Topic 8: Essential and Fundamental Matrices

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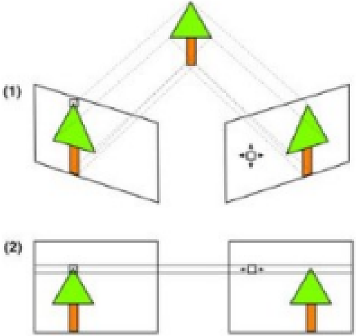
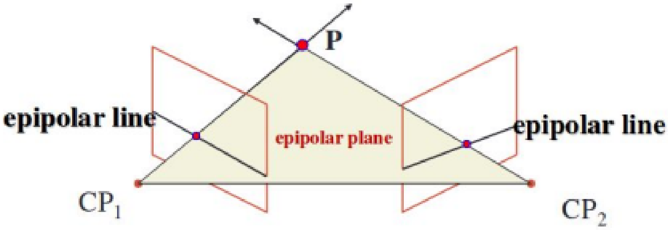
Overview

1. Stereo Correspondence
2. Stereo Correspondence with calibrated cameras
   - Essential Matrix
3. Stereo Correspondence with uncalibrated Cameras
   - Fundamental Matrix



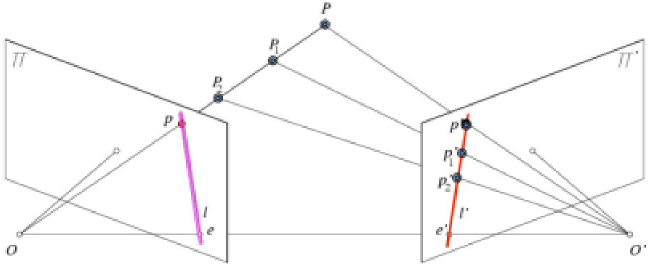
- **Stereo Correspondence Problem:** Given a point p in the left image, find the point p' such that 3D point P is mapped to p and p' by left and right cameras respectively
  - Which point from the right image to match
  - Which similarity measure to adopt
- Need of Stereo Correspondence: To find disparity value(from disparity, depth can be computed)

Stereo Correspondence (cont.)



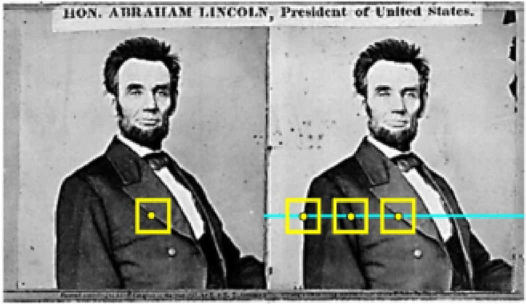
- Epipolar Plane for binocular vision: The plane that passes through the centers(COP1, COP2) of lens and the point(P) to be imaged
- Epipolar Line: The intersection between the epipolar plane and image plane

Epipolar Constraint



- Let p and p' be the projections of P
- Potential match for p has to lie on the corresponding epipolar line r
- Potential match for p' has to lie on the corresponding epipolar line l

Why is the epipolar constraint useful?

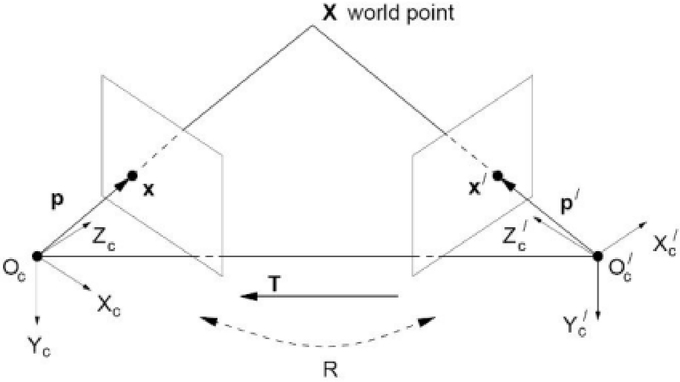


The epipolar constraint reduces the correspondence problem to a 1D search along the epipolar line.

