GAUS - AG

on

Rigid Meromorphic Cocycles

Notes for Talk 3 on

FACTORIZATION OF SINGULAR MODULI

§ NOTATION

- $\mathbb{H}$ denote the upper half plane
  $\tau = x + iy \in \mathbb{H}$ denote a point

- $d_1, d_2 < 0$ be relatively prime fundamental discriminants. $D = d_1 d_2 > 0$ and $F := \mathbb{Q}(\sqrt{D})$

For $i = 1, 2$

- $K_i = \mathbb{Q}(\sqrt{d_i})$ · $w_i = \#\left\{\begin{matrix}\text{roots of unity}\\ \text{inside } K_i\end{matrix}\right\}$ · $h_i = \# Cl_{K_i}$

For primes ℓ satisfying $\left(\frac{D}{\ell}\right) \neq -1$, define

$$\varepsilon(\ell) = \begin{cases} \left(\frac{d_1}{\ell}\right) & ; (d_1, \ell) = 1 \\ \left(\frac{d_2}{\ell}\right) & ; (d_2, \ell) = 1 \end{cases}$$

Extend 'ε' multiplicatively to all such primes.

- $j(\tau) = \frac{1}{q} + 744 + O(q)$ denote the j-invariant
  $(q = e^{2\pi i \tau})$

§1. Statement of main result

Recall from last time, that given $\tau \in \mathbb{H}^{CM}$, which generates an imaginary quadratic field, ~~so~~ and it has C.M. by the maximal order $\mathcal{O}$ of $K = \mathbb{Q}(\tau)$, ~~Then~~ the Hilbert class field $H_{\mathcal{O}} = K(j(\tau))$ and $[H_{\mathcal{O}} : K] = h(\mathcal{O})$. The minimal polynomial of $j(\tau)$ is defined over $\mathbb{Q}$, and conjugates of $j(\tau)$ are exactly $j(\tau')$, where τ' runs over equivalence class of lattices $([1, \tau'])$ having C.M. by $\mathcal{O}$ (or proper fractional $\mathcal{O}$-ideals). Hence

$$J(d_1, d_2) = \prod_{\substack{[\tau_i] \\ \mathrm{disc}(\tau_i) = d_i}} \left(j(\tau_1) - j(\tau_2)\right)^{\frac{4}{w_1 w_2}}$$

is the (generally) norm of $j(\tau_1) - j(\tau_2)$, from $\mathbb{Q}(j(\tau_1), j(\tau_2)) \longrightarrow \mathbb{Q}$.

($[\tau_i]$ runs over classes in $SL_2(\mathbb{Z}) \backslash \mathbb{H}$ s.t. $\mathrm{disc}(\tau_i) = d_i$; $i = 1, 2$)

- When $d_1, d_2 < -4$, then $w_i = 2$ and $J(d_1,d_2)$ is an integer (being the norm of an algebraic integer)
- When $d_i \geq -4$, $J(d_1,d_2)^2$ is an integer (follows from Weber's theorem) ~~that~~ on cube roots of j-invariants

Main theorem. [Thm. 1.3, GZ85]

$$J(d_1,d_2)^2 = \pm \prod_{\substack{4nn' = D - x^2 \\ x^2 < D \\ x,n,n' \in \mathbb{Z} \\ n,n' > 0}} n^{\varepsilon(n')}$$

Examples: (i) $j\left(\frac{1+i\sqrt{163}}{2}\right) - j(i) =$

$$-2^{18}.3^3.5^3.23^3.29^3 - 1728$$

$$= -2^6.3^6.7^2.11^2.19^2.127^2.163$$

(ii) $j\left(\frac{1+i\sqrt{67}}{2}\right) - j\left(\frac{1+i\sqrt{163}}{2}\right) =$

$$-2^{15}.3^3.5^3.11^3 + 2^{18}.3^3.5^3.23^3.29^3 =$$

$$2^{15}.3^7.5^3.7^2.13.139.331$$

In ~~both~~ examples, we have computed $J(-163,-4)$ and $J(-67,-163)$ respectively.

