

Greedy Algorithms: Main Ideas

Michael Levin

Higher School of Economics

Algorithmic Toolbox
Data Structures and Algorithms

Outline

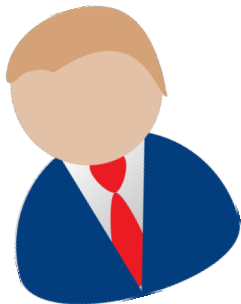
- 1 Largest Number
- 2 Car Fueling
- 3 Implementation and Analysis
- 4 Main Ingredients

Learning objectives

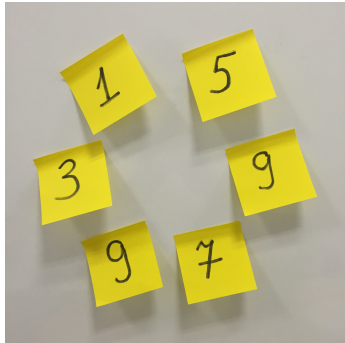
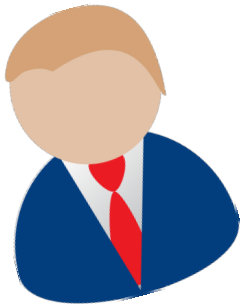
- Come up with a greedy algorithm yourself

Job Interview

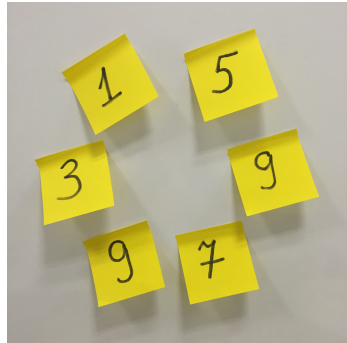
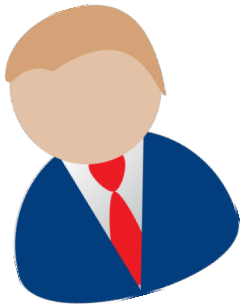
Job Interview



Job Interview



Job Interview



Largest Number

Toy problem

What is the largest number that consists of digits 3, 9, 5, 9, 7, 1? Use all the digits.

Largest Number

Toy problem

What is the largest number that consists of digits 3, 9, 5, 9, 7, 1? Use all the digits.

Examples

359179, 537991, 913579, ...

Correct answer

997531

Greedy Strategy

5	7	3	9	1	9
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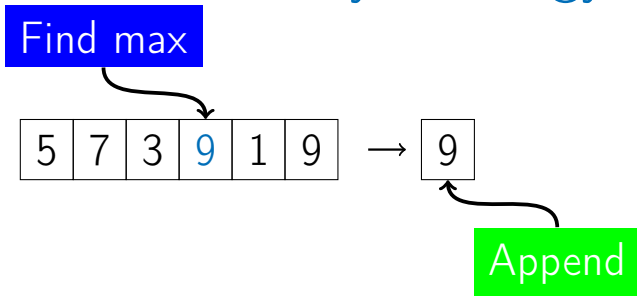
Greedy Strategy

Find max



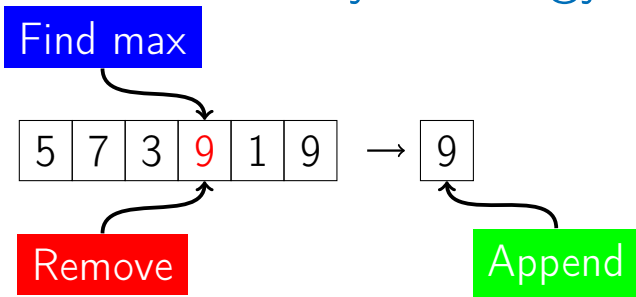
- Find max digit

Greedy Strategy



- Find max digit
- Append it to the number

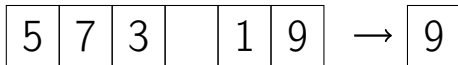
Greedy Strategy



- Find **max** digit
- **Append** it to the number
- **Remove** it from the list of digits

