01/08/24

Module – 05 :-

First order Recurrence Relations:-

First, we consider for solution of recurrence relation of the form $a_n = C \cdot a_{n-1} + f(n)$, where C is the constant and f(n) is known function. Such a relation is called a linear recurrance relation of first order with constant coefficient. If $f(n) = 0$, the relation is called Homogeneous otherwise it is called Non-Homogeneous.

Note :-

* if $f(n)=0$ that is if the recurriance relation is homogeneous then solution will be $a_n = c^n a_0$.

Problems :-

① For $a_{n+1} = 4a_n$, $n \geq 0$ given $a_0 = 3$.

Soln :- Given $a_{n+1} = 4a_n * f(n)$

replace n by n-1

$$\boxed{a_n = 4a_{n-1}}$$

$$a_n = 4^n * a_0$$

$$\boxed{a_n = 3 * 4^n}$$

$$[a_n = c^n a_0].$$

② For $a_n = 7a_{n-1}$, $n \geq 1$ given that $a_2 = 98$

Soln :- Given :-

$$a_n = 7a_{n-1}$$

$$a_n = 7^n a_0 \qquad [a_n = c^n a_0]$$

$$a_2 = 7^2 a_0$$

$$98 = 7^2 * a_0$$

$$a_0 = \frac{98}{49}$$

$$\boxed{a_0 = 2}$$

$$a_n = C^n a_0$$

$$\boxed{a_n = 7^n * 2}$$

(3) If a_n is a solution of the recurrence relation $a_{n+1} = k\, a_n$, for $n \geq 0$ and $a_3 = \frac{153}{49}$, $a_5 = \frac{1377}{2401}$ what is 'k'?

Sol^n:-

$$a_n = k^n a_0$$

$$\boxed{a_3 = k^3 a_0} \rightarrow (1) \text{ and}$$

$$\boxed{a_5 = k^5 a_0} \rightarrow (2)$$

$$\frac{a_5}{a_3} = \frac{k^5 \cancel{a_0}}{k^3 \cancel{a_0}}$$

$$\frac{a_5}{a_3} = \frac{k^5}{k^3}$$

$$\frac{a_5}{a_3} = k^{5-3}$$

$$\frac{a_5}{a_3} = k^2$$

$$k^2 = \frac{1377/249}{153/49}$$

$$\boxed{k = \pm \frac{3}{7}}$$

(4) Find recurrence relation and initial condition for the sequence 2, 10, 50, 250. Hence find general term of the sequence.

Soln :-

$a_0 = 2, \quad a_1 = 10, \quad a_2 = 50, \quad a_3 = 250.$

$a_n = c^n a_0$

$a_n = 5^n a_0.$

$a_1 = 10 = 2 \times 5 = 5 * a_0.$

$a_2 = 50 = 2 \times 5^2 = 5^2 * a_0$ $\qquad [\therefore a_n = c^n * a_0].$

$a_3 = 250 = 2 \times 5^3 = 5^3 * a_0$

$$\boxed{a_n = 2 * 5^n}$$

$\boxed{a_0 = 2}$

$\boxed{c = 5}$

(5) The number of virus affected files in a system is 1000 (to start with) and this increases 250% every 2 hours. Use a recurrence relation to determine number of virus affected files in a system after 1 day.

Soln :- Given $a_0 = 1000$ → Initial virus infected files be $a_0 = 1000$

let a_n denote the number of virus affected files after $2n$ hours. Then the number increases by $a_n * \frac{250}{100}$. & increases 250% every 2 hrs so, after 2 hr infected files are

After $2n+2$ hour the number will be

$a_{n+1} = a_n + a_n * \frac{250}{100}$ $\qquad a_2 = a_0 + \frac{250}{100} a_0$

$a_{n+1} = a_n + a_n (2.5)$ $\qquad a_2 = 3.5 a_0$

$\boxed{a_{n+1} = 3.5 a_n}$

replace n by $n-1$

$a_n = 3.5 a_{n-1}$

$a_n = (3.5)^n a_0$

$a_n = (3.5)^n * 1000$

$\boxed{x_2 = 3.5 x_0}$

So, the general sum be $x_n = 3.5 a_{n-2}$

$\therefore \boxed{a_n = c^n a_0}$

$n=12$ (because 24 hours should be grouped by 2 hours each so 12 hours)

$a_{12} = (3.5)^{12} * 1000$

$\boxed{a_{12} = 3,37,92,20,508}$

(6) A person invest ₹ 10,000 at 10.5 % interest (per year) compounded monthly. Find and solve recurrence relation for the value of investment at the end of 'n' months. what is the value of investment at the end of the 1st year. How long will it take to double the investment.

