TALK 3

TATE CONSTRUCTIONS – STRUCTURE OF $F_A^*(\mathcal{O})$

§ RECAP (BRIEF)

$\mathcal{O}$ a d.v.r., $c \geq 0$, $\overset{\pi}{\cancel{\varpi}} \in \mathcal{O}$ a fixed uniformizer.

$C_{\mathcal{O}}(c) :=$ category of pairs (A, λ_A); $\lambda_A : A \to \mathcal{O}$;
A regular at $P_A := \ker(\lambda_A)$; $ht(P_A) = c$.

$$F_A^i(\mathcal{O}) := \operatorname{Ext}_A^i(\mathcal{O}, \mathcal{O})^{tf}$$

~~Th.~~ ~~Wher~~ $F_A^c(\mathcal{O})$ is a free $\mathcal{O}$-module of rank 1 (Talk 1). In this talk, we aim at obtaining a generator for this $\mathcal{O}$-module. ~~and~~ ~~It~~ This consequently, also furnishes generators for all $\mathcal{O}$-modules $F_A^i(\mathcal{O})$ as we will see.

In §3 As in Talk 2, the rings A we will be interested in, are of the form $A = P/I$ where $P = \mathcal{O}[[t_1, \dots, t_n]]$, I is generated minimally by $f_1, \dots, f_m$ for

$$f_i = \varpi^{d_i} t_i + g_i \;;\; g_i \in (\mathbf{t})^2$$

$(\mathbf{t} = (t_1, \dots, t_n))$

§ DG ~~Algebras~~ (Differential Graded) Algebras

Let R be a ring.

Def'n A (homological) complex of R-modules $(X\cancel{A}_\bullet, \partial)$ is said to be a DG-algebra over R if (below $|x|$ is the integer λ s.t. $x \in A_\lambda$)
also called "degree of x"

(i) $X\cancel{A}^\# := \bigoplus X\cancel{A}_i$ is a graded R-algebra
i.e. $X\cancel{A}_0 = R$ ~~1~~, $X\cancel{A}_\lambda \cdot X\cancel{A}_\mu \subset X\cancel{A}_{\lambda+\mu}$ and $X\cancel{A}_i = 0$ $\forall\, i<0$.

(ii) Each $X\cancel{A}_i$ is finitely generated R-module;

(iii) $X\cancel{A}^\#$ is skew symmetric i.e.

$$x \cdot y = (-1)^{|x|\,|y|}\, y \cdot x$$

$$x^2 = 0 \quad \text{if } |x| \text{ odd}$$

(iv) $\partial(x \cdot y) = (\partial x)\cdot y + (-1)^{|x|}\, x \cdot \partial y$

$\Updownarrow$ equiv.

$\cdot : X\cancel{A}_\bullet \otimes_R X\cancel{A}_\bullet \longrightarrow X\cancel{A}_\bullet$ is a map of complexes.

Remark – $H(X\cancel{A}_\bullet, \partial) = Z(X\cancel{A}_\bullet, \partial)/B(X\cancel{A}_\bullet, \partial)$ is a graded R-algebra.

Example (1) ~~(KOSZUL COMPLEX)~~ $X_1, \dots, X_n$ be (of degree 1) indeterminates and $\underline{h} := (h_1, \dots, h_n) \in R^n$. Then the complex $K(\underline{h}, R) := (\Lambda^\bullet R[X_1, \dots, X_n], d)$ where 'd' is determined by $d\,T_i = h_i$, is called the <u>Koszul complex</u>. It has the property, $\sup\{i \mid H_i(K(\underline{h}, M)) \neq 0\} = n - \operatorname{depth}(\underline{h}, M)$

Example (2) Adjoining a variable to dissolve a homology class (Tate '56)

Let $(A_\bullet, \partial)$ be a DG-algebra over R and $t \in Z_{i-1}(A_\bullet, \partial)$ a cycle of degree $(i-1)$.

