CHAPTER ٢

POLYNOMIALS

٢.١ Introduction

You have studied algebraic expressions، their addition، subtraction، multiplication and division in earlier classes. You also have studied how to factorise some algebraic expressions. You may recall the algebraic identities :

$$(x + y)٢ = x٢ + ٢xy + y٢$$

$$(x - y)٢ = x٢ - ٢xy + y٢$$

and $$x٢ - y٢ = (x + y)(x - y)$$

and their use in factorisation. In this chapter، we shall start our study with a particular type of algebraic expression، called polynomial، and the terminology related to it. We shall also study the Remainder Theorem and Factor Theorem and their use in the factorisation of polynomials. In addition to the above، we shall study some more algebraic identities and their use in factorisation and in evaluating some given expressions.

٢.٢ Polynomials in One Variable

Let us begin by recalling that a variable is denoted by a symbol that can take any real value. We use the letters x، y، z، etc. to denote variables. Notice that ٢x، $\frac{١}{٢}$٣x، −x، −x are algebraic expressions. All these expressions are of the form (a constant) × x. Nov suppose we want to write an expression which is (a constant) × (a variable) and we do not know what the constant is. In such cases، we write the constant as a، b، c، etc. S the expression will be ax، say.

However، there is a difference between a letter denoting a constant and a letter denoting a variable. The values of the constants remain the same throughout a part situation، that is، the values of the constants do not change in a given problem، but t value of a variable can keep changing.

Now، consider a square of side ٣ units (see Fig. ٢.١). What is its perimeter؟ You know that the perimeter of a square is the sum of the lengths of its four sides. Here، each side is ٣ units. So، its perimeter is ٤ × ٣، i.e.، ١٢ units. What will be the perimeter if each side of the square is ١٠ units؟ The perimeter is ٤ × ١٠، i.e.، ٤٠ units. In case the length of each side is x units (see Fig. ٢.٢)، the perimeter is given by ٤x units. So، as the length of the side varies، the perimeter varies.

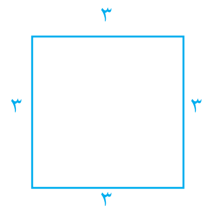

Fig. ٢.١

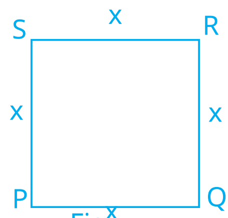

Fig. ٢.٢

Can you find the area of the square PQRS؟ It is $x \times x = x^{٢}$ square units. $x^{٢}$ is an algebraic expression. You are also familiar with other algebraic expressions like $٢x$، $x^{٢} + ٢x$، $x^{٣} - x^{٢} + ٤x + ٧$. Note that، all the algebraic expressions we have considered so far have only whole numbers as the exponents of the variable. Expressions of this form are called polynomials in one variable. In the examples above، the variable is x. For instance، $x^{٣} - x^{٢} + ٤x + ٧$ is a polynomial in x. Similarly، $٣y^{٢} + ٥y$ is a polynomial in the variable y and $t^{٢} + ٤$ is a polynomial in the variable t.

In the polynomial $x^{٢} + ٢x$، the expressions $x^{٢}$ and $٢x$ are called the terms of the polynomial. Similarly، the polynomial $٣y^{٢} + ٥y + ٧$ has three terms، namely، $٣y^{٢}$، $٥y$ and ٧. Can you write the terms of the polynomial $-x^{٣} + ٤x^{٢} + ٧x - ٢$؟ This polynomial has ٤ terms، namely، $-x^{٣}$، $٤x^{٢}$، $٧x$ and -٢.

Each term of a polynomial has a coefficient. So، in $-x^{٣} + ٤x^{٢} + ٧x - ٢$، the coefficient of $x^{٣}$ is -١، the coefficient of $x^{٢}$ is ٤، the coefficient of x is ٧ and -٢ is the coefficient of $x^{٠}$ (Remember، $x^{٠} = ١$). Do you know the coefficient of x in $x^{٢} - x + ٧$؟ It is -١.

٢ is also a polynomial. In fact، ٢، –٥، ٧، etc. are examples of constant polynomials. The constant polynomial ٠ is called the zero polynomial. This plays a very important role in the collection of all polynomials، as you will see in the higher classes.

Now، consider algebraic expressions such as $x + \frac{١}{x}$، $\sqrt{x} + ٣$ and $\sqrt[٣]{y} + y^{٢}$. Do you know that you can write $x + \frac{١}{x} = x + x^{-١}$؟ Here، the exponent of the second term، i.e.، $x^{-١}$ is -١، which is not a whole number. So، this algebraic expression is not a polynomial.

Again، $\sqrt{x} + ٣$ can be written as $x^{\frac{١}{٢}} + ٣$. Here the exponent of x is $\frac{١}{٢}$، which is not a whole number. So، is $\sqrt{x} + ٣$ a polynomial؟ No، it is not. What about $\sqrt[٣]{y} + y^{٢}$؟ It is also not a polynomial (Why؟).

If the variable in a polynomial is x، we may denote the polynomial by p(x)، or q(x)، or r(x)، etc. So، for example، we may write :

$$p(x) = ٢x^{٢} + ٥x - ٣$$
$$q(x) = x^{٣} - ١$$
$$r(y) = y^{٣} + y + ١$$
$$s(u) = ٢ - u - u^{٢} + ٦u^{٥}$$

A polynomial can have any (finite) number of terms. For instance، $x^{١٥٠} + x^{١٤٩} + ... + x^{٢} + x + ١$ is a polynomial with ١٥١ terms.

Consider the polynomials $٢x$، ٢، $٥x^{٣}$، $-٥x^{٢}$، y and $u^{٤}$. Do you see that each of these polynomials has only one term؟ Polynomials having only one term are called monom
('mono' means 'one').

