Alfven Wave Relations in Plasma

B-Tech Project Report

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1 Introduction

Plasma is essentially a gas of ions - a conducting medium produced when atoms in a gas become ionized. The subject of plasma physics can be thought of as an interaction between these particles and electromagnetic fields. These interactions and behaviour of the plasma consisting of large number of particles can be described by magnetohydrodynamics (MHD), if the plasma is sufficiently collisonal. The basic state of such a plasma can be locally defined by its mass density ρ, momentum density $\rho\vec{V}$, pressure p, electric field $\vec{E}$ and magnetic field $\vec{B}$ [1].

A simple example of an MHD motion is a wave, which is a perturbation of these quantities from smooth motion. The solutions to this MHD wave dispersion relation are the three *modes* by which MHD waves propagate. The intermediate mode (also called the Alfven mode) is the main interest for us. These waves have low frequencies and long wavelengths when compared to ion gyroradii or ion inertial lengths. They are transverse to their direction of propagation, incompressible or weakly compressible and can propagate long distances without heavy damping.

This MHD Alfven wave can turn into a kinetic Alfven wave when a large wavenumber $k_\perp$ develops perpendicular to the magnetic field. Kinetic Alfven waves, unlike their MHD counterparts, are linearly compressible. Electric field fluctuations are generated in the direction of magnetic fields, and consequently lead to damping. Knowledge of the kinetic Alfven wave is necessary in many areas, particularly in studying the solar coronal heating problem, the earth's Aurora and planetary magnetospheres.

The focus of this report is on understanding the wave dispersion relations in plasma, exploring the differences that arise between the MHD and the kinetic Alfven waves, and the deriving the necessary quantities to define the state of a quasi-neutral two-fluid plasma.

Section (2) lists Maxwell's Equations and Ideal MHD equations used throughout the report. In Section (3), we derive the dispersion relation and the Alfven mode solution for a static homogeneous medium in a low frequency and low wavelength limit. Section (4) deals with the quantities that may have been ignored in the MHD wave equation, and talks about their significance when they are no longer negligible. Section (5) covers the major focus of this project, dealing with the behaviour of different variables in a quasi-neutral two fluid plasma and deriving quantities that describe the state locally. Section (6) is a continuation of the previous section, and contains the derivation of the dispersion relation, and discussed the Alfven wave mode expressions.

2 Ideal MHD equations and Maxwell's equations

2.1 Maxwell's Equations

Faraday's Law of Induction:	$\frac{\partial \vec{B}}{\partial t} = -c(\nabla \times \vec{E})$
Ampere's Law:	$(\nabla \times \vec{B}) = 4\pi \vec{j} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$
Divergence Equation:	$\nabla.\vec{B} = 0$
Poisson's Equation:	$\nabla.\vec{E} = 4\pi q$

2.2 Ideal MHD Equations [1]

Continuity Equation:	$\frac{\partial \rho}{\partial t} + \nabla.(\rho V) = 0$
Equation of Motion:	$\rho\frac{d\vec{V}}{dt} = \vec{j} \times \vec{B} - \nabla p + \rho g$
Entropy or Pressure Equation:	$\frac{d}{dt}\left(\frac{p}{\rho\gamma}\right) = 0$
Magnetic Differential Equation:	$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{V} \times \vec{B})$
Magnetic Differential Equation with resistivity:	$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{V} \times \vec{B}) + ((\eta c)/4\pi)\nabla^2 \vec{B}$

3 MHD Alfven wave

Let us consider a static, homogeneous medium defined by mass density ρ_0, pressure p_0, velocity $V_0 = 0$ and uniform magnetic field B_0 in the z-direction. Small perturbations in these quantities create waves, and we denote these perturbations by δ.
By assuming small perturbations, we can use linearized MHD equations to derive the dispersion relation. For perturbations of the form $e^{i(\vec{k}.\vec{r}-\omega t)}$ denoted by δ, the MHD equations become:

$$\begin{aligned} -\omega\delta\rho + \rho_0\vec{k}.\delta\vec{V} = 0 \\ -\omega p_0\delta\vec{V} + \vec{k}\delta p = (\vec{k} \times \delta\vec{B}) \times \vec{B}_0 \\ -\omega\delta\vec{B} - \vec{k} \times (\delta\vec{V} \times \vec{B}_0) = 0 \\ -\omega\delta p - c_s^2\delta\rho_0\vec{k}.\delta\vec{V} = 0 \end{aligned} \tag{1}$$

where $c_s = \gamma p_0/\rho_0$.
On expanding the triple vector product on the second equation of 1 and setting $\alpha = \vec{k}.\vec{B}_0$, we get

$$\omega p_0\delta\vec{V} = \vec{k}\delta p + (B_0.\delta B)\vec{k} - (\alpha)\delta\vec{B}$$

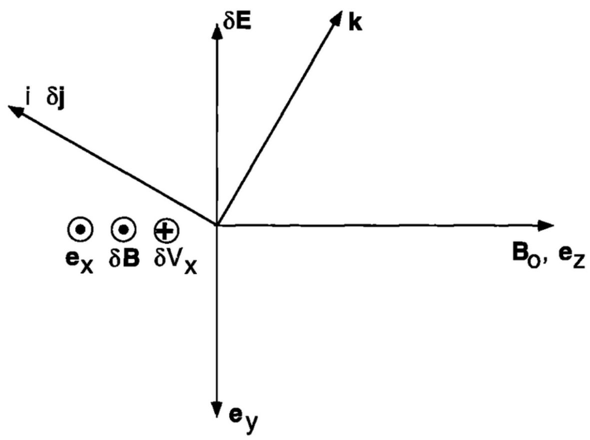

Figure 1: The directions of B_0 and k assumed without loss of generality

We replace δp by using the last relation of (1) and manipulate terms such that:

$$(\omega^2\rho_0 - \alpha^2)\delta\vec{V} = (\vec{k}.\delta\vec{V})\left[(c_s^2\rho_0 + B_0^2)\vec{k} - \alpha B_0\right] - \alpha(\vec{B}_0.\delta\vec{V})\vec{k} \tag{2}$$

The directions of the fields are as shown in the Figure 1. If θ is the angle between and , then:

$$\vec{k} = k_y\hat{y} + k_z\hat{z}$$
$$\alpha = k_z B_0$$

From solenoidal condition in Fourier space, we can say that there is no field perturbation in the direction of wave propagation, i.e., we have $\vec{k}.\delta\vec{B} = 0$. If we take $v_a^2 = B_0^2/\rho$, (2) becomes:

$$(\omega^2 - k_z^2v_a^2)\delta\vec{V} - \vec{k}.\delta\vec{V}\left[(c_s^2 + v_a^2)\vec{k} - k_z v_a^2\hat{z}\right] + k_z v_a^2\delta V_z\vec{k} = 0 \tag{3}$$

Equation (3) is a homogeneous linear equation in $\delta\vec{V}$ of the form $A\delta V = 0$, where A is a matrix (We can also solve it by taking x-, y- and z- components of the equation separately).

$$A = \begin{pmatrix} (\omega^2 - k_z^2v_a^2) & 0 & 0 \\ 0 & \omega^2 - k^2v_a^2 - c_s^2k_y^2 & \text{-k}_y k_z c_s^2 \\ 0 & \text{-k}_y k_z c_s^2 & (\omega^2 - k_z^2c_s^2) \end{pmatrix}. \tag{4}$$

For a non-zero solution of $\delta\vec{V}$, $det(A) = 0$ must be satisfied. This equation, after a little bit of rearranging, gives us:

$$\left[(\frac{w}{k})^2 - v_a^2cos^2\theta\right]\left[(\frac{w}{k})^4 - (\frac{w}{k})^2(v_a^2 + c_s^2) + v_a^2c_s^2cos^2\theta\right] = 0 \tag{5}$$