Soln :- 10.5 % (per year)

$\frac{10.5}{12} \% = 0.875\%$

0.875% per month $= 0.00875$

$S_0 = 10,000$ let $S_1, S_2, \ldots S_n$

For each month

$S_1 = S_0 + S_0 * (0.00875)$

$\boxed{S_1 = (1.00875) S_0}$

$S_2 = S_1 + (0.00875) S_1$

$$\boxed{S_2 = (1.00875) S_1}$$

$S_3 = S_2 + (0.00875) S_2$

$$\boxed{S_3 = (1.00875) S_2}$$

$\vdots$

$$\boxed{S_n = (1.00875) S_{n-1}}$$

$a_n = c^n a_0$

similarly

$S_n = c^n S_0$.

$S_n = (1.00875)^n S_0$

$\therefore$ for $n = 12$

$S_{12} = (1.00875)^{12} * 10000$

$$\boxed{S_{12} = 11.102}$$

In how many months will it double the investment

$S_n = 2 * S_0$.

$(1.00875)^n \cancel{S_0} = 2 * \cancel{S_0}$.

$(1.00875)^n = 2$.

Apply log on b/s

$\log (1.00875)^n = \log 2$

$n \log (1.00875) = \log 2$.

$$n = \frac{\log 2}{\log (1.00875)}$$

$$\boxed{n = 80}$$

$\therefore$ 80 months it will take to double the investment.//

Second Order Homogeneous Recurrence Relations

$$C_n a_n + C_{n-1} a_{n-1} + C_{n-2} a_{n-2} = 0 \longrightarrow (1), \quad n \geq 2$$

where C_n, C_{n-1} and C_{n-2} are constants with $C_n \neq 0$.

A Relation of this type is called second order linear Homogeneous Recurrence Relation with constant coefficient.

We seek a solution of relation 1 in the form $a_n = Ck^n$ where $C_0 = 0$ and $k_0 = 0$. Substituting in equation 1, reduces to

$$C_n(Ck^n) + C_{n-1}(Ck^{n-1}) + C_{n-2}(Ck^{n-2}) = 0$$

$$Ck^{n-2}(C_n k^2 + C_{n-1}k + C_{n-2}) = 0 \quad x_1 \quad x_2$$

$$\Rightarrow \boxed{C_n k^2 + C_{n-1}k + C_{n-2} = 0} \quad // \quad Ae^{x_1} + Be^{x_2}$$

Thus, $\boxed{a_n = Ck^n}$ is a solution of 1 if k satisfies equation 2. This Quadratic Equation is called auxilary equation (or) characteristic equation.

$A(x_1)^n + B(x_2)^n$

Case 1:-

Two roots k_1 and k_2 of equation 2 are real and distinct, then solution is $\boxed{a_n = Ak_1^n + Bk_2^n}$

Case 2:-

If k_1 and k_2 are two roots of equation 2 are real and equal, then solution is $\boxed{a_n = (A + Bn)k_1^n}$

(or)

$$\boxed{a_n = (A + Bn)k_2^n}$$

Case 3:-

Two roots k_1 and k_2 of equation 2 are complex, then k_1 & k_2 are complex conjugate to each other. If $k_1 = p + iq$ then $k_2 = p - iq$. Then $\boxed{a_n = r^n(A\cos n\theta + B\sin n\theta)}$.

where A and B are arbitrary real constants

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① Solve the Recurrance relation $a_n + a_{n-1} - 6a_{n-2} = 0$ $n \geq 2$
given that $a_0 = -1$, $a_1 = 8$.

Soln:-

$k^2 + k - 6 = 0$

$-6k^2$

$-2k \; +3k$

$\boxed{k = -3}$ $\boxed{k = 2}$

$\boxed{a_n = A(-3)^n + B \cdot 2^n}$

$n = 0$, $a_0 = A + B \Rightarrow A + B = -1$

$n = 1$, $a_1 = A(-3) + B(2) \Rightarrow -3A + 2B = 8$

$\boxed{a_n = -2(-3)^n + 2^n}$

$3A + 3B = -3$
$-3A + 2B = 8$
$5B = 5$

$\boxed{B = 1}$

② Solve the Recurrance relation $a_n = 3a_{n-1} - 2a_{n-2}$ for $n \geq 2$, given that $a_1 = 5$, $a_2 = 3$.