Now observe each of the following polynomials:

$p(x) = x + ١$، $q(x) = x^{٢} - x$، $r(y) = y^{٩} + ١$، $t(u) = u^{١٥} - u^{٢}$

How many terms are there in each of these؟ Each of these polynomials has only two terms. Polynomials having only two terms are called binomials ('bi' means 'two
Similarly، polynomials having only three terms are called trinomials ('tri' means 'three'). Some examples of trinomials are

$p(x) = x + x^{٢} + \pi$، $q(x) = \sqrt{٢} + x - x^{٢}$،

$r(u) = u + u^{٢} - ٢$، $t(y) = y^{٤} + y + ٥$.

Now، look at the polynomial $p(x) = ٣x^{٧} - ٤x^{٦} + x + ٩$. What is the term with the highest power of x ؟ It is $٣x^{٧}$. The exponent of x in this term is ٧. Similarly، in the polynomial $q(y) = ٥y^{٦} - ٤y^{٢} - ٦$، the term with the highest power of y is $٥y^{٦}$ and the exponent of y in this term is ٦. We call the highest power of the variable in a polynom as the degree of the polynomial. So، the degree of the polynomial $٣x^{٧} - ٤x^{٦} + x + ٩$ is ٧ and the degree of the polynomial $٥y^{٦} - ٤y^{٢} - ٦$ is ٦. The degree of a non-zero constant polynomial is zero.

Example ١ : Find the degree of each of the polynomials given below:

(i) $x^{٥} - x^{٤} + ٣$ (ii) $٢ - y^{٢} - y^{٣} + ٢y^{٨}$ (iii) ٢

Solution : (i) The highest power of the variable is ٥. So، the degree of the polynomial is ٥.

(ii) The highest power of the variable is ٨. So، the degree of the polynomial is ٨.

(iii) The only term here is ٢ which can be written as $٢x^{٠}$. So the exponent of x is ٠. Therefore، the degree of the polynomial is ٠.

Now observe the polynomials p(x) = ٤x + ٥, q(y) = ٢y, r(t) = t + √٢ and s(u) = ٣ – u. Do you see anything common among all of them؟ The degree of each of these polynomials is one. A polynomial of degree one is called a linear polynomial.

Some more linear polynomials in one variable are √٢x – ١, ٢y + ١, ٢ – u. Now, try and find a linear polynomial in x with ٣ terms؟ You would not be able to find it because a linear polynomial in x can have at most two terms. So, any linear polynomial in x will be of the form ax + b, where a and b are constants and a ≠ ٠ (why؟). Similarly, ay + b is a linear polynomial in y.

Now consider the polynomials:

$$٢x^{٢} + ٥,\ ٥x^{٢} + ٣x + \pi,\ x^{٢} \text{ and } x^{٢} + \frac{٢}{٥}x$$

Do you agree that they are all of degree two؟ A polynomial of degree two is called a quadratic polynomial. Some examples of a quadratic polynomial are ٥ – $y^{٢}$, ٤y + ٥$y^{٢}$ and ٦ – y – $y^{٢}$. Can you write a quadratic polynomial in one variable with four different terms؟ You will find that a quadratic polynomial in one variable will have at most ٣ terms. If you list a few more quadratic polynomials, you will find that any quadratic polynomial in x is of the form $ax^{٢} + bx + c$, where a ≠ ٠ and a, b, c are constants. Similarly, quadratic polynomial in y will be of the form $ay^{٢} + by + c$, provided a ≠ ٠ and a, b, c are constants.

We call a polynomial of degree three a cubic polynomial. Some examples of a cubic polynomial in x are ٤$x^{٣}$, ٢$x^{٣}$ + ١, ٥$x^{٣}$ + $x^{٢}$, ٦$x^{٣}$ – x, ٦ – $x^{٣}$, ٢$x^{٣}$ + ٤$x^{٢}$ + ٦x + ٧. How many terms do you think a cubic polynomial in one variable can have؟ It can have at most ٤ terms. These may be written in the form $ax^{٣} + bx^{٢} + cx + d$, where a ≠ ٠ and a, b, c and d are constants.

Now, that you have seen what a polynomial of degree ١, degree ٢, or degree ٣ looks like, can you write down a polynomial in one variable of degree n for any natural number n؟ A polynomial in one variable x of degree n is an expression of the form

$$a_n x^n + a_{n-١} x^{n-١} + \dots + a_١ x + a_٠$$

where $a_٠, a_١, a_٢, \dots, a_n$ are constants and $a_n \neq ٠$.

In particular, if $a_٠ = a_١ = a_٢ = a_٣ = \dots = a_n = ٠$ (all the constants are zero), we get the zero polynomial, which is denoted by ٠. What is the degree of the zero polynomial؟ The degree of the zero polynomial is not defined.

So far we have dealt with polynomials in one variable only. We can also have polynomials in more than one variable. For example, $x^{٢} + y^{٢} + xyz$ (where variables are x, y and z) is a polynomial in three variables. Similarly $p^{٢} + q^{١٠} + r$ (where the variables are p, q and r), $u^{٣} + v^{٢}$ (where the variables are u and v) are polynomials in three and two variables, respectively. You will be studying such polynomials in detail later.

EXERCISE ٢.١

١. Which of the following expressions are polynomials in one variable and which are not؟ State reasons for your answer.

(i) $٤x٢ - ٣x + ٧$ (ii) $y٢ + \sqrt{٢}$ (iii) $٣\sqrt{t} + t\sqrt{٢}$ (iv) $y + \frac{٢}{y}$

(v) $x١٠ + y٣ + t٥٠$

٢. Write the coefficients of x٢ in each of the following:

(i) $٢ + x٢ + x$ (ii) $٢ - x٢ + x٣$ (iii) $\frac{\pi}{٢}x٢ + x$ (iv) $\sqrt{٢}x - ١$

٣ Give one example each of a binomial of degree ٣٥، and of a monomial of degree ١٠٠.

. Write the degree of each of the following polynomials:

٤ (i) $٥x٣ + ٤x٢ + ٧x$ (ii) $٤ - y٢$

(iii) $٥t - \sqrt{٧}$ (iv) ٣

٥. Classify the following as linear، quadratic and cubic polynomials:

(i) $x٢ + x$ (ii) $x - x٣$ (iii) $y + y٢ + ٤$ (iv) $١ + x$

(v) ٣t (vi) r٢ (vii) ٧x٣

٢.٣ Zeroes of a Polynomial

Consider the polynomial $p(x) = ٥x٣ - ٢x٢ + ٣x - ٢$.