<u>Prop'n</u> ([Tate56, §2]) ~~There~~ Let T be an indeterminate. Then there exists a canonical procedure for constructing a DG algebra extension $(A_\bullet\langle T\rangle, \partial') \supset (A_\bullet, \partial)$ (where $|T| = i$) s.t. $(A_\bullet\langle T\rangle)_\lambda = A_\lambda \quad \forall \lambda < i$

$$\text{and } B_{i-1}(A_\bullet\langle T\rangle) = B_{i-1}(A_\bullet) + Rt$$

In particular, $H_{i-1}(A_\bullet\langle T\rangle) = \dfrac{H_{i-1}(A_\bullet)}{Rt}$

<u>Pf</u> :- Give it!

We denote the above ext'n by $[A_\bullet\langle T\rangle; dT = t]$

Doing this process iteratively gives us us an infinite free resolution of R/J

~~for~~ <u>Cor.</u> ([Thm 1, Tate56]) There exists a free, acyclic DG-algebra, denoted $R\langle U\rangle$, which is a free resolution of R/J.

<u>Remark</u> :- If $\underline{h} = (h_1, \dots, h_n) \in Z_0(A_\bullet)$, then $[R\langle T_1, \dots, T_n\rangle ; dT_i = h_i] = K(\underline{h}, R)$.

$R\langle U\rangle$ is called the acyclic closure of R/J.

Def'n: A ~~R-linear~~ derivation $\theta : \underset{R}{A}\langle U\rangle \to \underset{R}{A}\langle U\rangle$ is a R-linear map s.t.

(i) $\theta(xy) = \theta(x)y + (-1)^{|x|} x\theta(y)$;

(ii) $\theta(x^{(i)}) = \theta(x)\, x^{(i-1)}$ for $x \in A_{even}$, $i \geq 1$

(iii) $\partial \circ \theta = (-1)\, \theta \circ \partial$

Prop'n ([Iye01, Prop. 1.4]) (*) ~~If $f = \mathrm{rank}_R(J/J^2)$~~ ~~$\geq n$, then there~~ If $a_1, \dots, a_n$ form a R/J-basis of a free direct summand of J/J^2 & $\{x_1, \dots, x_n\}$ ~~satisfy $\partial x_i = a_i$.~~ of degree 1 in $R\langle U\rangle$ satisfy $\partial x_i = a_i$. Then $\exists$ a derivation $\theta_i : R\langle U\rangle \to R\langle U\rangle$ s.t. $\theta_i(x_j) = \delta_{ij}$.

~~We apply the~~ § Structure of $F_A^*(\mathcal{O})$

We apply the above discussion to the case of $R = A = \mathcal{O}[[t_1, \dots, t_n]]/(f_1, \dots, f_m)$

$f_i = \varpi t_i + g_i$; $g_i \in (t)^2$.

$J = P_A$, so that $R/J = \mathcal{O}$.

Let $\underline{X} = (~~X~~X_1, \dots, X_n)$ be indeterminates in degree 1 ~~s~~ forming $[A\langle \underline{X} \rangle; dX_i = t_i]$ $= K(\underline{t}, A)$.

If $g_i = \sum g'_{ij} t_j$ for $g'_{ij} \in (\underline{t})$,

$$z_i = \varpi^{d_i} X_i + \sum g'_{ij} X_j$$ are minimal generators of $H_1(A\langle \underline{X} \rangle)$. By adjoining ~~ex~~ divided power variables $\underline{Y} = \{Y_1, \dots, Y_m\}$ s.t. $dY_i = z_i$, we can dissolve these classes i.e. $A\langle \underline{X}, \underline{Y} \rangle := [A\langle \underline{X} \rangle \langle \underline{Y} \rangle; dY_i = z_i]$.

So, $H_1(A\langle \underline{X}, \underline{Y} \rangle) = 0$.

