Derived category and rational points

A study of Esnault's question

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by

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Notations

Throughout the thesis, by field we mean a perfect field. Hence we will not distinguish between smooth and regular varieties.
For a group G and a $\mathbb{Z}[G]-$module M we denote by $H^i(G, M)$ the $i-th$ group cohomology.

For a variety X over a field k by X^s we mean $X \times_k k^{sep}$.

Often by $D^b(X)$, we will mean the *bounded derived category* of coherent sheaves. When it is not so, it will be explicitly mentioned.

Abstract

The derived category of projective varieties has played an important role in the study of Algebraic Geometry in the past few decades, one instance of which is the celebrated work of Mukai et al in the study of moduli space of sheaves. Naturally it is important to study what properties of varieties are preserved under an equivalence of their derived categories. For example, S. Mukai [21] and D. Orlov [24] in their seminal work on $K3$ surfaces showed that certain unimodular lattices associated to $K3$ surfaces determine whether or not two such surfaces are derived equivalent.

One such question of arithmetic nature was posed by H. Esnault

> "**For smooth projective varieties over a non-algebraically closed field k, is the existence of a k-rational point preserved by a k-linear derived equivalence?**"

The main focus of this thesis will be to study Esnault's question with some negative as well as affirmative answers.

The first part of the thesis will present two examples which answers negatively to Esnault's question, following the paper by N. Addington et al [1]. They provide negative examples to the question with a class of Abelian varieties and torsors over them without a rational point and also two hyperkähler 4-folds that are moduli spaces of sheaves on an explicit $K3$ surface s.t. one of them has infinitely many rational points while the other has none. More precisely, following are the two main results :-

Theorem 1:- For every $n \geq 2$, there is an Abelian variety $X/\mathbb{Q}$ of dimension n and an X-Torsor $Y/\mathbb{Q}$, s.t. there is a $\mathbb{Q}-$exact equivalence of derived categories

$$D^b(X) \simeq D^b(Y)$$

where Y is a non-trivial $X-$ torsor or equivalently $Y(\mathbb{Q}) = \emptyset$.

Theorem 2:- There is an explicit $K3$ surface $S/\mathbb{Q}$, and two smooth projective 4 dimensional moduli spaces of sheaves X, Y over S s.t. X has infinitely many rational points whereas Y has no zero cycle of degree 1 and there is a $\mathbb{Q}-$linear exact equivalence

$$D^b(X) \simeq D^b(Y)$$

For proving Theorem 1 one first shows that there exists a derived equivalence between an Abelian variety and the connected component of it's Picard variety associated to any $\lambda \in NS(A^s)^{Gal(k^s/k)}$. Then one takes A to be the Jacobian of a smooth curve $C/\mathbb{Q}$ and λ to

be the class of the *Theta divisor*. This induces an exact equivalence between $D^b(Pic_C^0)$ and $D^b(Pic_C^{g-1})$ with $g = g(C)$. Then it is just a matter of exploring the literature for examples of curves C s.t. Pic_C^1 has no rational points for $g = 2$, for we can cross it with an appropriate power of some elliptic curve to get Theorem 1. That is, once we get hold of such a curve C of genus 2, we can take some elliptic curve $E/\mathbb{Q}$ and define $X = Pic_C^0 \times E^{n-2}, Y = Pic_C^1 \times E^{n-2}$. Since Pic_C^1 has no rational points, so $Pic_C^1 \times E^{n-2}$ is a non-trivial X-torsor. (Here Pic_C^i is the component of the Picard scheme Pic_C of C, parametrizing line bundles with degree i)

Proving Theorem 2 requires a combination of both arithmetic and geometric techniques. The construction of the $K3$ surface S as well as X, Y are centered on the notion of a *tritangent line* $L \subset \mathbb{P}^2$ to a sextic curve $B \subset \mathbb{P}^2$. Since surfaces which form a double cover of $\mathbb{P}^2$ branched over a sextic curve B are $K3$ surfaces, our S will be constructed using a curve $B \subset \mathbb{P}^2$ whose tritangent lines satisfy certain conditions. For such an S, we choose X(resp. Y) to be the moduli space parametrizing semi-stable sheaves on S with *Mukai vector* $(0, h, -1)$(resp. $(0, h, 0)$) (here h is the polarization of S coming from $\mathbb{P}^2$ under the double cover).

The X and Y have many interesting properties. Both can be realized as relative compactified Picard scheme, of the tautological family of curves over the linear system $|h|$, of degrees 0 and 1 respectively. This description has the advantage of using techniques from D. Arinkin's seminal work on autoduality of compactified Jacobians of curves with planar singularities [4], to give a $\mathbb{Q}-$exact equivalence between $D^b(X)$ and $D^b(Y)$. Secondly X, Y parametrize only geometrically stable sheaves. The Brauer class obstructing the existence of universal sheaf on Y, it turns out, gives a *Brauer Manin obstruction to local-global principle* for Y. Hence Y has no rational points. On the other hand X contains a copy of $\mathbb{P}^2$, so it has infinitely many rational points.

The second part of the thesis will be based on studying Esnault's question for $K3$ surfaces in positive characteristic. We will see that $K3$ surfaces over a finite field $\mathbb{F}_q$ give an affirmative answer to Esnault's question. This is meant to be a counterpoint to the first part of the thesis. We will be following the paper by M. Lieblich and M. Olsson [16].

For a $K3$ surface $X/\mathbb{C}$, Mukai introduced a pure weight-2 Hodge structure on the cohomology ring $H^*(X, \mathbb{Z})$ and showed that there is a unimodular lattice structure on it and any functor between $D^b(X)$ and $D^b(Y)$ (Y also a $K3$ surface) on descending to cohomology respects the Hodge and lattice structure. One aims to follow Mukai's treatment of $K3$ surfaces even in positive characteristic but since singular cohomolgy is not available to us here, we have to resort to either the crystalline cohomology or the l-adic cohomology. For the purpose of studying rational points of $X/\mathbb{F}_q$ one can employ either of the cohomology theories. But we will be following Lieblich and Olsson in using crystalline cohomology for imitating Mukai's arguments in positive characteristic. The main theorem is the following:-

Theorem 3:- Suppose X and Y are $K3$ surfaces over a finite field $\mathbb{F}_q$. If there is a derived equivalence $D^b(X) \simeq D^b(Y)$ of $\mathbb{F}_q$-linear derived categories then

$$\#X(\mathbb{F}_{q^n}) = \#Y(\mathbb{F}_{q^n})$$

$\forall n \geq 1$. In particular, $\zeta_X = \zeta_Y$.

The proof of this theorem essentially follows from the *Lefschetz Fixed Point theorem* applied to the Frobenius morphism, after descending the derived equivalence to the *Mukai isocrystal*.

Part 1
Negative examples to Esnault's Question

Chapter 1

Torsors over Abelian varieties and derived category

Mukai proved (in [20]) a derived equivalence between an Abelian variety and its dual using the associated Poincaré bundle. In this chapter we will generalize Mukai's result to show a derived equivalence between Abelian variety and torsors over the dual. This result is almost straightforward extension of Mukai's theorem. Nonetheless it has a wonderful corollary in giving a derived equivalence between the Jacobian of a smooth curve C of genus g and the component of Picard scheme of C parametrizing line bundles of degree $g-1$ i.e. between $D^b(Pic^0_C)$ and $D^b(Pic^{g-1}_C)$. Then it's just a matter of exploring curves for which $Pic^{g-1}_C(\mathbb{Q}) = \emptyset$, to give a negative answer to Esnault's question.

1.1 Abelian varieties - Generalities

Following is the definition of Abelian variety we will be working with:-

Definition 1.1.1. *Over a field k, a finite type $k-$scheme A is said to be an Abelian variety if it is a $k-$group scheme which is smooth, proper and geometrically connected.*

This section will be a brief recap of the various equivalent ways of describing the *Neron-Severi group* of an Abelian variety. Let k be a field and $G = Gal(k^s/k)$ be the absolute Galois group of k. Let A be an Abelian variety and denote $A^s := A \times_k k^s$.

Definition 1.1.2. *On an Abelian variety A/k, two line bundles L, M are said to be* ***algebraically equivalent*** *to each other if there is a smooth, geometrically connected $k-$scheme S and a line bundle $\mathcal{P}$ on $S \times A$ s.t. there are two geometric points $\{s_1\}, \{s_2\} \hookrightarrow S$ with $\mathcal{P}_{\{s_1\}} \simeq L_{|\{s_1\}\times A}$ and $\mathcal{P}_{\{s_2\}} \simeq M_{|\{s_2\}\times A}$.*

The isomorphism classes of ***algebraically trivial line bundles****, $Pic^0(A^s)$, is the group of all isomorphism classes of line bundles on A^s algebraically equivalent to the trivial line bundle.*

Remark 1.1.3. *In fact, the group $Pic^0(A^s)$ coincides with isomorphism classes of translation invariant line bundles on A^s.*

Then one has the **Neron-Severi group** defined on A^s as

$$NS(A^s) := Pic(A^s)/Pic^0(A^s)$$

Hence one has the canonical short exact sequence:-

$$0 \to Pic^0(A^s) \to Pic(A^s) \to NS(A^s) \to 0$$

Note that $G = Gal(k^s/k)$ acts canonically on each of $Pic^0(A^s), Pic(A^s)$ and $NS(A^s)$. This yields a long exact sequence in group cohomology

$$0 \to Pic^0(A^s)^G \to Pic(A^s)^G \to NS(A^s)^G \xrightarrow{c} H^1(G, Pic^0(A^s)) \to H^1(G, Pic(A^s)) \to \dots$$

Note that A being an Abelian variety has a k rational point i.e. $A(k) \neq \emptyset$ and hence $Pic(A^s)^G = Pic(A)$ and also $Pic^0(A^s) = Pic^0(A)$.[1]
Hence we have the long exact sequence

$$0 \to Pic^0(A) \to Pic(A) \to NS(A^s)^G \xrightarrow{c} H^1(G, Pic^0(A^s)) \to H^1(G, Pic(A^s)) \to \dots$$

For any class $\lambda \in NS(A^s)^G$ we get an element of $c(\lambda) \in H^1(G, Pic^0(A^s)) = H^1(G, Pic^0_A(Spec(k^s)))$ [2]

Recall the following:-

Definition 1.1.4. *For a scheme S and an $S-$group scheme G, an $S-$ scheme X is said to be a $S-$**torsor** if for every $S-$scheme T, there is a group action of $Hom_S(T, G)$ on $Hom_S(T, X)$ which is functorial in T.*

By [28, section 5.12.4], we know that there is a 1-1 correspondence between the isomorphism classes of torsors over a smooth algebraic group scheme A and the elements of the group $H^1(Gal(k^s/k), A(k^s))$. So $c(\lambda)$ represents a torsor over Pic^0_A.

Also we have the $1-1$ correspondence

$$NS(A^s) \leftrightarrow \{\text{connected components of } Pic_{A^s/k^s} = Pic_{A/k} \times_k k^s\}$$

which is an isomorphism of G-sets. We have the following by [see 32, Lemma 038D]

$$\{\text{connected components of } Pic_{A/k}\} \leftrightarrow \{\text{connected components of } Pic_{A^s/k^s}\}^G$$

given by

$$T \mapsto T^s$$

Hence a natural $1-1$ correspondence

$$NS(A^s)^G \leftrightarrow \{\text{connected components of } Pic_{A/k}\}$$

In fact the $1-1$ correspondence above is $\lambda \mapsto c(\lambda)$ (all components of the Picard scheme $Pic_{A/k}$ are torsors over $\hat{A} := Pic^0_A$). This follows from the description of the connecting homomorphism c and the construction of the 1 cocycle representing any $A-$torsor X in $H^1(G, A)$ [see 28, Rmk 5.12.13]. We will henceforth denote for any $\lambda \in NS(A^s)^G$ its associated connected component of $Pic_{A/k}$ (from the correspondence above) to be $Pic^\lambda_A := c(\lambda)$.

[1] This follows from [see 14, Thm 2.5]: The condition $\mathcal{O}_{Spec(k)} \simeq \pi_*\mathcal{O}_A$ is satisfied since A is an Abelian variety (here $\pi : A \to Spec(k)$ is the structure map). The natural map of functors $Pic_{A/k} \hookrightarrow Pic_{Zar,A/k} \hookrightarrow Pic_{ét,A/k}$ is an isomorphism whenever $A \to Spec(k)$ has a section i.e. $A(k) \neq \emptyset$ (here $Pic_*(A/k)$ is the sheafification of the relative Picard functor in $*$ site, $* = Zar, ét$). But $Pic_{A/k}(Spec(k)) = Pic(A)$ and $Pic_{ét,A/k}(Spec(k)) = Pic_{ét,A/k}(Spec(k^s))^G = Pic(A^s)^G$.

[2] $Pic^0(A^s) = Pic^0_A(Spec(k^s))$ since Pic^0_A represents the moduli functor $Pic^0_{A,ét}$(parametrizing line bundles algebraically equivalent to trivial line bundle) and hence $Pic^0_{A,ét}(k^s) = Pic^0(A \times k^s)$

Recall that for any Abelian variety B over an algebraically closed field $\bar{k}$ and a line bundle $\mathcal{L} \in Pic(B)$, there is a homomorphism given by

$$\phi_{\mathcal{L}} : B \to \hat{B}$$

defined

$$x \in B(k) \mapsto t_x^* \mathcal{L} \otimes \mathcal{L}^{-1}$$

Definition 1.1.5. *For Abelian variety A, a morphism of $k-$ varieties $\phi : A \to \hat{A}$ is said to be a polarization if the base change $\phi^s : A^s \to \hat{A}^s$ is given by $\phi_{\mathcal{L}}$ for some line bundle $\mathcal{L} \in Pic(A^s)$.*

We have a natural $1-1$ correspondence

$$NS(A^s) \leftrightarrow \{\text{polarizations } A^s \to \hat{A}^s\}$$

which is an isomorphism of $G-$sets. By Galois theory, we have $1-1$ correspondence

$$NS(A^s)^G \leftrightarrow \{\text{polarizations } A \to \hat{A}\}$$

Combining the above correspondences we have the following result:-

Lemma 1.1.6. *For an Abelian variety A over a field k we have the following bijective correspondence*

$$(i)\ NS(A^s)^G \leftrightarrow \{\textit{connected components of } Pic_{A/k}\}$$

given by $\lambda \mapsto Pic_A^\lambda$

$$(ii)\ NS(A^s)^G \leftrightarrow \{\textit{polarizations } A \to \hat{A}\}$$

given by $\mathcal{L} \mapsto \phi_{\mathcal{L}}$.

Remark 1.1.7. *Note that Pic_A^λ is trivial if and only if λ is in the image of $Pic(A)$ under the map $Pic(A) \to NS(A^s)^G$ of the long exact sequence above.*

In the next section we will use the following result stated in [see 29, Cor. 4]. Before stating the proposition we recall here the definition of *Theta divisor*:-

Definition 1.1.8. *For a smooth curve C/k the **Abel-Jacobi map** $AJ_C^{g-1} : Sym_C^{g-1} \to Pic_C^{g-1}$ sends a divisor $D \hookrightarrow Sym_C^{g-1}$ (as a geometric point) to $\mathcal{O}_C(D) \hookrightarrow Pic_C^{g-1}$.*
*The **Theta divisor** is defined as the image of the Abel-Jacobi map AJ_C^{g-1}, denoted Θ.*

Remark 1.1.9. *It is clear from the definition of Theta divisor that set theoritically it is the closed set $\{\eta \in Pic_C^{g-1} | H^1(\eta) \neq 0\}$. This follows from the fact that $H^1(\eta) = H^0(\eta)$ (for degree $g-1$ line bundle η) and that $\eta \simeq \mathcal{O}(D)$ for some effective divisor D iff $H^0(\eta) \neq 0$. It turns out that the scheme structure on Θ is the reduced structure on the closed set $\{\eta \in Pic_C^{g-1} | H^1(\eta) \neq 0\}$. (cf. [27, Sec. 17.3])*

Proposition 1.1.10. *If (J, λ) is the canonically polarized Jacobian of a smooth, geometrically connected curve C/k og genus g, then the element $c(\lambda)$ is represented by the $J-$torsor Pic_C^{g-1} (here $\lambda = \phi_\Theta$ under the 1-1 correspondence between $NS(A^s)^G$ and polarizations of A, of Lemma 1.1.6).*

Proof: See [29, Cor. 4].

$\square$

1.2 Derived category of Abelian variety

We will state and prove a theorem in the spirit of the celebrated derived equivalence between an Abelian variety and its dual (proved by Mukai).

Proposition 1.2.1. *There is a $k-$linear derived equivalence*

$$D^b(A) \simeq D^b(Pic_A^\lambda)$$

for $\lambda \in NS(A^s)^G$.

Proof: Since A has a $k-$rational point, a *universal line bundle* exists on $A \times Pic_{A/k}$[3] and let its restriction to the open subscheme $A \times Pic_A^\lambda$ be denoted $\mathcal{P}$. We aim to show that the Fourier Mukai transform given by the Fourier Mukai kernel $\mathcal{P}$

$$F = \Phi_{\mathcal{P}} : D^b(A) \to D^b(Pic_A^\lambda)$$

is an exact equivalence of derived categories sought for.

We will first reduce to the case where k is an algebraically closed field. Let

$$L, R : D^b(Pic_A^\lambda) \to D^b(A)$$

be the left, right adjoints to F induced by the Fourier Mukai kernels $\mathcal{P}^\vee[dim(A)]$ [4]. We want to show that the unit of adjuction $1 \to RF$ and co-unit of adjunction $FL \to 1$ are isomorphism or equivalently their mapping cones are acyclic. But any complex $\mathcal{F}^\bullet \in D^b(A)$ is acyclic iff

$$\mathcal{F}^\bullet \otimes_k \bar{k} \in D^b(A_{\bar{k}})$$

is acyclic (due to faithful flatness of $k \to \bar{k}$). Hence we may(and do) assume that k is an algebraically closed field.

So Pic_A^λ can be identified with the dual of A, denoted $\widehat{A}$ $(:= Pic_A^0)$, after choosing a base point $[M] \in Pic_A^\lambda$. Hence $\mathcal{P} \otimes \pi_1^* M^\vee$ is a Poincaré bundle on $A \times \widehat{A}$ ($\pi_1 : Pic_A^\lambda \times A \to Pic_A^\lambda$ is the first projection). The Fourier Mukai kernel, $\mathcal{P} \otimes \pi_1^* M^\vee$ being a Poincaré sheaf on $\hat{A} \times A$, induces an equivalence $D^b(A) \simeq D^b(\hat{A})$ [see 20], which is the composition of $D^b(A) \to D^b(Pic_A^\lambda)$ and tensoring with the line bundle $\check{M}$ (which is an equivalence). Hence $\Phi_{\mathcal{P}} : D^b(A) \to D^b(Pic_A^\lambda)$ is an equivalence of derived categories.

□

Proposition 1.2.2. *If C is a smooth , projective, geometrically connected curve of genus $g \geq 1$ over a field k, then there is a $k-$linear exact equivalence*

$$D^b(Pic_C^0) \simeq D^b(Pic_C^{g-1})$$

of derived categories.

Proof: For $A = Pic_C^0$, we denote by λ the element of $NS(A^s)^G$ corresponding to the canonical principal polarization of the Jacobian, induced by the *Theta divisor*, under the bijective correspondence of 1.1.6. By 1.1.10 we know that $Pic_A^\lambda \simeq Pic_C^{g-1}$, and hence by previous proposition, a derived equivalence $D^b(Pic_C^0) \simeq D^b(Pic_C^{g-1})$.

□

[3] This is because since $\pi_* \mathcal{O}_A \simeq \mathcal{O}_{Spec(k)}$ holds universally and π has a section ($\pi : A \to Spec(k)$ is the structure map), [14, Thm. 2.5] says that $\mathbf{Pic}_{A/k} \hookrightarrow \mathbf{Pic}_{ét,A/k}$ is an isomorphism and hence by [14, Thm. 4.8], $\mathbf{Pic}_{A/k}$ is representable or in other words there is a universal sheaf on $A \times Pic_A$.