If we replace x by ١ everywhere in p(x)، we get

$$p(١) = ٥ \times (١)٣ - ٢ \times (١)٢ + ٣ \times (١) - ٢$$
$$= ٥ - ٢ + ٣ - ٢$$
$$= ٤$$

So، we say that the value of p(x) at x = ١ is ٤.

Similarly، $p(٠) = ٥(٠)٣ - ٢(٠)٢ + ٣(٠) - ٢$

$= -٢$

Can you find p(−١)؟

Example ٢ : Find the value of each of the following polynomials at the indicated value of variables:

(i) $p(x) = ٥x٢ - ٣x + ٧$ at $x = ١$.

(ii) $q(y) = ٣y٣ - ٤y + \sqrt{١١}$ at $y = ٢$.

(iii) $p(t) = ٤t٤ + ٥t٣ - t٢ + ٦$ at $t = a$.

Solution : (i) $p(x) = ٥x^٢ - ٣x + ٧$

The value of the polynomial $p(x)$ at $x = ١$ is given by

$$p(١) = ٥(١)^٢ - ٣(١) + ٧$$
$$= ٥ - ٣ + ٧ = ٩$$

(ii) $q(y) = ٣y^٣ - ٤y + \sqrt{١١}$

The value of the polynomial $q(y)$ at $y = ٢$ is given by

$$q(٢) = ٣(٢)^٣ - ٤(٢) + \sqrt{١١} = ٢٤ - ٨ + \sqrt{١١} = ١٦ + \sqrt{١١}$$

(iii) $p(t) = ٤t^٤ + ٥t^٣ - t^٢ + ٦$

The value of the polynomial $p(t)$ at $t = a$ is given by

$$p(a) = ٤a^٤ + ٥a^٣ - a^٢ + ٦$$

Now، consider the polynomial $p(x) = x - ١$.

What is $p(١)$؟ Note that : $p(١) = ١ - ١ = ٠$.

As $p(١) = ٠$، we say that ١ is a zero of the polynomial $p(x)$.

Similarly، you can check that ٢ is a zero of $q(x)$، where $q(x) = x - ٢$.

In general، we say that a zero of a polynomial $p(x)$ is a number c such that $p(c) = ٠$.

You must have observed that the zero of the polynomial $x - ١$ is obtained by equating it to ٠، i.e.، $x - ١ = ٠$، which gives $x = ١$. We say $p(x) = ٠$ is a polynomial equation and ١ is the root of the polynomial equation $p(x) = ٠$. So we say ١ is the zero of the polynomial $x - ١$، or a root of the polynomial equation $x - ١ = ٠$.

Now، consider the constant polynomial ٥. Can you tell what its zero is؟ It has no zero because replacing x by any number in $٥x^٠$ still gives us ٥. In fact، a non-zero constant polynomial has no zero. What about the zeroes of the zero polynomial؟ By convention، every real number is a zero of the zero polynomial.

Example ٣ : Check whether -٢ and ٢ are zeroes of the polynomial $x + ٢$.

Solution : Let $p(x) = x + ٢$.

Then $p(٢) = ٢ + ٢ = ٤$، $p(-٢) = -٢ + ٢ = ٠$

Therefore، -٢ is a zero of the polynomial $x + ٢$، but ٢ is not.

Example ٤ : Find a zero of the polynomial $p(x) = ٢x + ١$.

Solution : Finding a zero of $p(x)$، is the same as solving the equation

$$p(x) = ٠$$

Now، $٢x + ١ = ٠$ gives us $x = -\frac{١}{٢}$

So، $-\frac{١}{٢}$ is a zero of the polynomial $٢x + ١$.

Now، if $p(x) = ax + b$، $a \neq ٠$، is a linear polynomial، how can we find a zero of $p(x)$؟ Example ٤ may have given you some idea. Finding a zero of the polynomial $p(x)$ amounts to solving the polynomial equation $p(x) = ٠$.

Now، $p(x) = ٠$ means $ax + b = ٠$، $a \neq ٠$

So، $ax = -b$

i.e.، $x = -\frac{b}{a}$.

So، $x = -\frac{b}{a}$ is the only zero of $p(x)$، i.e.، a linear polynomial has one and only one zero.

Now we can say that ١ is the zero of $x - ١$، and -٢ is the zero of $x + ٢$.

Example ٥ : Verify whether ٢ and ٠ are zeroes of the polynomial $x^{٢} - ٢x$.

Solution : Let $p(x) = x^{٢} - ٢x$

Then $p(٢) = ٢^{٢} - ٤ = ٤ - ٤ = ٠$

and $p(٠) = ٠ - ٠ = ٠$

Hence، ٢ and ٠ are both zeroes of the polynomial $x^{٢} - ٢x$.

Let us now list our observations:

(i) A zero of a polynomial need not be ٠.

(ii) ٠ may be a zero of a polynomial.

(iii) Every linear polynomial has one and only one zero.

(iv) A polynomial can have more than one zero.

