Coherent-mode representation of Gaussian Schell-model sources and of their radiation fields

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A recently formulated theory of partial coherence in the space-frequency domain is used to determine the mode structure of an important class of partially coherent sources and of the radiation fields generated by them. The effective number of modes is found to depend in a fundamental way on the ratio of the coherence length to the effective size of the source. The contribution of the effective modes to the far-field intensity is also analyzed.

1. INTRODUCTION

In recent papers[1,2] a new theory was developed for analyzing sources of any state of coherence and also the fields that the sources generate. The basic feature of the theory is a rigorous decomposition of the cross-spectral density function of the source into coherent modes.

In the present paper we apply this theory to determine the mode structure of a Gaussian Schell-model source and of the far field that such a source produces. Sources of this type have been playing an increasingly important role in optical coherence theory (see, for example, Refs. 3 and 4) because they exhibit the essential features of many sources encountered in practice and yet they can be analyzed mathematically with relative ease. For the sake of simplicity we consider only a one-dimensional version of such sources, and we obtain their mode decomposition in explicit form. The effective number of modes that a Gaussian Schell-model source possesses is found to depend only on the ratio β of the coherence length of the source to the size of the source. When $\beta \gg 1$, the source is effectively spatially coherent in the global sense and is then found to be well represented by a single mode. When $\beta \ll 1$, the source is effectively spatially incoherent in the global sense, and the number of modes needed to describe its behavior is of the order of $1/\beta$.

Several diagrams are also given, pertaining to the mode structure of the source and to the contributions of the lowest-order coherent modes to the far-field intensity distribution.

2. COHERENT MODES OF A ONE-DIMENSIONAL GAUSSIAN SCHELL-MODEL SOURCE

A one-dimensional Schell-model source is characterized by a cross-spectral density function of the form

$$W(x_1, x_2, \omega) = [I(x_1, \omega)]^{1/2}[I(x_2, \omega)]^{1/2}\mu(x_1 - x_2, \omega), \tag{2.1}$$

where $I(x, \omega)$ is the spectral intensity at a source point with coordinate x and $\mu(x_1 - x_2, \omega)$ is the complex degree of spatial coherence (also known as the complex degree of spectral coherence[5]) at frequency ω. The source may be a primary one, in which case W is the cross-spectral density of the true source distribution, or it may be a secondary one, in which case W is the cross-spectral density of the field distribution across the source.[6] Although the mode analysis that we develop in this paper is strictly similar for the two cases, we will regard Eq. (2.1) as representing the cross-spectral density of a secondary source.

For a Gaussian Schell-model source that we will consider, the intensity distribution and the degree of coherence are of the form

$$I(x, \omega) = A(\omega) \exp[-x^2/2\sigma_I{}^2(\omega)], \tag{2.2}$$

$$\mu(x_1 - x_2, \omega) = \exp[-(x_1 - x_2)^2/2\sigma_\mu{}^2(\omega)], \tag{2.3}$$

where $A(\omega)$, $\sigma_I(\omega)$, and $\sigma_\mu(\omega)$ are positive constants.

According to the theory developed in Refs. 1 and 2, the cross-spectral density $W(x_1, x_2, \omega)$ of a wide class of sources may be represented in the form

$$W(x_1, x_2, \omega) = \sum_n \lambda_n(\omega)\phi_n{}^*(x_1, \omega)\phi_n(x_2, \omega), \tag{2.4}$$

where $\lambda_n(\omega)$ are the eigenvalues and $\phi_n(x, \omega)$ are the orthonormal eigenfunctions of the homogeneous Fredholm integral equation

$$\int W(x_1, x_2, \omega)\phi_n(x_1, \omega)\mathrm{d}x_1 = \lambda_n(\omega)\phi_n(x_2, \omega) \tag{2.5}$$

and the integration extends over the source. Actually, only finite sources were considered in Refs. 1 and 2, but the theory also applies to many infinite sources, including Gaussian Schell-model sources.

In physical terms, formula (2.4) represents the cross-spectral density function of the source as a sum of contributions from modes that are spatially completely coherent.