Stereo correspondence with calibrated cameras

- Calibrated Camera: Camera with its parameters that are known
- Keep the CCS of left camera is the WCS
- X is measured w.r.t Left CCS, where as X' is measured w.r.t Right CCS.
- How to related X and X'?
  - Use WCS, and relate X and X'
  - Any two 3D-co-ordinate systems can be aliened using two rotations and one translation

Stereo correspondence with calibrated cameras (cont.)



$$X' = RX + T$$

Cross product with T on both sides..

$T \; x \; X' = T \; x \; (RX \; + \; T)$

T x X' = T x RX + T x T

T x X' = T x RX (since T x T = 0)

Dot product with X' on both sides..

X' . (T x X') = X' . (T x RX)

(X' . (T x X') = 0)

Essential Matrix

- Cross Product Vs Matrix Multiplication:
  Let $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$
  $a \times b = i(a_2 b_3 - b_2 a_3) - J(a_1 b_3 - b_1 a_3) + k(a_1 b_2 - b_1 a_2)$

$$= \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = [a\ x]\ b$$

$$X' . (T x R X) = 0$$

$$X' . ([T x] R X) = 0$$

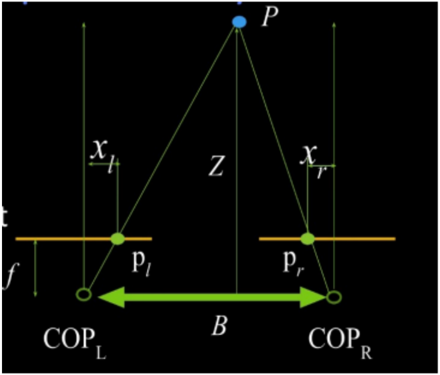
Let

$$E = [T x] R$$

$$X'^T E X = 0$$

- E is called the essential matrix and it relates corresponding image points between both cameras, given the rotation and translation.

Essential Matrix example : Parallel Cameras



R = Identity

T=[B,0,0]

$$E = [T\ x]\ R$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -B \\ 0 & B & 0 \end{pmatrix}$$

Essential Matrix example : Parallel Cameras (cont.)

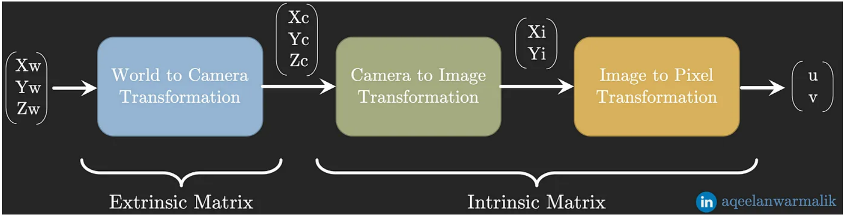
$$\begin{bmatrix} x' & y' & 1 \end{bmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -B \\ 0 & B & 0 \end{pmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} x' & y' & 1 \end{bmatrix} \begin{bmatrix} 0 \\ B \\ By \end{bmatrix} = 0$$

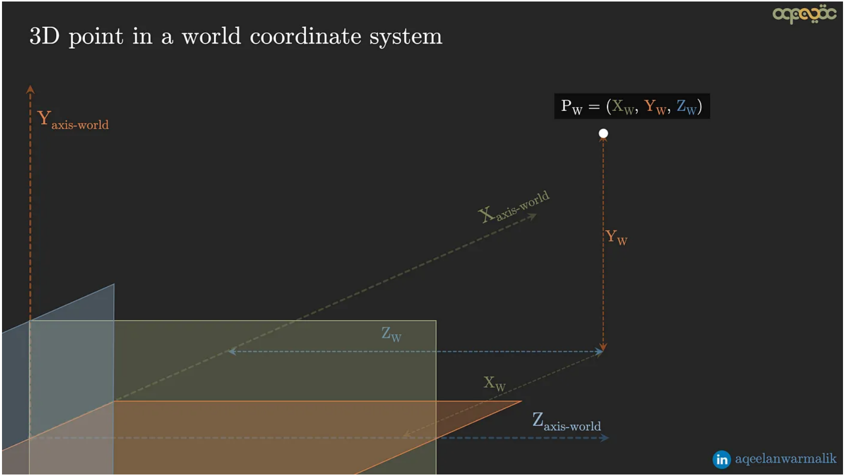
$$B\, y' \;=\; B\, y \;\Rightarrow y' \;=\; y$$

Given a point (x,y) from left image, the corresponding point (x',y') in right image lies on the line $y' = y$.