This is a cubic equation in ω^2 is the *dispersion relation* of MHD waves, and its solutions form the three MHD modes: fast, slow and Alfven (intermediate). From (5), the Alfven root is given by

$$\omega^2 = k^2 v_a^2 cos^2(\theta) \tag{6}$$

where $k^2 = k_y^2 + k_z^2$ and θ is the angle between k and B_0.
Its corresponding eigenvector is $\delta\ \vec{V} = (\delta V_x, 0, 0)$, i.e., velocity perturbations lie in the x-direction. From (1), we see that this wave carries no pressure or density perturbations (since $\delta V\ \perp\ k$), and thus has an incompressible nature. Thus, we can say that *Alfven mode is incompressible in a low-frequency, low-wavelength MHD limit.*

4 Approximations in MHD Alfven Wave

In the previous section, we have derived the wave dispersion relation from ideal Magneto-Hydrodynamic Equations. In this section we will try to identify and calculate some quantities which we have ignored or were very small in the MHD Alfven wave and the circumstances under which they are no longer ignorable. We will see how this plays a role in how the MHD Alfven wave undergoes a transition to the kinetic Alfven wave.

4.1 Density Perturbations

In the MHD Alfven wave, there exists a y- component of current[(2.2)] that is chiefly carried by ions via a polarization drift:

$$\delta V_y = -\frac{q}{m}\frac{i\omega}{\omega_{ci}^2}\delta E_y$$

where $\omega << \omega_c$. Thus, $k.\delta V$ is no longer zero. Therefore, our MHD Alfven wave is incompressible only when k_y^-1 is much lesser than ion-intertial length. When those two quantities become comparable, the Alfven mode is no longer incompressible. Instead, we see density compressions (1) like:

$$\frac{\partial n}{\partial t} + n_0 \nabla.\delta\vec{V} = 0$$

These compressions are associated with ion currents as

4.2 Electric Field Perturbations

The parallel electric field fluctuation, δE_z, is determined by the requirement that the electrons move along B_0 in such a way as to neutralize the ion density perturbations given by the above equation. The z- component of the linearized electron momentum equation is:

$$m_e\omega\delta V_{z_e} = -ie\delta E_z + k_z\delta p_e/n_{0_e}$$

The MHD Alfven wave does not have a significant electric field component parallel to its magnetic field. This leads to little damping of the wave and enables it to travel long distances. For the kinetic Alfven wave, on the other hand, its parallel component of the electric field will affect particles that are in Landau resonance with the wave. This leads to Landau damping of the wave.

5 Two Fluid Model

In this section, we try to define the state of a quasi-neutral two fluid plasma model with electron and proton species in terms of pressure, density, velocity, electric field and magnetic field perturbations.

5.1 Pressure and Velocity Perturbations

The linearised momentum equation for any species (as derived from the MHD equation of motion given in Section 2) is:

$$-i\omega m\delta\vec{V} = q(\delta\vec{E}' - \frac{\delta\vec{V}\times\vec{B_0}}{c})$$

On taking cross product with B_0 on both sides, we get

$$-i\omega m(\delta\vec{V}\times\vec{B_0}) = q(\delta\vec{E}'\times\vec{B_0} - \frac{\delta V_{\parallel}B_0^2\hat{B}_0 - B_0^2\delta\vec{V}}{c})$$

$$-i\omega m(-i\omega m\delta\vec{V} - q\delta\vec{E}') = q(\delta\vec{E}'\times\vec{B_0} - \frac{\delta\vec{V}_{\perp}B_0^2}{c})$$

$$\delta\vec{V_\perp} - \frac{\omega^2}{\omega_c^2}\delta\vec{V} = (\frac{\delta\vec{E}'\times\vec{B_0}}{B_0^2})c - \frac{q}{m}\frac{i\omega}{\omega_c^2}\delta\vec{E}' \tag{7}$$

where $\delta\vec{E}' = \delta\vec{E} - \frac{\nabla\delta p}{qn_0}$, $V_0 = 0$ and $\omega_c = \frac{qB_0}{mc}$.

As we've seen in Section 4, we need $\vec{V}$, $\vec{B}$, p, n and $\vec{E}$ to describe the state of plasma. We are now going to derive expressions of the perturbations in all those quantities and the wave dispersion relation for this Two-Fluid Model containing ion and electron species.