§2. MOTIVATION / IDEA FOR A PROOF

In the paper [GZ 86], Gross & Zagier give a strong evidence for the Birch & Swinnerton Dyer conjecture (B.S.D conjecture). Recall that it says that for an elliptic curve $E/\mathbb{Q}$,

$$\begin{array}{ccc} \text{algebraic rank} & \overset{\text{BSD}}{\underset{\text{conj}}{=\!=}} & \text{analytic rank} \\ \| & & \| \\ \dim_{\mathbb{Q}}(E(\mathbb{Q})\otimes_{\mathbb{Z}}\mathbb{Q}) & & \operatorname{ord}_{s=1}(L(E,s)) \end{array}$$

(The L-function $L(E,s)$ is defined at $s=1$ by virtue of modularity ~~lifting~~ theorem)

Thm ([GZ86]). Assume that $L(E,1)=0$. Then there is a rational point P in $E(\mathbb{Q})$ s.t.

$$L'(E,1) \sim \Omega \langle P,P \rangle, \text{ where}$$

$\sim$ means ~~it differs~~ the two terms differ by a constant $\alpha \in \mathbb{Q}^\times$ (which is "arithmatically computable")

Ω is the real period of regular differential on E,

$\langle \cdot,\cdot \rangle$ is Néron-Tate height pairing on $E(\mathbb{Q})\otimes\mathbb{R}$

If $L'(E,1) \neq 0 \Rightarrow$ 'P' has infinite order.

Main idea of the proof is to construct a weight 2 cusp form for $\Gamma_0(N)$ (N = conductor of E) "algebraically", and also alternately construct it "analytically" to relate it to derivatives of L-function. Brief ~~sketch~~ details are sketched below. For more details, please refer to [GZ86, §I] or [FLPSW23, §4.2].

By modularity theorem, E is a direct factor of $J_0(N) := Jac(X_0(N))$. ($N>1$)

Choose $d<0$ $\boxed{(d,N)=1}$ a fundamental discriminant and form $K=\mathbb{Q}(\sqrt{d})$. A point $\tau \in \mathbb{H}^{CM}$ is called a Heegner point, if the ~~$(\phi: E$~~ $(\phi: A \longrightarrow A')$ corresponding to 'τ' ~~satisfi~~ on $X_0(N)(\overline{\mathbb{Q}})$, satisfies that both A, A' have CM by $\mathcal{O}_K$. By abuse of notation, we also denote the point on $X_0(N)$ by 'τ'. Form the degree 0 divisor $C_\tau := (\tau)-(\infty)$

on $J_0(N)$. It is defined over $H := H_{\mathcal{O}_K}$

The divisor $c_d := \sum_{\substack{[\tau] \\ \mathrm{disc}(\tau)=d}} c_\tau$ is defined $/K$ (by Shimura reciprocity)

The point P appearing in the theorem ~~is~~ stated, ~~as~~ is defined as a suitable trace of c_d.

As alluded to before, the main idea hinges around a weight 2 cusp form for $\Gamma_0(N)$

This is $G_d(q) = \sum \langle c_d, T_n c_d \rangle q^n$

where $\langle \cdot , \cdot \rangle$ is the Néron-Tate height pairing on $J_0(N) \otimes \mathbb{R}$, T_n is the usual Hecke correspondence.

G_d is called the <u>modular generating series</u> and the crucial point is to express it alternately by exploiting the derivative of a suitable family $\mathcal{G}_s$ of real analytic Hilbert modular forms of constant parellel weight 1 for $\mathbb{Q}$-algebrea $\mathbb{Q} \times \mathbb{Q}$.

This is explicitly given as the holomorphic projection of the product of

- a weight 1 theta series Θ_1 attached to K
- the derivative at $s=0$ of a family of real analytic Eisenstein series $\mathcal{E}_{1,s}$ of constant weight 1, parametrized by $s \in \mathbb{C}$.