Greedy Strategy

Find max



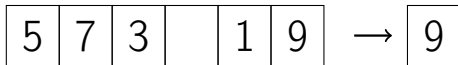
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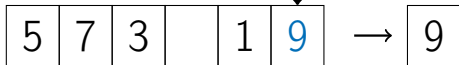
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Greedy Strategy

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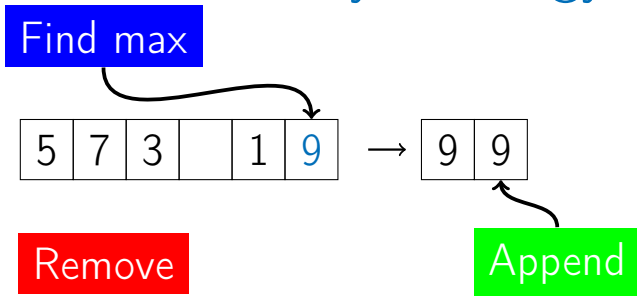


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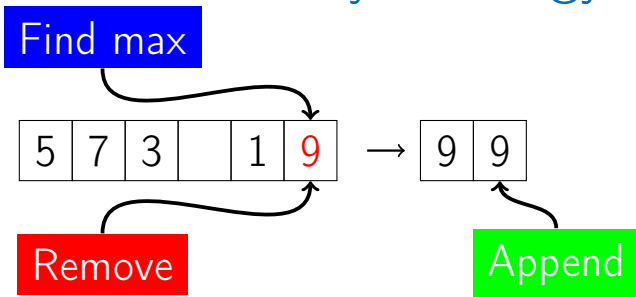
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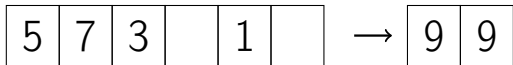
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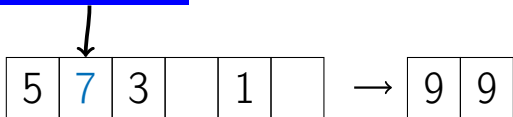
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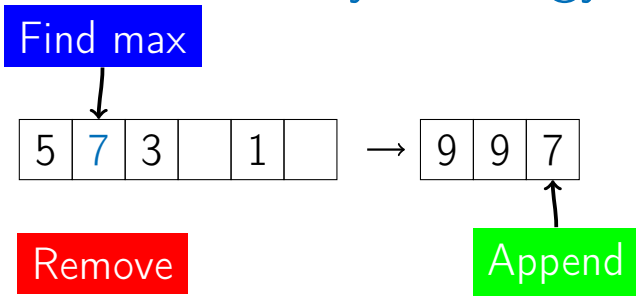


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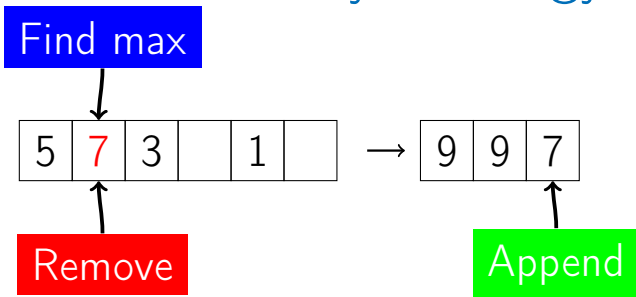
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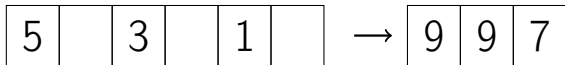
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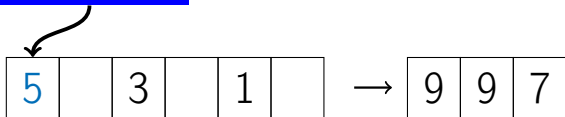
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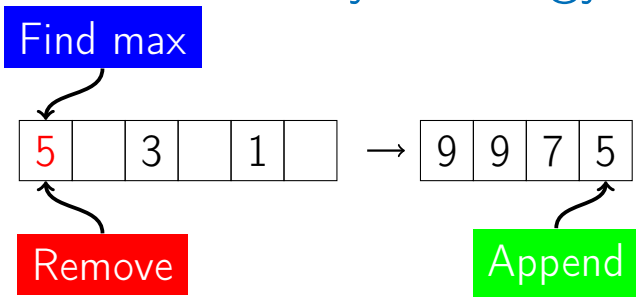


