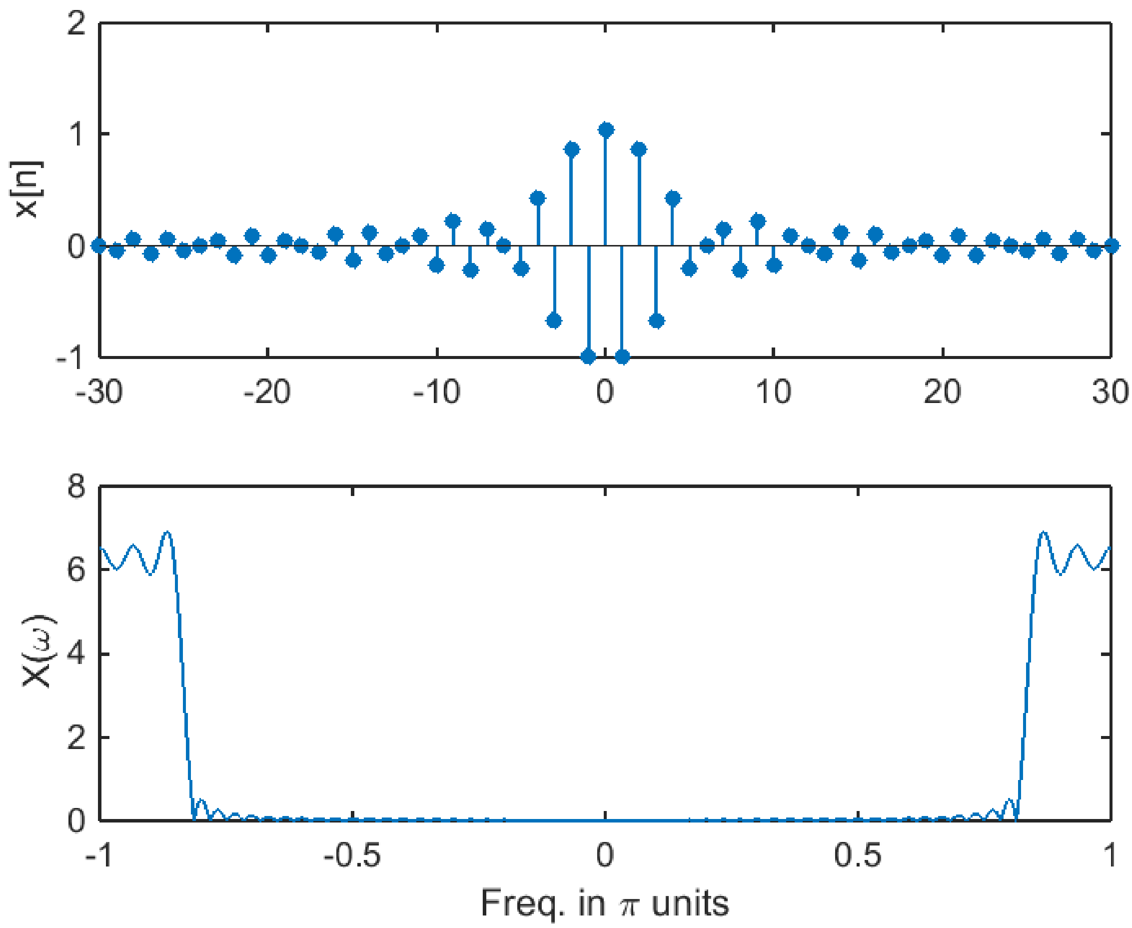
In [1]:
```
%plot -s 800,600

wc = pi/6;

N = 30;
n = -N:N;

h = zeros(1,length(n));
for i = 1:length(n)
    h(i) = 2*wc*sinc(1/pi*wc*(i-N-1));
end

w = linspace(-1,1,2^10)*pi;
%w = linspace(0,2,2^10)*pi;

X = dtft(h,n,w);

subplot(2,1,1), stem(n,h,'filled','markersize',4), ylabel('x[n]')
subplot(2,1,2), plot(w/pi,abs(X)), ylabel('X(\omega)')
xlabel('Freq. in \pi units'), ylabel('X(\omega)')
```



Out[1]:

In [2]:

```
%plot -s 800,600

wc = pi/6;

N = 30;
n = -N:N;

d = zeros(1,length(n));
h = zeros(1,length(n));
d(N+1) = d(N+1) + 1;

for i = 1:length(n)
    h(i) =  (-1)^(i-N-1)*2*wc*sinc(1/pi*wc*(i-N-1));
end

%h = sigadd(d,n,-h,n);

w = linspace(-1,1,2^12)*pi;
%w = linspace(0,2,2^10)*pi;

X = dtft(h,n,w);

subplot(2,1,1), stem(n,h,'filled','markersize',4), ylabel('x[n]')
subplot(2,1,2), plot(w/pi,abs(X)), ylabel('X(\omega)')
xlabel('Freq. in \pi units'), ylabel('X(\omega)')
```

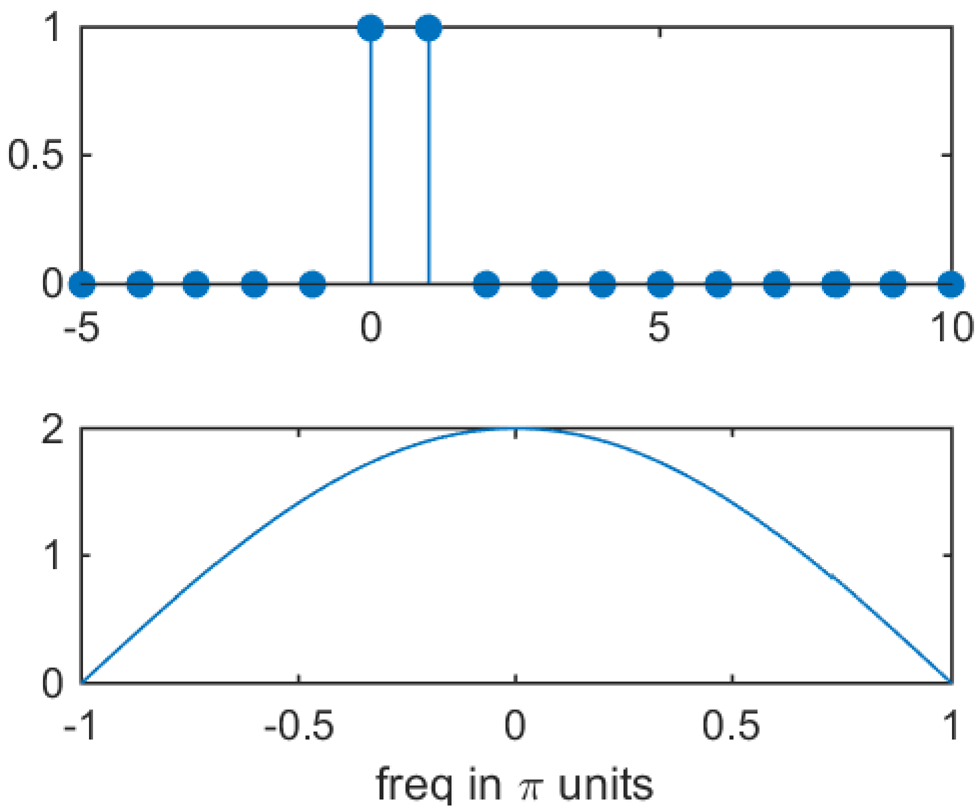


Out[2]:

Linear Filters: Low-Pass

In [20]:

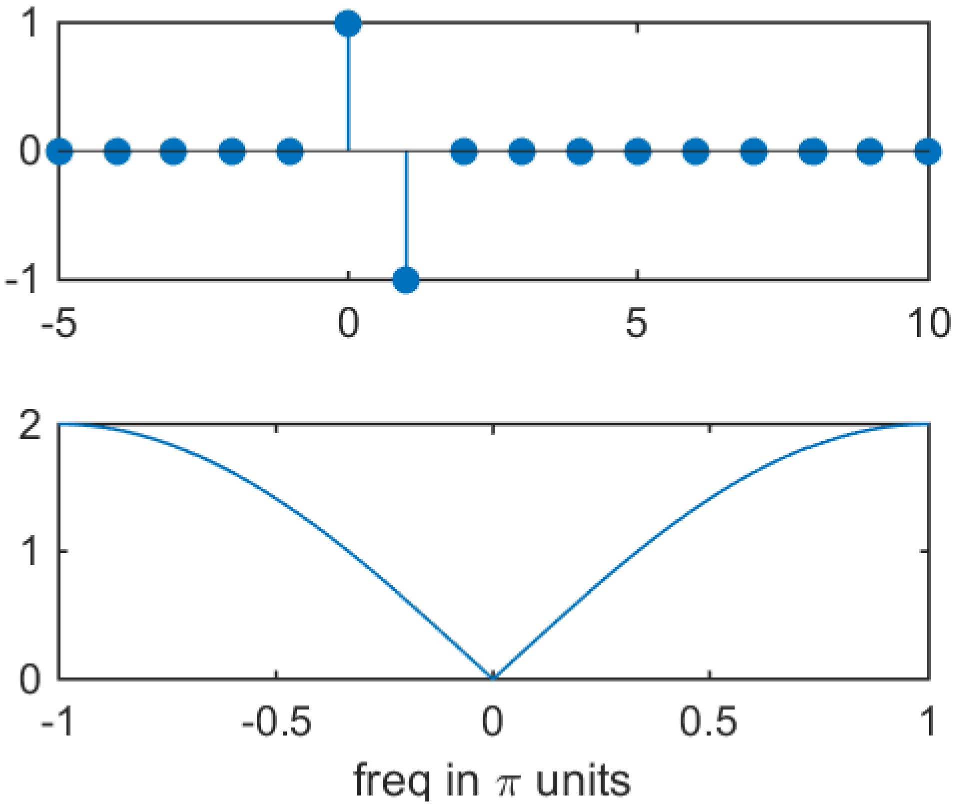
```
%plot -s 560,420
nd = -5:10;
hd = zeros(1,length(nd));

N = 2;
h = ones(1,N);
n = 0:N-1;

w = linspace(-1,1,2^10)*pi;
X = dtft(h,n,w);

[y,ny] = sigadd(hd,nd,h,n);
subplot(2,1,1), stem(ny,y,'filled')
subplot(2,1,2), plot(w/pi,abs(X)), xlabel('freq in \pi units')
```



Out[20]:

Linear Filters: High-Pass

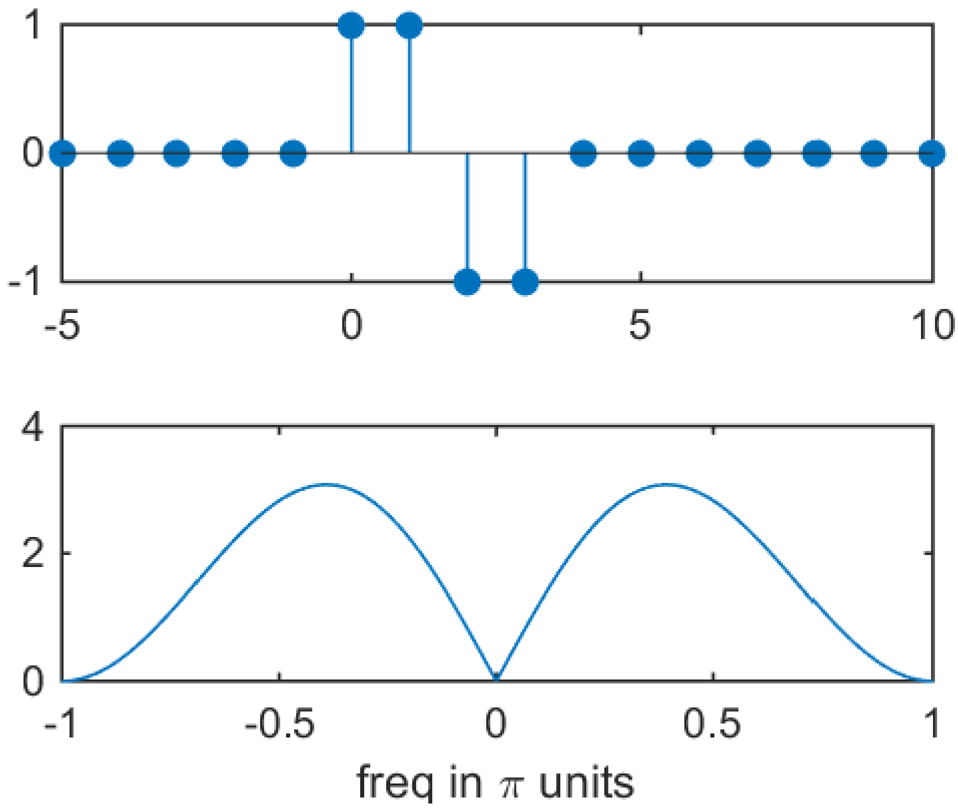
In [21]:
```
%plot -s 560,420
nd = -5:10;
hd = zeros(1,length(nd));

N = 2;
h = [1 -1];
n = 0:N-1;

w = linspace(-1,1,2^10)*pi;
X = dtft(h,n,w);

[y,ny] = sigadd(hd,nd,h,n);
subplot(2,1,1), stem(ny,y,'filled')
subplot(2,1,2), plot(w/pi,abs(X)), xlabel('freq in \pi units')
```



