## D-elliptic sheaves and Hasse principle NOTES FOR TALK 3 on STACKS II

§1- BRIEF RECAP OF TALK!

(for more details, refer to the notes for Talk 1 by Prof. Yu)

Generally moduli functors

F: Schip -> Sets are not representable by schemes, due to existence of non-trivial automorphisms of objects being classified by F.

Ex. M11: Scho Sets

THOSElliptic curves}/~

is not representable by a scheme, for instance, due to non-injectivity of  $M_{1,1}$  (Spec(C(t)))  $M_{1,1}$  (Spec(C(t)))  $M_{1,1}$  (Spec(C(t))) where two elliptic curves  $Y_z^2 = X^3 - t Z^3$  and  $Y_z^2 = X^3 - Z^3$  are non-isom. over

t(t) but isomorphic over t(t16).

To remedy this situation of non-representability, one allows
non-representability, one allows
LI CAMPADMISMS OF ODJECTS CO DE
be conded, thereby forming a 2 juice.
M <sub>1,1</sub> : Schop -> Grpd
TIN Elliptic curves
THY (Elliptic curives)
This torms a Stack. 10 avoid
a - cateagrical language, one
instead forms a category fibered
on groupoids (CFG)
on groups
$P: \mathcal{M}_{1,1} \longrightarrow Sch_{\mathcal{C}}$
(T, Elliptic ) H T
The defining condition for a stack
The detining conduction is that
P: M -> (Schk) fppf is that  P: M (Schk) fppf  W(T) the pre-
P: M TE Schk P X, y & M(T), the pre- (1) Y TE Schk P X, y & M(T), the pre- sheaf Hom (X, y) on T is a sheaf;
sheaf Hom (x, y) on 115 as silvery
(ii) Descent data is effective.
These two conditions can be
These wo

summed up as an equivalence of categories  $\mathcal{M}(T) \longrightarrow \mathcal{M}(T' \xrightarrow{f} T)$  $\chi \mapsto f^*\chi$ where T' => T is a fppf, swijective morphism and M(T/T) is a category nat wally encapsulating descent data w.r.t. T'+>T. (YONEDA) An equivalence Homstack, TM) ~MCT) GEOMETRY ON STACKS DEFN A Stack P: M-> Schk is said to be Deligne-Mumford (DM) if (i) the diagonal M -> M × M is representable by schemes, and is quasi-compact and separated. (i') Hom (x,y) is rep'ble by a quasicompact and separated scheme

X x,y & M(T) (=)(i") Any X -> M is replace by scheme and is quasi compact and separated (11) I a k-scheme U and an étale, surjective  $U \rightarrow M$  ( U is called an atlas)

Ex. M1,1 is a DM-stack. Proving it's a DM stack is quite non-trivial. We'll see below another example of a DM stack, namely quotient stacks, where it easier to show this.

## \$2. QUOTIENT STACKS

DEFN: ((PRINCIPAL) G-BUNDLE) Let Y be an S-scheme, G a smooth, affine Y-group scheme. Then a (principal) G-bundle is a pair of morphisms ( T.P → Y, T:G × P → P) s.t. It is flat, locally finitely presented and swijective, and

(i) G×G×P(id,p)G×P commutes;

(ii) P - G x P P is identity (iii) G × P (P, Pr2) P × P is an isom. Example: (Trivial G-bundle) (G->Y, P=m:GxG-3G) Fact: Every G-bundle is étale locally trivial. DEF'N: (QUOTIENT STACK) Let X/k be a scheme, G/k a smooth, affine group scheme. Then the quotient stack [X/G] - (Schk) fppf
is the CFG s.t. for any T & Schk

[X/G] (T) = (P - T a G-bundle)

[X/G] (T) = (P - X a G-equivariant)

map For Grétale, [X/Gr] is a DM stack -Effectiveness of descent data

This follows from the equivalence  $\{G-bundles\} \longleftrightarrow \{G-tonsors\}$  and the effectiveness of certain sheaves descent data for the with a suitable stack of sheaves.