The comparision of G_d and $\Theta_1 \cdot \frac{\partial}{\partial s}(\mathcal{E}_{1,s}|_{s=0})$ is done by computing the Archimedean heights using Green's functions, and non-Archimedean heights using intersection theory.

What about the case $N=1$?

~~The techniques~~ Here one would expect the cusp form constructed above to vanish identically, since ~~theo~~ there are no forms of weight 2 for $SL_2(\mathbb{Z})$!

In [GZ85], the authors consider the above techniques but in the context of the following diagram ($F = \mathbb{Q}(\sqrt{D})$)

$$\begin{tikzcd}[column sep=small]
& L = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2}) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
K_1 \arrow[dr, dash] & F \arrow[d, dash] & K_2 \arrow[dl, dash] \\
& \mathbb{Q} &
\end{tikzcd}$$

(Note that in the previous case, F corr. to the split $\mathbb{Q}$-algebra $\mathbb{Q} \times \mathbb{Q}$ and $L = K_1 \times K_1$)

~~#~~ Then one could form the modular generating series

$$G_{d_1, d_2}(q) = \sum_{n \geq 1} \langle c_{d_1}, T_n c_{d_2} \rangle q^n$$

and ~~obt~~ expect to obtain it alternately using ~~to~~ real analytic Hilbert Eisenstein series, $\mathcal{E}_s$, of parellel weight for F. More precisely,

(i) compute the derivative $\mathcal{E}'$ of $\mathcal{E}_s$ at $s = 0$.

(ii) pullback $\Delta^* \mathcal{E}'$ using the diagonal $\Delta : \mathbb{H} \to \mathbb{H}^2$

(iii) apply holomorphic projector $\pi_{hol}(\Delta^* \mathcal{E}')$.

And then G_{d_1, d_2} should be the same as $\pi_{hol}(\Delta^* \mathcal{E}')$.

But both of them should be zero since there are no weight 2 forms for $\Gamma_0(1)$!

The <u>algebraic proof</u> in [GZ85] could be summed up as trying to show the first Fourier coefficient of G_{d_1,d_2}, $\langle c_{d_1}, c_{d_2}\rangle = 0$, by computing explicitly the non-Archimedean contribution in $\langle c_{d_1}, c_{d_2}\rangle = \langle c_{d_1}, c_{d_2}\rangle_{Arch} +$

$$\langle c_{d_1}, c_{d_2}\rangle_{non\text{-}Arch}$$

using Deuring's theory.

The <u>analytic proof</u> concerns writing ~~computing~~ the 1st Fourier coefficient of $\pi_{hol}(\Delta^* \mathcal{G}')$, which is zero, using the Sturm projection formula as an expression of the Fourier coefficient of $\Delta^* \mathcal{G}'$ and some arithmetic terms and then use Green's function to complete proof.

Strategy of Analytic proof

$$-\log |J(d_1, d_2)|^2$$

1. Use "Green's function" to estimate
$$\log |j(\tau_1) - j(\tau_2)|^2 = \lim_{s \to 1} [G_s(\tau_1, \tau_2) + \cdots] - 24$$
This computation corr. to calculating Archimedean contribution of $\langle C_{d_1}, C_{d_2} \rangle$

2. Sum $\sum_{\substack{[\tau_1][\tau_2] \\ \mathrm{disc}(\tau_i) = d_i}}$ to get
$$-\log |J(d_1, d_2)|^2 = \lim_{s \to 1} [\text{blah}]$$

$$\sum_{\substack{n \mid (D - x^2)/4 \\ x^2 < D,\ n, x > 0}} \varepsilon(n) \log(n)$$