[4] Here $\mathcal{P}^\vee$ is the derived dual, $R\mathcal{H}om(\mathcal{P}, \mathcal{O}_{A \times A})$. The fact that left and right adjoints of F have $\mathcal{P}^\vee[dim(A)]$ as Fourier Mukai kernel follows from [20] or [10, Prop. 5.9]

1.3 Negative answer I to Esnault's Question

We get a class of negative examples to Esnault's question from the following theorem :-

Theorem 1.3.1. *For every $n \geq 2$, there is an Abelian variety $X/\mathbb{Q}$ of dimension n and an X-Torsor $Y/\mathbb{Q}$, s.t.*

$$D^b(X) \simeq D^b(Y)$$

is $\mathbb{Q}-$exact equivalence of derived categories and Y is a non-trivial $X-$torsor i.e. $Y(\mathbb{Q}) = \emptyset$.

The idea is to use the derived equivalence of Propositon 1.2.2 and explore curves $C/\mathbb{Q}$ of genus 2 with $Pic^1_C(\mathbb{Q}) = \emptyset$. For then Pic^1_C is a non-trivial torsor over Pic^0_C [see 28, Prop 5.12.14] and setting $X = Pic^0_C \times E^{n-2}$ and $Y = Pic^1_C \times E^{n-2}$ with E being an elliptic curve over $\mathbb{Q}$, we see that $Y(\mathbb{Q}) = \emptyset$ (otherwise Pic^1_C would have a rational point) and Y is an $X-$torsor (using the functor of points definition of torsors).

A curve $C/\mathbb{Q}$ having properties stated above can be found in the work of Poonen and Stoll [29, Prop. 26], where they give an explicit hyperelliptic curve C of even genus g s.t. $Pic^{g-1}_C(\mathbb{Q}) = \emptyset$. The curve determined by the equation $y^2 = -(x^{2g+2} + x + t)$ has $\mathbb{R}-$points (in particular $\mathbb{Q}-$points) only when $t > 0$.We choose one such hyperelliptic curve $C/\mathbb{Q}$, $g = 2$ and $t = 0$. This gives the theorem above by following the ideas of the previous paragraph.

$\square$

Chapter 2

A K3 surface over $\mathbb{Q}$ and moduli spaces of sheaves

Another negative answer to Esnault's question is provided by a pair of moduli space of semistable sheaves on an explicit K3 surface. This surface is gotten as the double cover of $\mathbb{P}^2$ branched over a smooth sextic curve. Associated to such $K3$ surface is the notion of tritangent lines. This makes it tractable to handle such $K3$ surfaces and associated varieties on it (like moduli spaces), as the construction of our explicit K3 surface itself imposes conditions on the its tritangent lines (Thm. 4.1.3 and Prop. 4.2.3).

In this chapter, we first define *tritangent line to a sextic curve* (as Prop. 2.1.4). Then we construct two moduli spaces of semistable sheaves on a $K3$ surface and show that both parametrize only geometrically stable sheaves if there are no tritangent lines (over algebraic closure) to the $K3$ surface. Also these two moduli spaces can be described as compactified Picard schemes, which will be used to give a derived equivalence between them.

2.1 K3 surfaces as branched double cover

We will use the following definition of $K3$ surfaces :-

Definition 2.1.1. *A K3 surface is a projective smooth surface over a field k s.t.*
(i) $H^1(X, \mathcal{O}_X) = 0$
(ii) The canonical bundle on X is trivial i.e. $\omega_X \simeq \mathcal{O}_X$.

It is a well known fact that varieties which form a double cover of $\mathbb{P}^2$ branched over a smooth sextic curve $B \subset \mathbb{P}^2$ are K3 surfaces.

Proposition 2.1.2. *Let $\pi : S \to \mathbb{P}^2$ be a double cover branched over a smooth sextic curve $V_+(f(x, y, z)) \subset \mathbb{P}^2$. Then S is a $K3$ surface.*

Proof: See [11, Example 1.3(iv)].

□

Given a smooth sextic curve in $\mathbb{P}^2$ we can always construct a $K3$ surface, branched over this curve, using the previous proposition.

Proposition 2.1.3. *For any smooth sextic curve $B = V_+(f(x, y, z)) \subset \mathbb{P}^2$ there exists a surface S which is a double cover of $\mathbb{P}^2$ branched over B. In particular S is a K3 surface.*

Proof: Consider the surface $S = V_+(w^2 - f(x,y,z))$ in the weighted projective space $\mathbb{WP}^3(3,1,1,1)$. Let $\pi : S \to \mathbb{P}^2$ be the map sending $[w : x : y : z] \mapsto [x : y : z]$ (on geometric points). Then it is evident that π is double cover which is ramified precisely over B (this is seen by computing the stalks of the sheaf of Kahler differentials for the double cover). Hence all the hypothesis of Prop.2.1.2 are satisfied.

$\square$

Throughout the rest of this chapter we fix $\pi : S \to \mathbb{P}^2$ to be the double cover branched over a smooth sextic curve $B \subset \mathbb{P}^2$, given by the closed embedding of S in the weighted projective space as in the proposition above. Since π is a finite map $h := \pi^*\mathcal{O}_{\mathbb{P}^2}(1)$ is an ample line bundle. Note that the intersection product $(h.h) = (\pi^*\mathcal{O}_{\mathbb{P}^2}(1).\pi^*\mathcal{O}_{\mathbb{P}^2}(1)) = deg(\pi)(\mathcal{O}_{\mathbb{P}^2}(1).\mathcal{O}_{\mathbb{P}^2}(1)) = 2$.

Consider the linear system $|h|$. It consists of curves in S, which are pre-images of elements in $|\mathcal{O}_{\mathbb{P}^2}(1)|$ i.e. hyperplanes in $\mathbb{P}^2$ (since $deg : Pic(\mathbb{P}^2) \to \mathbb{Z}$ is an isomorphism).

Let $C \subset \mathbb{P}^2$ be a hyperplane and by abuse of notation, $\pi : \pi^{-1}C \to C$ be the restriction of π to the closed subscheme $\pi^{-1}C \in |h|$. Since $\pi : S \to \mathbb{P}^2$ is a double cover and so is étale outside a dimension 1 subvariety (i.e. a curve). Hence for a general member $\pi^{-1}C \in |h|$ (i.e. for an element in a dense set of $|h|$) $\pi : \pi^{-1}C \to C$ is finite, flat and separable of degree 2. In other words $\pi^{-1}C$ will in general be a normal hyperelliptic curve. Since the ramification locus is given by a zero locus of a degree 6 homogeneous polynomial in C we have by [18, Sec. 7.4.3] that $2g_{\pi^{-1}C} + 2 = 6$ or $g_{\pi^{-1}C} = 2$. So a general member of $|h|$ is a smooth curve of genus 2.[1]

2.1.1 Tritangent lines

Proposition 2.1.4. *Let $L \subset \mathbb{P}^2_k$ be a hyperplane (we assume either k is algebraically closed or when non-algebraically closed $char(k) \neq 2$) and a sextic curve $B \subset \mathbb{P}^2_k$ given by a degree 6 homogeneous polynomial $f(x,y,z) \in k[x,y,z]$. The following conditions are equivalent:-*

(a) L is tangent to B at three points, in the scheme-theoretic sense;

(b) $f_{|L} = cg^2$ for some scalar $c \in k$ and cubic $g \in H^0(\mathcal{O}_L(3))$;

(c) the curve $C = \pi^{-1}(L)$ is not geometrically integral.

Proof: $(a) \implies (b)$
Without loss of generality, we may assume $L = V_+((z))$. We may also assume that the three points in $L \cap B$, say x_0, x_1, x_2, are in the affine open set $D_+(y)$ (after a linear change of coordinates) and let $t := x/y$ be the coordinate of the affine open set $D_+(y) = Spec(k[\frac{x}{y}])$. Let the point x_i be represented by a prime ideal $\mathfrak{p}_i \in D_+(y)$.
The closed set $L \cap B$ has a scheme structure coming from the scheme theoretic intersection. More precisely $L \cap B = V_+(f(x,y,0))$ and as a Weil divisor $L \cap B = a_0[x_0] + a_1[x_1] + a_2[x_2]$ where

$$a_i := len_{k[t]_{\mathfrak{p}_i}}\left(\frac{k[t]_{\mathfrak{p}_i}}{(f(t))_{\mathfrak{p}_i}}\right)$$

[1]The ramification locus of $\pi : S \to \mathbb{P}^2$ is given by a degree 6 polynomial $f(x,y,z) = ax^6 + by^6 + cz^6 + \ldots$ with $abc \neq 0$ by virtue of $V_+((f(x,y,z)))$ being a smooth curve. Hence the ramification locus of $\pi : \pi^{-1}C \to C$ is $C \cap V_+((f(x,y,z))) = V_+((f(x,y,0)))$ [assuming without loss generality that $C = V_+((z))$] and hence $f(x,y,0)$ is a degree 6 homogeneous polynomial.

Since L is tangent to B so

$$a_i > 1$$

$\forall i = 1, 2, 3$. Clearly $6 = deg(f) = deg(\sum_{i=1}^{3} a_i[x_i]) = \sum_{i=1}^{3} a_i[k(x_i) : k] \geq 2(\sum_{i=1}^{3}[k(x_i) : k])$. This is possible only when $k(x_i) = k; a_i = 2\ \forall 1 \leq i \leq 3$. Since x_i is a rational point it is given by a maximal ideal $(t - l_i)$. Hence the ideal $(f(t)) = ((t - l_1)(t - l_2)(t - l_3))^2$. [2] In other words, there is a constant $c \in k$ s.t. $f(t) = ch^2(t)$ where $h(t) = (t - l_1)^2(t - l_2)^2(t - l_3)^2$. Let the *homogeneization* of h be called g. Then we have $f_{|L} = cg^2$, whence (b).

$(b) \implies (a)$

By the previous part we know that $L \cap B = \sum_{i=1}^{n} a_i[x_i]$ (for some $n \in \mathbb{N}$) and $\sum_{i=1}^{3} a_i[k(x_i) : k] = 6$ (this is independent of the assumption in (a)). Also locally

$$a_i = len_{k[t]_{\mathfrak{p}_i}}\left(\frac{k[t]_{\mathfrak{p}_i}}{(g(t))^2_{\mathfrak{p}_i}}\right) > 1$$

for some prime ideal $\mathfrak{p}_i$ in a affine open set around x_i. Hence $n \leq 3$ and L intersects B tangentially in scheme-theoritic sense.

$(b) \implies (c)$

$\pi^{-1}L$ is the sub-variety given by $V_+(w^2 - cg^2) \subset \mathbb{WP}^2(3, 1, 1)$. Clearly over $\bar{k}$, $c = d^2$ for some scalar $d \in \bar{k}$ and hence $w^2 - cg^2 = (w - dg)(w + dg)$. In other words, $V_+(w^2 - cg^2)$ is not integral over $\bar{k}$ or $V_+(w^2 - cg^2)$ is not geometrically integral over k.

$(c) \implies (b)$
Suppose as before that $L = V_+((z))$; then the curve $\pi^{-1}(L) = V_+((w^2 - f(x, y, 0)))$ in $\mathbb{WP}^2(3, 1, 1)$ is not geometrically integral or $w^2 - f(x, y, 0)$ factors as $(w - g_1)(w - g_2)$ with $g_i \in \bar{k}[x, y]$. Simplifying the R.H.S. we get $g_1 = -g_2 = g'$ and $g'^2 = f$. Let $g'(t) = a_0t^3 + .. + a_3$. Then $a_0^2 \in k$ and $a_i/a_0 \in k$ (by assumption on k). Hence $f = cg^2$ where $c = a_0^2 \in k$ and $g = g'/a_0$.

$\square$

Remark 2.1.5. *The hypothesis $char(k) \neq 2$ if k is not algebraically closed is used in $(c) \implies (b)$ above. Indeed if $char(k) = 2$, then there exists monic polynomial $g[T] \in \overline{\mathbb{F}_2}[T]$ s.t. $g^2[T] \in \mathbb{F}_2[T]$ but $g[T] \notin \mathbb{F}_2[T]$.*

Definition 2.1.6. *Let $L \subset \mathbb{P}_k^2$ be a hyperplane (k is algebraically closed or when non-algebraically closed $char(k) \neq 2$) and a sextic curve $B \subset \mathbb{P}_k^2$. Then L is said to be a **tritangent line** to B if any of the equivalent conditions of the previous proposition holds true.*

2.2 Moduli spaces over $K3$ surfaces

2.2.1 Recap of moduli spaces

The theory of moduli spaces of semistable sheaves on a projective variety X/k was developed, with contributions from many mathemticians, in view of constructing a variety M/k which is

[2]here we are using three results from commutative algebra :- Krull's principle ideal theorem, existence of primary decomposition and intersection of co-prime ideals is same as product:- Krull's theorem and primary decompsition say that $(f) = \mathfrak{p}_1^2 \cap \mathfrak{p}_2^2 \cap \mathfrak{p}_3^2$ and hence $(f) = \mathfrak{p}_1^2\mathfrak{p}_2^2\mathfrak{p}_3^2 = ((t - l_1)(t - l_2)(t - l_3))^2$.

the universal object in Sch_k parametrizing all semistable sheaves on X with a fixed Hilbert polynomial. In other words, we ask for M to represent the functor of family of semistable sheaves with fixed Hilbert polynomial (called the *moduli functor*). Then M is said to be a *fine moduli space*. But fine moduli space may not always exist (for vaious reasons) and the weaker notion of *coarse moduli space* (or co-representabilty) is sought. We now state the important notions from the theory of moduli space. For more details regarding the construction and properties of moduli spaces, the reader is refered to [12].

Definition 2.2.1. *A coherent sheaf $\mathcal{F}$ is called a pure sheaf if for any non-trivial coherent subsheaf of $\mathcal{G} \subset \mathcal{F}$ we have $dimSupp(\mathcal{G}) = dimSupp(\mathcal{F})$.*

Definition 2.2.2. *For a polarized projective variety $(X/k, \mathcal{O}_X(1))$ and a coherent sheaf $\mathcal{F}$ on X, the function $P : \mathbb{Z} \to \mathbb{Z}$ defined as $n \mapsto \chi(\mathcal{F}(n))$ is a polynomial, denoted $P_{\mathcal{F}}(T) \in \mathbb{Q}[T]$, called the **Hilbert polynomial** of $\mathcal{F}$. The polynomial $p_{\mathcal{F}}(T) = P_{\mathcal{F}}(T)/\alpha_d$ (α_d is the leading coefficient of P) is called the **reduced Hilbert polynomial** of $\mathcal{F}$.*

Definition 2.2.3. *A pure coherent sheaf $\mathcal{F}$ on $(X, \mathcal{O}_X(1))$ is said to be a semistable sheaf if for any non-trivial coherent subsheaf $\mathcal{G} \subset \mathcal{F}$ we have $p_{\mathcal{G}}(n) \leq p_{\mathcal{F}}(n)\ \forall\ n \gg 0$. Moreover a semistable sheaf $\mathcal{F}$ is said to be stable if for all non-trivial coherent subsheaf $\mathcal{G} \subset \mathcal{F}$ we have $p_{\mathcal{G}}(n) < p_{\mathcal{F}}(n)\ \forall\ n \gg 0$.*

Definition 2.2.4. *For a coherent sheaf E on a scheme X, a subsheaf $F \subset E$ is called saturated if E/F is pure of dimension $dimSupp(E)$ or is zero.*

Lemma 2.2.5. *Let F be a coherent sheaf on a polarized variety X. To check if the condition of stability holds true i.e. if for every non-trivial coherent subsheaf $G \subset F$ whether $p_G(n) < p_F(n); n \gg 0$ ($p_*(-)$ is the reduced Hilbert polynomial), it is enough to check it for G a saturated subsheaf of F.*

Proof: For a proof see [12, Prop 1.2.6]

□

Definition 2.2.6. *Let $(X, \mathcal{O}_X(1))$ be a polarized projective variety and $P(T) \in \mathbb{Q}[T]$ be a polynomial. The moduli functor*

$$\mathcal{M}^P_{X/k}(-) : Sch_k^{opp} \to Sets$$

is defined as

$$[T] \mapsto \{T-\textit{flat coherent sheaf } \mathcal{F} \textit{ on } X \times_k T \textit{ s.t. for all geometric points } t \hookrightarrow T, \mathcal{F}_t$$
$$\textit{is semistable on } X_t \textit{ with Hilbert polynomial } P(T) \textit{ on } X\}/(\sim, q^*Pic(T))$$

$(q : X \times_k T \to T)$

Remark 2.2.7. *The subfunctor $\mathcal{M}^{P,s}_{X/k}(-) : Sch_k^{opp} \to Sets$ parametrizing geometrically stable sheaves is an open subfunctor of $\mathcal{M}^P_{X/k}(-)$ (this follows from Grothendieck's Quot scheme and semi-continuity arguments).*

Definition 2.2.8. *For a polarized projective variety $(X, \mathcal{O}_X(1))$. A $k-$scheme M is said to be a fine moduli space if $\mathcal{M}^P_{X/k}(-)$ is represented by M i.e. there is an isomorphism of functors $\mathcal{M}^P_{X/k}(-) \simeq Hom_k(-, M)$.*
M is said to be coarse moduli space if M co-represents the moduli functor $\mathcal{M}(-)$ and the natural map $\mathcal{M}^P_{X/k}(Spec(l)) \to M(l)$ (l an algebraically closed field) is a bijection of sets. More precisely, there is a natural transformation of functors $\mathcal{M}^P_{X/k}(-) \to Hom_k(-, M)$ which is universal i.e. for any $\mathcal{M}^P_{X/k}(-) \to Hom(-, M')$ there exists a unique map $M \to M'$ s.t. the natural triangle commutes.

Following is the main result in the theory of moduli spaces of sheaves (For details see [12, Chap. 4]):-

Theorem 2.2.9. *For a polarized projective variety* $(X, \mathcal{O}_X(1))$ *over an algebraically closed field in characteristic* 0 *the course moduli space* M *exists and is a projective variety. Moreover the open subfunctor* $\mathcal{M}^{P,s}_{X/k}$ *(see Remark2.2.7) is corepresented by an open subscheme of the projective variety* M*, denoted* M^s*.*

Remark 2.2.10. *More generally the theorem is true if we consider projective morphisms* $f : X \to S$ *with polarizations and suitably defining the moduli functor* $\mathcal{M}^P_{X/S} : Sch^{opp}_S \to (Sets)$*. [See 12, p. 4.3.7].*

2.2.2 Moduli space of sheaves on K3 surface

Throughout the rest of this chapter we fix $\pi : S \to \mathbb{P}^2_{\mathbb{Q}}$ to be the double cover as before branched over a sextic curve $B \subset \mathbb{P}^2_{\mathbb{Q}}$.
The Hilbert polynomial $P(n) = \chi(\mathcal{F}(n))$, for a coherent sheaf $\mathcal{F}$ on X, is equivalent to defining a triple of elements from $CH^*(X)$ called the *Mukai vector.*

Definition 2.2.11. *For a coherent sheaf* $\mathcal{F}$ *on a* $K3$ *surface* X *over an algebraically closed field* k*, the Mukai vector* $v(\mathcal{F}) := (rk([\mathcal{F}]), c_1([\mathcal{F}]), \chi(\mathcal{F}) + rk([\mathcal{F}])) \in CH^{2*}(X)$*.*

Remark 2.2.12. *For* $K3$ *surfaces Cher character induces a natural isomorphism* $K(X) \simeq CH^*(X)$ *(over an algebraically closed field) and hence the Mukai vector can be considered as an element of the Grothendieck Group* $K(X)$*.(cf. [11, Chap. 12, Cor. 1.5])*

The equivalence of the Mukai vector and Hilbert polynomial follows from the Hirzebruch-Riemann-Roch applied to the sheaf $\mathcal{F} \otimes \mathcal{O}(n)$ and using the expression of *Chern character* $ch(-)$ in terms of the Chern classes.

From the viewpoint of derived category the advantages of the Mukai vector are many, its definition itself rooted in defining the *cohomological* version of the Fourier Mukai transform. Hence for $K3$ surface X, it is natural to redefine the moduli functor of semistable sheaves in terms of a fixed Mukai vector $v \in K(X)$ instead of a polynomial $P(T) \in \mathbb{Q}[T]$. Then the coarse moduli space if it exists is denoted $M_h(v)$ (h is the polarization $\mathcal{O}(1)$).

Theorem 2.2.13. *For a* $K3$ *surface* X/k *(*k *not necessarily algebraically closed) with polarzation* h*, the course moduli space* $M_h(v)$ *exists and is a projective variety.*

Proof: See [11, Chap. 10, Pg. 205].