To determine the modes of a Gaussian Schell-model source, we must solve integral equation (2.5), with $W(x_1, x_2, \omega)$ having the form indicated by Eq. (2.1), where $I(x, \omega)$ and $\mu(x_1 - x_2, \omega)$ are given by Eqs. (2.2) and (2.3), respectively. For this purpose it is convenient to introduce two parameters $a(\omega)$ and $b(\omega)$ defined as

$$a(\omega) = \frac{1}{4\sigma_I{}^2(\omega)}, \qquad b(\omega) = \frac{1}{2\sigma_\mu{}^2(\omega)}. \tag{2.6}$$

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On substituting from Eqs. (2.2) and (2.3) into Eq. (2.1) and using Eqs. (2.6), the cross-spectral density of a Gaussian Schell-model source takes the form[7]

$$W(x_1, x_2, \omega) = A \exp\{-[(a+b)(x_1^2 + x_2^2) - 2bx_1x_2]\} \quad (2.7)$$

and integral equation (2.5) becomes

$$\int_{-\infty}^{\infty} A \exp\{-[(a+b)(x_1^2+x_2^2) - 2bx_1x_2]\}\phi_n(x_1)\mathrm{d}x_1 = \lambda_n \phi_n(x_2). \quad (2.8)$$

Integral equation (2.8) was not long ago encountered and solved by Gori[4] in another investigation involving Gaussian Schell-model sources. We will solve it by a different procedure. For this purpose we make use of the following bilinear generating function for Hermite polynomials[8] $H_n(x)$:

$$\sum_{n=0}^{\infty} \frac{H_n(x)H_n(y)}{n!} t^n = (1-4t^2)^{-1/2} \exp\left[y^2 - \frac{(y-2xt)^2}{1-4t^2}\right]. \quad (2.9)$$

Let us set $x = \sigma x_1$ and $y = \sigma x_2$ in Eq. (2.9), where σ is a constant, and let us then multiply both sides of the resulting expression by the factor $A(1-4t^2)^{1/2} \exp[-\sigma^2(x_1^2 + x_2^2)/2]$. We then obtain, after a straightforward calculation, the formula

$$\sum_{n=0}^{\infty} A(1-4t^2)^{1/2} t^n \frac{H_n(\sigma x_1)H_n(\sigma x_2)}{n!} \exp[-\sigma^2(x_1^2+x_2^2)/2] = A \exp\left\{-\frac{\sigma^2}{2(1-4t^2)}[(1+4t^2)(x_1^2+x_2^2) - 8tx_1x_2]\right\}. \quad (2.10)$$

Comparison of the right-hand sides of Eqs. (2.10) and (2.7) shows that the two expressions will be equal if

$$\frac{\sigma^2(1+4t^2)}{2(1-4t^2)} = a + b \quad (2.11a)$$

and

$$\frac{2t\sigma^2}{1-4t^2} = b. \quad (2.11b)$$

The pair of equations, (2.11a) and (2.11b), can be solved for σ and t, and one obtains (see Appendix A)

$$\sigma = \sqrt{2c}, \qquad t = \frac{b}{2(a+b+c)}, \quad (2.12)$$

where

$$c = (a^2 + 2ab)^{1/2}. \quad (2.13)$$

With σ and t related to the parameters a and b by means of Eqs. (2.11), the right-hand sides of Eqs. (2.7) and (2.10) will be equal, and hence the left-hand sides will then also be equal; consequently we obtain for the cross-spectral density function the expansion

$$W(x_1, x_2, \omega) = \sum_{n=0}^{\infty} \frac{A}{n!}\left[1 - \left(\frac{b}{a+b+c}\right)^2\right]^{1/2} \left[\frac{b}{2(a+b+c)}\right]^n \times H_n(x_1\sqrt{2c})H_n(x_2\sqrt{2c}) \exp[-c(x_1^2+x_2^2)]. \quad (2.14)$$

By making use of Eq. (2.13), one may readily express Eq. (2.14) in the form

$$W(x_1, x_2, \omega) = \sum_{n=0}^{\infty} \frac{A}{2^n n!}\left(\frac{2c}{a+b+c}\right)^{1/2}\left(\frac{b}{a+b+c}\right)^n \times H_n(x_1\sqrt{2c})H_n(x_2\sqrt{2c})\exp[-c(x_1^2+x_2^2)]. \quad (2.15)$$