I. Form the real analytic Hilbert Eisenstein series

$E_s(\tau_1, \tau_2) =$ ('χ' to be defined later)
$$\sum_{[\mathfrak{a}] \in Cl_F^+} \chi(\mathfrak{a}) N(\mathfrak{a})^{1+2s} \sum_{\substack{(m,n) \in \mathfrak{a}^2 / \mathcal{O}^\times \\ (m,n) \neq 0}} \frac{y_1^s \, y_2^{\xi s}}{(m\tau_1 + n)(m'\tau_2 + n') \, |m\tau_1 + n|^{2s} \, |m'\tau_2 + n'|^{2s}}$$
(Here $(\ )'$ ξ is the conjugation on $F = \mathbb{Q}(\sqrt{D})$)

2. Apply Sturm's holomorphic projection formula to $F(\tau) = \frac{\sqrt{D}}{8\pi^2} \frac{\partial}{\partial s} E_s(\tau, \tau)\Big|_{s=0}$ to get an estimate
$$\sum \varepsilon(n) \log(n) = \lim_{s \to 1} [\text{blah}']$$

3. Use Taylor expansion of analytic functions in blah, blah' to conclude
$$\lim_{s \to 1} [\text{blah}] = \lim_{s \to 1} [\text{blah}']$$

§3. Analytic proof (Sketch)

As pointed out previously, in the analytic proof in [GZ85], the authors try to show

$$-\log |J(d_1, d_2)|^2 = -\sum_{\substack{4nn' = D - x^2 \\ x^2 < D \\ D \equiv x^2 (4) \\ n, n' > 0}} \varepsilon(n') \log n$$

$$= \sum_{\substack{x^2 < D \\ D \equiv x^2 (4)}} \sum_{n \mid \frac{D - x^2}{4}} \varepsilon(n) \log(n)$$

(Here we used $\varepsilon\left(\frac{D-x^2}{4}\right) = -1$, as we'll see below)

The authors rewrite the last sum ~~below~~ in terms of the genus character χ associated to it. This helps them relate it to the Fourier coefficients of the Hilbert Eisenstein series (real analytic) which was alluded to before.

Let Cl_F^+ denote the narrow class group of F. It is defined as group of fractional ideals of F modulo principal ideals generated by positive norm elements.

Def'n :- The genus character, χ, associated to the decomposition of $D = (d_1).(d_2)$ is the quadratic character

$$\chi : Cl_F^+ \longrightarrow \{\pm 1\} \quad \text{s.t.}$$

$$[P] \longmapsto \begin{cases} 1 & ; \text{ if } P \text{ is inert} \\ \varepsilon(N(P)) & ; \text{ o.w.} \end{cases}$$

Analogous to the Hilbert class field, one can form the narrow Hilbert class field of F, ~~st~~. H_F^+, s.t. we have a Artin reciprocity map, which is an isomorphism

$$A : Cl_F^+ \xrightarrow{\sim} Gal(H_F^+ | F)$$

defined in the usual as lifting the Frobenius. In the ideal theoritic formulation of CFT, Cl_F^+ corresponds to the modulus $m = m_0 m_\infty$ of F s.t. $m_0 = 1$, $m_\infty = \sigma.\sigma'$, where σ, σ' are the two real embeddings of $F \underset{\sigma,\sigma'}{\rightarrow} \mathbb{R}$

Hence H_F^+ is unramified at all finite places.

One can show that if $\alpha \in F$ has $N(\alpha) < 0$ then $\chi((\alpha)) = -1$, in particular $\varepsilon\left(\frac{D-x^2}{4}\right) = -1$, for $D > x^2$.

The kernel of $\mathcal{X}$ has index 2 in Cl_F^+ and therefore fixes a quadratic extension of F. Using Artin reciprocity, ~~one ε~~ $\mathbb{A}$, one can show $(H_F^+)^{Ker(\mathcal{X})} = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$.