EXERCISE ٢.٢

١. Find the value of the polynomial $٥x - ٤x^{٢} + ٣$ at

(i) $x = ٠$ (ii) $x = -١$ (iii) $x = ٢$

٢. Find $p(٠)$، $p(١)$ and $p(٢)$ for each of the following polynomials:

(i) $p(y) = y^{٢} - y + ١$ (ii) $p(t) = ٢ + t + ٢t^{٢} - t^{٣}$

(iii) $p(x) = x^{٣}$ (iv) $p(x) = (x - ١)(x + ١)$

٣. Verify whether the following are zeroes of the polynomial، indicated against them.

(i) p(x) = ٣x + ١، x = $-\frac{١}{٣}$ (ii) p(x) = ٥x − π، x = $\frac{٤}{٥}$

(iii) p(x) = x٢ − ١، x = ١، −١ (iv) p(x) = (x + ١) (x − ٢)، x = − ١، ٢

(v) p(x) = x٢، x = ٠ (vi) p(x) = lx + m، x = $-\frac{m}{l}$

(vii) p(x) = ٣x٢ − ١، x = $-\frac{١}{\sqrt{٣}}$، $\frac{٢}{\sqrt{٣}}$ (viii) p(x) = ٢x + ١، x = $\frac{١}{٢}$

٤. Find the zero of the polynomial in each of the following cases:

(i) p(x) = x + ٥ (ii) p(x) = x − ٥ (iii) p(x) = ٢x + ٥

(iv) p(x) = ٣x − ٢ (v) p(x) = ٣x (vi) p(x) = ax، a ≠ ٠

(vii) p(x) = cx + d، c ≠ ٠، c، d are real numbers.

٢.٤ Factorisation of Polynomials

Let us now look at the situation of Example ١٠ above more closely. It tells us that sinc

the remainder، $q\left(-\frac{١}{2}\right)$

for some polynomial g(t). This is a particular case of the following theorem.

Factor Theorem: If p(x) is a polynomial of degree n ≥ ١ and a is any real number، then (i) x − a is a factor of p(x)، if p(a) = ٠، and (ii) p(a) = ٠، if x − a is a factor of p(x).

Proof: By the Remainder Theorem، p(x)=(x − a) q(x) + p(a).

(i) If p(a) = ٠، then p(x) = (x − a) q(x)، which shows that x − a is a factor of p(x).

(ii) Since x − a is a factor of p(x)، p(x) = (x − a) g(x) for same polynomial g(x).

In this case، p(a) = (a − a) g(a) = ٠.

Example ٦: Examine whether x + ٢ is a factor of x٣ + ٣x٢ + ٥x + ٦ and of ٢x + ٤.

Solution: The zero of x + ٢ is −٢. Let p(x) = x٣ + ٣x٢ + ٥x + ٦ and s(x) = ٢x + ٤

Then، p(−٢) =(−٢)٣ + ٣(−٢)٢ + ٥(−٢) + ٦

=−٨ + ١٢ − ١٠ + ٦

= ٠

So، by the Factor Theorem، $x + ٢$ is a factor of $x٣ + ٣x٢ + ٥x + ٦$.

Again، $s(-٢) = ٢(-٢) + ٤ = ٠$

So، $x + ٢$ is a factor of $٢x + ٤$. In fact، you can check this without applying the Factor Theorem، since $٢x + ٤ = ٢(x + ٢)$.

Example ٧ : Find the value of k، if $x - ١$ is a factor of $٤x٣ + ٣x٢ - ٤x + k$.

Solution : As $x - ١$ is a factor of $p(x) = ٤x٣ + ٣x٢ - ٤x + k$، $p(١) = ٠$

Now، $p(١) = ٤(١)٣ + ٣(١)٢ - ٤(١) + k$

So، $٤ + ٣ - ٤ + k = ٠$

i.e.، $k = -٣$

We will now use the Factor Theorem to factorise some polynomials of degree ٢ and ٣. You are already familiar with the factorisation of a quadratic polynomial like $x٢ + lx + m$. You had factorised it by splitting the middle term lx as $ax + bx$ so that $ab = m$. Then $x٢ + lx + m = (x + a)(x + b)$. We shall now try to factorise quadratic polynomials of the type $ax٢ + bx + c$، where $a \neq ٠$ and a، b، c are constants.

Factorisation of the polynomial $ax٢ + bx + c$ by splitting the middle term is as follows:

Let its factors be $(px + q)$ and $(rx + s)$. Then $\frac{٣x٢}{x} = ٣x$ = first term of quotient

$$ax٢ + bx + c = (px + q)(rx + s) = pr\,x٢ + (ps + qr)x + qs$$

Comparing the coefficients of $x٢$، we get $a = pr$.

Similarly، comparing the coefficients of x، we get $b = ps + qr$.

And، on comparing the constant terms، we get $c = qs$.

This shows us that b is the sum of two numbers ps and qr، whose product is $(ps)(qr) = (pr)(qs) = ac$.

Therefore، to factorise $ax٢ + bx + c$، we have to write b as the sum of two numbers whose product is ac. This will be clear from Example ١٣.

Example ٨ : Factorise $٦x٢ + ١٧x + ٥$ by splitting the middle term، and by using the Factor Theorem.

Solution ١ : (By splitting method) : If we can find two numbers p and q such that

$p + q = ١٧$ and $pq = ٦ \times ٥ = ٣٠$، then we can get the factors.

So، let us look for the pairs of factors of ٣٠. Some are ١ and ٣٠، ٢ and ١٥، ٣ and ١٠، ٥ and ٦. Of these pairs، ٢ and ١٥ will give us $p + q = ١٧$.

So، $٦x٢ + ١٧x + ٥ = ٦x٢ + (٢ + ١٥)x + ٥$

$$= ٦x٢ + ٢x + ١٥x + ٥$$

$$= ٢x(٣x + ١) + ٥(٣x + ١)$$

$$= (٣x + ١)(٢x + ٥)$$

Solution ٢ : (Using the Factor Theorem)

$٦x٢ + ١٧x + ٥ = ٦\left(x^{٢} + \frac{١٧}{٦}x + \frac{٥}{٦}\right) = ٦\,p(x)$، say. If a and b are the zeroes of p(x)، then $٦x٢ + ١٧x + ٥ = ٦(x - a)(x - b)$. So، $ab = \frac{٥}{٦}$. Let us look at some possibilities for a and b. They could be $\pm\frac{١}{٢}, \pm\frac{١}{٣}, \pm\frac{٥}{٣}, \pm\frac{٥}{٢}, \pm ١$. Now، $p\left(\frac{١}{٢}\right) = \frac{١}{٤} + \frac{١٧}{٦}\left(\frac{١}{٢}\right) + \frac{٥}{٦} \neq ٠$. But $p\left(\frac{-1}{3}\right) = ٠$. So، $\left(x + \frac{١}{٣}\right)$ is a factor of p(x). Similarly، by trial، you can find that $\left(x + \frac{5}{2}\right)$ is a factor of p(x).