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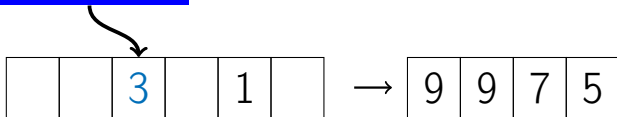
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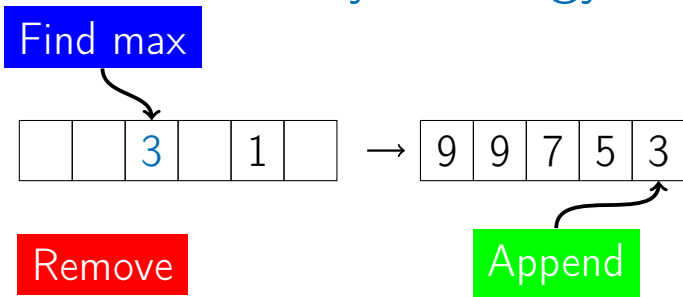


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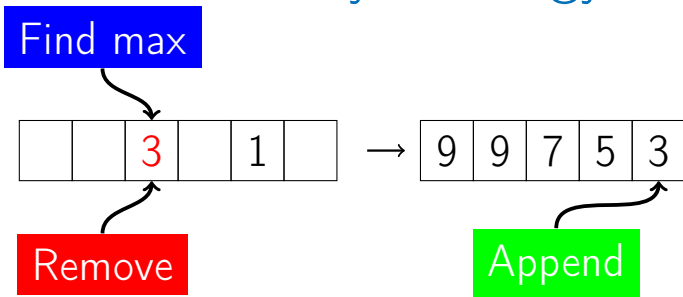
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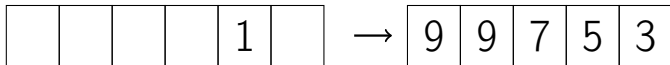
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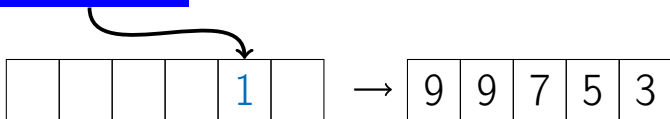
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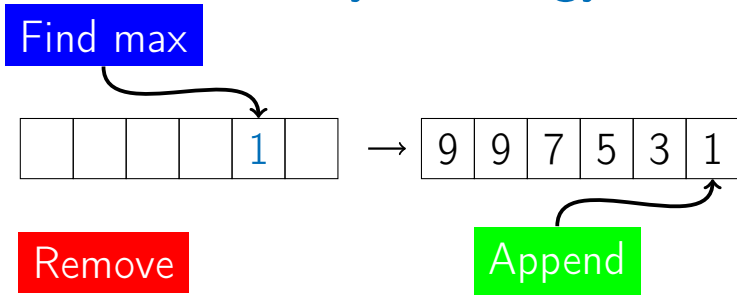