□

Recall that the line bundle $h = \pi^*\mathcal{O}_{\mathbb{P}^2}(1)$ is an ample line bundle. The two varieties which provide a negative answer to Esnault's question are defined as

$$X = M_h(0, h, -1)$$

and

$$Y = M_h(0, h, 0)$$

two moduli spaces of sheaves parametrizing semistable sheaves with rank 0, first Chern class h and Euler characteristic -1 and 0 respectively. The rest of this section and the next chapter will be a preparation to show that the pair X, Y is a negative answer to Esnault's question.

The moduli spaces X, Y have a host of important properties. Since these properties are detrimental in proving our claim that X, Y is a negative example, they occupy our attention for the rest of this section.

2.2.3 Properties of moduli spaces X, Y

The two important properties of X, Y, proving which will occupy the rest of this chapter, are that X parametrizes only geometrically stable sheaves and Y does so whenever there are no tritangent lines to the sextic B over $\overline{\mathbb{Q}}$. Moreover X, Y are compactification of relative Picard scheme of certain degree.

§When are X, Y moduli space of geometrically stable sheaves?

Following is the result we will prove :-

Theorem 2.2.14. *The moduli space $X = M_h(0, h, -1)$ always parametrizes geometrically stable sheaves and is smooth. If there are no tritangent lines to B over $\overline{\mathbb{Q}}$ then the moduli space $Y = M_h(0, h, 0)$ also parametrizes only geometrically stable sheaves and is smooth.*

We will need to prove some lemma's first.

Lemma 2.2.15. *If F, G are coherent sheaves on a smooth surface Z s.t. $\psi : F_{|U} \simeq G_{|U}$ for an open subscheme $U \xrightarrow{i} Z$ with U^c (the complement of U in Z) consisting of finitely many closed points of Z. Then $c_1(F) = c_1(G)$.*

Remark 2.2.16. *Here the first chern class of F is defined for F considered as an element of the Grothendieck Group $K(Z)$.*

Proof: The main idea is that the first Chern class $c_1(k(x)) = 0$ for a closed point $x \in Z$. So it is enough to find a homomorphism of coherent sheaves $\phi : F' \to G'$ s.t. $\phi_{|U} : F'_{|U} \to G'_{|U}$ is an ismorphism and $c_1(F') = c_1(F), c_1(G') = c_1(G)$ (for $ker(\phi)$ and $coker(\phi)$ are coherent sheaves supported on finite number of closed points hence will have zero first chern class. So the natural exact sequence $0 \to ker(\phi) \to F' \to G' \to coker(\phi) \to 0$ gives $c_1(F) = c_1(G)$).

We have the natural adjunction map given by $ad_F : F \to i_*F_{|U}$ and $ad_G : G \to i_*G_{|U}$. Since U^c is finite number of closed points, the inclusion $i : U \hookrightarrow Z$ is quasi compact and separated morphism, and hence $i_*F_{|U}, i_*G_{|U}$ are quasi coherent. The image of F in $i_*F_{|U}$ we denote by $F' =: ad_F(F)$ and define $G' =: ad_G(G) + i_*\psi(F')$ ($i_*\psi$ is the map $i_*F_{|U} \simeq i_*G_{|U}$). Clearly the kernel of the surjection $F \to F'$ and the cokernal of the injection $G \hookrightarrow G'$ are both coherent sheaves supported on finitely many closed points (since $F_{|U} \to F'_{|U}$ and $G_{|U} \to G'_{|U}$ are both isomorphism's under the maps constructed before). Hence $c_1(F') = c_1(F), c_1(G') = c_1(G)$. The natural map $F' \to G'$ is an isomorphism when restricted to U i.e. $F'_{|U} \to G'_{|U}$.
Hence as observed before $c_1(F) = c_1(F') = c_1(G') = c_1(G)$.

□

Lemma 2.2.17. *Let F be a rank 0 pure sheaf on $K3$ surface $S_{\overline{\mathbb{Q}}}$. Then $c_1(F) = \sum m_i[C_i]$ where C_i are the irreducible components of the reduced support of F and $m_i = rk_{k(\zeta_i)}(F_{\zeta_i})$, ζ_i being the generic point of C_i.*

Proof: F being a pure sheaf, all its irreducible components C_i must be of dimension 1. Consider the sheaf $H = \oplus_{i=1}^{r} \mathcal{O}_{C_i}^{\oplus m_i}$ (here $\mathcal{O}_{C_i}$ is the extension by zero of structure sheaf of the reduced closed subscheme C_i). The two sheaves F, H are isomorphic on an open set $U \subset S_{\overline{Q}}$ s.t. $S_{\overline{\mathbb{Q}}} - U$ consisting of only finitely many closed points. Such an open set U can be gotten by taking a union $\cup_i U_i$ of sufficiently small open neighbourhoods U_i containing $\zeta_i \in C_i$, $U_i \cap U_j = \{\eta\}; i \neq j$ (here η is the generic point of the S and ζ_i is the generic point of C_i for each i) and s.t. $F_{|U_i} \simeq \mathcal{O}_{C_i|U_i}^{\oplus m_i}$. Then an isomorphism $F_{|U} \simeq H_{|U}$ is obtained as the sum of all isomorphisms as above.

Hence by Lemma 2.2.15 $c_1(F) = c_1(H) = \sum_{i=1}^r m_i c_1(\mathcal{O}_{C_i})$. But we have $c_1(\mathcal{O}_{C_i}) = [C_i]$ (from the short exact sequence $0 \to \mathcal{O}(-C_i) \to \mathcal{O} \to \mathcal{O}_{C_i} \to 0$). The result follows.

□

Lemma 2.2.18. *Let F be a semistable sheaf on $S_{\overline{\mathbb{Q}}}$ s.t. $c_1(F) = [h]$. Then either F is supported on an irreducible curve C with generic rank 1 on C or, F is supported on a reducible curve $C \cup D$ with C, D irreducible s.t. F has generic rank 1 on each irreducible component.*

Proof: By Lemma 2.2.17, $c_1(F) = \sum_{i=1}^r m_i[C_i]$ where C_i's are the irreducible components of the reduced support of F, that are curves. But $c_1(F) = [h]$ (where the line bundle h is now considered as a Weil divisor). By Bertini's theorem we know that each $[C_i].[h] \neq 0 (1 \leq i \leq r)$ and hence using the fact that intersection product is a ring operation on the Chow ring $A^*(S_{\overline{\mathbb{Q}}})$ we have

$$2 = [h].[h] = c_1(F).[h] = \sum_{i=1}^r m_i[C_i.h]$$

The only two possibilites are :-

(i) $\underline{r = 2,\ c_1(F) = [C_1] + [C_2]}$

Then $m_1 = m_2 = 1$ and hence F is supported on the reducible curve $C_1 \cup C_2$ and has generic rank 1 on each of the irreducible curve C_i. Moreover $[C_1].[h] = [C_2].[h] = 1$(since they must be non-zero and their sum is 2.)

(ii) $\underline{r = 1, m_1 = 1}$

F is supported on an irreducible curve C with generic rank 1 there. (Note that the case $r = 1, m_1 > 1$ cannot happen since then $2 = [h].[h] = c_1(F).c_1(F) = m_1^2[C].[C]$ which is absurd).

□

Proof of 2.2.14: From [22, Thm 0.1] we know that any moduli space of geometrically stable sheaves on a $K3$ surface is smooth. It is enough to show that the moduli space under consideration parametrize only geometrically stable sheaves.
Let $F \in X(\overline{\mathbb{Q}})$ be a semistable sheaf on $S_{\overline{Q}}$ which satisfies $c_1(F) = [h]$, $rk(F) = 0$ and $\chi(F) = -1$. By Lemma 2.2.18 either F is supported on an irreducible curve $C \subset S_{\overline{\mathbb{Q}}}$ with generic rank 1 there or it is supported on a reducible curve $C \cup D$ with generic rank 1 on each of the irreducible component. We prove that in each of these cases F is stable.

<u>Case 1:-F is supported on an irreducible curve $C \subset S_{\overline{\mathbb{Q}}}$ with generic rank 1 on it</u>

To check stability it is enough to show that for all proper saturated subsheaves $G \subset F$ $p_G(n) < p_F(n); n \gg 0$ holds. Since F has generic rank 1 on C, any saturated subsheaf must be either 0 or all of F. Hence 0 is the only proper saturated subsheaf of F and hence F is stable.

Case 2:-F is supported on a reducible curve $C \cup D$ with generic rank 1 on each of the irreducible components C, D.

Note that since $[C] + [D] = [h]$ here, $C \cup D \in |h|$ and hence $C \cup D = \pi^{-1}L$ for some line $L \subset \mathbb{P}^2$ (recall $\pi : S \to \mathbb{P}^2$ is the double cover). By Proposition 2.1.4 we note that, since $C \cup D$ is not integral (over $\overline{\mathbb{Q}}$), this case cannot occur if there are no tritangent line to B over $\overline{\mathbb{Q}}$.
Let $G \subset F$ be a proper saturated subsheaf of F. G might be supported on only C or D with generic 1 there. Without loss of generality assume G is supported on C. By Hirzebruch-Riemann-Roch theorem, we have $\chi(G(n))c_1(G).[h]n + \chi(G)$ (where $G(n) = G \otimes h^n$. Recall h is ample). By Lemma 2.2.18 $c_1(G) = [C]$. Hence $c_1(G).[h] = [C].[h] = 1$ and so $P_G(n) = n + \chi(G) = p_G(n)$ (i.e. the Hilbert polynomial is already reduced).
On the other hand, $\chi(F(n)) = c_1(F).[h]n + \chi(F) = 2n - 1$. Hence $p_F(n) = n - \frac{1}{2}$. Due to semistability of F we know that $n + \chi(G) = p_G(n) \leq p_F(n) = n - \frac{1}{2}; n \gg 0$. But $\chi(G) \in \mathbb{Z}$ and hence $\chi(G) < -\frac{1}{2}$ or $p_G(n) < p_F(n); n \gg 0$. Hence F is stable.

For any semistable sheaf in $Y(\overline{\mathbb{Q}})$ only Case (i) can occur if we assume that there are no tritangent lines to B over $\overline{\mathbb{Q}}$. But the proof of case (i) here is identical to above.

$\square$

§Description of X, Y as compactified of Picard schemes

We will closely follow [30] in showing the following result:-

Theorem 2.2.19. *Let $\mathcal{C} \to |h|$ be the tautological family of curves over the linear system $|h|$. Then the moduli space X is the compactification $\overline{Pic}^0_{\mathcal{C}/|h|}$ and Y is the compactification $\overline{Pic}^1_{\mathcal{C}/|h|}$.*

We will first define the terminologies and notions stated in the theorem and then prove it.

Definition 2.2.20. *Let $X \to S$ be a flat, projective morphism with geometric fibers being integral curves. Let $\mathcal{O}(1)$ be a relative very ample line bundle on X for $X \to S$. Then the compactified Picard functor, denoted $\overline{\mathcal{P}ic}_{X/S}$, is defined*

$$\overline{\mathcal{P}ic}_{X/S} : Sch_S^{opp} \to (Sets)$$

as

$$(T \to S) \mapsto \{T-\text{ flat coherent sheaf } \mathcal{F} \text{ on } X \times_S T \text{ s.t. for all geometric points } t \hookrightarrow T, \mathcal{F}_t$$
$$\text{is torsion free of rank 1 on } X_t\}/\sim, Pic(T)$$

Similarly the compactified Picard functor of degree n, denoted $\overline{\mathcal{P}ic}^n_{X/S}$, is defined

$$\overline{\mathcal{P}ic}^n_{X/S} : Sch_S^{opp} \to (Sets)$$

as

$$(T \to S) \mapsto \{T-\text{ flat coherent sheaf } \mathcal{F} \text{ on } X \times_S T \text{ s.t. for all geometric points } t \hookrightarrow T, \mathcal{F}_t$$
$$\text{is torsion free of rank 1 on } X_t \text{ with } deg(F_t) = n\}/\sim, Pic(T)$$

Here $deg(F) = \chi(F) - \chi(\mathcal{O})$.

By semi-continuity theorem the functors $\overline{\mathcal{P}ic}^n_{X/S}$ are open subfunctors of $\overline{\mathcal{P}ic}_{X/S}$. Denote the étale sheafification of these functors by $\overline{\mathcal{P}ic}^n_{X/S,ét}$ and $\overline{\mathcal{P}ic}^n_{X/S,ét}$. Then all these functors are representable and schemes representing $\overline{\mathcal{P}ic}^n_{X/S,ét}$ form a disjoint open cover of the schemes representing $\overline{\mathcal{P}ic}^n_{X/S,ét}$.

Theorem 2.2.21. *Let $X \to S$ be a relative curve as in the defintion above. Then $\overline{\mathcal{P}ic}^n_{X/S,ét}$ is represented by the disjoint union $\sqcup_n P_n$ where P_n are $S-$schemes representing the étale sheaves $\overline{\mathcal{P}ic}^n_{X/S,ét}$. The $S-$scheme $\sqcup_n P_n$ is called the* ***compactified Picard scheme****, denoted $\overline{Pic}_{X/S}$.*

Proof: See [3, Thm. 8.1]

□

For the linear system of curves $|h|$ we have the *universal* divisor $\mathcal{C} \subset S \times |h|$ (or also called the *tautological family of curves*) which is a family of divisors on S with obvious universal property. On geometric points $C \to |h|$, it has the geometric fiber $\{(x, C)|x \in C\}$.

There are natural maps $X \to |h|$ and $Y \to |h|$ which are heuristically defined as "sending a sheaf to its support". But to make sure it is a morphism one has to use the *Fitting support* rather than just support. For more details about the notion of Fitting support and the description of the maps $X, Y \to |h|$ see [30, Sec. 1.2]. By the universal property of compactified Jacobians we have a natural $|h|$ morphism of schemes $X \to \overline{Pic}^0_{|h|}$ and

$$Y \to \overline{Pic}^1_{|h|}$$

defined as send any sheaf to its pull back on its support.
We claim that
<u>**For each geometric point $C \to |h|$, we have $X_C \simeq \overline{Pic}^0_C$ and $Y_C \simeq \overline{Pic}^1_C$.**</u>

We show the isomorphism for X, the method showing it for Y being similar.
Let F be a sheaf on X_C, then i^*F $(i : C \to S_{\overline{\mathbb{Q}}})$ has generic rank 1 and is torsion free. (by Lemma 2.2.18). Conversely for any generic rank 1 and torsion free sheaf G on C with $deg(G) : \chi(G) - \chi(\mathcal{O}_C) = 0$, we have i_*G has rank 0 and has $\chi(i_*G) = -1$ and $c_1(i_*G) = h$ or $v(i_*G) = (0, h, -1)$. In other words we have shown the following :-

(2.2.19)

The natural morphism of $|h|-$schemes $X \to \overline{Pic}^0_{\mathcal{C}/|h|}$ and $Y \to \overline{Pic}^1_{\mathcal{C}/|h|}$ is an isomorphism.

In conclusion, we have obtained the following descriptions of the two moduli spaces $X = M_h(0, h, -1)$ and $Y = M_h(0, h, 0)$:-
(1): If there are no tritangent lines to the sextic curve B over $\overline{\mathbb{Q}}$, then both the moduli spaces X and Y parametrize only geometrically stable sheaves.

(2): On the other hand, the $|h|-$scheme structure on X, Y (induced by the map sending a sheaf to its support) gives $X = \overline{Pic}^0_{\mathcal{C}/|h|}$ and $Y = \overline{Pic}^1_{\mathcal{C}/|h|}$

Chapter 3

Brauer group and twisted sheaves

In this chapter we discuss the theory of twisted sheaves, in particular twisted universal sheaves. The Brauer group plays a pivotal in the definition and development of the theory of twisted sheaves. So we will first discuss briefly the basic notions from the theory of Brauer groups over fields and subsequently over a scheme. Then we define twisted sheaves and discuss generalizations of usual operations on sheaves like push forward, pull back etc. to twisted sheaves. We then state the main result of this chapter, namely the existence of *twisted universal sheaf* for a moduli space parametrizing simple sheaves M, i.e. a twisted sheaf which is locally a universal sheaf. This immediately shows the existence of twisted universal sheaves on other moduli spaces as well. The class in Brauer group defining the twisted universal sheaf is called the *Brauer class obstructing the universal sheaf* on M. In the next chapter, such a Brauer class plays a crucial role in proving that $Y = M_h(0, h, 0)$ has no rational points.

3.1 Brauer group

We recall here the absolute minimum theory of the Brauer groups required for us. The two excellent texts which will be followed by us are [28, Chap. 1] and [19, Chap. 4]

Brauer group of fields

Definition 3.1.1. *(**Central Simple Algebras**) Let k be a field and A be a finite dimensional $k-$algebra. A is said to be **simple** if it has no two sided ideals. A is said to **central** if the center of A (i.e. those $z \in A$ s.t. $za = az$ for all $a \in A$) is precisely the image of k in A.*
*A finite dimensional k algebra A is said to be a **central simple k-algebra** if it is both central and simple.*

Following proposition provides equivalent ways for defining central simple algebras, which can be generalized to rings as well.

Proposition 3.1.2. *The following are equivalent for a $k-$algebra :-*
(i) There exists a finite separable extension $k \subset L$ s.t. the L algebra $A \otimes_k L$ is isomorphic to the matrix algebra $M_n(L)$ for some $n \geq 1$.
(ii) The k^s-algebra $A \otimes_k k^s$ is isomorphic to $M_n(k^s)$ for some $n \geq 1$.
(iii) The $k-$algebra A is a finite dimensional central simple $k-$algebra.
(iv) There exists a finite dimensional central division $k-$algebra D and an integer $n \geq 1$ s.t. $A \simeq M_r(D)$ is an isomorphism of $k-$algebras. Moreover r and D are unique.

Proof: See [28, Prop. 1.5.2].

□

Henceforth we will denote by

$$CSA_k := \{\text{central simple algebras over } k\}/\text{k-algebra isomorphisms}$$

Definition 3.1.3. *(**Brauer group of a field** k) Let $A, B \in CSA_k$. We call A and B **similar** (or **Brauer equivalent**), denoted $A \sim B$, if the following equivalent conditions holds true:-*
(i) There exists $m, n \geq 1$ and a division algebra $D \in CSA_k$ s.t. $A \simeq M_m(D)$ and $B \simeq M_n(D)$ as $k-$algebras.
(ii) There exists $m, n \geq 1$ s.t. $M_n(A) \simeq M_m(B)$ as $k-$algebras.
(the equivalence of (i) and (ii) is due to Prop. 3.1.2(iv))

Define the Brauer group, denoted $Br(k)$, as

$$Br(k) := CSA_k/\sim$$

The Brauer group has a natural group structure arising out of the tensor product. More precisely, define the group operation on $Br(k)$ as $[A].[B] := [A \otimes B]$ for $[A], [B] \in Br(k)$ (here $[A], [B]$ are the representatives of central simple algebras A, B in $Br(k)$). Well definedness of the operation follows from (ii) of the definition above. The inverse of $[A] \in Br(k)$ is given by the opposite algebra $[A^{opp}] \in Br(k)$, since $A \otimes_k A^{opp} \simeq End_{k-vect}(A, A) = M_n(k)$. Also for any extension field $k \subset L$ and $A \in CSA_k$ we have $A \otimes_k L \in CSA_L$. Hence we have a covariant functor

$$Br : (Fields) \to (Groups)$$

Cohomological description

We now give a cohomological interpretation of the Brauer group $Br(k)$.