As in Tate's theorem above, we can keep doing this process to obtain an acyclic closure $\varepsilon : A\langle \mathcal{U} \rangle \xrightarrow[q.i.s]{} \mathcal{O}$, where $\mathcal{U} = (U_1, U_2, \dots)$ and $\mathcal{U}_i = \begin{cases} X_i & ; 1 \le i \le n \\ Y_{i-n} & ; n+1 \le i \le n+m \end{cases}$

~~The diff~~ $\Rightarrow \mathrm{Hom}_A(A\langle \mathcal{U} \rangle, A\langle \mathcal{U} \rangle) \xrightarrow[q.i.s]{} \mathrm{Hom}_A(A\langle \mathcal{U} \rangle, \mathcal{O})$

Hence two ways to interpret $\mathrm{Ext}^*_A(\mathcal{O}, \mathcal{O})$.

$\mathrm{End}_A(A\langle \mathcal{U} \rangle)$ is again a DG-algebra, hence $\mathrm{Ext}^*_A(\mathcal{O}, \mathcal{O})$ gets $\mathcal{O}$-algebra structure (not comm.).

$t_{n-c+1}, \dots, t_n$ form a basis of $\left(P_A/P_A^2\right)^{tf}$ (Talk 2)

& Consequently by Prop'n $(\ast)$, we get $\theta_i : A\langle U\rangle \to A\langle U\rangle$

$n-c+1 \le i \le n$

with the properties

(a) θ is a Γ- derivation & $\theta_i(X_j) = \delta_{ij}$

(b) $d\,\theta_i + \theta_i d = 0$

$\Rightarrow \{\theta_i\}$ is a subset of $Z^1(\mathrm{End}_A(A\langle U\rangle, A\langle U\rangle))$

$\Rightarrow \quad \theta_A := \varepsilon \circ \theta_n \circ \cdots \circ \theta_{n-c+1} : A\langle U\rangle \longrightarrow \mathcal{O}$

is a class in $\mathrm{Ext}_A^c(\mathcal{O}, \mathcal{O})$

Also $\theta_A(X_{n-c+1} \cdots X_n) = 1$

<u>Lemma</u>. Let $\theta : A\langle U\rangle \longrightarrow \mathcal{O}$ be any A-linear chain map of (upper) degree 'c' s.t. $\theta(A\langle X\rangle) = \mathcal{O}$. The class $[\theta] \in \mathrm{Ext}_A^c(\mathcal{O}, \mathcal{O})$ generates the free $\mathcal{O}$-module $F_A^c(\mathcal{O})$. In particular, it is true for θ_A above.

<u>Pf</u> ÷ Suppose not. Then there exists $\alpha : A\langle U\rangle \to \mathcal{O}$ a chain map of upper degree 'c' s.t. $\theta - \varpi\alpha$ is zero in $F_A^c(\mathcal{O})$ or $\varpi^m(\theta - \varpi\alpha) = 0$ in $\mathrm{Ext}_A^1(\mathcal{O}, \mathcal{O})$. This means there is a A-linear homotopy $\beta : A\langle U\rangle \longrightarrow A\langle U\rangle$

of upper degree $(c-1)$ s.t. $\varpi^m(\theta - \varpi\alpha) = \beta d$

By hypothesis, $\exists\, a \in A\langle X\rangle$ s.t. $\theta(a) = 1$.

$\Rightarrow \varpi^m(\theta(a) - \varpi\alpha(a)) = \beta d(a) = 0$

$\Rightarrow \varpi^m = \varpi^{m+1}\alpha(a) \in \varpi^{m+1}\mathcal{O}$, $\Rightarrow\Leftarrow$ ∎

This can be upgraded to a description of $F_A^*(\mathcal{O})$

Fact: $\mathrm{Ext}_A^*(k(P), k(P))$ is the exterior algebra over $\mathrm{Ext}_A^1(k(P), k(P))$. Since $\mathrm{Ext}_A^*(\mathcal{O},\mathcal{O})_P = \mathrm{Ext}_A^*(k(P), k(P))$ & $F_A^*(\mathcal{O}) \subset \mathrm{Ext}_A^*(\mathcal{O},\mathcal{O})_P$, hence $F_A^*(\mathcal{O})$ is also a strictly graded commutative, hence inducing a map, $\xi_A : \Lambda^* F_A^1(\mathcal{O}) \to F_A^*(\mathcal{O})$

