PROOF OF RIGID ANALOGUE OF GAGA

§ Introduction

Given here is sketch of a proof of rigid analytic version of Serre's Celebrated GAGA theorem, following essentially the structure of proof given by Serre in his original paper.

The theorem states that,

(*) If X is a projective variety, we have X^{an} the corresponding rigid projective variety, then

(a) $\left\{ \begin{matrix} \text{~~Cate~~ Coherent Sheaves} \\ \text{on } X \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{Coherent Sheaves} \\ \text{on } \cancel{\mathcal{F}^{an}} \; X^{an} \end{matrix} \right\}$

$$\mathcal{F} \longmapsto \mathcal{F}^{an}$$

The above functor is an equivalence of categories.

(b) Cohomologies are the "same", i.e.

$$H^q(X, \mathcal{F}) \xrightarrow{\sim} H^q(X^{an}, \mathcal{F}^{an}),$$ $\mathcal{F}$ being a coherent sheaf on X $(q \geq 0)$.

To make sense of the statement, we briefly describe rigid analytification of a scheme (details can be found in Bosch's book [1]). We will apply it to obtain the rigid projective space, $\mathbb{P}^{n,an}_K$ and quickly observe that it can be covered by 'n+1' unit balls, as an admissible affinoid covering. This observation is very crucial for what follows, since we can use the Čech-cohomology (of the covering by 'n+1' unit balls) to compute the K-vector space $H^q(X^{an}, \mathcal{F}^{an})$. Kiehl's theorem's (proper mapping theorem etc.) will be very fundamental to us, for it helps us follow Serre's proof very smoothly.

In the proof of (*) (b), the general idea is to first show, $H^q(X, \mathcal{O}_X(m)) \xrightarrow{\sim} H^q(X^{an}, \mathcal{O}_{X^{an}}(m))$. ~~and~~ For the proof of this fact, we closely follow Neeman [5].

§ Notation and Terminology

K, will denote a complete, non-archimedean, non-trivial, normed field.

We will only consider schemes over k, separated of finite type.

Rigid K-spaces for us will be locally ringed spaces, endowed with strong Grothendieck topology, s.t. it has an admissible open affinoid covering.

For a scheme $(X, \mathcal{O}_X)$ over K, $(X^{an}, \mathcal{O}_{X^{an}})$ will denote the rigid analytification.

$\mathbb{P}^n_K = \operatorname{Proj} K[X_0, \dots, X_n]$, the ~~po~~ projective K-space.

$K[\underline{\xi}]$, will denote the polynomial ring $K[\xi_1, \dots, \xi_n]$

$K\langle \underline{\xi} \rangle$, will denote the Tate Algebra, $K\langle \xi_1, \dots, \xi_n \rangle$

$\{X_i\}_{i=0}^n$ will denote the cover by unit balls, i.e.

$X_i = \operatorname{Sp} K\left\langle \frac{\xi_0}{\xi_i}, \dots, \widehat{\frac{\xi_i}{\xi_i}}, \dots, \frac{\xi_n}{\xi_i} \right\rangle$

$\{\mathcal{U}_j\}_{j=0}^n$ will denote the covering of $\mathbb{P}^n_K$ by affine planes

i.e. $\mathcal{U}_j = \operatorname{Spec} K\left[\frac{\xi_0}{\xi_i}, \dots, \frac{\xi_n}{\xi_i}\right]$

Symbols $\mathcal{F}$, $\mathcal{G}$, $\mathcal{M}$ etc. will be reserved for coherent sheaves (unless ~~or~~ otherwise stated).

§ Rigid Analytification

Since, K is non-trivial, choose $c \in K$ s.t. $|c| > 1$.

Set $T_n^{(i)} = K\langle c^{-i}\zeta_1, \dots, c^{-i}\zeta_n \rangle$ and we have sequence of inclusions, $K[\zeta_1, \dots, \zeta_n] \hookrightarrow \cdots \hookrightarrow T_n^{(1)} \hookrightarrow T_n^{(0)}$

Lemma (Bosch [1], Pg 109-110) The inclusions $T_n^{(0)} \hookleftarrow T_n^{(1)} \hookleftarrow \dots \hookleftarrow K[\zeta]$ induce inclusions of spectra of max'l ideals,

$$\operatorname{Max} K[\underline{\zeta}] \hookleftarrow \cdots \hookleftarrow \operatorname{Sp} T_n^{(1)} \hookleftarrow \operatorname{Sp} T_n^{(0)}, \quad \text{s.t.}$$

$$\bigcup_{i \geq 0} \operatorname{Sp} T_n^{(i)} = \operatorname{Max} K[\underline{\zeta}].$$

We want the following notion of rigid analytification:-

For any scheme over k (separated, finite type), $(X, \mathcal{O}_X)$ a rigid K-space $(X^{an}, \mathcal{O}_{X^{an}})$ is called it's analytification if there is a natural map, $(X^{an}, \mathcal{O}_{X^{an}}) \xrightarrow{i} (X, \mathcal{O}_X)$ of locally ringed spaces, s.t. it is universal w.r.t this property, i.e. for any morphism of locally ringed spaces, $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$, where Y is a rigid K-space, there is a unique map $Y \to X^{an}$ as rigid K-space s.t.

$$\begin{tikzcd}
(Y, \mathcal{O}_Y) \arrow[rr] \arrow[dr, dashed] & \circlearrowleft & (X, \mathcal{O}_X) \\
& (X^{an}, \mathcal{O}_{X^{an}}) \arrow[ur] &
\end{tikzcd}$$

Note that if a rigid analytification exists, it is unique upto unique isomorphism.