Expansion (2.15) has the form of the required mode representation [Eq. (2.4)]. Moreover, by making use of the well-known orthogonality relations for Hermite polynomials [Ref. 8, Sec. 110, Eqs. (5) and (6)], it follows that the functions

$$\phi_n(x) = \left(\frac{2c}{\pi}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(x\sqrt{2c})e^{-cx^2} \quad (2.16)$$

form an orthonormal set over the domain $-\infty < x < \infty$. Hence the eigenfunctions ϕ_n of integral equation (2.5) of a Gaussian Schell-model source are precisely the functions given by Eq. (2.16), i.e., the Hermite functions of order n, with argument $x\sqrt{2c}$. When this fact is used, the comparison of Eq. (2.15) and Eq. (2.4) then shows that the eigenvalues are given by

$$\lambda_n = A\left(\frac{\pi}{a+b+c}\right)^{1/2}\left(\frac{b}{a+b+c}\right)^n. \quad (2.17)$$

3. DISTRIBUTION OF THE EIGENVALUES

From expression (2.17) for the eigenvalues of the Gaussian Schell-model source, we may readily derive an interesting formula relating to the "strength" with which the different modes contribute to the cross-spectral density. We see at once that the ratio of the eigenvalue λ_n to the lowest eigenvalue λ_0 is given by

$$\frac{\lambda_n}{\lambda_0} = \left(\frac{b}{a+b+c}\right)^n. \quad (3.1)$$

Recalling the definitions of the quantities a, b, and c [Eqs. (2.6) and (2.13)], we may readily express Eq. (3.1) in the form

$$\frac{\lambda_n}{\lambda_0} = \left[\frac{1}{\beta^2/2 + 1 + \beta[(\beta/2)^2 + 1]^{1/2}}\right]^n, \quad (3.2)$$

where the parameter β is the ratio of the rms widths of the degree of coherence and of the intensity of the source:

$$\beta = \frac{\sigma_\mu}{\sigma_I}. \quad (3.3)$$

Clearly β is a measure of the "degree of global coherence" of the source.

When $\sigma_\mu \gg \sigma_I$, the source may be said to be rather *coherent in the global sense*. In this case, $\beta \gg 1$, and it then follows from Eq. (3.2) that

$$\frac{\lambda_n}{\lambda_0} \approx \frac{1}{\beta^{2n}}. \quad (3.4)$$

This formula implies that, for all $n \neq 0$, $\lambda_n \ll \lambda_0$, and hence in the coherent limit the behavior of the source is well approximated by the lowest-order mode.

When $\sigma_\mu \ll \sigma_I$, the source may be said to be rather *incoherent in the global sense*. (It then belongs into the class of the so-called quasi-homogeneous sources.[9]) In this case $\beta \ll$

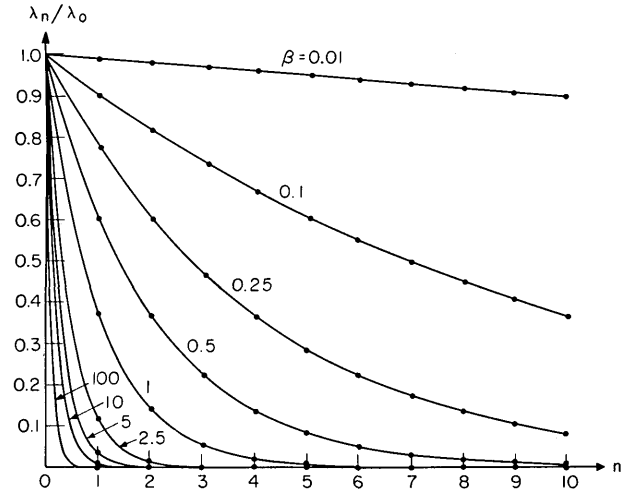

Fig. 1. The ratio of the $(n+1)$th eigenvalue λ_n to the lowest-order eigenvalue λ_0 of a Gaussian Schell-model source as a function of n for selected values of the degree of global coherence $\beta \equiv \sigma_\mu/\sigma_I$. The extreme values $\beta \gg 1$ and $\beta \ll 1$ correspond, respectively, to spatial coherence and spatial incoherence, in the global sense.

1, and one readily deduces from Eq. (3.2) that the ratio λ_n/λ_0 is now well approximated by the formula

$$\frac{\lambda_n}{\lambda_0} \approx 1 - n\beta, \tag{3.5}$$

indicating that a large number of modes (of the order $1/\beta$) is now needed to represent the source properly.

In Fig. 1 the behavior of the ratio λ_n/λ_0 as function of n is shown for various selected values of the parameter β.

