AP Calculus BC Extra Credit Exam

Points are given as **extra credit** in the **tests category**

December 2022

A maximum of 15 extra credit points can be earned. Questions of higher point value are not necessarily more difficult. Partial credit will be given.

Question 1 (3 pts.). *Let $C_0, \ldots, C_n$ be real numbers. Show that if*

$$C_0 + \frac{C_1}{2} + \ldots + \frac{C_n}{n+1} = 0,$$

there exists at least one real $0 < x < 1$ such that

$$C_0 + C_1 x + \ldots + C_n x^n = 0.$$

Question 2 (2 pts.). *Let x be a real number and suppose $|x| < 1$. Show that*

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

You may assume that the Taylor series for $f(x) = \frac{1}{1-x}$ converges to $f(x)$ for $|x| < 1$. However, you are not obligated to use the Taylor series to complete the proof.

Question 3 (2 pt.). *Show that if f is a differentiable function defined for all real numbers which satisfies*

$$|f(x) - f(y)| \leq (x-y)^2$$

for all real numbers x, y, then $f(x) = c$ for some constant c.

Question 4 (4 pt.). *The gamma function is a special function in mathematics. It is defined by the improper integral*

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t}\, dt$$

for all real numbers x. The symbol "Γ" is read "gamma." Use integration by parts and L'Hospital's rule to show that for all integers $n > 0$,

$$\Gamma(n+1) = n\Gamma(n).$$

Question 5 (2 pt.). *Use Question 4 to show that for integers $n > 0$,*

$$\Gamma(n) = (n-1)!$$

This property of the Gamma function makes it important in many areas of mathematics and physics (e.g. quantum mechanics, probability/statistics, complex analysis).