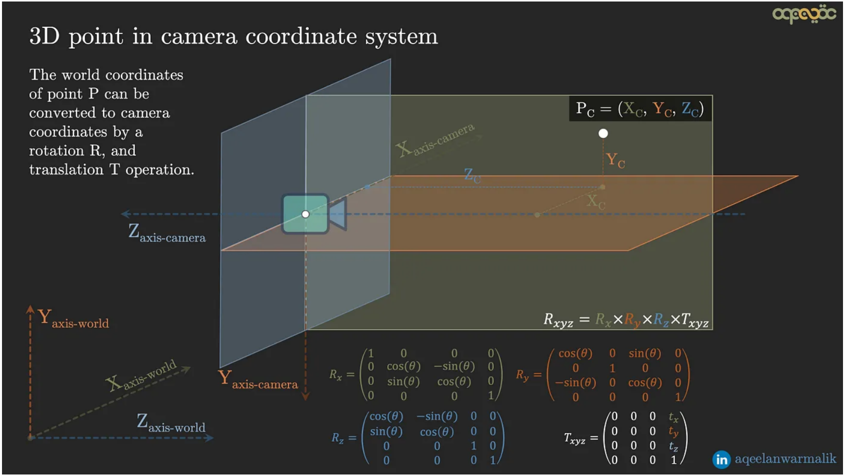
Stereo Correspondence with uncalibrated Cameras



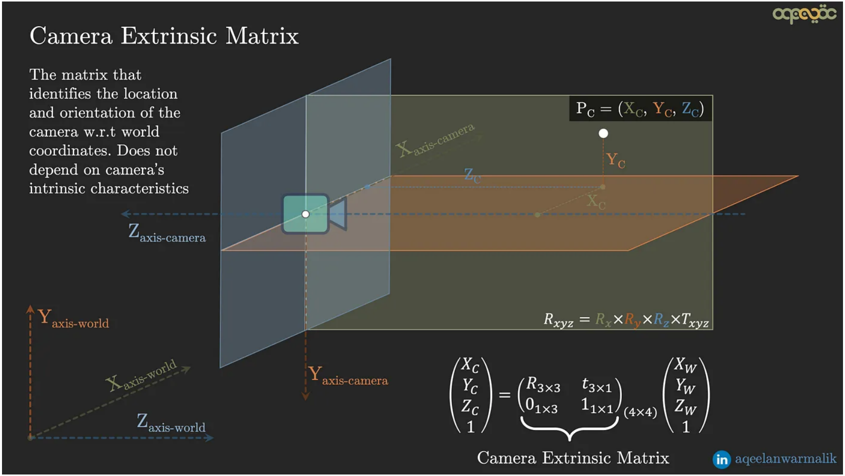
Stereo Correspondence with uncalibrated Cameras



