Linear Classifier: Linear Discriminant Function

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Discriminant Function

- We know the proper forms for the discriminant functions and use the samples to estimate the values of parameters of the discriminant function
- Although it estimates the parameters of the discriminant function, it is said to be **non-parametric form** as it does require the knowledge about the probability distributions.
- Linear Discriminant function will be formulated as a problem of minimizing a criterion function.
- **Criterion function:** the obvious criterion function for classification purpose is the sample risk or training error.

Discriminant function

- **Training error:** The average loss incurred in classifying the set of training samples.
- **No probability form is assumed:** If the parametric form of the class-density function is not known; then we have to design the decision boundary using samples which are available with us.
- Here, we don't assume any parametric form of any probability distribution function.
- But, what we know is that, the classes are linearly separable

Linear Discriminant Function

- Non parametric form
- Supervised Learning
- Classes are linearly separable
- Classes : ω_1 and ω_2
- Using this information, as the classes are linearly separable, we can formulate the linear equation as $g(x) = W^t X + w_0$ X - d-dimensional vector W - d-dimensional weight vector $W^t X$ - Inner product of two vectors w_0 - bias/threshold weight

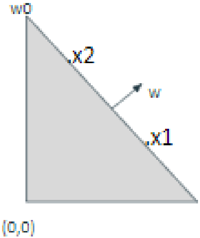
Decision criteria

- $g(x) > 0; x \epsilon \omega_1$
- $g(x) < 0; x \epsilon \omega_2$
- $g(x) = 0$; then x on the decision boundary

Now let us analyze the significance of each attribute in the equation, $g(x) = W^t X + w_0$

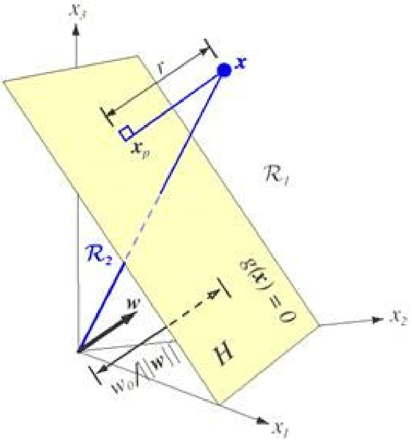
- Nature of weight vector W
- What does g(x) represents?

1. Nature of weight vector w



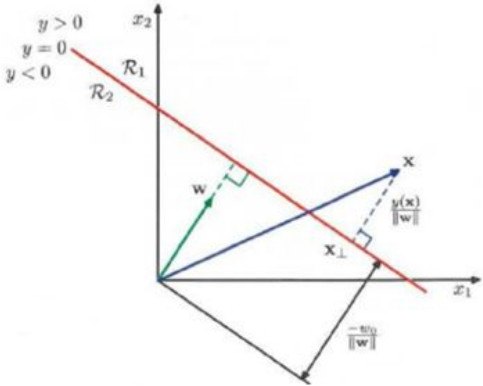
- $g(X_1) = g(X_2)$
- $W^t X_1 + w_0 = W^t X_2 + w_0$
- $W^t(X_1 - X_2) = 0$
- We know that, $A.B = |A|.|B| cos\Theta$;
- If $A.B = 0$, then A is perpendicular to B
- Likewise, $W^t(X_1 - X_2)$ is the inner product of weight vector W with $(X_1 - X_2)$.
- As it is zero, it indicates that vector 'W' is orthogonal to any vector lying on decision surface.
- In d-dimensional space, this surface is called as Hyper plane 'H'.