Out[21]:

Linear Filters: Band-Pass

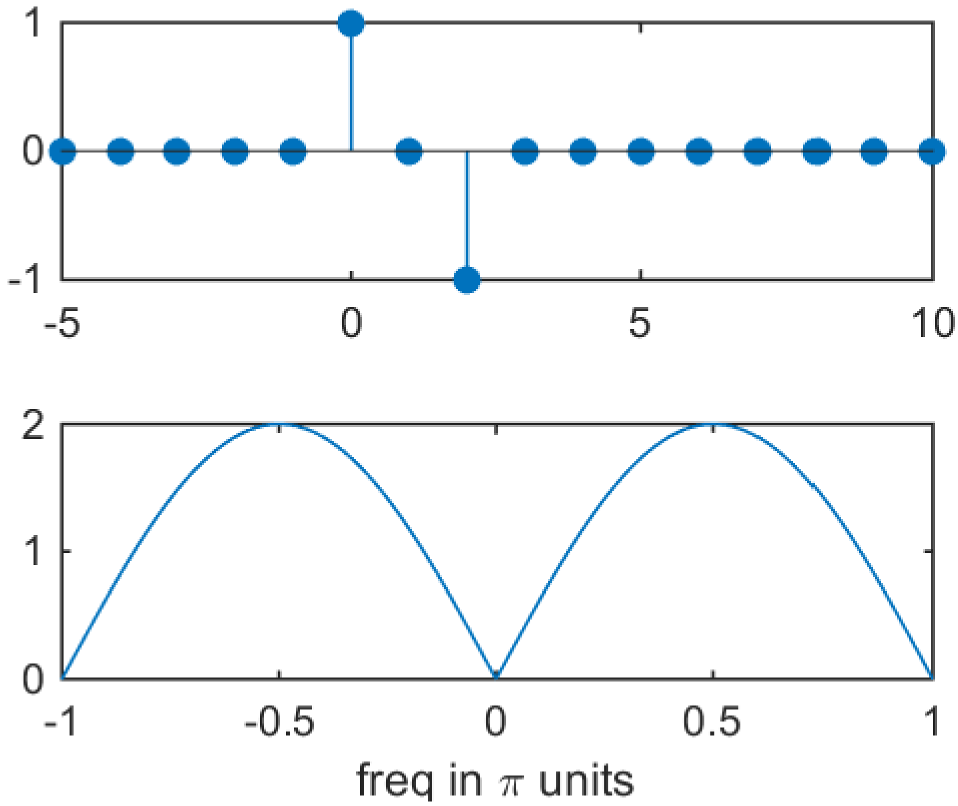
In [22]:
```
%plot -s 560,420
nd = -5:10;
hd = zeros(1,length(nd));

N = 4;
h = [1 1 -1 -1];
n = 0:N-1;

w = linspace(-1,1,2^10)*pi;
X = dtft(h,n,w);

[y,ny] = sigadd(hd,nd,h,n);
subplot(2,1,1), stem(ny,y,'filled')
subplot(2,1,2), plot(w/pi,abs(X)), xlabel('freq in \pi units')
```



Out[22]:

Linear Filters: Band-Stop

In [23]:
```
%plot -s 560,420
nd = -5:10;
hd = zeros(1,length(nd));

N = 3;
h = [1 0 -1];
n = 0:N-1;

w = linspace(-1,1,2^10)*pi;
X = dtft(h,n,w);

[y,ny] = sigadd(hd,nd,h,n);
subplot(2,1,1), stem(ny,y,'filled')
subplot(2,1,2), plot(w/pi,abs(X)), xlabel('freq in \pi units')
```



Out[23]:

5. High-density spectrum and high-resolution spectrum

$$x[n] = \cos(0.48\pi n) + \cos(0.52\pi n)$$

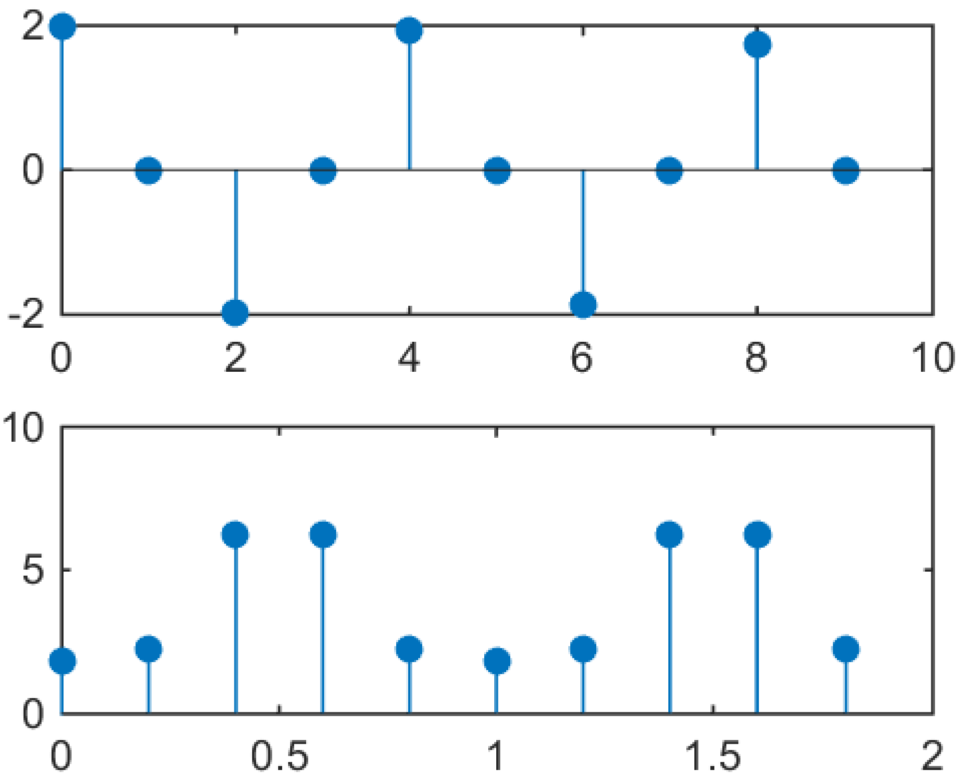
In [24]:
```
N = 100;
n = 0:N-1;
x = cos(0.48*pi*n) + cos(0.52*pi*n);
```

Out[24]:

use only 10-point DFT of $x[n]$

In [25]:

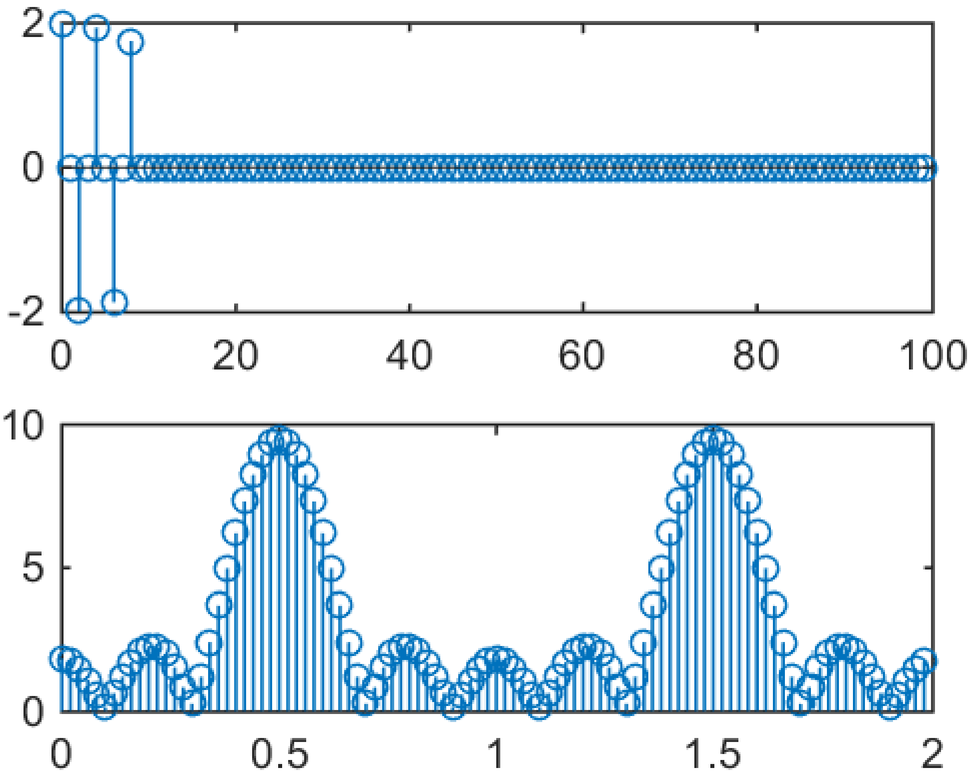
```
%plot -s 560,420

n1 = 0:9;
y1 = x(1:10);
Y1 = dft(y1,10);
k1 = n1;
w1 = 2*pi/10*k1;

subplot(2,1,1), stem(n1,y1,'filled')
subplot(2,1,2), stem(w1/pi,abs(Y1),'filled')
```