Therefore، $$٦x٢ + ١٧x + ٥ = ٦\left(x + \frac{١}{٣}\right)\left(x + \frac{٥}{٢}\right)$$

$$= ٦\left(\frac{٣x + ١}{٣}\right)\left(\frac{٢x + ٥}{٢}\right)$$

$$= (٣x + ١)(٢x + ٥)$$

For the example above، the use of the splitting method appears more efficient. How let us consider another example.

Example ٩ : Factorise $y٢ - ٥y + ٦$ by using the Factor Theorem.

Solution : Let $p(y) = y٢ - ٥y + ٦$. Now، if $p(y) = (y - a)(y - b)$، you know that the constant term will be ab. So، $ab = ٦$. So، to look for the factors of p(y)، we look at the factors of ٦.

The factors of ٦ are ١، ٢ and ٣.

Now، $p(٢) = ٢٢ - (٥ \times ٢) + ٦ = ٠$

So، $y - ٢$ is a factor of p(y).

Also، $p(٣) = ٣^٢ - (٥ \times ٣) + ٦ = ٠$

So، $y - ٣$ is also a factor of $y^٢ - ٥y + ٦$.

Therefore، $y^٢ - ٥y + ٦ = (y - ٢)(y - ٣)$

Note that $y^٢ - ٥y + ٦$ can also be factorised by splitting the middle term $-٥y$.

Now، let us consider factorising cubic polynomials. Here، the splitting method will n be appropriate to start with. We need to find at least one factor first، as you will see i the following example.

Example ١٠ : Factorise $x^٣ - ٢٣x^٢ + ١٤٢x - ١٢٠$.

Solution : Let $p(x) = x^٣ - ٢٣x^٢ + ١٤٢x - ١٢٠$

We shall now look for all the factors of -١٢٠. Some of these are $\pm ١، \pm ٢، \pm ٣،$ $\pm ٤، \pm ٥، \pm ٦، \pm ٨، \pm ١٠، \pm ١٢، \pm ١٥، \pm ٢٠، \pm ٢٤، \pm ٣٠، \pm ٦٠$.

By trial، we find that $p(١) = ٠$. So $x - ١$ is a factor of $p(x)$.

Now we see that $x^٣ - ٢٣x^٢ + ١٤٢x - ١٢٠ = x^٣ - x^٢ - ٢٢x^٢ + ٢٢x + ١٢٠x - ١٢٠$

$$= x^٢(x - ١) - ٢٢x(x - ١) + ١٢٠(x - ١) \qquad \text{(Why؟)}$$

$$= (x - ١)(x^٢ - ٢٢x + ١٢٠) \qquad \text{(Taking } (x - ١) \text{ common)}$$

We could have also got this by dividing $p(x)$ by $x - ١$.

Now $x^٢ - ٢٢x + ١٢٠$ can be factorised either by splitting the middle term or by using the Factor theorem. By splitting the middle term، we have:

$$x^٢ - ٢٢x + ١٢٠ = x^٢ - ١٢x - ١٠x + ١٢٠$$

$$= x(x - ١٢) - ١٠(x - ١٢)$$

$$= (x - ١٢)(x - ١٠)$$

So، $$x^٣ - ٢٣x^٢ - ١٤٢x - ١٢٠ = (x - ١)(x - ١٠)(x - ١٢)$$

EXERCISE ٢.٣

١. Determine which of the following polynomials has $(x + ١)$ a factor:

(i) $x^٣ + x^٢ + x + ١$ (ii) $x^٤ + x^٣ + x^٢ + x + ١$

(iii) $x^٤ + ٣x^٣ + ٣x^٢ + x + ١$ (iv) $x^٣ - x^٢ - (٢ + \sqrt{٢})x + \sqrt{٢}$

٢. Use the Factor Theorem to determine whether $g(x)$ is a factor of $p(x)$ in each of the following cases:

(i) $p(x) = ٢x^٣ + x^٢ - ٢x - ١،\ g(x) = x + ١$

(ii) $p(x) = x٣ + ٣x٢ + ٣x + ١$، $g(x) = x + ٢$

(iii) $p(x) = x٣ - ٤x٢ + x + ٦$، $g(x) = x - ٣$

٣. Find the value of k، if $x - ١$ is a factor of $p(x)$ in each of the following cases:

(i) $p(x) = x٢ + x + k$ (ii) $p(x) = ٢x٢ + kx + \sqrt{٢}$

(iii) $p(x) = kx٢ - \sqrt{٢}x + ١$ (iv) $p(x) = kx - ٣x + k$

٤. Factorise:

(i) $١٢x٢ - ٧x + ١$ (ii) $٢x٢ + ٧x + ٣$

(iii) $٦x٢ + ٥x - ٦$ (iv) $٣x٢ - x - ٤$

٥. Factorise:

(i) $x٣ - ٢x٢ - x + ٢$ (ii) $x٣ - ٣x٢ - ٩x - ٥$

(iii) $x٣ + ١٣x٢ + ٣٢x + ٢٠$ (iv) $٢y٣ + y٢ - ٢y - ١$

٢.٥ Algebraic Identities

From your earlier classes، you may recall that an algebraic identity is an algebraic equation that is true for all values of the variables occurring in it. You have studied the following algebraic identities in earlier classes:

Identity I : $(x + y)٢ = x٢ + ٢xy + y٢$

Identity II : $(x - y)٢ = x٢ - ٢xy + y٢$

Identity III: $x٢ - y٢ = (x + y)(x - y)$

Identity IV: $(x + a)(x + b) = x٢ + (a + b)x + ab$

You must have also used some of these algebraic identities to factorise the algeb expressions. You can also see their utility in computations.