2. What does g(x) represents?



- Draw a perpendicular line from a point x to the Hyper plane 'H' which is X_p
- Let the distance of X and Xp is 'r' Then, $X = X_p + r.\frac{W}{||W||}$

2. What does g(x) represents?



- As seen earlier, W is orthogonal to the hyper plane 'H'.
- So, the direction of 'W' is same direction of from X_p to X.
- Hence, Both X_p to X and 'W' is orthogonal to hyper plane 'H'

2. What does g(x) represents?

$$\boxed{\frac{W}{||W||} = \frac{\sum_{i=1}^{d} w_i}{\sqrt{\sum_{i=1}^{d}(w_i)^2}}}$$

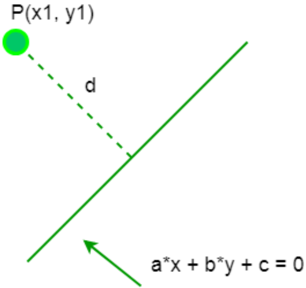
$$\boxed{X = X_P + r.\frac{w}{||W||}}$$

- $g(X) = W^t X + w_0$
- $g(X) = W^t[X_p + r.\frac{w}{||W||}] + w_0$
- $g(X) = W^t X_p + w_0 + r.\frac{W^t.W}{||W||}$

The point X_p that lies on the decision surface so $W^t X_p + w_0$ is zero.

- $g(X) = 0 + r.\frac{W^t.W}{||W||}$
- $g(X) = 0 + r.\frac{||W||^2}{||W||}$
- $g(X) = r.||W||$

2. Why g(x) is algebraic measure?

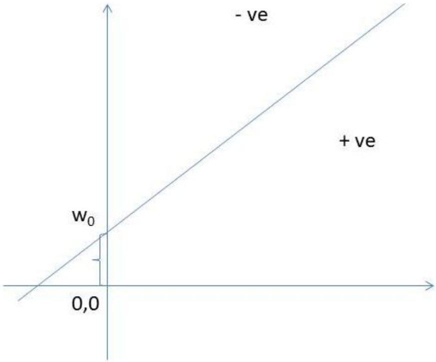


- If $ax + by + c = 0$ is the equation of the straight line and (x_1, y_1) is a point, then distance of (x_1, y_1) to the line is nothing but
- $d = \frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}$ In, 2-dimension.
- $r = \frac{g(x)}{||w||^2}$ In, d-dimension.

3. Distance of origin from the hyperplane H

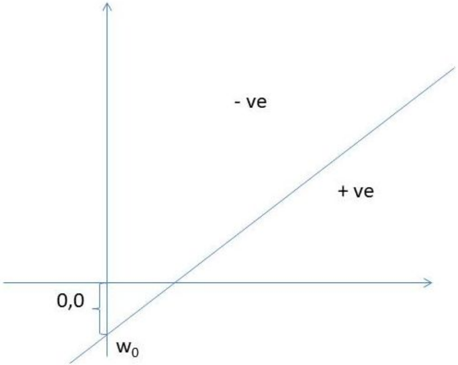
- Distance of origin from the huperplane H is $\frac{w_0}{||W||}$; w_0 is Bias/Threshold.
- If w_0 is $+$ve, then origin lies on the $+$ve side of the hyper plane 'H'.
- If w_0 is -ve, then origin lies on the -ve side of the hyper plane 'H'.
- If w_0 is zero, then the hyper plane passes through origin. And also,

3. Distance of origin from the hyperplane H



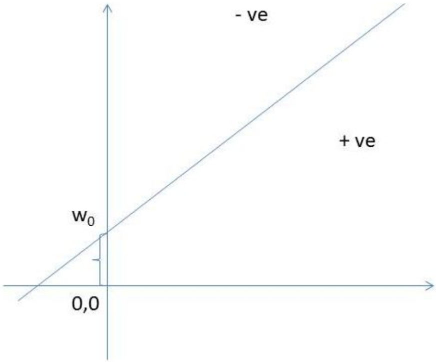
If w_0 is positive then origin lies in positive side

3. Distance of origin from the hyperplane H



If w_0 is negative then origin lies in negative side

3. Distance of origin from the hyperplane H



If w_0 is positive then origin lies in positive side

3. Distance of origin from the hyperplane H

- If w_0 is zero, discriminant function $g(x)$ takes the particular form $g(x) = W^T X$; in this case we don't have any bias because $w_0 = 0$
- $g(x) = W^T X$ is said to be in Homogeneous form
- In mathematics, It is convenient, If we represent the equation in Homogeneous form.
- So, in order to design a linear classifier we should estimate two parameters such as weight vector W and bias w_0.
- Since it is supervised learning W and w_0 are supposed to be estimated based on the samples that are available.