So we have the following diagram of quadratic extensions. L/F is unramified at finite places of F.

$$\begin{tikzcd}
 & L = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2}) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
K_1 \arrow[dr, dash] & F \arrow[d, dash] & K_2 \arrow[dl, dash] \\
 & \mathbb{Q} &
\end{tikzcd}$$

Using this and the fact that the <u>different of F</u>, $\boxed{\delta = (\sqrt{D})}$, we have

$$\sum_{\substack{x^2 < D \\ D \equiv x^2 (4)}} \sum_{n \mid \frac{D - x^2}{4}} \varepsilon(n) \log(n) = \sum_{\substack{\nu \in \delta^{-1} \\ \nu \gg 0 \\ Tr(\nu) = 1}} \sum_{\mathfrak{n} \mid (\nu)\delta} \mathcal{X}(\mathfrak{n}) \log N(\mathfrak{n})$$

We obtain this by setting $\nu = \frac{x + \sqrt{D}}{2\sqrt{D}}$ and noting that $\mathfrak{n} \leftrightarrow N(\mathfrak{n}) =: n$, is a bijection between integral divisors of $(\nu\sqrt{D})$ and positive divisors of $\frac{D - x^2}{4}$, with $\mathcal{X}(\mathfrak{n}) = \varepsilon(n)$.

The authors observe that the R.H.S. is similar to the following formula of Siegel :

$$30\,\zeta_F(-k+1) = \sum_{\substack{\nu\in\delta^{-1}\\ \nu\gg 0\\ \mathrm{Tr}(\nu)=1}} \sum_{\mathfrak{n}\mid(\nu)\delta} N(\mathfrak{n})^{k-1} \quad (k=2,4)$$

See [Sie 68].

These formulas are obtained by restricting to the diagonal $\tau_1=\tau_2$, the following Hilbert Eisenstein series:

$$E_{F,k}(\tau_1,\tau_2) = \sum_{[\mathfrak{a}]\in Cl_F^+} N(\mathfrak{a})^k \sum_{\substack{(m,n)\in\mathfrak{a}^2/\mathcal{O}^\times\\ \neq(0,0)}} \frac{1}{(m\tau_1+n)^k(m'\tau_2+n')^k}$$

where $(\ )': F\to F$ is $x+y\sqrt{D}\mapsto x-y\sqrt{D}$.

So one can expect an analogous formula, with $N(\mathfrak{a})^{k-1}$ replaced by $\chi(\mathfrak{a})\log N(\mathfrak{a})$, to be related to the function $\frac{\partial}{\partial s}E_s(\tau,\tau)|_{s=0}$, where

$$E_s(\tau_1,\tau_2) = E_{F,\chi,1,s}(\tau_1,\tau_2) \quad \text{(Hecke trick!)}$$

$$= \sum_{[\mathfrak{a}]\in Cl_F^+} \chi(\mathfrak{a})N(\mathfrak{a})^{1+2s} \sum_{\substack{(m,n)\in\mathfrak{a}^2/\mathcal{O}^\times\\ \neq(0,0)}} \frac{y_1^s\,y_2^s}{(m\tau_1+n)(m'\tau_2+n')|m\tau_1+n|^{2s}|m'\tau_2+n'|^{2s}}$$

a real analytic Hilbert-Eisenstein series of parellel weight 1 for the $\mathbb{Q}$-algebra $F=\mathbb{Q}(\sqrt{D})$. $(\mathrm{Re}(s)>0)$

One computes the Fourier expansion of $E_s(\tau_1, \tau_2)$ using the Poisson summation formula as

$$E_s(\tau_1,\tau_2) = L_F(1+2s, \chi)\, y_1^s y_2^s + D^{-\frac{1}{2}} L_F(s,\chi)\, \Phi_s(0)^2 y_1^s y_2^s$$
$$+ D^{-\frac{1}{2}} y_1^{-s} y_2^{-s} \sum \sigma_{-2s,\chi}((\nu)\delta)\, \Phi_s(\nu y_1)\, \Phi_s(\nu' y_2)\, \nu\, e^{2\pi i(\nu x_1 + \nu' x_2)}$$