4. MODE CONTRIBUTIONS TO THE FAR-ZONE INTENSITY DISTRIBUTION

At a typical point P(x, z) in the half-plane $z > 0$ (see Fig. 2), each source mode $\phi_n(x)$ will produce a field given by[10]

$$\psi_n(x, z) = -\frac{i}{2}\frac{\partial}{\partial z}\int_{-\infty}^{\infty} \phi_n(x')H_0^{(1)}(kR)\mathrm{d}x', \tag{4.1}$$

where $H_0^{(1)}$ is the Hankel function of the first kind and zero order and

$$R = [(x - x')^2 + z^2]^{1/2}. \tag{4.2}$$

Suppose now that the point P is in the far zone. We may then make the approximation[11]

$$R \approx r - x' \sin\theta, \tag{4.3}$$

where $r = (x^2 + z^2)^{1/2}$ is the distance of the field point P from the origin and θ is the angle that the direction OP makes with the z axis. If we make use of formula (4.3) and of the well-known asymptotic approximation for the Hankel function $H_0^{(1)}(kR)$ for large values of the argument kR, viz.,[12]

$$H_0^{(1)}(kR) \sim \left(\frac{2}{\pi kR}\right)^{1/2} \exp[i(kR - \pi/4)], \tag{4.4}$$

we readily obtain from Eq. (4.1) the following expression for the field [which we now denote by $\psi_n^{(\infty)}(r, \theta)$] at a typical point P in the far zone, generated by the source mode $\phi_n(x)$:

$$\psi_n^{(\infty)}(r, \theta) = k\sqrt{2\pi}\cos\theta\tilde{\phi}_n(k\sin\theta)\frac{\exp[i(kr - \pi/4)]}{\sqrt{kr}}, \tag{4.5}$$

where

$$\tilde{\phi}_n(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi_n(x')e^{-iux'}\mathrm{d}x' \tag{4.6}$$

is the Fourier transform of $\phi_n(x')$.

If we recall the definition [Eq. (2.16)] of the mode function $\phi_n(x)$ and make use of the well-known formula[13]

$$\int_{-\infty}^{\infty} H_n(x)e^{-x^2/2}e^{ixy}\mathrm{d}x = i^n\sqrt{2\pi}H_n(y)e^{-y^2/2}, \tag{4.7}$$

we readily find that

$$\tilde{\phi}_n(u) = \frac{(-i)^n}{2\pi\sqrt{2^n n!}}\left(\frac{2\pi}{c}\right)^{1/4}H_n\left(\frac{u}{\sqrt{2c}}\right)e^{-u^2/4c}. \tag{4.8}$$

On substituting from Eq. (4.8) into Eq. (4.5), we finally obtain the required expression for the far field generated by the source mode $\phi_n(x)$:

$$\psi_n^{(\infty)}(r, \theta) = A_n\cos\theta \times H_n\left(\frac{k\sin\theta}{\sqrt{2c}}\right)\exp[-(k\sin\theta)^2/4c]\frac{\exp[i(kr - \pi/4)]}{\sqrt{kr}}, \tag{4.9}$$

where

$$A_n = \frac{(-1)^n}{\sqrt{2^n n!}}\frac{k}{(2\pi c)^{1/4}}. \tag{4.10}$$

It is of interest to note that the variances of the intensity distribution across the source and of the degree of spatial coherence of the source enter Eq. (4.9) only through the parameter c, which, according to Eqs. (2.13) and (2.6), is equal to

$$c = \frac{1}{2\sigma_I}\left[\frac{1}{(2\sigma_I)^2} + \frac{1}{\sigma_\mu^2}\right]^{1/2}. \tag{4.11}$$

Hence corresponding modes of two Gaussian Schell-model sources for which the parameter c has the same value will generate identical far fields.

When the cross-spectral density across the source is represented in the form of mode expansion (2.4), one can at once obtain, by an argument similar to one given in Ref. 1, an expression for the cross-spectral density of the field that the source generates at any two points (x_1, z_1), (x_2, z_2) in the half-plane $z > 0$. One only has to replace $\phi_n(x)$ in Eq. (2.4) with the corresponding value $\psi_n(x, z)$ of the field that the source mode ϕ_n generates at point (x, z). In particular, the cross-spectral density of the field vibrations at points (r_1, θ_1),

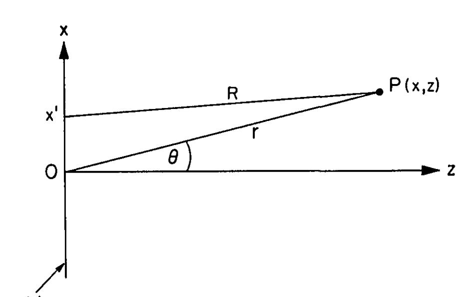

Fig. 2. Illustrating the notation used in calculating the contribution of a source mode to the far-zone intensity distribution.

