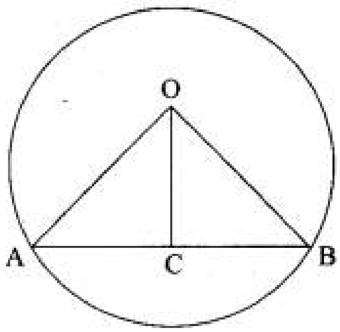
Applications of Vector Algebra

Ex 6.1

Question 1.
Prove by vector method that if a line is drawn from the centre of a circle of a circle to the midpoint of a chord, then the line is perpendicular to the chord.
Solution:
Let 'C' be the mid point of the chord AB
Take 'O' on the centre of the circle.
Since, OA = OB (Radii)



Let $\overrightarrow{OA} = \vec{a}$; $\overrightarrow{OB} = \vec{b}$; $\overrightarrow{OC} = \frac{\vec{a}+\vec{b}}{2}$

To prove: $\overrightarrow{OC} \perp^r \overrightarrow{AB}$

$$\overrightarrow{OC} \cdot \overrightarrow{AB} = \left(\frac{a+\vec{b}}{2}\right) \cdot \left(\overrightarrow{OB} - \overrightarrow{OA}\right) = \left(\frac{\vec{b}+\vec{a}}{2}\right) \cdot \left(\vec{b} - \vec{a}\right)$$

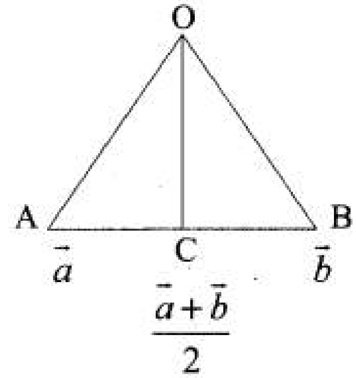
$$= \frac{1}{2}\left[\left(\vec{b}\right)^2 - \left(\vec{a}\right)^2\right] = \frac{1}{2}\left(\overrightarrow{OB}^2 - \overrightarrow{OA}^2\right) = \frac{1}{2}\left(OB^2 - OA^2\right)$$

$$= 0$$

$\therefore$ $\overrightarrow{OC}$ $\perp^r$ to $\overrightarrow{AB}$. Hence the result.

Question 2.
Prove by vector method that the median to the base of an isosceles triangle is perpendicular to the base.
Solution:
Let OAB be an isosceles triangle with OA = OB
Let OC be the median to the base AB
C is the midpoint of AB

Take O as origin.



Let $\overrightarrow{OA} = \vec{a}$; $\overrightarrow{OB} = \vec{b}$; $\overrightarrow{OC} = \frac{\vec{a}+\vec{b}}{2}$

To prove $\overrightarrow{OC} \cdot \overrightarrow{AB} = 0$

$$\overrightarrow{OC} . \overrightarrow{AB} = \left(\frac{\vec{a}+\vec{b}}{2}\right) . \left(\overrightarrow{OB} \quad \overrightarrow{OA}\right) = \left(\frac{\vec{a}+\vec{b}}{2}\right) . \left(\vec{b}-\vec{a}\right)$$

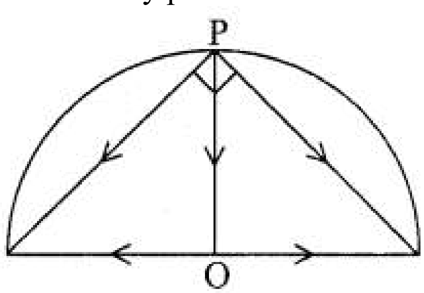
$$= \left(\frac{\vec{b}+\vec{a}}{2}\right) . \left(\vec{b}-\vec{a}\right) = \frac{1}{2}\left[\left(\vec{b}\right)^2 - \left(\vec{a}\right)^2\right]$$

$$= \frac{1}{2}\left[\left(\overrightarrow{OB}\right)^2 - \left(\overrightarrow{OA}\right)^2\right] = \frac{1}{2}\left(OB^2 - OA^2\right)$$

$$\overrightarrow{OC} \cdot \overrightarrow{AB} = \frac{1}{2}(0)$$

$\therefore \overrightarrow{OC} \perp^r$ to $\overrightarrow{AB}$. Hence the result.

Question 3.
Prove by vector method that an angle in a semi-circle is a right angle.
Solution:
Let AB be the diameter of the circle with centre 'O'
Let P be any point on the semi-circle.



• To prove $\angle APB = 90^\circ$

We have OA = OB = OP (radii)

Now
$$\overrightarrow{PA} = \overrightarrow{PO} + \overrightarrow{OA}$$
$$\overrightarrow{PB} = \overrightarrow{PO} + \overrightarrow{OB}$$
$$= \overrightarrow{PO} - \overrightarrow{OA} \quad (\text{since } \overrightarrow{OB} = -\overrightarrow{OA})$$

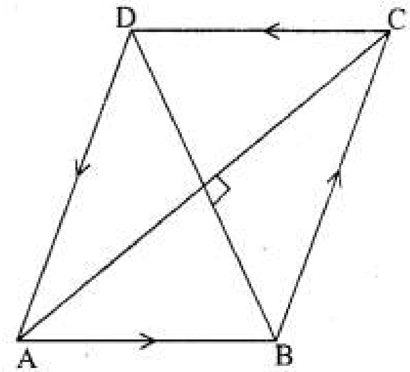
$$\therefore \quad \overrightarrow{PA} \cdot \overrightarrow{PB} = \left(\overrightarrow{PO} + \overrightarrow{OA}\right) \cdot \left(\overrightarrow{PO} - \overrightarrow{OA}\right) = \left(\overrightarrow{PO}\right)^2 - \left(\overrightarrow{OA}\right)^2 = (PO)^2 - (OA)^2$$

$$\overrightarrow{PA} \cdot \overrightarrow{PB} = 0$$

$\therefore \overrightarrow{PA} \perp^r$ to $\overrightarrow{PB}$

This gives ∠APB = 90°. Hence the result.

Question 4.
Prove by vector method that the diagonals of a rhombus bisect each other at right angles.
Solution:
Let ABCD be a rhombus



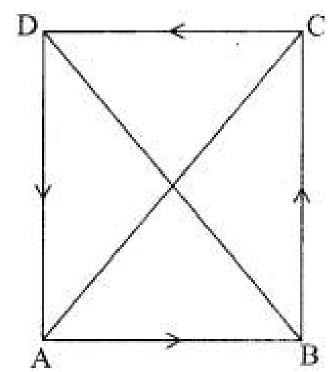
To prove $\overrightarrow{AC} \cdot \overrightarrow{BD} = 0$

We have AB = BC = CD = DA

Now
$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$
$$\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD}$$
$$= \overrightarrow{BC} - \overrightarrow{AB} \text{ (since } \overrightarrow{CD} = -\overrightarrow{AB}\text{)}$$
$$\overrightarrow{AC} \cdot \overrightarrow{BD} = \left(\overrightarrow{BC} + \overrightarrow{AB}\right) \cdot \left(\overrightarrow{BC} - \overrightarrow{AB}\right)$$
$$= \left(\overrightarrow{BC}\right)^2 - \left(\overrightarrow{AB}\right)^2 = (BC)^2 - (AB)^2$$
$$\overrightarrow{AC} \cdot \overrightarrow{BD} = 0$$

$\therefore$ $\overrightarrow{AC}$ $\perp^r$ to $\overrightarrow{BD}$. Hence the result.

Question 5.
Using vector method, prove that if the diagonals of a parallelogram are equal, then it is a rectangle.
Solution:
Let ABCD be a parallelogram
To prove ABCD be a rectangle provided the diagonals are equal.



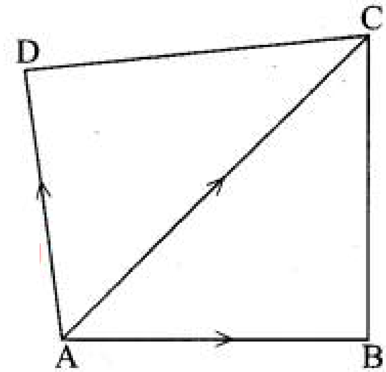
Now
$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$
$$\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{BC} - \overrightarrow{AB}$$

But
$$\left(\overrightarrow{AC}\right)^2 = \left(\overrightarrow{BD}\right)^2$$
$$\left(\overrightarrow{AB} + \overrightarrow{BC}\right)^2 = \left(\overrightarrow{BC} - \overrightarrow{AB}\right)^2$$
$$\left(\overrightarrow{AB}\right)^2 + \left(\overrightarrow{BC}\right)^2 + 2\overrightarrow{AB} \cdot \overrightarrow{BC} = \left(\overrightarrow{BC}\right)^2 + \left(\overrightarrow{AB}\right)^2 - 2\overrightarrow{AB} \cdot \overrightarrow{BC}$$
$$4\,\overrightarrow{AB} \cdot \overrightarrow{BC} = 0$$
$$\overrightarrow{AB} \cdot \overrightarrow{BC} = 0$$

$\overrightarrow{AB} \perp^r$ to $\overrightarrow{BC}$
⇒ ABCD is a rectangle.

Question 6.
Prove by vector method that the area of the quadrilateral ABCD having diagonals AC and BD is
$\frac{1}{2}|\overrightarrow{AC} \times \overrightarrow{BD}|$
Solution:

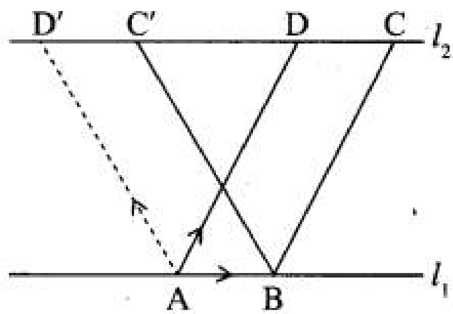
Vector area of quadrilateral ABCD = {Vector area of ΔABC} + {Vector area of Δ ACD}



$$= \frac{1}{2}\left(\overrightarrow{AB}\times\overrightarrow{AC}\right)+\frac{1}{2}\left(\overrightarrow{AC}\times\overrightarrow{AD}\right)$$
$$= -\frac{1}{2}\left(\overrightarrow{AC}\times\overrightarrow{AB}\right)+\frac{1}{2}\left(\overrightarrow{AC}\times\overrightarrow{AD}\right)$$
$$= \frac{1}{2}\overrightarrow{AC}\times\left[-\overrightarrow{AB}+\overrightarrow{AD}\right]$$
$$= \frac{1}{2}\overrightarrow{AC}\times\left[\overrightarrow{BA}+\overrightarrow{AD}\right] = \frac{1}{2}\overrightarrow{AC}\times\overrightarrow{BD}$$

$\therefore$ The area of the quadrilateral ABCD $= \frac{1}{2}\left|\overrightarrow{AC}\times\overrightarrow{BD}\right|$

Question 7.
Prove by vector method that the parallelogram on the same base and between the same parallels are equal in area.
Solution:
Let ABCD and ABC'D' be two parallelogram between the parallels with same base
To prove: Area of ABCD = Area of ABC'D'



$$\text{Area of } ABC'D' = \left|\overrightarrow{AB}\times\overrightarrow{AD'}\right|$$
$$= \left[\overrightarrow{AB}\times\left(\overrightarrow{AD}+\overrightarrow{DD'}\right)\right]$$
$$= \left|\left(\overrightarrow{AB}\times\overrightarrow{AD}\right)+\left(\overrightarrow{AB}\times\overrightarrow{DD'}\right)\right|$$
$$= \left|\overrightarrow{AB}\times\overrightarrow{AD}\right|+0 \qquad \text{since } (\overrightarrow{AB} \text{ and } \overrightarrow{DD'} \text{ are parallel}).$$
$$= \text{Area of } ABCD$$

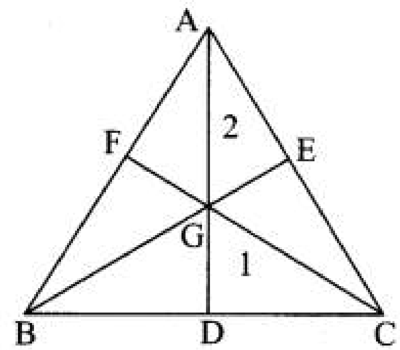
Question 8. If G is the centroid of a AABC, prove
that.
(area of ΔGAB) = (area of ΔGBC) = (area of ΔGAC) = $\frac{1}{3}$ [area of ΔABC]
Solution:

Area of $\Delta GAB = \frac{1}{2}\left(\overrightarrow{AB} \times \overrightarrow{AG}\right)$

$$= \frac{1}{2}\left[\left(\overrightarrow{OB} - \overrightarrow{OA}\right) \times \left(\overrightarrow{OG} - \overrightarrow{OA}\right)\right]$$

$$= \frac{1}{2}\left\{\left(\vec{b} - \vec{a}\right) \times \left[\frac{\vec{a} + \vec{b} + \vec{c}}{3} - a\right]\right\}$$

$$= \frac{1}{2}\left\{\left(\vec{b} - \vec{a}\right) \times \left[\frac{\vec{b} + \vec{c} - 2\vec{a}}{3}\right]\right\}$$



$$= \frac{1}{6}\left\{\vec{b} \times \vec{c} - 2\vec{b} \times \vec{a} - \vec{a} \times \vec{b} - \vec{a} \times \vec{c}\right\} = \frac{1}{6}\left\{\vec{b} \times \vec{c} + \vec{c} \times \vec{a} + \vec{a} \times \vec{b}\right\}$$

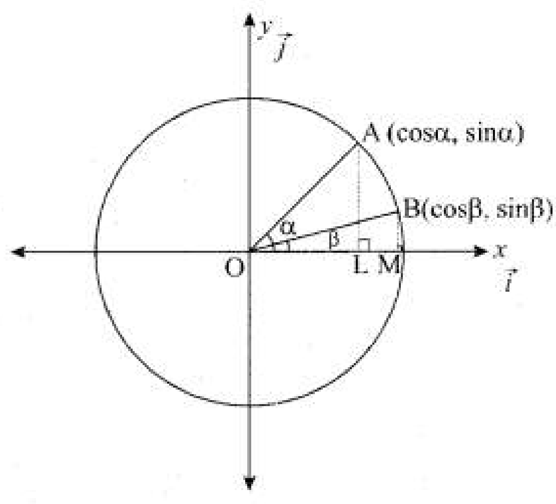
$\left(\text{Since } \vec{b} \times \vec{b} = 0 \text{ and } \vec{a} \times \vec{a} = 0\right)$

$$= \frac{1}{3} \times \frac{1}{2}\left\{\vec{b} \times \vec{c} + \vec{c} \times \vec{a} + \vec{a} \times \vec{b}\right\} = \frac{1}{3} \text{ [area of } \Delta ABC]$$

Similarly we can prove
Area of ΔGBC = Area of ΔGAC = $\frac{1}{3}$ [Area of ΔABC]

Question 9.
Using vector method, prove that $\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$.
Solution:

- Take two points A and B on the unit circle with centre as origin 'o'. so $\left|\overrightarrow{OA}\right| = \left|\overrightarrow{OB}\right| = 1$
- $\underline{|AOx} = \alpha$; $\underline{|BOx} = \beta \Rightarrow \underline{|AOB} = \alpha - \beta$
- Let $\vec{i}$ and $\vec{j}$ be the unit vectors along the x, y respectively.
- The co-ordinates of A and B be $(\cos \alpha, \sin \alpha)$ and $(\cos \beta, \sin \beta)$ respectively.



$$\overrightarrow{OA} = \overrightarrow{OL} + \overrightarrow{LA}$$

$$\overrightarrow{OA} = \cos\alpha\vec{i} + \sin\alpha\vec{j}$$

$$\overrightarrow{OB} = \overrightarrow{OM} + \overrightarrow{MB} \qquad \Rightarrow \overrightarrow{OB} = \cos\beta\vec{i} - \sin\beta\vec{j}$$

So, $\quad \overrightarrow{OB} \cdot \overrightarrow{OA} = \cos \alpha \cos \beta + \sin \alpha \sin \beta \qquad \text{...(1)}$

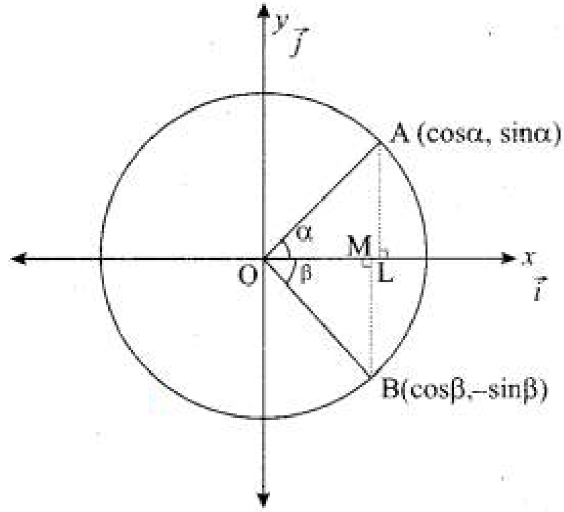
But $\quad \overrightarrow{OB} \cdot \overrightarrow{OA} = \left|\overrightarrow{OB}\right| \left|\overrightarrow{OA}\right| \cos(\alpha + \beta)$

$$\overrightarrow{OB} \cdot \overrightarrow{OA} = \cos(\alpha + \beta) \qquad \text{...(2)}$$

From (1) and (2), we get
$\cos(\alpha + \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$

Question 10.
Prove by vector method that sin(α + β) = sin α cos β + cos α sin β.
Solution:
Take two points A and B on the unit circle with centre as origin 'O', so $|\overrightarrow{OA}| = |\overrightarrow{OB}| = 1$

$\underline{|AOX} = \alpha$; $\underline{|BOX} = \beta \Rightarrow \underline{|AOB} = \alpha + \beta$



Let $\vec{i}$ and $\vec{j}$ be the unit vectors along the x, y direction respectively.
The co-ordinates of A and B be (cos α, sin α) and (cos β, –sin β) respectively.

$$\overline{OA} = \overline{OL} + \overline{LA}$$
$$\overline{OA} = \cos\alpha\vec{i} + \sin\alpha\vec{j}$$
$$\overline{OB} = \overline{OM} + \overline{MB}$$
$$\overline{OB} = \cos\beta\vec{i} - \sin\beta\vec{j}$$

$$\overline{OB}\times\overline{OA} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\beta & -\sin\beta & 0 \\ \cos\alpha & \sin\alpha & 0 \end{vmatrix}$$

So, $$\overline{OB}\times\overline{OA} = (\sin\alpha\cos\beta + \cos\alpha\sin\beta)\vec{k} \quad ...(1)$$

But, $$\overline{OB}\times\overline{OA} = |\overline{OB}|\,|\overline{OA}|\sin(\alpha+\beta)\,\vec{k}$$
$$\overline{OB}\times\overline{OA} = \sin(\alpha+\beta)\,\vec{k} \quad ...(2)$$

From (1) & (2), we get
sin (α + β) = sin α cos β + cos α sin β

Question 11.

A particle acted on by constant forces $8\vec{i} + 2\vec{j} - 6\vec{k}$ and $\vec{6i} + 2\vec{j} - 2\vec{k}$ is displaced from the point (1, 2, 3) to the point (5, 4, 1). Find the total work done by the forces.
Solution:

$$\vec{F} = \vec{F}_1 + \vec{F}_2$$
$$\vec{F} = \left(8\vec{i}+2\vec{j}-6\vec{k}\right) + \left(6\vec{i}+2\vec{j}-2\vec{k}\right)$$
$$\vec{F} = \left(14\vec{i}+4\vec{j}-8\vec{k}\right) \quad ...(1)$$
$$\vec{d} = \overline{OB} - \overline{OA}$$
$$\vec{d} = \left(5\vec{i}+4\vec{j}+\vec{k}\right) - \left(\vec{i}+2\vec{j}+3\vec{k}\right)$$
$$\vec{d} = \left(4\vec{i}+2\vec{j}-2\vec{k}\right) \quad ...(2)$$

From (1) & (2), we get

Work done by the force = $\vec{F}\cdot\vec{d}$ = 56 + 8 + 16 = 80 units.

Question 12.

Forces of magnitude $5\sqrt{2}$ and $10\sqrt{2}$ units acting in the directions $(3\vec{i}+4\vec{j}+5\vec{k})$ and $(10\vec{i}+6\vec{j}-8\vec{k})$, respectively, act on a particle which is displaced from the point with position vector $(4\vec{i}-3\vec{j}-2\vec{k})$ to the point with position vector $(\overrightarrow{6i}+\vec{j}-3\vec{k})$. Find the work done by the forces.

Solution:

$$\vec{a} = 3\vec{i}+4\vec{j}+5\vec{k} \qquad (\text{since } \hat{a} = \frac{\vec{a}}{|\vec{a}|})$$

$$|\vec{a}| = \sqrt{9+16+25} = \sqrt{50} = 5\sqrt{2}$$

$$\vec{F_1} = 5\sqrt{2}\hat{a} = 5\sqrt{2}\ \frac{(3\vec{i}+4\vec{j}+5\vec{k})}{5\sqrt{2}} = 3\vec{i}+4\vec{j}+5\vec{k}$$

$$\vec{b} = 10\vec{i}+6\vec{j}-8\vec{k}$$

$$|\vec{b}| = \sqrt{100+36+64} = \sqrt{200} = 10\sqrt{2}$$

$$\vec{F_2} = 10\sqrt{2}\hat{b}$$

$$= 10\sqrt{2}\ \frac{(10\vec{i}+6\vec{j}-8\vec{k})}{10\sqrt{2}} \quad (\text{since } \hat{b} = \frac{\vec{b}}{|\vec{b}|})$$

$$= 10\vec{i}+6\vec{j}-8\vec{k}$$

$\therefore$ So,

$$\vec{F} = \vec{F}_1+\vec{F}_2$$

$$\vec{F} = (3\vec{i}+4\vec{j}+5\vec{k})+(10\vec{i}+6\vec{j}-8\vec{k})$$

$$\vec{F} = 13\vec{i}+10\vec{j}-3\vec{k} \qquad ...(1)$$

$$\vec{d} = \overrightarrow{OB}-\overrightarrow{OA}$$

$$\vec{d} = (6\vec{i}+\vec{j}-3\vec{k})-(4\vec{i}-3\vec{j}-2\vec{k})$$

$$\vec{d} = 2\vec{i}+4\vec{j}-\vec{k} \qquad ...(2)$$

From (1) & (2), we get

Work done by the force = $\vec{F}\cdot\vec{d} = (13\vec{i}+10\vec{j}-3\vec{k})\cdot(2\vec{i}+4\vec{j}-\vec{k}) = 26+40+3 = 69$ Units

Question 13.

Find the magnitude and direction cosines of the torque of a force represented by $3\vec{i}+4\vec{j}-5\vec{k}$ about the point with position vector $2\vec{i}-3\vec{j}+4\vec{k}$ acting through a point whose position vector is $\overrightarrow{4i}+2\vec{j}-3\vec{k}$.

Solution:

$$\vec{F} = -3\vec{i}+6\vec{j}-6\vec{k}$$

$$\vec{r} = \text{(through the point)} - \text{(about the point)}$$

$$\vec{r} = (4\vec{i}+2\vec{j}-3\vec{k}) - (2\vec{i}-3\vec{j}+4\vec{k})$$

$$\vec{r} = 2\vec{i}+5\vec{j}-7\vec{k}$$

$$\text{Torque } (\vec{M}) = \vec{r} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 5 & -7 \\ 3 & 4 & -5 \end{vmatrix}$$

$$= \vec{i}\ [-25+28] - \vec{j}\ [-10+21] + \vec{k}\ [8-15]$$

$$\vec{M} = 3\vec{i}-11\vec{j}-7\vec{k}$$

$$\text{Magnitude} = |\vec{M}| = \sqrt{9+121+49} = \sqrt{179}$$

$$\text{Direction cosines } \left(\frac{3}{\sqrt{179}}, \frac{-11}{\sqrt{179}}, \frac{-7}{\sqrt{179}}\right)$$

Question 14.

Find the torque of the resultant of the three forces represented by $-3\vec{i}+6\vec{j}-3\vec{k}$, $\vec{4i}-10\vec{j}+12\vec{k}$ and $\vec{4i}+7\vec{j}$ acting at the point with position vector $8\vec{i}-\vec{6}\vec{j}-4\vec{k}$, about the point with position vector $18\vec{i}+3\vec{j}-9\vec{k}$

Solution:

$$\vec{F_1} = -3\vec{i}+6\vec{j}-6\vec{k}$$

$$\vec{F_2} = 4\vec{i}-10\vec{j}+12\vec{k} \text{ and } \vec{F_3} = 4\vec{i}+7\vec{j}$$

But

$$\vec{F} = \vec{F}_1+\vec{F}_2+\vec{F}_3$$

$$\vec{F} = 5\vec{i}+3\vec{j}+9\vec{k}$$

$$\vec{r} = \text{(act at the point)} - \text{(about the point)} = (8\vec{i}-6\vec{j}-4\vec{k}) - (18\vec{i}+3\vec{j}-9\vec{k})$$

$$\vec{r} = -10\vec{i}-9\vec{j}+5\vec{k}$$

$$\text{Torque } (\vec{M}) = \vec{r} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -10 & -9 & 5 \\ 5 & 3 & 9 \end{vmatrix}$$

$$= [-81-15] - \vec{j}\ [-90-25] + \vec{k}\ [-30+45]$$

$$= -96\vec{i}+115\vec{j}+15\vec{k}$$

Additional Problems

Question 1.

The work done by the force $\overrightarrow{\mathrm{F}} = a\vec{i} + \vec{j} + \vec{k}$ in moving the point of application from (1, 1, 1) to (2, 2, 2) along a straight line is given to be 5 units. Find the value of a.

Solution:

$\vec{\mathrm{F}} = a\vec{i} + \vec{j} + \vec{k}$; $\overrightarrow{\mathrm{OA}} = \vec{i} + \vec{j} + \vec{k}$; $\overrightarrow{\mathrm{OB}} = 2\vec{i} + 2\vec{j} + 2\vec{k}$

Work done = 5 units

$$\vec{d} = \overrightarrow{\mathrm{AB}} = \overrightarrow{\mathrm{OB}} - \overrightarrow{\mathrm{OA}} = \vec{i} + \vec{j} + \vec{k}$$

Work done = $\vec{\mathrm{F}} \cdot \vec{d}$

$$5 = \left(a\vec{i} + \vec{j} + \vec{k}\right) \cdot \left(\vec{i} + \vec{j} + \vec{k}\right)$$

$$5 = a + 1 + 1 \qquad \Rightarrow a = 3$$

Question 2.