where $L_F(s,\chi) = L\left(s, \left(\frac{d_1}{\cdot}\right)\right) L\left(s, \left(\frac{d_2}{\cdot}\right)\right)$ &

$$\Phi_s(t) = \int_{-\infty}^{\infty} \frac{e^{-2\pi i x t}}{(x+i)(x^2+1)^s}\, dx \quad (t \in \mathbb{R}),$$

$$\sigma_{s,\chi} = \sum_{\mathfrak{m} \mid \mathfrak{a}} \chi(\mathfrak{m})\, N(\mathfrak{m})^s.$$ One can analytically continue $E_s(\tau_1,\tau_2)$ to all $s \in \mathbb{C}$, since all the coefficients of expansion can be done so. $E_s(\tau_1,\tau_2)$ vanishes at $s=0$, but one can compute

$$F(\tau) := \frac{\sqrt{D}}{8\pi^2} \frac{\partial}{\partial s} E_s(\tau,\tau)\Big|_{s=0} = \frac{\sqrt{D}}{2\pi^2}\left(L_F(1,\chi)\log(y_1 y_2) + C_\chi\right)$$
$$+ \sum_{\substack{\nu \in \delta^{-1} \\ \nu \gg 0}} \sigma'_\chi((\nu)\delta)\, e^{2\pi i \operatorname{Tr}(\nu)\tau}$$
$$- \sum_{\substack{\nu \in \delta^{-1} \\ \nu > 0 > \nu'}} \sigma_{0,\chi}((\nu)\delta)\, \Phi(|\nu'| y_1)\, e^{2\pi i \operatorname{Tr}(\nu)\tau}$$

where $C_\chi = L_F'(1,\chi) + (\log\sqrt{D} - \log\pi - \gamma)L_F(1,\chi)$

$\cancel{\sigma_{s,\chi}(\mathfrak{a})}$ (γ = Euler's cnst)

$$\sigma_\chi'(\mathfrak{a}) = \frac{\partial}{\partial s}\sigma_{s,\chi}(\mathfrak{a})\Big|_{s=0} = \sum_{\mathfrak{p}\mid\mathfrak{a}} \chi(\mathfrak{p})\log N(\mathfrak{p})$$

<u>Sturm's ~~pre~~ holomorphic projection</u> – a quick recap

If $\varphi : \mathbb{H} \to \mathbb{C}$ is a C^∞-function which transforms like a weight ~~3~~ 'k' modular form ($k>2$) for $SL_2(\mathbb{Z})$ and has polynomial growth near the cusp (i.e. $O(\mathrm{Im}(\tau)^c)$ for some 'c') then 'φ' defines a non-zero linear functional $S_k \longrightarrow \mathbb{C}$, sending $[f \mapsto (f,\varphi)_{Pet.}]$.

Since the Petersson inner product $(\cdot,\cdot)_{Pet}$ is non-degenerate, $\exists!\ \pi_{hol}(\varphi) \in S_k$ s.t.

$$(f,\varphi) = (f,\pi_{hol}(\varphi)) \quad \forall\ f \in S_k.$$

One could give an explicit description of the coefficients of the q-expansion of $\pi_{hol}(\varphi)$ in terms of coefficients of the Fourier expansion of 'φ'. This is achieved by the m-th Poincaré series ($m \in \mathbb{N}$)

$$P_m(\tau) := \sum_{\gamma = \left(\begin{smallmatrix} \cdot & \cdot \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash SL_2(\mathbb{Z})} \frac{e^{2\pi i m \gamma(\tau)}}{(c\tau+d)^k}$$

This converges absolutely for $k > 2$ and defines a cusp form.