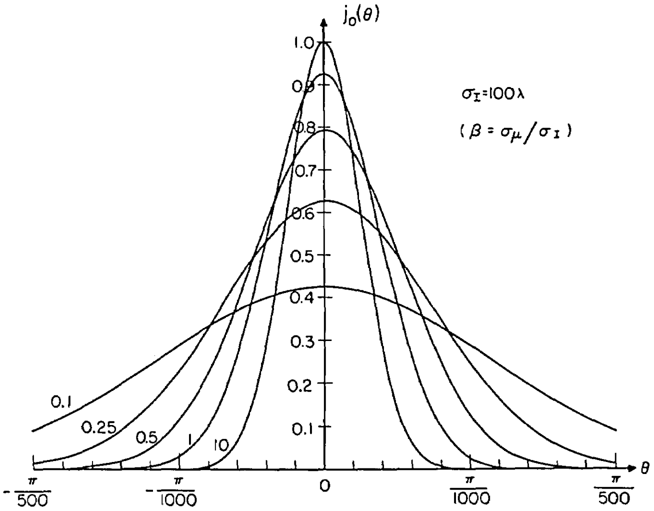

Fig. 3. The normalized far-zone intensity distribution $j_0(\theta) \equiv I_0^{(\infty)}(r, \theta)/I^{(\infty)}(r, 0)$, generated by the lowest-order mode of a Gaussian Schell-model source, for selected values of the degree of global coherence, β.

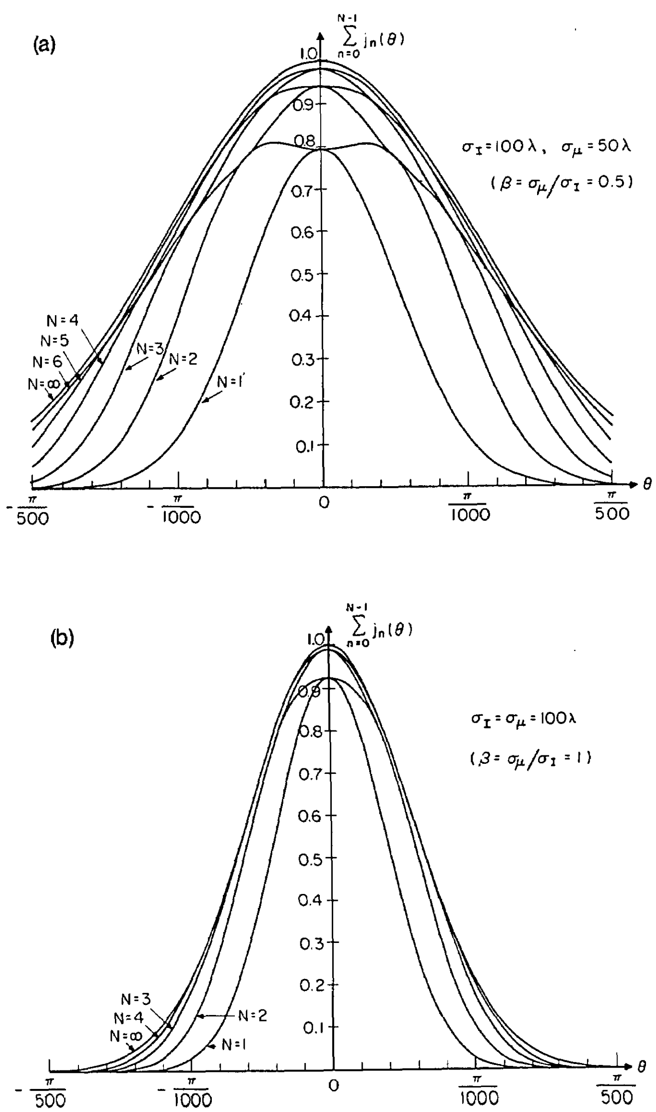

Fig. 4. The contribution $\sum_{n=0}^{N-1} j_n(\theta) \equiv \sum_{n=0}^{N-1} I_n^{(\infty)}(r, \theta)/I^{(\infty)}(r, 0)$ of the N lowest-order modes to the normalized far-zone intensity distribution generated by a Gaussian Schell-model source.