- Representability of the diagonal Sufficient to show that for scheme T and P1, P2 & [X/G](T), the sheaf Hom (P1, P2) is representable by a scheme. Enough to show this étale locally, where there are isomorphisms of: P1 = GT, 2: P2 - GT, fre have P1: GT -XT, P2: GT XT. the corresponding G-equivariant maps Consequently, Hom (P2, P2) is rep'ble by the fiber product?, ?= GT 1(P,(e), P2)  $X_T \xrightarrow{X} X_T \stackrel{*}{\uparrow} X_T$ 

- An étale, surjective atlas Let X -> [X/G] be the map corresponding to the trivial Gibundle. Then we claim this is the required atlas. Indeed, if T -> [X/G,] corresponds to a Gipbundle T.P - Tand P:P->X, then one can easily show a contesian diagram P -> X Since P->T

Is étale4

surjective, T -> [X/G] we're done Example -ii)Let X = Spec (A), for a finite type k-algebra A and G be a finite group acting on A. From the fact that G-bundles

on Spec (K), for an algebraic closure k c, k, are trivial, we conclude [Spec(A)/G]( $\overline{k}$ ) = {G-equivariant maps}  $G \rightarrow Spec(A)(\overline{k})$  } In particular,  ${\rm Isom.\ classes}$  in  ${\rm [Spec\,(A)/G](K)}$ J-1 Spec(A)(R)/G

(orboits of Gi-)

(action But we know from the work of Mumford that Spec (AG)(K) = Spec (A)(K)/G. => Spec(AG) = Spec(A)/G is a coarse moduli space for [Spec(A)/G] (see [Ch.6 & 11, Olsson]). Warning. It is not true in general that Spec (AG) represents

[Spec(A)/G] (happens when the G-action on Spec (A) has non-trivial isotropy groups). This is showcased in the following simple example: let  $G = \mathbb{Z}_{(2)}$  act on  $X = \operatorname{Spec}(k[T,Y])$ via (T,Y) +→(Y,T)-Exercise (!) Show that X/G ~ A'k. To show that [X/G] is not representable by 1/6 = Ax, it suffices to show (X/G) is not a smooth stack. Indeed, by what we saw previously, the atlas X - [X/G] is an étale map. But since X

is not smooth (being union of two lines), [X/G,] is also not smooth.

(ii) The stack of elliptic curves

M1,1 is also a DM stack

and admits the coarse moduli space IH/ (see Talk 1) via

the j-invariant of elliptic curves.

(see also [Introduction, Olsson]).

The pattern above is encaptured by the following deep result, which will not used by us in this seminar.

Theorem (Keel-Mori). A DM-stack with finite diagonal (for instance, a separated stack) admits a coarse moduli space.