If $\varphi = \sum_{n \in \mathbb{Z}} c_n(y) e^{2\pi i n x}$, $\pi_{hol}(\varphi) = \sum_{n=1}^{\infty} c_n e^{2\pi i n \tau}$
$(c_0(y) = 0)$

then using

$$\frac{(k-2)!}{(4\pi m)^{k-1}} c_m = (\pi_{hol}(\varphi), P_m) = (\varphi, P_m) = \int_0^\infty c_m(y) e^{-2\pi m y} y^{k-2} \, dy$$

(the behaviour of φ, P_m at ∞ allows one to compute the last integral)

Now back to our real analytic $F(\tau)$, which transforms like a weight 2 form the holomorphic projection is trivial since $S_2(SL_2(\mathbb{Z})) = 0$. But the technique above, with P_m replaced by $P_{m,s}(\tau) =$

(Hecke's trick, again!!)

$$\sum_{\left(\begin{smallmatrix} \cdot & \cdot \\ c & d \end{smallmatrix}\right) = \gamma \in \Gamma_\infty \backslash SL_2(\mathbb{Z})} \frac{1}{(c\tau+d)^2} \frac{y^s}{|c\tau+d|^{2s}} e^{2\pi i m \gamma(\tau)} \qquad (\mathrm{Re}(s) > 0)$$

~~and s→0 gives~~

gives us the following result (letting $s \to 0$ in previous technique)

Prop'n ÷ Let $F(\tau)$ be real analytic function on $\mathbb{H}$ which transforms under $SL_2(\mathbb{Z})$ like a modular form of weight 2 and satisfies

$$F(\tau) = A \log y + B + O(y^{-\varepsilon}) \quad \text{as } y \to \infty$$

$(A, B \in \mathbb{R}$ & $\varepsilon > 0)$. If $F(\tau) = \sum_{m=-\infty}^{\infty} a_m(y) e^{2\pi i m\tau}$. Then

$$\lim_{s \to 0} \left(4\pi \int_0^\infty a_1(y) e^{-4\pi y} y^s dy + \frac{24A}{s} \right) = 24A\left(2\frac{\zeta'}{\zeta}(2) + 1 + \log 4\right) - 24B.$$

For us, $A = \frac{\sqrt{D}}{2\pi^2} L_F(1, \chi)$, $B = \frac{\sqrt{D}}{2\pi^2} C_\chi$ and

$$a_1(y) = \sum_{\substack{\nu \in \delta^{-1} \\ \nu \gg 0 \\ Tr(\nu) = 1}} \sigma'_\chi((\nu)\delta) - \sum_{\substack{\nu \in \delta^{-1} \\ \nu > 0 > \nu' \\ Tr(\nu) = 1}} \sigma_{0,\chi}((\nu)\delta)\, \Phi(|\nu'| y)$$

$$\Rightarrow 4\pi \int_0^\infty a_1(y) e^{-4\pi y} y^s dy = \frac{\Gamma(s+1)}{(4\pi)^s} \left(M - \sum_{\substack{n > \sqrt{D} \\ n \equiv D(2)}} f(n) \psi_s\left(\frac{n - \sqrt{D}}{2\sqrt{D}}\right) \right)$$

where, $M = \sum_{\substack{\nu \in \delta^{-1} \\ \nu \gg 0 \\ Tr(\nu) = 1}} \sigma'_\chi((\nu)\delta) = \sum_{\substack{\nu \in \delta^{-1} \\ \nu \gg 0 \\ Tr(\nu) = 1}} \sum_{n \mid (\nu)\delta} \chi(n) \log N(n)$

and Ψ_s is a well defined holomorphic func.

The proposition gives

$$\sum_{\substack{\nu \in \delta^{-1} \\ \nu \gg 0 \\ Tr(\nu)=1}} \sum_{\mathfrak{n} \mid (\nu)\delta} \chi(\mathfrak{n}) \log N(\mathfrak{n}) =$$

$$\lim_{s \to 0} \left(\sum_{\substack{n > \sqrt{D} \\ n \equiv D (2)}} \rho(n) \Psi_s\left(\frac{n-\sqrt{D}}{2\sqrt{D}}\right) - \frac{12\sqrt{D}}{\pi^2} L_F(1,\chi) s^{-1} \right) +$$

$$\frac{12\sqrt{D}}{\pi^2} L_F(1,\chi) \left(2\frac{\zeta'}{\zeta}(2) + 1 + \log(4) \right) - \frac{12\sqrt{D}}{\pi^2} C_\chi$$