(r_2, θ_2) in the far zone generated by a Gaussian Schell-model source is, therefore, given by

$$W^{(\infty)}(r_1, \theta_1, r_2, \theta_2; \omega) = \sum_{n=0}^{\infty} \lambda_n \psi_n^{(\infty)*}(r_1, \theta_1)\psi_n^{(\infty)}(r_2, \theta_2), \quad (4.12)$$

and the optical intensity at the far zone is given by

$$I^{(\infty)}(r, \theta; \omega) = W^{(\infty)}(r, \theta, r, \theta; \omega) = \sum_{n=0}^{\infty} \lambda_n |\psi_n^{(\infty)}(r, \theta)|^2. \quad (4.13)$$

The series on the right-hand side of Eq. (4.13), with $\psi_n^{(\infty)}$ given by Eq. (4.9), may be summed by making use once again of the bilinear generating function [Eq. (2.9)] for the Hermite polynomials. An alternative derivation of the expression for $I^{(\infty)}$ is given in Appendix B. The result is that

$$I^{(\infty)}(r, \theta; \omega) = \frac{Ak}{2c} \frac{\cos^2\theta}{r} \exp[-a(k \sin\theta)^2/2c^2]. \quad (4.14)$$

The contribution of the source mode labeled by the suffix n to the total intensity at a point (r, θ) in the far zone is, according to Eq. (4.13), given by

$$I_n^{(\infty)}(r, \theta; \omega) = \lambda_n |\psi_n^{(\infty)}(r, \theta)|^2. \quad (4.15)$$

In Fig. 3 the (normalized) contribution of the lowest-order mode is shown for some typical cases. Some normalized contributions of the first N lowest-order modes are shown in Fig. 4. It is of interest to note that the modes of odd orders (i.e., those labeled by the indices $n = 1, 3, 5, \ldots$) do not contribute to the far-zone intensity in the forward direction $\theta = 0$.

One may well ask about the new facts that may be learned from a mode representation of partially coherent sources and partially coherent fields of the kind that was obtained in this paper. This question was briefly discussed in Refs. 1 and 2. In the context of the present paper the representation elucidates in a clear manner the gradual change that a typical radiation field undergoes as the state of coherence of the source changes from complete coherence to complete incoherence.

Finally, we wish to mention that mode representations of partially coherent fields were previously considered by other authors, notably by Gamo[15] and by Martínez-Herrero.[16] However, these authors based their analyses on a mode decomposition of the mutual intensity $J(\mathbf{r}_1, \mathbf{r}_2) = \Gamma(\mathbf{r}_1, \mathbf{r}_2, 0)$ rather than on a mode decomposition of the cross-spectral density $W(\mathbf{r}_1, \mathbf{r}_2, \omega)$. A theory based on a mode decomposition of the cross-spectral density is preferable because, unlike the mutual intensity, the cross-spectral density and consequently the modes associated with it obey precise propagation laws.[17]

APPENDIX A: SOLUTIONS OF EQS. (2.11)

In this Appendix we solve Eqs. (2.11a) and (2.11b), viz.,

$$\frac{\sigma^2(1 + 4t^2)}{2(1 - 4t^2)} = a + b, \quad (A1)$$

$$\frac{2t\sigma^2}{1 - 4t^2} = b. \quad (A2)$$

If we add and subtract these two equations from each other, we obtain the formulas

$$\frac{\sigma^2}{2}\frac{1+2t}{1-2t}=a+2b \tag{A3}$$

and

$$\frac{\sigma^2}{2}\frac{1-2t}{1+2t}=a. \tag{A4}$$

If we multiply the left-hand-side of Eq. (A3) by the left-hand-side of Eq. (A4) and equate the resulting expression to the product of the right-hand sides, we find that

$$\sigma=[4a(a+2b)]^{1/4}=\sqrt{2c}, \tag{A5}$$

where

$$c=(a^2+2ab)^{1/2}. \tag{A6}$$

Next we substitute for σ from Eq. (A5) into Eq. (A4) and solve for t. This gives

$$t=\frac{1}{2}\frac{c-a}{c+a}. \tag{A7}$$

On multiplying both the numerator and the denominator on the right-hand side of Eq. (A7) by $(c+a)$ and recalling the definition [Eq. (A6)] of the parameter c, we readily find that

$$t=\frac{b}{2(a+b+c)}. \tag{A8}$$

Expressions (A5) and (A8), together with definition (A6), are the solutions of Eqs. (A1) and (A2).