Let $A \in CSA_k$ of dimension r. By Prop. 3.1.2(ii), there exists a k^s-algebra isomorphism $\phi : M_r(k^s) \simeq A \otimes_k k^s$. The absolute Galois group $G = Gal(k^s/k)$ acts on ϕ in a natural way. Define

$$\xi_\sigma := \phi^{-1}({}^\sigma\phi) \in Aut(M_r(k^s)) = PGL_r(k^s)$$

(the automorphisms of the matrix group are inner, due to *Skolem-Noether theorem*).
One easily checks that $\{\xi_\sigma\}$ satisfies the $1-$cocycle condition for group cohomology and hence represents an element in $H^1(G, PGL_r(k^s))$. This gives a map

$$\{\text{elements of } CSA_k \text{ of rank } r^2\} \to H^1(G, PGL_r(k^s))$$

Consider the short exact sequence

$$1 \to k^{s\times} \to GL_r(k^s) \to PGL_r(k^s) \to 1$$

which induces the following injective (due to *Hilbert's 90-theorem*) homomorphism

$$H^1(G, PGL_r(k^s)) \hookrightarrow H^2(G, k^{s\times}) = H^2_{et}(k, \mathbb{G}_m)$$

Theorem 3.1.4. *The map taking $A \in CSA_k$ to the associated element of $H^2_{et}(k, \mathbb{G}_m)$ induces an isomorphism of groups $Br(k) \simeq H^2_{\acute{e}t}(k, \mathbb{G}_m)$.*

Proof: See [31, Chap. X, §5]

□

§Properties of Brauer groups, $Br(k)$

This section is just a recap of well known properties of Brauer groups like the existence of maps $inv : Br(k) \to \mathbb{Q}/\mathbb{Z}$ for local fields k, the fact that Brauer group is zero for C_1-fields and other related results.

Definition 3.1.5. *Let $r \in \mathbb{R}_{\geq 0}$. A field k is said to be a C_r-**field** if every non-zero homogeneous polynomial $f(x_1, ..., x_n)$ of degree $d > 0$ with $n > d^r$ has a non-trivial zero in k^n.*

The fact that Brauer group of finite fields is zero follows from Chevalley-Warning theorem and Tsen's theorem stated below.

Proposition 3.1.6. *(**Chevalley-Warning theorem**) Let $k = \mathbb{F}_q$ be the finite field containing q elements. Then $\mathbb{F}_q$ is a C_1-field.*

Proof: For proof see [13, Chap. 10, §2].

□

Let $A \in CSA_k$ and by Prop. 3.1.2(ii) we have an isomorphism $A \otimes_k k^s \xrightarrow{i} M_n(k^s)$ and let nr_i be the composition $A \times_k k^s \xrightarrow{i} M_n(k^s) \xrightarrow{det} k^s$. Note that nr_i does not depend on the choice of i since any automorphism of $M_n(k^s)$ is inner. Hence we have $nr_{\sigma_i}(x) = nr_i(x)$ for all $\sigma \in Gal(k^s/k)$. Hence by Galois theory we have that nr_i descends to a ring homomorphism $nr : A \to k$.

Definition 3.1.7. *The ring homomorphism $nr : A \to k$ is called the reduced norm.*

Proposition 3.1.8. *(**Tsen's theorem**) If k is a C_1-field then $Br(k) = 0$.*

Proof: Let D be a finite dimensional central division algebra over k s.t. $[D : k] = r^2$ ($D \in CSA_k$ and hence its $k-$dimension is that of a matrix ring [see Prop. 3.1.2] which is a perfect square).
The reduced norm nr for $D \in CSA_k$ is a homogeneous polynomial in r^2 variables and of degree r. Indeed the base change $D \otimes_k k^s \xrightarrow{nr \otimes 1} k^s$ is just the determinant map, which is a homogeneous polynomial of degree r and in r^2 variables. If $r > 1$ then nr will have a non-trivial zero (by C_1-field property) say $a \in D$. Let $b \in D$ be s.t. $ab = 1$. Then $1 = nr(1) = nr(ab) = nr(a)nr(b) = 0$, which is absurd. Hence $r = 1$. In other words, the only central division algebra over k is 1 dimensional or by Prop. 3.1.2(iv) all central simple algebras over k are isomorphic to matrix rings $M_r(k)$ or $Br(k) = 0$.

□

Corollary 3.1.9. *For finite fields $\mathbb{F}_q$, $Br(\mathbb{F}_q) = 0$.*

Proof: Follows from Prop. 3.1.6 and Prop. 3.1.8.

□

Now we give the result which computes the Brauer groups of local fields, using the map $inv : Br(k) \to \mathbb{Q}/\mathbb{Z}$.

Remark 3.1.10. *Recall that local field is a normed field $(K, |.|)$ s.t. it is complete w.r.t the norm on it. One has the following characterization of local fields :-*
(i) If the normed field $(K, |.|)$ has an Archimedean norm $|.|$, then $(K, |.|) \simeq (\mathbb{R},$ std. inner product$)$ or $(\mathbb{C},$ standard inner product$)$, isomorphic as normed fields.
(ii) If the normed field $(K, |.|)$ has a non-Archimedean norm $|.|$, then $(K, |.|) \simeq (F, |.|')$ where $(F, |.|')$ is a finite extension field of $(\mathbb{Q}_p, |.|_p)$ (p-adic numbers with p-adic metric) or is a finite extension field of $(\mathbb{F}_q((t)), |.|_t)$(rational power series field with standard $t-$adic metric), isomorphic as normed fields.

Theorem 3.1.11. *(**Brauer group of local fields**) Suppose that k is a local field.*
(i) There is an injective map of groups $inv : Br(k) \to \mathbb{Q}/\mathbb{Z}$ whose image is

$$\begin{cases} \frac{1}{2}\mathbb{Z}/\mathbb{Z} & \text{if } k = \mathbb{R} \\ 0 & \text{if } k = \mathbb{C} \\ \mathbb{Q}/\mathbb{Z} & \text{if } k \text{ is non-archimedean} \end{cases}$$

(ii) If L is a finite extension of k then we have a commutative diagram

$$\begin{tikzcd} Br(k) \arrow[r, "inv"] \arrow[d] & \mathbb{Q}/\mathbb{Z} \arrow[d, "\times[L:k]"] \\ Br(L) \arrow[r, "inv"] & \mathbb{Q}/\mathbb{Z} \end{tikzcd}$$

Proof: See [28, Thm. 1.5.34]

□

The following result helps us compute the Brauer group of a global field from the Brauer group of its local fields, using a short exact sequence.

Theorem 3.1.12. *(**Brauer group of a global field**) Let k is a global field and v a place on it (or a discrete valuation on k), denote the completion of k w.r.t v by k_v. Also denote by $inv_v : Br(k_v) \to \mathbb{Q}/\mathbb{Z}$ for the map inv defined for the local field k_v in Thm.3.1.11.*
(i) The following sequence is exact

$$0 \to Br(k) \to \oplus_v Br(k_v) \xrightarrow{\oplus_v inv_v} \mathbb{Q}/\mathbb{Z} \to 0$$

Proof: See [28, Thm. 1.5.36]

□

Brauer group of schemes

The analogue of central simple algebra over a commutative local ring $(R, \mathfrak{m})$ is called a *Azumaya algebra* and more generally called a *sheaf of Azumaya algebras over X* for a scheme X.

Definition 3.1.13. *(**Azumaya algebras**) Let $(R, \mathfrak{m})$ be a commutative local ring. An $R-$algebra A is called an **Azumaya algebra** if it is a free module of finite rank and the natural map*

$$A \otimes_k A^{opp} \to End_{R-mdl}(A)$$

is a isomorphism of $k-$algebras.
*More generally if X is a variety over a field k and $\mathcal{A}$, an $\mathcal{O}_X-$algebra, is called an **Azumaya algebra on** X if for each point $x \in X$ the $\mathcal{O}_{X,x}-$algebra $\mathcal{A}_x$ is an Azumaya algebra. Denote the category of Azumaya algebras over a scheme X by Az_X.*

Following proposition characterizes Azumaya algebras over schemes.

Proposition 3.1.14. *Let X be a variety over field k and $\mathcal{A}$ be an $\mathcal{O}_X$ algebra of finite type. Then the following statements are equivalent:-*
(i) $\mathcal{A}$ is an Azumaya algebra over X.
(ii) $\mathcal{A}$ is locally free and the canonical map $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A}^{opp} \xrightarrow{\sim} \mathcal{E}nd_{\mathcal{O}_X-mdl}(\mathcal{A})$ is an isomorphism of $\mathcal{O}_X-$algebras.
(iii) There exists étale cover $(U_i \to X)$ s.t. $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{O}_{U_i} \simeq M_{r_i}(\mathcal{O}_{U_i})$ for all i.

Proof: For a proof see [19, Chap.4, Prop.2.1]

□

Definition 3.1.15. *(**Brauer group of a scheme**) Two Azumaya algebras $\mathcal{A}$ and $\mathcal{B}$ over X are said to be **similar** (or **Brauer equivalent**), denoted $\sim$, if there exists a locally free $\mathcal{O}_X$ modules E and E' of finite rank over $\mathcal{O}_X$ s.t. $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{E}nd_{\mathcal{O}_X-mod}(E) \simeq \mathcal{B} \otimes_{\mathcal{O}_X} \mathcal{E}nd_{\mathcal{O}_X-mod}(E')$.*

Define the Brauer group of X, denoted $Br(X)$, as

$$Br(X) := Az_X / \sim$$

For local ring, we denote $(R, \mathfrak{m})$ $Br(R) := Br(Spec(R))$.

For scheme X, unlike the case of fields, the two groups $Br(X)$ and $H^2_{ét}(X, \mathbb{G}_m)$ need not coincide. Nevertheless one has the following result by [19, Chap.4, Thm.2.5]

Proposition 3.1.16. *There is a canonical injective homomorphism $Br(X) \hookrightarrow H^2_{ét}(X, \mathbb{G}_m)$.*

Remark 3.1.17. *The group $H^2_{ét}(X, \mathbb{G}_m)$ is called the **cohomological Brauer group**.*

3.2 Twisted Sheaves

Instead of the structure sheaf $\mathcal{O}_X$ and sheaf of modules over it, if we consider a Azumaya algebra $\mathcal{A}$ over X and sheaf of modules over $\mathcal{A}$, the natural object that arises is called a *twisted sheaf*. More precisely, there is an equivalence of the category of $\mathcal{A}-$modules and the "category of twisted sheaves"(Thm.3.2.8). As the name suggests, *twisted sheaves* are sheaves only locally and there is a Brauer class which obstructs the patching of transition functions on triple intersections. The reference we will follow for this section is A. Căldăraru's Ph.D thesis [6].

We will consider our schemes X to be of finite type over k. Hence we can compute the etale cohomology groups of the etale sheaves on X using the Čech cohomology. [see 19, Thm. 2.17, Chap. 3].

Definition 3.2.1. *(**Twisted sheaf**) Let $\alpha \in \check{C}^2_{ét}(\mathcal{U}, \mathbb{G}_m)$ be a Čech 2-cocycle (in the étale topology) given by means of an étale cover $\mathcal{U} := (U_i \to X)_{i\in I}$ and let $\{\alpha_{ijk}\} \subset \Gamma(U_i \cap U_j \cap U_k, \mathbb{G}_m)$ be the representative of α.*
*Then define the $\alpha-$**twisted sheaf** ($\alpha-$sheaf in short) to consist of a pair $(\{F_i\}_{i\in I}, \{\phi_{ij}\}_{i,j\in I})$ with F_i being a sheaf of $\mathcal{O}_X-$modules on U_i and $\phi_{ij} : F_{j|_{U_i\cap U_j}} \xrightarrow{\sim} F_{i|_{U_i\cap U_j}}$ being isomorphism s.t.*
(i) ϕ_{ii} is identity.
(ii) $\phi_{ij} = \phi_{ji}^{-1}$.
(iii) $\phi_{ij} \circ \phi_{jk} \circ \phi_{ki} = \alpha_{ijk} Id_{F_{U_{ijk}}}$.

Homomorphism between two $\alpha-$twisted sheaves $(\{F_i\}_{i\in I}, \{\phi_{ij}\}_{i,j\in I})$ and $(\{G_i\}_{i\in I}, \{\phi'_{ij}\}_{i,j\in I})$ is defined in an obvious manner: It is a family of homomorphisms $\psi_i : F_i \to G_i$ where ψ_i are $\mathcal{O}_{U_i}-$module morphisms s.t. on intersection $U_i \cap U_j$, ψ_i and ψ_j commute with the transition maps ϕ_{ij} and ϕ'_{ij}.

For an étale cover $\mathcal{U} = (U_i \to X)_{i\in I}$ and $\alpha \in \check{C}^2_{ét}(\mathcal{U}, \mathbb{G}_m)$ a Čech 2-cocycle, we denote by $\mathfrak{Mod}(X, \alpha, \mathcal{U})$ as the category of twisted sheaves (in the sense of definition above) with morphisms described above. Clearly it is an abelian category.

Note that $\mathfrak{Mod}(X, \alpha, \mathcal{U})$ depends on the covering $(U_i \to X)$. The following lemma removes this restriction by allowing the $2-$cocycle to be a class in the cohomological Brauer group.

Lemma 3.2.2. *Let $\mathcal{U}$ and $\mathcal{U}'$ be two étale covers of X s.t. $\mathcal{U}'$ is a refinement of $\mathcal{U}$. Let $\alpha \in \check{C}^2_{ét}(\mathcal{U}, \mathbb{G}_m)$ be a Čech 2-cocyle and by abuse of notation also denote its restriction to $\check{C}^2(\mathcal{U}', \mathbb{G}_m)$ by α. Then there is a canonical equivalence of abelian category $\mathfrak{Mod}(X, \alpha, \mathcal{U}) \xrightarrow{\sim} \mathfrak{Mod}(X, \alpha, \mathcal{U}')$.*

Proof: See [6, Lemma 1.2.3]

□

By above lemma any two Čech 2-cocyles $\alpha \in \check{C}^2_{ét}(\mathcal{U}, \mathbb{G}_m)$ and $\beta \in \check{C}^2_{ét}(\mathcal{U}', \mathbb{G}_m)$ which represent the same element in the cohomological Brauer group $H^2_{ét}(X, \mathbb{G}_m) = \lim_{\mathcal{U}} \check{C}^2_{ét}(\mathcal{U}, \mathbb{G}_m) = \lim_{\mathcal{U}'} \check{C}^2_{ét}(\mathcal{U}', \mathbb{G}_m)$ determine the same category $\mathfrak{Mod}(X, \alpha, \mathcal{U}) \sim \mathfrak{Mod}(X, \beta, \mathcal{U}')$ (by taking the common refinement of $\mathcal{U}$ and $\mathcal{U}'$).

Definition 3.2.3. *For any element $a \in H^2_{ét}(X, \mathbb{G}_m)$ we may define the category of $a-$twisted sheaves (or $a-$sheaves in short) by $\mathfrak{Mod}(X, a) := \mathfrak{Mod}(X, \alpha, \mathcal{U})$ for any representative $(\mathcal{U}, \alpha) \in \check{C}^2_{ét}(\mathcal{U}, \mathbb{G}_m)$ for $a \in H^2_{ét}(X, \mathbb{G}_m)$.*

Remark 3.2.4. *In fact $Br(X)$ is contained in the torsion part $H^2_{ét}(X, \mathbb{G}_m)_{tor}$ under the inclusion in Remark 3.1.16. We will be concerned mostly with $\alpha-$twisted sheaves with $\alpha \in Br(X)$.*

We have a natural short exact sequence of étale group sheaves

$$1 \to \mathbb{G}_m \to GL_n \to PGL_n \to 1$$

for $n \geq 1$. The long exact sequence in cohomology gives

$$\ldots \to Pic(X) \to H^1(X, GL_n) \to H^1(X, PGL_n) \xrightarrow{\delta} H^2_{ét}(X, \mathbb{G}_m) \to \ldots$$

Since for geometric points $H^i_{ét}$ vanish for $i > 0$, we have from the above exact sequence that the principle PGL_n-bundle $Y \to X$ (or $\mathbb{P}^n-$fiber bundles) are locally projectivization

of principle GL_n bundles (or vector bundles) $(\{V_i, \phi_{ij}\}_{i,j\in I})$ s.t. transition maps on triple intersection gives a Čech 2-cocyle α s.t. $\delta(Y) = \alpha$ (or Y is $\alpha-$twisted).
In fact, α is an element of $Br(X)$ and every element of $Br(X)$ arises this way. Equivalently $\delta(H^1(X, PGL_n)) = Br(X)$.
Following lemma gives us information regarding the torsion of elements in $Br(X)$.

Lemma 3.2.5. *If $\alpha \in Br(X)$ is in the image of $\delta : H^1(X, PGL_n) \to H^2_{\acute{e}t}(X, \mathbb{G}_m)$ (as above) for some n, then $n\alpha = 0$.*

Proof: See [6, Chap.1]

□

Following proposition generalizes the notion of functors, push forward and pull back, to the category of twisted sheaves.

Proposition 3.2.6. *Let $a \in H^2_{\acute{e}t}(Y, \mathbb{G}_m)$ and $f : X \to Y$ be a finite type morphism of $k-$varieties. There is a natural pull-back map, denoted f^*, defined as $f^* : H^2_{\acute{e}t}(Y, \mathbb{G}_m) \to H^2_{\acute{e}t}(X, \mathbb{G}_m)$. We have the pull back functor between the category of twisted sheaves, denoted f^* by abuse of notation,*

$$f^* : \mathfrak{Mod}(Y, a) \to \mathfrak{Mod}(X, f^*a)$$

The functor is unique with the property that when $a \in H^2_{\acute{e}t}(Y, \mathbb{G}_m)$ is trivial then it coincides with the usual pull-back functor. Similarly there is the push-forward functor

$$f_* : \mathfrak{Mod}(X, f^*a) \to \mathfrak{Mod}(Y, a)$$

which conincides with the usual push-forward functor when $a \in H^2_{\acute{e}t}(Y, \mathbb{G}_m)$ is trivial. Moreover (f^*, f_*) is an adjoint pair.

Proof: See [6, Prop. 1.2.13]

□

Following sequence of results will be useful when discussing the theory of twisted universal sheaves. These results help us construct a twisted sheaf whenever we're given a sheaf of modules over an Azumaya algebra.

Theorem 3.2.7. *Let $\mathcal{A}$ be an Azumaya algebra over a scheme X and $\alpha := [\mathcal{A}]$ be the class in Brauer group of $\mathcal{A}$[see Defn.3.1.15]. Then there exists a locally free $\alpha-$twisted sheaf E (not necessarily unique), in short called $\alpha-$lffr, s.t. $\mathcal{A} = \mathcal{E}nd(E)$. Conversely for any $\alpha-$lffr sheaf E, $\mathcal{E}nd(E)$ is an Azumaya algebra whose class in $Br(X)$ is α.*

Proof: See [6, Thm.1.3.5].

□

Let $\mathcal{A}$ be an Azumaya algebra and E be some $\alpha-$lffr associated to $\mathcal{A}$ as above. Then there exists a natural functor

$$F : \mathfrak{Mod}(X, \alpha) \to \mathfrak{Mod} - \mathcal{A}$$

defined $\mathcal{F} \mapsto \mathcal{F} \otimes_{\mathcal{O}_X} E^\vee$.

Theorem 3.2.8. *The functor F defined above is an equivalence of abelian categories.*

Proof: See [6, Thm. 1.3.7]

□

3.2.1 Twisted Universal Sheaves

From the theory of moduli spaces of sheaves we know that for any projective morphism $X \to S$ together with a polarization $\mathcal{O}_S(1)$ and a fixed polynomial $P(T) \in \mathbb{Q}[T]$ the moduli functor of semi-stable sheaves $\mathcal{M}_{X/S}$ is co-representable by a projective $S-$ scheme M. Also the open subfunctor $\mathcal{M}^s_{X/S}$ of geometrically stable sheaves is co-represented by an open subscheme of M [see Remark2.2.7]. It is in general not expected that these moduli spaces are fine i.e. the respective moduli functors need not be representable or a universal sheaf may not exist on $X \times_S M$ (resp. $X \times_S M^s$).

But *Luna's étale slice theorem* together with its corollaries shows that in fact in one has an locally universal sheaf in **étale topology** on the **moduli space of geometrically stable sheaves**, i.e. there exists an étale cover $(U_i \to M^s)$ and a family of universal sheaves F_i on $U_i \times_S M^s$. It turns out that one can get hold of a Brauer class $\alpha \in Br(M^s)$ s.t. the family sheaves $\{F_i\}$ can be regarded as giving an $p_2^*\alpha-$sheaf i.e. an element of $\mathfrak{Mod}(X \times_S M^s, p_2^*\alpha)$. The idea of the existence of twisted universal sheaf is contained in Mukai's work on vector bundles on $K3$ surfaces [21, Appendix 2]. Mukai proves the existence of a *quasi universal* family of sheaves on a moduli space of simple sheaves but this is equivalent to getting a twisted universal sheaf by Theorem 3.2.8.

