Non-regular languages

We have hinted before that not all languages are regular. E.g.

- The language $\{a^n b^n \mid n \geq 0\}$.
- The language of all *well-matched* sequences of brackets (,). N.B. A sequence x is well-matched if it contains the same number of opening brackets '(' and closing brackets ')', and no initial subsequence y of x contains more)'s than ('s.
- The language of all prefixes of well-matched sequences of brackets (,). A string x is in this language if no initial subsequence y of x contains more)'s than ('s.

But how do we know these languages aren't regular?

And can we come up with a general technique for proving the non-regularity of languages?

The basic intuition: DFAs can't count!

Consider $L = \{a^n b^n \mid n \geq 0\}$. Just suppose, hypothetically, there *were* some DFA M with $\mathcal{L}(M) = L$.

Suppose furthermore that M had just processed a^n, and some continuation b^m was to follow.

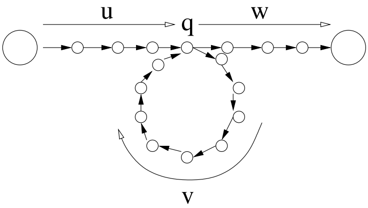
Intuition: M would need to have *counted* the number of a's, in order to know how many b's to require.

More precisely, let q_n denote the state of M after processing a^n. Then for any $m \neq n$, the states q_m, q_n must be different, since b^m takes us to an accepting state from q_m, but not from q_n.

In other words, M would need infinitely many states, one for each natural number. Contradiction!

Loops in DFAs

Let M be a DFA with k states. Suppose, starting from any state of M, we process a string y of length $|y| \geq k$. We then pass through a sequence of $|y| + 1$ states. So there must be some state q that's visited *twice or more*:



(Note that u and w might be ϵ, but v definitely isn't.)

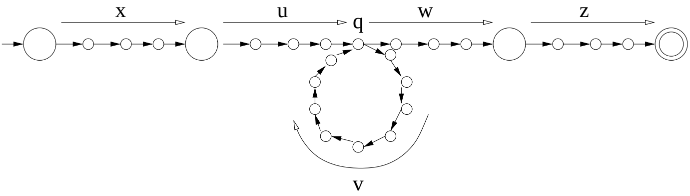
So any string y with $|y| \geq k$ can be decomposed as uvw, where

- u is the prefix of y that leads to the first visit of q
- v takes us once round the loop from q to q,
- w is whatever is left of y after uv.

A general consequence

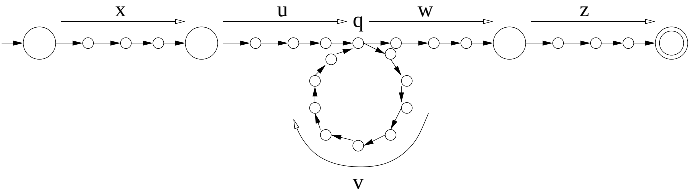
If L is *any* regular language, we can pick *some* corresponding DFA M, and it will have some number of states, say k.

Suppose we run M on a string $xyz \in L$, where $|y| \geq k$. There must be at least one state q visited twice in the course of processing y:



(There may be other 'revisited states' not indicated here.)

The idea of 'pumping'



So y can be decomposed as uvw, where

- xu takes M from the initial state to q,
- $v \neq \epsilon$ takes M once round the loop from q to q,
- wz takes M from q to an accepting state.

But now M will be oblivious to whether, or how many times, we go round the v-loop!

So we can 'pump in' as many copies of the substring v as we like, knowing that we'll still end in an accepting state.

The pumping lemma: official form

The pumping lemma basically summarizes what we've just said.

Pumping Lemma. Suppose L is a regular language. Then L has the following property.

> *(P) There exists $k \geq 0$ such that, for all strings x, y, z with $xyz \in L$ and $|y| \geq k$, there exist strings u, v, w such that $y = uvw$, $v \neq \epsilon$, and for every $i \geq 0$ we have $xuv^iwz \in L$.*

The pumping lemma: contrapositive form

Since we want to use the pumping lemma to show a language *isn't* regular, we usually apply it in the following equivalent but back-to-front form.

Suppose L is a language for which the following property holds:

> ($\neg P$) *For all* $k \geq 0$, *there exist strings* x, y, z *with* $xyz \in L$ *and* $|y| \geq k$ *such that, for every decomposition of* y *as* $y = uvw$ *where* $v \neq \epsilon$, *there is some* $i \geq 0$ *for which* $xuv^iwz \notin L$.

Then L is not a regular language.

N.B. The pumping lemma can only be used to show a language *isn't* regular. Showing L satisfies (P) doesn't prove L is regular!

To show that a language *is* regular, give some DFA or NFA or regular expression that defines it.

The pumping lemma: a user's guide

So to show some language L is not regular, it's enough to show that L satisfies (¬P).

Note that (¬P) is quite a complex statement: $\forall \cdots \exists \cdots \forall \cdots \exists \cdots$.

We'll look at a simple example first, then offer some advice on the general pattern of argument.

Example 1

Consider $L = \{a^n b^n \mid n \geq 0\}$.

We show that L satisfies $(\neg\mathrm{P})$.

Suppose $k \geq 0$. (We can't choose the value of k. The argument has to work for all numbers.)

Consider the strings $x = \epsilon$, $y = a^k$, $z = b^k$. Note that $xyz \in L$ and $|y| \geq k$ as required. (We make a cunning choice of x, y, z.)

Suppose now we're given a decomposition of y as uvw with $v \neq \epsilon$. (We can't choose the strings u, v, w. The argument has to work for all possibilities.)

Let $i = 0$. (We make a cunning choice of i.)

Then $uv^i w = uw = a^l$ for some $l < k$. So $xuv^i wz = a^l b^k \notin L$.

Thus L satisfies $(\neg\mathrm{P})$, so L isn't regular.

Use of pumping lemma: general pattern

On the previous slide, the full argument is in black, whereas the parenthetical comments in blue are for pedagogical purposes only.

The comments emphasise the care that is needed in dealing with the quantifiers in the property ($\neg$P). In general:

- You are not allowed to choose the number $k \geq 0$. Your argument has to work for every possible value of k.
- You have to choose the strings x, y, z, which might depend on k. You must choose these to satisfy $xyz \in L$ and $|y| \geq k$. Also, y should be chosen cunningly to 'disallow pumping' …
- You are not allowed to choose the strings u, v, w. Your argument has to work for every possible decomposition of y as uvw with $v \neq \epsilon$.
- You have to choose the number i ($\neq 1$) such that $xuv^iwz \notin L$. Here i might depend on all the previous data.

Example 2

Consider $L = \{a^{n^2} \mid n \geq 0\}$.
We show that L satisfies ($\neg$P):

Suppose $k \geq 0$.
Let $x = a^{k^2-k}$, $y = a^k$, $z = \epsilon$, so $xyz = a^{k^2} \in L$.
Given any splitting of y as uvw with $v \neq \epsilon$, we have $1 \leq |v| \leq k$.
So taking $i = 2$, we have $xuv^2wz = a^n$ where $k^2 + 1 \leq n \leq k^2 + k$.
But there are no perfect squares between k^2 and $k^2 + 2k + 1$.
So n isn't a perfect square. Thus $xuv^2wz \notin L$.

Thus L satisfies ($\neg$P), so L isn't regular.