If the position vectors of three points A, B and C are respectively $\vec{i} + 2\vec{j} + 3\vec{k}$, $4\vec{i} + \vec{j} + 5\vec{k}$ and $7(\vec{i} + \vec{k})$. Find $\overrightarrow{\mathrm{AB}} \times \overrightarrow{\mathrm{AC}}$. Interpret the result geometrically.

Solution:

$$\overrightarrow{\mathrm{OA}} = \vec{i} + 2\vec{j} + 3\vec{k} \text{ ; } \overrightarrow{\mathrm{OB}} = 4\vec{i} + \vec{j} + 5\vec{k} \text{ ; } \overrightarrow{\mathrm{OC}} = 7\vec{i} + 7\vec{k}$$

$$\overrightarrow{\mathrm{AB}} = \overrightarrow{\mathrm{OB}} - \overrightarrow{\mathrm{OA}} = \left(4\vec{i} + \vec{j} + 5\vec{k}\right) - \left(\vec{i} + 2\vec{j} + 3\vec{k}\right)$$

$$\overrightarrow{\mathrm{AB}} = 3\vec{i} - \vec{j} + 2\vec{k}$$

$$\overrightarrow{\mathrm{AC}} = \overrightarrow{\mathrm{OC}} - \overrightarrow{\mathrm{OA}} = 6\vec{i} - 2\vec{j} + 4\vec{k}$$

$$\overrightarrow{\mathrm{AB}} \times \overrightarrow{\mathrm{AC}} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & 2 \\ 6 & -2 & 4 \end{vmatrix} = \vec{0}$$

The vectors $\overrightarrow{\mathrm{AB}}$ and $\overrightarrow{\mathrm{AC}}$ are parallel. But they have the point A as a common point.

$\overrightarrow{\mathrm{AB}}$ and $\overrightarrow{\mathrm{AC}}$ are along the same line.

$\therefore$ A, B, C are collinear.

Question 3.

A force given by and $3\vec{i} + 2\vec{j} - 4\vec{k}$ is applied at the point (1, – 1, 2). Find the moment of the force about the point (2, – 1, 3).

Solution:

$$\vec{F} = 3\vec{i}+2\vec{j}-4\vec{k}$$

$$\overrightarrow{OP} = \vec{i}-\vec{j}+2\vec{k}$$

$$\overrightarrow{OA} = 2\vec{i}-\vec{j}+3\vec{k}$$

$$\vec{r} = \overrightarrow{AP}=\overrightarrow{OP}-\overrightarrow{OA} = \left(\vec{i}-\vec{j}+2\vec{k}\right)-\left(2\vec{i}-\vec{j}+3\vec{k}\right)$$

$$\vec{r} = -\vec{i}-\vec{k}$$

The moment $\overline{M}$ of the force $\vec{F}$ about the point A is given by

$$\overline{M} = \vec{r}\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 0 & -1 \\ 3 & 2 & -4 \end{vmatrix} = 2\vec{i}-7\vec{j}-2\vec{k}$$

Question 4.

Show that the area of a parallelogram having diagonals $3\vec{i}+\vec{j}-2\vec{k}$ **and** $\vec{i}-3\vec{j}+4\vec{k}$ is $5\sqrt{3}$

Solution:

$$\text{Let } \vec{d}_1 = 3\vec{i}+\vec{j}-2\vec{k} \text{ and } \vec{d}_2 = \vec{i}-3\vec{j}+4\vec{k}$$

$$\text{Area of the parallelogram} = \frac{1}{2}\left|\vec{d}_1\times\vec{d}_2\right|$$

$$\vec{d}_1 \times \vec{d}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix} = -2\vec{i}-14\vec{j}-10\vec{k}$$

$$\Rightarrow \quad \left|\vec{d}_1\times\vec{d}_2\right| = \sqrt{(-2)^2+(-14)^2+(-10)^2} = \sqrt{300}=10\sqrt{3}$$

$$\text{Area of the parallelogram} = \frac{1}{2}\left|\vec{d}_1\times\vec{d}_2\right| = \frac{1}{2}10\sqrt{3} = 5\sqrt{3} \text{ Sq. units.}$$

Ex 6.2

Question 1.

If $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$, $\vec{b} = 2\hat{i} + \hat{j} - 2\hat{k}$, $\vec{c} = 3\hat{i} + 2\hat{j} + \hat{k}$, find $\vec{a} \cdot \left(\vec{b} \times \vec{c}\right)$.

Solution:

Given $\vec{a} = \vec{i} - 2\vec{j} + 3\vec{k}$, $\vec{b} = 2\vec{i} + \vec{j} - 2\vec{k}$, $\vec{c} = 3\vec{i} + 2\vec{j} + \vec{k}$

$$\vec{b} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{vmatrix}$$

$$\vec{b} \times \vec{c} = 5\vec{i} - 8\vec{j} + \vec{k}$$

$$\vec{a} \cdot \left(\vec{b} \times \vec{c}\right) = 5 + 16 + 3 = 24$$

Aliter: $$\vec{a} \cdot \left(\vec{b} \times \vec{c}\right) = \left[\vec{a}\vec{b}\vec{c}\right] = \begin{vmatrix} 1 & -2 & 3 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{vmatrix} = 1(5) + 2(8) + 3(1) = 5 + 16 + 3 = 24$$

Question 2.

Find the volume of the parallelepiped whose coterminous edges are represented by the vectors $-6\hat{i} + 14\hat{j} + 10\hat{k}$, $14\hat{i} - 10\hat{j} - 6\hat{k}$ and $2\hat{i} + 4\hat{j} - 2\hat{k}$.

Solution:

Volume of the parallelepiped = $||\vec{a}, \vec{b}, \vec{c}]$

$$\left[\vec{a}\,\vec{b}\,\vec{c}\right] = \begin{vmatrix} -6 & 14 & 10 \\ 14 & -10 & -6 \\ 2 & 4 & -2 \end{vmatrix} = -6\,[20 + 24] - 14\,[-28 + 12] + 10\,[56 + 20]$$

= -264 + 224 + 760 = 720 cubic units

Question 3.

The volume of the parallelepiped whose coterminus edges are $7\vec{i}+\lambda\vec{j}-3\vec{k}$, $\vec{i}+2\vec{j}-\vec{k}$, $-3\vec{i}+7\vec{j}+5\vec{k}$ is 90 cubic units. Find the value of λ

Solution:

Given, Volume of the parallelepiped = 90 cubic units

(i.e), $\left[\vec{a}\vec{b}\vec{c}\right] = 90$

$$\begin{vmatrix} 7 & \lambda & -3 \\ 1 & 2 & -1 \\ -3 & 7 & 5 \end{vmatrix} = 90 \qquad \Rightarrow 7[10+7] - \lambda[5-3] - 3[7+6] = 90$$

$$\Rightarrow 119 - 2\lambda - 39 = 90$$

$$\Rightarrow -2\lambda = 10 \qquad \Rightarrow \lambda = \frac{-10}{2} = -5$$

Question 4.

If $\vec{a},\vec{b},\vec{c}$ are three non-coplanar vectors represented by concurrent edges of a parallelepiped of volume 4 cubic units, find the value of $(\vec{a}+\vec{b})\cdot(\vec{b}\times\vec{c})+(\vec{b}+\vec{c})\cdot(\vec{c}\times\vec{a})+(\vec{c}+\vec{a})\cdot(\vec{a}\times\vec{b})$

Solution:

Let $\vec{a},\vec{b},\vec{c}$ be the concurrent edges of parallelepiped

Given volume of parallelepiped = 4 cubic units

$$\left|\left[\vec{a},\vec{b},\vec{c}\right]\right| = 4$$

But, $\left[\vec{a}\,\vec{b}\,\vec{c}\right] = \pm 4$

$$(\vec{a}+\vec{b})\cdot(\vec{b}\times\vec{c}) = \vec{a}\cdot(\vec{b}\times\vec{c})+\vec{b}\cdot(\vec{b}\times\vec{c}) = \left[\vec{a}\,\vec{b}\,\vec{c}\right] + \left[\vec{b}\,\vec{b}\,\vec{c}\right] = \left[\vec{a}\,\vec{b}\,\vec{c}\right] + 0$$

$$= \left[\vec{a}\,\vec{b}\,\vec{c}\right] \qquad \text{...(1)}$$

Similarly $(\vec{b}+\vec{c})\cdot(\vec{c}\times\vec{a}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \left[\vec{b}\,\vec{c}\,\vec{a}\right] = \left[\vec{a}\,\vec{b}\,\vec{c}\right]$ ...(2)

$$(\vec{c}+\vec{a})\cdot(\vec{a}\times\vec{b}) = \vec{c}\cdot(\vec{a}\times\vec{b}) = \left[\vec{c}\,\vec{a}\,\vec{b}\right] = \left[\vec{a}\,\vec{b}\,\vec{c}\right] \qquad \text{...(3)}$$

So, $(1)+(2)+(3) = 2\left[\vec{a}\,\vec{b}\,\vec{c}\right] = 3(\pm 4) \Rightarrow \pm 12$

Question 5.

Find the altitude of a parallelepiped determined by the vectors $\vec{a} = -2\hat{i}+5\hat{j}+3\hat{k}$, $\vec{b} = \hat{i}+3\hat{j}-2\hat{k}$ and $\vec{c} = -3\vec{i}+\vec{j}+4\vec{k}$ if the base is taken as the parallelogram determined by b and c.

Solution:

Volume = Base Area × Height

$|[\vec{a}, \vec{b}, \vec{c}]| = |\vec{b} \times \vec{c}| \times$ Height

$$\left|\left[\vec{a}, \vec{b}, \vec{c}\right]\right| = \begin{vmatrix} -2 & 5 & 3 \\ 1 & 3 & -2 \\ -3 & 1 & 4 \end{vmatrix} = -2(12+2) - 5(4-6) + 3(1+9)$$

$$= -28 + 10 + 30 = 12$$

$$\vec{b} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & -2 \\ -3 & 1 & 4 \end{vmatrix} = \vec{i}(14) - \vec{j}(-2) + \vec{k}(10)$$

$$\vec{b} \times \vec{c} = 14\vec{i} + 2\vec{j} + 10\vec{k}$$

$$\left|\vec{b} \times \vec{c}\right| = \sqrt{196+4+100} = \sqrt{300}$$

$$\left(\sqrt{300}\right) \times (\text{height}) = 12$$

$$\text{Height} = \frac{12}{\sqrt{300}} = \frac{12}{10\sqrt{3}} = \frac{6}{5\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{6\sqrt{3}}{5 \times 3} = \frac{2\sqrt{3}}{5}$$

Question 6.

Determine whether the three vectors $2\hat{i} + 3\hat{j} + \hat{k}, \hat{i} - 2\hat{j} + 2\hat{k}$ and $3\hat{i} + \hat{j} + 3\hat{k}$ are coplanar.

Solution:

$$\text{Let } \vec{a} = 2\vec{i} + 3\vec{j} + \vec{k}$$
$$\vec{b} = \vec{i} - 2\vec{j} + 2\vec{k}$$
$$\vec{c} = 3\vec{i} + \vec{j} + 3\vec{k}$$

we know that $\vec{a}, \vec{b}, \vec{c}$ are coplanar if and only if $\left[\vec{a}\,\vec{b}\,\vec{c}\right] = 0$

$$\left[\vec{a}\,\vec{b}\,\vec{c}\right] = \begin{vmatrix} 2 & 3 & 1 \\ 1 & -2 & 2 \\ 3 & 1 & 3 \end{vmatrix} = 2(-6-2) - 3(3-6) + (1+6) = -16 + 9 + 7 = 0$$

$\therefore$ Gives vectors are coplanar

Question 7.

Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$. If $c_1 = 1$ and $c_2 = 2$, find c_3 such that $\vec{a}, \vec{b}$ and $\vec{c}$ and c are coplanar.

Solution:

Given $\vec{a} = \vec{i} + \vec{j} + \vec{k}$

$\vec{b} = \vec{i}$

$\vec{c} = c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$ are coplanar.

But $c_1 = 1$ and $c_2 = 2$

So $\vec{c} = \vec{i} + 2\vec{j} + c_3\vec{k}$

We know that $\left[\vec{a}\,\vec{b}\,\vec{c}\right] = 0$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 2 & c_3 \end{vmatrix} = 0$$

$1[0] - 1[c_3] + 1[2] = 0 \qquad \Rightarrow -c_3 + 2 = 0 \qquad \Rightarrow c_3 = 2$

Question 8.

If $\vec{a} = \hat{i} - \hat{k}, \vec{b} = x\hat{i} + \hat{j} + (1-x)\hat{k}, \vec{c} = y\hat{i} + x\hat{j} + (1+x-y)\hat{k}$, show that $[\vec{a}\vec{b}\vec{c}]$ depends neither x nor y.

Solution:

Given $\vec{a} = \vec{i} - \vec{k}$

$\vec{b} = x\vec{i} + \vec{j} + (1-x)\vec{k}$

$\vec{c} = y\vec{i} + x\vec{j} + (1+x-y)\vec{k}$

$$\left[\vec{a}\,\vec{b}\,\vec{c}\right] = \begin{vmatrix} 1 & 0 & -1 \\ x & 1 & 1-x \\ y & x & 1+x-y \end{vmatrix} = 1(1+x-y) - x(1-x) - 1(x^2 - y)$$

$$= 1 + x - y - x + x^2 - x^2 + y$$

$$\left[\vec{a}\,\vec{b}\,\vec{c}\right] = 1$$

$\therefore$ Clearly $\left[\vec{a}\,\vec{b}\,\vec{c}\right]$ depends on neither x nor y

Question 9.

If the vectors

$a\hat{i} + a\hat{j} + c\hat{k}$, $\hat{i} + \hat{k}$ **and** $c\hat{i} + c\hat{j} + b\hat{k}$

are coplanar, prove that c is the geometric mean of a and b.

Solution:

$$\text{Let } \vec{a}_1 = a\vec{i} + a\vec{j} + c\vec{k}$$
$$\vec{a}_2 = \vec{i} + \vec{k}$$
$$\vec{a}_3 = c\vec{i} + c\vec{j} + b\vec{k}$$

But $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$ are coplanar (Given)

So, $[\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3] = 0$

$$\begin{vmatrix} a & a & c \\ 1 & 0 & 1 \\ c & c & b \end{vmatrix} = 0 \qquad \Rightarrow a[0-c] - a[b-c] + c[c-0] = 0$$

$$\Rightarrow -ac - ab + ac + c^2 = 0$$
$$\Rightarrow c^2 = ab \qquad \Rightarrow c = \sqrt{ab}$$

∴ c is the geometric means of 'a' and 'b'.

Question 10.
Let $\vec{a}, \vec{b}, \vec{c}$ be three non-zero vectors such that $\vec{c}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$. If the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{6}$ show that $\left[\vec{a}, \vec{b}, \vec{c}\right]^2 = \frac{1}{4}|\vec{a}|^2|\vec{b}|^2$.
Solution:
Given $\vec{c}$ is perpendicular to both $\vec{a}$ and $\vec{b}$. So $\vec{c}$ is parallel to $\vec{a} \times \vec{b}$

$$\left[\vec{a}\,\vec{b}\,\vec{c}\right] = \vec{a} \cdot \left(\vec{b} \times \vec{c}\right)$$
$$\left|\left[\vec{a}, \vec{b}, \vec{c}\right]\right| = |\vec{a}||\vec{b} \times \vec{c}| = |\vec{a}||\vec{b}||\vec{c}|\sin\left(\frac{\pi}{6}\right)$$
$$\left|\left[\vec{a}, \vec{b}, \vec{c}\right]\right| = |\vec{a}||\vec{b}||\vec{c}|\left(\frac{1}{2}\right)$$

Squaring on both sides $\left[\vec{a}, \vec{b}, \vec{c}\right]^2 = |\vec{a}|^2|\vec{b}|^2|\vec{c}|^2\frac{1}{4}$ (Since $|\vec{c}| = 1$)

$$\left[\vec{a}, \vec{b}, \vec{c}\right]^2 = \frac{1}{4}|\vec{a}|^2|\vec{b}|^2$$

Additional Problems

Question 1.
If the edges $\vec{a} = -3\vec{i} + 7\vec{j} + 5\vec{k}$, $\vec{b} = -5\vec{i} + 7\vec{j} - 3\vec{k}$, $\vec{c} = 7\vec{i} - 5\vec{j} - 3\vec{k}$ meet a vertex, find the volume of the parallelepiped.
Solution:

$$\left[\vec{a},\vec{b},\vec{c}\right] = \begin{vmatrix} -3 & 7 & 5 \\ -5 & 7 & -3 \\ 7 & -5 & -3 \end{vmatrix} = -264$$

Volume of the parallelepiped =
The volume cannot be negative
∴ Volume of parallelepiped = 264 cu. units

Question 2.
If $\vec{x}\cdot\vec{a}=0$, $\vec{x}\cdot\vec{b}=0$, $\vec{x}\cdot\vec{c}=0$ and $\vec{x} \neq \vec{0}$ then show that $\vec{a},\vec{b},\vec{c}$ are coplanar.
Solution:
$\vec{x}\cdot\vec{a}=0$, $\vec{x}\cdot\vec{b}=0$ implies $\vec{a}$ and $\vec{b}$ are ⊥r to $\vec{x}$
∴ $\vec{a} \times \vec{b}$ is parallel to $\vec{x}$

$$\vec{x} = \lambda(\vec{a} \times \vec{b})$$

Now $\vec{x}\cdot\vec{c} = 0 \Rightarrow \lambda\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}=0$

$\Rightarrow \left[\vec{a}\,\vec{b}\,\vec{c}\right] = 0 \Rightarrow \vec{a},\vec{b},\vec{c}$ are coplanar.

Question 3.
The volume of a parallelepiped whose edges are represented by $-12\vec{i}+\lambda\vec{k},\, 3\vec{j}-\vec{k}, 2\vec{i}+\vec{j}-15\vec{k}$ is 546. Find the value of λ.
Solution:

$$\left[\vec{a}\,\vec{b}\,\vec{c}\right] = \begin{vmatrix} -12 & 0 & \lambda \\ 0 & 3 & -1 \\ 2 & 1 & -15 \end{vmatrix}$$

Volume of the parallelepiped =
= -12 [-45 + 1] – 0 () + λ [0 – 6] = -12 (-44) -6 λ
= 528 – 6λ = 546 (given)
⇒ -6λ = 546 – 528 = 18
∴ $\lambda = \frac{18}{-6} = -3$

Question 4.
Prove that $|\vec{a}\vec{b}\vec{c}|$ = abc if and only if $\vec{a},\vec{b},\vec{c}$ are mutually perpendicular.
Solution:

(*i*) Let $\vec{a}, \vec{b}, \vec{c}$ be three mutually perpendicular vectors.

$$i.e., \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{c} = \vec{a}\cdot\vec{c} = 0$$

$$\text{Now } \left[\vec{a}\,\vec{b}\,\vec{c}\right] = \vec{a}\cdot\left(\vec{b}\times\vec{c}\right) = \left|\vec{a}\right|\,\left|\vec{b}\times\vec{c}\right| \cos\phi \qquad ...(1)$$

where ϕ is the angle between $\vec{a}$ and $\vec{b}\times\vec{c}$.

Squaring on both sides $\left[\vec{a},\vec{b},\vec{c}\right]^2 = \left|\vec{a}\right|^2\left|\vec{b}\right|^2\left|\vec{c}\right|^2\frac{1}{4}$ (Since $|\vec{c}| = 1$)

$$\left[\vec{a},\vec{b},\vec{c}\right]^2 = \frac{1}{4}\left|\vec{a}\right|^2\left|\vec{b}\right|^2$$

Question 5.

Show that the points (1, 3, 1), (1, 1, -1), (-1, 1, 1), (2, 2, -1) are lying on the same plane.

Solution:

To prove that the points A, B, C, D are coplanar, we have to prove that $\left[\overline{AB}\ \overline{AC}\ \overline{AD}\right] = 0$.

$$\overrightarrow{AB} = \overrightarrow{OB}-\overrightarrow{OA} = \left(\vec{i}+\vec{j}-\vec{k}\right)-\left(\vec{i}+3\vec{j}+\vec{k}\right) = -2\vec{j}-2\vec{k}$$

$$\overrightarrow{AC} = \overrightarrow{OC}-\overrightarrow{OA} = \left(-\vec{i}+\vec{j}+\vec{k}\right)-\left(\vec{i}+3\vec{j}+\vec{k}\right) = -2\vec{i}-2\vec{j}$$

$$\overrightarrow{AD} = \overrightarrow{OD}-\overrightarrow{OA} = \left(2\vec{i}+2\vec{j}-\vec{k}\right)-\left(\vec{i}+3\vec{j}+\vec{k}\right)=\vec{i}-\vec{j}-2\vec{k}$$

$$\left[\overline{AB}, \overline{AC}, \overline{AD}\right] = \begin{vmatrix} 0 & -2 & -2 \\ -2 & -2 & 0 \\ 1 & -1 & -2 \end{vmatrix} = 0(\ ) + 2[4-0] - 2[2+2] = 8 - 8 = 0$$

$\Rightarrow$ The points A, B, C, D are lying on the same plane.

Question 6.

If $\vec{a} = 2\vec{i}+3\vec{j}-5\vec{k}$, $\vec{b} = -\vec{i}+\vec{j}+2\vec{k}$ **and** $\vec{c} = 4\vec{i}-2\vec{j}+3\vec{k}$, **show that** $\left(\vec{a}\times\vec{b}\right)\times\vec{c} \neq \vec{a}\times\left(\vec{b}\times\vec{c}\right)$.

Solution:

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -5 \\ -1 & 1 & 2 \end{vmatrix} = \vec{i}(6+5)-\vec{j}(4-5)+\vec{k}(2+3) = 11\vec{i}+\vec{j}+5\vec{k}$$

So, $$\left(\vec{a}\times\vec{b}\right)\times\vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 11 & 1 & 5 \\ 4 & -2 & 3 \end{vmatrix} = \vec{i}(3+10)-\vec{j}(33-20)+\vec{k}(-22-4)$$

$$= 13\vec{i}-13\vec{j}-26\vec{k} \qquad \text{...(1)}$$

$$\vec{b}\times\vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 2 \\ 4 & -2 & 3 \end{vmatrix} = \vec{i}(3+4)-\vec{j}(-3-8)+\vec{k}(2-4) = 7\vec{i}+11\vec{j}-2\vec{k}$$

So, $$\vec{a}\times\left(\vec{b}\times\vec{c}\right) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -5 \\ 7 & 11 & -2 \end{vmatrix} = \vec{i}(-6+55)-\vec{j}(-4+35)+\vec{k}(22-21)$$

$$= 49\vec{i}-31\vec{j}+\vec{k} \qquad \text{...(2)}$$

$$(1)\neq(2) \qquad \Rightarrow \left(\vec{a}\times\vec{b}\right)\times\vec{c} \neq \vec{a}\times\left(\vec{b}\times\vec{c}\right)$$

Ex 6.3

Question 1.

If $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$, $\vec{b} = 2\hat{i} + \hat{j} - 2\hat{k}$, $\vec{c} = 3\hat{i} + 2\hat{j} + \hat{k}$, **find (*i*)** $(\vec{a} \times \vec{b}) \times \vec{c}$ **(*ii*)** $\vec{a} \times (\vec{b} \times \vec{c})$

Solution:

Given $\vec{a} = \vec{i} - 2\vec{j} + 3\vec{k}$, $\vec{b} = 2\vec{i} + \vec{j} - 2\vec{k}$, $\vec{c} = 3\vec{i} + 2\vec{j} + \vec{k}$

(*i*) $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{b} \cdot \vec{c})\vec{a} = (3 - 4 + 3)\vec{b} - (6 + 2 - 2)\,\vec{a} = 2\vec{b} - 6\vec{a}$

$= 4\vec{i} + 2\vec{j} - 4\vec{k} - 6\vec{i} + 12\vec{j} - 18\vec{k} = -2\vec{i} + 14\vec{j} - 22\vec{k}$

(*ii*) $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

$= (3 - 4 + 3)\,\vec{b} - (2 - 2 - 6)\,\vec{c} = 2\vec{b} + 6\,\vec{c}$

$= 4\vec{i} + 2\vec{j} - 4\vec{k} + 18\vec{i} + 12\vec{j} + 6\vec{k} = 22\vec{i} + 14\vec{j} + 2\vec{k}$

Question 2.

For any vector $\vec{a}$, prove that $\hat{i} \times (\vec{a} \times \hat{i}) + \hat{j} \times (\vec{a} \times \hat{j}) + \hat{k} \times (\vec{a} \times \hat{k}) = 2\vec{a}$.

Solution:

Let $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

$\vec{i} \times (\vec{a} \times \vec{i}) = (\vec{i} \cdot \vec{i})\vec{a} - (\vec{i} \cdot \vec{a})\vec{i} = \vec{a} - a_1\vec{i}$ ...(1)

Similarly, $\vec{j} \times (\vec{a} \times \vec{j}) = \vec{a} - a_2\vec{j}$ ...(2)

$\vec{k} \times (\vec{a} \times \vec{k}) = \vec{a} - a_3\vec{k}$ ...(3)

$(1) + (2) + (3) \Rightarrow \hat{i} \times (\vec{a} \times \hat{i}) + \hat{j} \times (\vec{a} \times \hat{j}) + \hat{k} \times (\vec{a} \times \hat{k}) = 3\vec{a} - (a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) = 3\vec{a} - \vec{a} = 2\vec{a}$

Question 3.