Out[25]:

pad 90 zeros to obtain a dense spectrum

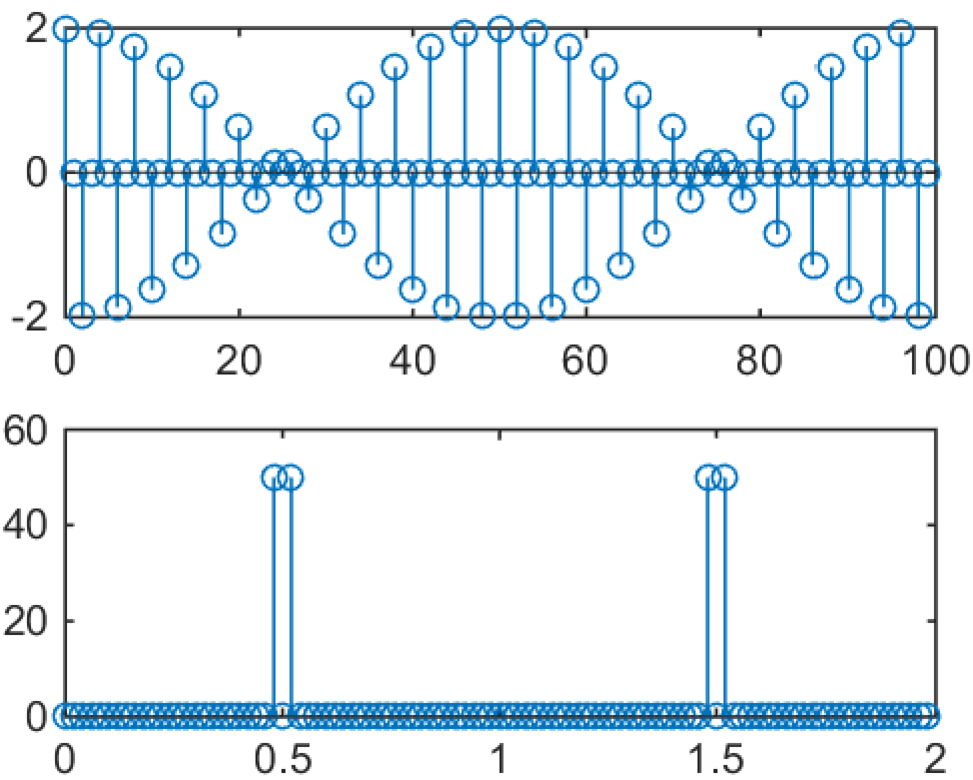
In [26]:
```
%plot -s 560,420

n2 = 0:99;
y2 = [x(1:10), zeros(1,90)];
Y2 = dft(y2,N);
k2 = n2;
w2 = 2*pi/100*k2;

subplot(2,1,1), stem(n2,y2)
subplot(2,1,2), stem(w2/pi,abs(Y2))
```



Out[26]:

use 100 samples of $x[n]$

In [27]:
```
%plot -s 560,420

n3 = n;
y3 = x;
Y3 = dft(y3,100);
k3 = n3;
w3 = 2*pi/100*k3;

subplot(2,1,1), stem(n3,y3)
subplot(2,1,2), stem(w3/pi,abs(Y3))
```



Out[27]:

conclusion:

padding more zeros to the 100-point sequence will result in a smoother rendition of the spectrum but will not reveal any new information.

In [2]:
```
%%javascript
$.getScript('https://kmahelona.github.io/ipython_notebook_goodies/ipython_notebook_to
c.js')
```