Definition 3.2.9. *The moduli functor $\mathcal{S}pl_{X/S} : Sch_S^{opp} \to (Sets)$ of simple sheaves is defined as*

$$\overline{\mathcal{S}pl}_{X/S} : Sch_S^{opp} \to (Sets)$$

as

$$(T \to S) \mapsto \{T- \text{ flat coherent sheaf } \mathcal{F} \text{ on } X \times_S T \text{ s.t. for all geometric points } t \hookrightarrow T, \mathcal{F}_t$$
$$\text{is a simple sheaf on } X_t\}/(\sim, q^*Pic(T))$$

(here $q : X \times_S T \to T$)

Remark 3.2.10. *We will be concerned only with the case $S = Spec(k)$.*

Mukai proved the existence of a *quasi-universal sheaf* on the moduli space of simple sheaves, M. We will prove below that this is equivalent to giving a twisted universal sheaf on $X \times M$. Since both compactified Picard schemes and moduli space of geometrically stable sheaves are open subschemes of M, we also have twisted universal sheaves on them.

Definition 3.2.11. *Let $X \to k$ be a projective morphism and let $\mathcal{M}$ be a connected component of moduli space functor $\mathcal{S}pl : Sch_k^{opp} \to (Sets)$ of simple sheaves. Suppose that the étale sheafification of $\mathcal{M}$ is representable by a finite type scheme, M, over k. Then :-*
*(i) Let T be a $k-$scheme and $\mathcal{E}$ be a flat $T-$flat coherent sheaf on $X \times_k T$. The family of sheaves $\mathcal{E}_t$ for geometric point $t \hookrightarrow T$, is said be **quasi family** if there exists $E \in \mathcal{M}(Spec(t))$ s.t. $\mathcal{E}_t \simeq E^r$ for some $r \in \mathbb{N}$, for all geometric points $t \hookrightarrow T$.*

*(ii) Two quasi families $\mathcal{E}$ and $\mathcal{E}'$ on $X \times_k T$ are said to be **equivalent** if there exists vector bundles V, V' on T s.t. $\mathcal{E} \otimes \pi_2^*(V) \simeq \mathcal{E}' \otimes \pi_2^*(V')$ where $\pi_2 : X \times_k T \to T$ is the second projection.*

*(iii) A **quasi-universal sheaf** $\mathcal{E}$ on $X \times_k M$ if for any scheme $T \to S$ and quasi family $\mathcal{F}$ on $X \times_k T$, there exists a unique map $f : T \to M$ s.t. $f^*\mathcal{F}$ and $\mathcal{E}$ are equivalent.*
(cf. [21, 12, Appendix 2])

Theorem 3.2.12. *Suppose $\mathcal{M} : Sch_k^{opp} \to (Sets)$ be as in the definition above. Then there exists a quasi-universal sheaf on $X \times_k M$.*

Proof: See [21, Thm. A.5]

$\square$

From the proof of previous theorem we can deduce the following result:-

Proposition 3.2.13. *A quasi universal sheaf $\mathcal{E}$ on $X \times_k M$ as constructed above is a sheaf of modules of an Azumaya algebra $\mathcal{A}$, which by Theorem 3.2.8, induces a $\pi_2^*\alpha$-twisted sheaf on $X \times_k M$ which is locally universal for the étale topology ($\alpha = [\mathcal{A}]$ in $Br(X)$). Conversely for some $\alpha \in Br(M)$, a $p_2^*\alpha-$twisted sheaf $\mathcal{E}$ which is locally universal for the étale topology induces naturally a quasi-universal sheaf on M*

Proof: From Mukai's proof in [21, Theorem A.2], we see that the quasi universal sheaf $\mathcal{E}$ is the gluing data $(\mathcal{E}_i \otimes p_{i,2}^*V_i^\vee, f_{ij} \otimes (p_{i,2}^*\bar{f_{ij}}^{-1})^\vee)$ where :
(i) $(U_i \to M)$ is an étale cover.
(ii) $\mathcal{E}_i$ are universal sheaves on $X \times U_i$ with an isomorphism $f_{ij} : \mathcal{E}_{j_{|U_{ij}}} \simeq \mathcal{E}_{i_{|U_{ij}}}$.
(iii) By semi-continuity arguments, for some sufficiently high power of the ample line bundle L on $X \times U_i$, $\mathcal{E}_i(n) =: V_i$ is a vector bundle and f_{ij} induces an isomorphism of vector bundles $\bar{f}_{ij} : V_{j_{|U_{ij}}} \simeq V_{i_{|U_{ij}}}$.
The projectivization of the vector bundles patches to give a projective bundle hence $E = (V_i, \bar{f}_{ij})$ is a $\alpha-$twisted sheaf for some $\alpha \in H^2_{\acute{e}t}(M, \mathbb{G}_m)$. Also $(\mathcal{E}_i \otimes p_{i,2}^*V_i^\vee, f_{ij} \otimes (p_{i,2}^*\bar{f_{ij}}^{-1})^\vee) = (\mathcal{E}_i, f_{ij}) \otimes_{\mathcal{O}} p_2^*(V_i, \bar{f}_{ij})^\vee = (\mathcal{E}_i, f_{ij}) \otimes_{\mathcal{O}} E^\vee$. $\mathcal{E}$ is evidently a sheaf of modules over the Azumaya algebra $\mathcal{A} := \mathcal{E}nd(E)$. By Thm. 3.2.8 $(\mathcal{E}_i, f_{ij})$ is an $p_2^*\alpha-$twisted sheaf with E an $\alpha-$lffr and Thm. 3.2.7 tells that α is the class of $\mathcal{A}$ in $Br(X)$. Also by definition the twisted sheaf $(\mathcal{E}_i, f_{ij})$ is locally universal sheaf.
The converse follows immediately from Thm.3.2.7 since for a $p_2^*\alpha-$twisted universl sheaf $\mathcal{E}$ let F be a $\alpha - lffr$. Then the (untwisted) sheaf $\mathcal{E} \otimes F^\vee$ is a quasi-universal sheaf.

$\square$

Definition 3.2.14. *The Brauer class $\alpha \in Br(M)$ constructed above is called the* ***obstruction class to the existence of a universal sheaf****. The $p_2^*\alpha-$twisted sheaf constructed above is called twisted universal sheaf.(We will follow similar terminology for other moduli spaces as well).*

Remark 3.2.15. *The obstruction $\alpha \in Br(M)$ is unique. Indeed, suppose $\mathcal{E}, \mathcal{E}'$ are two $p_2^*\alpha, p_2^*\beta-$twisted universal sheaves and suppose $\mathfrak{U}, \mathfrak{U}'$ be two trivializing covers for $\mathcal{E}, \mathcal{E}'$ respectively. Consider a refinement $\mathfrak{U}'' = \mathfrak{U} \times \mathfrak{U}'$ of both these étale covers. Then on $\mathfrak{U}''$ there are $\alpha-$lffr V and $\beta-$lffr W associated to these twisted universal sheaf such that $[\mathcal{E}nd(V)] = \alpha$ and $[\mathcal{E}nd(W)] = \beta$ ($[*]$ is the class in Brauer group of a Azumaya algebra). But $\mathcal{E}, \mathcal{E}'$ differ by a line bundle L on $\mathfrak{U}''$ and hence $V \otimes L \simeq W$ snd so $\mathcal{E}nd(W) \simeq \mathcal{E}nd(V \otimes L) \simeq \mathcal{E}nd(V)$. Hence $\alpha = \beta$.*

The moduli functor $\mathcal{S}pl$ is a disjoint union of functors $\mathcal{S}pl^P_{X/k}$, where $\mathcal{S}pl^P_{X/k}$ are moduli functors of simple sheaves with Hilbert polynomial P. In fact by [3, Lemma 5.8] $\mathcal{S}pl^P_{X/k}$ are both open and closed subfunctors of $\mathcal{S}pl_{X/k}$ and hence any scheme representing $\mathcal{S}pl^P_{X/k}$ is a connected component of Spl_X (which corepresents $\mathcal{S}pl_{X/k}$). The functors $\mathcal{P}ic^P_{X/k}$, $\overline{\mathcal{P}ic}^P_{X/k}$ and $\mathcal{M}^{P,s}_{X/k}$ are open subfunctors of $\mathcal{S}pl^P_{X/k}$ (by [3, Lemma 5.12(i), (ii)] and [12, Prop. 2.3.1]). This leads to the following result:-

Proposition 3.2.16. *Let $P \in \mathbb{Q}[T]$ be a polynomial and $M = Spl^P_{X/k}$ be the scheme representing the moduli functor $\mathcal{S}pl^P_{X/k}$ and let $\overline{Pic}^P_{X/k}$ be the scheme representing moduli functor $\overline{\mathcal{P}ic}^P_{X/k}$. Then there exists an obstruction $\alpha \in Br(\overline{Pic}^P_{X/k})$ to the existence of an universal sheaf on $X \times \overline{Pic}^P_{X/k}$. Same result holds with $\overline{\mathcal{P}ic}^P_{X/k}$ replaced by $\mathcal{P}ic^P_{X/k}$ and $\mathcal{M}^{P,s}_{X/k}$.*

Proof: By Prop. 3.2.13 we have a class a in $Br(M) = Br(Spl^P_{X/k})$ obstructing the existence of an universal sheaf on $X \times M$. Since $\overline{Pic}^P_{X/k}$ is an open subscheme of M the pull-back $i^* : \mathfrak{Mod}(X \times Spl^P_{X/k}, a) \to \mathfrak{Mod}(X \times \overline{Pic}^P_{X/k}, i^*a)$ $(i : \overline{Pic}^P_{X/k} \hookrightarrow Spl^P_{X/k})$ gives an obstruction to the existence of universal sheaf on $X \times \overline{Pic}^P_{X/k}$ with $\alpha = i^*a$ being the obstruction class.

$\square$

Remark 3.2.17. *Note that the coarse moduli space $M^{P,s}$ parametrizing geometrically stable sheaves represents the étale sheafification of the moduli functor of geometrically stable sheaves. Indeed for any geometric point $t \hookrightarrow S$, $\mathcal{M}^{P,s}_{X/S,\acute{e}t}(k(t)) = \mathcal{M}^{P,s}_{X/S}(k(t))$ and hence the two étale sheaves $\mathcal{M}^{P,s}_{X/S,\acute{e}t}(-)$ and $Hom_S(-, M^{P,s})$ have identical stalks at geometric points and hence are isomorphic. In other words, $\mathcal{M}^{P,s}_{X/S,\acute{e}t}(-)$ is representable.*

Zero cycle of degree 1 and Brauer group

Let X/k be a projective, geometrically integral variety which is Cohen-Macaulay (for example projective, geometrically integral curves). Let $\lambda \in NS(X^s)^G$ for $G = Gal(k^s/k)$ the absolute Galois group of k. There is a natural map from $NS(X^s)^G \to K_{num}(X^s)^G$ and denote by c the image of λ in $K_{num}(X^s)^G$. Let $P \in \mathbb{Q}[T]$ be the Hilbert polynomial associated to c. Then we denote by

$$\overline{Pic}^\lambda_{X/k} := \overline{Pic}^P_{X/k}$$

We denote by

$$\alpha_\lambda \in Br(\overline{Pic}^P_{X/k})$$

the obstruction class to the existence of a universal sheaf on $X \times \overline{Pic}^\lambda_{X/k}$.
The next two propositions will be used later in conjunction to prove that certain Brauer classes are trivial.

Lemma 3.2.18. *Let X/k be a projective, geometrically integral variety over a field k which is Cohen-Macaulay. Let $\lambda \in NS(X^s)^G$, $\overline{Pic}^P_{X/k}$ and α_λ be as above. Let E be a vector bundle on X and $c \in K_{num}(X^s)^G$ be the image of $\lambda \in NS(X^s)^G$. Then there exists an integer n_0 s.t. for all $n \geq n_0$ the integer $N_E(n) := \chi(c.[E(n)])$ annihilates α_λ (here $E(n)$ is the tensor product of E w.r.t to the $n-th$ power of a very ample line bundle on X).*

Proof: Let U_λ on $X \times \overline{Pic}^\lambda_{X/k}$ be the $p_2^*\alpha_\lambda$−twisted universal sheaf. By definition there is a trivializing étale cover $(U_i \to \overline{Pic}^\lambda_{X/k})$ s.t. $\mathcal{E}_i := U_{\lambda_{|X \times U_i}}$ is a universal sheaf. Since $\mathcal{E}_i$ is U_i−flat coherent sheaf hence so is $\mathcal{E}_i \otimes p_1^*E$ a U_i−flat sheaf. So there exists an integer n_0 s.t. for all $n \geq n_0$ $W_i^n := p_{2,*}(\mathcal{E}_i \otimes p_1^*E(n))$ is a vector bundle (by semi-continuity arguments). Since $\overline{Pic}^\lambda_{X/k}$ is quasi compact, we can choose a finite cover and hence a global n_0 can be chosen which works for all U_i. Hence $p_{2,*}(U_\lambda \otimes p_1^*E(n))$ is an α_λ−lffr. The rank of this α_λ−twisted sheaf is $\chi(p_{2,*}(\mathcal{E}_i \otimes p_1^*E(n))) = \chi(X_t, (\mathcal{E}_i \otimes p_1^*E(n))_t)$ (follows from semi-continuity theorem) where $\{t\} \to X$ is a geometric point. But $\chi(X_t, (\mathcal{E}_i \otimes p_1^*E(n))_t) = \chi(c.[E(n)]) = N_E(n)$ $\forall n \geq n_0$ (since $[U_{\lambda,t}] = c$). Hence by Lemma 3.2.5 we have $N_E(n).\alpha_\lambda = 0$.

$\square$

Proposition 3.2.19. *Let X/k be a projective, geometrically integral variety over a field k which is Cohen-Macaulay. Let $c \in K_{num}(X^s)^G, \lambda \in NS(X^s)^G$, $\overline{Pic}^P_{X/k}$ and α_λ be as above. Suppose that X has a smooth k−point or more generally has a zero-cycle of degree 1 contained in the smooth locus of X. Then $\alpha_\lambda = 0$.*

Remark 3.2.20. *Note that the smooth locus is non-empty by [18, Lemma 3.2.21]. Also note the similarity between this Proposition and [21, Thm. A.6].*

Proof: First observe that for any zero dimensional subscheme of $\xi \subset X$ of length d satisfies $\chi(c.[\mathcal{O}_\xi]) = d$. Indeed c is the class of a line bundle and hence in the numerical Grothendieck group $c.[\mathcal{O}_\xi] = [\mathcal{O}_\xi]$ (since tensoring a skyscraper sheaf with a line bundle leaves it invariant). But $\chi([\mathcal{O}_\xi]) = d$. If moreover, $\xi \subset X^{smooth}$ then $\mathcal{O}_\xi$ has finite resolution by vector bundles, say $E^\bullet$. Since tensoring by line bundles doesn't affect $\mathcal{O}_\xi$, we can choose the resolution to be $E^\bullet(n)$ (tensor by a power of an ample line bundle). Then we have by above argument $\sum_i (-1)^i \chi(c.[E^i(n)]) = \sum_i (-1)^i N_{E_i}(n) = \chi(c.[\mathcal{O}_\xi]) = d$.

Let $Z = \sum_t n_t[t]$ be a zero cycle of degree 1 i.e. $\{t\} \hookrightarrow X$ are closed points of X (the sum running over finitely many of them) and $\sum n_t[k(t) : k] = 1$. Now $\{t\}$ is zero dimensional subscheme of length $[k(t) : k]$. We have finite resolution by vector bundles for every structure sheaf $\mathcal{O}_t$. Since all the non-zero quantities involved are only finitely many, choose an integer n_0 which satisfies the hypothesis of previous lemma for all vector bundles involved in the resolution of $\mathcal{O}_t$ for all closed points t having $n_t \neq 0$. We then have $1 = \sum_t \sum_i (-1)^i n_t N_{E_t^i}(n_0)$ and by previous lemma $1.\alpha_\lambda = \sum_{t,i} n_t (N_{E_i^t}(n_0).\alpha_\lambda) = 0$.

$\square$

Proposition 3.2.21. *Let X be a projective variety over a non-Archimedean local field $k = k_v$ with ring of integers $\mathcal{O}_v$ and residue field κ_v (which is a finite field). Suppose that X has a model $\chi \to Spec(\mathcal{O}_v)$ whose special fiber is geometrically integral or more generally has a component Y of multiplicity 1 that is geometrically integral. Then X has a zero-cycle of degree 1 supported in its smooth locus.*

Proof: By the Lang-Weil bounds Y has a smooth point defined over a degree $r-$extension of κ_v for all $r \gg 0$ [see 28, Thm 7.7.1]. By Hensel's lemma this lifts to a smooth point on X defined over a degree $r-$extension of k_v (the lift will be smooth by Jacobian criterion). Consider the difference of a smooth point of degree $r+1$ and degree r on X. It will give a zero-cycle of degree 1 contained in the smooth locus.

$\square$

Chapter 4

Derived equivalence between hyperkähler varieties and rational points

This chapter concludes the first part of the thesis. We shall show that the pair of moduli spaces of sheaves previously constructed, $X = M_h(0, h, -1)$ and $Y = M_h(0, h, 0)$ is a negative answer to Esnault's question, under some conditions on the K3 surface S over which these moduli spaces are defined. We show that a derived equivalence exists between X and Y with the help of a sheaf defined on $X \times_{|h|} Y$. We construct a functor between the derived categories $D^b(X)$ and $D^b(Y)$ using this sheaf and then use Arinkin's theorem on autoduality of compactified Jacobian of curves with planar singularities [4] to prove that the functor is in fact an equivalence.

On the one hand, X contains a copy of $\mathbb{P}^2$ and hence has infinitely many rational points. On the other, the Brauer class obstructing the universal sheaf on $S \times Y$ gives a Brauer Manin obstruction to the Hasse principle, forcing Y to have no rational points.

4.1 Derived equivalence of compactified Picard schemes

The two moduli spaces X and Y can be described as $X = \overline{Pic}^0_{\mathcal{C}/|h|}$ and $Y = \overline{Pic}^1_{\mathcal{C}/|h|}$ for the tautological family of curves $\mathcal{C} \to |h|$ (Thm. 2.2.19). The aim of this section is to prove that a $\mathbb{Q}-$linear derived equivalence exists between $D^b(X)$ and $D^b(Y)$ (here by abuse of notation $D^b(T)$ denotes the bounded derived category of *quasi coherent sheaves* on T). The motivation to construct a derived equivalence between X and Y is rooted in D. Arinkin's seminal work on autoduality of compactified Jacobians of curves with planar singularities (cf. [4]).

Before we proceed, let us introduce some notation and assumptions. As before, let $\pi : S \to \mathbb{P}^2$ be the double cover defining a K3 surface branched over a sextic curve $B \subset \mathbb{P}^2$. Let B be defined by a homogeneous degree 6 polynomial in three variables $f(x, y, z) \in \mathbb{Z}[x, y, z]$. **Assume henceforth that the sextic curve B doesn't have any tritangent lines over $\overline{\mathbb{Q}}$.** By X^o we denote the open subscheme $Pic^o_{\mathcal{C}/|h|} \hookrightarrow \overline{Pic}^0_{\mathcal{C}/|h|} = X$ and similarly denote Y^o for the open subscheme $Pic^1_{\mathcal{C}/|h|} \hookrightarrow \overline{Pic}^1_{\mathcal{C}/|h|} = Y$. Let $\alpha \in Br(X^o)$ (resp. $\bar{\alpha} \in Br(X)$) be the obstruction class to the existence of universal sheaf $\mathcal{L}$ on $\mathcal{C} \times_{|h|} X^o$ (resp. $\bar{\mathcal{L}}$ on $\mathcal{C} \times_{|h|} X$). Note that $\mathcal{L}$(resp. $\bar{\mathcal{L}}$) is $1 \boxtimes \alpha-$twisted sheaf(resp. $1 \boxtimes \bar{\alpha}-$twisted sheaf). Similarly define $\beta \in Br(Y), \bar{\beta} \in Br(Y)$ and $1 \boxtimes \beta-$twisted universal sheaf $\mathcal{M}$ and $1 \boxtimes \bar{\beta}-$twisted universal sheaf $\bar{\mathcal{M}}$ on $\mathcal{C} \times_{|h|} Y^o$ and $\mathcal{C} \times_{|h|} Y$ respectively.