Prove that $[\vec{a} - \vec{b}, \vec{b} - \vec{c}, \vec{c} - \vec{a}] = 0$

Solution:

$$\text{LHS} = \left[\vec{a}-\vec{b}, \vec{b}-\vec{c}, \vec{c}-\vec{a}\right]$$

$$= (\vec{a}-\vec{b})\cdot\left[\left(\vec{b}-\vec{c}\right)\times\left(\vec{c}-\vec{a}\right)\right]$$

$$= (\vec{a}-\vec{b})\cdot\left[\left(\vec{b}\times\vec{c}\right)-\left(\vec{b}\times\vec{a}\right)-\left(\vec{c}\times\vec{c}\right)+\left(\vec{c}\times\vec{a}\right)\right] \qquad [\text{Since } \vec{c}\times\vec{c}=0]$$

$$= \vec{a}\cdot\left(\vec{b}\times\vec{c}\right)-\vec{a}\cdot\left(\vec{b}\times\vec{a}\right)+\vec{a}\cdot(\vec{c}\times\vec{a})-\vec{b}\cdot\left(\vec{b}\times\vec{c}\right)+\vec{b}\cdot\left(\vec{b}\times\vec{c}\right)-\vec{b}\cdot(\vec{c}\times\vec{a})$$

$$= \left[\vec{a}\ \vec{b}\ \vec{c}\right]-\left[\vec{b}\ \vec{c}\ \vec{a}\right] = \left[\vec{a}\ \vec{b}\ \vec{c}\right]-\left[\vec{a}\ \vec{b}\ \vec{c}\right]$$

$$= 0 = \text{RHS}$$

Question 4.

If $\vec{a}=2\hat{i}+3\hat{j}-\hat{k}$, $\vec{b}=3\hat{i}+5\hat{j}+2\hat{k}$, $\vec{c}=-\hat{i}-2\hat{j}+3\hat{k}$, verify that

(i) $(\vec{a}\times\vec{b})\times\vec{c}=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{b}\cdot\vec{c})\vec{a}$ ***(ii)*** $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$

Solution:

Given $\vec{a} = 2\vec{i}+3\vec{j}-\vec{k}$, $\vec{b} = 3\vec{i}+5\vec{j}-2\vec{k}$, $\vec{c} = -\vec{i}-2\vec{j}+3\vec{k}$

(*i*) $(\vec{a}\times\vec{b})\times\vec{c}=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{b}\cdot\vec{c})\vec{a}$

$$\vec{a}\times\vec{b} = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2 & 3 & -1\\ 3 & 5 & 2\end{vmatrix} = 11\vec{i}-7\vec{j}+\vec{k}$$

$$\left(\vec{a}\times\vec{b}\right)\times\vec{c} = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 11 & -7 & 1\\ -1 & -2 & 3\end{vmatrix} = -19\vec{i}-34\vec{j}-29\vec{k} \qquad \text{...(1)}$$

$$\left(\vec{a}\cdot\vec{c}\right)\vec{b}-\left(\vec{b}\cdot\vec{c}\right)\vec{a} = (-2-6-3)\ \vec{b}\ -(-3-10+6)\vec{a} = -11\vec{b}+7\vec{a}$$

$$= -33\vec{i}-55\vec{j}-22\vec{k}+14\vec{i}+21\vec{j}-7\vec{k}$$

$$= -19\vec{i}-34\vec{j}-29\vec{k} \qquad \text{...(2)}$$

From (1) & (2), we get

$$\left(\vec{a}\times\vec{b}\right)\times\vec{c} = \left(\vec{a}\cdot\vec{c}\right)\vec{b}-\left(\vec{b}\cdot\vec{c}\right)\vec{a}$$

(ii) $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$

$$\vec{b}\times\vec{c}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 3 & 5 & 2\\ -1 & -2 & 3\end{vmatrix}=19\vec{i}-11\vec{j}-\vec{k}$$

$$\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2 & 3 & -1\\ 19 & -11 & -1\end{vmatrix}=-14\vec{i}-17\vec{j}-79\vec{k} \quad ...(1)$$

$$\left(\vec{a}\cdot\vec{c}\right)\vec{b}-\left(\vec{a}\cdot\vec{b}\right)\vec{c}=(-2-6-3)\ \vec{b}-(6+15-2)\vec{c}=-11\vec{b}-19\vec{c}$$

$$=-33\vec{i}-55\vec{j}-22\vec{k}+19\vec{i}+38\vec{j}-57\vec{k}=-14\vec{i}-17\vec{j}-79\vec{k} \quad ...(2)$$

From (1) & (2), we get

$$\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\left(\vec{a}\cdot\vec{c}\right)\vec{b}-\left(\vec{a}\cdot\vec{b}\right)\vec{c}$$

Question 5.

$\vec{a}=2\hat{i}+3\hat{j}-\hat{k},\ \vec{b}=-\hat{i}+2\hat{j}-4\hat{k},\ \vec{c}=\hat{i}+\hat{j}+\hat{k}$ **then find the value of** $\left(\vec{a}\times\vec{b}\right)\cdot\left(\vec{a}\times\vec{c}\right)$**.**

Solution:

$$\text{Given } \vec{a} = 2\vec{i}+3\vec{j}-\vec{k},\ \vec{b} = -\vec{i}+2\vec{j}-4\vec{k},\ \vec{c} = \vec{i}+\vec{j}+\vec{k}$$

$$(\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{c}) = \begin{vmatrix}\vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{c}\\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{c}\end{vmatrix} = \begin{vmatrix}(4+9+1) & (2+3-1)\\ (-2+6+4) & (-1+2-4)\end{vmatrix}$$

$$= \begin{vmatrix}14 & 4\\ 8 & -3\end{vmatrix} = -42-32 = -74$$

Question 6.

If $\vec{a},\vec{b},\vec{c},\vec{d}$ are coplanar vectors, show that $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = \vec{0}$

Solution:

Given $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are coplanar vectors.

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a}\ \vec{b}\ \vec{d}]\vec{c}-[\vec{a}\ \vec{b}\ \vec{c}]\vec{d} = 0(\vec{c})-0(\vec{d})$$

$$= \vec{0}$$

Question 7.

If $\vec{a}=\hat{i}+2\hat{j}+3\hat{k}$, $\vec{b}=2\hat{i}-\hat{j}+\hat{k}$, $\vec{c}=3\hat{i}+2\hat{j}+\hat{k}$ **and** $\vec{a}\times(\vec{b}\times\vec{c})=l\vec{a}+m\vec{b}+n\vec{c}$, find the values of l, m, n

Solution:

$$\text{Given } \vec{a} = \vec{i}+2\vec{j}+3\vec{k},\ \vec{b} = 2\vec{i}-\vec{j}+\vec{k},\ \vec{c} = 3\vec{i}+2\vec{j}+\vec{k}$$

$$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c} = (3+4+3)\ \vec{b} - (2-2+3)\ \vec{c}$$

$$= 10\vec{b}-3\vec{c} = 20\vec{i}-10\vec{j}+10\vec{k}-9\vec{i}-6\vec{j}-3\vec{k} = 11\hat{i}-16\hat{j}+7\hat{k} \quad ...(1)$$

$$l\vec{a}+m\vec{b}+n\vec{c} = l(\vec{i}+2\vec{j}+3\vec{k})+m(2\vec{i}-\vec{j}+\vec{k})+n(3\vec{i}+2\vec{j}+\vec{k}) \quad ...(2)$$

From (1) & (2) compare the $\vec{i},\vec{j},\vec{k}$ co efficients.

$$l+2m+3n = 11 \quad ...(3)$$

$$2l-m+2n = -16 \quad ...(4)$$

$$3l+m+n = 7 \quad ...(5)$$

On solving (3) & (4)

$$\begin{array}{lrl}
(3) \Rightarrow & l + \cancel{2m} + 3n = & 11 \\
(4) \times 2 \Rightarrow & 4l - \cancel{2m} + 4n = & -32 \\
\hline
 & 5l \qquad + 7n = & -21 \\
\hline
\end{array}$$

$$5l + 7n = -21 \qquad \text{...(6)}$$

On solving (4) & (5)

$$\begin{array}{lrl}
(4) \Rightarrow & 2l - \cancel{m} + 2n = & -16 \\
(5) \Rightarrow & 3l + \cancel{m} + n = & 7 \\
\hline
 & 5l \qquad + 3n = & -9 \\
\hline
\end{array}$$

$$5l + 3n = -9 \qquad \text{...(7)}$$

On solving (6) & (7)

$$\begin{array}{lrl}
(6) \Rightarrow & \cancel{5l} + 7n = & -21 \\
(7) \Rightarrow & \cancel{5l} + 3n = & -9 \\
 & (-) \quad (-) & (+) \\
\hline
 & 4n = & -12 \\
\hline
\end{array} \quad \Rightarrow n = -3$$

Substituting $n = -3$ in (6)

$$(6) \Rightarrow \quad 5l + 7(-3) = -21 \qquad \Rightarrow 5l - 21 = -21$$

$$5l = 0 \qquad \Rightarrow l = 0$$

Substitute l & n values in (3)

$$(3) \Rightarrow \quad l + 2m + 3n = 11 \qquad \Rightarrow 2m - 9 = 11$$

$$2m = 20$$

$$m = \frac{20}{2} \qquad \Rightarrow m = 10$$

$\therefore \; l = 0; \; m = 10; \; n = -3$

Question 8.

If $\hat{a}, \hat{b}, \hat{c}$ are three unit vectors such that $\hat{b}$ and $\hat{c}$ are non-parallel and $\hat{a} \times (\hat{b} \times \hat{c}) = \frac{1}{2}\hat{b}$, find the angle between $\hat{a}$ and $\hat{c}$.

Solution:

Given $\vec{a}\times(\vec{b}\times\vec{c}) = \frac{1}{2}\vec{b}$ $\Rightarrow (\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c} = \frac{1}{2}\vec{b}$

Since $\hat{b}$ and $\hat{c}$ are non collinear vectors. So, equating corresponding coefficients on both sides.

$$\vec{a}\cdot\vec{c} = \frac{1}{2}$$

$$|\vec{a}|\cdot|\vec{c}|\cos\theta = \frac{1}{2} \qquad \Rightarrow \cos\theta = \frac{1}{2}$$

$$\theta = \cos^{-1}\left(\frac{1}{2}\right) \qquad \Rightarrow \theta = \left(\frac{\pi}{3}\right)$$

$\therefore$ $\hat{a}$ makes an angle with $\hat{c}$ is $\frac{\pi}{3}$

Additional Problems

Question 1.

If $\vec{a}=3\vec{i}+2\vec{j}-4\vec{k}$, $\vec{b}=5\vec{i}-3\vec{j}+6\vec{k}$, $\vec{c}=5\vec{i}-\vec{j}+2\vec{k}$, **find (*i*)** $\vec{a}\times(\vec{b}\times\vec{c})$ **(*ii*)** $(\vec{a}\times\vec{b})\times\vec{c}$ and show that they are not equal.

Solution:

(*i*)
$$\vec{b}\times\vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5 & -3 & 6 \\ 5 & -1 & 2 \end{vmatrix} = 20\vec{j}+10\vec{k}$$

$$\therefore \quad \vec{a}\times(\vec{b}\times\vec{c}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 2 & -4 \\ 0 & 20 & 10 \end{vmatrix} = 100\vec{i}-30\vec{j}+60\vec{k}$$

(*ii*)
$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 2 & -4 \\ 5 & -3 & 6 \end{vmatrix} = -38\vec{j}-19\vec{k}$$

$$(\vec{a}\times\vec{b})\times\vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -38 & -19 \\ 5 & -1 & 2 \end{vmatrix} = -95\vec{i}-95\vec{j}+190\vec{k}$$

Question 2.

. If $\vec{a}=2\vec{i}+3\vec{j}-\vec{k}$ **,** $\vec{b}=-2\vec{i}+5\vec{k}$ **,** $\vec{c}=\vec{j}-3\vec{k}$ **verify that** $\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\left(\vec{a}\cdot\vec{c}\right)\vec{b}-\left(\vec{a}\cdot\vec{b}\right)\vec{c}$ **.**

Solution:

$$\vec{b}\times\vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 0 & 5 \\ 0 & 1 & -3 \end{vmatrix} = \vec{i}(-5)-\vec{j}(6)+\vec{k}(-2)=-5\vec{i}-6\vec{j}-2\vec{k}$$

$$\text{So, } \vec{a}\times\left(\vec{b}\times\vec{c}\right) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -1 \\ -5 & -6 & -2 \end{vmatrix} = \vec{i}(-6-6)-\vec{j}(-4-5)+\vec{k}(-12+15)$$

$$= -12\vec{i}+9\vec{j}+3\vec{k} \quad \text{...(1)}$$

$$\vec{a}\cdot\vec{c} = \left(2\vec{i}+3\vec{j}-\vec{k}\right)\cdot\left(\vec{j}-3\vec{k}\right) = (2)\,(0)+3\,(1)+(-1)\,(-3)=3+3=6$$

$$\vec{a}\cdot\vec{b} = \left(2\vec{i}+3\vec{j}-\vec{k}\right)\cdot\left(-2\vec{i}+5\vec{k}\right)$$

$$= (2)\,(-2)+\ (3)\,(0)+(-1)\,(5)=-4+0-5=-9$$

$$\text{So, } (\vec{a}\cdot\vec{c})\,\vec{b}-(\vec{a}\cdot\vec{b})\,\vec{c} = 6\left[-2\vec{i}+5\vec{k}\right]-(-9)\left[\vec{j}-3\vec{k}\right] = 6\left[-2\vec{i}+5\vec{k}\right]+9\left[\vec{j}-3\vec{k}\right]$$

$$= -12\vec{i}+30\vec{k}+9\vec{j}-27\vec{k} = -12\vec{i}+9\vec{j}+3\vec{k} \quad \text{...(2)}$$

$$(1)=(2) \Rightarrow \quad \vec{a}\times\left(\vec{b}\times\vec{c}\right) = \left(\vec{a}\cdot\vec{c}\right)\vec{b}-\left(\vec{a}\cdot\vec{b}\right)\vec{c}$$

Ex 6.4

Question 1.
Find the non-parametric form of vector equation and Cartesian equations of the straight line passing through the point with position vector $4\hat{i}+3\hat{j}-7\hat{k}$ and parallel to the vector $2\hat{i}-6\hat{j}+7\hat{k}$
Solution:

Given $\quad \vec{a} = 4\vec{i}+3\vec{j}-7\vec{k}\,,\ \vec{b} = 2\vec{i}-6\vec{j}+7\vec{k}$

$\therefore$ Non- Parametric vector equation of the straight line is $(\vec{r}-\vec{a})\times\vec{b}=\vec{0}$

$$\left(\vec{r}-\left(4\vec{i}+3\vec{j}-7\vec{k}\right)\right)\times\left(2\vec{i}-6\vec{j}+7\vec{k}\right)=\vec{0}$$

Cartesian equation:

$$\frac{x-x_1}{b_1}=\frac{y-y_1}{b_2}=\frac{z-z_1}{b_3}$$

$$\frac{x-4}{2}=\frac{y-3}{-6}=\frac{z+7}{7}$$

$$\begin{pmatrix} x_1 & y_1 & z_1 \\ 4 & 3 & -7 \end{pmatrix} \quad \begin{pmatrix} b_1 & b_2 & b_3 \\ 2 & -6 & 7 \end{pmatrix}$$

Question 2.
Find the parametric form of vector equation and Cartesian equations of the straight line passing through the point (-2, 3, 4) and parallel to the straight line $\frac{x-1}{-4}=\frac{y+3}{5}=\frac{8-z}{6}$
Solution:

Given $\quad \vec{a} = -2\vec{i}+3\vec{j}+4\vec{k}\,,\ \vec{b} = -4\vec{i}+5\vec{j}+6\vec{k}$

$\therefore$ Parametric vector equation of the straight line is

$$\vec{r} = \vec{a}+t\vec{b}$$

$$\vec{r} = (-2\vec{i}+3\vec{j}+4\vec{k}) + t\,(-4\vec{i}+5\vec{j}+6\vec{k}),\ t\in\mathbb{R}.$$

Cartesian equation:

$$\frac{x-x_1}{b_1}=\frac{y-y_1}{b_2}=\frac{z-z_1}{b_3}$$

$$\frac{x+2}{-4}=\frac{y-3}{5}=\frac{z-4}{6}$$

$$\begin{pmatrix} x_1 & y_1 & z_1 \\ -2 & 3 & 4 \end{pmatrix} \quad \begin{pmatrix} b_1 & b_2 & b_3 \\ -4 & 5 & 6 \end{pmatrix}$$

Question 3.
Find the points where the straight line passes through (6, 7, 4) and (8, 4, 9) cuts the xz and yz planes.
Solution:
Given straight line passing through the points (6, 7, 4) and (8, 4, 9).
Direction ratio of the straight line joining these two points 2, -3, -5.
Cartesian equation:

$$\frac{x-6}{2} = \frac{y-7}{-3} = \frac{z-4}{5} \qquad \Rightarrow \frac{x-6}{2} = \frac{y-7}{-3} = \frac{z-4}{5} = t \text{ (say)}$$

$(2t+6, -3t+7, 5t+4)$

$\therefore$ (*i*) The straight line cuts the xz – plane.

So we get $y = 0$

$$-3t + 7 = 0$$

$$-3t = -7 \qquad \Rightarrow t = \frac{7}{3}$$

$$5t + 4 = 5\left(\frac{7}{3}\right) + 4 = \frac{35+12}{3} = \frac{47}{3}$$

$$2t + 6 = 2\left(\frac{7}{3}\right) + 6 = \frac{14+18}{3} = \frac{32}{3}$$

The required point $\left(\frac{32}{3}, 0, \frac{47}{3}\right)$

(ii) The straight Line cuts yz-plane
So we get x = 0
$2t + 6 = 0 \Rightarrow 2t = -6$
$t = -3$
$-3t + 7 = -3(-3) + 7 = 9 + 7 = 16$
$5t + 4 = 5(-3) + 4 = -15 + 4 = -11$
The required point (0, 16, -11).

Question 4.
Find the direction cosines of the straight line passing through the points (5, 6, 7) and (7,9,13). Also, find the parametric form of vector equation and Cartesian equations of the straight line passing through two given points.
Solution:
Given straight line passing through the points (5, 6, 7) and (7, 9, 13)
$\therefore$ d.r.s : 2, 3, 6

$$\text{d.c.s} = \left(\frac{2}{\sqrt{2^2+3^2+6^2}}, \frac{3}{\sqrt{2^2+3^2+6^2}}, \frac{6}{\sqrt{2^2+3^2+6^2}}\right)$$

$$\text{d.c.s} = \left(\frac{2}{7}, \frac{3}{7}, \frac{6}{7}\right)$$

∴ Parametric form of vector equation passing through give two points.

$$\vec{r} = \vec{a} + t\left(\vec{b} - \vec{a}\right)$$

Given $\vec{a} = 5\vec{i} + 6\vec{j} + 7\vec{k}$, $\vec{b} = 7\vec{i} + 9\vec{j} + 13\vec{k}$

$$\vec{r} = \left(5\vec{i} + 6\vec{j} + 7\vec{k}\right) + t\left(2\vec{i} + 3\vec{j} + 6\vec{k}\right)$$

Cartesian equation:

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

$$\frac{x-5}{2} = \frac{y-6}{3} = \frac{z-7}{6}$$

Note: Selection of $\vec{a}$ and $\vec{b}$ is your choice.

Question 5.
Find the acute angle between the following lines.

(i) $\vec{r} = (4\hat{i} - \hat{j}) + t(\hat{i} + 2\hat{j} - 2\hat{k})$, $\vec{r} = (\hat{i} - 2\hat{j} + 4\hat{k}) + s(-\hat{i} - 2\hat{j} + 2\hat{k})$

(ii) $\frac{x+4}{3} = \frac{y-7}{4} = \frac{z+5}{5}$, $\vec{r} = 4\hat{k} + t(2\hat{i} + \hat{j} + \hat{k})$.

(iii) 2x = 3y = -z and 6x = -y = -4z
Solution:

(*i*) Given $\vec{b} = \vec{i}+2\vec{j}-2\vec{k}$, $\vec{d} = -\vec{i}-2\vec{j}+2\vec{k}$

$$\cos\theta = \frac{|\vec{b}\cdot\vec{d}|}{|\vec{b}|\ |\vec{d}|} = \frac{|(-1-4-4)|}{\sqrt{1+4+4}\ \sqrt{1+4+4}} = \frac{9}{3\times 3}$$

$$\cos\theta = 1$$

$$\theta = \cos^{-1}(1) \qquad \Rightarrow \theta = 0^{\circ}$$

(*ii*) Given $\vec{b} = 3\vec{i}+4\vec{j}+5\vec{k}$, $\vec{d} = 2\vec{i}+\vec{j}+\vec{k}$

$$\cos\theta = \frac{|\vec{b}\cdot\vec{d}|}{|\vec{b}|\ |\vec{d}|} \qquad \text{...(1)}$$

$$\vec{b}\cdot\vec{d} = 6+4+5 = 15$$

$$|\vec{b}| = \sqrt{9+16+25} = \sqrt{50} = 5\sqrt{2}$$

$$|\vec{d}| = \sqrt{4+1+1} = \sqrt{6}$$

(1) $\Rightarrow$ $$\cos\theta = \frac{15}{(5\sqrt{2})(\sqrt{6})} = \frac{3}{2\sqrt{3}} \Rightarrow \theta = \cos^{-1}\left(\frac{3}{2\sqrt{3}}\right)$$

(*iii*)

$2x = 3y = -z$	$6x = -y = -4z$
$\dfrac{x}{1/2} = \dfrac{y}{1/3} = \dfrac{z}{-1}$	$\dfrac{x}{1/6} = \dfrac{y}{-1} = \dfrac{z}{-1/4}$
$\lvert\vec{b}\rvert = \frac{1}{2}\vec{i}+\frac{1}{3}\vec{j}-\vec{k}$	$\vec{d} = \frac{1}{6}\vec{i}-\vec{j}-\frac{1}{4}\vec{k}$

$$\vec{b}\cdot\vec{d} = \frac{1}{12}-\frac{1}{3}+\frac{1}{4} = \frac{1-4+3}{12} = 0$$

$$\cos\theta = \frac{|\vec{b}\cdot\vec{d}|}{|\vec{b}|\ |\vec{d}|} = \frac{0}{\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{1}{3}\right)^2+(-1)^2}\sqrt{\left(\frac{1}{6}\right)^2+(-1)^2+\left(\frac{-1}{4}\right)^2}}$$

$$\cos\theta = 0 \qquad \Rightarrow \theta = \cos^{-1}(0)$$

$$\theta = \frac{\pi}{2}$$

Question 6.
The vertices of ΔABC are A(7, 2, 1), B(6, 0, 3), and C(4, 2, 4) . Find $\angle$ABC.

Solution:

Let $\overrightarrow{OA} = 7\vec{i}+2\vec{j}+\vec{k}$, $\overrightarrow{OB} = 6\vec{i}+0\vec{j}+3\vec{k}$, $\overrightarrow{OC} = 4\vec{i}+2\vec{j}+4\vec{k}$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -\vec{i}-2\vec{j}+2\vec{k}$$

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = -2\vec{i}+2\vec{j}+\vec{k}$$

$$\cos\theta = \frac{\overrightarrow{AB}\cdot\overrightarrow{BC}}{|\overrightarrow{AB}|\cdot|\overrightarrow{BC}|} \quad \Rightarrow \cos\theta = \frac{2-4+2}{\sqrt{1+4+4}\sqrt{4+4+1}}$$

$$\cos\theta = 0$$

$$\theta = \cos^{-1}(0) \quad \Rightarrow \theta = \frac{\pi}{2}$$

Question 7.
If the straight line joining the points (2, 1, 4) and (a – 1, 4, -1) is parallel to the line joining the points (0, 2, b – 1) and (5, 3, -2), find the values of a and b.
Solution:

Let $\overrightarrow{OA} = 2\hat{i}+\hat{j}+4\vec{k}$, $\overrightarrow{OB} = (a-1)\vec{i}+4\vec{j}-\vec{k}$

So, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (a-3)\,\vec{i}+3\vec{j}-5\vec{k}$

Let $\overrightarrow{OC} = 2\vec{i}+(b-1)\vec{k}$

$$\overrightarrow{OD} = 5\vec{i}+3\vec{j}-2\vec{k}$$

$$\overrightarrow{CD} = \overrightarrow{OD}-\overrightarrow{OC} = 5i+j+(-b-1)\vec{k}$$

Given $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are parallel.

$$\overrightarrow{AB} = m\,\overrightarrow{CD}$$

$$(a-3)\;\vec{i}+3\vec{j}-5\vec{k} = m\left(5\vec{i}+\vec{j}+(-b-1)\vec{k}\right)$$

Equating $\vec{i},\vec{j},\vec{k}$ coefficient on both side

$a-3 = 5m$
$a-3 = 5(3)$
$a-3 = 15$
$a = 18$

$3 = m$

$m(-b-1) = -5$
$3(-b-1) = -5 \quad \Rightarrow -b-1 = \frac{-5}{3}$
$-b = \frac{-5}{3}+1 = \frac{-5+3}{3} = \frac{-2}{3} \Rightarrow b = \frac{2}{3}$

$\therefore\ a = 18;\ b = \frac{2}{3}$

Question 8.
If the straight lines $\frac{x-5}{5m+2} = \frac{2-y}{5} = \frac{1-z}{-1}$ **and** $x = \frac{2y+1}{4m} = \frac{1-z}{-3}$ are perpendicular to each other, find the value of m.

Solution:

$$\frac{x-5}{5m+2} = \frac{2-y}{5} = \frac{1-z}{-1} \text{ and } x = \frac{2y+1}{4m} = \frac{1-z}{-3} \text{ are perpendicular.}$$

Rewrite the above equations.

$$\frac{x-5}{5m+2} = \frac{y-2}{-5} = \frac{z-1}{1} \text{ and } x = \frac{y+\frac{1}{2}}{2m} = \frac{z-1}{3}$$

So, we get,
$$\vec{b} = (5m+2)\vec{i} - 5\vec{j} + \vec{k}$$
$$\vec{d} = \vec{i} + 2m\vec{j} + 3\vec{k}$$
$$\vec{b} \cdot \vec{d} = 0 \text{ (Given)}$$
$$5m + 2 - 10m + 3 = 0 \quad \Rightarrow -5m + 5 = 0$$
$$-5m = -5 \quad \Rightarrow m = 1$$

Question 9.
Show that the points (2, 3, 4),(-1, 4, 5) and (8, 1, 2) are collinear.
Solution:
Given points are (2, 3, 4), (-1, 4, 5) and (8, 1, 2) Equation of the line joining of the first and second point is
$\frac{x-2}{-3} = \frac{y-3}{1} = \frac{z-4}{1}$ = m (say)
(-3m + 2, m + 3, m + 4)
On putting m = -2, we get the third point is (8, 1, 2)
∴ Given points are collinear.