Since, each $\mathrm{Sp}\, T_n^{(i)}$ has a strong G-topology, by gluing lemma for Strong Grothendieck topologies (Bosch[1.], Pg 98-99) ~~and~~ and lemma above, gets us

$$\mathbb{A}_K^{n,an} := \bigcup_{i \geq 0} \mathrm{Sp}\, T_n^{(i)}, \text{ where the identification}$$

$$\mathrm{Sp}\, T_n^{(i)} \hookrightarrow \mathrm{Sp}\, T_n^{(j)} \quad , \ i<j \ , \text{ is by the Lemma above.}$$

Observe that points of $\mathbb{A}_K^{n,an}$ is the set of closed points of $K[\underline{\xi}]$ (by Lemma).

For any affine scheme, $X = \mathrm{Spec}\, \frac{K[\underline{\xi}]}{\mathfrak{a}}$, we have

$$X^{an} := \bigcup_{i \geq 0} \mathrm{Sp}\left(\frac{T_n^{(i)}}{\mathfrak{a}^e}\right), \text{ and pt's of } X^{an} = \mathrm{Max}\left(\frac{K[\underline{\xi}]}{\mathfrak{a}}\right)$$

It can also be checked that $X^{an} \xhookrightarrow{i} X$ is the rigid analytification, in the sense of the universal property.

(Note that X^{an} adopts a strong G-topology) hence becomes a Rigid K-Space

~~For aw any~~

For any scheme over K, ∃ we have the following (Bosch[1], Pg 113) :-

<u>Proposition</u> :- Every K-scheme Z, admits an analytification, $Z^{an} \longrightarrow Z$ (in the sense of universal property). Furthermore, the underlying map of sets of Z^{an}, identifies with closed points of Z.

§ <u>Rigid Projective Space</u>, $\underline{\mathbb{P}_K^{n,an}}$

$\mathbb{P}_K^{n,an}$ is the rigid analytification of the projective K-space. We recall how the projective K-space $\mathbb{P}_K^n$ is obtained. Set $U_i = \operatorname{Spec} K[\zeta_{0i}, \ldots \zeta_{ni}]$, $(\zeta_{ii} = 1)$

$U_{ij} = \operatorname{Spec} K[\zeta_{0i}, \ldots, \zeta_{ni}, \zeta_{ij}^{-1}]$. Glue the U_i's, with identifications, $\varphi_{ij} : U_{ij} \longrightarrow U_{ji}$, given by

$\varphi_{ij}(\zeta_{kj}) = \zeta_{ki}(\zeta_{ji})^{-1}$.

So that rigid analytification $\mathbb{P}_K^{n,an}$ admits an admissible affinoid covering, $U_i^{an} = \bigcup_{j \geq 0} \operatorname{Sp} K\langle c^{-j} \underline{\zeta}_{\bullet i}\rangle$

s.t. $\operatorname{Sp} K\langle \underline{\zeta}_{\bullet i}\rangle \cap \operatorname{Sp} K\langle \underline{\zeta}_{\bullet j}\rangle = (\operatorname{Sp} K\langle \underline{\zeta}_{\bullet i}\rangle)(\zeta_{ji}^{-1})$

where $(\operatorname{Sp} K\langle \zeta_{\bullet i}\rangle)(\zeta_{ji}^{-1})$ is an ~~Laurent Domain~~ Affinoid subdomain.

Claim :- $\mathbb{P}_K^{n,an}$ has an admissible affinoid covering by the "unit balls", $\{Sp\, K\langle \underline{\xi}_{\cdot i}\rangle\}_{i=0}^{n}$.

Proof :- Since, set of points of $\mathbb{P}_K^{n,an}$ are the closed points of ~~K[ξ~~ $\mathbb{P}_K^n$, let $x \in \mathbb{P}_K^n$ be a closed point. 'm_x' be the corresponding homogeneous max'l ideal in $\mathrm{Proj}(K[\xi_0, \ldots, \xi_n])$.

$K \hookrightarrow \frac{K[\bar{\xi}_0, \ldots, \bar{\xi}_n]}{m} = L$, is a finite field extension (by virtue of Nullstellensatz).

Since, K is complete, the norm $|\cdot|$ on K extends uniquely to L. By abuse of notation, we call that too $|\cdot|$ (the norm on L). Choose 'i' s.t.