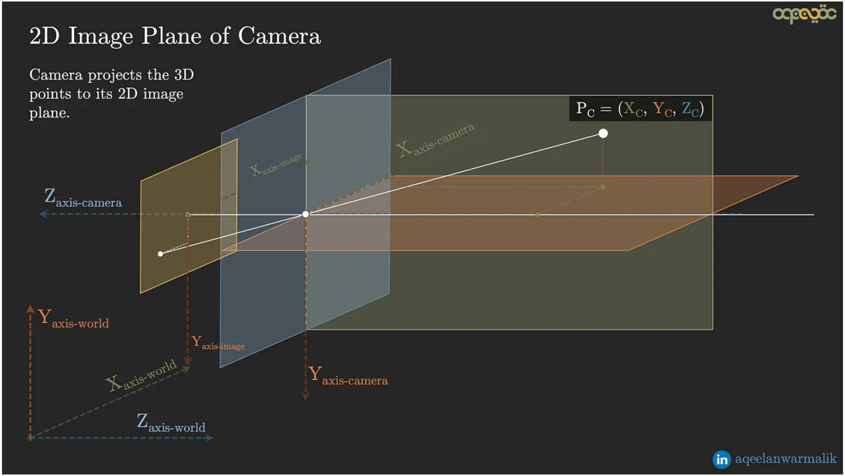
Stereo Correspondence with uncalibrated Cameras



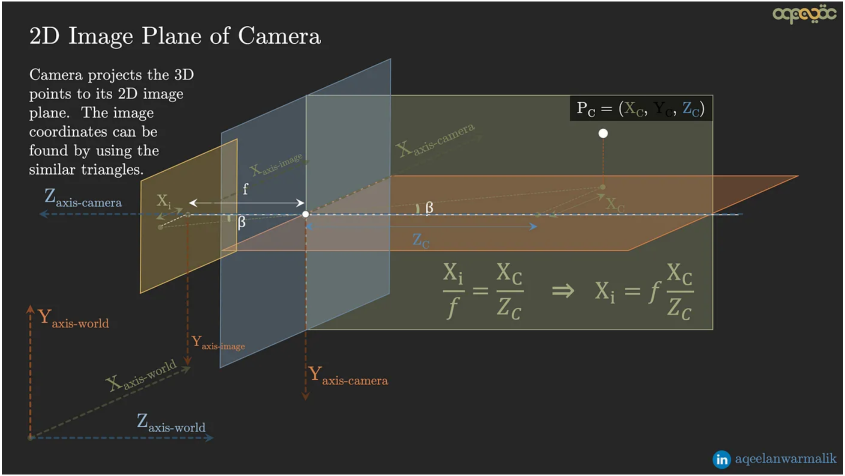
Stereo Correspondence with uncalibrated Cameras



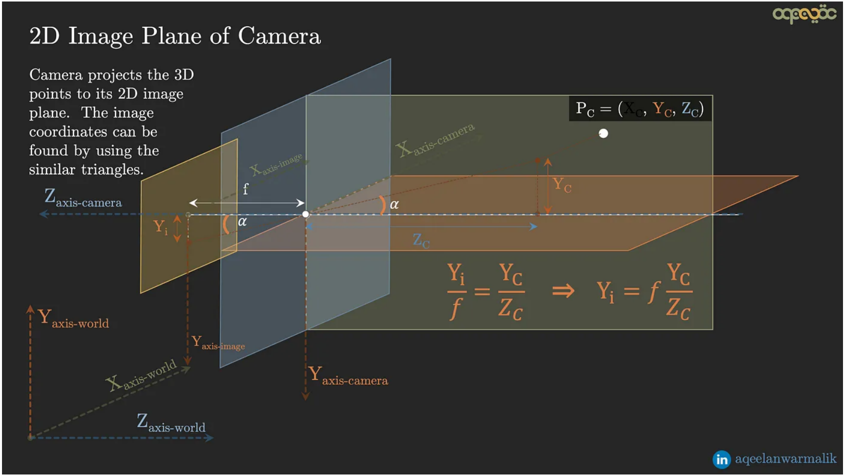
Stereo Correspondence with uncalibrated Cameras