Additional Problems

Question 1.
Find the vector and cartesian equations of the straight line passing through the point A with position vector $3\vec{i} - \vec{j} + 4\vec{k}$ and parallel to the vector $-5\vec{i} + 7\vec{j} + 3\vec{k}$.
Solution:
We know that vector equation of the line through the point with position vector $\vec{a}$ and parallel to $\vec{v}$ is given by $\vec{r} = \vec{a} + t\vec{v}$ where t is a scalar.
Here $\vec{a} = 3\vec{i} - \vec{j} + 4\vec{k}$ and $\vec{v} = -5\vec{i} + 7\vec{j} + 3\vec{k}$
Vector equation of the line is
$\vec{r} = (3\vec{i} - \vec{j} + 4\vec{k}) + t(-5\vec{i} + 7\vec{j} + 3\vec{k})$ ……………… (1)
The cartesian equation of the line passing through (xp yx, zx) and parallel to a vector whose d.r.s are l, m, n

is

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$

Here $(x_1, y_1, z_1) = (3, -1, 4)$ and $(l, m, n) = (-5, 7, 3)$

$\therefore$ The required equation is $\frac{x-3}{-5}=\frac{y+1}{7}=\frac{z-4}{3}$

Question 2.
Find the vector and cartesian equations of the straight line passing through the points (- 5, 2, 3) and (4, – 3, 6).
Solution:
Vector equation of the straight line passing through two points with position vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{r} = \vec{a}+t(\vec{b}-\vec{a})$$

Here $\vec{a} = -5\vec{i}+2\vec{j}+3\vec{k}$ and $\vec{b} = 4\vec{i}-3\vec{j}+6\vec{k}$

$$\vec{b}-\vec{a} = 9\vec{i}-5\vec{j}+3\vec{k}$$

$\therefore$ Vector equation of the line is

$$\vec{r} = -5\vec{i}+2\vec{j}+3\vec{k}+t(9\vec{i}-5\vec{j}+3\vec{k})$$

(or) $\vec{r} = (1-t)-5\vec{i}+2\vec{j}+3\vec{k}+t(4\vec{i}-3\vec{j}+6\vec{k})$

Cartesian Form:

The required equation is $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$

Here $(x_1, y_1, z_1) = (-5, 2, 3)$; $(x_2, y_2, z_2) = (4, -3, 6)$

$\therefore \frac{x+5}{9}=\frac{y-2}{-5}=\frac{z-3}{3}$ is the cartesian equation of the line.

Question 3.
Find the angle between the lines.

$\vec{r}=3\vec{i}+2\vec{j}-\vec{k}+t(\vec{i}+2\vec{j}+2\vec{k})$ **and** $\vec{r}=5\vec{j}+2\vec{k}+s(3\vec{i}+2\vec{j}+6\vec{k})$

Solution:

Let the given lines be in the direction of $\vec{u}$ and $\vec{v}$

Then $\vec{u} = \vec{i} + 2\vec{j} + 2\vec{k}$, $\vec{v} = 3\vec{i} + 2\vec{j} + 6\vec{k}$

Let θ be the angle between the given lines

$$\therefore \quad \cos\theta = \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}|\;|\vec{v}|}\right)$$

$$\vec{u}\cdot\vec{v} = 19;\ |\vec{u}| = 3;\ |\vec{v}| = 7$$

$$\therefore \quad \cos\theta = \frac{19}{21} \qquad \Rightarrow \theta = \cos^{-1}\left(\frac{19}{21}\right)$$

Question 4.

Find the angle between the following lines $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-4}{6}$ and $x+1 = \frac{y+2}{2} = \frac{z-4}{2}$

Solution:

Angle between two lines is the same as angle between their parallel vectors.

The parallel vectors are: $a = 2\vec{i} + 3\vec{j} + 6\vec{k}$ and $b = \vec{i} + 2\vec{j} + 2\vec{k}$.

If θ is the angle between $\vec{a}$ and $\vec{b}$, then $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}$.

$$\vec{a}\cdot\vec{b} = (2)(1) + (3)(2) + (6)(2) = 2 + 6 + 12 = 20$$

$$|\vec{a}| = \sqrt{4+9+36} = \sqrt{49} = 7$$

$$|\vec{b}| = \sqrt{1+4+4} = \sqrt{9} = 3$$

$$\text{So, } \cos\theta = \frac{20}{7\times 3} = \frac{20}{21} \qquad \Rightarrow \theta = \cos^{-1}\frac{20}{21}$$

Ex 6.5

Question 1.
Find the parametric form of vector equation and Cartesian equations of a straight line passing through (5, 2, 8) and is perpendicular to the straight lines

$\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + s(2\hat{i} - 2\hat{j} + \hat{k})$ **and** $\vec{r} = (2\hat{i} - \hat{j} - 3\hat{k}) + t(\hat{i} + 2\hat{j} + 2\hat{k})$.

Solution:

Given point $\quad \vec{a} = 5\vec{i} + 2\vec{j} + 8\vec{k}$, $\vec{b} = 2\vec{i} + \vec{j} - 2\vec{k}$, $\vec{d} = \vec{i} + 2\vec{j} + 2\vec{k}$

$$\vec{b} \times \vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -2 & 1 \\ 1 & 2 & 2 \end{vmatrix} = -6\vec{i} - 3\vec{j} + 6\vec{k}$$

∴ This' vector is perpendicular to both the given straight lines.
∴ The required straight line is

$\vec{r} = \vec{a} + t(\vec{b} \times \vec{d})$

$$\vec{r} = (5\vec{i} + 2\vec{j} + 8\vec{k}) + t(-6\vec{i} - 3\vec{j} + 6\vec{k}) \text{ (OR)}$$

Cartesian equation:

$$\vec{r} = (5\vec{i} + 2\vec{j} + 8\vec{k}) + t(2\vec{i} + \vec{j} - 2\vec{k}),\ t \in \mathbb{R}$$

$$\frac{x-5}{-6} = \frac{y-2}{-3} = \frac{z-8}{6} \text{ (OR)}$$

$$\frac{x-5}{2} = \frac{y-2}{1} = \frac{z-8}{-2}$$

Question 2.

Show that the lines $\vec{r} = (6\hat{i} + \hat{j} + 2\hat{k}) + s(\hat{i} + 2\hat{j} - 3\hat{k})$ **and** $\vec{r} = (3\hat{i} + 2\hat{j} - 2\hat{k}) + t(2\hat{i} + 4\hat{j} - 5\hat{k})$ are skew lines and hence find the shortest distance between them.

Solution:

Given
$$\vec{a} = 6\vec{i}+\vec{j}+2\vec{k}\,,\ \vec{b} = \vec{i}+2\vec{j}-3\vec{k}\,,$$
$$\vec{c} = 3\vec{i}+2\vec{j}-2\vec{k}\,,\ \vec{d} = 2\vec{i}+4\vec{j}-5\vec{k}$$
$$\vec{c}-\vec{a} = -3\vec{i}+\vec{j}-4\vec{k}$$
$$\left[(\vec{c}-\vec{a})\ \vec{b}\ \vec{d}\right] = \begin{vmatrix} -3 & 1 & -4 \\ 1 & 2 & -3 \\ 2 & 4 & -5 \end{vmatrix} = -3\,(-10+12) - 1\,(-5+6) - 4\,(4-4)$$
$$= -6 - 1 \quad \Rightarrow -7 \neq 0$$

Given lines are skew lines

$$\vec{b}\times\vec{d} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -3 \\ 2 & 4 & -5 \end{vmatrix} = \vec{i}(-10+12) - \vec{j}(-5+6) + \vec{k}(4-4)$$
$$\vec{b}\times\vec{d} = 2\vec{i}-\vec{j}$$
$$|\vec{b}\times\vec{d}| = \sqrt{4+1} = \sqrt{5}$$

Shortest distance between skew lines $= \dfrac{|[(\vec{c}-\vec{a})\vec{b}\ \vec{d}]|}{\sqrt{5}} = \dfrac{7}{\sqrt{5}}$ units.

Question 3.

If the two lines $\dfrac{x-1}{2} = \dfrac{y+1}{3} = \dfrac{z-1}{4}$ **and** $\dfrac{x-3}{1} = \dfrac{y-m}{2} = z$ intersect at a point, find the value of m.

Solution:

$$\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4} = s \text{ (say)}$$

$(2s + 1, 3s - 1, 4s + 1)$

$$\frac{x-3}{1} = \frac{y-m}{2} = z = t \text{ (say)}$$

$(t + 3, 2t + m, t)$

Given two lines are intersecting lines. So, equate the corresponding co-ordinates.

$2s + 1 = t + 3$ $2s - t = 2 \quad ...(1)$	$4s + 1 = t \quad ... (2)$	$3s - 1 = 2t + m \quad ... (3)$

Substitute (2) in (1)

$$2s - 4s - 1 = 2$$

$$-2s = 2 + 1 \qquad \Rightarrow s = \frac{-3}{2}$$

$$\text{sub } s = -\frac{3}{2} \text{ in (1)}$$

$$(1) \Rightarrow \quad 2\left(-\frac{3}{2}\right) - t = 2$$

$$-t = 2 + 3 \qquad \Rightarrow t = -5$$

Substitute $s = \frac{-3}{2}$ and $t = -5$ in (3)

$$(3) \Rightarrow \quad 3\left(-\frac{3}{2}\right) - 1 = 2(-5) + m$$

$$\frac{-9}{2} - 1 = -10 + m \qquad \Rightarrow \frac{-9}{2} + 9 = m$$

$$\frac{9}{2} = m$$

Question 4.

Show that the lines $\frac{x-3}{3} = \frac{y-3}{-1}$, $z - 1 = 0$ and $\frac{x-6}{2} = \frac{z-1}{3}$, $y - 2 = 0$ intersect. Also find the point of intersection

Solution:

$$\frac{x-3}{3} = \frac{y-3}{-1}, z-1=0 \qquad \Rightarrow z=1$$

$$\frac{x-6}{2} = \frac{z-1}{3}, y-2=0 \qquad \Rightarrow y=2$$

$$(x_1, y_1, z_1) = (3, 3, 1) \text{ and } (x_2, y_2, z_2) = (6, 2, 1)$$

$$(b_1, b_2, b_3) = (3, -1, 0) \text{ and } (d_1, d_2, d_3) = (2, 0, 3)$$

Condition for intersection of two lines

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$$

$$\begin{vmatrix} 3 & -1 & 0 \\ 3 & -1 & 0 \\ 2 & 0 & 3 \end{vmatrix} = 0 \qquad \text{Since } (R_1 \equiv R_2)$$

$\therefore$ Given two lines are intersecting lines.

Any point on the first line

$$\frac{x-3}{3} = \frac{y-3}{-1} = \lambda \text{ and } z = 1$$

$(3\lambda + 3, -\lambda + 3, 1)$

Any point on the Second line

$$\frac{x-6}{2} = \frac{z-1}{3} = \mu \text{ and } y = 2$$

$(2\mu + 6, 2, 3\mu + 1)$

$$\therefore 3\mu + 1 = 1 \qquad -\lambda + 3 = 2$$

$$3\mu = 0 \qquad -\lambda = -1$$

$$\mu = 0 \qquad \lambda = 1$$

$\therefore$ The required point of intersection is (6, 2, 1)

Question 5.

Show that the straight lines x + 1 = 2y = -12z and x = y + 2 = 6z – 6 are skew and hence find the shortest distance between them.

Solution:

$$x+1=2y=-12z \qquad x=y+2=6z-6$$

$$x+1=\frac{y}{1/2}=\frac{z}{-1/12} \qquad x=y+2=\frac{z-1}{1/6}$$

$(x_1, y_1, z_1) = (-1, 0, 0)$ and $(x_2, y_2, z_2) = (0, -2, 1)$

$(b_1, b_2, b_3) = (1, 1/2, -1/12)$ and $(d_1, d_2, d_3) = (1, 1, 1/6)$

Condition for skew lines

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} \neq 0 \text{ (OR) } \left[\left(\vec{c}-\vec{a}\right)\,\vec{b}\,\vec{d}\right] \neq 0$$

$$\left[\left(\vec{c}-\vec{a}\right)\,\vec{b}\,\vec{d}\right] = \begin{vmatrix} 1 & -2 & 1 \\ 1 & \frac{1}{2} & \frac{-1}{12} \\ 1 & 1 & \frac{1}{6} \end{vmatrix} = 1\left[\frac{1}{12}+\frac{1}{12}\right]+2\left[\frac{1}{6}+\frac{1}{12}\right]+1\left[1-\frac{1}{2}\right]$$

$$=\frac{1}{6}+\frac{2}{6}+\frac{2}{12}+1-\frac{1}{2}=\frac{1}{6}+\frac{2}{6}+\frac{1}{6}+1-\frac{1}{2}=\frac{2}{3}+\frac{1}{2}=\frac{7}{6}\neq 0$$

Given Lines are skew lines.

$$\vec{b}\times\vec{d} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & \frac{1}{2} & -\frac{1}{12} \\ 1 & 1 & \frac{1}{6} \end{vmatrix} = \vec{i}\left(\frac{1}{6}\right)-\vec{j}\left(\frac{3}{12}\right)+\vec{k}\left(\frac{1}{2}\right)=\frac{1}{6}\vec{i}-\frac{1}{4}\vec{j}+\frac{1}{2}\vec{k}$$

$$|\vec{b}\times\vec{d}| = \sqrt{\frac{1}{36}+\frac{1}{16}+\frac{1}{4}} = \sqrt{\frac{16+36+144}{576}} = \sqrt{\frac{196}{576}} = \frac{14}{24} = \frac{7}{12}$$

$$\text{Shortest distance between skew lines} = \frac{|[\vec{c}-\vec{a}\ \vec{b}\ \vec{d}]|}{|\vec{b}\times\vec{d}|} = \frac{7/6}{7/12} = \frac{7}{6}\times\frac{12}{7} = 2 \text{ units.}$$

Question 6.
Find the parametric form of vector equation of the straight line passing through (-1, 2, 1) and parallel to the straight line $\vec{r}=(2\hat{i}+3\hat{j}-\hat{k})+t(\hat{i}-2\hat{j}+\hat{k})$ and hence find the shortest distance between the lines.

Solution:

$$\text{Given point } \vec{a} = -\vec{i}+2\vec{j}+\vec{k}\text{ , } \vec{b} = \vec{i}-2\vec{j}+\vec{k}$$

$$\vec{r} = \vec{a}+t\vec{b}$$

$$\vec{r} = (-\vec{i}+2\vec{j}+\vec{k})+t(\vec{i}-2\vec{j}+\vec{k})$$

Parallel to the straight line

$$\vec{r} = (2\vec{i}+3\vec{j}-\vec{k})+t(\vec{i}-2\vec{j}+\vec{k})$$

$$\text{Here } \vec{c} = 2\vec{i}+3\vec{j}-\vec{k}$$

$$\vec{c}-\vec{a} = 3\vec{i}+\vec{j}-2\vec{k}$$

$$(\vec{c}-\vec{a})\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -2 \\ 1 & -2 & 1 \end{vmatrix} = -3\vec{i}-5\vec{j}-7\vec{k}$$

$$|(\vec{c}-\vec{a})\times\vec{b}| = \sqrt{9+25+49} = \sqrt{83}$$

$$|\vec{b}| = \sqrt{1+4+1} = \sqrt{6}$$

$$\text{Shortest distance between parallel lines} = \frac{|(\vec{c}-\vec{a})\times\vec{b}|}{|\vec{b}|} = \frac{\sqrt{83}}{\sqrt{6}} \text{ units.}$$

Question 7.

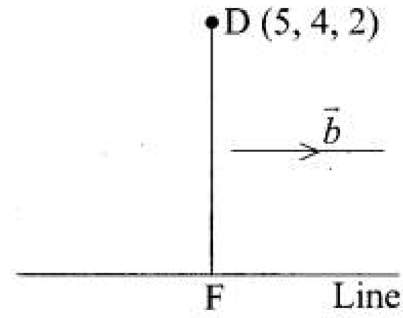
Find the foot of the perpendicular drawn from the point (5, 4, 2) to the line $\frac{x+1}{2} = \frac{y-3}{3} = \frac{z-1}{-1}$. Also, find the equation of the perpendicular.

Solution:

. Compare the given equation $\frac{x+1}{2}=\frac{y-3}{3}=\frac{z-1}{-1}$ in to $\frac{x-x_1}{b_1}=\frac{y-y_1}{b_2}=\frac{z-z_1}{b_3}$

$$\text{Here } \vec{b} = b_1\vec{i}+b_2\vec{j}+b_3\vec{k}$$

$$\text{So, } \vec{b} = 2\hat{i}+3\hat{j}-\vec{k}$$



Any point on the line

$$\frac{x+1}{2}=\frac{y-3}{3}=\frac{z-1}{-1} = t \text{ (say)}$$

The point F $(2t-1, 3t+3, -t+1)$

$$\overrightarrow{DF} = \overrightarrow{OF}-\overrightarrow{OD}$$

$$\overrightarrow{DF} = (2t-6)\vec{i}+(3t-1)\vec{j}+(-t-1)\vec{k}$$

Since $\vec{b}$ is perpendicular to $\overrightarrow{DF}$, we have

$$\vec{b}\cdot\overrightarrow{DF} = 0$$

$$2(2t-6)+3(3t-1)-1(-t-1) = 0$$

$$4t-12+9t-3+t+1 = 0$$

$$14t-14 = 0$$

$$t = 1$$

The co-ordinate of F is (1, 6, 0). The equation of the perpendicular is

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1} \text{ and D(5, 4, 2), F(1, 6, 0)}$$

$$\frac{x-5}{-4} = \frac{y-4}{2} = \frac{z-2}{-2}$$

Additional Problems

Question 1.
Find the shortest distance between the parallel line

$\vec{r}=(\vec{i}-\vec{j})+t(2\vec{i}-\vec{j}+\vec{k})$ **and** $\vec{r}=(2\vec{i}+\vec{j}+\vec{k})+s(2\vec{i}-\vec{j}+\vec{k})$

Solution:

Compare the given equations with $\vec{r} = \vec{a}_1 + t\vec{u}$ and $\vec{r} = \vec{a}_2 + s\vec{u}$,

$\vec{a}_1 = \vec{i} - \vec{j}$; $\vec{a}_2 = 2\vec{i} + \vec{j} + \vec{k}$ and $\vec{u} = 2\vec{i} - \vec{j} + \vec{k}$

$$\vec{a}_2 - \vec{a}_1 = \vec{i} + 2\vec{j} + \vec{k}$$

$$\vec{u} \times (\vec{a}_2 - \vec{a}_1) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = -3\vec{i} - \vec{j} + 5\vec{k}$$

$$|\vec{u} \times (\vec{a}_2 - \vec{a}_1)| = \sqrt{9+1+25} = \sqrt{35}$$

$$|\vec{u}| = \sqrt{4+1+1} = \sqrt{6}$$

The Distance between the parallel lines $= \dfrac{|\vec{u} \times (\vec{a}_2 - \vec{a}_1)|}{|\vec{u}|} = \sqrt{\dfrac{35}{\sqrt{6}}}$

Question 2.

Show that the two lines $\vec{r} = (\vec{i} - \vec{j}) + t(2\vec{i} + \vec{k})$ and $\vec{r} = (2\vec{i} - \vec{j}) + s(\vec{i} + \vec{j} - \vec{k})$ skew lines and find the distance between them.

Solution:

Compare the given equations with $\vec{r} = \vec{a}_1 + t\vec{u}$ and $\vec{r} = \vec{a}_2 + s\vec{v}$

$\vec{a}_1 = \vec{i} - \vec{j}$; $\vec{a}_2 = 2\vec{i} - \vec{j}$ and $\vec{u} = 2\vec{i} + \vec{k}$, $\vec{v} = \vec{i} + \vec{j} - \vec{k}$

$$\vec{a}_2 - \vec{a}_1 = \vec{i}$$

$$\left[\left(\vec{a}_2 - \vec{a}_1\right)\ \vec{u}\ \vec{v}\right] = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & -1 \end{vmatrix} = -1 \neq 0$$

$\therefore$ They are skew lines.

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & 1 \\ 1 & 1 & -1 \end{vmatrix} = -\vec{i} + 3\vec{j} + 2\vec{k}$$

$$|\vec{u} \times \vec{v}| = \sqrt{14}$$

Shortest distance between the lines $= \left[\dfrac{\left(\vec{a}_2 - \vec{a}_1\right)\ \vec{u}\ \vec{v}}{|\vec{u} \times \vec{v}|}\right]$ ...(1)

From (1) shortest distance between them is $\dfrac{1}{\sqrt{14}}$

Question 3.

Show that the lines $\frac{x-1}{3}=\frac{y-1}{-1}=\frac{z+1}{0}$ **and** $\frac{x-4}{2}=\frac{y}{0}=\frac{z+1}{3}$ intersect and hence find the point of intersection

Solution:

The condition for intersecting is $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$

Compare with $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ and $\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$, we get

$(x_1, y_1, z_1)=(1, 1, -1)$; $(x_2, y_2, z_2)=(4, 0, -1)$

$(l_1, m_1, n_1)=(3, -1, 0)$; $(l_2, m_2, n_2)=(2, 0, 3)$

The determinant becomes

$$\begin{vmatrix} 3 & -1 & 0 \\ 3 & -1 & 0 \\ 2 & 0 & 3 \end{vmatrix}=0$$

Note that $\vec{u}$ and $\vec{v}$ are not parallel.

$\therefore$ The lines are intersecting lines.

Point of intersection:

Take $\frac{x-1}{3}=\frac{y-1}{-1}=\frac{z+1}{0}=\lambda$

$\therefore$ Any point on the line is of the form $(3\lambda+1, -\lambda+1, -1)$

Take $\frac{x-4}{2}=\frac{y}{0}=\frac{z+1}{3}=\mu$

Any point on this line is of the form $(2\mu+4, 0, 3\mu-1)$

Since they are intersecting, for some λ, μ

$(3\lambda+1, -\lambda+1, -1)=(2\mu+4, 0, 3\mu-1) \Rightarrow \lambda=1$ and $\mu=0$

To find the point of intersection either take $\lambda=1$ or $\mu=0$

$\therefore$ The point of intersection is $(4, 0, -1)$,

Question 4.

Find the shortest distance between the skew lines.

$\vec{r}=(\vec{i}-\vec{j})+\lambda(2\vec{i}+\vec{j}+\vec{k})$ **and** $\vec{r}=(\vec{i}+\vec{j}-\vec{k})+\mu(2\vec{i}-\vec{j}-\vec{k})$

Solution:

Compare the given equation with $\vec{r} = \vec{a}_1 + t\vec{u}$ and $\vec{r} = \vec{a}_2 + s\vec{v}$

$\vec{a}_1 = \vec{i} - \vec{j}$; $\vec{a}_2 = \vec{i} + \vec{j} - \vec{k}$; $\vec{u} = 2\vec{i} + \vec{j} + \vec{k}$; $\vec{v} = 2\vec{i} - \vec{j} - \vec{k}$

$$\vec{a}_2 - \vec{a}_1 = 2\vec{j} - \vec{k} \text{ and } \vec{u} \times \vec{v} = 4\vec{j} - 4\vec{k}$$

$$\left[\left(\vec{a}_2 - \vec{a}_1\right) \vec{u}\ \vec{v}\right] = \begin{vmatrix} 0 & 2 & -1 \\ 2 & 1 & 1 \\ 2 & -1 & 1 \end{vmatrix} = 12$$

$$|\vec{u} \times \vec{v}| = 4\sqrt{2}$$

$$\text{distance} = \frac{|[(\vec{a}_2 - \vec{a}_1)\vec{u}\ \vec{v}]|}{|\vec{u}\ \vec{v}|} = \frac{12}{4\sqrt{2}} = \frac{3}{\sqrt{2}}$$

Question 5.

Find the shortest distance between the parallel lines.

$\vec{r} = (2\vec{i} - \vec{j} - \vec{k}) + t(\vec{i} - 2\vec{j} + 3\vec{k})$ **and** $\vec{r} = (\vec{i} - 2\vec{j} + \vec{k}) + s(\vec{i} - 2\vec{j} + 3\vec{k})$

Solution:

Comparing the given equations with $\vec{r} = \vec{a}_1 + t\vec{u}$ and $\vec{r} = \vec{a}_2 + s\vec{u}$ we get,

$$\vec{a}_1 = 2\vec{i} - \vec{j} - \vec{k}$$

$$\vec{a}_2 = \vec{i} - 2\vec{j} + \vec{k} \text{ and } \vec{u} = \vec{i} - 2\vec{j} + 3\vec{k}$$

$$\vec{a}_2 - \vec{a}_1 = (\vec{i} - 2\vec{j} + \vec{k}) - (2\vec{i} - \vec{j} - \vec{k}) = -\vec{i} - \vec{j} + 2\vec{k}$$

$$\vec{u} \times (\vec{a}_2 - \vec{a}_1) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 3 \\ -1 & -1 & 2 \end{vmatrix} = \vec{i}[-4+3] - \vec{j}[2+3] + \vec{k}[-1-2] = -\vec{i} - 5\vec{j} - 3\vec{k}$$

$$|\vec{u} \times (\vec{a}_2 - \vec{a}_1)| = \sqrt{1+25+9} = \sqrt{35}$$

$$|\vec{u}| = \sqrt{1+4+9} = \sqrt{14}$$

The distance between the parallel lines $= \dfrac{|\vec{u} \times (\vec{a}_2 - \vec{a}_1)|}{|\vec{u}|} = \dfrac{\sqrt{35}}{\sqrt{14}} = \sqrt{\dfrac{35}{14}} = \sqrt{\dfrac{5}{2}}$ units.

Ex 6.6

Question 1.
Find a parametric form of vector equation of a plane which is at a distance of 7 units from the origin having 3, -4, 5 as direction ratios of a normal to it.
Solution:
Given p = 7

Direction ratios : 3, –4, 5 and Direction cosines $\left(\frac{3}{5\sqrt{2}}, \frac{-4}{5\sqrt{2}}, \frac{5}{5\sqrt{2}}\right)$

$$\hat{d} = \frac{\vec{d}}{|\vec{d}|} = \frac{1}{5\sqrt{2}}\left(3\vec{i} - 4\vec{j} + 5\vec{k}\right)$$

∴ The parametric form of vector equation of the plane is

$$\vec{r} \cdot \hat{d} = p$$

$$\vec{r} \cdot \left(\frac{3\vec{i} - 4\vec{j} + 5\vec{k}}{5\sqrt{2}}\right) = 7$$

Question 2.
Find the direction cosines of the normal to the plane 12x + 3y – 4z = 65. Also, find the non-parametric form of vector equation of a plane and the length of the perpendicular to the plane from the origin.
Solution:
12x + 3y – 4z = 65

$$\vec{r} \cdot \left(\frac{12\vec{i} + 3\vec{j} - 4\vec{k}}{\sqrt{144+9+16}}\right) = \frac{65}{\sqrt{144+9+16}}$$

$$\vec{r} \cdot \left(\frac{12\vec{i} + 3\vec{j} - 4\vec{k}}{13}\right) = \frac{65}{13} \Rightarrow \vec{r} \cdot \left(\frac{12\vec{i} + 3\vec{j} - 4\vec{k}}{13}\right) = 5$$

(*i*) Direction cosines $\left(\frac{12}{13}, \frac{3}{13}, \frac{4}{13}\right)$

(*ii*) Non-parametric form of vector equation of a plane

$$\vec{r} \cdot \left(\frac{12\vec{i} + 3\vec{j} - 4\vec{k}}{13}\right) = 5$$

(iii) Length of the perpendicular to the plane from the origin is 5 units.