~~$|\bar{x}_i| = \max\{|\bar{x}_0$~~

$|\bar{\xi}_i| = \max\{|\bar{\xi}_0|, \ldots, |\bar{\xi}_n|\}$. By universal property of Tate Algebra's, there is an unique map, $K\langle \xi_{0i}, \ldots, \xi_{ni}\rangle \to L$, s.t.

$$\boxed{\xi_{ki} \mapsto \frac{\bar{\xi}_k}{\bar{\xi}_i}}, \text{ for } \left|\frac{\bar{\xi}_k}{\bar{\xi}_i}\right| \le 1$$

∴ This induces maps,

$$\begin{tikzcd} \{x\} \arrow[rr] \arrow[dr, hook] & & U_i^{an} \arrow[r, hook] & \mathbb{P}_K^{n,an} \\ & \mathrm{Sp}\, K\langle \underline{\xi}_{*i} \rangle \arrow[ur] & & \end{tikzcd}$$

$\Rightarrow x \in \mathrm{Sp}\, K\langle \underline{\xi}_{*i} \rangle$.

Hence, $\mathbb{P}_K^{n,an} = \bigcup_{i=0}^{n} \mathrm{Sp}\, K\langle \underline{\xi}_{*i} \rangle$.

Also, this is an admissible cover for $\mathbb{P}_K^{n,an}$, by virtue of the Strong G-topology endowed on $\mathbb{P}_K^{n,an}$. For let, $X_i = \mathrm{Sp}\, K\langle \xi_{*i} \rangle$. Then by completeness properties of the Strong Grothendieck topology (Bosch[1], Pg 98, Prop. 10), $(X_i)_{i=0}^{n}$ is an admissible cover iff $(X_i \cap U_j^{an})_{i=0}^{n}$ is an admissible cover for each U_j^{an}. But again due to the strong G-topology on U_j^{an}, $(X_i \cap U_j^{an})_{i=0}^{n}$ is admissible iff $\{X_i \cap \mathrm{Sp}\, T_n^{(\ell)}\}_{i=0}^{n}$ is admissible for all $\ell \geq 0$

But each $X_i \cap \mathrm{Sp}\, T_n^{(\ell)}$ is an affinoid subdomain, hence. $\square$

We will henceforth, call the above covering by unit balls as $\{X_i\}_{i=0}^{n}$ or simply $\{X_i\}$

§ Coherent Sheaves On Rigid Spaces

The concept of coherent sheaf is similar to that of it's complex analytic analogue. We have take care of the admissible covering's of the Grothendieck topology. (Bosch [1], Pg 118-119)

Definition Let X be a rigid K-space, $\mathcal{F}$ an $\mathcal{O}_X$-module

(i) $\mathcal{F}$ is called finite type if there exists an admissible covering $(Y_i)_{i\in I}$ of X together with exact sequences of type

$$\mathcal{O}_X^{s_j}|_{X_j} \longrightarrow \mathcal{F}|_{X_j} \longrightarrow 0 \quad , \; j \in I$$

(ii) $\mathcal{F}$ is called a finite presentation, if there exists an admissible covering $(Y_i)_{i\in I}$ of X together with exact sequences of type

$$\mathcal{O}_X^{r_i}|_{Y_i} \longrightarrow \mathcal{O}_X^{l_i}|_{Y_i} \longrightarrow \mathcal{F}|_{Y_i} \longrightarrow 0 \, , \; i \in I$$

(iii) $\mathcal{F}$ is coherent if it is of finite type and if for all admissible open $U \subset X$, the kernel of a morphism $\mathcal{O}_{X|U}^{s} \longrightarrow \mathcal{F}_{|U}$ is of finite type.

Rmk :- An $\mathcal{O}_X$-module $\mathcal{F}$ for a rigid space $(X, \mathcal{O}_X)$ is coherent iff $\exists$ admissible affinoid covering $\mathcal{U} = (Y_j)_{j\in I}$ s.t. $\mathcal{F}|_{Y_j} = \widetilde{M_j}$, where M_j is a finite $\mathcal{O}_X|_{Y_j}(Y_j)$-module.

More clearly,

Theorem (Kiehl) 1:- $X = \mathrm{Sp}A$ be an affinoid K-space, then an $\mathcal{O}_X$-module $\mathcal{F}$ is coherent iff $\mathcal{F} = \widetilde{M}$, M a finite A-module.

By virtue of the above theorem, we have that $H^q(\mathrm{Sp}A, \mathcal{F}) = 0$, for $q > 0$ and $\mathcal{F}$-coherent sheaf on $\mathrm{Sp}A$. Hence, by the remark above, if $\mathcal{F}$ is any coherent sheaf on $\mathbb{P}_K^{n,an}$, then, $\mathcal{F}$ is $\{X_i\}_{i\geq 0}^{n}$ acyclic, hence

We have, $\check{H}^q(\mathcal{U}, \mathcal{F}) \xrightarrow{\sim} H^q(X, \mathcal{F})$, where $\mathcal{U} = \{X_i\}_{i=0}^n$. From now on we will only be concerned with Čech-cohomologies.

(Note that $\mathcal{F}$ is $\mathcal{U}$-acyclic, for $X_{i_0,\dots,i_k} = X_{i_0} \cap \dots \cap X_{i_k}$ is an affinoid space, and Thm 1 and Tate's Acyclicity gives us that $\mathcal{F}$ is $\mathcal{U}$-acyclic.)