Remove

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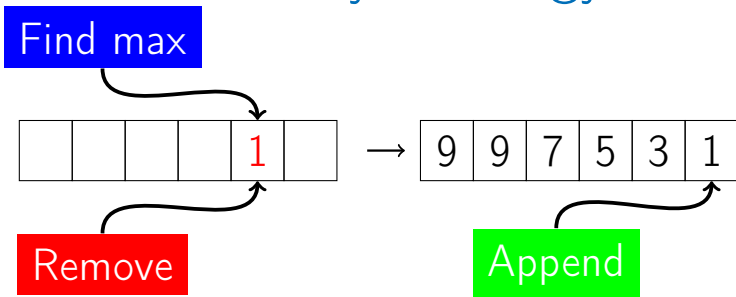
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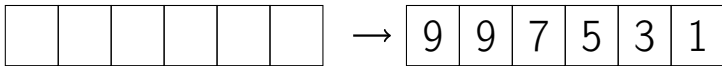
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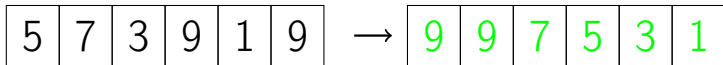


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Outline

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Car Fueling

Distance with full tank = 400km

Car Fueling

Distance with full tank = 400km

0km

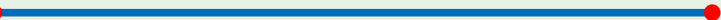


A

950km

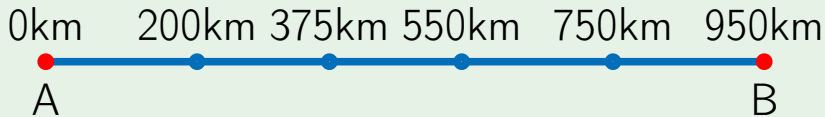


B



Car Fueling

Distance with full tank = 400km



Car Fueling

Distance with full tank = 400km



Car Fueling

Distance with full tank = 400km



Car Fueling

Distance with full tank = 400km



Minimum number of refills = 2

Car Fueling

Input: A car which can travel at most L kilometers with full tank, a source point A , a destination point B and n gas stations at distances $x_1 \leq x_2 \leq x_3 \leq \dots \leq x_n$ in kilometers from A along the path from A to B .

Output: The minimum number of refills to get from A to B , besides refill at A .

Greedy Strategy

- Make some greedy choice
- Reduce to a smaller problem
- Iterate

Greedy Choice

- Refill at the the closest gas station
- Refill at the farthest reachable gas station
- Go until there is no fuel

Greedy Choice

- Refill at the the closest gas station
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Greedy Algorithm

- Start at A

Greedy Algorithm

- Start at A
- Refill at the farthest reachable gas station G

Greedy Algorithm

- Start at A
- Refill at the farthest reachable gas station G
- Make G the new A

Greedy Algorithm

- Start at A
- Refill at the farthest reachable gas station G
- Make G the new A
- Get from new A to B with minimum number of refills

Definition

Subproblem is a similar problem of smaller size.

Subproblem

Examples

■ `LargestNumber(3, 9, 5, 9, 7, 1) =`

Subproblem

Examples

- `LargestNumber(3, 9, 5, 9, 7, 1) =`
“9” +

Subproblem

Examples

- $\text{LargestNumber}(3, 9, 5, 9, 7, 1) =$
“9” + $\text{LargestNumber}(3, 5, 9, 7, 1)$

Subproblem

Examples

- $\text{LargestNumber}(3, 9, 5, 9, 7, 1) =$
“9” + $\text{LargestNumber}(3, 5, 9, 7, 1)$
- Min number of refills from A to $B =$

Subproblem

Examples

- $\text{LargestNumber}(3, 9, 5, 9, 7, 1) =$
“9” + $\text{LargestNumber}(3, 5, 9, 7, 1)$
- Min number of refills from A to $B =$
first refill at $G +$

Subproblem

Examples

- $\text{LargestNumber}(3, 9, 5, 9, 7, 1) =$
“9” + $\text{LargestNumber}(3, 5, 9, 7, 1)$
- Min number of refills from A to $B =$
first refill at G + min number of refills
from G to B

Safe Move

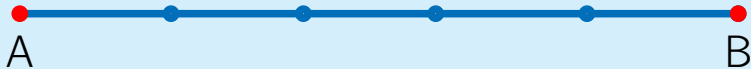
Definition

A greedy choice is called **safe move** if there is an optimal solution consistent with this first move.