Thm: (i) $\mathcal{O}^c \simeq \mathrm{Hom}_{\mathcal{O}}(P/P^2, \mathcal{O}) \simeq \mathrm{Ext}_A^1(\mathcal{O},\mathcal{O})$

(ii) ξ_A is bijective.

Pf: (i) $0 \to P \to A \to \mathcal{O} \to 0 \rightsquigarrow$

$$0 \to \mathrm{Hom}_A(\mathcal{O},\mathcal{O}) \xrightarrow{\sim} \mathrm{Hom}_A(A,\mathcal{O}) \to \mathrm{Hom}_A(P,\mathcal{O}) \to \mathrm{Ext}_A^1(\mathcal{O},\mathcal{O}) \to 0$$

$$\mathrm{Hom}_A(P,\mathcal{O}) \underset{\text{adj.}}{\simeq} \mathrm{Hom}_{\mathcal{O}}(P/P^2,\mathcal{O}) \underset{\text{Talk 2}}{\simeq} \mathcal{O}^c$$

(ii) Key idea: By previous lemma,

$\xi_A(\Lambda^c F_A^1(\mathcal{O})) = F_A^c(\mathcal{O})$, since $\theta_{n-c+1}, \dots, \theta_n$ form a basis of $F_A^1(\mathcal{O}) = \mathrm{Ext}_A^1(\mathcal{O},\mathcal{O})$. ∎

Then use ξ_A is an isom. after localizing at P. ∎

ξ_A is functorial :- If $\varphi : A \twoheadrightarrow B$ in $C_{\mathcal{O}}(c)$

then

$$\begin{tikzcd} \Lambda^* F_A^1(\mathcal{O}) \arrow[r, "\xi_A"] & F_A^*(\mathcal{O}) \\ \Lambda^* F_B^1(\mathcal{O}) \arrow[u, "\Lambda^* F_\varphi^1(\mathcal{O})"] \arrow[r, "\xi_B"] \arrow[ur, phantom, "\circlearrowright"] & F_B^*(\mathcal{O}) \arrow[u, "F_\varphi^*(\mathcal{O})"'] \end{tikzcd}$$

Prop'n :- If φ is an isom. at P_A, then $F_\varphi^*(\mathcal{O})$ is bijective.

Pf :-

$$0 \to I \to P_A \to P_B \to 0 \quad \xrightarrow{\otimes_A \mathcal{O}}$$

$$\frac{I}{P_A I} \to \frac{P_A}{P_A^2} \to \frac{P_B}{P_B^2} \to 0 \quad \xrightarrow{\mathrm{Hom}_{\mathcal{O}}(-,\mathcal{O})}$$

$$\mathrm{Hom}_{\mathcal{O}}\left(\frac{P_B}{P_B^2}, \mathcal{O}\right) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}}\left(\frac{P_A}{P_A^2}, \mathcal{O}\right) \quad \text{since}$$

$\frac{I}{P_A I}$ is a torsion $\mathcal{O}$-module $\square$

[IKM 24] S.B. IYENGAR, C.B. KHARE & J. MANNING, "CONGRUENCE MODULES AND THE WILES-LENSTRA-DIAMOND NUMERICAL CRITERION IN HIGHER CODIMENSIONS".

[IYE 01] S.B. IYENGAR, "FREE SUMMANDS OF CONORMAL MODULES AND CENTRAL ELEMENTS IN HOMOTOPY LIE ALGEBRAS OF LOCAL RINGS."

[TATE 56] J. TATE, "HOMOLOGY OF NOETHERIAN RINGS AND LOCAL RINGS".