(Here and above, $\rho(n) = \sum_{d \mid \frac{n^2-D}{4}} \varepsilon(d)$, which also can be described as the number of inequivalent representations of the binary quadratic form $[d_1, -2n, d_2]$ by the form $\Delta = b^2 - 4ac$ on $\mathcal{Q} = \{[a,b,c] \mid a,b,c \in \mathbb{Z}\}$)

In §5 of [GZ85], the authors use "Green's function" to give an estimate of $\log |J(d_1,d_2)|^2$ as

$$-\log |J(d_1, d_2)|^2 = \lim_{s \to 1} \Big[2 \sum_{\substack{n > \sqrt{D} \\ n \equiv D(2)}} f(n)\, Q_{s-1}\left(\frac{n}{\sqrt{D}}\right) +$$

$$\frac{4\pi}{\zeta(2s)} \left(h_2' \left|\frac{d_1}{4}\right|^{\frac{s}{2}} \zeta_{K_1}(s) + h_1' \left|\frac{d_2}{4}\right|^{\frac{s}{2}} \zeta_{K_2}(s) - h_1' h_2' \frac{\sqrt{\pi}\,\Gamma(s-\frac{1}{2})}{\Gamma(s)} \right.$$

$$\left. \zeta(2s-1) \right) \Big] - 24 h_1' h_2' ,$$

where $h_j' = \frac{2}{w_j} h_j$ and $Q_{s-1}(t) = \int_0^\infty (t + \sqrt{t^2-1} \cosh y)^{-s} dy$

$(t > 1)$

In the proof of above formula, first one forms the automorphic Green's function

$$G_s(\tau_1, \tau_2) = \sum_{\gamma \in PSL_2(\mathbb{Z})} g_s(\tau_1, \gamma \tau_2), \text{ where}$$

$$g_s(\tau_1, \tau_2) = -2 Q_{s-1}(\cosh d(\tau_1, \tau_2))$$

$$= -2 Q_{s-1}\left(1 + \frac{|\tau_1 - \tau_2|^2}{2 y_1 y_2}\right)$$

(These functions arise out of an attempt to compute Archimedean contribution of Néron - Tate heights (c.f. see [Ch. II, GZ86]))

Then one shows that

$$\log|j(\tau_1)-j(\tau_2)|^2 = \lim_{s\to 1}\Big(G_s(\tau_1,\tau_2)+4\pi E(\tau_1,s)+4\pi E(\tau_2,s) - 4\pi\varphi(s)\Big) - 24$$

where $E(\tau,s) = \sum_{\gamma\in\Gamma_\infty\backslash SL_2(\mathbb{Z})} \operatorname{Im}(\gamma\tau)^s$ and

$$\varphi(s) = \frac{\Gamma(\frac{1}{2})\,\Gamma(s-\frac{1}{2})}{\Gamma(s)}\,\zeta(2s-1).$$

Here $\boxed{\tau_1 \notin SL_2(\mathbb{Z})\tau_2}$. Summing over $[\tau_i]$, with $\operatorname{disc}(\tau_i)=d_i$, we get the desired formula.

Then it is just a matter of using the Taylor expansions of all the functions involved in the two formulae ($\zeta_{K_i}(s)$, $L_F(s,\chi)$, $\zeta(s)$ $\cancel{\zeta}$, $\Gamma(s)$, $L(1,(\frac{d_i}{\cdot}))$), to show that

$$-\log|J(d_1,d_2)|^2 = \sum_{\substack{\nu\in\delta^{-1}\\ \nu\gg 0\\ \operatorname{Tr}(\nu)=1}} \sum_{\mathfrak{n}\mid(\nu)\delta} \chi(\mathfrak{n})\log N(\mathfrak{n})$$

$\square$

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