Question 3.
Find the vector and Cartesian equation of the plane passing through the point with position vector $2\hat{i}+6\hat{j}+3\hat{k}$ and normal to the vector $\hat{i}+3\hat{j}+5\hat{k}$
Solution:

$$\text{Given, } \vec{a} = 2\hat{i}+6\hat{j}+3\hat{k} \text{ and } \vec{n} = \hat{i}+3\hat{j}+5\hat{k}$$

Vector equation of the plane

$$\vec{r}\cdot\vec{n} = \vec{a}\cdot\vec{n}$$

$$\vec{r}\cdot\vec{n} = \left(2\hat{i}+6\hat{j}+3\hat{k}\right)\left(\hat{i}+3\hat{j}+5\hat{k}\right)$$

$$\vec{r}\cdot\left(\vec{i}+3\vec{j}+5\vec{k}\right) = 2+18+15 \Rightarrow \vec{r}\cdot\left(\vec{i}+3\vec{j}+5\vec{k}\right) = 35$$

Cartesian equation of the plane

$$\left(x\vec{i}+y\vec{j}+z\vec{k}\right)\cdot\left(\vec{i}+3\vec{j}+5\vec{k}\right) = 35$$

$$x+3y+5z = 35$$

Question 4.
A plane passes through the point (-1, 1, 2) and the normal to the plane of magnitude $3\sqrt{3}$ makes equal acute angles with the coordinate axes. Find the equation of the plane.
Solution:
Given magnitude = $3\sqrt{3}$ and $\vec{a} = -\vec{i}+\vec{j}+2\vec{k}$
Then, the normal vector makes equal acute angle with the coordinate axes.

We know that $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$ (But $\alpha = \beta = \gamma$)

$$3 \cos^2 \alpha = 1 \qquad \Rightarrow \cos^2 \alpha = 1/3$$

$$\cos \alpha = \frac{1}{\sqrt{3}}$$

$$\therefore \qquad \text{so } \vec{n} = 3\sqrt{3}\left[\frac{1}{\sqrt{3}}\vec{i} + \frac{1}{\sqrt{3}}\vec{j} + \frac{1}{\sqrt{3}}\vec{k}\right]$$

$$\vec{n} = 3\vec{i} + 3\vec{j} + 3\vec{k}$$

Vector equation of the plane

$$\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$$

$$\vec{r} \cdot \vec{n} = \left(-\vec{i} + \vec{j} + 2\vec{k}\right)\left(3\vec{i} + 3\vec{j} + 3\vec{k}\right)$$

$$\vec{r} \cdot \left(3\hat{i} + 3\hat{j} + 3\hat{k}\right) = -3 + 3 + 6$$

$$\vec{r} \cdot \left(3\vec{i} + 3\vec{j} + 3\vec{k}\right) = 6 \text{ (or) } \vec{r} \cdot \left(\vec{i} + \vec{j} + \vec{k}\right) = 2$$

Cartesian equation of the plane

$$\left(x\vec{i} + y\vec{j} + z\vec{k}\right) \cdot \left(3\vec{i} + 3\vec{j} + 3\vec{k}\right) = 6 \qquad \Rightarrow 3x + 3y + 3z = 6$$

(or) $\qquad x + y + z = 2$

Question 5.
Find the intercepts cut off by the plane $\vec{r} \cdot (6\hat{i} + 4\hat{j} - 3\hat{k}) = 12$ on the coordinate axes.
Solution:

$$\vec{r} \cdot \left(6\vec{i} + 4\vec{j} - 3\vec{k}\right) = 12$$

Compare the above equations into $\vec{r} \cdot \vec{n} = q$ so $q = 12$

Let a, b, c are intercepts of x-axis, y-axis and z-axis respectively.

Clearly

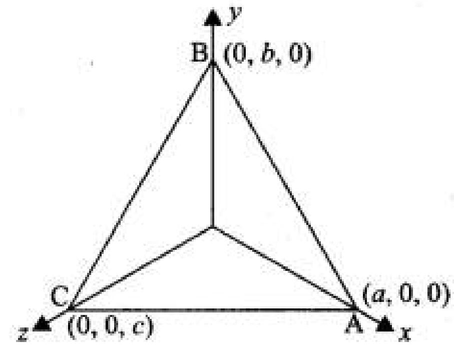
$\frac{q}{a} = 16$	$\frac{q}{b} = 4$	$\frac{q}{c} = -3$
$\frac{12}{6} = a$	$\frac{12}{4} = b$	$\frac{12}{-3} = c$
$2 = a$	$3 = b$	$-4 = c$

x-intercept = 2; y-intercept = 3; z-intercept = -4

Question 6.
If a plane meets the coordinate axes at A, B, C such that the centroid of the triangle ABC is the point (u, v, w), find the equation of the plane.
Solution:
Let A (a, 0, 0), B(0, b, 0), C(0, 0, c)

centroid of ΔABC = $\left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right)$

Given centroid = (u, v, w)



Equate these two centroids we get

$$\frac{a}{3} = u \quad \left| \quad \frac{b}{3} = v \quad \right| \quad \frac{c}{3} = w$$

$$a = 3u \quad \left| \quad b = 3v \quad \right| \quad c = 3w$$

We know that intercept form the plane is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \qquad \Rightarrow \frac{x}{3u} + \frac{y}{3v} + \frac{z}{3w} = 1$$

$$\frac{x}{u} + \frac{y}{v} + \frac{z}{w} = 3$$

Additional Problems

Question 1.
Find the vector and cartesian equations of a plane which is at a distance of 18 units from the origin and which is normal to the vector $2\vec{i} + 7\vec{j} + 8\vec{k}$
Solution:

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{2\vec{i} + 7\vec{j} + 8\vec{k}}{\sqrt{4 + 49 + 64}} = \frac{2\vec{i} + 7\vec{j} + 8\vec{k}}{\sqrt{117}}$$

$$= \frac{2\vec{i} + 7\vec{j} + 8\vec{k}}{3\sqrt{13}}$$

Hence, the required vector equation of the plane is $\vec{r} \cdot \hat{n} = p$.

i.e. $$\frac{\vec{r} \cdot \left(2\vec{i} + 7\vec{j} + 8\vec{k}\right)}{3\sqrt{13}} = 18 \text{ i.e. } \vec{r} \cdot \left(2\vec{i} + 7\vec{j} + 8\vec{k}\right) = 54\sqrt{13}$$

Taking $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ we get the cartesian equation as

$$\left(x\vec{i} + y\vec{j} + z\vec{k}\right) \cdot \left(2\vec{i} + 7\vec{j} + 8\vec{k}\right) = 54\sqrt{13}$$

i.e. $$2x + 7y + 8z = 54\sqrt{13}$$

Question 2.
Find the unit vector to the plane 2x – y + 2z = 5.
Solution:

Writing the plane in normal form we get,

$$\left(x\vec{i}+y\vec{j}+z\vec{k}\right)\cdot\left(2\vec{i}-\vec{j}+2\vec{k}\right)=5 \quad i.e.\ \vec{r}\cdot\vec{n}=\vec{p}$$

$$\text{Here normal vector} = \vec{n}=2\vec{i}-\vec{j}+2\vec{k}$$

$$|\vec{n}|=\sqrt{4+1+4}=3$$

So, the unit vector normal to the plane $=\hat{n}=\frac{\vec{n}}{|\vec{n}|}=\pm\frac{2\vec{i}-\vec{j}+2\vec{k}}{3}$

Question 3.
Find the length of the perpendicular from the origin to the plane $\vec{r}\cdot(3\vec{i}+4\vec{j}+12\vec{k})=26$.
Solution:
Taking the equation of the plane in cartesian form we get,
$(x\vec{i}+y\vec{j}+z\vec{k})\cdot(3\vec{i}+4\vec{j}+12\vec{k})=26$
i.e., $3x + 4y + 12z – 26 = 0$
The length of the perpendicular from (0, 0, 0) to the above plane is
$\pm\frac{-26}{\sqrt{9+16+144}}=\frac{+26}{13}=2$ units

Question 4.
The foot of the perpendicular drawn from the origin to a plane is (8, -4, 3). Find the equation of the plane.
Solution:
The required plane passes through the point A(8, -4, 3) and is perpendicular to $\overrightarrow{OA}$.

$\therefore$ $\vec{a}=8\vec{i}-4\vec{j}+3\vec{k}$ and $\vec{n}=\overrightarrow{OA}=8\vec{i}-4\vec{j}+3\vec{k}$

The required equation of the plane is $\vec{r}\cdot\vec{n}=\vec{a}\cdot\vec{n}$

$$\vec{r}\cdot\left(8\vec{i}-4\vec{j}+3\vec{k}\right)=\left(8\vec{i}-4\vec{j}+3\vec{k}\right)\cdot\left(8\vec{i}-4\vec{j}+3\vec{k}\right)=64+16+9=89$$

The cartesian equation is $8x – 4y + 3z = 89$.

Question 5.
Find the equation of the plane through the point whose position vector is $2\vec{i}-\vec{j}+\vec{k}$ and perpendicular to the vector $4\vec{i}+2\vec{j}-3\vec{k}$.
Solution:
The required plane is perpendicular to $4\vec{i}+2\vec{j}-3\vec{k}$
So, it is parallel to the plane $4x + 2y – 3z = k$
$\therefore$ the equation of the plane is $4x + 2y – 3z = k$
The plane passes through the point (2, -1, 1)
$\Rightarrow (4)(2) + 2(-1) – 3(1) = \lambda$ i.e. $\lambda = 8 – 2 – 3 = 3$
So, the equation of the plane is $4x + 2y – 3z = 3$.

Question 6.
Find the vector and cartesian equations of the plane passing through the point (2, -1, 4) and parallel to the plane $\vec{r}\cdot(4\vec{i}-12\vec{j}-3\vec{k})=7$.
Solution:
The given plane is $\vec{r}\cdot(4\vec{i}-12\vec{j}-3\vec{k})=7$

i e. $(x\vec{i} + y\vec{j} + z\vec{k}) \cdot (4\vec{i} - 12\vec{j} - 3\vec{k})$ = 7
i.e. 4x – 12y – 3z = 1
The required plane is parallel to the above plane. So, the equation of the required plane is 4x – 12y – 3z – k.
The plane passes through (2, -1, 4).
⇒ 4(2) – 12(-1) – 3(4) = k i.e. k = 8 + 12 – 12 = 8
So, the equation of the plane is 4x – 12y – 3z = 8.

Ex 6.7

Question 1.
Find the non-parametric form of vector equation, and Cartesian equation of the plane passing through the point (2, 3, 6) and parallel to the straight lines $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-3}{1}$ and $\frac{x+3}{2}=\frac{y-3}{-5}=\frac{z+1}{-3}$
Solution:

The required plane passing through the point $\vec{a}=2\vec{i}+3\vec{j}+6\vec{k}$ and parallel to the vectors $\vec{b}=2\vec{i}+3\vec{j}+\vec{k}$ and $\vec{c}=2\vec{i}-5\vec{j}-3\vec{k}$

$$\vec{b}\times\vec{c}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2 & 3 & 1\\ 2 & -5 & -3\end{vmatrix}=\vec{i}(-9+5)-\vec{j}(-6-2)+\vec{k}(-10-6)$$

$$\vec{b}\times\vec{c}=-4\vec{i}+8\vec{j}-16\vec{k}$$

Non-parametric form of vector equation

$$(\vec{r}-\vec{a})\cdot(\vec{b}\times\vec{c})=0 \qquad \text{(or)} \quad \vec{r}\cdot(\vec{b}\times\vec{c})=\vec{a}\cdot(\vec{b}\times\vec{c})$$

$$\vec{r}(-4\vec{i}+8\vec{j}-16\vec{k})=(2\vec{i}+3\vec{j}+6\vec{k})\cdot(-4\vec{i}+8\vec{j}-16\vec{k})$$

$$\vec{r}\cdot(-4\vec{i}+8\vec{j}-16\vec{k})=-8+24-96$$

$$\vec{r}\cdot(-4\vec{i}+8\vec{j}-16\vec{k})=-80 \qquad \Rightarrow -4\left[\vec{r}\cdot(\vec{i}-2\vec{j}+4\vec{k})\right]=-80$$

$$\vec{r}\cdot(\vec{i}-2\vec{j}+4\vec{k})=20$$

Cartesian equation

$$(x\vec{i}+y\vec{j}+z\vec{k})\cdot(\vec{i}-2\vec{j}+4\vec{k})=20$$

$$x-2y+4z=20$$

Question 2.
Find the parametric form of vector equation, and Cartesian equations of the plane passing through the points (2, 2, 1), (9, 3, 6) and perpendicular to the plane 2x + 6y + 6z = 9.
Solution:

The required plane passes through the points

$$\vec{a} = 2\vec{i} + 2\vec{j} + \vec{k}\ ,\ \vec{b} = 9\vec{i} + 3\vec{j} + 6\vec{k}$$

and parallel to the vector $\vec{c} = 2\vec{i} + 6\vec{j} + 6\vec{k}$

$$\vec{b} - \vec{a} = \left(9\vec{i} + 3\vec{j} + 6\vec{k}\right) - \left(2\vec{i} + 2\vec{j} + \vec{k}\right) = 7\vec{i} + \vec{j} + 5\vec{k}$$

Non-parametric form of vector equation

$$(\vec{r} - \vec{a}) \cdot \left(\left(\vec{b} - \vec{a}\right) \times \vec{c}\right) = 0 \qquad \text{...(1)}$$

$$\left(\vec{b} - \vec{a}\right) \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 7 & 1 & 5 \\ 2 & 6 & 6 \end{vmatrix} = \vec{i}\,(6-30) - \vec{j}\,(42-10) + \vec{k}\,(42-2)$$

$$= -24\hat{i} - 32\hat{j} + 40\hat{k}$$

$$(1) \Rightarrow \quad (\vec{r} - \vec{a}) \cdot \left(-24\hat{i} - 32\hat{j} + 40\hat{k}\right) = 0$$

$$\vec{r} \cdot \left(-24\hat{i} - 32\hat{j} + 40\hat{k}\right) = \vec{a} \cdot \left(-24\vec{i} - 32\vec{j} + 40\vec{k}\right)$$

$$\vec{r} \cdot \left(24\vec{i} - 32\vec{j} + 40\vec{k}\right) = \left(2\vec{i} + 2\vec{j} + \vec{k}\right)\left(-24\vec{i} - 32\vec{j} + 40\vec{k}\right)$$

$$\vec{r} \cdot \left(-24\hat{i} - 32\hat{j} + 40\hat{k}\right) = -48 - 64 + 40$$

$$-8\left[\vec{r} \cdot \left(3\hat{i} + 4\hat{j} - 5\hat{k}\right)\right] = -72$$

$$\vec{r} \cdot \left(3\hat{i} + 4\hat{j} - 5\hat{k}\right) = 9$$

Cartesian equation

$$\left(x\vec{i} + y\vec{j} + z\vec{k}\right) \cdot \left(3\vec{i} + 4\vec{j} - 5\vec{k}\right) = 9$$

$$3x + 4y - 5z - 9 = 0$$

Question 3.

Find the parametric form of vector equation and Cartesian equations of the plane passing through the points (2, 2, 1), (1, -2, 3) and parallel to the straight line passing through the points (2, 1, -3) and (-1, 5, -8).

Solution:

Equation of the straight line passing through the points (2, 1, -3) and (-1, 5, -8) is

$$\frac{x-2}{-3} = \frac{y-1}{4} = \frac{z+3}{-5}$$

$\therefore$ The required plane passing through the points

$$\vec{a} = 2\vec{i} + 2\vec{j} + \vec{k} \text{ and } \vec{b} = \vec{i} - 2\vec{j} + 3\vec{k}$$

and parallel to $\vec{c} = -3\vec{i} + 4\vec{j} - 5\vec{k}$

Parametric form of vector equation

$$\vec{r} = \vec{a} + s\left(\vec{b} - \vec{a}\right) + t\vec{c}$$

$$\vec{r} = \left(2\vec{i} + 2\vec{j} + \vec{k}\right) + s\left(\vec{i} - 4\vec{j} + 2\vec{k}\right) + t\left(-3\vec{i} + 4\vec{j} - 5\vec{k}\right), s, t \in \mathbb{R}$$

Cartesian equation:

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

$$\begin{vmatrix} x-2 & y-2 & z-1 \\ -1 & -4 & 2 \\ -3 & 4 & -5 \end{vmatrix} = 0$$

(x – 2) [20 – 8] – (y – 2) [5 + 6] + (z – 1) [-4 – 12] = 0
12(x – 2) – 11(y – 2) – 16(z – 1) = 0
12x – 24 – 11y + 22 – 16z + 16 = 0
12x – 11y – 16z + 14 = 0

Question 4.
Find the non-parametric form of vector equation of the plane passing through the point (1, -2, 4) and perpendicular to the plane x + 2y – 3z = 11 and parallel to the line $\frac{x+7}{3} = \frac{y+3}{-1} = \frac{z}{1}$
Solution:
The required plane passing through the point $\vec{a} = \vec{i} - 2\vec{j} + 4\vec{k}$ and parallel to the plane $\vec{b} = \vec{i} + 2\vec{j} - 3\vec{k}$

and parallel to the line $\vec{c} = 3\vec{i} - \vec{j} + \vec{k}$

$$\vec{b} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -3 \\ 3 & -1 & 1 \end{vmatrix} = \vec{i}(2-3) - \vec{j}(1+9) + \vec{k}(-1-6)$$

$$\vec{b} \times \vec{c} = -\vec{i} - 10\vec{j} - 7\vec{k}$$

Non-parametric form of vector equation

$$(\vec{r} - \vec{a}) \cdot (\vec{b} \times \vec{c}) = 0$$

$$(\vec{r} - \vec{a}) \cdot (-\vec{i} - 10\vec{j} - 7\vec{k}) = 0$$

$$\vec{r} \cdot (-\vec{i} - 10\vec{j} - 7\vec{k}) = \vec{a} \cdot (-\vec{i} - 10\vec{j} - 7\vec{k})$$

$$\vec{r} \cdot (-\vec{i} - 10\vec{j} - 7\vec{k}) = (\vec{i} - 2\vec{j} + 4\vec{k})(-\vec{i} - 10\vec{j} - 7\vec{k})$$

$$\vec{r} \cdot (-\vec{i} - 10\vec{j} - 7\vec{k}) = -1 + 20 - 28$$

$$\vec{r} \cdot (-\vec{i} - 10\vec{j} - 7\vec{k}) = -9 \qquad \text{(or) } \vec{r} \cdot (\vec{i} + 10\vec{j} + 7\vec{k}) = 9$$

Cartesian equation

$$(x\vec{i} + y\vec{j} + z\vec{k}) \cdot (\vec{i} + 10\vec{j} + 7\vec{k}) = 9$$

$$x + 10y + 7z - 9 = 0$$

Question 5.

Find the parametric form of vector equation, and Cartesian equations of the plane containing the line $\vec{r} = (\hat{i} - \hat{j} + 3\hat{k}) + t(2\hat{i} - \hat{j} + 4\hat{k})$ and perpendicular to the plane $\vec{r} \cdot (\hat{i} + 2\hat{j} + \hat{k}) = 8$.

Solution:

The required plane passing through the point $\vec{a} = \vec{i} - \vec{j} + 3\vec{k}$ and parallel to $\vec{b} = 2\vec{i} - \vec{j} + 4\vec{k}$ and $\vec{c} = \vec{i} + 2\vec{j} + \vec{k}$

Parametric form of vector equation

$$\vec{r} = \vec{a} + s\vec{b} + t\vec{c}$$

$$\vec{r} = (\vec{i} - \vec{j} + 3\vec{k}) + s(2\vec{i} - \vec{j} + 4\vec{k}) + t(\vec{i} + 2\vec{j} + \vec{k}), s, t \in \mathrm{R}$$

Cartesian equation

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

$$\begin{vmatrix} x - 1 & y + 1 & z - 3 \\ 2 & -1 & 4 \\ 1 & 2 & 1 \end{vmatrix} = 0$$

(x – 1) [-1 – 8] – (y + 1)[z – 4] + (z – 3) [4 + 1] = 0
-9(x- 1) + 2(y + 1) + 5(z- 3) = 0
-9x + 9 + 2y + 2 + 5z – 15 = 0
-9x + 2y + 5z – 4 = 0
9x – 2y – 5z + 4 = 0

Question 6.
Find the parametric vector, non-parametric vector and Cartesian form of the equation of the plane passing through the point (3, 6, -2), (-1, -2, 6) and (6, 4, -2).
Solution:
The required plane passing through the points

$$\vec{a} = 3\vec{i} + 6\vec{j} - 2\vec{k}$$

$$\vec{b} = -\vec{i} - 2\vec{j} + 6\vec{k} \quad \text{and } \vec{c} = 6\vec{i} + 4\vec{j} - 2\vec{k}$$

$$\vec{b} - \vec{a} = -4\vec{i} - 8\vec{j} + 8\vec{k} \quad \text{and } \vec{c} - \vec{a} = 3\vec{i} - 2\vec{j}$$

Parametric form of vector equation

$$\vec{r} = \vec{a} + s\left(\vec{b} - \vec{a}\right) + t\left(\vec{c} - \vec{a}\right)$$

$$\vec{r} = \left(3\vec{i} + 6\vec{j} - 2\vec{k}\right) + s\left(-4\vec{i} - 8\vec{j} + 8\vec{k}\right) + t\left(3\vec{i} - 2\vec{j}\right), s, t \in \mathbb{R}$$

Non-parametric form of vector equation

$$\left(\vec{b} - \vec{a}\right) \times \left(\vec{c} - \vec{a}\right) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -4 & -8 & 8 \\ 3 & -2 & 0 \end{vmatrix} = \vec{i}(0+16) - \vec{j}(0-24) + \vec{k}(8+24)$$

$$= 16\vec{i} + 24\vec{j} + 32\vec{k}$$

$$\therefore \quad \left(\vec{r} - \vec{a}\right) \cdot \left(\left(\vec{b} - \vec{a}\right) \times \left(\vec{c} - \vec{a}\right)\right) = 0$$

$$\left(\vec{r} - \vec{a}\right) \cdot \left(16\vec{i} + 24\vec{j} + 32\vec{k}\right) = 0$$

$$\vec{r} \cdot \left(16\vec{i} + 24\vec{j} + 32\vec{k}\right) = \vec{a} \cdot \left(16\vec{i} + 24\vec{j} + 32\vec{k}\right)$$

$$\vec{r} \cdot \left(16\vec{i} + 24\vec{j} + 32\vec{k}\right) = \left(3\vec{i} + 6\vec{j} - 2\vec{k}\right)\left(16\vec{i} + 24\vec{j} + 32\vec{k}\right)$$

$$\vec{r} \cdot \left(16\vec{i} + 24\vec{j} + 32\vec{k}\right) = 48 + 144 - 64$$

$$\vec{r} \cdot \left(16\vec{i} + 24\vec{j} + 32\vec{k}\right) = 128 \quad \Rightarrow 8\left[\vec{r} \cdot \left(2\vec{i} + 3\vec{j} + 4\vec{k}\right)\right] = 128$$

$$\vec{r} \cdot \left(2\vec{i} + 3\vec{j} + 4\vec{k}\right) = 16$$

Cartesian equation

$$\left(x\vec{i} + y\vec{j} + z\vec{k}\right) \cdot \left(2\vec{i} + 3\vec{j} + 4\vec{k}\right) = 16$$

$$2x + 3y + 4z - 16 = 0$$

Question 7.
Find the non-parametric form of vector equation, and Cartesian equations of the plane
$\vec{r}=(6\hat{i}-\hat{j}+\hat{k})+s(-\hat{i}+2\hat{j}+\hat{k})+t(-5\hat{i}-4\hat{j}-5\hat{k})$
Solution:

$$\text{Given } \vec{a} = 6\hat{i}-\hat{j}+\hat{k},\ \vec{b}=-\vec{i}+2\vec{j}+\vec{k} \text{ and } \vec{c}=-5\vec{i}-4j-5\vec{k}$$

$$\vec{b}\times\vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & 1 \\ -5 & -4 & -5 \end{vmatrix} = \vec{i}(-10+4)-\vec{j}(5+5)+\vec{k}(4+10)$$

$$\vec{b}\times\vec{c} = -6\vec{i}-10\vec{j}+14\vec{k}$$

Non-parametric form of vector equation

$$(\vec{r}-\vec{a})\cdot(\vec{b}\times\vec{c}) = 0$$

$$(\vec{r}-\vec{a})\cdot(-6\vec{i}-10\vec{j}+14\vec{k}) = 0$$

[÷ by –2]

$$(\vec{r}-\vec{a})\cdot(3\vec{i}+5\vec{j}-7\vec{k}) = 0$$

$$\vec{r}\cdot(3i+5\vec{j}-7\vec{k}) = \vec{a}\cdot(3\vec{i}+5\vec{j}-7\vec{k})$$

$$\vec{r}\cdot(3i+5\vec{j}-7\vec{k}) = (6\vec{i}-\vec{j}+\vec{k})(3\vec{i}+5\vec{j}-7\vec{k})$$

$$\vec{r}\cdot(3\vec{i}+5\vec{j}-7\vec{k}) = 18-5-7 \quad \Rightarrow \vec{r}\cdot(3\vec{i}+5\vec{j}-7\vec{k}) = 6$$