Example ١١ : Find the following products using appropriate identities:

(i) $(x + ٣)(x + ٣)$ (ii) $(x - ٣)(x + ٥)$

Solution : (i) Here we can use Identity I : $(x + y)٢ = x٢ + ٢xy + y٢$. Putting $y = ٣$ in it، we get

$$(x + ٣)(x + ٣) = (x + ٣)٢ = x٢ + ٢(x)(٣) + (٣)٢$$

$$= x٢ + ٦x + ٩$$

(ii) Using Identity IV above، i.e.، $(x + a)(x + b) = x٢ + (a + b)x + ab$، we have

$$(x - ٣)(x + ٥) = x٢ + (-٣ + ٥)x + (-٣)(٥)$$

$$= x٢ + ٢x - ١٥$$

Example ١٢ : Evaluate ١٠٥×١٠٦ without multiplying directly.

Solution :

$$
\begin{aligned}
١٠٥ \times ١٠٦ &= (١٠٠ + ٥) \times (١٠٠ + ٦) \\
&= (١٠٠)^٢ + (٥ + ٦)(١٠٠) + (٥ \times ٦), \text{ using Identity IV} \\
&= ١٠٠٠٠ + ١١٠٠ + ٣٠ \\
&= ١١١٣٠
\end{aligned}
$$

You have seen some uses of the identities listed above in finding the product of some given expressions. These identities are useful in factorisation of algebraic expressions also, as you can see in the following examples.

Example ١٣ : Factorise:

(i) $٤٩a^٢ + ٧٠ab + ٢٥b^٢$ (ii) $\frac{٢٥}{٤}x^٢ - \frac{y^٢}{٩}$

Solution : (i) Here you can see that

$$٤٩a^٢ = (٧a)^٢, \quad ٢٥b^٢ = (٥b)^٢, \quad ٧٠ab = ٢(٧a)(٥b)$$

Comparing the given expression with $x^٢ + ٢xy + y^٢$, we observe that $x = ٧a$ and $y = ٥b$.

Using Identity I, we get

$$٤٩a^٢ + ٧٠ab + ٢٥b^٢ = (٧a + ٥b)^٢ = (٧a + ٥b)(٧a + ٥b)$$

(ii) We have $\frac{٢٥}{٤}x^٢ - \frac{y^٢}{٩} = \left(\frac{٥}{٢}x\right)^٢ - \left(\frac{y}{٣}\right)^٢$

Now comparing it with Identity III, we get

$$
\begin{aligned}
\frac{٢٥}{٤}x^٢ - \frac{y^٢}{٩} &= \left(\frac{٥}{٢}x\right)^٢ - \left(\frac{y}{٣}\right)^٢ \\
&= \left(\frac{٥}{٢}x + \frac{y}{٣}\right)\left(\frac{٥}{٢}x - \frac{y}{٣}\right)
\end{aligned}
$$

So far, all our identities involved products of binomials. Let us now extend the Identity I to a trinomial $x + y + z$. We shall compute $(x + y + z)^٢$ by using Identity I.

Let $x + y = t$. Then,

$$
\begin{aligned}
(x + y + z)^٢ &= (t + z)^٢ \\
&= t^٢ + ٢tz + t^٢ && \text{(Using Identity I)} \\
&= (x + y)^٢ + ٢(x + y)z + z^٢ && \text{(Substituting the value of t)}
\end{aligned}
$$

$=x^{٢} + ٢xy + y^{٢} + ٢xz + ٢yz + z^{٢} = x^{٢}$ (Using Identity I)

$+ y^{٢} + z^{٢} + ٢xy + ٢yz + ٢zx$ (Rearranging the terms)

So، we get the following identity: Identity V : $(x + y + z)^{٢} = x^{٢} + y^{٢} + z^{٢} + ٢xy + ٢yz + ٢zx$ Remark : We call the right hand side expression the expanded form of the left hand side expression. Note that the expansion of $(x + y + z)^{٢}$ consists of three square terms and three product terms.

Example ١٤ : Write $(٣a + ٤b + ٥c)^{٢}$ in expanded form.

Solution : Comparing the given expression with $(x + y + z)^{٢}$، we find that

$x = ٣a$، $y = ٤b$ and $z = ٥c$.

Therefore، using Identity V، we have

$$(٣a + ٤b + ٥c)^{٢} = (٣a)^{٢} + (٤b)^{٢} + (٥c)^{٢} + ٢(٣a)(٤b) + ٢(٤b)(٥c) + ٢(٥c)(٣a)$$

$$= ٩a^{٢} + ١٦b^{٢} + ٢٥c^{٢} + ٢٤ab + ٤٠bc + ٣٠ac$$

Example ١٥ : Expand $(٤a - ٢b - ٣c)^{٢}$.

Solution : Using Identity V، we have

$$(٤a - ٢b - ٣c)^{٢} = ﵁٤a + (-٢b) + (-٣c)ﷺ^{٢}$$

$$= (٤a)^{٢} + (-٢b)^{٢} + (-٣c)^{٢} + ٢(٤a)(-٢b) + ٢(-٢b)(-٣c) + ٢(-٣c)(٤a)$$

$$= ١٦a^{٢} + ٤b^{٢} + ٩c^{٢} - ١٦ab + ١٢bc - ٢٤ac$$

Example ١٦ : Factorise $٤x^{٢} + y^{٢} + z^{٢} - ٤xy - ٢yz + ٤xz$.