We have the following theorem of Kiehl, which will useful to us later

<u>Theorm (Kiehl) 2</u> :- If X is a proper rigid K-space, then the K-vector ~~spec~~ space $H^q(X, \mathcal{F})$ are <u>finite dimensional</u>, for a coherent sheaf $\mathcal{F}$.

We observe that $\mathbb{P}_K^{n,an}$ is a proper K-space, we have two covers $\{X_i\}_{i=0}^n$ and $\{X_i'\}_{i=0}^n$, where $X_i' = \operatorname{Sp} K\langle c^{-1} \xi_{*i} \rangle$. Hence, $X_i \subset\subset_K X_i'$.

(Note that $\{X_i'\}$ is an ~~cov~~ admissible cover, since $\{X_i\}$ is one and $X_i' \supseteq X_i$)

$(**)$ For a coherent sheaf $\mathcal{F}$ on $\mathbb{P}_K^{n,an} = X$, $H^q(X, \mathcal{F})$ are finite dimensional K-vector spaces. $(q \geq 0)$

§ Analytification of ~~Sha~~ Scheme's Sheaves and Properties

If $(X, \mathcal{O}_X)$ is a scheme over K, $\mathcal{F}$ a $\mathcal{O}_X$-module then, define $\mathcal{F}^{an} := i^{-1}\mathcal{F} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{O}_{X^{an}}$, where $(X^{an}, \mathcal{O}_{X^{an}}) \xrightarrow{i} (X, \mathcal{O}_X)$. We have that, '$i$' is a ~~faithfully~~ flat morphism, which boils down to showing $\frac{K[\underline{\xi}]}{\mathfrak{m}^r} \xrightarrow{\sim} \frac{K\langle\underline{\xi}\rangle}{(\mathfrak{m}')^r}$, where

$\mathfrak{m}' \cap K[\underline{\xi}] = \mathfrak{m}$ (for, this shows, $\widehat{K[\underline{\xi}]} \cong \widehat{K\langle\underline{\xi}\rangle}$, $\Rightarrow K[\underline{\xi}] \hookrightarrow K\langle\underline{\xi}\rangle$ is flat).

Also, for coherent sheaves $\mathcal{F}$ on X, there is a natural map, $\Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X^{an}, \mathcal{F}^{an})$ and hence, a natural map $C^q(\mathcal{U}, \mathcal{F}) \longrightarrow C^q(\mathcal{U}^{an}, \mathcal{F}^{an})$ and hence a map $\check{H}^q(\mathcal{U}, \mathcal{F}) \longrightarrow \check{H}^q(\mathcal{U}^{an}, \mathcal{F}^{an})$.

Also, due to the flatness of $i : X^{an} \longrightarrow X$, we have, if $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{M} \to 0$ is an exact sequence of coherent sheaves on X, then, so is $0 \to \mathcal{F}^{an} \to \mathcal{G}^{an} \to \mathcal{M}^{an} \to 0$ exact on X^{an}. Note that $\mathcal{F}^{an}$ is coherent if $\mathcal{F}$ is coherent.

§ Proof of Rigid GAGA ($X \xhookrightarrow[\text{immersion}]{} \mathbb{P}^n_K$ is a closed)

- The natural map $H^q(X, \mathcal{F}) \xrightarrow{\sim} H^q(X^{an}, \mathcal{F}^{an})$ is an isomorphism

We use the following Lemma to reduce to coherent sheaves on $\mathbb{P}^n_K$:-

Lemma (Y. Tian [3], Pg 62) If $j: X \hookrightarrow Y$ is a closed embedding of rigid K-spaces, $\mathcal{F}$ coherent sheaf on X, then $j_* \mathcal{F}$ is coherent on Y and

$$H^q(X, \mathcal{F}) \cong H^q(Y, j_* \mathcal{F}), \quad q \geq 0.$$

$\therefore$ Since, $X^{an} \hookrightarrow \mathbb{P}^{n,an}_K$ is a closed immersion, we can restrict ourselves to coherent sheaves on $\mathbb{P}^n_K$. Henceforth, $X = \mathbb{P}^n_K$.

Proposition :- $H^q(X, \mathcal{O}_X(m)) \xrightarrow{\sim} H^q(X^{an}, \mathcal{O}_{X^{an}}(m))$
$\left(= H^q(X^{an}, \mathcal{O}_X^{an}(m))\right)$
$q \geq 0, m \in \mathbb{Z}$.