Remark 4.1.1. *We are obtaining a twisted universal sheaf from the fact that X and Y are* *moduli spaces of geometrically stable sheaves (Thm. 2.2.14 and Prop. 3.2.16). Indeed, there is a twisted universal sheaf on $X \times_k S$ and we get a twisted universal sheaf on $X \times_{|h|} \mathcal{C}$ as the pull back under the map $X \times_{|h|} \mathcal{C} \hookrightarrow X \times_{|h|} (S \times |h|) = X \times_{|h|} S \to X \times_k S$.*

Define the set

$$D := \{(L, M) | H^1(C, L \otimes M) \neq 0\} \subset \bar{X}^o \times_{|h|} \bar{Y} \cup \bar{X} \times_{|h|} \bar{Y}^o$$

For any variety T/k by $\bar{T}$ we mean the base change to $\bar{k}$ and the curve $C \in |h|$ is uniquely determined by choice of the point $(L, M) \hookrightarrow \bar{X}^o \times_{|h|} \bar{Y} \cup \bar{X} \times_{|h|} \bar{Y}^o$. We have the following result:-

Lemma 4.1.2. *D is a codimension 1 closed subscheme in $\bar{X}^o \times_{|h|} \bar{Y} \cup \bar{X} \times_{|h|} \bar{Y}^o$ defined over $\mathbb{Q}$ i.e. there is a codimension 1 closed subscheme D' on $X^o \times_{|h|} Y \cup X \times_{|h|} Y^o$ s.t. $\bar{D}' = D$.*

Proof: *We will use interchangeably the terms divisor and closed subscheme of codimension* 1. Consider the sets $D_1 := D \cap \bar{X}^o \times_{|h|} \bar{Y}$ and $D_2 := D \cap \bar{X} \times_{|h|} \bar{Y}^o$. It is enough to show that D_1, D_2 have the properties stated in the lemma. Consider the $\alpha \boxtimes \beta-$twisted sheaf $R^1\pi_{23,*}(\pi_{12}^*\mathcal{L} \otimes \pi_{13}^*\bar{\mathcal{M}})$ on $X^o \times_{|h|} Y$. By semicontinuity theorem, the set of geometric points of the support of this twisted sheaf is D_1. Indeed for any geometric point $(L, M) \hookrightarrow X^o \times_{|h|} Y$, we have $R^1\pi_{23,*}(\pi_{12}^*\mathcal{L} \otimes \pi_{13}^*\bar{\mathcal{M}}) \otimes k(L, M) = H^1(C, L \otimes M)$. Here the curve C is uniquely determined by the choice of the geometric point $(L, M) \hookrightarrow \bar{X}^o \times_{|h|} \bar{Y} \cup \bar{X} \times_{|h|} \bar{Y}^o$.
Hence D_1 is a closed subscheme which is defined over $\mathbb{Q}$ being the base change of the support of the twisted sheaf $R^1\pi_{23,*}(\pi_{12}^*\mathcal{L} \otimes \pi_{13}^*\bar{\mathcal{M}})$.
The reason D_1 is a divisor (on $\bar{X}^o \times_{|h|} \bar{Y}$) follows from the fact that on the open subscheme $\bar{X}^o \times_{|h|} \bar{Y}^o$ the fibers (in the dense open subset of $|h|$ consisting of smooth curves) of D_1 are translations of the Theta divisors (cf. 1.1.9), which are codimension 1 subvarieties.
Similarly D_2 is a divisor in $\bar{X} \times_{|h|} \bar{Y}^o$ and defined over $\mathbb{Q}$.

□

By abuse of notation we will also denote the divisor $D' \subset X^o \times_{|h|} Y \cup X \times_{|h|} Y^o$ also by D.

By theorem 2.2.14, we know that X, Y are smooth under the hypothesis above, on tritangent lines. Hence we can form the line bundle $\mathcal{O}(D)$ for the divisor $D \subset X^o \times_{|h|} Y \cup X \times_{|h|} Y^o$. Let $j : X^o \times_{|h|} Y \cup X \times_{|h|} Y^o \hookrightarrow X \times_{|h|} Y$ be the open immersion. Note that j is quasi compact and separated. Therefore $j_*\mathcal{O}(D)$ is a quasi coherent sheaf on $X \times_{|h|} Y$. The rest of this section will focus on giving a derived equivalence between X and Y using the sheaf $j_*\mathcal{O}(D)$.

We recall here, for reader's convenience, how Arinkin [4] showed a autoduality between $D^b(\bar{J})$ and $D^b(\overline{Pic}^0(\bar{J})) \simeq D^b(\bar{J})$. Here by $\bar{J}$ we mean the compactified Jacobian of an integral curve (or a family of curves) with planar singularities [1].
For smooth curve C/k, P. Deligne gave a description of the Poincaré line bundle $\mathcal{P}$ on $J \times J$ ($J := Pic^0_{C/k}$) as the unique family of line bundles with fiber over $(L, M) \hookrightarrow J \times J$ given by

$$\mathcal{P}_{(L,M)} = det(R\Gamma(L \otimes M))^{-1} \otimes det(R\Gamma(L)) \otimes det(R\Gamma(M)) \otimes det(R\Gamma(\mathcal{O}_C))^{-1}$$

For integral curves C with planar singularities, Arinkin defined his Poincaré sheaf as $\bar{P} = j_*P$ where $j : J \times \bar{J} \cup \bar{J} \times J \hookrightarrow \bar{J} \times \bar{J}$ is the open immersion and P is obtained using Deligne's description given above. Arinkin then goes on to show that $\bar{P}$ induces a derived equivalence $D^b(\bar{J}) \xrightarrow{\Phi_{j_*\bar{P}}} D^b(\bar{J})$ (Φ_- is the Fourier Mukai transform) [see 4, Thm C] .

[1] A curve $C/k (k = \bar{k})$ is said to have **planar singularity** if the maximal ideal $\mathfrak{m}_p$ of any closed point $p \in C$ is generated by atmost two elements.

Theorem 4.1.3. *Let X, Y be as before and suppose B has no tritangent lines over $\overline{\mathbb{Q}}$. Then there is a $\mathbb{Q}-$linear derived equivalence*

$$D^b(X) \simeq D^b(Y)$$

Proof: We claim that the derived equivalence is obtained as the functor

$$\mathbf{\Phi}^{|h|}_{j_*\mathcal{O}(D)} : D^b(X) \to D^b(Y)$$

defined as $\mathbf{\Phi}^{|h|}_{j_*\mathcal{O}(D)}(-) := Rq_*(Lp^*(-) \otimes j_*\mathcal{O}(D))$ (here $p(q) : X \times_{|h|} Y \to X(Y)$ is the first(second) projection).
To show that this is a derived equivalence it is enough to assume that the base field is $\overline{\mathbb{Q}}$ [see 25, Lem. 2.12].

Note that to prove the functor $F := \mathbf{\Phi}^{|h|}_{j_*\mathcal{O}(D)}$ is a derived equivalence, it is enough to show that F is fully faithful. Indeed then by [10, Prop. 7.6], since ω_X, ω_Y are trivial [10, Prop. 10.24], we have that F is in fact an equivalence. To prove that F is fully faithful we will use the fact that $\{k(x)|\}_{x \text{ closed points}}$ is a spanning class of the derived category of smooth varieties and so it is enough to show that

$$Hom_{D^b(X)}(k(x_1), k(x_2)[i]) \to Hom_{D^b(Y)}(F(k(x_1), F(k(x_2))[i]))$$

is an isomorphism for all closed points $x_1, x_2 \in X$ and all $i \in \mathbb{Z}$ [10, Prop. 1.49, Prop. 3.17].

Step 1: For a closed point $x \in X$, the support of the complex $F(k(x))$ is contained in the f
$\pi_2^{-1}(\pi_1(x))$ ($\pi_1 : X \to |h|, \pi_2 : Y \to |h|$ are the maps sending a sheaf to its support).

By definition $F(k(x)) = Rq_*(Lp^*(k(x)) \otimes j_*\mathcal{O}(D))$. Since $j_*\mathcal{O}(D)$ is defined on $X \times_{|h|} Y$, $Lp^*(k(x)) \otimes j_*\mathcal{O}(D)$ is supported on the fiber $\{x\} \times_{|h|} Y \hookrightarrow Y$ and so $Rq_*(Lp^*(k(x)) \otimes j_*\mathcal{O}(D))$ is supported on $\{x\} \times_{|h|} Y \subset Y$ and $\{x\} \times_{|h|} Y = \pi_2^{-1}(\pi_1(x))$.

Step 2: If x_1, x_2 are two closed points of X with $\pi_1(x_1) \neq \pi_1(x_2)$, then both $Hom_{D^b(X)}(k(x_1), k(x_2)[i]), Hom_{D^b(Y)}(F(k(x_1), F(k(x_2))[i]))$ **are zero**

This is immediate from step 1 which says that all the sheaves(or complexes) concerned in this step have disjoint support.

Step 3: If $\pi_1(x_1) = \pi_1(x_2)$ then the map $Hom_{D^b(X)}(k(x_1), k(x_2)[i]) \to Hom_{D^b(Y)}(F(k(x_1)), F(k(x_2))[i])$ **is an isomorphism**

First we make some observations. If $U \to |h|$ is an étale neighbourhood of $\pi_1(x_1) = \pi_1(x_2)$, then $Hom_{D^b(X)}(k(x_1), k(x_2)[i]) = Hom_{D^b(X_{|U})}(k(x_1), k(x_2)[i])$ and $Hom_{D^b(Y)}(F(k(x_1), F(k(x_2))[i])) = Hom_{D^b(Y_{|U})}(F(k(x_1), F(k(x_2))[i]))$, where $X_{|U} = X \times_{|h|} U = \overline{Pic}^0_{\mathcal{C}_U/U}$ and $Y_{|U} = Y \times_{|h|} U = \overline{Pic}^1_{\mathcal{C}_U/U}$. Hence it is enough to restrict our attention to an étale neighbourhood of $\pi_1(x_1) = \pi_1(x_2)$ s.t. $\mathcal{C} \to |h|$ has a section i.e. $\mathcal{C}_U \to U$ has a section. By abuse of notation, we denote again by D the pullback of the support of $R^1\pi_{23,*}(\pi_{12}^*\mathcal{L} \otimes \pi_{13}^*\bar{\mathcal{M}})$ to $X^o_{|U} \times_U X_{|U} \cup X_{|U} \times_U X^o_{|U}$. We identify $X_{|U}$ with $Y_{|U}$ using this section.
Now recall that all the theorems in Arinkin's paper holds true equally well for families of integral curves with planar singularities (cf. Remark (2) after Theorem C [4]). Our family

of curves $\mathcal{C}_U \to U$ is a family of integral curves with planar singularities since each curve $C \in |h|$ is contained in the regular surface S and the hypothesis on tritangent lines says that each curve $C \in |h|$ is geometrically integral (Prop.2.1.4(c)). Hence the following observation gives away the proof:-

(*) The sheaf corresponding to $\mathcal{O}(D)$ on $X^o_{|U} \times_U X_{|U} \cup X_{|U} \times_U X^o_{|U}$ (under the identification $X_{|U} \leftrightarrow Y_{|U}$) is the same as Arinkin's Poincaré line bundle upto line bundles from the base

Once we admit **(*)** the proof is clear, for then $j_*\mathcal{O}(D)$ is Arinkin's Poincaré sheaf on $X_{|U} \times_U Y_{|U}$ upto line bundles from base. By [4, Thm C], we have that $j_*\mathcal{O}(D)$ induces an equivalence between $D^b(X_{|U})$ and $D^b(Y_{|U})$, in particular is fully faithful and there is an isomorphism $Hom_{D^b(X_{|U})}(k(x_1), k(x_2)[i]) \simeq Hom_{D^b(Y_{|U})}(F(k(x_1)), F(k(x_2))[i])$.
(*) is proved in Appendix B.

□

Remark 4.1.4. *Note that we are considering bounded derived category of quasi coherent sheaves while all the results we have cited in the proof above from [10] are proved for derived category of coherent sheaves. But all the results can be generalized in a straightforward way to bounded derived category of quasi coherent sheaves. For instance, the Grotendieck duality is true for $D^b(Qcoh(Z))$ (for Z smooth projective variety), which can then be used to show that the right (and left) adjoint of a Fourier Mukai $\mathcal{P}$ has the form $\mathcal{P}^\vee \otimes p^*\omega_X[dim(X)]$ (and $\mathcal{P}^\vee \otimes q^*\omega_Y[dim(Y)]$) ($p(q) : X \times Y \to X(Y)$ is the first(second) projection) [see 10, Prop. 5.9].*

4.2 Rational points on X and Y

In this section we will show that X has infinitely many rational points while, under some conditions on tritangent lines to B, Y has no rational points. In [1, Section 4.2] it is shown using the magma computer algebra that there is a $K3$ surface S, satisfying this list of conditions on tritangent lines. Combining this with last section's result shows that X and Y (over this K3 surface S) give a negative answer to Esnault's question.

Proposition 4.2.1. *The moduli space $X = M_h(0, h, -1)$ contains a copy of $\mathbb{P}^2$, hence has infinitely many rational points.*

Proof: We will construct a section to the map $X \to |h|$. Recall that it was defined as sending a sheaf to its support. Now consider the section $s : |h| \to X$ given by $C \mapsto \mathcal{O}_C$ (on geometric points). Since $v(\mathcal{O}_C) = (0, h, -1)$ this is a well defined map. The fact that this induces a morphism from $|h| \to X$, which becomes then a section of $X \to |h|$, is by the usual argument involving Yoneda lemma. Since $|h| \simeq \mathbb{P}^2$, there is a copy of $\mathbb{P}^2$ in X.

□

The following two propositions show that Y is an example of the *Brauer Manin obstruction to the local global principle.* For a discussion of this theory see the Appendix A.

Proposition 4.2.2. *The moduli space Y has points over $\mathbb{R}$ and $\mathbb{Q}_p$ for all primes p.*

Proof: By Lichtenbaum's paper [15] we know that for any smooth curve C, Pic^1_C has both $\mathbb{R}$ as well as $\mathbb{Q}_p$ points. Choose a smooth curve $C \in |h|$, so the $Pic^1_C \subset Y$ has both $\mathbb{R}, \mathbb{Q}_p$ points.

$\square$

Proposition 4.2.3. *Let S be a K3 surface as before and let f be the degree 6 homogeneous polynomial defining $B \subset \mathbb{P}^2_{\mathbb{Q}}$. We choose an f satisfying the following conditions:-*
(a) $S(\mathbb{R}) = \emptyset$,
(b) there are no tritangent lines to B defined over $\overline{\mathbb{F}_2}$,
(c) for every tritangent line L to B defined over $\mathbb{F}_q$ with q odd, the curve $C = \pi^{-1}(L) \subset S_{\mathbb{F}_q}$ consists of two reduced rational curves defined over $\mathbb{F}_q$ rather than $\mathbb{F}_{q^2}$. [2].

Then $Y(k) = \emptyset$ for every number field k of odd degree over $\mathbb{Q}$. In particular, there is no zero cycle of degree 1 on Y.

Remark 4.2.4. *In [1, Sec. 4.2], N. Addington et al give the example of the following degree 6 homogeneous polynomial*

$$f(x,y,z) := -(x^6+x^5z+x^4y^2+x^4z^2+x^3yz^2+x^2y^2z^2+xy^5+xy^4z+xz^5+y^6+y^3z^3+y^2z^4+yz^5+z^6)$$

satisfying all the three conditions imposed above. They use the Magma computer algebra to produce it. For more details regarding the program, refer to the same section of [1].

Proof: First assume that $k = \mathbb{Q}$. We will show at the end how to change the proof to apply for all odd degree number fields.
Observe that S has no tritangent lines defined over $\overline{\mathbb{Q}}$ since there are no such lines over $\overline{\mathbb{F}_2}$. [3] Hence by Thm. 2.2.14 we have that Y parametrizes only geometrically stable sheaves. By Prop. 3.2.16 we have a class $\alpha \in Br(Y)$ obstructing the existence of an universal sheaf on $S \times Y$. Let $\mathfrak{U}$ be the $p_2^*\alpha-$twisted universal sheaf on $S \times Y$. We will show that Y is a Brauer Manin obstruction to the existence of rational points using the Brauer class α. In other words,

$$Y(\mathbf{A})^\alpha = \{(x_v) \in \prod Y(k_v))| \sum inv_v\alpha(x_v) = 0\} = \emptyset$$

We claim that the (1) and (2) below are true:-

(1) **Let $y \in Y(\mathbb{R})$. Then $\alpha_{|y} \neq 0$.**

On the contrary suppose that $\alpha_{|y} = 0$, Then $F = \mathfrak{U}_{S\times\{y\}}$ is a $p_2^*\alpha_{|y}-$sheaf on $S_{\mathbb{R}}$ or is a (untwisted) sheaf. Hence $v(F) = (0, h, 0)$ i.e. it has rank 0, first Chern class h and Euler characteristic 0. We have by definition that $c_1(F)^2 = h^2 = 2$ and $\chi(F) = 0$. Hence by Hirzebruch-Riemann-Roch theorem we have $\chi(F) = deg(\frac{c_1(F)^2}{2} - c_2(F))$ or $deg(c_2(F)) = 1$. But $c_2(F) \in CH_0(S_{\mathbb{R}})$ and hence consists of a sum of $\mathbb{R}-$closed points and $\mathbb{C}-$closed points. But since $S(\mathbb{R}) = \emptyset$, we must have $c_2(F)$ is a sum of $\mathbb{C}-$points or $deg(c_2(F))$ is an even number, a contradiction. Hence $\alpha_{|y} \neq 0$ for $y \in S(\mathbb{R})$.

(2) **Let v be a non-Archimedean place on k . For $C \in |h|(k_v)$ we have $\alpha_{|\overline{Pic^1_C}} = 0$ (in particular $\alpha_{|x_v} = 0$ for any $x_v \in Y(k_v)$)**

Let $\mathcal{O}_v$ be the ring of integers of k_v and $\kappa_v \simeq \mathbb{F}_q$ be the residue field. Let L be a line defined over k_v s.t. $\pi^{-1}(L) = C$. Consider the obvious models $\chi \to Spec(\mathcal{O}_v)$ and $\chi' \to Spec(\mathcal{O}_v)$ of

[2]This means that we can take $c = 1$ and g defined over $\mathbb{F}_q$ in Prop. 2.1.4(b)

[3]If there is a tritangent line over $\overline{\mathbb{Q}}$, say L, s.t. $f_{|L} = cg^2$. Then the coefficients of f, g and c lies in the ring of integers $\mathcal{O}_K$ for a suitable number field K (after clearing denominators). Choose a prime $\mathfrak{p}$ s.t. $\mathcal{O}_K/\mathfrak{p}$ is char. 2 finite field. Reduce the equation above w.r.t. this prime to get a tritangent line over $\overline{\mathbb{F}_2}$, a contradiction.

both C and L respectively i.e. $C \to Spec(k_v) \to Spec(\mathcal{O}_v)$ and $L \to Spec(k_v) \to Spec(\mathcal{O}_v)$ respectively. We have a map of models $\chi \to \chi'$. Since f is defined over $\mathbb{Z}$ the map of special fibers means the reduction $C_{\kappa_v} = \pi^{-1}(L_{\kappa_v})$. We have the two cases (i) either C_{κ_v} is reduced union of two rational curves (if L_{κ_v} is a tritangent line defined over κ_v); or (ii) C_{κ_v} is geometrically integral (by Prop. 2.1.4(c)). In both cases we have the hypothesis of Prop.'s 3.2.21 and 3.2.19 being satisfied. Both of them together yield $\alpha_{|\overline{Pic_C^1}} = 0$.

Now let $(x_v) \in \prod Y(k_v)$. Then $\sum_v inv_v(\alpha_{|x_v}) = \sum_v inv_{v\text{-real place}}(\alpha_{|x_v})$ (by (2) above $inv_v(\alpha_{|y})$ for non- Archimedean places vanish and by Prop. 3.1.11 it vanishes for $\mathbb{C}$ places as well). By (1) above since $\alpha_{|y}$ is non-zero for real places y we have $inv_v(\alpha_{|x_v}) = 1/2$ (by Prop. 3.1.11) or $\sum_{v \text{ real place}} inv_v(\alpha_{|x_v}) = n/2$ where n is odd number (since the number of real places is odd for k). Hence $Y(\mathbf{A})^\alpha = \emptyset$ or Y is a Brauer-Manin obstruction to local global principle (since $\prod Y(\mathbb{Q}_v) \neq \emptyset$ by Prop. 4.2.2). In particular, $Y(\mathbb{Q}) = \emptyset$.