Solution : We have $٤x^{٢} + y^{٢} + z^{٢} - ٤xy - ٢yz + ٤xz = (٢x)^{٢} + (-y)^{٢} + (z)^{٢} + ٢(٢x)(-y)$

$+ ٢(-y)(z) + ٢(٢x)(z)$

$= ﵁٢x + (-y) + zﷺ^{٢}$ (Using Identity V)

$= (٢x - y + z)^{٢} = (٢x - y + z)(٢x - y + z)$

So far، we have dealt with identities involving second degree terms. Now let us extend Identity I to compute $(x + y)^{٣}$. We have:

$$(x + y)^{٣} = (x + y)(x + y)^{٢}$$

$$= (x + y)(x^{٢} + ٢xy + y^{٢})$$

$$= x(x^{٢} + ٢xy + y^{٢}) + y(x^{٢} + ٢xy + y^{٢})$$

$$= x^{٣} + ٢x^{٢}y + xy^{٢} + x^{٢}y + ٢xy^{٢} + y^{٣}$$

$$= x^{٣} + ٣x^{٢}y + ٣xy^{٢} + y^{٣}$$

$$= x^{٣} + y^{٣} + ٣xy(x + y)$$

So، we get the following identity:

Identity VI : $(x + y)^٣ = x^٣ + y^٣ + ٣xy\,(x + y)$

Also، by replacing y by −y in the Identity VI، we get

Identity VII : $(x - y)^٣ = x^٣ - y^٣ - ٣xy(x - y)$

$$= x^٣ - ٣x^٢y + ٣xy^٢ - y^٣$$

Example ١٧ : Write the following cubes in the expanded form:

(i) $(٣a + ٤b)^٣$ (ii) $(٥p - ٣q)^٣$

Solution : (i) Comparing the given expression with $(x + y)^٣$، we find that

$x = ٣a$ and $y = ٤b$.

So، using Identity VI، we have:

$$(٣a + ٤b)^٣ = (٣a)^٣ + (٤b)^٣ + ٣(٣a)(٤b)(٣a + ٤b)$$
$$= ٢٧a^٣ + ٦٤b^٣ + ١٠٨a^٢b + ١٤٤ab^٢$$

(ii) Comparing the given expression with $(x - y)^٣$، we find that

$x = ٥p$، $y = ٣q$.

So، using Identity VII، we have:

$$(٥p - ٣q)^٣ = (٥p)^٣ - (٣q)^٣ - ٣(٥p)(٣q)(٥p - ٣q)$$
$$= ١٢٥p^٣ - ٢٧q^٣ - ٢٢٥p^٢q + ١٣٥pq^٢$$

Example ١٨ : Evaluate each of the following using suitable identities:

(i) $(١٠٤)^٣$ (ii) $(٩٩٩)^٣$

Solution : (i) We have

$$(١٠٤)^٣ = (١٠٠ + ٤)^٣$$
$$= (١٠٠)^٣ + (٤)^٣ + ٣(١٠٠)(٤)(١٠٠ + ٤)$$

(Using Identity VI)

$$= ١٠٠٠٠٠٠ + ٦٤ + ١٢٤٨٠٠$$
$$= ١١٢٤٨٦٤$$

(ii) We have

$$(٩٩٩)^٣ = (١٠٠٠ - ١)^٣$$
$$= (١٠٠٠)^٣ - (١)^٣ - ٣(١٠٠٠)(١)(١٠٠٠ - ١)$$

(Using Identity VII)

$$= ١٠٠٠٠٠٠٠٠٠ - ١ - ٢٩٩٧٠٠٠$$
$$= ٩٩٧٠٠٢٩٩٩$$

Example ١٩ : Factorise $٨x^٣ + ٢٧y^٣ + ٣٦x^٢y + ٥٤xy^٢$

Solution : The given expression can be written as

$$(٢x)^٣ + (٣y)^٣ + ٣(٤x^٢)(٣y) + ٣(٢x)(٩y^٢)$$
$$=(٢x)^٣ + (٣y)^٣ + ٣(٢x)^٢(٣y) + ٣(٢x)(٣y)^٢$$
$$=(٢x + ٣y)^٣ \qquad \text{(Using Identity VI)}$$
$$=(٢x + ٣y)(٢x + ٣y)(٢x + ٣y)$$

Now consider $(x + y + z)(x^٢ + y^٢ + z^٢ - xy - yz - zx)$

On expanding، we get the product as

$$x(x^٢ + y^٢ + z^٢ - xy - yz - zx) + y(x^٢ + y^٢ + z^٢ - xy - yz - zx)$$
$$+ z(x^٢ + y^٢ + z^٢ - xy - yz - zx) = x^٣ + xy^٢ + xz^٢ - x^٢y - xyz - zx^٢ + x^٢y$$
$$+ y^٣ + yz^٢ - xy^٢ - y^٢z - xyz + x^٢z + y^٢z + z^٣ - xyz - yz^٢ - xz^٢$$
$$= x^٣ + y^٣ + z^٣ - ٣xyz \qquad \text{(On simplification)}$$

So، we obtain the following identity:

Identity VIII : $x^٣ + y^٣ + z^٣ - ٣xyz = (x + y + z)(x^٢ + y^٢ + z^٢ - xy - yz - zx)$

Example ٢٠ : Factorise : $٨x^٣ + y^٣ + ٢٧z^٣ - ١٨xyz$

Solution : Here، we have

$$٨x^٣ + y^٣ + ٢٧z^٣ - ١٨xyz$$
$$=(٢x)^٣ + y^٣ + (٣z)^٣ - ٣(٢x)(y)(٣z)$$
$$=(٢x + y + ٣z)\text{رضي الله عنه}(٢x)^٢ + y^٢ + (٣z)^٢ - (٢x)(y) - (y)(٣z) - (٢x)(٣z)\text{ﷺ}$$
$$=(٢x + y + ٣z)(٤x^٢ + y^٢ + ٩z^٢ - ٢xy - ٣yz - ٦xz)$$