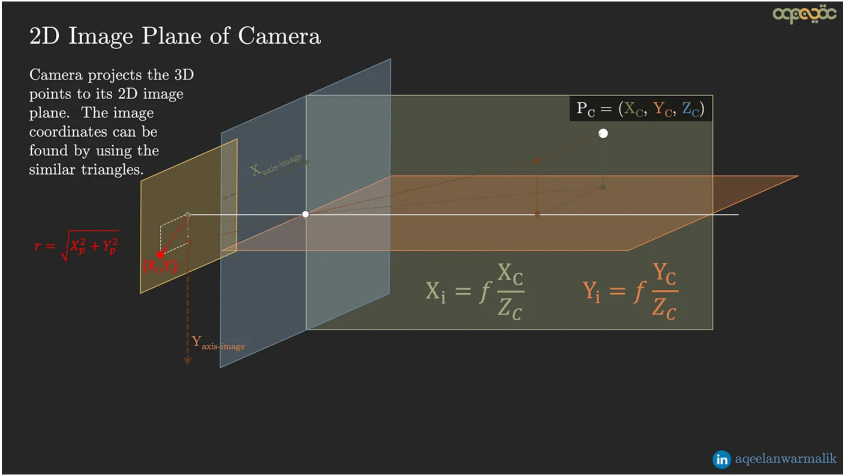
Stereo Correspondence with uncalibrated Cameras



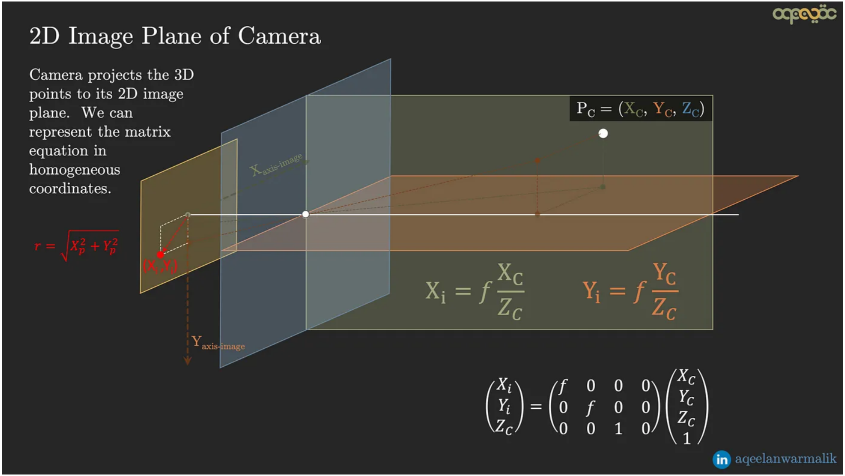
Stereo Correspondence with uncalibrated Cameras



Stereo Correspondence with uncalibrated Cameras



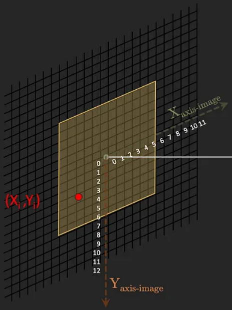
Stereo Correspondence with uncalibrated Cameras



- Pixel Coordinate System(PCS): Top-Left cell of 2D array is the origin; Row and Column are axes

Stereo Correspondence with uncalibrated Cameras

Discretizing the 2D Image Plane of Camera



Pixel coordinates of an image are discrete values withing a range that can be achieved by dividing the image coordinates with pixel width and height.

$$X_i = f\frac{X_C}{Z_C} \qquad Y_i = f\frac{Y_C}{Z_C}$$

Each pixel is $\rho_u \times \rho_v$ meters wide (meter/pixel)

$$x_i = \frac{1}{\rho_u} f\frac{X_C}{Z_C} \qquad y_i = \frac{1}{\rho_v} f\frac{Y_C}{Z_C}$$

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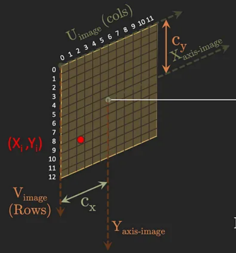
Pixel coordinates

The origin of the image coordinates lie at the center of the image plane, while the pixel coordinates has their origin defined at the top-left corner of the image. Hence a translation operator is required



$$x_i = \frac{1}{\rho_u} f \frac{X_C}{Z_C} \qquad y_i = \frac{1}{\rho_v} f \frac{Y_C}{Z_C}$$