Remark :- Modulo the above proposition, it is easy to see that $H^q(X, \mathcal{F}) \xrightarrow{\sim} H^q(X^{an}, \mathcal{F}^{an})$

Since, any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n_K$ has an exact sequence, $0 \to \mathcal{R} \to \mathcal{E} \to \mathcal{F} \to 0$, where $\mathcal{E} = \bigoplus_{n \in \mathbb{Z}} \mathcal{O}_X(n)$ (a finite direct sum); and $\mathcal{R}$ a coherent sheaf on X. Since, the covers $\mathcal{U}' = \{U_i\}_{i=0}^n$ has 'n+1' elements, and $\mathcal{U} = \{X_i\}_{i=0}^n$ also covers X^{an}, we have

$$H^q(X, \mathcal{F}) = H^q(X^{an}, \mathcal{F}^{an}) = 0 \text{ for, } \forall q >> 0.$$

Hence, by descending induction and usual Five lemma arguments, give us that

$$H^q(X, \mathcal{F}) \xrightarrow{\sim} H^q(X^{an}, \mathcal{F}^{an}) \quad \forall \; q \geq 0.$$

Here we're using the co-chain complexe's,

$$\begin{tikzcd}
\cdots \arrow[r] & H^q(X,\mathcal{R}) \arrow[r] \arrow[d] & H^q(X,\mathcal{E}) \arrow[r] \arrow[d, "\wr", "\text{By a proposition}"'] & H^q(X,\mathcal{F}) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H^q(X^{an},\mathcal{R}^{an}) \arrow[r] & H^q(X^{an},\mathcal{E}^{an}) \arrow[r] & H^q(X^{an},\mathcal{F}^{an}) \arrow[r] & \cdots
\end{tikzcd}$$

These are ~~o~~ obtained as the long exact sequence in cohomology corresponding to the exact sequence $0 \to \mathcal{R} \to \mathcal{E} \to \mathcal{F} \to 0$ and $0 \to \mathcal{R}^{an} \to \mathcal{E}^{an} \to \mathcal{F}^{an} \to 0$

§ Proof of Proposition

We will closely follow Neeman's [5] proof of the same proposition [5, Pg 356]. Some terminology first :-

$\left(U_i = \mathrm{Spec}\left(K\left[\frac{\zeta_0}{\zeta_i}, \dots, \frac{\zeta_n}{\zeta_i}\right]\right)\right)$

$J = \{i_0, \dots, i_k\} \subset \{0, 1, \dots, n\}$ and $U_J = U_{i_0} \cap \dots \cap U_{i_k}$

and $X_J = X_{i_0} \cap \dots \cap X_{i_k}$. We know from standard scheme theory, that

$$\Gamma(U_J, \mathcal{O}_X(m)) = \Big\{ \sum_{\substack{\forall \zeta^{\underline{a}} \\ \text{of deg. } m}} \lambda_{\underline{a}} \zeta^{\underline{a}} \Big\} \subset K[\zeta_0, \dots, \zeta_n]_{\zeta_J}$$

where $\zeta_J = \zeta_{i_0} \cdots \zeta_{i_k}$. Also the composite maps given by,

$$\Gamma(U_J, \mathcal{O}_X(m)) \longrightarrow \Gamma(U_J^{an}, \mathcal{O}_{X^{an}}(m)) \xrightarrow{\mathrm{res}^{U_J^{an}}_{X_J}} \Gamma(X_J, \mathcal{O}_{X^{an}}(m))$$

is the completion w.r.t ~~to~~ the norm on $\Gamma(U_J, \mathcal{O}_X(m))$ given by $q_J\left(\sum \lambda_{\underline{a}} \zeta^{\underline{a}}\right) := \max_{\underline{a}} \{|\lambda_{\underline{a}}|\}$

"Hence, the Čech complex of $\mathcal{O}_{X^{an}}(m)$ can be obtained as completion of Čech complex of $\mathcal{O}_X(m)$".

Now some Homological algebra.

We will denote $\{A_i \xrightarrow{\varphi_i} B_i\}$ for the complex, $\ldots \to 0 \to A_i \xrightarrow{\varphi_i} B_i \to 0 \to \ldots$. Here, all A_i's and B_j's are K- vector spaces.

Then we have the notion of "Tensor Product" of Complex's. $\{A_i \to B_i\}_{i=0}^{n}$ or <u>the Koszul complex</u> associated to it, with following properties:-

Let $C = \bigotimes_{i=0}^{n}{}_K \{A_i \longrightarrow B_i\}$

(i) $\otimes_K$ is distributive

(ii) If for some 'i', $A_i \xrightarrow{\varphi_i} B_i$ is an isomorphism, then C is contractible.

For us $A_i =$ ~~$K[X_i$~~ $K[\cancel{X_i}\,\xi_i]$, $B_i = K[\cancel{X}\,\xi_i, \xi_i^{-1}]$

$$A_i = K[\xi_i] \hookrightarrow K[\xi_i, \xi_i^{-1}] = B_i$$

By above two properties, we have

$$C = \otimes_K \{K[\xi_i] \hookrightarrow K[\xi_i, \xi_i^{-1}]\}$$ can written as, $C = C_1 \oplus C_2$, where C_1 is contractible and C_2 is concentrated in degree 'n'. This can be realised by decomposing

$$\{\to 0 \to K[\xi_i] \hookrightarrow K[\xi_i, \xi_i^{-1}] \to 0 \to \ldots\} = \begin{array}{c} \{\to 0 \to K[\xi_i] \xrightarrow{1} K[\xi_i] \to 0 \to \ldots\} \\ \oplus \\ \{\ldots \to 0 \to 0 \longrightarrow \xi_i^{-1} K[\xi_i^{-1}] \to 0 \ldots\} \end{array}$$