If δn denotes density perturbations, we can express pressure perturbations as:

$$\delta p = \gamma\kappa T_0\delta n \tag{8}$$

$$\omega\delta n = n_0(\vec{k}.\delta\vec{V}) \tag{9}$$

We calculate the x-, y- and z- components of $\delta\vec{V}$ from Equation (7). If $G = \gamma\kappa T$, then:

$$\delta\vec{V}_{\perp} - \frac{\omega^2}{\omega_c^2}\delta\vec{V} = \frac{\delta\vec{E}\times\vec{B}_0}{B_0^2}c - \frac{iG\delta n}{qn_0}\frac{\vec{k}\times\vec{B}_0}{B_0^2}c - i\frac{q\omega}{m\omega_c^2}\delta\vec{E} - \frac{\omega}{m\omega_c^2}\frac{G\delta n}{n_0}\vec{k}$$

The $\frac{\omega^2}{\omega_c^2}$ term on the left can be ignored for transverse components of velocity. Therefore,

$$\delta V_x = \frac{q}{mc}(\delta E_y - i\frac{\omega}{\omega_c}\delta E_x) - i\frac{Gk_y}{m\omega_c}\frac{\delta n}{n_0} \tag{10}$$

$$\delta V_y = -\frac{q}{m\omega_c}(\delta E_x - i\frac{\omega}{\omega_c}\delta E_y) - \frac{Gk_y\omega}{m\omega_c^2}\frac{\delta n}{n_0} \tag{11}$$

$$\delta V_z = i\frac{q}{m\omega}\delta E_z + \frac{Gk_z}{m\omega}\frac{\delta n}{n_0} \tag{12}$$

We must note that polarization drifts in (7) may be ignored for electron species when compared to ions.

5.2 Density Perturbations for Ions and Electrons

We have expressed δV as a function of δE and δn. We can now derive an expression for δn by substituting equations (11) and (12) in equation (9). For an ion species:

$$\delta n_i = n_{0_i}\frac{[m_i c\omega\omega_{c_i}^2 k_y\delta E_x + iqB_{0_z}(k_y\omega^2\delta E_y - k_z\omega_{c_i}^2\delta E_z)]}{B_{0_z}[G_i(k_z^2\omega_{c_i}^2 - k_y^2\omega^2) - m_i\omega^2\omega_{c_i}^2]} \tag{13}$$

To find density perturbations for electron species, the term $\frac{\omega^2}{\omega_{c_e}^2}$ can be neglected. Futhermore, if we take

$$\left|\frac{k_y}{k_z}\frac{\omega}{\omega_{c_e}}\frac{\delta E_x}{\delta E_z}\right| << 1$$

The density perturbations in electron species will be:

$$\delta n_e = n_{0_e}\frac{ie\delta E_z}{k_z(G_e - m_e\omega^2/k_z^2)} \tag{14}$$

5.3 Electric Field perturbation relations

Assuming the conditions for a quasi-neutral electron-proton plasma, we have

$$\left|\frac{q}{e}\right| = \frac{n_{0_e}}{n_{0_i}} = 1$$

$$\text{and} \qquad \frac{\delta n_i}{n_{0_i}} = \frac{\delta n_e}{n_{0_e}} \tag{15}$$

Let us substitute equations (13) and (14) in equation (15). If we drop terms involving m_e/m_i, we get a relation between all three components of $\delta\vec{E}$

$$\delta E_y = i\frac{\omega}{\omega_c}\delta E_x + \frac{k_z\omega_c^2}{k_y\omega^2}\frac{(\omega_c^2(G_tk_z^2 - m_i\omega^2) - G_tk_y^2\omega^2)}{(G_ek_z^2 - m_e\omega^2)}\delta E_z \tag{16}$$

where $G_t = G_i + G_e$.
To obtain relations between two components of $\delta\vec{E}$, we start with equations (2.2) and (2.2) from Maxwell's Equations and get relations between $\delta\vec{j}$ and $\delta\vec{E}$:

$$-c^2(\nabla\times(\nabla\times\vec{E})) = \frac{4\pi}{c}\frac{\partial\vec{j}}{\partial t} + \frac{1}{c}\frac{\partial^2\vec{E}}{\partial t^2}$$
$$-i4\pi\omega\delta\vec{j} = c^2((k_y\delta E_y + k_z\delta E_z)\vec{k} - (k^2-\omega^2)\delta\vec{E}) \tag{17}$$

where $k^2 = k_y^2 + k_z^2$ and δj is the current density vector. Now,

$$\delta\vec{j} = \delta\vec{j}_{ions} + \delta\vec{j}_{electrons}$$
$$\text{i.e.,}\quad \delta\vec{j} = n_{0_i}q\delta\vec{V}_i + n_{0_e}e\delta\vec{V}_e$$