Pixel coordinate system has origin at the top left corner of the image

$$u = \frac{1}{\rho_u} f \frac{X_C}{Z_C} + c_y \qquad v = \frac{1}{\rho_v} f \frac{Y_C}{Z_C} + c_x$$

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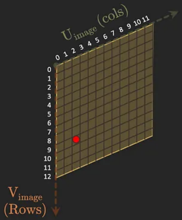
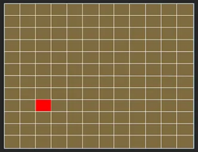
Stereo Correspondence with uncalibrated Cameras





$$u = \frac{1}{\rho_u} f \frac{X_C}{Z_C} + c_y \qquad v = \frac{1}{\rho_v} f \frac{Y_C}{Z_C} + c_x$$

In the example above, (u, v) = (2, 9) and the image size is 12x11

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 1/\rho_u & 0 & c_x \\ 0 & 1/\rho_v & c_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_i \\ Y_i \\ Z_C \end{pmatrix}$$

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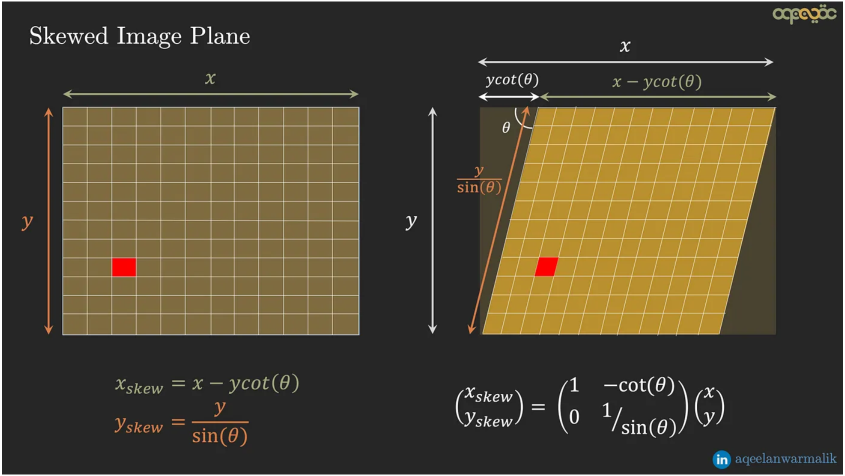
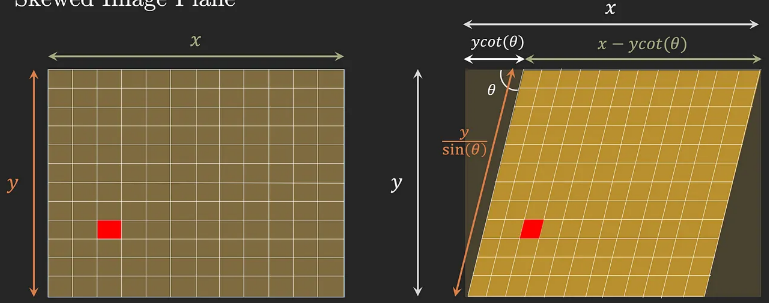
Stereo Correspondence with uncalibrated Cameras



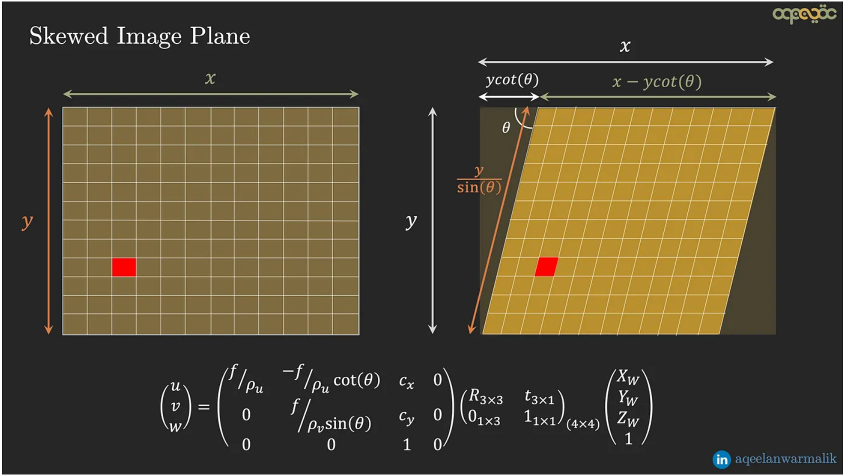
Stereo Correspondence with uncalibrated Cameras