EXERCISE ٢.٤

١. Use suitable identities to find the following products:

(i) $(x + ٤)(x + ١٠)$ (ii) $(x + ٨)(x - ١٠)$ (iii) $(٣x + ٤)(٣x - ٥)$

(iv) $(y^٢ + \frac{٣}{٢})(y^٢ - \frac{٣}{٢})$ (v) $(٣ - ٢x)(٣ + ٢x)$

٢. Evaluate the following products without multiplying directly:

(i) ١٠٣×١٠٧ (ii) ٩٥×٩٦ (iii) ١٠٤×٩٦

٣. Factorise the following using appropriate identities:

(i) $٩x^٢ + ٦xy + y^٢$ (ii) $٤y^٢ - ٤y + ١$ (iii) $x^٢ - \frac{y^٢}{١٠٠}$

٤. Expand each of the following، using suitable identities:

(i) $(x + ٢y + ٤z)٢$ (ii) $(٢x - y + z)٢$ (iii) $(-٢x + ٣y + ٢z)٢$

(iv) $(٣a - ٧b - c)٢$ (v) $(-٢x + ٥y - ٣z)٢$ (vi) $\left[\frac{1}{٤}a - \frac{١}{2}b + 1\right]^{٢}$

٥. Factorise:

(i) $٤x٢ + ٩y٢ + ١٦z٢ + ١٢xy - ٢٤yz - ١٦xz$

(ii) $٢x٢ + y٢ + ٨z٢ - ٢\sqrt{٢}xy + ٤\sqrt{٢}yz - ٨xz$

٦. Write the following cubes in expanded form:

(i) $(٢x + ١)٣$ (ii) $(٢a - ٣b)٣$ (iii) $\left[\frac{3}{٢}x + ١\right]^{٣}$ (iv) $\left[x - \frac{٢}{٣}y\right]^{٣}$

٧. Evaluate the following using suitable identities:

(i) $(٩٩)٣$ (ii) $(١٠٢)٣$ (iii) $(٩٩٨)٣$

٨. Factorise each of the following:

(i) $٨a٣ + b٣ + ١٢a٢b + ٦ab٢$ (ii) $٨a٣ - b٣ - ١٢a٢b + ٦ab٢$

(iii) $٢٧ - ١٢٥a٣ - ١٣٥a + ٢٢٥a٢$ (iv) $٦٤a٣ - ٢٧b٣ - ١٤٤a٢b + ١٠٨ab٢$

(v) $٢٧p٣ - \frac{١}{٢١٦} - \frac{٩}{٢}p^{٢} + \frac{١}{٤}p$

٩. Verify: (i) $x٣ + y٣ = (x + y)(x٢ - xy + y٢)$ (ii) $x٣ - y٣ = (x - y)(x٢ + xy + y٢)$

١٠. Factorise each of the following:

(i) $٢٧y٣ + ١٢٥z٣$ (ii) $٦٤m٣ - ٣٤٣n٣$

﵁Hint: See Question ٩.ﷺ

١١. Factorise: $٢٧x٣ + y٣ + z٣ - ٩xyz$

١٢. Verify that $x٣ + y٣ + z٣ - ٣xyz = \frac{١}{٢}(x + y + z)\left[(x - y)^{٢} + (y - z)^{٢} + (z - x)^{٢}\right]$

١٣. If $x + y + z = ٠$، show that $x٣ + y٣ + z٣ = ٣xyz$.

١٤. Without actually calculating the cubes، find the value of each of the following:

(i) $(-١٢)٣ + (٧)٣ + (٥)٣$

(ii) $(٢٨)٣ + (-١٥)٣ + (-١٣)٣$

١٥. Give possible expressions for the length and breadth of each of the following rectangles، in which their areas are given:

(i)	(ii)
Area: $٢٥a٢ - ٣٥a + ١٢$	Area: $٣٥y٢ + ١٣y - ١٢$

١٦. What are the possible expressions for the dimensions of the cuboids whose volumes are given below؟

Volume : $٣x^٢ - ١٢x$	Volume : $١٢ky^٢ + ٨ky - ٢٠k$
(i)	(ii)

٢.٦ Summary

In this chapter، you have studied the following points:

١. A polynomial $p(x)$ in one variable x is an algebraic expression in x of the form
$p(x) = a_n x^n + a_{n-١} x^{n-١} + \ldots + a_٢ x^٢ + a_١ x + a_٠$،
where $a_٠، a_١، a_٢، \ldots، a_n$ are constants and $a_n \neq ٠$.
$a_٠، a_١، a_٢، \ldots، a_n$ are respectively the coefficients of $x^٠، x، x^٢، \ldots، x^n$، and n is called the deg
of the polynomial. Each of $a_n x^n، a_{n-١} x^{n-١}، \ldots، a_٠$، with $a_n \neq ٠$، is called a term of the polynomial $p(x)$.

٢. A polynomial of one term is called a monomial.

٣. A polynomial of two terms is called a binomial.

٤. A polynomial of three terms is called a trinomial.

٥. A polynomial of degree one is called a linear polynomial.

٦. A polynomial of degree two is called a quadratic polynomial.

٧. A polynomial of degree three is called a cubic polynomial.

٨. A real number 'a' is a zero of a polynomial $p(x)$ if $p(a) = ٠$. In this case، a is also called a root of the equation $p(x) = ٠$.

٩. Every linear polynomial in one variable has a unique zero، a non-zero constant polynom has no zero، and every real number is a zero of the zero polynomial.

١٠. Factor Theorem : $x - a$ is a factor of the polynomial $p(x)$، if $p(a) = ٠$. Also، if $x - a$ is a factor of $p(x)$، then $p(a) = ٠$.

١١. $(x + y + z)^٢ = x^٢ + y^٢ + z^٢ + ٢xy + ٢yz + ٢zx$

١٢. $(x + y)^٣ = x^٣ + y^٣ + ٣xy(x + y)$

١٣. $(x - y)^٣ = x^٣ - y^٣ - ٣xy(x - y)$

١٤. $x^٣ + y^٣ + z^٣ - ٣xyz = (x + y + z)(x^٢ + y^٢ + z^٢ - xy - yz - zx)$