Since we know $\delta\vec{V}$ from (10), (11) and (12), we can find $\delta\vec{j}$ from the above equation. On substituting δj_x in the x- component of equation (17):

$$\frac{\delta E_z}{\delta E_x} = i\frac{k_z(\omega^2 - k^2v_a^2)}{k_y\omega\omega_{c_i}}\frac{(G_e - m_e\omega^2/k_z^2)}{G_t} \tag{18}$$

where terms involving $\frac{m_e}{m_i}$ and $\frac{\omega^2v_a^2}{c^2}$ have been dropped.
Similarly, from the y- component of (17) gives:

$$\frac{\delta E_z}{\delta E_y} = \frac{k_z}{k_y}\frac{(G_e - \frac{m_e\omega^2}{k_z^2})(\Omega^2-1)}{(\Omega^2(m_ev_a^2 - G_i) - G_e)} \tag{19}$$

where $\Omega = \frac{\omega}{k_zv_a}$ and $\frac{\omega^2}{k_z^2c^2}$ is ignored.
From (18) and (19):

$$\frac{\delta E_y}{\delta E_x} = i\frac{(\omega^2 - k^2v_a^2)(\Omega^2(m_ev_a^2 - G_i) - G_e)}{G_t(\Omega^2-1)\omega\omega_{c_i}} \tag{20}$$

5.4 Magnetic Field Perturbations

We simply use Faraday's Law of Induction((2.1)) to express the magnetic field in terms of δE_x, δE_y and δE_z. Therefore,

$$\delta B_x = \frac{c}{\omega}(k_y\delta E_z - k_z\delta E_y) \tag{21}$$
$$\delta B_y = \frac{c}{\omega}k_z\delta E_x \tag{22}$$
$$\delta B_z = -\frac{c}{\omega}k_y\delta E_x \tag{23}$$

5.5 Summary

Thus, the state of a quasi-neutral two species plasma model can be given by pressure perturbations δp ((8)), density perturbations δn ((13) and (14)), velocity perturbations $\delta\vec{V}$ ((10), (11) and (12)), magnetic field perturbations $\delta\vec{B}$ ((21), (22) and (23)) and electric field field perturbations $\delta\vec{E}$ ((18) and (20)), where all these quantities can be expressed as a function of δE_x.

6 Dispersion Relation and Results

6.1 Derivation

From (18) and (19), we get:

$$\frac{\delta E_z}{\delta E_x} = -i\frac{k_y k_z \omega\omega_{c_i}(\omega^2 - k_z^2 v_a^2)(G_e - m_e\omega^2/k_z^2)}{[(m_i\omega^2 - G_t k_z^2)(k_z^2 v_a^2 - \omega^2)\omega_{ci}^2 + k_y^2\omega^2 v_a^2(G_t^2 k_z^2 - m_e\omega^2)]} \tag{24}$$

On equating RHS of (18) and (24), we get the dispersion relation:

$$(\frac{\omega^2}{k_z^2 v_a^2} - 1)(\omega^2(\omega^2 - k^2 v_a^2) - \beta k^2 v_a^2(\omega^2 - k_z^2 v_a^2)] = \omega^2(\omega^2 - k^2 v_a^2)k_y^2[L^2 - \frac{\omega^2 L_e^2}{k_z^2 v_a^2}] \tag{25}$$

where:

$$\begin{aligned}
v_a^2 &= \frac{B_0^2}{4\pi m_i n_0}\\
G_t &= \gamma_i\kappa_i T_{0i} + \gamma_e\kappa_e T_0 e\\
\beta &= \frac{G_t}{m_i v_a^2}\\
\omega_{pe}^2 &= 4\pi q_e^2 n_{0_e}/m_e\\
L &= \frac{1}{|\omega_{c_i}|}\sqrt{\frac{\gamma_i\kappa_i T_{0i} + \gamma_e\kappa_e T_0 e}{m_i}} = \frac{v_s}{|\omega_{c_i}|}\\
L_e &= \frac{c}{\omega_{pe}} = \frac{L}{\sqrt{\frac{m_i}{m_e}\beta}}
\end{aligned}$$