Cartesian equation

$$(x\vec{i}+y\vec{j}+z\vec{k})\cdot(3\vec{i}+5\vec{j}-7\vec{k}) = 6$$

$$3x+5y-7z-6=0$$

Additional Problem

Question 1.
Find the vector and cartesian equations of the plane containing the line $\frac{x-2}{2}=\frac{y-2}{3}=\frac{z-1}{3}$ and parallel to the line $\frac{x+1}{3}=\frac{y-1}{2}=\frac{z+1}{1}$
Solution:
The required plane passes through the point A(2, 2, 1) and parallel to

$$\vec{u} = 2\vec{i} + 3\vec{j} + 3\vec{k} \text{ and } \vec{v} = 3\vec{i} + 2\vec{j} + \vec{k}$$

The required equation is $\vec{r} = \vec{a} + s\vec{u} + t\vec{v}$

i.e., $x\vec{i} + y\vec{j} + z\vec{k} = \left(2\vec{i} + 2\vec{j} + \vec{k}\right) + s\left(2\vec{i} + 3\vec{j} + 3\vec{k}\right) + t\left(3\vec{i} + 2\vec{j} + \vec{k}\right)$

Cartesian form: The equation of the plane is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0 \qquad i.e. \begin{vmatrix} x-2 & y-2 & z-1 \\ 2 & 3 & 3 \\ 3 & 2 & 1 \end{vmatrix} = 0$$

i.e., (x – 2)[3 – 6] – (y – 2)[2 – 9] + (z – 1)[4 – 9] = 0
i.e., (x – 2)(-3) + (y – 2)(7) – 5(z – 1) = 0
-3x + 6 + 7y – 14 – 5z + 5 = 0
-3x + 7y – 5z – 3 = 0 i.e. 3x – 7y + 5z + 3 = 0

Question 2.
Find the vector and cartesian equation of the plane passing through the point (1, 3, 2) and parallel to the lines

$$\frac{x+1}{2} = \frac{y+2}{-1} = \frac{z+3}{3} \text{ and } \frac{x-2}{1} = \frac{y+1}{2} = \frac{z+2}{2}.$$

Solution:
The required plane passes through the point A = (1, 3, 2) and parallel to the vectors

$$\vec{u} = 2\vec{i} - \vec{j} + 3\vec{k} \text{ and } \vec{v} = \vec{i} + 2\vec{j} + 2\vec{k}$$

The required equation is $\vec{r} = \vec{a} + s\vec{u} + t\vec{v}$

i.e. $x\vec{i} + y\vec{j} + z\vec{k} = \left(\vec{i} + 3\vec{j} + 2\vec{k}\right) + s\left(2\vec{i} - \vec{j} + 3\vec{k}\right) + t\left(\vec{i} + 2\vec{j} + 2\vec{k}\right)$

Cartesian equation of the plane is $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$

So, the equation of the plane is $\begin{vmatrix} x-1 & y-3 & z-2 \\ 2 & -1 & 3 \\ 1 & 2 & 2 \end{vmatrix} = 0$

i.e. (x – 1)(-2 – 6) – (y – 3)(4 – 3) + (z – 2)(4 + 1) = 0
—8(x – 1) – 1(y – 3) + 5(z – 2) = 0
-8x + 8 – y + 3 + 5z – 10 = 0
-8x – y + 5z + 1 = 0 i.e. 8x + y – 5z – 1 = 0

Question 3.
Find the vector and cartesian equations of the plane passing through the point (-1, 3, 2) and perpendicular to the planes x + 2y + 2z = 5 and 3x + y + 2z = 8
Solution:
The normal vector to the given planes x + 2y + 2z = 5 and 3x + y + 2z = 8 are respectively $\vec{i} + 2\vec{j} + 2\vec{k}$ and $3\vec{i} + \vec{j} + 2\vec{k}$. These vectors are parallel to the required plane
The required plane passes through the point A(-1, 3, 2) and parallel to the vectors

$$\vec{u} = \vec{i} + 2\vec{j} + 2\vec{k} \text{ and } \vec{v} = 3\vec{i} + \vec{j} + 2\vec{k}$$

The required equation is $\vec{r} = \vec{a} + s\vec{u} + t\vec{v}$

i.e. $$\vec{r} = -\vec{i} + 3\vec{j} + 2\vec{k} + s\left(\vec{i} + 2\vec{j} + 2\vec{k}\right) + t\left(3\vec{i} + \vec{j} + 2\vec{k}\right)$$

$$\text{Cartesian form is } \begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

$$\text{So, the equation of the plane is } \begin{vmatrix} x+1 & y-3 & z-2 \\ 1 & 2 & 2 \\ 3 & 1 & 2 \end{vmatrix} = 0$$

i.e., $(x + 1)(4 - 2) - (y - 3)(2 - 6) + (z - 2)(1 - 6) = 0$

$2(x + 1) + 4(y - 3) - 5(z - 2) = 0$

$2x + 2 + 4y - 12 - 5z + 10 = 0$

$2x + 4y - 5z = 0$

Question 4.

Find the vector and cartesian equations of the plane passing through the points A(1, -2, 3) and B(-1, 2, -1) and is parallel to the line $\frac{x-2}{2} = \frac{y+1}{3} = \frac{z-1}{4}$

Solution:

The vector equation of the plane is

$$\vec{r} = (1-s)\vec{a} + s\vec{b} + t\vec{c} \text{ where } s \text{ and } t \text{ are scalars}$$

$$\therefore \quad \vec{r} = (1-s)\left(\vec{i} - 2\vec{j} + 3\vec{k}\right) + s\left(-\vec{i} + 2\vec{j} - \vec{k}\right) + t\left(2\vec{i} + 3\vec{j} + 4\vec{k}\right)$$

$$\text{The cartesian equation of the plane is } \begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l & m & n \end{vmatrix} = 0$$

$$\text{So, the equation of the plane is } \begin{vmatrix} x-1 & y+2 & z-3 \\ -2 & 4 & -4 \\ 2 & 3 & 4 \end{vmatrix} = 0$$

i.e, $(x - 1)[16 + 12] - (y + 2)(-8 + 8) + (z - 3)(-6 - 8) = 0$

$28(x - 1) - 14(z - 3) = 0$

$28x - 28 - 14z + 42 = 0$

$28x - 14z + 14 = 0$

$(\div \text{ by } 14) \Rightarrow 2x - z + 1 = 0$

Question 5.

Find the vector and cartesian equations of the plane through the points (1, 2, 3) and (2,3,1) and perpendicular to the plane $3x - 2y + 4z - 5 = 0$.

Solution:

The vector normal to the plane $3x - 2y + 4z - 5 = 0$ is $3\vec{i} - 2\vec{j} + 4\vec{k}$

The required plane is parallel to the vector $3\vec{i} - 2\vec{j} + 4\vec{k}$
∴ Vector equation of the plane passing through two points and parallel to one vector is

$$\vec{r} = (1-s)\vec{a} + s\vec{b} + t\vec{c} \text{ where } s \text{ and } t \text{ are scalars.}$$

So, the vector equation of the required plane is

$$\vec{r} = (1-s)(\vec{i} + 2\vec{j} + 3\vec{k}) + s(2\vec{i} + 3\vec{j} + \vec{k}) + t(3\vec{i} - 2\vec{j} + 4\vec{k})$$

The cartesian equation is $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l & m & n \end{vmatrix} = 0$

So, the equation of the plane is $\begin{vmatrix} x-1 & y-2 & z-3 \\ 1 & 1 & -2 \\ 3 & -2 & 4 \end{vmatrix} = 0$

i.e. (x – 1)[4 – 4] – (y – 2)[4 + 6] + (z – 3)[-2 – 3] = 0
-10(y – 2) – 5(z – 3) = 0
-10y + 20 – 5z + 15 = 0
-10y – 5z + 35 = 0
10y + 5z – 35 = 0
(÷ by 5) ⇒ 2y + z – 1 = 0

Question 6.
Find the vector and cartesian equations of the plane containing the line $\frac{x-2}{2} = \frac{y-2}{3} = \frac{z-1}{-2}$ and passing through the point (-1, 1, -1).
Solution:
The required plane passes through the points A(-1, 1,-1) and B(2, 2, 1) and parallel to the vectors $\vec{c} = 2\vec{i} + 3\vec{j} - 2\vec{k}$.
The required equation is

$$\vec{r} = (1-s)\vec{a} + s\vec{b} + t\vec{c} \text{ where } s \text{ and } t \text{ are scalars.}$$

$$\therefore \quad \vec{r} = (1-s)(-\vec{i} + \vec{j} - \vec{k}) + s(2\vec{i} + 2\vec{j} + \vec{k}) + t(2\vec{i} + 3\vec{j} - 2\vec{k})$$

Cartesian equation of the plane is $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l & m & n \end{vmatrix} = 0$

So, the equation of the plane is $\begin{vmatrix} x+1 & y-1 & z+1 \\ 3 & 1 & 2 \\ 2 & 3 & -2 \end{vmatrix} = 0$

i.e. (x + 1)[-2 – 6] – (y – 1)[-6 – 4] + (z + 1)[9 – 2] = 0
-8(x + 1) + 10(y – 1) + 7(z + 1) = 0
-8x – 8 + 10y – 10 + 7z + 7 = 0
-8x + 10y + 7z – 11 = 0
i.e., 8x – 10y – 7z + 11 = 0

Question 7.
Find the vector and cartesian equations of the plane passing through the points with position vectors

$3\vec{i} + 4\vec{j} + 2\vec{k}$, $2\vec{i} - 2\vec{j} - \vec{k}$ and $7\vec{i} + \vec{k}$.

Solution:

Vector equation of the plane passing through three given non-collinear points is

$$\vec{r} = (1-s-t)\vec{a} + s\vec{b} + t\vec{c} \text{ where } s \text{ and } t \text{ are scalars}$$

$$\therefore \quad \vec{r} = (1-s-t)\left(3\vec{i} + 4\vec{j} + 2\vec{k}\right) + s\left(2\vec{i} - 2\vec{j} - \vec{k}\right) + t\left(7\vec{i} + \vec{k}\right)$$

$$\text{Cartesian equation of the plane is } \begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-3 & y-4 & z-2 \\ -1 & -6 & -3 \\ 4 & -4 & -1 \end{vmatrix} = 0$$

i.e. (x – 3)[6 – 12] – (y – 4)[1 + 12] + (z – 2)[4 + 24] = 0

-6(x – 3) – 13(y – 4) + 28(z – 2) = 0

-6x + 18 – 13y + 52 + 28z – 56 = 0

-6x – 13y + 28z + 14 = 0 .

i.e. 6x + 13y – 28z – 14 = 0

Question 8.

Derive the equation of the plane in the intercept form.

Solution:

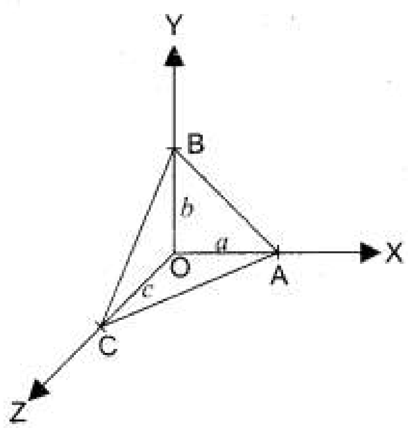
Let the required plane makes intercepts on X, Y, Z-axes respectively as a, b and c.

i.e. A = (a, 0, 0)

B = (0, b, 0)

C = (0, 0, c)

The equation of the plane passing through three non-collinear points A, B and C is



$$\vec{r} = (1-s-t)a + s\vec{b} + t\vec{c}$$

Here $\vec{a} = a\vec{i}, \vec{b} = b\vec{j}$ and $\vec{c} = c\vec{k}$

$\therefore$ $\vec{r} = (1-s-t)a\vec{i} + sb\vec{j} + tc\vec{k}$

Cartesian equation of the plane is $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} x-a & y-0 & z-0 \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix} = 0$$

$$(x-a)(bc-0) - y(-ac-0) + z(ab) = 0$$

$$\text{i.e. } xbc - abc + yac + zab = 0$$

$$xbc + yac + zab = abc$$

$$(\div \text{ by } abc) \Rightarrow \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

Question 9.
Find the cartesian form of the following plane:
$\vec{r} = (s-2t)\vec{i} + (3-t)\vec{j} + (2s+t)\vec{k}$
Solution:

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

Now $x\vec{i} + y\vec{j} + z\vec{k} = (0+s-2t)\vec{i} + (3-t)\vec{j} + (0+2s+t)\vec{k}$

$$(x-0)\vec{i} + (y-3)\vec{j} + (z-0)\vec{k} = (s-2t)\vec{i} + (-t)\vec{j} + (2s+t)\vec{k}$$

Equating $\vec{i}, \vec{j}, \vec{k}$ components and eliminating s and t we get,

$$\begin{vmatrix} x-0 & y-3 & z-0 \\ s & 0 & 2s \\ -2t & -t & t \end{vmatrix} = 0 \text{ ; i.e., } \begin{vmatrix} x & y-3 & z \\ 1 & 0 & 2 \\ -2 & -1 & 1 \end{vmatrix} = 0$$

$x(0 + 2) - (y - 3)(1 + 4) + z(-1) = 0$
$2x - 5(y - 3) - z = 0$
$2x - 5y + 15 - z = 0$
$2x - 5y - z + 15 = 0$
which is the cartesian equation of the plane.

Ex 6.8

Question 1.

Show that the straight lines $\vec{r}=\left(5\hat{i}+7\hat{j}-3\hat{k}\right)+s\left(4\hat{i}+4\hat{j}-5\hat{k}\right)$ and $\vec{r}=\left(8\hat{i}+4\hat{j}+5\hat{k}\right)+t\left(7\hat{i}+\hat{j}+3\hat{k}\right)$ are coplanar. Find the vector equation of the plane in which they lie.

Solution:

$$\text{Let } \vec{a} = 5\vec{i}+7\vec{j}-3\vec{k} \text{ and } \vec{b}=4\vec{i}+4\vec{j}-5\vec{k}$$
$$\vec{c} = 8\vec{i}+4\vec{j}+5\vec{k} \text{ and } \vec{d}=7\vec{i}+\vec{j}+3\vec{k}$$

We know that given two lines are coplanar if

$$(\vec{c}-\vec{a})\cdot\left(\vec{b}\times\vec{d}\right) = 0 \qquad \text{...(1)}$$

$$\vec{b}\times\vec{d} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 4 & -5 \\ 7 & 1 & 3 \end{vmatrix} = \vec{i}(12+5)-\vec{j}(12+35)+\vec{k}(4-28)$$

$$\vec{b}\times\vec{d} = 17\vec{i}-47\vec{j}-24\vec{k}$$

$$\vec{c}-\vec{a} = \left(8\vec{i}+4\vec{j}+5\vec{k}\right)-\left(5\vec{i}+7\vec{j}-3\vec{k}\right)=3\vec{i}-3\vec{j}+8\vec{k}$$

$$(1) \Rightarrow \left(3\vec{i}-3\vec{j}+8\vec{k}\right)\cdot\left(17\vec{i}-47\vec{j}-24\vec{k}\right)=51+141-192=0$$

∴ The two given lines are coplanar so, the non-parametric vector equation is

$$(\vec{r}-\vec{a})\cdot\left(\vec{b}\times\vec{d}\right) = 0$$

$$\vec{r}\cdot\left(\vec{b}\times\vec{d}\right) = \vec{a}\cdot\left(\vec{b}\times\vec{d}\right)$$

$$\vec{r}\cdot\left(17\vec{i}-47\vec{j}-24\vec{k}\right) = \left(5\vec{i}+7\vec{j}-3\vec{k}\right)\left(17\vec{i}-47\vec{j}-24\vec{k}\right)$$

$$\vec{r}\cdot\left(17\vec{i}-47\vec{j}-24\vec{k}\right) = 85-329+72 \qquad \Rightarrow \vec{r}\cdot\left(17\vec{i}-47\vec{j}-24\vec{k}\right) = -172$$

Question 2.

Show that lines $\frac{x-2}{1}=\frac{y-3}{1}=\frac{z-4}{3}$ and $\frac{x-1}{-3}=\frac{y-4}{2}=\frac{z-5}{1}$ are coplanar. Also, find the plane containing these lines.

Solution:

From the lines we have,

$(x_1, y_1, z_1) = (2, 3, 4)$ and $(x_2, y_2, z_2) = (1, 4, 5)$

$(b_1, b_2, b_3) = (1, 1, 3)$ and $(d_1, d_2, d_3) = (-3, 2, 1)$

Condition for coplanarity

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$$

$$= \begin{vmatrix} -1 & 1 & 1 \\ 1 & 1 & 3 \\ -3 & 2 & 1 \end{vmatrix} = -(1-6)-1(1+9)+1(2+3)$$

$$= 5 - 10 + 5 = 0$$

∴ The given two lines are coplanar.

Cartesian form of equation of the plane containing the two given coplanar lines.

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$$

$$\begin{vmatrix} x-2 & y-3 & z-4 \\ 1 & 1 & 3 \\ -3 & 2 & 1 \end{vmatrix} = 0$$

$(x-2)[1-6]-(y-3)[1+9]+(z-4)[2+3]=0$

$-5(x-2)-10(y-3)+5(z-4)=0$

$-5x+10-10y+30+5z-20=0$

$-5x-10y+5z+20=0$

$(\div \text{ by } -5) \Rightarrow x+2y-2z-4=0$

Question 3.

If the straight lines $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{m^2}$ and $\frac{x-3}{1}=\frac{y-2}{m^2}=\frac{z-1}{2}$ are coplanar, find the district real values of m.

Solution:

From the lines we have,

$(x_1, y_1, z_1) = (1, 2, 3)$ and $(x_2, y_2, z_2) = (3, 2, 1)$

$(b_1, b_2, b_3) = (1, 2, m^2)$ and $(d_1, d_2, d_3) = (1, m^2, 2)$

Condition for coplanarity

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$$

$$\begin{vmatrix} 2 & 0 & -2 \\ 1 & 2 & m^2 \\ 1 & m^2 & 2 \end{vmatrix} = 0$$

$$2[4 - m^4] - 2[m^2 - 2] = 0$$

$$(\div \text{ by } 2) \Rightarrow \quad 4 - m^4 - m^2 + 2 = 0 \qquad \Rightarrow m^4 + m^2 - 6 = 0$$

$$(m^2 + 3)(m^2 - 2) = 0$$

$$\begin{array}{c|c} m^2 + 3 = 0 & m^2 - 2 = 0 \\ m^2 = -3 & m^2 = 2 \\ \text{(not possible)} & m = \pm\sqrt{2} \end{array}$$

Question 4.

If the straight lines $\frac{x-1}{2} = \frac{y+1}{\lambda} = \frac{z}{2}$ and $\frac{x+1}{5} = \frac{y+1}{2} = \frac{z}{\lambda}$ are coplanar, find λ and equation of the planes containing these two lines.

Solution:

From the lines we have,

$(x_1, y_1, z_1) = (1, -1, 0)$ and $(x_2, y_2, z_2) = (-1, -1, 0)$

$(b_1, b_2, b_3) = (2, \lambda, 2)$ and $(d_1, d_2, d_3) = (5, 2, \lambda)$

Condition for coplanarity

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$$

$$\begin{vmatrix} -2 & 0 & 0 \\ 2 & \lambda & 2 \\ 5 & 2 & \lambda \end{vmatrix} = 0$$

$$-2(\lambda^2 - 4) = 0 \qquad \Rightarrow \lambda^2 = 4$$

$$\lambda = \pm 2$$

(*i*) If $\lambda = 2$

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$$

$$\begin{vmatrix} x-1 & y+1 & z \\ 2 & 2 & 2 \\ 5 & 2 & 2 \end{vmatrix} = 0$$

$$(x-1)[0] - (y+1)[4-10] + z[4-10] = 0$$

$$6(y+1) - 6(z) = 0$$

$$6y + 6 - 6z = 0$$

$$(\div \text{ by } 6) \Rightarrow (y - z + 1) = 0$$

(*ii*) If $\lambda = -2$

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$$

$$\begin{vmatrix} x-1 & y+1 & z \\ 2 & -2 & 2 \\ 5 & 2 & -2 \end{vmatrix} = 0$$

$(x - 1)[0] - (y + 1)[-4 - 10] + z[4 + 10] = 0$
$14(y + 1) + 14z = 0 \Rightarrow 14y + 14 + 14z = 0$
$(\div$ by $14) \Rightarrow y + z + 1 = 0$

Additional Problems

Question 1.
Show that the straight lines.

$$\vec{r} = \left(\vec{i} + \vec{j} - \vec{k}\right) + \lambda\left(3\vec{i} - \vec{j}\right)$$

$$\vec{r} = \left(4\vec{i} - \vec{k}\right) + \mu\left(2\vec{i} + 3\vec{k}\right)$$

are coplanar. Find the vector equation of the plane in which they lie.

Solution:

$$\text{Given } \vec{a} = \vec{i} + \vec{j} - \vec{k}; \text{ and } \vec{b} = 3\vec{i} - \vec{j}$$
$$\vec{c} = 4\vec{i} - \vec{k}; \text{ and } \vec{d} = 2\vec{i} + 3\vec{k}$$

Conditions for coplanar

$$(\vec{c} - \vec{a}) \cdot (\vec{b} \times \vec{d}) = 0$$
$$\vec{c} - \vec{a} = (4\vec{i} - \vec{k}) - (\vec{i} + \vec{j} - \vec{k}) = 3\vec{i} - \vec{j}$$
$$\vec{b} \times \vec{d} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & 0 \\ 2 & 0 & 3 \end{vmatrix} = \vec{i}(-3) - \vec{j}(9) + \vec{k}(2) = -3\vec{i} - 9\vec{j} + 2\vec{k}$$
$$(\vec{c} - \vec{a}) \cdot (\vec{b} \times \vec{d}) = (3\vec{i} - \vec{j})(-3\vec{i} - 9\vec{j} + 2\vec{k}) = -9 + 9 + 0 = 0$$

Given two lines are coplanar.

Vector equation of the plane:

$$\vec{r} = \vec{a} + t\vec{b} + s\vec{d} \quad \Rightarrow \vec{r} = (\vec{i} + \vec{j} - \vec{k}) + t(3\vec{i} - \vec{j}) + s(2\vec{i} + 3\vec{k})$$

(or)

$$\vec{r} = \vec{c} + t\vec{b} + s\vec{d} \quad \Rightarrow \vec{r} = (4\vec{i} - \vec{k}) + t(3\vec{i} - \vec{j}) + s(2\vec{i} + 3\vec{k})$$

Question 2.

If the straight lines $\frac{x-1}{1} = \frac{y-1}{\lambda} = \frac{z-1}{1}$ **and** $\frac{x}{2} = \frac{y-4}{\lambda} = \frac{z-2}{3}$ are coplanar. Find λ.

Solution:

From the lines we have,

$(x_1, y_1, z_1) = (1, 1, 1)$ and $(b_1, b_2, b_3) = (1, \lambda, 1)$

$(x_2, y_2, z_2) = (0, 4, 2)$ and $(d_1, d_2, d_3) = (2, \lambda, 3)$

Condition for coplanarity

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$$

$$\begin{vmatrix} -1 & 3 & 1 \\ 1 & \lambda & 1 \\ 2 & \lambda & 3 \end{vmatrix} = 0$$

$-1(3\lambda - \lambda) - 3(3 - 2) + 1(\lambda - 2\lambda) = 0 \Rightarrow -2\lambda - 3 - \lambda = 0$

$-3\lambda = 3 \Rightarrow \lambda = -1$

Question 3.

If the lines $\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{-k}$ **and** $\frac{x-1}{k} = \frac{y-4}{2} = \frac{z-5}{1}$ are coplanar, then find the value of k.