Soln:

$a_n - 3a_{n-1} + 2a_{n-2} = 0$.

$k^2 - 3k + 2 = 0$.

$+2k^2$

$-2k \; -k$.

$k^2 - 2k - k + 2 = 0$

$k(k-2) - (k-2) = 0$.

$(k-2)(k-1) = 0$

$k = 1$ (or) $k = 2$

$\Rightarrow a_n = A(1)^n + B(2)^n$

$a_n = A + B(2)^n$

$n = 1$, $a_1 = A + B \times (2)^1$

$\Rightarrow A + 2B = 5$

$n = 2$, $a_2 = A + B(2)^2$

$\Rightarrow A + 4B = 3$

$\boxed{B = -1}$ $\boxed{A = 7}$

$\boxed{a_n = 7 - (2)^n}$

③ Solve the Recurrance Relation $F_{n+2} = F_{n+1} + F_n$
$n \geq 0$ given that $F_0 = 0$, $F_1 = 1$.

Sol^n:- $F_{n+2} = F_{n+1} + F_n$

$F_{n+2} - F_{n+1} - F_n = 0$

$F_n - F_{n-1} - F_{n-2} = 0$. $\therefore C_n a_n + C_{n-1} a_{n-1} + C_{n-2} a_{n-2} = 0$.

$\Rightarrow k^2 - k - 1 = 0$.

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$ax^2 + bx + c = 0$

$a = 1$

$b = -1$

$c = -1$

$$= \frac{1 \pm \sqrt{1+4}}{2 \times 1}$$

$$= \frac{1 \pm \sqrt{5}}{2}$$

$$k_1 = \frac{1+\sqrt{5}}{2}, \quad k_2 = \frac{1-\sqrt{5}}{2}$$

$$\boxed{F_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n}$$

$n = 0$, $F_0 = A\left(\frac{1+\sqrt{5}}{2}\right)^0 + B\left(\frac{1-\sqrt{5}}{2}\right)^0$

$\Rightarrow A + B = 0$

$n = 1$, $F_1 = A\left(\frac{1+\sqrt{5}}{2}\right)^1 + B\left(\frac{1-\sqrt{5}}{2}\right)^1$

$\Rightarrow A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right) = 1$

From above equation,

$A + B = 0$

$\boxed{A = -B}$

$$A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right) = 1$$

$$-B\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right) = 1$$

$$B\left(\frac{-\cancel{1}}{\cancel{2}} - \frac{\sqrt{5}}{2} + \frac{\cancel{1}}{\cancel{2}} - \frac{\sqrt{5}}{2}\right) = 1$$

$$B\left(\frac{-\cancel{2}\sqrt{5}}{\cancel{2}}\right) = 1$$

$$\boxed{B = \frac{-1}{\sqrt{5}}}$$

$$\therefore \boxed{A = \frac{1}{\sqrt{5}}}$$

$$\boxed{F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n}$$

(4) Solve the recurrance relation, $D_n = bD_{n-1} - b^2 D_{n-2}$ for $n \geq 2$ given that $D_1 = b > 0$, $D_2 = 0$.

Soln :- Given :- $D_n = bD_{n-1} - b^2 D_{n-2}$

$$\Rightarrow D_n - bD_{n-1} + b^2 D_{n-2} = 0$$

$$\Rightarrow k^2 - bk + b^2 = 0$$

$$ax^2 + bx + c = 0$$

$a = 1$
$b = -b$
$c = b^2$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\cancel{x = \ldots}$$

$$k = \frac{+b \pm \sqrt{b^2 - 4 * 1 * b^2}}{2 * 1}$$

$$k = \frac{b \pm \sqrt{b^2 - 4b^2}}{2}$$

$$= \frac{b \pm \sqrt{-3b^2}}{2}$$

$$= \frac{b \pm b\sqrt{-3}}{2}$$

$$k = b\left(\frac{1 \pm i\sqrt{3}}{2}\right)$$

$$D_n = r^n (A \cos n\theta + B \sin n\theta)$$

$$z = \frac{b \pm ib\sqrt{3}}{2}$$

$$z = x + iy$$
$$r = \sqrt{x^2 + y^2}$$
$$\theta = \tan^{-1}(y/x)$$

$$x = \frac{b}{2} \quad y = \frac{b\sqrt{3}}{2}$$

$$r = \sqrt{\frac{b^2}{4} + \frac{b^2 * 3}{4}}$$

$$r = \sqrt{\frac{b^2 + 3b^2}{4}}$$

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$$= \sqrt{\frac{\cancel{4} b^2}{\cancel{4}}}$$