Design of weight vector W

- **Assumption: Two classes and linearly separable case**
- We have two classes and it is linearly separable
- We should have the discriminant function which separates these two classes
- It is of the form $g(X) = W^t X + w_0$
- This expression is not in homogeneous form.
- Hence, converting this homogeneous form makes the analysis easier.

Converting to Homogeneous form

- $g(X) = W^t X + w_0$
- $g(X) \approx a^t y$
- $g(X) \approx \begin{bmatrix} w_1 & w_2 & \dots & \dots & w_d & w_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ .. \\ x_d \\ 1 \end{bmatrix}$
- $g(X) \approx \sum_{i=1}^{d} w_i x_i + w_0$
- $g(X) \approx W^t X + w_0$

Decision rule in Homogeneous form

- **The decision rule remains the same, for** $a^t y$
- If $a^t y > 0$ then decide $y \epsilon \omega_1$
- If $a^t y < 0$ then decide $y \epsilon \omega_2$
- If $a^t y = 0$ then no decision can be taken.

How to design weight vector 'W' and w_0 using the samples?

- We have n- no of samples (or) training samples $y_1, y_2, ..., y_n$
- $y_1 = \begin{bmatrix} x_{11} \\ x_{12} \\ .. \\ .. \\ x_1 d \\ 1 \end{bmatrix}$ $y_2 = \begin{bmatrix} x_{21} \\ x_{22} \\ .. \\ .. \\ x_{2d} \\ 1 \end{bmatrix}$ $y_3 = \begin{bmatrix} .. \\ .. \\ .. \\ .. \\ .. \\ .. \end{bmatrix}$ $y_4 = \begin{bmatrix} .. \\ .. \\ .. \\ .. \\ .. \\ .. \end{bmatrix}$ $y_n = \begin{bmatrix} x_{n1} \\ x_{n2} \\ .. \\ .. \\ x_{nd} \\ 1 \end{bmatrix}$
- These are the samples which are useful to train the classifier.
- Some of the samples are labeled as ω_1 and some are labeled as ω_2.
- Let's consider the i^{th} sample as y_i.

Two Criterion Decision rule in Homogeneous form

- **The decision rule remains the same, for** $a^t y_i$
- If $a^t y_i > 0$ then decide $y_i \epsilon \omega_1$
- If $a^t y_i < 0$ then decide $y_i \epsilon \omega_2$
- If $a^t y_i = 0$ then no decision can be taken.

Two Criterion Decision rule in Homogeneous form

- Given a weight vector 'a'; If we take all the samples which are labelled as ω_1
- If for each of the samples, $a^t y_i > 0$; then that weight vector 'a' is correctly classifying all the samples which are labelled as ω_1
- If we also find, for the same weight vector 'a' all the samples belonging to class ω_2;
- If $a^t y_i < 0$; then the weight vector 'a' is also classified correctly for all samples belongs to class ω_2
- That particular weight vector 'a' is the correct weight vector, because it is correctly classified all the samples labelled as ω_1, also it is correctly classified all the samples labelled as ω_2.

Single Criterion

- Instead of two conditions $a^t y_i > 0$ and $a^t y_i < 0$, Can't we have a single criterion to classify correctly.
- $a^t y_i > 0$ true, irrespective of class label.
- We can say that, y_i is correctly classified if $a^t y_i > 0$. Otherwise, y_i is mis-classified.
- Otherwise, include < 0 and $= 0$.

Single Criterion: How can we do that?