The argument for number fields k with odd degree is almost the same as above except one needs to replace Y by Y_k and consider the respective Brauer classes.

□

Part 2
Affirmative examples to Esnault's Question in positive characteristic

Chapter 5

Even dimensional varieties in positive characteristic

In this part of thesis we will prove that for $K3$ surfaces over finite fields rational points are preserved under an equivalence of their derived categories. The proof essentially follows from *Lefschetz fixed point formula* once we descend the derived equivalence to either the $l-adic$ or *crystalline*(or $p-adic$) cohomology. Atleast for computing rational points, there is no distinction between either of these cohomology theories but we will follow M. Lieblich and M. Olsson [16] in using the *crystalline cohomology*.

This chapter introduces basic machinery needed to descend a Fourier Mukai transform to cohomology, imitating Mukai's treatment of the same over $\mathbb{C}$ [see 20]. We also state here the basic properties of crystalline cohomology, which will be used.

5.1 Mukai Hodge structure

Generalities on Hodge structure

For more details on Hodge theory the reader is referred to [2].

Definition 5.1.1. *A pure Hodge structure of weight $n \in \mathbb{Z}$ denoted $(H_{\mathbb{Z}}, H^{p,q})$ consists of a finitely generated, free abelian group $H_{\mathbb{Z}}$ and complex vector spaces $H^{p,q}$ s.t. $H_{\mathbb{C}} \simeq H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} \simeq \oplus_{p+q=n} H^{p,q}$ with $H^{p,q} \simeq \overline{H^{q,p}}$.*

Following are some commonly used examples of Hodge structures.

Example 5.1.2. *(**Tate-Hodge structure**) Take $H_{\mathbb{Z}} := 2\pi i\mathbb{Z}$ considered as a subgroup of $\mathbb{C}$ and let $n \in \mathbb{Z}$. Set $H^{-n,-n} := H_{\mathbb{C}} = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$. This is the unique $1-$dimensional pure Hodge structure of weight $-2n$ called the Tate Hodge structure and denoted $\mathbb{Z}(n)$.*

Example 5.1.3. *(**Tate twist**) Suppose $H = (H_{\mathbb{Z}}, H^{p,q})$ is a pure Hodge structure of weight n. The $r-th$ Tate twist for $r \in \mathbb{Z}$, denoted $H(r)$, is a pure Hodge structure of weight $n-2r$ defined as*

$$H(r)_{\mathbb{Z}} := H_{\mathbb{Z}}; H(r)^{p,q} := H^{p-r,q-r}$$

Example 5.1.4. *(**Direct sum of Hodge structures**) Let $H = (H_{\mathbb{Z}}, H^{p,q})$ and $L = (L_{\mathbb{Z}}, L^{p,q})$ be a pure Hodge structure of weight n. Then the direct sum of Hodge structures H and L denoted $H \oplus L$, is defined as $(H \oplus L)_{\mathbb{Z}} := H_{\mathbb{Z}} \oplus L_{\mathbb{Z}}$ and $(H \oplus L)^{p,q} := H^{p,q} \oplus L^{p,q}$.*

Example 5.1.5. *For a compact Kähler manifold X we have the* ***Hodge decomposition***

$$H^n(X,\mathbb{C}) \simeq \oplus_{p+q=n} H^q(X,\Omega_X^p)$$

with $H^p(X,\Omega_X^q) \simeq \overline{H^q(X,\Omega_X^p)}$. (here $H^n(X,\mathbb{C})$ is the $n-th$ singular cohomology with coefficients in $\mathbb{C}$ and Ω_X^p is the sheaf of holomorphic $p-$forms on X).
Taking $H_{\mathbb{Z}} := H^n(X,\mathbb{Z})/H^n(X,\mathbb{Z})_{tor}$ and $H^{p,q} := H^q(X,\Omega_X^p)$ we have a pure Hodge structure of weight n associated to singular cohomology of X.

A pure Hodge structure of weight 2

For a complex $K3$ surface X, Mukai introduced a pure Hodge structure of weight 2 on $H^*(X,\mathbb{Z})$ together with the structure of a $\mathbb{Z}-$lattice with a bilinear form. Indeed, each $H^i(X,\mathbb{Z})$ is zero for $i=1,3$ (by Hodge decomposition) and the even singular cohomolgies are free abelian groups ($H^2(X,\mathbb{Z})$ is torsion free since X is simply connected. For more details regarding this see [10, Sec. 10.1]). Hence, by Example 5.1.5 there is a weight $2i$ pure Hodge structure on $H^{2i}(X,\mathbb{Z})$. Also intersection product on $H^2(X,\mathbb{Z})$ induces a bilinear form on it (in fact with this bilinear form it is an *even unimodular lattice*). Thus we can introduce a weight $2-$Hodge structure $\widetilde{H}(X,\mathbb{Z}) = (H_{\mathbb{Z}}, H^{p,q})$ on $\oplus_i H^{2*}(X,\mathbb{Z})$ as

$$H_{\mathbb{Z}} := \oplus_i H^{2*}(X,\mathbb{Z})$$

$$\widetilde{H}^{1,1} := H^0(X,\mathbb{C}) \oplus H^{1,1}(X) \oplus H^4(X,\mathbb{C})$$
$$\widetilde{H}^{2,0} := H^{2,0}(X)$$
$$\widetilde{H}^{0,2} := H^{0,2}(X)$$

More succintly, the above Hodge structure can be written as the direct sum of Hodge structure $\widetilde{H(X,\mathbb{Z})} = H^0(X,\mathbb{Z})(-1) \oplus H^2(X,\mathbb{Z}) \oplus H^4(X,\mathbb{Z})(1)$ (see Example 5.1.4).
Mukai extended the bilinear form on $H^2(X,\mathbb{Z})$ (induced by the intersection product) to $\widetilde{H}(X,\mathbb{Z})$ by

$$\langle (a_0,b_0,c_0),(a_1,b_1,c_1)\rangle := b_0.b_1 - a_0.c_1 - a_1.c_0$$

$a_i \in H^0(X,\mathbb{Z}), b_i \in H^2(X,\mathbb{Z}), c_i \in H^4(X,\mathbb{Z})$.

Any Fourier- Mukai transform $\mathbf{\Phi}_P : D^b(X) \to D^b(Y)$ descends to a homomorphism of abelian groups $\widetilde{H}(X,\mathbb{Z}) \to \widetilde{H}(Y,\mathbb{Z})$ respecting both Hodge and lattice structure as follows:-
Define the *Mukai vector*

$$v(P) := ch(P)\sqrt{td(X\times Y)}$$

where $ch : D^b(X\times Y) \to K(X\times Y) \to H^*(X\times Y,\mathbb{Z})$ is the **Chern character** and $\sqrt{td(X\times Y)}$ is the **Todd class** of $X\times Y$. Define $\mathbf{\Phi}_{v(P)} : H^*(X.\mathbb{Z}) \to H^*(Y,\mathbb{Z})$ as $q_*(p^*(-)\cup v(P))$ where $p^* : H^*(X,\mathbb{Z}) \to H^*(X\times Y,\mathbb{Z})$ is the pull back, $q_* : H^*(X\times Y,\mathbb{Z}) \to H^*(Y,\mathbb{Z})$ is the push forward(defined using Poincaré duality}) and $\cup$ is the cup product on the cohomology ring. Then one has the result:-

Lemma 5.1.6. *For $P \in D^b(X\times Y)$ we have a commutative diagram*

$$\begin{tikzcd} D^b(X) \arrow[r, "\mathbf{\Phi}_P"] \arrow[d, "v(-)"] \arrow[dr, phantom, "\square"] & D^b(Y) \arrow[d, "v(-)"] \\ H^*(X,\mathbb{Z}) \arrow[r, "\mathbf{\Phi}_{v(P)}"] & H^*(Y,\mathbb{Z}) \end{tikzcd}$$

Proof: [See 10, Def. 5.28 and Cor. 5.29].

□

Remark 5.1.7. *The Hodge and lattice structure on $\oplus_i H^{2i}(X,\mathbb{Z})$ discussed above for $K3$ surfaces can be generalized to arbitrary proper smooth varieties of dimension $d = 2\delta$ over $\mathbb{C}$.* ***Let*** X, Y ***etc. be a smooth projective variety over*** $\mathbb{C}$ ***of dimension*** $d = 2\delta$.
Define the following pure Hodge structure of weight 2δ (called the Mukai-Hodge structure),

$$\widetilde{H}(X,\mathbb{Z}) := \oplus_{i=-\delta}^{\delta} H^{2\delta+2i}(X,\mathbb{Z})(i)$$

(here $H^j(X,\mathbb{Z})$ gets a pure Hodge structure of weight j given by Hodge decomposition).
The cup product and the identification $H^{2d}(X,\mathbb{Z}) \simeq \mathbb{Z}(-d)$ (via Poincaré duality) induces a bilinear form

$$\cup : \widetilde{H}(X,\mathbb{Z}) \times \widetilde{H}(X,\mathbb{Z}) \to \mathbb{Z}(-d)$$

defined $\langle (a_{-\delta},\ldots,a_\delta),(a'_{-\delta},\ldots,a'_\delta)\rangle := \sum_{i=-\delta}^{\delta}(-1)^i a_i a'_{-i}$.

A natural map $\psi_P : \widetilde{H}(X,\mathbb{Z}) \to \widetilde{H}(Y,\mathbb{Z})$ (dim$(Y) = d$) can be defined. It is described as adding the maps

$$\psi_P^{i,j} : H^{2\delta+2i}(X,\mathbb{Z})(i) \to H^{2\delta+2j}(Y,\mathbb{Z})(j)$$

, where each $\psi_P^{i,j}$ is described as the composite $H^{2\delta+2i}(X,\mathbb{Z})(i) \xrightarrow{p_1^*} H^{2\delta+2i}(X\times Y,\mathbb{Z})(i) \xrightarrow{\cup ch^{j-i+d}(P)} H^{2\delta+2j+2d}(X\times Y,\mathbb{Z})(j+d) \xrightarrow{p_{2*}} H^{2\delta+2j}(Y,\mathbb{Z})(j)$.

To descend a Fourier Mukai transform $\mathbf{\Phi}_P : D^b(X) \to D^b(Y)$ (dim$(Y) = d$) to cohomology we need only replace $ch^{j-i+d}(P)$ by $v^{j-i+d}(P)$ where $v(P) := ch(P).\sqrt{td(X\times Y)}$.

5.2 Mukai isocrystal

We're interested in generalizing Mukai's construction of the Hodge structure and a bilinear pairing of the cohomology groups, to characteristic $p > 0$. We will do it for crystalline cohomology (or $p-$adic cohomology) but as pointed out in the introduction of this chapter, one could use $l-$adic cohomology as well. Notice from the above construction of the Mukai-Hodge structure, that the properties, a cohomology theory requires to generalize Mukai's argument are Poincare duality theorem, that there is a cup product on the cohomology ring and characteristic classes and cohomology ring interact well via a class map.
First we discuss some properties of crystalline cohomology and then we define the *Mukai isocrystal* of a proper smooth variety over a perfect field in characteristic p.

§Crystalline cohomology

Crystalline cohomology was developed to account for the inability of $l-$adic cohomology to explain the $p-$torsion phenomenon. Grothendieck developed this cohomology theory and complete account of it was provided by P. Berthelot in his Ph.D thesis. All the properties of crystalline cohomology like Poincare duality, Lefschetz fixed point formula etc. have been verified in Berthelot's book [26]. Alternately one can also look at [5] and [9].

Let X be a smooth projective variety over an algebraically closed field in characteristic p. The basic step in defining the crystalline cohomology is to construct a *crystalline site* $Crys(X/W_n)$ (where $W_n := W_n(k)$ is n-th truncated ring of witt vectors) and a sheaf on it, denoted $\mathcal{O}_{X/W_n}$. Then one defines

$$H^i_{crys}(X/W_n) := H^i(Crys(X,W_n),\mathcal{O}_{X/W_n})$$

and passing to the projective limit gives

$$H^i_{crys}(X/W) := lim_{\leftarrow n} H^i_{crys}(X/W_n)$$

which becomes a $W := lim_{\leftarrow n} W_n$ module. Similarly one defines

$$H^i(X/K) := H^i_{crys}(X/W) \otimes_W K$$

where $K = Frac(W)$.

Definition 5.2.1. *Let k be a perfect field of characteristic $p > 0$. Let $\sigma : k \to k$ be the Frobenius morphism $x \mapsto x^p$. A finitely generated module M over the ring of Witt vectors $W := W(k)$ is called an **F-crystal** if there is an injective $\sigma-$linear map $\phi : M \to M$. A finite dimensional vector space V over the fraction field K of W is called an **F-isocrystal** if there is an injective $\sigma-$linear map $\phi : V \to V$.*

Following properties are true for crystalline cohomology:-

• The groups $H^i_{crys}(X/K)$ are F-isocrystals and contravariant in X. $H^i_{crys}(X/K) = 0$ for $i \notin [0, 2dim(X)]$
•(**Poincaré Duality**) There is an isomorphism of F-isocrystals $H^{2d}(X/K)(d) \simeq K$ and there is a cup product structure $\cup_{i,j} : H^i_{crys}(X/W) \times H^j_{crys}(X/W) \to H^{i+j}(X/K)$, which induces a perfect pairing $H^i_{crys}(X/W) \times H^{2d-i}_{crys}(X/W) \to H^{2d}(X/K) \simeq K(-d)$.
• (**Lefschetz fixed point formula**) Let $f : X \to X$ be a morphism of k varieties s.t. for each fixed point x is nondegenerate i.e. $1 - df_x$ is invertible on Zariski tangent space (for example, f could be the relative Frobenius). Then

$$\#\{\text{fixed points of } f\} = \sum_i (-1)^i tr(f_{|H^i_{crys}(X/W)})$$

§Imitating Mukai's argument

As above let W be the ring of Witt vectors of k and K be its field of fractions. Henceforth we will be considering only crystalline cohomology groups and denote for simplicity, $H^i_{crys}(X/K)$ by $H^i(X/K)$. Analogous to the Tate twist in the context of Hodge structure there is a Tate twist for F-isocrystals as well. Henceforth we will always mean this when mentioning a Tate twist.

Definition 5.2.2. *(**Tate twist of F-isocrytals**) $K(1)$ denotes the $F-$ isocrystal whose underlying vector space is K and it has a Frobenius action $K \to K$ given by multiplication by $1/p$. For any $F-$isocrystal M, $M(n) := M \otimes K(1)^{\otimes n}$.*

Remark 5.2.3. *Note that $W(k)$ is an integral domain in characteristic 0 and hence K contains a copy of $\mathbb{Q}$ and so multiplication by $1/p$ is well defined.*

Denote by $\phi_X : H^i(X/K) \to H^i(X/K)$ the Frobenius action on $H^i(X/K)$ induced by the relative Frobenius $F : X \to X$, which induces a F-isocrystal structure on $H^i(X/K)$.

There is a cycle class map $\eta : A^*(X) \to H^{2*}_{crys}(X/K)$ [see 7] which upon composing with the Chern character $ch : K(X) \to A^*(X)$ is called the *crystalline Chern character*, denoted $ch_{crys} : K(X) \to H^{2*}_{crys}(X/K)$. We will denote by ch^i_{crys} the $2i-$th component of ch_{crys}. We have the following important property of the crystalline Chern character.

Proposition 5.2.4. *Let X be a smooth proper variety over k. For an integer i and $E \in K(X)$ we have*

$$\phi_X(ch^i_{crys}(E)) = p^i ch^i_{crys}(E)$$

Proof: We may assume that E is a vector bundle and let $p : X' \to X$ be the full flag scheme of E. Note that the relative Frobenius of X and X' commutes with p, hence by functoriality of $H^*_{crys}(-/K)$ it is enough to show the proposition for the vector bundle p^*E. By definition of flag scheme p^*E has a filtration $0 \subset F_1 \subset ... \subset F_{n-1} \subset F_n = p^*E$ s.t. $L_j := F_j/F_{j-1}$ is a line bundle. So we may assume E has such a filtration. Hence by additivity of ch_{crys}, if $\alpha_j = c_1(L_j)$ then $ch^i_{crys}(E) = \sum_{\sum i_k = i} c_{\underline{i}} \alpha_1^{i_1}...\alpha_n^{i_n}$. Since pull-back of morphisms commutes with intersection product it is enough to show the proposition for each $\alpha_1^{i_1}...\alpha_n^{i_n}$. Now for any line bundle L on X, we have $F^*L \simeq L^{\otimes p}$. Hence $F^*(\alpha_1^{i_1}...\alpha_n^{i_n}) = p^i\alpha_1^{i_1}...\alpha_n^{i_n}$.

□

Let $f : X \to Y$ be a map of proper smooth varieties X, Y of dimension d_X, d_Y respectively. The pull back map $f^* : H^i(Y/K) \to H^i(X/K)$ is the adjoint to the map $H^{2d_X - i}(X/K)(d_X) \to H^{2d_Y - i}(Y/K)(d_Y)$ (this is obtained by Poincaré duality theorem, which gives an isomorphism $H^i(X/K) \simeq (H^{2d-i}(X/K)(d))^\vee$). Hence we define the push forward $f_* : H^s(X/K) \to H^{s+2d_Y - i}(Y/K)(d_Y - d_X)$. We have the following *projection formula*

$$f_*(\alpha \cup f^*\beta) = f_*\alpha \cup \beta$$

Suppose X is proper and smooth of dimension $d = 2\delta$. We define

$$\widetilde{H}^i(X/K) := H^{2\delta+2i}(X/K)(-i)$$

and define the ***Mukai isocrystal*** as

$$\widetilde{H}(X/K) := \oplus_{i=-\delta}^{\delta} \widetilde{H}^i(X/K)$$

Similar to the case of Muaki Hodge structure, we can define the bilinear form

$$\langle .,. \rangle : \widetilde{H}(X/K) \times \widetilde{H}(X/K) \to K(-d)$$

given by $\langle (a_{-\delta}, \dots, a_\delta), (a'_{-\delta}, \dots, a'_\delta) \rangle := \sum_{i=-\delta}^{\delta} (-1)^i a_i \cup a'_{-i}$. Here $a_i, a'_i \in H^{2\delta+2i}(X/K)(-i)$ and we identify $H^{4\delta}(X/K)$ with $K(-d)$. The pairing is clearly compatible with the F-isocrystal structure.
Let $P \in K(X \times Y)$. We define here a map $\psi_P : \widetilde{H}(X/K) \to \widetilde{H}(Y/K)$ with $dim(X) = dim(Y) = d$. It is described as adding the maps

$$\psi_P^{i,j} : H^{2\delta+2i}(X/K)(i) \to H^{2\delta+2j}(Y/K)(j)$$

where each $\psi_P^{i,j}$ is described as the composite $H^{2\delta+2i}(X/K)(i) \xrightarrow{p_1^*} H^{2\delta+2i}(X \times Y/K)(i) \xrightarrow{\cup ch^{j-i+d}(P)_{crys}} H^{2\delta+2j+2d}(X \times Y/K)(j+d) \xrightarrow{p_{2*}} H^{2\delta+2j}(Y/K)(j)$. Since both p_1^*, p_{2*} are compatible with the F-isocrystal on the respective cohomology rings and we have $(1/p)^{-j-d}(\cup ch_{crys}^{j-i+d}(P)) \circ \phi_{X\times Y} = (1/p)^{-i}\phi_{X \times Y} \circ (\cup ch_{crys}^{j-i+d}(P))$, and so ψ_P is compatible with the F-isocrystal structure of $\widetilde{H}(X/K), \widetilde{H}(Y/K)$.

To descend any Fourier Mukai transform $\mathbf{\Phi}_P : D^b(X) \to D^b(Y)$ to the Mukai isocrystal $\mathbf{\Phi}_P^H : \widetilde{H}(X/K) \to \widetilde{H}(Y/K)$ we just need to replace ch^{j-i+d} by v^{j-i+d}, where $v = ch_{crys}(P)\sqrt{(}td(X \times Y))$. Since the map $ch : K(X)_{\mathbb{Q}} \to A^*(X \times Y)_{\mathbb{Q}}$ is an isomorphism, we have that $\mathbf{\Phi}_P^H$ is also compatible with the F-isocrystal structure.