By definition of a Koszul complex, C is given by the co-chain complex

$$C_{[-1]} \longrightarrow C_{[0]} \longrightarrow \cdots \longrightarrow C_{[n]}$$, where

$C_{[i]}$ is direct sum of C_J, where $(|J| = i+1)$

$$C_J = \bigotimes_{i=0}^{n} C_i^J \;,\; C_i^J = \begin{cases} \in K[\xi_i] & \text{if } i \notin J \\ K[\xi_i, \xi_i^{-1}] & \text{if } i \in J \end{cases}$$

It is easy to see that,

$$\begin{tikzcd} C_J \arrow[r] \arrow[d, "\wr"] & C_{J'} \arrow[d, "\wr"] \\ S_J \arrow[r, hook] & S_{J'} \end{tikzcd}$$

$J \subset J'$, and

$S_J = K[\xi_0, \ldots, \xi_n]_{\xi_J}$, where

$\xi_J = \xi_{i_0} \cdots \xi_{i_k}$, $J = \{i_0, \ldots, i_k\}$

Hence, $C \simeq S'$ as complexes, where

$$S' : S'_{[-1]} \longrightarrow S'_{[0]} \longrightarrow \cdots \longrightarrow S'_{[n]} .$$

and $S'_{[i]} = \bigoplus_{|J| = i+1} S_J$.

Also, $S_J = \bigoplus_{m \in \mathbb{Z}} S_J(m)$, where $S_J(m)$ are the laurent polynomials of degree 'm'. We have observed before that

$$\begin{tikzcd} S_J(m) \arrow[r, hook] \arrow[d, "\wr", "\shortparallel"'] \arrow[dr, phantom, "\circlearrowleft"] & S_{J'}(m) \arrow[d, "\shortparallel"] \\ \Gamma(U_J, \mathcal{O}_X(m)) \arrow[r] & \Gamma(U_{J'}, \mathcal{O}_X(m)) \end{tikzcd}$$

Hence, the degree m-part of complex S'

$~~S'_{[-1]}~~ \to S'_{[0]} \longrightarrow \cdots \longrightarrow S'_{[n]}$ iadentifies with

complex

$$\mathcal{O}_X(m)_{[0]} \longrightarrow \mathcal{O}_X(m)_{[1]} \longrightarrow \cdots \longrightarrow \mathcal{O}_X(m)_{[n]}$$

From all the above discussion we have the following conclusion :-

- $H^i(\mathcal{O}(m))$ is isomorphic to the cohomology of degree m-part of the complex C', given by,

$$0 \longrightarrow C_{[0]} \longrightarrow C_{[1]} \longrightarrow \cdots \longrightarrow C_{[n]} \longrightarrow 0$$

- $H^i(C') = 0$, for $0 < i < n$
- $H^n(C')$ is isomorphic to $\bigotimes_{i=0}^{n} \xi_i^{-1} K[\xi_i^{-1}]$
- $H^0(C') = C_{[-1]} = \bigotimes_{i=0}^{n} K[\xi_i] = K[\xi_0, \ldots, \xi_n]$.

, hence $H^0(\mathcal{O}(m)) =$ K-span of $\{\xi_0^{b_0} \ldots \xi_n^{b_n} \mid \sum b_i = m\}$

- $H^n(\mathcal{O}(m))$ is K-subspace of $\bigotimes_{i=0}^{n} \xi_i^{-1} K[\xi_i^{-1}]$, of degree 'm'.
- $H^i(\mathcal{O}(m)) = 0$, for $0 < i < n$.

We form the completed complex $\hat{C}$, of complex C, $\hat{C}_{[-1]} \longrightarrow \hat{C}_{[0]} \longrightarrow \cdots \longrightarrow \hat{C}_{[n]}$. If we truncate $\hat{C}$ by leaving out $\hat{C}_{[-1]}$, we get $\hat{C}'$

Degree m-part of $H^i(\hat{C}')$ is $H^i(\mathcal{O}^{an}(m))$

Note that we complete C' in each degree m-part, by considering formal power series

$$\sum_{\langle \underline{a} \rangle = m} \lambda_{\underline{a}} \xi^{\underline{a}}, \ \text{s.t. } |\lambda_{\underline{a}}| \to 0.$$

~~$\xi^{\underline{a}} \in S$~~ It boils down to showing $C' \longrightarrow \widehat{C}'$ is a quasi-isomorphism. We have a short exact sequence of complexes

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & C_{[0]} \arrow[r] \arrow[d] & C_{[1]} \arrow[r] \arrow[d] & \cdots \arrow[r] & C_{[n]} \arrow[r] \arrow[d] & 0 & C' \arrow[d] \\
0 \arrow[r] & C_{[-1]} \arrow[r] \arrow[d] & \cdots \arrow[d] & \arrow[d] & \arrow[r] & C_{[n]} \arrow[r] & 0 & C \arrow[d] \\
0 \arrow[r] & C_{[-1]} \arrow[r] & 0 \arrow[r] & \cdots & \arrow[r] & 0 \arrow[r] & 0. & C_{[-1]}
\end{tikzcd}$$