APPENDIX B: DERIVATION OF FORMULA (4.14) FOR $I^{(\infty)}$

Let $\{U(r,\theta;\omega)\}$ be an ensemble of monochromatic fields, all of the same frequency ω, that represents the fluctuating field at the point $x=r\sin\theta$, $z=r\cos\theta$. If $U^{(\infty)}(r,\theta;\omega)$ is the far-zone value of a typical realization of the ensemble, the optical intensity in the far zone is given by

$$I^{(\infty)}(r,\theta;\omega)=\langle U^{(\infty)*}(r,\theta;\omega)U^{(\infty)}(r,\theta;\omega)\rangle, \tag{B1}$$

where the angle brackets denote the ensemble average. Now[7] $U^{(\infty)}(r,\theta)$ can be expressed in terms of the values $U^{(0)}(x)$ across the source by a formula of the form of Eq. (4.5), viz.,

$$U^{(\infty)}(r,\theta)=k\sqrt{2\pi}\cos\theta\,\tilde{U}^{(0)}(k\sin\theta)\,\frac{\exp[i(kr-\pi/4)]}{\sqrt{kr}}, \tag{B2}$$

where

$$\tilde{U}^{(0)}(u)=\frac{1}{2\pi}\int_{-\infty}^{\infty}U^{(0)}(x)e^{-iux}\mathrm{d}x \tag{B3}$$

is the Fourier transform of $U^{(0)}(x)$. On substituting from Eq. (B2) into Eq. (B1), we obtain the following expression for $I^{(\infty)}$:

$$I^{(\infty)}(r,\theta;\omega)=\frac{2\pi k\cos^2\theta}{r}\langle\tilde{U}^{(0)*}(k\sin\theta)\tilde{U}^{(0)}(k\sin\theta)\rangle. \tag{B4}$$

Now, according to Eq. (B3),

$$\langle\tilde{U}^{(0)*}(u)\tilde{U}^{(0)}(u)\rangle=\frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty}W(x_1,x_2,\omega)\exp[iu(x_1-x_2)]\mathrm{d}x_1\mathrm{d}x_2, \tag{B5}$$

where

$$W(x_1,x_2,\omega)=\langle\tilde{U}^{(0)*}(x_1)\tilde{U}^{(0)}(x_2)\rangle \tag{B6}$$

is the cross-spectral density function of the field across the source.

For a Gaussian Schell-model source, $W(x_1,x_2,\omega)$ is given by Eq. (2.7). Hence, if we substitute from Eq. (2.7) into Eq. (B5) and then from the resulting expression into Eq. (B4), we obtain for $I^{(\infty)}$ the expression

$$I^{(\infty)}(r,\theta;\omega)=\frac{Ak\cos^2\theta}{2\pi r}\iint_{-\infty}^{\infty}\exp\{-[(a+b)(x_1{}^2+x_2{}^2)-2bx_1x_2]\}\times\exp[ik(x_1-x_2)\sin\theta]\mathrm{d}x_1\,\mathrm{d}x_2. \tag{B7}$$

The integral on the right-hand side of Eq. (B7) may be evaluated by elementary methods, by an argument similar to that employed in an appendix of Ref. 14 in connection with three-dimensional, primary, Gaussian Schell-model sources; one then obtains for the far-zone intensity distribution formula (4.14) of the text, viz.,

$$I^{(\infty)}(r,\theta;\omega)=\frac{Ak}{2c}\frac{\cos^2\theta}{r}\exp[-a(k\sin\theta)^2/2c^2], \tag{B8}$$

where c is defined by Eqs. (2.13) and (2.6).

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6. Actually a third case is possible: Eq. (2.1) may represent the cross-spectral density of the field across a primary source.
7. To keep the notation as simple as possible, we have suppressed in Eqs. (2.7) and (2.8), and also in some subsequent equations, the explicit dependence of some of the quantities on the frequency ω.
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13. H. Hochstadt, *Integral Equations* (Wiley, New York, 1973), p. 146, Eq. (4).
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