Chapter 6

$K3$ surfaces over $\mathbb{F}_q$

In this chapter we prove that two derived equivalent $K3$ surfaces over a finite field have the same number of rational points. The proof is straightforward application of Lefschetz fixed point theorem to the theory of descending Fourier Mukai transform to Mukai isocrystals discussed in the last chapter (in the case of $K3$ surfaces).

Theorem 6.0.1. *Let X, Y be $K3$ surfaces over a finite field $\mathbb{F}_q$ (of characteristic p) and let $\mathbf{\Phi} : D^b(X) \to D^b(Y)$ be an $\mathbb{F}_q-$exact equivalence of derived categories. Then $\#X(\mathbb{F}_{q^n}) = \#Y(\mathbb{F}_{q^n})$ for all $n \geq 1$. In particular $\zeta_X = \zeta_Y$.*

Proof: First make a base change of X, Y and $\mathbf{\Phi}$ to $\overline{\mathbb{F}_q}$ and denote them by $\bar{X}$,$\bar{Y}$ and $\bar{\mathbf{\Phi}}$. By Orlov's result [10, Thm. 5.14] $\bar{\mathbf{\Phi}}$ is a Fourier Mukai transform i.e. there is a perfect complex $P \in D^b(\bar{X} \times \bar{Y})$ s.t. $\bar{\mathbf{\Phi}} = \mathbf{\Phi}_P$. Denote by F the Frobenius action on the Mukai isocrystal $\widetilde{H}(\bar{X}/K)$ i.e. the Frobenius action on X together with Frobenius on the Tate twist. As we discussed in previous chapter, one can descend the equivalence $\mathbf{\Phi}_P$, to a map of F-isocrystals $\mathbf{\Phi}_P^H : \widetilde{H}(\bar{X}/K) \to \widetilde{H}(\bar{Y}/K)$ and hence we have $Tr(F_{|\widetilde{H}(\bar{X}/K)}) = Tr(F_{|\widetilde{H}(\bar{Y}/K)})$. For $K3$ surfaces S, we have $H^i(S/K) = 0$ for $i = 1, 3$. Also $H^4(S/K) = K(-2)$ and $H^0(S/K) = H^4(S/K)(2)^\vee = K$. Hence for both $\bar{X}, \bar{Y}$ the relative Frobenius ϕ_X, ϕ_Y acts identically on $H^j(X/K), H^j(Y/K)$ respectively for $j = 0, 1, 3, 4$. By virtue of Lefschetz fixed point formula it suffices to show that the trace of Frobenius is same on $H^2(\bar{X}/K)$ and $H^2(\bar{Y}/K)$.

We have
$Tr(F|\widetilde{H}(\bar{X}/K)) = Tr(F|H^0(\bar{X}/K)(-1)) + Tr(F|H^2(\bar{X}/K)) + Tr(F|H^4(\bar{X}/K)(1)) = Tr(\phi_X|H^2(\bar{X}/K)$
$Tr(F|K(-1)) + Tr(F|K(-1)) = Tr(H^2(\phi_X|\bar{X}/K)) + 2p$.
Here we used Poincaré duality to get an isomorphism of $F-$isocrystals $H^4(\bar{X}/K)(1) = K(-2)(1) = K(-1)$.
Similarly $Tr(F|\widetilde{H}(\bar{Y}/K)) = Tr(\phi_Y|H^2(\bar{Y}/K)) + 2p$.
Hence $Tr(\phi_X|H^2(\bar{X}/K)) = Tr(\phi_Y|H^2(\bar{Y}/K))$.
By Lefschetz fixed point formula, we have $\#\{\text{fixed points of } \phi_X\} = \sum_{i=1}^4 (-1)^i Tr(\phi_X|H^i(X/K)) = \sum_{i=1}^4 (-1)^i Tr(\phi_Y|H^i(Y/K)) = \#\{\text{fixed points of } \phi_Y\}$. But fixed points of ϕ_X (resp. ϕ_Y) are $\mathbb{F}_q-$rational points on the respective varieties. Hence $\#X(\mathbb{F}_q) = \#Y(\mathbb{F}_q)$.

Instead of ϕ_X (resp. ϕ_Y) consider the composition of ϕ_X (resp. ϕ_Y) with itself $n-$times and the same argument as above yields $\#\{\text{fixed points of } \phi_X^n\} = \#\{\text{fixed points of } \phi_Y^n\}$. But $\#\{\text{fixed points of } \phi_X^n\} = \#X(\mathbb{F}_{q^n})$ and $\#\{\text{fixed points of } \phi_Y^n\} = \#Y(\mathbb{F}_{q^n})$. Hence $\#X(\mathbb{F}_{q^n}) = \#Y(\mathbb{F}_{q^n})$

$\square$

Appendices

Appendix A

Brauer Manin obstruction

For a global field k, a functor $F : Sch_k^{opp} \to (Sets)$ induces an obstruction to existence of rational points. A **Brauer Manin obstruction to local global principle** is the special case of $F = Br$ a functor taking schemes to their Brauer groups. The inv_v for any valuation v on k makes it easier to handle such obstructions. We will follow [28, Chap. 8] to define these concepts.

A.1 F-obstruction to local-global principle

Let $F : Sch_k^{opp} \to (Sets)$ be a functor. X/k be a scheme and $A \in F(X)$. Then there is a natural *evaluation* map $ev_A : X(L) \to F(L) := F(Spec(L))$ defined as $[x : Spec(L) \to X] \mapsto im(A) \in F(L)$ under the map $F(X) \to F(L)$.

Let $\mathbf{A}$ be the adèle ring of k. Then we have a natural commutative diagram
$$\begin{tikzcd} X(k) \arrow[r, hook] \arrow[d, "ev_A"] & X(\mathbf{A}) \arrow[d, "ev_A"] \\ F(k) \arrow[r] & F(\mathbf{A}) \end{tikzcd}$$

Definition A.1.1. *For $A \in F(X)$ define $X(\boldsymbol{A})^A := \{y \in X(\boldsymbol{A}) | ev_A(y) \in im(F(K) \to F(L))\}$. Note that, $X(k) \subset X(\boldsymbol{A})^A$.*
*If $X(\boldsymbol{A}) \neq \emptyset$ but $X(\boldsymbol{A})^F := \cap_{A \in F(X)} X(\boldsymbol{A})^A = \emptyset$, we say there is an **F-obstruction to local-global principle** (in particular we will have $X(k) = \emptyset$).*

A.2 Brauer Manin Obstruction

We will assume that X/k is a proper scheme. Then we have an equality $X(\mathbf{A}) = \prod_v X(k_v)$ v running over valuations of k [see 28, Ex. 3.4]. For $A \in Br(X)$ we will denote $ev_A(x)$ by $A(x)$ for $x \in X(L)$ and L a $k-$algebra. We have the following proposition:-

Proposition A.2.1. *If $(x_v) \in X(\boldsymbol{A})$, then $A(x_v) = 0$ for almost all valuations v.*

Proof: See [28, Prop. 8.2.2].

□

The above proposition helps us define a map as follows

$$X(\mathbf{A}) \to \mathbb{Q}/\mathbb{Z}$$

given by

$$(x_v) \mapsto (A, (x_v)) := \sum_v inv_v(A(x_v))$$

Definition A.2.2. *For $A \in Br(X)$ define*

$$X(\boldsymbol{A})^A := \{(x_v) \in X(\boldsymbol{A}) := \prod X(k_v) | (A, (x_v)) = \sum inv_v(A(x_v)) = 0\}$$

Also define $X(\boldsymbol{A})^{Br} := \cap_{A \in Br(X)} X(\boldsymbol{A})^A$.

Note that $x \in X(k) \subset X(\mathbf{A})$, then $(A, x) = A(x) = 0$. Indeed one has the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & X(k) \arrow[r] \arrow[d, "ev_A"] & X(\mathbf{A}) \arrow[d, "ev_A"] & & \\
0 \arrow[r] & Br(k) \arrow[r] & \oplus_v Br(k_v) \arrow[r, "\sum inv_v"] & \mathbb{Q}/\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

Both rows of the commutative diagram are exact (by 3.1.12). Hence $X(k) \subset X(\mathbf{A})^{Br}$.

Definition A.2.3. *One says that there is a* ***Brauer Manin obstruction to local global principle*** *for X if $X(\boldsymbol{A}) \neq \emptyset$ but $X(\boldsymbol{A})^{Br} = \emptyset$.*

Appendix B

Proof of Theorem 4.1.3

This appendix is meant to complete the proof of Theorem 4.1.3, in particular **(*)** stated in step 3 of the proof.

Recall that the situation in step 3 of the proof of Theorem 4.1.3 was that the base field was $\overline{\mathbb{Q}}$ and the family of integral curves with planar singularities, $\mathcal{C}_{|U} \to U$, had a section which induces a natural identification between $X_{|U}$ with $Y_{|U}$. Our aim is to show that the family of line bundles $\mathcal{O}(D)$ coincides with the Poincaré line bundle considered by Arinkin in [4]. Arinkin constructed his Poincaré line bundle on $X^o_{|U} \times_U X_{|U} \cup X_{|U} \times_U X^o_{|U}$ as a line bundle P (unique upto line bundles from the base) satisfying Deligne's description of a Poincaré line bundle

$$\mathcal{P}_{(L,M)} = det(R\Gamma(L \otimes M))^{-1} \otimes det(R\Gamma(L)) \otimes det(R\Gamma(M)) \otimes det(R\Gamma(C, \mathcal{O}_C))^{-1}$$

for $(L, M) \in X^o_{|U} \times_U X_{|U} \cup X_{|U} \times_U X^o_{|U}$. Hence it suffices to show that the line bundle $\mathcal{O}(D)$ satisfies Deligne's description on $X^o_{|U} \times_U Y_{|U} \cup X_{|U} \times_U Y^o_{|U}$.

We need to recall the theory of Cohen-Macaulay sheaves.

Definition B.0.1. *Let X be a scheme of pure dimension. A coherent sheaf $\mathcal{F}$ on X is said to be* ***Cohen-Macaulay*** *if each stalk $\mathcal{F}_x$ as an $\mathcal{O}_{X,x}$ is a Cohen-Macaulay module. It is said to be* ***maximal Cohen-Macaulay*** *sheaf if it is Cohen-Macaulay and pure.*

We have the following result of maximal Cohen-Macaulay sheaves which says that they are uniquely determined by their restriction to the complement of a closed subscheme codimension atleast two.

Lemma B.0.2. *Let X be a scheme of pure dimension and M a maximal Cohen-Macaulay sheaf on X. Then for a closed subscheme $Z \subset X$ with $codim(Z, X) \geq 2$ we have an isomorphism $M \xrightarrow{\simeq} i_* M_{|X-Z}$, where $i : X - Z \hookrightarrow X$ is the open immersion.*

Proof: For a proof see [8, Thm. 5.10.5].

∎

Hence it is enough to show that $\mathcal{O}(D)$ satisfies Deligne's description on $X^o_{|U} \times_U Y^o_{|U} \subset X^o_{|U} \times_U Y_{|U} \cup X_{|U} \times_U Y^o_{|U}$. Indeed, if P is Arinkin's Poincaré line bundle on $X^o_{|U} \times_U Y_{|U} \cup X_{|U} \times_U Y^o_{|U}$ (after identification of $X_{|U}$ with $Y_{|U}$) and $P_{|X^o_{|U} \times_U Y^o_{|U}}$ coincides with $\mathcal{O}(D)_{|X^o_{|U} \times_U Y^o_{|U}}$ (upto line bundles from base schemes), then by the lemma above(since P and $\mathcal{O}(D)$ being line bundles on a Cohen-Macaulay scheme are Cohen-Macaulay sheaves), P and $\mathcal{O}(D)$ are isomorphic to each other, upto line bundles from base (here $X^o_{|U} \times_U Y^o_{|U} \subset X^o_{|U} \times_U Y_{|U} \cup X_{|U} \times_U Y^o_{|U}$ is the complement of a codimension at least two closed subscheme).

Hence it suffices to show that $\mathcal{O}(D)_{|X^o_{|U} \times_U Y^o_{|U}}$ satisfies Deligne's description

Let $\mathcal{L}$ be the universal sheaf on $\mathcal{C}_U \times_U X^o_U$ and $\mathcal{M}$ be the universal sheaf on $\mathcal{C}_U \times_U Y^o_U$ (universal sheaves exists since our family of curves $\mathcal{C}_U \to U$ has a section). Let p_{ij} be the $ij-$th projection from $\mathcal{C}_U \times_U X^o_U \times_U Y^o_U$.
Now consider the perfect complex $\mathcal{F} := Rp_{23,*}(p^*_{12}\mathcal{L} \otimes p^*_{13}\mathcal{M})$. For any geometric point $(L, M) \hookrightarrow X^o_{|U} \times_U Y^o_{|U}$ we have

$$\mathcal{F} \otimes^{\mathbf{L}} k(L, M) = R\Gamma(L \otimes M)$$

(by base change theorem [see 17, Thm. 3.10.3]). From this equality we make two observations:-

(1) supp$(\mathcal{F}) \subset \{(L, M)|H^0(L \otimes M) = H^1(L \otimes M) \neq 0\}$ ($H^0(L \otimes M) = H^1(L \otimes M)$ by Riemann-Roch theorem). This is because $R\Gamma(L \otimes M)$ is the complex $H^*(L \otimes M)$ and so Nakayama shows that supp$(\mathcal{F})$ is contained in $\{(L, M)|H^0(L \otimes M) = H^1(L \otimes M) \neq 0\}$.

(2) Also $supp(\mathcal{F}) = supp(R^0p_{23,*}(p^*_{12}\mathcal{L} \otimes p^*_{13}\mathcal{M})) \cup supp(R^1p_{23,*}(p^*_{12}\mathcal{L} \otimes p^*_{13}\mathcal{M}))$. Hence, while $R^0p_{23,*}(p^*_{12}\mathcal{L} \otimes p^*_{13}\mathcal{M})$ does not have full support $X^o_{|U} \times_U Y^o_{|U}$, it is also a torsion free sheaf, being the push forward of a torsion free sheaf. Hence $R^0p_{23,*}(p^*_{12}\mathcal{L} \otimes p^*_{13}\mathcal{M}) = 0$.

Hence we have $Rp_{23,*}(p^*_{12}\mathcal{L} \otimes p^*_{13}\mathcal{M}) \simeq R^1p_{23,*}(p^*_{12}\mathcal{L} \otimes p^*_{13}\mathcal{M})$.
Hence by above discussion we have that the Poincaré line bundle on $X^o_{|U} \times_U Y^o_{|U}$ satisfying Deligne's description is the same $det(R^1p_{23,*}(p^*_{12}\mathcal{L} \otimes p^*_{13}\mathcal{M}))$ upto line bundles from the base schemes (i.e. the sheaves corresponding to fibers $R\Gamma(L), R\Gamma(M)$ and $R\Gamma(\mathcal{O}_C)$ are pull-back of line bundles from the $X_{|U}, Y_{|U}$ and $\mathcal{C}$).
It is enough to show that $det(R^1p_{23,*}(p^*_{12}\mathcal{L} \otimes p^*_{13}\mathcal{M})) = \mathcal{O}(D)$.

To see this we make the following observations:-

(1) $R^1p_{23,*}(p^*_{12}\mathcal{L} \otimes p^*_{13}\mathcal{M})$ has generic rank 1 on its support.

It is enough to show that for all smooth curve $C \in U$ (they form a dense set as geometric points) and a line bundle L of degree 1 on C s.t. $h^0(C, L) = h^1(C, L) = 1$ (since curves in the linear system $|h|$ have genus 2, Riemann-Roch theorem tells us that $h^0(C, L) = h^1(C, L)$ for degree 1 line bundle).

Indeed this is sufficient to show that $R^1p_{23,*}(p^*_{12}\mathcal{L} \otimes p^*_{13}\mathcal{M})$ has generic rank 1 on its support, for by semi-continuity theorem [see 23, pg.50, Cor.(a)] the set $\{z|dim_{k(z)}H^1((p^*_{12}\mathcal{L} \otimes p^*_{13}\mathcal{M})_z) < 2\} \subset X^o_{|U} \times_U Y^o_{|U}$ will be a non-empty open set containing a dense set of closed points of $supp(R^1p_{23,*}(p^*_{12}\mathcal{L} \otimes p^*_{13}\mathcal{M}))$ and in particular it will contain all the generic point of $supp(R^1p_{23,*}(p^*_{12}\mathcal{L} \otimes p^*_{13}\mathcal{M}))$ (if an open set contains a dense set of a closed set, then it must contain all the generic points of the closed set). By [see 23, pg.51, Cor. 2] for $y \in X^o_{|U} \times_U Y^o_{|U}$, a generic point of $supp(R^1p_{23,*}(p^*_{12}\mathcal{L} \otimes p^*_{13}\mathcal{M}))$, we have

$$R^1p_{23,*}(p^*_{12}\mathcal{L} \otimes p^*_{13}\mathcal{M}) \otimes k(y) \simeq H^1((X^o_{|U} \times_U Y^o_{|U})_y, (p^*_{12}\mathcal{L} \otimes p^*_{13}\mathcal{M})_y)$$

and also by above $y \in supp(R^1 p_{23,*}(p_{12}^*\mathcal{L} \otimes p_{13}^*\mathcal{M})) \cap \{z | dim_{k(z)} H^1((p_{12}^*\mathcal{L} \otimes p_{13}^*\mathcal{M})_z) < 2\}$. Hence $dim_{k(y)} H^1((X^o_{|U} \times_U Y^o_{|U})_y, (p_{12}^*\mathcal{L} \otimes p_{13}^*\mathcal{M})_y) = 1$ or $R^1 p_{23,*}(p_{12}^*\mathcal{L} \otimes p_{13}^*\mathcal{M}) \otimes k(y)$ has rank 1 as a $k(y)-$vector space or by Nakayama Lemma $R^1 p_{23,*}(p_{12}^*\mathcal{L} \otimes p_{13}^*\mathcal{M})$ will have generic rank 1 on its support.

To get hold of a degree 1 line bundle on a smooth curve C with $h^0(L) = h^1(L) = 1$, just choose two points $x_0, x_1 \in C$ s.t. $\mathcal{O}(x_0) \nsim \mathcal{O}(x_1)$ (in $Pic(C)$). Indeed one can choose two such points, for otherwise C would be a rational curve. Consider the long exact sequence in cohomology of the short exact sequence $0 \to L(-x_0) \to L \to k(x_0) \to 0$ ($L = \mathcal{O}(x_1)$) which yields $0 \to H^0(L(-x_0)) \to H^0(L) \to H^0(k(x_0))(\simeq k) \to H^1(L(-x_0)) \to H^1(L) \to H^1(k(x_0)) = 0$.

Now $H^0(L(-x_0)) = 0$ for otherwise $L(-x_0)$ is trivial. Hence by Riemann Roch $h^1(L(-x_0)) = 1$ and the only possibility of $h^0(L) = h^1(L)$ is that they be one-dimensional.

(2) $det(R^1 p_{23,*}(p_{12}^*\mathcal{L} \otimes p_{13}^*\mathcal{M})) \simeq \mathcal{O}(D)$

By Lemma 4.1.2, we know that D is the support of the sheaf $R^1 p_{23,*}(p_{12}^*\mathcal{L} \otimes p_{13}^*\mathcal{M})$ and, by (1) above, it has generic rank 1 on D. Hence $R^1 p_{23,*}(p_{12}^*\mathcal{L} \otimes p_{13}^*\mathcal{M})$ coincides with $\mathcal{O}_D$ over an open set V s.t. V^c (the complement of V in $X^o_{|U} \times_U Y^o_{|U}$) has codimension at least 2. Since sheaves $\mathcal{G}$ supported in closed subscheme of codimension at least 2 have trivial determinant, we have $det(R^1 p_{23,*}(p_{12}^*\mathcal{L} \otimes p_{13}^*\mathcal{M})) \simeq det(\mathcal{O}_D)$. But one has the short exact sequence $0 \to \mathcal{O}(-D) \to \mathcal{O} \to \mathcal{O}_D \to 0$. Hence $det(\mathcal{O}_D) = \mathcal{O}(D)$ and hence $det(R^1 p_{23,*}(p_{12}^*\mathcal{L} \otimes p_{13}^*\mathcal{M})) = \mathcal{O}(D)$.

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