$0 \longrightarrow C' \longrightarrow C \longrightarrow C_{[-1]} \longrightarrow 0$. Completing this also is an exact sequence of complexes, we have,

$$\begin{tikzcd}
0 \arrow[r] & C' \arrow[r] \arrow[d, "i'"'] & C \arrow[r] \arrow[d, "i"'] & C_{[-1]} \arrow[r] \arrow[d, "i_{[-1]}"] & 0 \\
0 \arrow[r] & \widehat{C}' \arrow[r] & \widehat{C} \arrow[r] & \widehat{C}_{[-1]} \arrow[r] & 0
\end{tikzcd}$$

To show i' is a quasi-isomorphism, it suffices to show i, $i_{[-1]}$ are quasi-isomorphism.

$C_{[-1]}(m)$ is finite dimensional K-vector space.

Since K is complete, $C_{[-1]}(m) = \widehat{C_{[-1]}(m)}$.

Here, $C_{[-1]}(m)$ is degree m-part of $C_{[-1]} = \cancel{K[\xi_0, \dots, \xi_n]}$ $K[\xi_0, \dots, \xi_n]$

Hence, $i_{[-1]} = id$ is a quasi-isomorphism.

It remains to show, $i : C \longrightarrow \widehat{C}$ is a quasi-isomorphism. Here, $\widehat{C} = \bigoplus_{J \in 2^{[n]}} \widehat{C(J)}$

Hence, since for $J = \{0, 1, \ldots, n\}$, $C(J)(m)$ is finite dimensional K-vector space, hence

$C(J)(m) = \widehat{C(J)(m)}$ $\forall\, m \in \mathbb{Z}$. ($C(J)(m)$ is deg 'm' part of $\bigotimes_{i \geq 0}^{n} \xi_i^{-1} K[\xi_i^{-1}]$)

For $J \subsetneq \{0, 1, \ldots, n\}$, we have the following

<u>Lemma</u> $\equiv$ (Neeman [5, Pg 367]) :- Suppose $J \subsetneq \{0, 1, \ldots, n\}$, then $\widehat{C(J)}$ is contractible.

From the above lemma we have, $C(J) \longrightarrow \widehat{C(J)}$ is a quasi-isomorphism for $J \subsetneq \{0, 1, \ldots, n\}$, since $C(J)$ is also contractible in this case. $\square$

§ Equivalence Of Categories

The proof of full faithfullness of the functor, $(\cdot)^{an} : \mathcal{F} \longmapsto \mathcal{F}^{an}$, is almost direct from Serre's original paper (Serre [4]). We will briefly sketch it (see Y. Tian [3, Pg 67]).

- $\underline{Mor(\mathcal{F} \to \mathcal{G})^{an} \text{ "=" } Mor(\mathcal{F}^{an} \to \mathcal{G}^{an})}$

  This is immediate from part (b) of the Statement i.e. $H^q(X, \mathcal{F}) \xrightarrow{\sim} H^q(X^{an}, \mathcal{F}^{an})$, once we observe that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is a coherent sheaf (given $\mathcal{F}, \mathcal{G}$ are coherent) and that

  $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})^{an} = \mathcal{H}om_{\mathcal{O}_{X^{an}}}(\mathcal{F}^{an}, \mathcal{G}^{an})$

$\Rightarrow$ By taking, $q = 0$ and $\mathcal{F}$ as $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$, we get $\Gamma(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})) \xrightarrow{\sim} \Gamma(X^{an}, \mathcal{H}om_{\mathcal{O}_{X^{an}}}(\mathcal{F}^{an}, \mathcal{G}$

hence, every $\Psi : \mathcal{F}^{an} \to \mathcal{G}^{an}$ comes from $\varphi : \mathcal{F} \to \mathcal{G}$, i.e. $\Psi = \varphi^{an}$.

- $\underline{\text{Every coherent sheaf } \mathcal{M} \text{ on } X^{an} \text{ is } \mathcal{F}^{an} \text{ for some coherent } \mathcal{F} \text{ on } X}$

$\underline{\text{Lemma 1}}$:- H be a hyperplane in $\mathbb{P}^n_k$, $\mathcal{E}$ be any coherent sheaf on H^{an}. Then, $H^q(H^{an}, \mathcal{E}(n)) = 0$ for $n \gg 0$, $q > 0$.

Pf. By changing ~~co~~ coordinates (if necessary) we may assume $H = \mathbb{P}_K^{n-1}$. By induction hypothesis, $\exists\ \mathcal{F}$ coherent on H s.t. $\mathcal{F}^{an} = \mathcal{E}$. Hence, $\mathcal{F}^{an}(n) = \mathcal{E}(n)$. Now,

$$H^q(H, \mathcal{F}(n)) \cong H^q(H^{an}, \mathcal{F}^{an}(n)) = H^q(H^{an}, \mathcal{E}(n))$$

By Serre Vanishing Thm, $H^q(H, \mathcal{F}(n)) = 0$, $n >> 0$

$$\Rightarrow H^q(H^{an}, \mathcal{F}^{an}(n)) = H^q(H^{an}, \mathcal{E}(n)) = 0 \text{ for } n >> 0$$

Lemma 2:- $\mathcal{M}$ be coherent analytic sheaf on $X^{an} = \mathbb{P}_K^{n,an}$. Then $\exists\ n \in \mathbb{Z}$ s.t. $\mathcal{M}$ is generated by $H^0(X^{an}, \mathcal{M}(n))$.

