AP Calculus BC Extra Credit Exam (Answers)

Points are given as **extra credit** in the **tests category**

December 2022

A maximum of 15 extra credit points can be earned. Questions of higher point value are not necessarily more difficult. Partial credit will be given.

Question 1 (3 pts.). *Let $C_0, \ldots, C_n$ be real numbers. Show that if*

$$C_0 + \frac{C_1}{2} + \ldots + \frac{C_n}{n+1} = 0,$$

there exists at least one real $0 < x < 1$ such that

$$C_0 + C_1 x + \ldots + C_n x^n = 0.$$

Solution. Let $f(x) = C_0 + C_1 x + \ldots + C_n x^n$ and let F be an antiderivative of f defined by

$$F(x) = C_0 x + \frac{C_1 x^2}{2} + \ldots + \frac{C_n x^{n+1}}{n+1}.$$

It is easy to verify that $F' = f$. Since $F(0) = F(1) = 0$, we may invoke the mean value theorem to see that there exists at least one $0 < x_0 < 1$ such that

$$f(x_0) = F'(x_0) = \frac{F(1) - F(0)}{1 - 0} = 0.$$

Note that the application of the mean value theorem is justified by the fact that F is a polynomial and is thus differentiable (and hence continuous) for all real numbers. A similar application of Rolle's theorem will yield the same result. $\square$

Question 2 (2 pts.). *Let x be a real number and suppose $|x| < 1$. Show that*

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

You may assume that the Taylor series for $f(x) = \frac{1}{1-x}$ converges to $f(x)$ for $|x| < 1$. However, you are not obligated to use the Taylor series to complete the proof.

Solution. Let $f(x) = \frac{1}{1-x}$. Since $f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}$, the Maclaurin series for f is

$$\sum_{n=0}^{\infty} = \frac{f^{(n)}(0) x^n}{n!} = \sum_{n=0}^{\infty} x^n.$$

For grading purposes, the proof is now complete. However, we are technically not done. We must show that the Taylor series for f actually converges to f for $|x| < 1$. A straightforward proof involves Taylor's theorem and the Cauchy form of the remainder, which is beyond the scope of this exam (and its solutions). $\square$

Question 3 (2 pt.). *Show that if f is a differentiable function defined for all real numbers which satisfies*

$$|f(x) - f(y)| \leq (x-y)^2$$

for all real numbers x, y, then $f(x) = c$ for some constant c.

Solution. Let x_0 be any real number. Observe that

$$0 \leq \left| \frac{f(x) - f(x_0)}{x - x_0} \right| \leq \frac{(x-x_0)^2}{|x-x_0|} = |x - x_0|.$$

Since $\lim_{x \to x_0} |x - x_0| = 0$,

$$0 = \lim_{x \to x_0} \left| \frac{f(x) - f(x_0)}{x - x_0} \right| = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

by the definition of the derivative and the squeeze theorem. Because x_0 is arbitrary, we have shown that $f'(x) = 0$ for all x.

For grading purposes, the proof is complete. However, we are technically not done. We must show that a $f'(x) = 0$ for all x implies that $f(x) = c$ for some constant c. To do this, we invoke the mean value theorem. For any real numbers x, y, there exists some z between x and y so that

$$\frac{f(x) - f(y)}{x - y} = f'(z) = 0.$$

Thus $f(x) = f(y)$ for all real numbers x, y and the proof is complete. □

Question 4 (4 pt.). *The gamma function is a special function in mathematics. It is defined by the improper integral*

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt$$

for all real numbers x. The symbol "Γ" is read "gamma." Use integration by parts and L'Hospital's rule to show that for all integers $n > 0$,

$$\Gamma(n+1) = n\Gamma(n).$$

Solution. By integration by parts, we see that

$$\Gamma(n+1) = \left[-t^n e^{-t}\right]_0^\infty + n \int_0^\infty t^{n-1} e^{-t} \, dt.$$

Since $\lim_{x \to \infty} -x^n e^{-x} = 0$ by L'Hospital's rule, $[-x^n e^{-x}]_0^\infty = 0$ and

$$\Gamma(n+1) = n \int_0^\infty x^{n-1} e^{-x} \, dx = n\Gamma(n)$$

as required. □

Question 5 (2 pt.). *Use Question 4 to show that for integers $n > 0$,*

$$\Gamma(n) = (n-1)!$$

This property of the Gamma function makes it important in many areas of mathematics and physics (e.g. quantum mechanics, probability/statistics, complex analysis).

Solution. We first recognize that $\Gamma(1) = \int_0^\infty e^{-x} = 1$. If $n = 1$, we are done since $0! = 1$. For $n > 1$, it follows from Question 4 that

$$\Gamma(n) = (n-1)\Gamma(n-1) = \ldots = (n-1)\ldots(1)\Gamma(1) = (n-1)!$$

□