GAUS AG on D-elliptic sheaves and Hasse principle

NOTES FOR TALK 3 on STACKS II

§1. BRIEF RECAP OF TALK 1

(for more details, refer to the notes for Talk 1 by Prof. Yu)

Generally moduli functors

$$F : Sch_k^{op} \longrightarrow Sets$$

are not representable by schemes, due to existence of non-trivial automorphisms of objects being classified by F.

<u>Ex.</u>

$$M_{1,1} : Sch_{\mathbb{C}}^{op} \longrightarrow Sets$$

$$T \longmapsto \left\{ \begin{array}{c} \text{Elliptic curves} \\ \text{over } T \end{array} \right\} / \sim$$

is not representable by a scheme, for instance, due to non-injectivity of

$$M_{1,1}(Spec(\mathbb{C}(t))) \longmapsto M_{1,1}(Spec(\mathbb{C}(t^{1/6})))$$

where two elliptic curves $Y^2Z = X^3 - tZ^3$ and $Y^2Z = X^3 - Z^3$ are non-isom. over $\mathbb{C}(t)$ but isomorphic over $\mathbb{C}(t^{1/6})$.

To remedy this situation of non-representability, one allows the isomorphisms of objects to be recorded, thereby forming a 2-functor

$$M_{1,1} : Sch_{\mathbb{C}}^{op} \longrightarrow Grpd$$
$$T \longmapsto \left\langle \begin{matrix} \text{Elliptic curves} \\ \text{over } T \end{matrix} \right\rangle$$

This forms a stack. To avoid 2-categorical language, one instead forms a category fibered on groupoids (CFG)

$$p : \mathcal{M}_{1,1} \longrightarrow Sch_{\mathbb{C}}$$
$$\left(T, \begin{matrix} \text{Elliptic} \\ \text{curve } E/T \end{matrix}\right) \longmapsto T$$

The defining condition for a stack $p : \mathcal{M} \longrightarrow (Sch_k)_{fppf}$ is that

(i) $\forall\, T \in Sch_k$ & $x, y \in \mathcal{M}(T)$, the pre-sheaf $\underline{Hom}(x,y)$ on T is a sheaf;

(ii) Descent data is effective.

These two conditions can be

summed up as an equivalence of categories

$$\mathcal{M}(T) \longrightarrow \mathcal{M}(T' \xrightarrow{f} T)$$
$$x \longmapsto f^*x$$

where $T' \xrightarrow[f]{} T$ is a fppf, surjective morphism and $\mathcal{M}(T' \xrightarrow[f]{} T)$ is a category naturally encapsulating descent data w.r.t. $T' \xrightarrow{f} T$.

(YONEDA) An equivalence $\mathrm{Hom}_{\mathrm{Stacks}}(\underline{T}, \mathcal{M}) \simeq \mathcal{M}(T)$

GEOMETRY ON STACKS

DEF'N A stack $p: \mathcal{M} \longrightarrow \mathrm{Sch}_k$ is said to be Deligne-Mumford (DM) if

(i) the diagonal $\mathcal{M} \longrightarrow \mathcal{M} \times \mathcal{M}$ is representable by schemes, and is quasi-compact and separated.

$\Leftrightarrow$ (i′) $\underline{\mathrm{Hom}}(x, y)$ is rep'ble by a quasi-compact and separated scheme $\forall\ x, y \in \mathcal{M}(T)$

$\Leftrightarrow$ (i″) Any $\underline{X} \longrightarrow \mathcal{M}$ is rep'ble by scheme and is quasi compact and separated

(ii) $\exists$ a k-scheme U and an étale, surjective $\underline{U} \to \mathcal{M}$ (U is called an atlas)

is a DM-stack.

Ex: $\mathcal{M}_{1,1}$ is a DM-stack.
Proving it's a DM stack is quite non-trivial. We'll see below another example of a DM stack, namely quotient stacks, where it easier to show this.

§2. QUOTIENT STACKS

DEF'N: ((PRINCIPAL) G-BUNDLE) Let Y be an S-scheme, G a smooth, affine Y-group scheme. Then a (principal) G-bundle is a pair of morphisms $(\pi: P \to Y,\ \rho: G \times_Y P \to P)$ s.t. π is flat, locally finitely presented and surjective, and

(i)
$$\begin{tikzcd} G \times_Y G \times_Y P \arrow[r, "(id,\rho)"] \arrow[d, "(m,id)"'] \arrow[dr, phantom, "\square"] & G \times_Y P \arrow[d, "\rho"] \\ G \times_Y P \arrow[r, "\rho"'] & P \end{tikzcd}$$
commutes;

(ii) $P \xrightarrow[(e\pi, id)]{} G \times_Y P \xrightarrow[\rho]{} P$ is identity

(iii) $G \times_Y P \xrightarrow[(\rho, pr_2)]{} P \times_Y P$ is an isom.