Pf. Apriori, we donot have a reason to work with stalks in the Grothendieck Topology, but with coherent sheaves on X^{an}, (M coherent) ~~stalks~~ $\mathcal{M}_x = 0\ \forall x \in X^{an} \Rightarrow \mathcal{M} = 0$, so that we can as well work with stalks.

(Kiehl's Thm on Coherent Sheaves on X^{an} and Bosch [1, Pg 67, Corollary 3], imply the above statement)

Since X^{an} can be covered by finitely many affinoid spaces and hence any admissible covering has a finite refinement by affinoid's, hence it is enough to show

$\exists\, n \in \mathbb{Z}$ s.t. $H^0(X^{an}, \mathcal{M}(n))$ generates $\mathcal{M}(n)_x$.

(Here $\mathcal{M}(n) := \mathcal{M} \otimes_{\mathcal{O}_{X^{an}}} \mathcal{O}_{X^{an}}(n)$ and $\mathcal{O}_{X^{an}}(n) = \mathcal{O}_X(n)^{an}$).

Let $H \subset \mathbb{P}^n_K$ be a hyperplane, and $x \in H$. Consider

$0 \to \mathcal{O}_{X^{an}}(-1) \longrightarrow \mathcal{O}_{X^{an}} \longrightarrow \mathcal{O}_{H^{an}} \longrightarrow 0$. After sufficient manipulations, we get

$$0 \to C(n) \to \mathcal{M}(n-1) \to \mathcal{M}(n) \to \mathcal{M}_H(n) \to 0$$

$C = \mathrm{Tor}_1^{\mathcal{O}_{X^{an}}}(\mathcal{O}_H, \mathcal{M})$. $C(n)$ and $\mathcal{M}_H(n)$ are supported on H. Above sequence splits as

$$0 \to C(n) \to \mathcal{M}(n-1) \to P_n \to 0$$

$$0 \to P_n \longrightarrow \mathcal{M}(n) \longrightarrow \mathcal{M}_H(n) \to 0$$

From the resulting long exact sequence of cohomologies we get $\dim H^1(X^{an}, \mathcal{M}(n-1)) \geq \dim H^1(X^{an}, \mathcal{M}(n))$.

By Kiehl's finiteness theorem, $\dim H^i(X^{an}, \mathcal{M}(n')) < \infty$ $\forall\, n' \in \mathbb{Z}$ $\Rightarrow$ for $n_0 \gg 0$, $\dim H^1(X^{an}, \mathcal{M}(n)) = \dim H^1(X^{an}, \mathcal{M}(n_0))$ $\forall\, n \geq n_0$. Hence, $\dim H^0(X^{an}, \mathcal{M}(n)) \geq \dim H^0(X^{an}, \mathcal{M}_H(n))$ $\forall\, n \geq n_0$.

(In concluding the above from the l.e.s, lemma 1 is used on $C(n)$, by induction.)

By induction, $H^0(X^{an}, \mathcal{M}_H(n))$ generates $\mathcal{M}_H(n)_x$ $\exists\, n \in \mathbb{Z}$.

The same 'n' works for $\mathcal{M}(n)$ as wells, i.e. $H^0(X^{an}, \mathcal{M}(n))$ generates $\mathcal{M}(n)_x$. (This is by usual Nakayama argument for the surjection $H^0(\mathcal{M}(n)) \twoheadrightarrow H^0(\mathcal{M}_H(n))$)

Now by Lemma 2, $\exists\ n \in \mathbb{Z}$ s.t.

$$\oplus\, \mathcal{O}^r_{X^{an}} \longrightarrow \mathcal{M}(n) \longrightarrow 0 \quad \text{is exact}$$

or, $\oplus\, \mathcal{O}^r_{X^{an}}(-n) \longrightarrow \mathcal{M} \longrightarrow 0$ is exact.

The kernel is also coherent, hence by precisely the same argument, we get

$$\mathcal{O}^s_{X^{an}}(-m) \xrightarrow{\Psi} \mathcal{O}^r_{X^{an}}(-n) \longrightarrow \mathcal{M} \longrightarrow 0 \quad \text{is exact}$$

Now, since, $\Psi = \varphi^{an}$ for $\varphi : \mathcal{O}^s_X(-m) \longrightarrow \mathcal{O}^r_X(-n)$,

Hence, $(\operatorname{coker}\varphi)^{an} = \operatorname{coker}(\varphi^{an})$

$\Rightarrow \mathcal{M} = (\operatorname{coker}\varphi)^{an}$ and $\operatorname{coker}\varphi$ is coherent $\mathcal{O}_X$-module $\square$

§ References

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