- Samples belonging to class ω_1, we can take them as it is.
- For the samples belonging to class ω_2, we augment them by appending 1 and then take negative of it.
- Take all the samples which are labelled as ω_2 and then negate it.
- Instead of considering y_i, consider $-y_i$
- If we take negative, this $a^t y_i$ which is supposed to be < 0, now it will be > 0.
- So, we get single (uniform) decision criterion which is $a^t y_i > 0$ for both the classes.

Single Criterion: How can we do that?

- If $a^t y_i > 0$, all samples are correctly classified, irrespective of class labels.
- Now, for this what will be the weight vector 'a'?
- We take some criteria Function, $J(a)$.
- $J(a)$ has to be minimized, if 'a' is a solution (correct weight) vector.
- $J(a)$ will be minimum, If it classifies all the training samples correctly, for the weight vector 'a' which is obtained.
- For minimization of $J(a)$, we can make use of **Gradient Descent Procedure**.

Gradient Descent Procedure

Gradient Descent

Gradient descent is an iterative optimization algorithm for finding the **minimum** of a function.

4

next position

opposite direction

gradient at current position

-10

$$x_{i+1} = x_i - \alpha \nabla f(x_i)$$

current position

-5

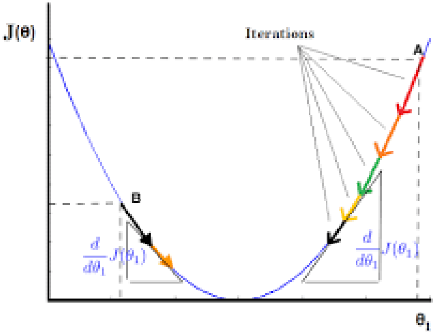
step size (learning rate)

set to 0.9 here, usually much smaller

Gradient Descent Procedure

- Initialize with weight vector $a(k)$ with some random values and try to minimize the training error for every iteration.
- At the k^{th} iteration, we know the values of a(k).
- We should update the weight vector for a(k+1).
- $a(k+1) = a(k) - \eta(k) \bigtriangledown J(a(k))$
- This is called **Gradient Descent Procedure** or **Steepest Descent Procedure**.

Algorithm: Gradient Descent



- Initialize a, threshold $\theta, \eta(.), k \leftarrow 0$
- **do** $k \leftarrow k + 1$
- $a \leftarrow a - \eta(k) \bigtriangledown J(a)$
- **until** $\eta(k) \bigtriangledown J(a) < \theta$
- return a

Perceptron Criterion Function

- Our aim will be to find out weight vector 'a' which will classify all the training samples correctly.
- So, we can try to design a criterion function which will make use of samples which are **not correctly classified**.
- If the samples are not correctly classifies by a(k), then update weight vector 'a' in a(k+1)
- So, accordingly we can define the criterion function
- **Criterion function** can be defined as,
- $J_p(a) = \sum_{\forall y \text{ misclassified}} (-a^t y)$, here p refers to perceptron criterion.

Perceptron Criterion Function

- $J_p(a) = \sum_{\forall y \text{ misclassified}} (-a^t y)$,
- Here, $(-a^t y)$ is positive.
- As a result, The criterion for $J_p(a)$, never have a negative value.
- It can always have a positive value.
- The minimum value can be 0.
- So, we have a global minimum for $J_p(a)$ and this can be find by Gradient Decent Procedure.

Perceptron Criterion Function

- According to Gradient Descent Procedure, take gradient of $J_p(a)$ w.r.t weight vector a.
- $J_p(a) = \sum_{\forall y \text{ misclassified}}(-a^t y)$,
- $\triangledown .\ J_p(a) = \sum_{\forall y \text{ misclassified}}(-y)$
- The update rule is,

  $\boxed{a(0) \Rightarrow \text{Initial weight vector; arbitrary}}$.

  $\boxed{a(k+1) = a(k) + \eta(k) \sum_{\forall y \text{ misclassified}}(y)}$
- This is the algorithm to design weight vector 'a' if the samples are linearly separable.

THANK YOU