Lemma

To refill at the farthest reachable gas station is a **safe move**.

Proof



Proof



Proof

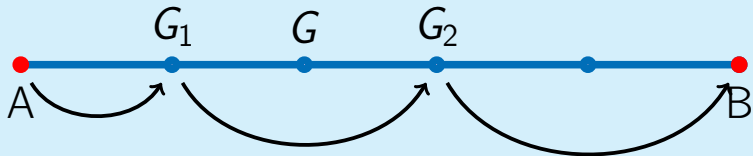


Proof



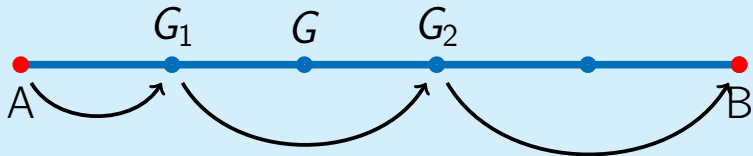
First case: G is closer than G_2

Proof



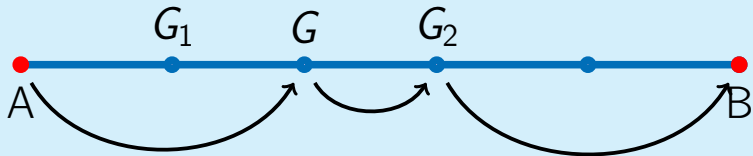
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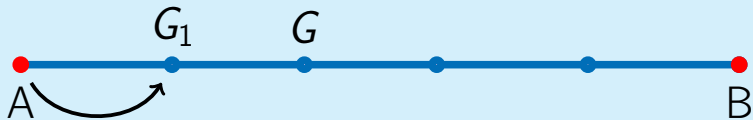
First case: G is closer than G_2
Refill at G instead of G_1

Proof



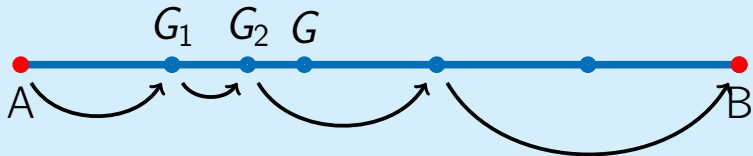
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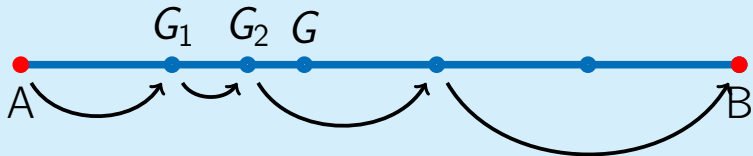
Second case: G_2 is closer than G

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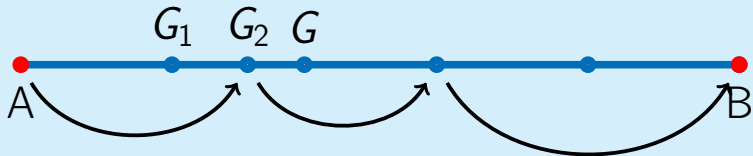
Second case: G_2 is closer than G

Proof



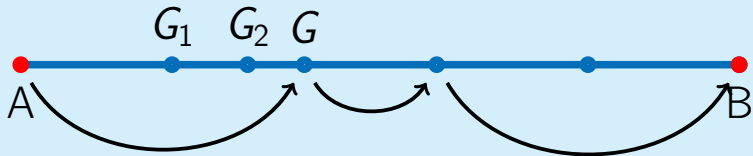
Second case: G_2 is closer than G
Avoid refill at G_1

Proof



Second case: G_2 is closer than G
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Proof



Second case: G_2 is closer than G
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Proof

- Route R with the minimum number of refills

Proof

- Route R with the minimum number of refills
- G_1 — position of first refill in R

Proof

- Route R with the minimum number of refills
- G_1 — position of first refill in R
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Proof