Solution:

From the lines we have,
$(x_1, y_1, z_1) = (2, 3, 4)$ and $(b_1, b_2, b_3) = (1, 1, -1)$
$(x_2, y_2, z_2) = (1, 4, 5)$ and $(d_1, d_2, d_3) = (k, 2, 1)$
Condition for coplanarity

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$$

$$\begin{vmatrix} -1 & 1 & 1 \\ 1 & 1 & -k \\ k & 2 & 1 \end{vmatrix} = 0$$

$-1(1 + 2k) - 1(1 + k^2) + 1(2 - k) = 0$
$-1 - 2k - 1 - k^2 + 2 - k = 0$
$-k^2 - 3k = 0$
$k^2 + 3k = 0$
$k(k + 3) = 0$
$k = 0$ or $k = -3$

Question 4.
Show that the lines $\frac{x+3}{-3} = \frac{y-1}{1} = \frac{z-5}{5}$ **and** $\frac{x+1}{-1} = \frac{y-2}{2} = \frac{z-5}{5}$ are coplanar, Also find the equation of the plane containing these two lines.
Solution:
From the lines we have,
$(x_1, y_1, z_1) = (-3, 1, 5)$ and $(b_1, b_2, b_3) = (-3, 1, 5)$
$(x_2, y_2, z_2) = (-1, 2, 5)$ and $(d_1, d_2, d_3) = (-1, 2, 5)$
Condition for coplanarity

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$$

$$\begin{vmatrix} 2 & 1 & 0 \\ -3 & 1 & 5 \\ -1 & 2 & 5 \end{vmatrix} = 2(5 - 10) - 1(-15 + 5) + 0(-6 + 1) = -10 + 10 = 0$$

Given two lines are coplanar

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$$

$$\begin{vmatrix} x+3 & y-1 & z-5 \\ -3 & 1 & 5 \\ -1 & 2 & 5 \end{vmatrix} = 0$$

(x + 3)[5 – 10] – (y – 1)[-15 + 5] + (z – 5)[-6 + 1] = 0

5(x + 3) + 10(y – 1) – 5(z – 5) = 0

(÷ by 5) ⇒ (x + 3) -2(y – 1) + (z – 5) = 0

x + 3 – 2y + 2 + z – 5 = 0

x – 2y + z = 0

or

$$\begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$$

$$\begin{vmatrix} x+1 & y-2 & z-5 \\ -3 & 1 & 5 \\ -1 & 2 & 5 \end{vmatrix} = 0$$

(x + 1)(5 – 10) – (y – 2)(-15 + 5) + (z – 5)(-6 + 1) = 0

-5(x + 1) + 10(y – 2) – 5(z – 5) = 0

(x + 1) – 2(y – 2) + (z – 5) = 0

x + 1 – 2y + 4 + z – 5 = 0

x – 2y + z = 0

Ex 6.9

Question 1.
Find the equation of the plane passing through the line of intersection of the planes $\vec{r} \cdot (2\hat{i} - 7\hat{j} + 4\hat{k}) = 3$ and 3x – 5y + 4z + 11 = 0, and the point (-2, 1, 3).
Solution:
Given planes are
$\vec{r} \cdot (2\hat{i} - 7\hat{j} + 4\hat{k}) = 3$
2x – 7y + 4z – 3 = 0 and 3x – 5y + 4z + 11 = 0
Equation of a plane which passes through the line of intersection of the planes
(2x – 7y + 4z – 3) + λ(3x – 5y + 4z + 11) = 0 …………… (1)
This passes through the point (-2, 1, 3).
(1) ⇒ (-4 – 7 + 12 – 3) + λ(-6 – 5 + 12 + 11) = 0
-2 + λ(12) = 0 ⇒ 12λ = 2
$\lambda = \frac{2}{12} \Rightarrow \lambda = \frac{1}{6}$
The required equation is
(1) ⇒ (2x – 7y + 4z – 3) + $\frac{1}{6}$ (3x – 5y + 4z + 11) = 0
12x – 42y + 24z – 18 + 3x – 5y + 4z + 11 = 0
15x – 47y + 28z – 7 = 0

Question 2.
Find the equation of the plane passing through the line of intersection of the planes x + 2y + 3z = 2 and x – y + z + 11 = 3, and at a distance $\frac{2}{\sqrt{3}}$ from the point (3, 1, -1).
Solution:
Equation of a plane which passes through the line of intersection of the plane
(x + 2y + 3z – 2) + λ(x – y + z – 3) = 0

$(1 + \lambda)x + (2 - \lambda)y + (3 + \lambda)z + (-2 - 3\lambda) = 0$ ………….. (1)

Given $$\left|\frac{3(\lambda+1)+1(2-\lambda)-1(3+\lambda)-2-3\lambda}{\sqrt{(\lambda+1)^2+(2-\lambda)^2+(\lambda+3)^2}}\right| = 0$$

$$\left|\frac{\cancel{3\lambda}+\cancel{3}+\cancel{2}-\lambda-\cancel{3}-\lambda-\cancel{2}-\cancel{3\lambda}}{\sqrt{\lambda^2+1+2\lambda+4+\lambda^2-4\lambda+\lambda^2+9+6\lambda}}\right| = \frac{2}{\sqrt{3}}$$

$$\left|\frac{-2\lambda}{\sqrt{3\lambda^2+4\lambda+14}}\right| = \frac{2}{\sqrt{3}}$$

Squaring on both sides $$\frac{\cancel{4}\lambda^2}{3\lambda^2+4\lambda+14} = \frac{\cancel{4}}{3}$$

$$\cancel{3\lambda^2} = \cancel{3\lambda^2} + 4\lambda + 14$$

$$4\lambda = -14$$

$$\lambda = \frac{-7}{2}$$

Substituting in (1)

$$(1) \Rightarrow \left(1-\frac{7}{2}\right)x+\left(2+\frac{7}{2}\right)y+\left(3-\frac{7}{2}\right)z-2-3\left(\frac{-7}{2}\right) = 0$$

$$\left(\frac{-5}{2}\right)x+\left(\frac{11}{2}\right)y+\left(\frac{-1}{2}\right)z+\frac{17}{2} = 0$$

$$-5x + 11y - z + 17 = 0$$

(or) $$5x - 11y + z - 17 = 0$$

Question 3.
Find the angle between the line $\vec{r} = (2\hat{i} - \hat{j} + \hat{k}) + t(\hat{i} + 2\hat{j} - 2\hat{k})$ and the plane $\vec{r} \cdot (6\hat{i} + 3\hat{j} + 2\hat{k}) = 8$.
Solution:

Angle between the line and a plane

$$\sin\theta = \frac{|\vec{b}\cdot\vec{n}|}{|\vec{b}||\vec{n}|} \quad \text{...(1)}$$

$$\vec{b} = \vec{i} + 2\vec{j} - 2\vec{k}$$

$$\vec{n} = 6\vec{i} + 3\vec{j} + 2\vec{k}$$

$$\vec{b}\cdot\vec{n} = 6 + 6 - 4 = 8$$

$$|\vec{b}| = \sqrt{1+4+4} = \sqrt{9} = 3$$

$$|\vec{n}| = \sqrt{36+9+4} = \sqrt{49} = 7$$

$$(1) \Rightarrow \quad \sin\theta = \frac{8}{(3)(7)} \quad \Rightarrow \theta = \sin^{-1}\left(\frac{8}{21}\right)$$

Question 4.
Find the angle between the planes $\vec{r}\cdot(\hat{i} + \hat{j} - 2\hat{k}) = 3$ and $2x - 2y + z = 2$.
Solution:
Angle between given two planes

$$\cos\theta = \frac{|\vec{n}_1\cdot\vec{n}_2|}{|\vec{n}_1||\vec{n}_2|} \quad \text{...(1)}$$

$$\vec{n}_1 = \vec{i} + \vec{j} - 2\vec{k}\,,\ \vec{n}_2 = 2\vec{i} - 2\vec{j} + \vec{k}$$

$$\vec{n}_1\cdot\vec{n}_2 = 2 - 2 - 2 = -2$$

$$|\vec{n}_1| = \sqrt{1+1+4} = \sqrt{6}$$

$$|\vec{n}_2| = \sqrt{4+4+1} = 3$$

$$(1) \Rightarrow \quad \cos\theta = \frac{|-2|}{3\sqrt{6}} \quad \Rightarrow \theta = \cos^{-1}\left(\frac{2}{3\sqrt{6}}\right)$$

Question 5.
Find the equation of the plane which passes through the point (3, 4, -1)and is parallel to the plane $2x - 3y + 5z + 7 = 0$. Also, find the distance between the two planes.
Solution:
The required equation parallel to the plane
$2x - 3y + 5z + 7 = 0$ ………….. (1)
$2x - 3y + 5z + \lambda = 0$ ………….. (2)
This passes through (3, 4, -1)
(2) $\Rightarrow 2(3) - 3(4) + 5(-1) + \lambda = 0$
$6 - 12 - 5 + 1 = 0$
$\lambda = 11$
(2) $\Rightarrow$ The required equation is $2x - 3y + 5z + 11 = 0$ …………… (3)
∴ Now, distance between the above parallel lines (1) and (3)

a = 2, b = -3, c = 5, d_1 = 7, d_2 = 11

$$\text{Distance} = \left|\frac{d_2 - d_1}{\sqrt{a^2 + b^2 + c^2}}\right| = \left|\frac{11-7}{\sqrt{(2)^2 + (-3)^2 + (5)^2}}\right| = \left|\frac{4}{\sqrt{4+9+25}}\right|$$

$$= \frac{4}{\sqrt{38}}$$

Question 6.
Find the length of the perpendicular from the point (1, -2, 3) to the plane x – y + z = 5.
Solution:
Perpendicular length from the point (x_1, y_1, z_1) to the plane ax + by + cz + d = 0 is

$$\left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right|$$

Given point (1, –2, 3) and the plane $x - y + z = 5$

$$\therefore \text{Length of the perpendicular} = \left|\frac{1-(-2)+3-5}{\sqrt{(1)^2 + (-1)^2 + (1)^2}}\right| = \left|\frac{1+2+3-5}{\sqrt{3}}\right| = \frac{1}{\sqrt{3}} \text{ units}$$

Question 7.
Find the point of intersection of the line x – 1 = $\frac{y}{2}$ = z + 1 with the plane 2x – y + 2z = 2. Also, find the angle between the line and the plane.
Solution:
Any point on the line x – 1 = $\frac{y}{2}$ = z + 1 is
x – 1 = $\frac{y}{2}$ = z + 1 = λ,(say)
(λ + 1, 2λ, λ – 1)
This passes through the plane 2x – y + 2z = 2
2(λ + 1) – 2λ + 2(λ – 1) = 2
2λ + 2 – 2λ + 2λ – 2 = 2
λ = 1
∴ The required point of intersection is (2, 2, 0)

Angle between the line and the plane is

$$\sin\theta = \frac{|\vec{b}\cdot\vec{n}|}{|\vec{b}||\vec{n}|} \quad \ldots(1)$$

$$\vec{b} = \vec{i}+2\vec{j}+\vec{k} \text{ and } \vec{n} = 2\vec{i}-\vec{j}+2\vec{k}$$

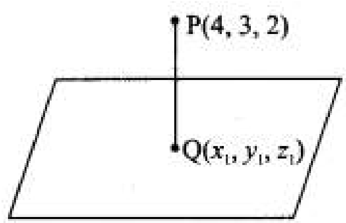
$$\vec{b}\cdot\vec{n} = 2-2+2=2$$

$$|\vec{b}| = \sqrt{1+4+1}=\sqrt{6}$$

$$|\vec{n}| = \sqrt{4+1+4}=\sqrt{9}=3$$

$$(1) \Rightarrow \quad \sin\theta = \frac{2}{3\sqrt{6}} \quad \Rightarrow \theta = \sin^{-1}\left(\frac{2}{3\sqrt{6}}\right)$$

Question 8.
Find the coordinates of the foot of the perpendicular and length of the perpendicular from the point (4, 3, 2) to the plane x + 2y + 3z = 2
Solution:



Direction of the normal plane (1, 2, 3)
d.c.s of the PQ is $\frac{x_1-4}{1} = \frac{y_1-3}{2} = \frac{z_1-2}{3}$ = k
x_1 = k + 4, y_1 = 2k + 3, z_1 = 3k + 2
This passes through the plane x + 2y + 3z = 2
k + 4 + 2(2k + 3) + 3(3k + 2) = 2
k + 4 + 4k + 6 + 9k + 6 = 2
14k = 2 – 16 ⇒ 14k = -14
k = -1
∴ The coordinate of the foot of the perpendicular is (3, 1, -1)
∴ Length of the perpendicular to the plane is

$$= \left|\frac{4+2(3)+3(2)-2}{\sqrt{(1)^2+(2)^2+(3)^2}}\right| = \left|\frac{4+6+6-2}{\sqrt{1+4+9}}\right| = \frac{14}{\sqrt{14}} = \sqrt{14} \text{ units}$$

Additional Problems

Question 1.
Find the point of intersection of the line passing through the two points (1, 1, – 1); (-1, 0,1) and the xy-plane.
Solution:

The equation of the line passing through (1, 1,-1) and (-1, 0, 1) is

$$\frac{x-1}{2}=\frac{y-1}{1}=\frac{z+1}{-2}$$

It meets the *xy*-plane *i.e.* $z=0$

$$\therefore \quad \frac{x-1}{2}=\frac{y-1}{1}=\frac{1}{-2} \qquad \Rightarrow x=0, y=\frac{1}{2}$$

The required point is $\left(0,\frac{1}{2},0\right)$.

Question 2.
Find the co-ordinates of the point where the line $\vec{r}=(\vec{i}+2\vec{j}-5\vec{k})+t(2\vec{i}-3\vec{j}+4\vec{k})$ meets the plane $\vec{r}\cdot(2\vec{i}+4\vec{j}-\vec{k})$ = 3.
Solution:
The equation of the straight line in the cartesian form is
$\frac{x-1}{2}=\frac{y-2}{-3}=\frac{z+5}{4}$ = λ (say)
∴ Any point on this line is of the form (2λ + 1, -3λ, + 2, 4λ – 5)
The cartesian equation of the plane is 2x + 4y – z – 3 = 0
But the required point lies on this plane.
∴ 2(2λ + 1) + 4(-3λ + 2) – (4λ – 5) – 3 = 0 ⇒ λ = 1
∴ The required point is (3, -1, -1).

Question 3.
Find the point of intersection of the line $\vec{r}=(\vec{j}-\vec{k})+s(2\vec{i}-\vec{j}+\vec{k})$ and xz-plane.
Solution:
The given point is (0, 1, -1); parallel vector is $2\vec{i}-\vec{j}+\vec{k}$.
The equation of the line passing through the point (0, 1, -1) and parallel to the vector $2\vec{i}-\vec{j}+\vec{k}$ is

$$\frac{x-0}{2}=\frac{y-1}{-1}=\frac{z+1}{1}$$

It meets the *xz*-plane *i.e.* $y=0$

$$\Rightarrow \quad \frac{x}{2}=1=\frac{z+1}{1} \qquad \Rightarrow \frac{x}{2}=1 \Rightarrow x=2$$

$$\frac{z+1}{1}=1 \qquad \Rightarrow z+1=1 \Rightarrow z=0$$

∴ The required point is (2, 0, 0)

Question 4.
Find the meeting point of the line $\vec{r}=(2\vec{i}+\vec{j}-3\vec{k})+t(2\vec{i}-\vec{j}-\vec{k})$ and the plane x – 2y + 3z + 7 = 0.
Solution:
The equation of the straight line in cartesian form is
$\frac{x-2}{2}=\frac{y-1}{-1}=\frac{z+3}{-1}$ = λ (say)
Any point on the line is (2λ, + 2, -λ + 1, λ – 3)
The point lies on the plane x – 2y + 3z + 7 = 0

⇒ 2λ, + 2 – 2(-λ + 1) + 3(-λ, – 3) + 7 = 0 .
2λ + 2 + 2λ – 2 – 3λ – 9 + 7 = 0
λ – 2 = 0 ⇒ λ = 2
So 2λ + 2 = 6; -λ + 1 = -1; -λ – 3 = – 5
∴ The required point is (6, -1, -5)

Question 5.
Show that the following planes are at right angles:

$$\vec{r}\cdot\left(2\vec{i}-\vec{j}+\vec{k}\right)=15 \textbf{ and } \vec{r}\cdot\left(\vec{i}-\vec{j}-3\vec{k}\right)=3$$

Solution:
The normal vectors are
$\vec{n}_1 = 2\vec{i} - \vec{j} + \vec{k}$ and $\vec{n}_2 = +\vec{i} - \vec{j} - 3\vec{k}$
$\vec{n}_1 \cdot \vec{n}_2$ = (2)(1) + (-1)(-1) + (1)(-3) = 2 + 1 – 3 = 0
⇒ $\vec{n}_1$ ⊥r to $\vec{n}_2$, i.e., the normals to the planes are at right angles. So, the planes are at right angles.

Question 6.
The planes $\vec{r} \cdot (2\vec{i} + \lambda\vec{j} - 3\vec{k})$ = 10 and $\vec{r} \cdot (\lambda\vec{i} + 3\vec{j} + \vec{k})$ = 5 are perpendicular. Find λ.
Solution:
Since the planes are perpendicular, the angle between the normals = 90°.
The normals are $\vec{n}_1 = 2\vec{i} + \lambda\vec{j} - 3\vec{k}$ and $\vec{n}_2 = \lambda\vec{i} + 3\vec{j} + \vec{k}$
$\vec{n}_1 \cdot \vec{n}_2$ = 0 [∵ θ = π/2]
⇒ (2)(λ) + (λ)(3) + (-3)(1) = 0 ⇒ 2λ + 3λ – 3 = 0
5λ – 3 = 0 ⇒ 5λ = 3 ⇒ λ = 3/5

Question 7.
Find the angle between the line $\frac{x-2}{3} = \frac{y+1}{-1} = \frac{z-3}{-2}$ and the plane 3x + 4y + z = 0.
Solution:
The angle between the line $\vec{r} = \vec{a} + t\vec{b}$ and the plane $\vec{r} \cdot \vec{n}$ = p is given by the formula

$$\sin\theta = \frac{\left|\vec{b}\cdot\vec{n}\right|}{\left|\vec{b}\right|\left|\vec{n}\right|}$$

$$\text{So, } \sin\theta = \frac{\left(3\vec{i}-\vec{j}-2\vec{k}\right)\cdot\left(3\vec{i}+4\vec{j}+\vec{k}\right)}{\left|3\vec{i}-\vec{j}-2\vec{k}\right|\left|3\vec{i}+4\vec{j}+\vec{k}\right|} = \frac{9-4-2}{\sqrt{9+1+4}\sqrt{9+16+1}}$$

$$= \frac{3}{\sqrt{14}\sqrt{26}} = \frac{3}{\sqrt{364}} = \frac{3}{\sqrt{4\times 91}} = \frac{3}{2\sqrt{91}}$$

$$\text{So, } \theta = \sin^{-1}\frac{3}{2\sqrt{91}}$$

Question 8.
Find the angle between the line $\vec{r} = \vec{i} + \vec{j} + 3\vec{k} + \lambda(2\vec{i} + \vec{j} - \vec{k})$ and the plane $\vec{r} \cdot (\vec{i} + \vec{j})$ = 1,

Solution:

$$\text{Here } \vec{b} = 2\vec{i} + \vec{j} - \vec{k} \text{ and } \vec{n} = \vec{i} + \vec{j}$$

The angle between the line and the plane is given by

$$\sin\theta = \frac{\left|\vec{b}\cdot\vec{n}\right|}{\left|\vec{b}\right|\left|\vec{n}\right|}$$

$$\vec{b}\cdot\vec{n} = \left(2\vec{i} + \vec{j} - \vec{k}\right)\cdot\left(\vec{i} + \vec{j}\right) = 2 + 1 = 3$$

$$\vec{b} = \sqrt{4+1+1} = \sqrt{6}$$

$$\vec{n} = \sqrt{1+1} = \sqrt{2}$$

$$\therefore \qquad \sin\theta = \frac{3}{\sqrt{6}\sqrt{2}} = \frac{3}{\sqrt{12}} = \frac{3}{\sqrt{4\times 3}} = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{}. \text{ So}\theta = \frac{\pi}{3}$$

Ex 6.10

Choose the correct or the most suitable answer from the given four alternatives:
Question 1.
If $\vec{a}$ and $\vec{b}$ are parallel vector, then $[\vec{a}, \vec{c}, \vec{b}]$ is equal to ………………..
(a) 2
(b) -1
(c) 1
(d) 0
Solution:
(d) 0
Hint:
$\vec{a}$ and $\vec{b}$ are parallel vectors, so $\vec{a} \times \vec{b} = 0$
then $[\vec{a}\vec{c}\vec{b}] = -[\vec{a}\vec{b}\vec{c}] = -(\vec{a} \times \vec{b}) \cdot \vec{c} = 0$

Question 2.
If a vector $\vec{\alpha}$ lies in the plane of $\vec{\beta}$ and $\vec{\gamma}$, then ………..
(a) $[\vec{\alpha}, \vec{\beta}, \vec{\gamma}] = 1$ (b) $[\vec{\alpha}, \vec{\beta}, \vec{\gamma}] = -1$ (c) $[\vec{\alpha}, \vec{\beta}, \vec{\gamma}] = 0$ (d) $[\vec{\alpha}, \vec{\beta}, \vec{\gamma}] = 2$
Solution:
(c) $[\vec{\alpha}, \vec{\beta}, \vec{\gamma}] = 0$
Hint:
Since $[\vec{\alpha}, \vec{\beta}, \vec{\gamma}]$ are lie in the same plane
so $[\vec{\alpha}, \vec{\beta}, \vec{\gamma}] = 0$

Question 3.
If $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = \vec{c} \cdot \vec{a} = 0$, then the value of $|[\vec{a}, \vec{b}, \vec{c}]|$ is ………………
(a) $|\vec{a}||\vec{b}||\vec{c}|$
(b) $\frac{1}{3}|\vec{a}||\vec{b}||\vec{c}|$
(c) 1
(d) -1
Solution:
(a) $|\vec{a}||\vec{b}||\vec{c}|$

Hint:

$$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{c} = \vec{c}\cdot\vec{a} = 0$$

$\vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular vectors

$$\left|\left[\vec{a}, \vec{b}, \vec{c}\right]\right| = |\vec{a}|\,|\vec{b}|\,|\vec{c}| \text{ is volume of cuboid where } \vec{a}, \vec{b}, \vec{c} \text{ are coterminus edges.}$$

Question 4.
If $\vec{a}, \vec{b}, \vec{c}$ are three unit vectors such that $\vec{a}$ is perpendicular to $\vec{b}$, and is parallel to $\vec{c}$ then $\vec{a} \times (\vec{b} \times \vec{c})$ is equal to …………………
(a) $\vec{a}$
(b) $\vec{b}$
(c) $\vec{c}$
(d) $\vec{0}$
Solution:
(b) $\vec{b}$
Hint:

$\vec{a}$ is perpendicular to $\vec{b} \quad \Rightarrow \quad \vec{a}\cdot\vec{b} = 0$

$\vec{a}$ is parallel to $\vec{c} \quad \Rightarrow \quad \vec{a}\times\vec{c} = 0$

then
$$\begin{aligned}\vec{a}\times\left(\vec{b}\times\vec{c}\right) &= (\vec{a}\cdot\vec{c})\vec{b} - \left(\vec{a}\cdot\vec{b}\right)\vec{c} \\ &= (1)\vec{b} - 0\,\vec{c} \qquad \left[\vec{a}\times\vec{c} \text{ are unit vectors}\right] \\ &= \vec{b}\end{aligned}$$

Question 5.
If $\left[\vec{a}, \vec{b}, \vec{c}\right] = 1$, then the value of $\dfrac{\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)}{(\vec{c}\times\vec{a})\cdot\vec{b}} + \dfrac{\vec{b}\cdot(\vec{c}\times\vec{a})}{\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}} + \dfrac{\vec{c}\cdot\left(\vec{a}\times\vec{b}\right)}{\left(\vec{c}\times\vec{b}\right)\cdot\vec{a}}$ is ....

(a) 1
(b) -1
(c) 2
(d) 3
Solution:
(a) 1
Hint:

$$\left[\vec{a}, \vec{b}, \vec{c}\right] = 1 \text{ (Given)}$$

$$\frac{\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)}{(\vec{c}\times\vec{a})\cdot\vec{b}} + \frac{\vec{b}\cdot(\vec{c}\times\vec{a})}{\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}} + \frac{\vec{c}\cdot\left(\vec{a}\times\vec{b}\right)}{\left(\vec{c}\times\vec{b}\right)\cdot\vec{a}} = \frac{\left[\vec{a}\,\vec{b}\,\vec{c}\right]}{\left[\vec{a}\,\vec{b}\,\vec{c}\right]} + \frac{\left[\vec{a}\,\vec{b}\,\vec{c}\right]}{\left[\vec{a}\,\vec{b}\,\vec{c}\right]} + \frac{\left[\vec{a}\,\vec{b}\,\vec{c}\right]}{-\left[\vec{a}\,\vec{b}\,\vec{c}\right]} = 1+1-1 = 1$$

Question 6.
The volume of the parallelepiped with its edges represented by the vectors $\hat{i} + \hat{j}, i + 2\hat{j}$, $\hat{i} + \hat{j} + \pi\hat{k}$ is

……………

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) π (d) $\frac{\pi}{4}$

Solution:
(c) π
Hint:

$$\text{Volume} = [\vec{a}\vec{b}\vec{c}] = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & \pi \end{vmatrix}$$

$= \pi(2 - 1) = \pi$ cubic units

Question 7.
If $\vec{a}$ and $\vec{b}$ are unit vectors such that $[\vec{a}, \vec{b}, \vec{a} \times \vec{b}] = \frac{\pi}{4}$ then the angle between $\vec{a}$ and $\vec{b}$ is ………..

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

Solution:
(a) $\frac{\pi}{6}$
Hint:

$$\left[\vec{a}, \vec{b}, \vec{a}\times\vec{b}\right] = \frac{1}{4}$$
$$\left(\vec{a}\times\vec{b}\right)\cdot\left(\vec{a}\times\vec{b}\right) = \frac{1}{4}$$
$$\left(\vec{a}\times\vec{b}\right)^2 = \frac{1}{4}$$
$$\left|\vec{a}\times\vec{b}\right|^2 = \frac{1}{4}$$
$$\left|\vec{a}\times\vec{b}\right| = \frac{1}{2}$$

$$|\vec{a}||\vec{b}|\sin\theta = \frac{1}{2}$$
$$(1)(1)\sin\theta = \frac{1}{2}$$
$$\sin\theta = \frac{1}{2}$$
$$\theta = \sin^{-1}\left(\frac{1}{2}\right)$$
$$\theta = \frac{\pi}{6}$$

Question 8.
If $\vec{a} = \hat{i} + \hat{j} + \hat{k}, \vec{b} = \hat{i} + \hat{j}, \vec{c} = \hat{i}$ and $\left(\vec{a}\times\vec{b}\right)\times\vec{c} = \lambda\vec{a} + \mu\vec{b}$, then the value of $\lambda + \mu$ is ....
(a) 0
(b) 1
(c) 6
(d) 3
Solution:
(a) 0
Hint:

$$
\begin{aligned}
(\vec{a}\times\vec{b})\times\vec{c} &= (\vec{a}\cdot\vec{c})\vec{b}-(\vec{b}\cdot\vec{c})\vec{a}=\vec{b}-\vec{a}\\
\text{But}\quad (\vec{a}\times\vec{b})\times\vec{c} &= \lambda\vec{a}+\mu\vec{b}\\
\vec{b}-\vec{a} &= \lambda\vec{a}+\mu\vec{b}\\
-\vec{k} &= \lambda\vec{i}+\lambda\vec{j}+\lambda\vec{k}+\mu\vec{i}+\mu\vec{j}
\end{aligned}
$$

Equate corresponding coefficients on both sides
$\lambda + \mu = 0$ and $\lambda = -1$ this gives $\mu = 1$
∴ Then the value of $\lambda + \mu = 0$.

Question 9.
If $\vec{a},\vec{b},\vec{c}$ are three non-coplanar vectors such that $\vec{a}\times(\vec{b}\times\vec{c})=\frac{\vec{b}+\vec{c}}{\sqrt{2}}$, then the angle between $\vec{a}$ and $\vec{b}$ is …………
(a) 81
(b) 9
(c) 27
(d) 18
Solution:
(a) 81
Hint:

$$\left\{\left[\vec{a}\times\vec{b}\ \ \vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}\right]\right\}^2=\left\{\left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right]^2\right\}^2=\left\{(3)^2\right\}^2=\{9\}^2=81$$

Question 10.
If $\vec{a}, \vec{b}, \vec{c}$ are three non-coplanar vector such that $\vec{a}\times(\vec{b}\times\vec{c})=\frac{\vec{b}+\vec{c}}{\sqrt{2}}$, then the angle between $\vec{a}$ and $\vec{b}$ is ....