After a little algebraic manipulation, we can rewrite this (25) as:

$$\Omega^6\frac{k_z^2}{k^2}(1 + k_y^2 L_e^2) - \Omega^4[\frac{k_z^2}{k^2}(1 + k_y^2 L^2) + (1 + \beta + k_y^2 L_e^2)] + \Omega^2(1 + 2\beta + k_y^2 L^2) - \beta = 0 \tag{26}$$

where $\Omega = \frac{\omega}{k_z v_a}$.
This is a cubic equation in Ω^2 and can be expressed as:

$$a(\Omega^2)^3 + b(\Omega^2)^2 + c(\Omega^2)^1 + d = 0$$

with:

$$\begin{aligned}
a &= \frac{k_z^2}{k^2}(1 + k_y^2 L_e^2) \\
b &= -[\frac{k_z^2}{k^2}(1 + k_y^2 L^2) + (1 + \beta + k_y^2 L_e^2)] \\
c &= (1 + 2\beta + k_y^2 L^2) \\
d &= -\beta
\end{aligned}$$

6.2 Solutions of the Dispersion Relation

The three roots of (26) represent the three linear wave modes in plasma: fast, Alfven (intermediate) and slow modes. We will chiefly focus on the Alfven wave mode and its behaviour.

6.2.1 With approximation $k_y^2 >> k_z^2$

In his paper [2], J. V. Hollweg argued that if $k_y^2 >> k_z^2$, then $(\omega^2 - k^2 v_a^2) \approx -k^2 v_a^2$, resulting in eqn. (25) becoming quadratic in Ω^2 (eliminating the root for the fast mode):

$$\Omega^4(1 + \beta + k_y^2 L_e^2) - \Omega^2(1 + 2\beta + k_y^2 L^2) + \beta = 0 \tag{27}$$

The Aflven root for this dispersion relation is now given by:

$$\Omega^2 \approx \frac{1 + 2\beta + k_y^2 L^2 + \sqrt{(1 + k_y^2 L^2)^2 + 4\beta k_y^2 (L^2 - L_e^2)}}{2(1 + \beta + k_y^2 L_e^2)} \tag{28}$$

6.2.2 Without approximation $k_y^2 >> k_z^2$

If we do not want to use the approximation J. V. Hollweg did, we are faced with the task of solving a cubic equation. The solutions for a cubic [3] equations with coefficients a, b, c and d are:

$$\begin{aligned}
x_1 &= -term1 + (r13)cos(q^3/3) && (29) \\
x_2 &= -term1 + (r13)cos(q^3 + (2\pi)/3) && (30) \\
x_3 &= -term1 + (r13)cos(q^3 + (4\pi)/3) && (31) \\
& && (32)
\end{aligned}$$

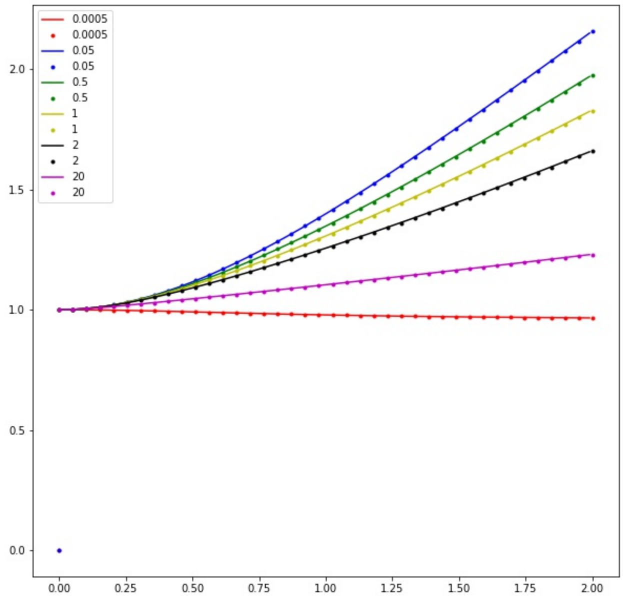

Figure 2: Plot of Ω vs $k_y L$ where $k_z = 0.01$. Dotted lines represent the Alfven root from the cubic equation, whereas the solid lines represent the root from the quadratic approximation.