- Route R with the minimum number of refills
- G_1 — position of first refill in R
- G_2 — next stop in R (refill or B)
- G — farthest refill reachable from A
- If G is closer than G_2 , refill at G instead of G_1

Proof

- Route R with the minimum number of refills
- G_1 — position of first refill in R
- G_2 — next stop in R (refill or B)
- G — farthest refill reachable from A
- If G is closer than G_2 , refill at G instead of G_1
- Otherwise, avoid refill at G_1



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$$A = x_0 \leq x_1 \leq x_2 \leq \dots \leq x_n \leq x_{n+1} = B$$

MinRefills(x, n, L)

```
numRefills  $\leftarrow$  0, currentRefill  $\leftarrow$  0
while currentRefill  $\leq$  n:
    lastRefill  $\leftarrow$  currentRefill
    while (currentRefill  $\leq$  n and
            $x[\text{currentRefill} + 1] - x[\text{lastRefill}] \leq L$ ):
        currentRefill  $\leftarrow$  currentRefill + 1
    if currentRefill == lastRefill:
        return IMPOSSIBLE
    if currentRefill  $\leq$  n:
        numRefills  $\leftarrow$  numRefills + 1
return numRefills
```

Lemma

The running time of $\text{MinRefills}(x, n, L)$ is $O(n)$.

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- *currentRefill* changes from 0 to $n + 1$, one-by-one

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- *currentRefill* changes from 0 to $n + 1$, one-by-one
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Lemma

The running time of $\text{MinRefills}(x, n, L)$ is $O(n)$.

Proof

- *currentRefill* changes from 0 to $n + 1$, one-by-one
- *numRefills* changes from 0 to at most n , one-by-one
- Thus, $O(n)$ iterations □

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Reduction to Subproblem

- Make a first move
- Then solve a problem of the same kind
- Smaller: fewer digits, fewer fuel stations
- This is called a “subproblem”

Safe move

- A move is called **safe** if there is an optimal solution consistent with this first move

Safe move

- A move is called **safe** if there is an optimal solution consistent with this first move
- Not all first moves are safe

Safe move

- A move is called **safe** if there is an optimal solution consistent with this first move
- Not all first moves are safe
- Often greedy moves are not safe

General Strategy

Problem

General Strategy

Problem $\xrightarrow{\text{greedy choice}}$

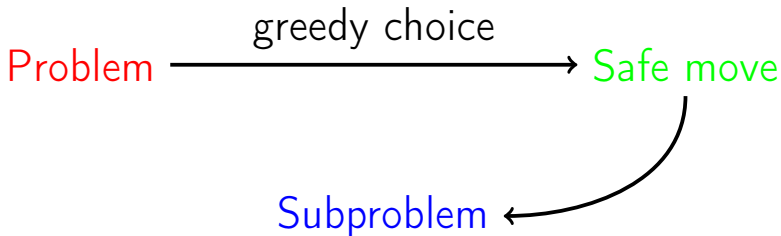
- Make a greedy choice

General Strategy

Problem $\xrightarrow{\text{greedy choice}}$ Safe move

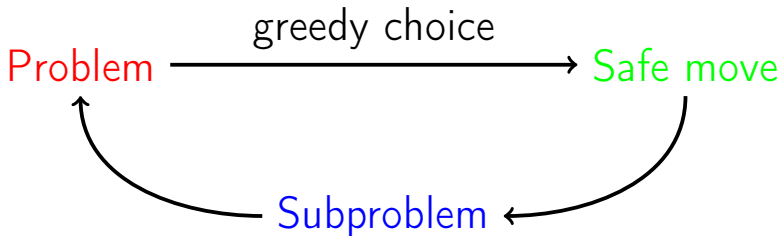
- Make a greedy choice
- Prove that it is a safe move

General Strategy



- Make a greedy choice
- **Prove** that it is a **safe move**
- Reduce to a **subproblem**

General Strategy



- Make a greedy choice
- **Prove** that it is a **safe move**
- Reduce to a **subproblem**
- Solve the **subproblem**