(a) $\frac{\pi}{2}$ (b) $\frac{3\pi}{4}$ (c) $\frac{\pi}{4}$ (d) π

Solution:
(b) $\frac{3\pi}{4}$

Hint:

$$\vec{a} \times (\vec{b} \times \vec{c}) = \frac{\vec{b}+\vec{c}}{\sqrt{2}}$$

$$(\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c} = \frac{\vec{b}+\vec{c}}{\sqrt{2}}$$

$\therefore$

$$\vec{a}\cdot\vec{c} = \frac{1}{\sqrt{2}}$$

$$\vec{a}\cdot\vec{b} = -\frac{1}{\sqrt{2}}$$

$$|\vec{a}|\,|\vec{b}|\cos\theta = -\frac{1}{\sqrt{2}}$$

$$\cos\theta = -\frac{1}{\sqrt{2}}$$

$$\theta = \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \frac{3\pi}{4}$$

Question 11.
If the volume of the parallelepiped with $\vec{a} \times \vec{b}, \vec{b} \times \vec{c}, \vec{c} \times \vec{a}$ as coterminous edges is 8 cubic units, then the volume of the parallelepiped with $(\vec{a} \times \vec{b}) \times (\vec{b} \times \vec{c}), (\vec{b} \times \vec{c}) \times (\vec{c} \times \vec{a})$ and $(\vec{c} \times \vec{a}) \times (\vec{a} \times \vec{b})$ as coterminous edges is ………………
(a) 8 cubic units
(b) 512 cubic units
(c) 64 cubic units
(d) 24 cubic units
Solution:
(c) 64 cubic units
Hint:

$$\left[\vec{a}\times\vec{b},\ \vec{b}\times\vec{c},\ \vec{c}\times\vec{a}\right] = 8 \text{ (Given)}$$

$$\left[(\vec{a}\times\vec{b})\times(\vec{b}\times\vec{c}),(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a}),(\vec{c}\times\vec{a})\times(\vec{a}\times\vec{b})\right] = \left[(\vec{a}\times\vec{b}),(\vec{b}\times\vec{c}),(\vec{c}\times\vec{a})\right]^2$$
$$= (8)^2 = 64$$

Question 12.
Consider the vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ such that $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \vec{0}$. Let P_1 and P_2 be the planes determined by the pairs of vectors $\vec{a}, \vec{b}$ and $\vec{c}, \vec{d}$ respectively. Then the angle between P_1 and P_2 is ……………..
(a) 0°
(b) 45°
(c) 60°
(d) 90°
Solution:
(a) 0°

Hint:

$$P_1 \to \vec{a}, \vec{b} \qquad P_2 \to \vec{c}, \vec{d}$$
$$\vec{n}_1 \to \vec{a} \times \vec{b} \qquad \vec{n}_2 \to \vec{c} \times \vec{d}$$

Given. $\vec{n}_1 \times \vec{n}_2 = 0$ $(\vec{n}_1, \vec{n}_2$ are normal unit vectors$)$

$$|\vec{n}_1||\vec{n}_2|\sin\theta = 0$$
$$\sin\theta = 0 \qquad \Rightarrow \theta = 0°$$

Question 13.

If $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c}$, where $\vec{a}, \vec{b}, \vec{c}$ are any three vectors such that $\vec{b} \cdot \vec{c} \neq 0$ and $\vec{a} \cdot \vec{b} \neq 0$, then $\vec{a}$ and $\vec{c}$ are ....

(a) perpendicular
(b) parallel
(c) inclined at an angle $\frac{\pi}{3}$
(d) inclined at an angle $\frac{\pi}{6}$

Solution:
(b) parallel
Hint:

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c}$$
$$\cancel{(\vec{a} \cdot \vec{c})\vec{b}} - (\vec{a} \cdot \vec{b})\vec{c} = \cancel{(\vec{a} \cdot \vec{c})\vec{b}} - (\vec{b} \cdot \vec{c})\vec{a}$$
$$\vec{a} = \left(\frac{\vec{a} \cdot \vec{b}}{\vec{b} \cdot \vec{c}}\right)\vec{c}$$
$$\vec{a} = \lambda\vec{c}$$

$\therefore$ $\vec{a}$ and $\vec{c}$ are parallel.

Question 14.

If $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}, \vec{b} = \hat{i} + 2\hat{j} - 5\hat{k}, \vec{c} = 3\hat{i} + 5\hat{j} - \hat{k}$, then a vector perpendicular to $\vec{a}$ and lies in the plane containing $\vec{b}$ and $\vec{c}$ is

(a) $-17\hat{i} + 21\hat{j} - 97\hat{k}$
(b) $-17\hat{i} + 21\hat{j} - 123\hat{k}$
(c) $-17\hat{i} - 21\hat{j} + 97\hat{k}$
(d) $-17\hat{i} - 21\hat{j} - 97\hat{k}$

Solution:
(d) $-17\hat{i} - 21\hat{j} - 97\hat{k}$

Hint:

A vector $\perp$r to $\vec{a}$ and lies in the plane containing $\vec{b}$ and $\vec{c}$ is $\vec{a}\times\left(\vec{b}\times\vec{c}\right)$

$$\vec{b}\times\vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -5 \\ 3 & 5 & -1 \end{vmatrix} = 23\vec{i} - 14\vec{j} - \vec{k}$$

$$\vec{a}\times\left(\vec{b}\times\vec{c}\right) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -1 \\ 23 & -14 & -1 \end{vmatrix} = -17\vec{i} - 21\vec{j} - 97\vec{k}$$

Question 15.

The angle between the lines $\frac{x-2}{3} = \frac{y+1}{-2}, z = 2$ and $\frac{x-1}{1} = \frac{2y+3}{3} = \frac{z+5}{2}$ is ………..

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

Solution:
(d) $\frac{\pi}{2}$
Hint:

$$\frac{x-2}{3} = \frac{y+1}{-2}, z = 2 \text{ and } \frac{x-1}{1} = \frac{2y+3}{3} = \frac{z+5}{2}$$

$$\frac{x-1}{1} = \frac{y+\frac{3}{2}}{\frac{3}{2}} = \frac{z+5}{2}$$

d.r.s : (3, –2, 0) and d.c.s $\left(1, \frac{3}{2}, 2\right)$

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$\cos\theta = \frac{3-3+0}{\sqrt{(3)^2 + (-2)^2 + (0)^2}\sqrt{(1)^3 + \left(\frac{3}{2}\right)^2 + (2)^2}}$$

$$\cos\theta = 0$$

$$\theta = \frac{\pi}{2}$$

Question 16.

If the line $\frac{x-2}{3}=\frac{y-1}{-5}=\frac{z+2}{2}$ lies in the plane x + 3 + αz + β = 0, then (α, β) is …………..

(a) (-5, 5)
(b) (-6, 7)
(c) (5, -5)
(d) (6, -7)

Solution:
(d) (-6, 7)

Hint:
d.c.s of the first line = (3, -5, 2)
d.c.s of the line perpendicular to plane = (1, 3, -α)
$a_1a_2 + b_1b_2 + c_1c_2 = 0$
3 – 15 – 2α = 0 => -12 – 2α = 0
-2α =12 => α = -6
Plane passes through the point (2, 1, -2) so
2 + 3 + 6(-2) + β = 0 => β = 7
(α, β) = (-6, 7)

Question 17.

The angle between the line $\vec{r}=(\hat{i}+2\hat{j}-3\hat{k})+t(2\hat{i}+\hat{j}-2\hat{k})$ and the plane $\vec{r}\cdot(\hat{i}+\hat{j})+4=0$ is ……………

(a) 0°
(b) 30°
(c) 45°
(d) 90°

Solution:
(c) 45°

Hint:

$$\vec{b} = 2\vec{i}+\vec{j}-2\vec{k}\ ;\ \vec{n}=\vec{i}+\vec{j}$$

$$\sin\theta = \frac{\vec{b}\cdot\vec{n}}{|\vec{b}||\vec{n}|}$$

$$\sin\theta = \frac{2+1}{\sqrt{4+1+4}\sqrt{1+1}} = \frac{\cancel{3}}{\cancel{3}\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\theta = 45^\circ$$

Question 18.

The coordinates of the point where the line $\vec{r}=(6\hat{i}-\hat{j}-3\hat{k})+t(-\hat{i}+4\hat{k})$ meets the plane $\vec{r}\cdot(\hat{i}+\hat{j}-\hat{k})$ are

(a) (2, 1, 0)
(b) (7, -1, -7)
(c) (1, 2, -6)
(c) (5, -1, 1)

Solution:
(d) (5, -1, 1)

Hint:

Cartesian equation of the line

$\frac{x-6}{-1} = \frac{y+1}{0} + \frac{z+3}{4} = \lambda$

(-λ + 6, -1, 4λ – 3)

This meets the plane x + y – z = 3

-λ + 6 – 1 – 41 + 3 = 3 ⇒ -5λ = -5

λ = 1

The required point (5, -1, 1).

Question 19.

Distance from the origin to the plane 3x – 6y + 2z + 7 = 0 is ………………

(a) 0
(b) 1
(c) 2
(d) 3

Solution:

(b) 1

Hint:

Distance from the origin (0, 0, 0) to the plane

$$= \left|\frac{3(0)-6(0)+2(0)+7}{\sqrt{9+36+4}}\right| = \frac{7}{\sqrt{49}} = \frac{7}{7} = 1$$

Question 20.

The distance between the planes x + 2y + 3z + 7 = 0 and 2x + 4y + 6z + 7 = 0 is ……………

(a) $\frac{\sqrt{7}}{2\sqrt{2}}$ (b) $\frac{7}{2}$ (c) $\frac{\sqrt{7}}{2}$ (d) $\frac{7}{2\sqrt{2}}$

Solution:

(a) $\frac{\sqrt{7}}{2\sqrt{2}}$

Hint:

x + 2y + 3z + 1 = 0; 2x + 4y + 6z + 7 = 0

Multiplying 2 on both sides

2x + 4y + 6z + 14 = 0 .

a = 2, b = 4, c = 6, d_1 = 14, d_2 = ?

$$\text{Distance} = \left|\frac{d_1-d_2}{\sqrt{a^2+b^2+c^2}}\right| = \left|\frac{14-7}{\sqrt{4+16+36}}\right| = \left|\frac{7}{\sqrt{56}}\right| = \frac{\cancel{7}\sqrt{7}}{2\sqrt{2}\cancel{\sqrt{7}}} = \frac{\sqrt{7}}{2\sqrt{2}}$$

Question 21.

If the direction cosines of a line are $\frac{1}{c}, \frac{1}{c}, \frac{1}{c}$, then ………………

(a) c = ±3
(b) c = ± $\sqrt{3}$
(c) c > 0
(d) 0 < c < 1

Solution:

(b) c = ± $\sqrt{3}$

Hint:

We know that sum of the squares of direction cosines = 1

$$\frac{1}{c^2}+\frac{1}{c^2}+\frac{1}{c^2}=1 \qquad \Rightarrow \frac{3}{c^2}=1$$

$$c^2=3 \qquad \Rightarrow c=\pm\sqrt{3}$$

Question 22.
The vector equation $\vec{r}=(\hat{i}-2\hat{j}-\hat{k})+t(6\vec{j}-\hat{k})$ represents a straight line passing through the points …………… (a) (0, 6, -1) and (1, -2, -1) (b) (0, 6, -1) and (-1, -4, -2) (c) (1, -2, -1) and (1, 4, -2) (d) (1, -2, -1) and (0, -6, 1)
Solution:
(c) (1, -2, -1) and (1, 4, -2)
Hint:
The required vector equation is $\vec{r}=\vec{a}+t(\vec{b}-\vec{a})$

$$\therefore \quad \vec{a}=\vec{i}-2\vec{j}-\vec{k} \qquad \text{...(1)}$$

$$\vec{b}-\vec{a}=6\vec{j}-\vec{k}$$

$$\vec{b}=6\vec{j}-\vec{k}+\vec{a} \qquad \Rightarrow \vec{b}=6\vec{j}-\vec{k}+\vec{i}-2\vec{j}-\vec{k}$$

$$\vec{b}=\vec{i}+4\vec{j}-2\vec{k} \qquad \text{...(2)}$$

From (1) and (2) The points are (1, -2, -1) and (1, 4, -2)

Question 23.
If the distance of the point (1, 1, 1) from the origin is half of its distance from the plane x + y + z + k= 0, then the values of k are ………….. (a) ±3 (b) ±6 (c) -3, 9 (d) 3, -9
Solution:
(d) 3, -9
Hint:

$$\sqrt{3}=\frac{1}{2}\left[\frac{3+k}{\sqrt{3}}\right]$$

$$3=\pm\frac{1}{2}(3+k)$$

$$3+k=\pm 6$$

$$3+k=6 \qquad\qquad 3+k=-6$$

$$k=3 \qquad\qquad k=-9$$

Question 24.
If the planes $\vec{r}\cdot(2\hat{i}-\lambda\hat{j}+\hat{k})=3$ and $\vec{r}\cdot(4\hat{i}+\hat{j}-\mu\hat{k})$ = 5 are parallel, then the value of λ and μ are ……………
Solution:
(c) $-\frac{1}{2}$, -2

Hint:

$$\vec{a} = 2\vec{i} - \lambda\vec{j} + \vec{k} \text{ and } \vec{b} = 4\vec{i} + \vec{j} - \mu\vec{k}$$

Given $\vec{a}$ and $\vec{b}$ are parallel

$$\vec{a} = m\vec{b}$$

$$2\vec{i} - \lambda\vec{j} + \vec{k} = 4m\vec{i} + m\vec{j} - m\mu\vec{k}$$

$4m = 2$	$-\lambda = m$	$-m\mu = 1$
$m = 1/2$	$-\lambda = 1/2$	$\mu = -1/m$
	$\lambda = -1/2$	$\mu = -2$

so $(\lambda, \mu) = (-1/2, -2)$

Question 25.
If the length of the perpendicular from the origin to the plane 2x + 3y + λz = 1, λ > 0 is $-\frac{1}{5}$, then the value of λ is …………..
(a) $2\sqrt{3}$
(b) $3\sqrt{2}$
(c) 0
(d) 1
Solution:
(a) $2\sqrt{3}$
Hint:
Given length of perpendicular from origin to the plane = $-\frac{1}{5}$

$$\left|\frac{2(0)+3(0)+\lambda(0)-1}{\sqrt{4+9+\lambda^2}}\right| = \frac{1}{5}$$

$$\frac{1}{\sqrt{13+\lambda^2}} = \frac{1}{5} \qquad \Rightarrow 5 = \sqrt{13+\lambda^2}$$

Squaring on both sides

$$25 = 13 + \lambda^2 \qquad \Rightarrow \lambda^2 = 12$$

$$\lambda = \sqrt{12} \qquad \Rightarrow \lambda = 2\sqrt{3}$$

Additional Problems

Question 1.
If $\vec{a} = 2\hat{i} + \hat{j} - \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$ and $\vec{c} = \hat{i} - \hat{j} + 2\hat{k}$ then $\vec{a} \cdot (\vec{b} \times \vec{c}) =$ ....
(a) 6
(b) 10
(c) 12
(d) 24
Solution:
(c) 12

Hint:

$$\vec{a}\cdot\left(\vec{b}\times\vec{c}\right) = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & -1 & 2 \end{vmatrix} = 2(4+1)-1(2-1)-1(-1-2)$$
$$= 10 - 1 + 3 = 12$$

Question 2.

Let $\vec{a} = \hat{i} - \hat{k}, \vec{b} = x\hat{i} + \hat{j} + (1-x)\hat{k},\ \vec{c} = y\hat{i} + x\hat{j} + (1+x-y)\hat{k}$, then $\left[\vec{a}\ \vec{b}\ \vec{c}\right]$ depend on ....

(a) only x
(b) only y
(c) Neither x nor y
(d) Both x and y
Solution:
(c) Neither x nor y
Hint:

$$\left[\vec{a}\ \vec{b}\ \vec{c}\right] = \begin{vmatrix} 1 & 0 & -1 \\ x & 1 & 1-x \\ y & x & 1+x-y \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 1 \\ y & x & 1+x \end{vmatrix} \left[\text{Applying } C_3 \to C_3 + C_1\right]$$

Expanding along R_1, we get

$$\begin{vmatrix} 1 & 1 \\ x & 1+x \end{vmatrix} = 1 + x - x = 1$$

Question 3.

If $\vec{a}, \vec{b}, \vec{c}$ are three non-coplanar vectors and $\vec{p}, \vec{q}, \vec{r}$ are defined by the relations $\vec{p} = \frac{\vec{b}\times\vec{c}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}, \vec{q} = \frac{\vec{c}\times\vec{a}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}, \vec{r} = \frac{\vec{a}\times\vec{b}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}$, then $\vec{p}\cdot\left(\vec{a}+\vec{b}\right) + \vec{q}\cdot\left(\vec{b}+\vec{c}\right) + \vec{r}\left(\vec{c}+\vec{a}\right) =$ ....

(a) 0
(b) 1
(c) 2
(d) 3
Solution:
(b) 3

Hint:

$$\vec{p}\cdot\left(\vec{a}+\vec{b}\right) = \vec{p}\cdot\vec{a}+\vec{p}\cdot\vec{b}$$

$$\frac{\left(\vec{b}\times\vec{c}\right)\cdot\vec{a}}{\left[\vec{a}\,\vec{b}\,\vec{c}\right]}+\frac{\left[\vec{b}\times\vec{c}\right]\cdot\vec{b}}{\left[\vec{a}\,\vec{b}\,\vec{c}\right]} = \frac{\left[\vec{a}\,\vec{b}\,\vec{c}\right]}{\left[\vec{a}\,\vec{b}\,\vec{c}\right]}+0=1+0=1$$

Similarly $\quad \vec{q}\cdot\left(\vec{b}+\vec{c}\right) = 1$ and $\vec{r}\cdot\left(\vec{c}+\vec{a}\right)=1$

Hence, $\vec{p}\cdot\left(\vec{a}+\vec{b}\right)+\vec{q}\left(b+\vec{c}\right)+\vec{r}\cdot\left(\vec{c}+\vec{a}\right)=1+1+1=3$

Question 4.

The value of $\hat{i}\cdot\left(\hat{j}\times\hat{k}\right)+\hat{j}\cdot\left(\hat{k}\times\hat{i}\right)+\hat{k}\cdot\left(\hat{i}\times\hat{j}\right)$ = ………………
(a) 1
(b) 3
(c) -3
(d) 0
Solution:
(b) 3
Hint:

$$\hat{i}\cdot\left(\hat{j}\times\hat{k}\right)+\hat{j}\cdot\left(\hat{k}\times\hat{i}\right)+\hat{k}\cdot\left(\hat{i}\times\hat{j}\right) = \hat{i}\cdot\hat{i}+\hat{j}\cdot\hat{j}+\hat{k}\cdot\hat{k} \qquad \left[\because \hat{j}\times\hat{k}=\hat{i},\ \hat{k}\times\hat{i}=\hat{j} \text{ and } \hat{i}\times\hat{j}=\hat{k}\right]$$

$$\hat{i}^2+\hat{j}^2+\hat{k}^2 = 1+1+1=3$$

Question 5.

Let a, b, c be distinct non-negative numbers. If the vectors $a\hat{i}+a\hat{j}+c\hat{k},\ \hat{i}+\hat{k}$ and $c\hat{i}+c\hat{j}+b\hat{k}$ lie in a plane, then c is ……………..
(a) the A.M. of a and b
(b) the G.M. of a and b
(c) the H.M. of a and b
(d) equal to zero.
Solution:
(b) the G.M. of a and b
Hint:

$a\hat{i}+a\hat{j}+c\hat{k},\ \hat{i}+\hat{k}$ and $c\hat{i}+c\hat{j}+b\hat{k}$ are coplanar

$$\therefore \quad \begin{vmatrix} a & a & c \\ 1 & 0 & 1 \\ c & c & b \end{vmatrix} = 0 \qquad \Rightarrow a(0-c)-a(b-c)+c(c-0)=0$$

$$\Rightarrow -ac-ab+ac+c^2=0 \qquad \Rightarrow c^2=ab$$

Question 6.
The value of $\hat{i} \cdot (\hat{j} \times \hat{k}) + (\hat{i} \times \hat{k}) \cdot \hat{j}$ …………
(a) 1
(b) -1
(c) 0
(d) $\hat{j}$
Solution:
(c) 0
Hint:

$$\hat{i} \cdot (\hat{j} \times \hat{k}) + (\hat{i} \times \hat{k}) \cdot \hat{j} \Rightarrow \hat{i} \cdot \hat{i} + (-\hat{j})\hat{j} \qquad [\because \hat{j} \times \hat{k} = \hat{i} \text{ and } \hat{i} \times \hat{k} = -\hat{j}]$$

$$\Rightarrow \qquad \hat{i}^2 - \hat{j}^2 = 1 - 1 = 0$$

Question 7.
The value of $(\hat{i} - \hat{j}, \hat{j} - \hat{k}, \hat{k} - \hat{i})$ is …………..
(a) 0
(b) 1
(c) 2
(d) 3
Solution:
(a) 0
Hint:

$$(\hat{i} - \hat{j}) \cdot \left[\left(\hat{j} - \hat{k} \times (\hat{k} - \hat{i})\right)\right]$$

$$(\hat{i} - \hat{j}) \cdot \left[\hat{j} \times \hat{k} - \hat{j} \times \hat{i} - \hat{k} \times \hat{k} + \hat{k} \times \hat{i}\right] = \hat{i} \cdot \hat{i} + \hat{i} \cdot \hat{k} + \hat{i} \cdot \hat{j} - \hat{j} \cdot \hat{i} - \hat{j} \cdot \hat{k} - \hat{j} \cdot \hat{j}$$

$$= \hat{i}^2 + 0 + 0 - 0 - 0 - \hat{j}^2 = 1 - 1 = 0$$

Question 8.
If $\vec{u} = \vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b})$, then ……….
(a) $\vec{u}$ is a unit vector (b) $\vec{u} = \vec{a} + \vec{b} + \vec{c}$ (c) $\vec{u} = \vec{0}$ (d) $\vec{u} \neq \vec{0}$
Solution:
(c) $\vec{u} = \vec{0}$
Hint:

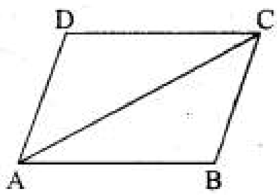
$$\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = \vec{0}$$

$$\Rightarrow \qquad \vec{u} = \vec{0}$$

Question 9.
The area of the parallelogram having a diagonal $3\vec{i} + \vec{j} - \vec{k}$ and a side $\vec{i} - 3\vec{j} + 4\vec{k}$ is ………………
(a) $10\sqrt{3}$ (b) $6\sqrt{30}$ (c) $\frac{3}{2}\sqrt{30}$ (d) $3\sqrt{30}$
Solution:
(d) $3\sqrt{30}$

Solution:

$$\text{Area of } \Delta ABC = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$$



$\therefore$ Area of parallelogram = 2 × Area of ΔABC

$$= 2 \times \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -3 & 4 \\ 3 & 1 & -1 \end{vmatrix} = -\vec{i} + 13\vec{j} + 10\vec{k}$$

$$\therefore \quad \left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \sqrt{1+69+100} = \sqrt{270} = 3\sqrt{30}$$

Question 10.

If $\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (a \times \vec{b}) = \vec{x} \times \vec{y}$, then ………………

(a) $\vec{x} = \vec{0}$

(b) $\vec{y} = \vec{0}$

(c) $\vec{x}$ and $\vec{y}$ are parallel

(d) $\vec{x} = \vec{0}$ or $\vec{y} = \vec{0}$ or $\vec{x}$ and $\vec{y}$ are parallel

Solution:

(d) $\vec{x} = \overrightarrow{0}$ or $\vec{y} = \overrightarrow{0}$ or $\vec{x}$ and $\vec{y}$ are parallel

Hint:

$$\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = \vec{0}$$

So $\quad \vec{x} \times \vec{y} = \vec{0} \quad \Rightarrow \vec{x} = \vec{0} \quad$ or $\vec{y} = \vec{0}$

or $\quad \vec{x} \parallel \vec{y}$

Question 11.

If $\overrightarrow{PR} = 2\vec{i} + \vec{j} + \vec{k}, \overrightarrow{\text{Question}} = -\vec{i} + 3\vec{j} + 2\vec{k}$, then the area of the quadrilateral PQRS is ………………

(a) $5\sqrt{3}$

(b) $10\sqrt{3}$

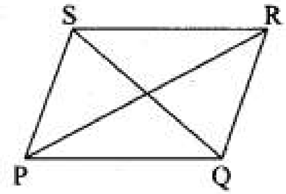
(c) $\frac{5\sqrt{3}}{2}$

(d) $\frac{3}{2}$

Solution:

(c) $\frac{5\sqrt{3}}{2}$

Hint:



$$\text{Area of PQRS} = \frac{1}{2}\left|\overline{PR} \times \overline{QS}\right|$$

$$\overline{PR} \times \overline{QS} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 1 \\ -1 & 3 & 2 \end{vmatrix} = \vec{i}(2-3) - \vec{j}(4+1) + \vec{k}(6+1)$$

$$= -\vec{i} - 5\vec{j} + 7\vec{k}$$

$$\left|\overline{PR} \times \overline{QS}\right| = \sqrt{1+25+49} = \sqrt{75} = 5\sqrt{3}$$

$$\text{Area of PQRS} = \frac{1}{2}\left(5\sqrt{3}\right) = \frac{5\sqrt{3}}{2}$$

Question 12.

If $\vec{a} \times \left(\vec{b} \times \vec{c}\right) = \left(\vec{a} \times \vec{b}\right) \times \vec{c}$ for non-coplanar vectors $\vec{a}, \vec{b}, \vec{c}$ then ....

(*a*) $\vec{a}$ parallel to $\vec{b}$ (*b*) $\vec{b}$ parallel to $\vec{c}$ (*c*) $\vec{c}$ parallel to $\vec{a}$ (*d*) $\vec{a} + \vec{b} + \vec{c} = \vec{0}$

Solution:

(c) $\vec{c}$ parallel to $\vec{a}$

Hint:

Given $\vec{a} \times \left(\vec{b} \times \vec{c}\right) = \left(\vec{a} \times \vec{b}\right) \times \vec{c}$

$\Rightarrow \left(\vec{a} \cdot \vec{c}\right)\vec{b} - \left(\vec{a} \cdot \vec{b}\right)\vec{c} = \left(\vec{a} \cdot \vec{c}\right)\vec{b} - \left(\vec{b} \cdot \vec{c}\right)\vec{a}$

i.e. $-\left(\vec{a} \cdot \vec{b}\right)\vec{c} = -\left(\vec{b} \cdot \vec{c}\right)\vec{a}$ $\Rightarrow t\vec{c} = s\vec{a}$ $\Rightarrow \vec{a} \parallel \vec{c}$