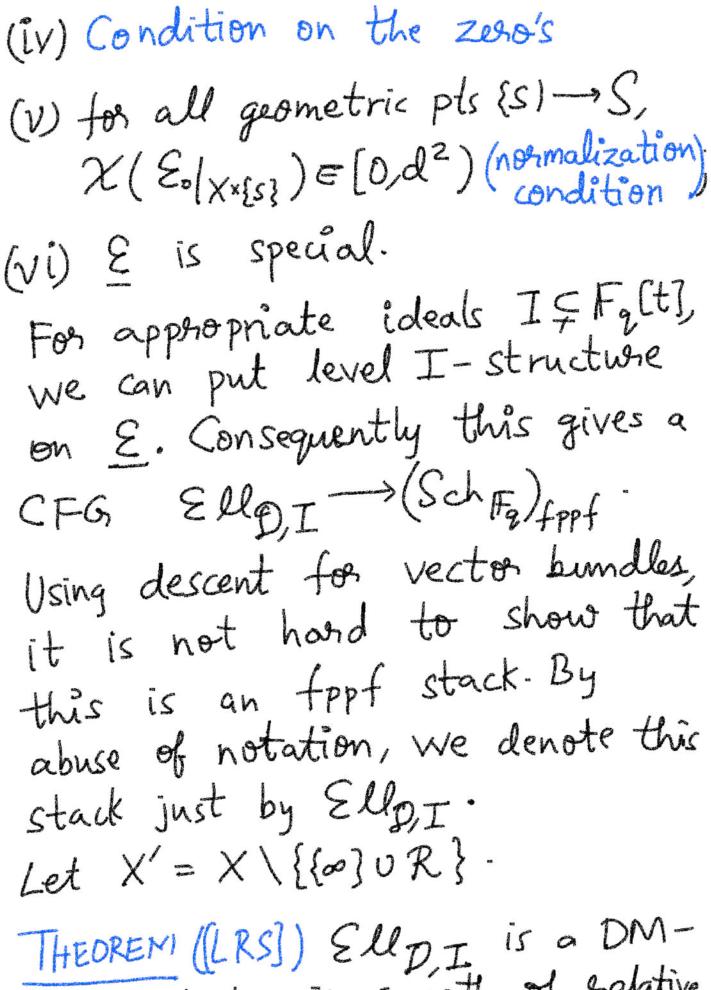
§3. The stack Ellp, I-

Let D be a central division algebra of dimension  $d^2$  over algebra of dimension  $d^2$  over  $F=F_q(t)$ . Let  $X=P_{F_q}^1$  and D be the locally free  $O_X$ -algebra s.t.

Dy = D (generic fiber) and Dx = D& Oxx C Fx (xeX) is a maximal order. Recall the definition of a D-elliptic sheaf &= (Ei, Ji, ti)iell over an Fq-scheme S. (for the full definition see the notes of Talk 2): E; are locally free Oxxs-modules of rank de with an action by D s.t. (i) Si Ji Si+1

Tti-1 Tti

Commundes τε<sub>14</sub> τε<sub>1</sub> ; (ll)  $\varepsilon_{i} \rightarrow \varepsilon_{i+1} \rightarrow \cdots \rightarrow \varepsilon_{i+d} = \varepsilon_{i}(\infty)$ is the canonical injection; (iii) Condition on the pole oo



THEOREM ([LRS]) Ellp, I is a DM-stack which is smooth of relative dimension d-1 over X'\I. Moreover,

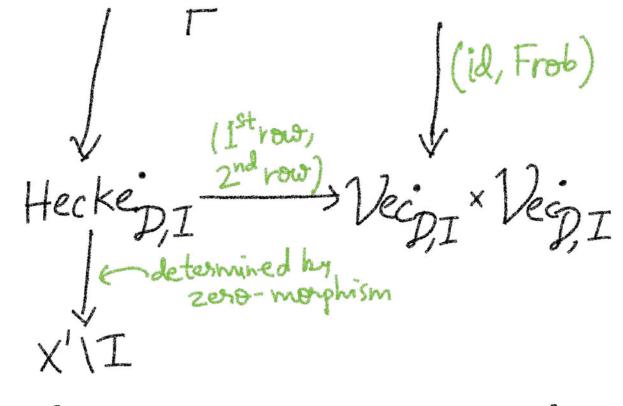
if  $I \neq \emptyset$ , then  $\mathcal{E}ll_{D,I}$  is representable by a smooth, projective scheme over  $X' \mid I$ .

Remark. Note that (LRS) donot impose the conditions (w, (vi) above for a D-elliptic sheaf. From the work of Spiess, (VI) is equivalent to (iv), and (v) being a normalization condition, we get EllDI is the quotient by Z of the corresponding stack in [LRS]. The proof of the above theorem seems to work even after the quotient by Z.

Sketch of a proof

The key to the proof is the following Cartesian diagram of Stacks:

 $\mathcal{E}_{p,I} \longrightarrow \mathcal{V}_{p,I}$ 

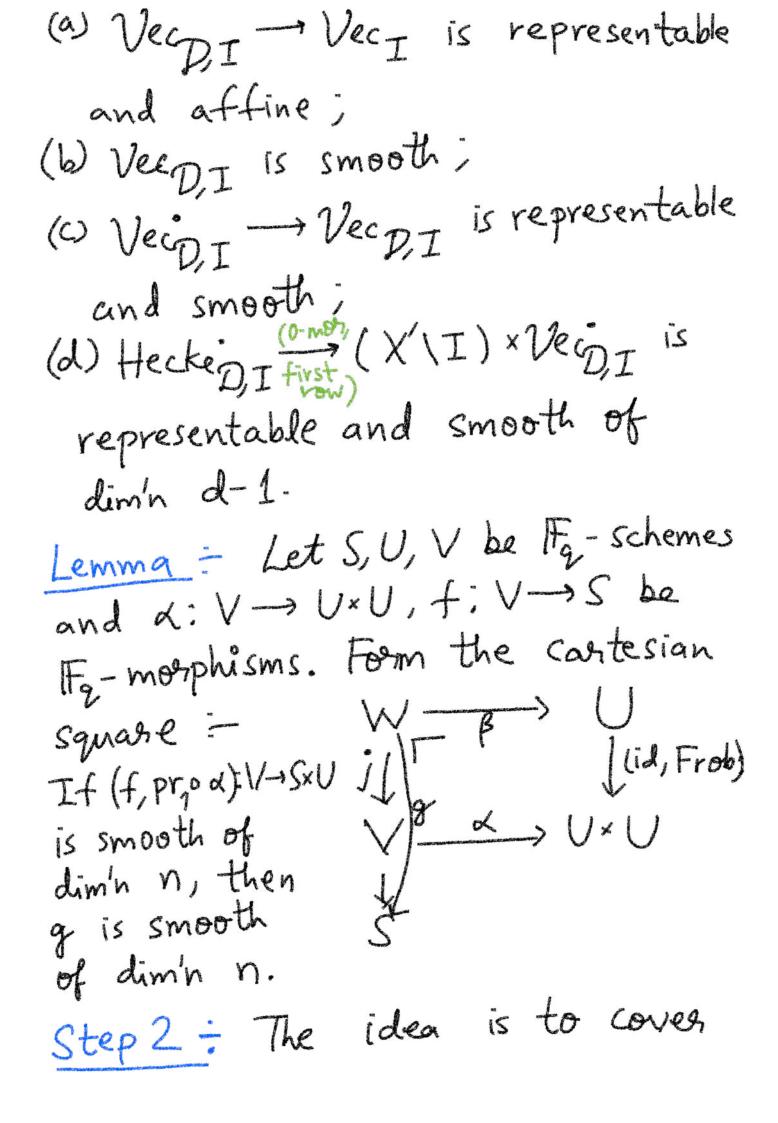


as in the defh of D-ell-sheaves, together with conditions (ii) & (ii) and level I - structure.

together with (ii),(11),(11),(11) and level I-structure.

Step 1. Let Vec<sup>st</sup> denote the stack classifying rank d2 vectors bundles & which are Seshadri-I-

stable, i-e- & has a level I - structure and for all subbundles FCE, deg(F) - deg(I) < deg(E) - deg(I)tk(E)rk(F) ([Seshadri, 4-I. Définition 2]) If deg(I)>0 then VecI is representable by a disjoint union of smooth quasi-projective schemes. Let now Ellpi denote D-ell. sheaves (E) s.t. Eo is Seshadoù-Istable. Then we can conclude using the following, as well as the Cartesian diagram above that Ellet is a disjoint union of smooth quasi-projective schemes if deg(I)>0.



EllD, I by finite quotients of EllD, I for JPI. Consider an ideal J?I s.t. deg(J)>0. f VJ.I: EllD,J→EllD,I be the quotient of the level J-str. by I. This is a  $G_{I,J} := \text{Ker}([\mathcal{O}_{D/J})^* \longrightarrow (\mathcal{O}_{D/I})^*)^$ torson. Hence, EllDJ C EllD, I being stable under the action of GI, J, we get a smooth, open. OM-substack [EUD, J/GI, J] C EUDI. Since any vector bundle is Seshadri- J-stable for deg(I)>0 such finite quotient stacks cover EllD, I and we're done. 10

## §4- REFERENCES

[OLSSON] Algebraic spaces of Stacks. [LRS] D-elliptic sheaves and the local Langlands correspondence. [Seshadri] Fibrés vectoriels sur les courbes algébriques.