Skewed Image Plane

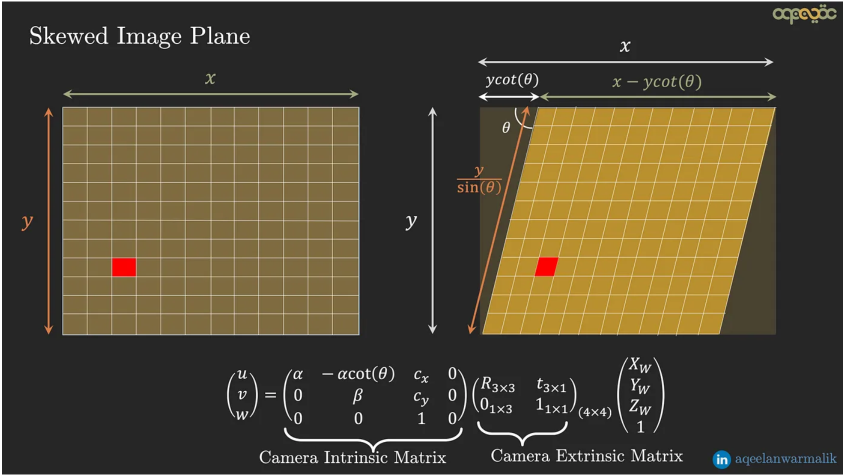


$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} f/\rho_u & 0 & c_x & 0 \\ 0 & f/\rho_v & c_y & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -\cot(\theta) & 0 & 0 \\ 0 & 1/\sin(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} R_{3\times3} & t_{3\times1} \\ 0_{1\times3} & 1_{1\times1} \end{pmatrix}_{(4\times4)} \begin{pmatrix} X_W \\ Y_W \\ Z_W \\ 1 \end{pmatrix}$$

Stereo Correspondence with uncalibrated Cameras



Stereo Correspondence with uncalibrated Cameras



Camera Matrix

- Camera Matrix or Camera Projection Matrix or Projection Matrix P is the matrix that transforms each point of object in WCS to Pixel Coordinate System
- Camera matrix $P = K_{int}\phi_{ext}$, where
  - K_{int} is the Camera Intrinsic Matrix
  - ϕ_{ext} is the Camera Extrinsic Matrix
  - K_{int} has only the internal parameters(f, scaling and translation to align to pixel coordinates) of P
  - ϕ_{ext} has only the external parameters(Translation and rotation to align CCS with WCS) of P

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \mathrm{K}_{int}\ \Phi_{ext} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

Stereo Correspondence using Fundamental Matrix

For a given camera:

$$Let\ P_p = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$P_p = K_{int}\ P_c$$

$$where \quad P_c = \phi_{int} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

So, for two cameras (left and right):

$$P_{c,left} = K_{int,left}^{-1}\ P_{p,left}$$

$$P_{c,right} = K_{int,right}^{-1}\ P_{p,right}$$

From before the essential matrix E

$$P^T_{c,right}\ E\ P_{c,left}\ =\ 0$$

$$\left(\ K^{-1}_{int,right}\ P_{im,right}\ \right)^T\ E\ \left(\ K^{-1}_{int,left}\ P_{im,left}\ \right)\ =\ 0$$

$$P^T_{im,right}\ \left(\ (K^{-1}_{int,right})^T\ E\ K^{-1}_{int,left}\ \right)\ P_{im,left}\ =\ 0$$