where

$$q = (3c - b^2)/9$$
$$r = (-27d + b(9c - 2b^2))/54$$
$$discriminant(\Delta) = q^3 + r^2$$
$$r13 = 2\sqrt{q}$$

if $\Delta > 0$:

$$term1 = b/3$$

if $\Delta <= 0$:

$$s = r + \sqrt{\Delta}$$
$$t = r - \sqrt{\Delta}$$
$$term1 = \sqrt{3}\Big(\frac{s-t}{2}\Big)$$

Out of the three solutions of the equations, x_2 is the root of the Alfven wave.

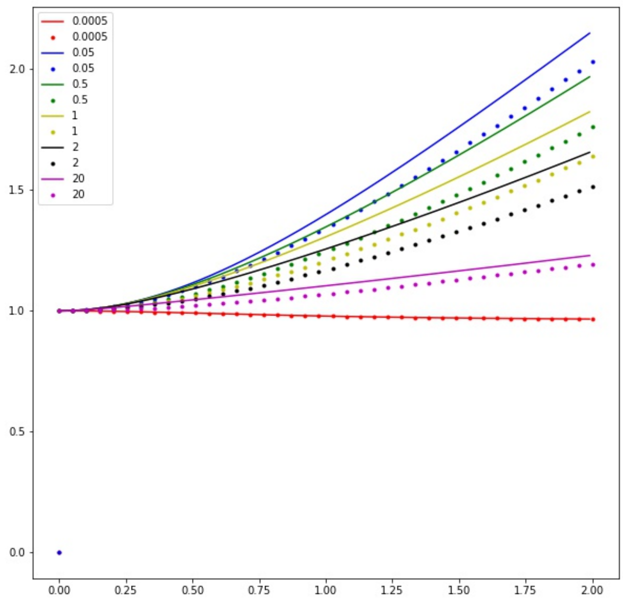

Figure 3: Plot of Ω vs $k_y L$ where $k_z = 1$. Dotted lines represent the Alfven root from the cubic equation, whereas the solid lines represent the root from the quadratic approximation.

6.3 Comparison of Solutions

In Figure 2, we plotted the graph of Ω vs $k_y L$ for equations (28) and (30) for small values of $k_z L$ when compared to $k_y L$ for a variety of values for β. The values taken for $k_z L$ were of the order 10^-2. As we can see from this graph, these approximations for the cubic equations agree quite well with the solutions of the quadratic equations, thus making the quadratic dispersion relation viable for small k_z values.

But from Figure 3, even for $k_z \quad k_y$, the the graphs seem very different. Therefore, we can say that the roots of the quadratic equations are good approximations for k_z strictly less than k_y, but cannot be used when the two quantities are comparable.

7 Conclusions

Over the course of this project, we have attempted to understand and derive expressions for various quantities related to Alfven wave relations. We started by deriving the Alfven wave relation given ideal MHD equations. Then we discussed the conditions under which the MHD Alfven wave becomes more compressible and dampened, i.e., becomes the kinetic Alfven wave. We derived the relations for the state of a quasi-neutral two-fluid plasma and got the wave dispersion relation. We solved for the Alfven wave relation from the cubic equation and also examined an approximation where $k_y >> k_z$.

We have tried to provide a very general form of the Alfven wave relation by removing approximations when we could. The relations derived in this report will be useful for further simulation of waves in plasma and understanding of the kinetic Alfven wave. We can now examine the waves under various β, k_y, k_z and mass ratios other than m_i/m_e, instead of confining ourselves to only low wavelength electron-proton plasmas.

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