Introduction to Stone Duality - Proofs

Sri Pranav Kunda

1 Structure Preserving Maps and Duality

Definition 1. *Let $f : X \to Y$. Then $f^* : 2^Y \to 2^X$ defined by, for $b \in 2^Y$,*

$$f^*(b) = \{x \in X : f(x) \in b\}$$

is called the inverse image function of f.

Remark: The inverse image function is also called the preimage and can be denoted by $f^{-1}(x)$ on other writings.

Proposition 1. *Given a function $f : X \to Y$, the inverse image function f^* preserves unions, intersections, and complements.*

Proof. Let $S \subseteq 2^Y$ and suppose $x \in \bigcup_{s \in S} f^*(s)$. This means x is in at least one of $f^*(s)$ so that, by definition, $f(x)$ is in at least one s and $x \in f^*(\bigcup_{s \in S} s)$.

Since the converse follows similarly, $f^*(\bigcup_{s \in S} s) = \bigcup_{s \in S} f^*(s)$. It is easy to see that f^* preserves intersections in the same way.

Finally, $x \in f^*(\neg a) \iff f(x) \notin a \iff x \notin f^*(a) \iff x \in \neg f^*(a)$, completing the proof. □

Definition 2 (Adjoints)**.** *Let $R : 2^Y \to 2^X$. Then $L : 2^X \to 2^Y$ is left-adjoint to R (and R is right-adjoint to L) if for all $a \in 2^X$ and $b \in 2^Y$,*

$$L(a) \subseteq b \iff a \subseteq R(b). \tag{1}$$

Remark: The phrases L is a left adjoint and L has a right adjoint are equivalent.

Lemma 1. *If L is left-adjoint to R, $a \subseteq R(L(a))$ and $L(R(b)) \subseteq b$.*

Proof. Since $L(a) \subseteq L(a)$, $a \subseteq R(L(a))$ by (1). Similar reasoning shows that $L(R(b)) \subseteq b$. □

Definition 3 (Monotone Function)**.** *A function $F : 2^X \to 2^Y$ is monotone if for any $a, a' \in 2^X$, $a \subseteq a' \Rightarrow F(a) \subseteq F(a')$.*

Lemma 2. *If $F : 2^X \to 2^Y$ has a left or right adjoint, F is monotone.*

Proof. For concreteness we assume that F has a right adjoint R. We show that both F and R are monotone. Define a, a' as in Definition 3.

By Lemma 1, $a \subseteq a' \subseteq R(F(a'))$. From (1) we see that if $a \subseteq R(F(a'))$, $F(a) \subseteq F(a')$. A similar argument invoking the latter portion of Lemma 1 shows that R is monotone. $\square$

Lemma 3. *Let $L : 2^X \to 2^Y$ and $R : 2^Y \to 2^X$. If L and R are monotone and $a \subseteq R(L(a))$ and $L(R(b)) \subseteq b$ for all $a \in 2^X$ and $b \in 2^Y$, then L is left-adjoint to R.*

Proof. Suppose $L(a) \subseteq b$. Since L is monotone, $a \subseteq R(L(a)) \subseteq R(b)$. Conversely, suppose $a \subseteq R(b)$. Because R is monotone, $L(a) \subseteq L(R(b)) \subseteq b$. Thus L is left-adjoint to R (and R is right-adjoint to L). $\square$

Lemma 4. *Left and right adjoints are unique.*

Proof. We omit the proof for right-adjoints because it is similar to the proof for left-adjoints. Let $F : 2^X \to 2^Y$ and suppose L, L' are left-adjoint to F. Then $a \subseteq R(L'(a))$ by Lemma 1. It follows from (1) that $L(a) \subseteq L'(a)$. Likewise, $a \subseteq R(L(a)) \Rightarrow L'(a) \subseteq L(a)$. Thus $L = L'$. $\square$

Proposition 2. *Any function $g : 2^Y \to 2^X$ which preserves intersections and unions has a left adjoint $g_\exists : 2^X \to 2^Y$ and a right adjoint $g_\forall : 2^X \to 2^Y$. Moreover, $g_\exists$ and $g_\forall$ preserve unions and intersections respectively.*

Proof. We prove more generally that (i) g has a left adjoint $g_\exists$ if and only if g preserves intersections and that (ii) g has a right adjoint $g_\forall$ if and only if g preserves unions. The proof of (ii) is excluded for brevity since it is similar to (i).