Example: (Trivial G-bundle)

$(G \to Y, \rho = m : G \times_Y G \to G)$

Fact: Every G-bundle is étale locally trivial.

DEF'N: (QUOTIENT STACK) Let X/k be a scheme, G/k a smooth, affine group scheme. Then the quotient stack $[X/G] \to (Sch_k)_{fppf}$ is the CFG s.t. for any $T \in Sch_k$,

$$[X/G](T) := \left\langle \begin{array}{l} P \to T \text{ a } G_T\text{-bundle}, \\ P \to X \text{ a } G\text{-equivariant} \\ \text{map} \end{array} \right\rangle$$

For G étale, $[X/G]$ is a DM stack

- Effectiveness of descent data

This follows from the equivalence

$$\{G\text{-bundles}\} \longleftrightarrow \{G\text{-torsors}\}$$

certain sheaves with a suitable action of h_G

and the effectiveness of descent data for the stack of sheaves.

- Representability of the diagonal

Sufficient to show that for scheme T and $P_1, P_2 \in [X/G](T)$, the sheaf $\underline{\mathrm{Hom}}(P_1, P_2)$ is representable by a scheme. Enough to show this étale locally, where there are isomorphisms $\sigma_1 : P_1 \simeq G_T$, $\sigma_2 : P_2 \simeq G_T$, & we have $\rho_1 : G_T \to X_T$, $\rho_2 : G_T \to X_T$. the corresponding G-equivariant maps.

Consequently, $\underline{\mathrm{Hom}}(P_1, P_2)$ is rep'ble by the fiber product $?$,

$$\begin{tikzcd} ? \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & G_T \arrow[d, "{(\rho_1(e), \rho_2)}"] \\ X_T \arrow[r, "\Delta"'] & X_T \times_T X_T \end{tikzcd}$$

– An étale, surjective atlas

Let $\underline{X} \longrightarrow [X/G]$ be the map corresponding to the trivial G-bundle. Then we claim this is the required atlas. Indeed, if $\underline{T} \longrightarrow [X/G]$ corresponds to a G_T-bundle $\pi : P \longrightarrow T$ and $\rho : P \longrightarrow X$, then one can easily show a cartesian diagram

$$\begin{tikzcd}
\underline{P} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner"] & \underline{X} \arrow[d] \\
\underline{T} \arrow[r] & {[X/G]}
\end{tikzcd}$$

Since $P \longrightarrow T$ is étale & surjective, we're done

Example :- (i) Let $X = \mathrm{Spec}(A)$, for a finite type k-algebra A and G be a finite group acting on A. From the fact that G-bundles on $\mathrm{Spec}(\bar{k})$, for an algebraic closure $k \hookrightarrow \bar{k}$, are trivial, we conclude that

$$[\mathrm{Spec}(A)/G](\bar{k}) = \left\{ \begin{array}{c} G\text{-equivariant maps} \\ G \longrightarrow \mathrm{Spec}(A)(\bar{k}) \end{array} \right\}$$

In particular,

$$\left\{ \begin{array}{l} \text{Isom. classes} \\ \text{in } [\mathrm{Spec}(A)/G](\bar{k}) \end{array} \right\} \overset{1-1}{\longleftrightarrow} \mathrm{Spec}(A)(\bar{k})/G$$

(orbits of G-action)

But we know from the work of Mumford that

$$\mathrm{Spec}(A^G)(\bar{k}) = \mathrm{Spec}(A)(\bar{k})/G.$$

$\Rightarrow \mathrm{Spec}(A^G) = \mathrm{Spec}(A)/G$ is a coarse moduli space for $[\mathrm{Spec}(A)/G]$ (see [Ch. 6 § 11, Olsson]).

Warning: It is not true in general that $\mathrm{Spec}(A^G)$ represents

$[\mathrm{Spec}(A)/G]$ (happens when the G-action on $\mathrm{Spec}(A)$ has non-trivial isotropy groups). This is showcased in the following simple example: let $G = \mathbb{Z}/(2)$ act on $X = \mathrm{Spec}\left(\frac{k[T,Y]}{(TY)}\right)$

via $(T,Y) \longmapsto (Y,T)$.

<u>Exercise</u> (!) Show that $X/G \simeq \mathbb{A}^1_k$.

To show that $[X/G]$ is not representable by $X/G = \mathbb{A}^1_k$, it suffices to show $[X/G]$ is not a smooth stack. Indeed, by what we saw previously, the atlas $\underline{X} \longrightarrow [X/G]$ is an étale map. But since X is not smooth (being union of two lines), $[X/G]$ is also not smooth.