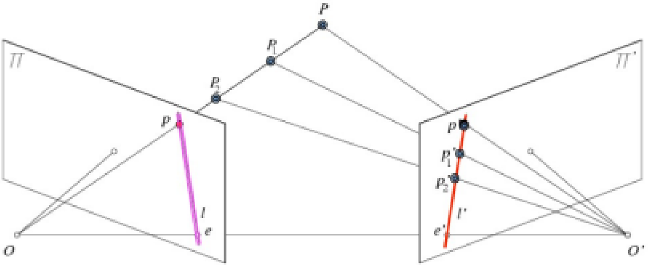
Fundamental matrix

$$F\ =\ (K^{-1}_{int,right})^T\ E\ K^{-1}_{int,left}$$

$$P^T_{p,right}\ F\ P_{p,left}\ =\ 0$$

or

$$P^T\ F\ P'\ =\ 0$$



$$P^T \; F \; P' \; = \; 0$$

How to find epipolar line, given a point in left image or right image

- l $=$ F P' is the epipolar line associated with P'
- l' $= F^T$ P is the epipolar line associated with P

Computing F from point correspondences

Each point correspondence generates one constraint on F in the eqn.
$P_{im,right}^{T} \; F \; P_{im,left} \; = \; 0$

- For instance, for one point correspondence,

$$\begin{bmatrix} u_1' & v_1' & 1 \end{bmatrix} \begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{pmatrix} \begin{bmatrix} u_1 \\ v_1 \\ 1 \end{bmatrix} = 0$$

Which can be written as

$$\begin{bmatrix} u_1' u_1 & u_1' v_1 & u_1' & v_1' u_1 & v_1' v_1 & v_1' & u_1 & v_1 & 1 \end{bmatrix} \begin{bmatrix} f_{11} \\ f_{12} \\ f_{13} \\ f_{21} \\ f_{22} \\ f_{23} \\ f_{31} \\ f_{32} \\ f_{33} \end{bmatrix} = 0$$

- For n point correspondences,

$$\begin{bmatrix} u_1'u_1 & u_1'v_1 & u_1' & v_1'u_1 & v_1'v_1 & v_1' & u_1 & v_1 & 1 \\ u_2'u_2 & u_2'v_2 & u_2' & v_2'u_2 & v_2'v_2 & v_2' & u_2 & v_2 & 1 \\ . & & & & & & & & \\ . & & & & & & & & \\ . & & & & & & & & \\ u_n'u_n & u_n'v_n & u_n' & v_n'u_n & v_n'v_n & v_n' & u_n & v_n & 1 \end{bmatrix} \begin{bmatrix} f_{11} \\ f_{12} \\ f_{13} \\ f_{21} \\ f_{22} \\ f_{23} \\ f_{31} \\ f_{32} \\ f_{33} \end{bmatrix} = 0$$

solve for f, vector of parameters.

Problems

- Let $X = (1, 2)$ be the projection of a 3D point p by the camera 1(in CCS), which is aligned with world coordinate system. Suppose the second camera is translated by $(1, 1, 0)$ and rotated with $30, 60, 90$ degrees about X, Y, Z axes respectively. Find the stereo correspondence for X, and then find the 3D point p. Finally, Verify your answer by projecting p using each of the cameras.
- Is it possible to compute the essential matrix E, given two cameras, when the camera parameter are not known
- Will the parallel lines be transformed into parallel lines when each of the following transforms are performed
  - Translation
  - Rotation
  - Scaling
  - Sheering
  - Perspective

Problems (cont.)

- A transform T is said to be isometric or distance preserving if $d(p_1, p_2) = d(T(p_1), T(p_2))$ for all points (P_1, p_2). Which of the following are isometric transforms. Justify your answers
  - Translation
  - Rotation
  - Scaling
  - Sheering
  - Perspective
- A transform of the form $T(X) = AX + t$ is called as affine transform, where A is a matrix. Which of the following are affine transforms. Justify your answers. Consider 3D Cartesian coordinates for X
  - Translation
  - Rotation
  - Scaling

Problems (cont.)

- Sheering
- Perspective

- Is every linear transform an affine? justify your answer
- Is every affine transform linear? justify your answer