Define $g_\exists(a) = \bigcap\{y : a \subseteq g(y)\}$ and suppose g preserves intersections. Let $a \subseteq a'$. Then $g(a \cap a') = g(a) = g(a) \cap g(a') \Rightarrow g(a) \subseteq g(a')$. Thus g is monotone. Hence, $g_\exists(a) \subseteq b \Rightarrow g(b) \supseteq g(g_\exists(a)) \supseteq a$ since g preserves intersections and $a \subseteq g(b) \Rightarrow g_\exists(a) \subseteq b$. It is easy to see that $g_\exists$ preserves unions from the definition, completing the proof.

It is useful to note that if g preserves both unions and intersections, then we may write equivalently that $g_\exists(a) = \{y \in Y \mid \exists x \in g(\{y\}) : x \in a\}$. $\square$

Definition 4 (Atom)**.** *$a \subseteq X$ is an atom if for all $S \subseteq 2^X$, $a \subseteq \bigcup S$ implies that there exists an $a' \in S$ such that $a \subseteq a'$.*

Proposition 3. *$a \in 2^X$ is an atom if and only if there exists $x \in X$ s.t. $a = \{x\}$.*

Proof. Suppose there exists $x \in X$ s.t. $a = \{x\}$. Then for any $S \subseteq 2^X$, we have that if $a \subseteq \bigcup S$, there exists some $a' \in S$ s.t. $x \in a'$. Then $a \subseteq a'$ since a is a singleton containing x.

For the converse, suppose a is an atom. First, observe that a is nonempty, otherwise we may choose $S = \emptyset$ so that there exists no $a' \in S$ s.t. $a \subseteq a'$ although $a \subseteq \bigcup S$.

For a contradiction, suppose $|a| > 1$. Choosing $S = \{\{x\} : x \in a\}$, we see that $a \subseteq \bigcup S$. However, a is not contained in any $a' \in S$. Thus $|a| = 1$ and the proof is complete. $\square$

Lemma 5. *Let $g : 2^Y \to 2^X$ preserve unions and intersections. Then the left adjoint of g (which exists and is unique by Proposition 2 and Lemma 4) maps atoms to atoms.*

Proof. Let $a \in 2^X$ be an atom, let $S \subseteq 2^Y$ s.t. $g_\exists(a) \subseteq \bigcup S$, and let $T = \{g(s) : s \in S\}$. Then $a \subseteq g(\bigcup S) = \bigcup T$ by the definition of the adjoint. Since a is an atom and $T \subseteq 2^X$, there exists $a' \in T$ s.t. $a \subseteq a'$. Because $a' = g(s)$ for some $s \in S$, we have that $g_\exists(a) \subseteq g_\exists(g(s)) \subseteq s$ by Lemmas 1 and 2. $\square$

Proposition 4. *Every function $g : 2^Y \to 2^X$ that preserves unions and intersections is the inverse image function for a unique $g_* : X \to Y$.*

Proof. By Lemma 5, g has a unique left adjoint $g_\exists$ which maps atoms to atoms. Define $g_*(x) = g_\exists(\{x\})$. We show that $(g_*)^*$ is right adjoint to $g_\exists$ so that g is the inverse image function of $g_* : X \to Y$ by the uniqueness of adjoints.

It has already been shown that both $g_\exists$ and $(g_*)^*$ are monotone and preserve unions. Hence, $(g_*)^*(g_\exists(a)) = \bigcup_{x \in a} (g_*)^*(g_\exists(\{x\})) \supseteq a$ and $g_\exists((g_*)^*(b)) = \bigcup_{y \in b} g_\exists((g_*)^*(\{y\})) \subseteq b$.

We retain the notation g_* to denote the function for which g is the inverse image function. $\square$

Theorem 1. *There is a bijection between functions $2^Y \to 2^X$ which preserve unions and intersections and functions $X \to Y$.*

Proof. By Proposition 4, there exists a unique $g_* : X \to Y$ for every $g : 2^Y \to 2^X$. Similarly, every $f : X \to Y$ has a unique inverse image function f^* by definition. Thus we have a bijection. $\square$