(ii) The stack of elliptic curves $\mathcal{M}_{1,1}$ is also a DM stack and admits the coarse moduli space $\mathbb{H}/_{SL_2(\mathbb{Z})}$ (see Talk 1) via the j-invariant of elliptic curves. (see also [Introduction, Olsson]).

The pattern above is encaptured by the following deep result, which will not used by us in this seminar.

Theorem (Keel–Mori). A DM-stack with finite diagonal (for instance, a separated stack) admits a coarse moduli space.

§3. The stack $\mathcal{E}ll_{\mathcal{D},I}$

Let D be a central division algebra of dimension d^2 over $F = \mathbb{F}_q(t)$. Let $X = \mathbb{P}^1_{\mathbb{F}_q}$ and $\mathcal{D}$ be the locally free $\mathcal{O}_X$-algebra s.t.

$\mathcal{D}_\eta = D$ (generic fiber) and

$\mathcal{D}_x = \mathcal{D} \otimes \widehat{\mathcal{O}}_{X,x} \subset F_x \ (x \in X)$

is a maximal order.

Recall the definition of a $\mathcal{D}$-elliptic sheaf $\underline{\mathcal{E}} = (\mathcal{E}_i, j_i, t_i)_{i \in \mathbb{Z}}$ over an $\mathbb{F}_q$-scheme S.

(for the full definition see the notes of Talk 2): $\mathcal{E}_i$ are locally free $\mathcal{O}_{X \times S}$-modules of rank d^2 with an action by $\mathcal{D}$ s.t.

(i)
$$\begin{tikzcd} \mathcal{E}_i \arrow[r, "j_i"'] & \mathcal{E}_{i+1} \\ {}^\tau\mathcal{E}_{i-1} \arrow[u, "t_{i-1}"] \arrow[r, "{}^\tau j_{i-1}"'] & {}^\tau\mathcal{E}_i \arrow[u, "t_i"] \end{tikzcd}$$
commutes;

(ii) $\mathcal{E}_i \xrightarrow[j_i]{} \mathcal{E}_{i+1} \longrightarrow \cdots \longrightarrow \mathcal{E}_{i+d} = \mathcal{E}_i(\infty)$

is the canonical injection;

(iii) Condition on the pole ∞

(iv) Condition on the zero's

(v) for all geometric pts $\{s\} \to S$,
$\chi(\mathcal{E}_0|_{X\times\{s\}}) \in [0, d^2)$ (normalization condition)

(vi) $\underline{\mathcal{E}}$ is special.

For appropriate ideals $I \subsetneq \mathbb{F}_q[t]$, we can put level I-structure on $\underline{\mathcal{E}}$. Consequently this gives a CFG $\mathcal{E}\ell\ell_{\mathcal{D},I} \longrightarrow (Sch_{\mathbb{F}_q})_{fppf}$.

Using descent for vector bundles, it is not hard to show that this is an fppf stack. By abuse of notation, we denote this stack just by $\mathcal{E}\ell\ell_{\mathcal{D},I}$.

Let $X' = X \setminus \{\{\infty\} \cup R\}$.

<u>THEOREM</u> ([LRS]) $\mathcal{E}\ell\ell_{\mathcal{D},I}$ is a DM-stack which is smooth of relative dimension $d-1$ over $X' \setminus I$. Moreover,

if $I \neq \phi$, then $\mathcal{E}ll_{D,I}$ is representable by a smooth, projective scheme over $X' \setminus I$.

Remark. Note that [LRS] do not impose the conditions (v), (vi) above for a D-elliptic sheaf. From the work of Spiess, (vi) is equivalent to (iv), and (v) being a normalization condition, we get $\mathcal{E}ll_{D,I}$ is the quotient by $\mathbb{Z}$ of the corresponding stack in [LRS]. The proof of the above theorem seems to work even after the quotient by $\mathbb{Z}$.

Sketch of a proof

The key to the proof is the following Cartesian diagram of stacks:

$$\begin{tikzcd} \mathcal{E}ll_{D,I} \arrow[r] \arrow[d, dash] & Vec^{\bullet}_{D,I} \arrow[d, dash] \\ {} & {} \end{tikzcd}$$

$$
\begin{tikzcd}
{} \arrow[d] \arrow[r, phantom, "\ulcorner"] & {} \arrow[d, "{(\mathrm{id},\, \mathrm{Frob})}"] \\
\mathrm{Hecke}^{\bullet}_{\mathcal{D},I} \arrow[r, "{(1^{st}\text{ row},\ 2^{nd}\text{ row})}"] \arrow[d, "\text{determined by zero-morphism}"'] & \mathcal{V}ec^{\bullet}_{\mathcal{D},I} \times \mathcal{V}ec^{\bullet}_{\mathcal{D},I} \\
X' \setminus I &
\end{tikzcd}
$$

where

(i) $\mathcal{V}ec^{\bullet}_{\mathcal{D},I}$ classifies $\{\mathcal{E}_i \xrightarrow[j_i]{} \mathcal{E}_{i+1}\}_{i\in\mathbb{Z}}$, as in the def'n of $\mathcal{D}$-ell. sheaves, together with conditions (ii) & (iii) and level I-structure.

(ii) $\mathrm{Hecke}^{\bullet}_{\mathcal{D},I}$ classifies
$$
\left\{
\begin{tikzcd}
\mathcal{E}_i \arrow[r, "j_i"'] & \mathcal{E}_{i+1} \\
\mathcal{E}'_{i-1} \arrow[u, "t_{i-1}"] \arrow[r, "j'_{i-1}"'] & \mathcal{E}'_i \arrow[u, "t_i"']
\end{tikzcd}
\right\}
$$
together with (ii), (iii), (iv) and level I-structure.

Step 1. Let $\mathcal{V}ec^{st}_{I}$ denote the stack classifying rank d^2 vector bundles $\mathcal{E}$ which are Seshadri-I-

stable, i.e. $\mathcal{E}$ has a level I-structure and for all subbundles $\mathcal{F} \subset \mathcal{E}$,

$$\frac{\deg(\mathcal{F}) - \deg(I)}{\mathrm{rk}(\mathcal{F})} < \frac{\deg(\mathcal{E}) - \deg(I)}{\mathrm{rk}(\mathcal{E})}$$

([Seshadri, 4.I, Définition 2]) If $\deg(I) > 0$, then $\mathcal{V}ec_I^{st}$ is representable by a disjoint union of smooth quasi-projective schemes.

Let now $\mathcal{E}ll_{D,I}^{st}$ denote D-ell. sheaves $(\underline{\mathcal{E}})$ s.t. $\mathcal{E}_0$ is Seshadri-I-stable. Then we can conclude using the following, as well as the cartesian diagram above that $\mathcal{E}ll_{D,I}^{st}$ is a disjoint union of smooth quasi-projective schemes if $\deg(I) > 0$.

(a) $\mathrm{Vec}_{D,I} \longrightarrow \mathrm{Vec}_I$ is representable and affine;

(b) $\mathrm{Vec}_{D,I}$ is smooth;

(c) $\mathrm{Vec}^{\bullet}_{D,I} \longrightarrow \mathrm{Vec}_{D,I}$ is representable and smooth;

(d) $\mathrm{Hecke}^{\bullet}_{D,I} \xrightarrow[\text{first row})]{(0\text{-mor},} (X' \setminus I) \times \mathrm{Vec}^{\bullet}_{D,I}$ is representable and smooth of dim'n $d-1$.

Lemma: Let S, U, V be $\mathbb{F}_q$-schemes and $\alpha : V \to U \times U$, $f : V \to S$ be $\mathbb{F}_q$-morphisms. Form the cartesian square:

$$
\begin{tikzcd}
W \arrow[r, "\beta"] \arrow[d, "j"'] \arrow[dd, bend left, "g"] \arrow[dr, phantom, "\ulcorner", very near start] & U \arrow[d, "{(\mathrm{id}, \mathrm{Frob})}"] \\
V \arrow[r, "\alpha"] \arrow[d] & U \times U \\
S &
\end{tikzcd}
$$

If $(f, pr_1 \circ \alpha) : V \to S \times U$ is smooth of dim'n n, then g is smooth of dim'n n.

Step 2: The idea is to cover

$\mathcal{E}\ell\ell_{D,I}$ by finite quotients of $\mathcal{E}\ell\ell^{st}_{D,J}$ for $J \supsetneq I$. Consider an ideal $J \supsetneq I$ s.t. $\deg(J) > 0$. & $r_{J,I} : \mathcal{E}\ell\ell_{D,J} \longrightarrow \mathcal{E}\ell\ell_{D,I}$ be the quotient of the level J-str. by I. This is a $G_{I,J} := \mathrm{Ker}((\mathcal{O}_{D/J})^* \longrightarrow (\mathcal{O}_{D/I})^*)$-torsor. Hence, $\mathcal{E}\ell\ell^{st}_{D,J} \subset \mathcal{E}\ell\ell_{D,I}$ being stable under the action of $G_{I,J}$, we get a smooth, open, DM-substack $[\mathcal{E}\ell\ell^{st}_{D,J}/G_{I,J}] \subset \mathcal{E}\ell\ell_{D,I}$. Since any vector bundle is Seshadri-J-stable for $\deg(J) \gg 0$, such finite quotient stacks cover $\mathcal{E}\ell\ell_{D,I}$ and we're done. ∎

§4. REFERENCES

[OLSSON] Algebraic spaces & Stacks.

[LRS] D-elliptic sheaves and the local Langlands correspondence.

[Seshadri] Fibrés vectoriels sur les courbes algébriques.