$$r = \sqrt{b^2}$$

$$\boxed{r = b}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$= \tan^{-1}\left(\frac{\frac{\cancel{b}\sqrt{3}}{\cancel{2}}}{\frac{\cancel{b}}{\cancel{2}}}\right)$$

$$\theta = \tan^{-1}(\sqrt{3})$$

$$\boxed{\theta = \frac{\pi}{3}} \quad (\text{or}) \quad \boxed{\theta = 60^\circ}$$

$$D_n = b^n \left(A \cos \frac{n\pi}{3} + B \sin \frac{n\pi}{3}\right)$$

$n=1$

$$D_1 = b\left(A \cos \frac{\pi}{3} + B \sin \frac{\pi}{3}\right)$$

$$\cancel{b} = \cancel{b}\left(\frac{A}{2} + \frac{B \times \sqrt{3}}{2}\right)$$

$$\boxed{A + \sqrt{3}B = 2} \longrightarrow (1)$$

$n=2$

$$D_2 = b^2\left(A \cos \frac{2\pi}{3} + B \sin \frac{2\pi}{3}\right)$$

$$0 = b^2\left(A \cos(120^\circ) + B \sin(120^\circ)\right)$$

$$= b^2\left(A \cos(90^\circ + 30^\circ) + B \sin(90^\circ + 30^\circ)\right)$$

$$0 = b^2\left(A \times \left(-\frac{1}{2}\right) + B \frac{\sqrt{3}}{2}\right)$$

$$\Rightarrow -\frac{A}{2} + \frac{\sqrt{3}B}{2} = 0$$

$$\Rightarrow \boxed{-A + \sqrt{3}B = 0} \longrightarrow (2)$$

$$A + \sqrt{3}B = 2$$

$$-A + \sqrt{3}B = 0$$

$$2\sqrt{3}B = 2$$

$$\boxed{B = \frac{-1}{\sqrt{2}}}$$

from ②, $-A+\sqrt{3}B=0$.

$$-A+\sqrt{3}*\frac{1}{\sqrt{3}}=0.$$

$$-A+1=0$$

$$\Rightarrow \cancel{-}A=\cancel{-}1$$

$$\boxed{A=1}$$

$$\therefore \boxed{D_n = 6^n \quad \cos n\frac{\pi}{3} + \frac{1}{\sqrt{3}} \sin n\frac{\pi}{3}}$$

Non homogeneous Recurrance Relation of Second Orders:-

$$C_n a_n + C_{n-1} a_{n-1} + C_{n-1} a_{n-2} + \cdots + C_{n-k} a_{n-k} = f(n) \rightarrow ①$$

$f(n)=0$ [homogeneous]

where $C_n, C_{n-1}, C_{n-2} \ldots C_{n-k}$ are real constant with $C_n \neq 0$. and $f(n)$ is a even, $f(n) \neq 0$ [Non-homogeneous] $f(n)$ is a real valued function of n.

A general solution of the Recurriance Relation ① is given by,

$$\boxed{a_n = a_n^{(h)} + a_n^{(P)}}$$

where $a_n^{(h)}$ is the general solution of homogeneous part of the recurriance relation ①. Namely, the Relation ① with $f(n)=0$. and $a_n^{(P)}$ is any particular solution of the Relation ①.

Case ①:

Suppose $f(n)$ is a polynomial of degree q then $a_n^{(P)}$ is taken in the form $a_n^{(P)} = A_0 + A_1 n + A_2 n^2 + \cdots + A_q n^q$.

where $A_0, A_1, A_2, \ldots, A_q$ are constants. To be evaluated using the fact that $a_n = a_n^{(P)}$ to satisfy the relation ①.

Problems:-

① Solve the recurriance relation, $a_n - a_{n-1} - 2a_{n-2} = 4$ given $a_0 = 0$, $a_1 = -1$

Soln:

$$a_n = a_n^{(h)} + a_n^{(P)}$$

$$a_n - a_{n-1} - 2a_{n-2} = 0.$$

$\Rightarrow k^2 - k - 2 = 0.$ $\qquad -2k^2$

$\Rightarrow k^2 - 2k + k - 2 = 0.$ $\qquad -2k \quad 1k$

$k(k-2) + (k-2) = 0$

$(k-2)(k+1) = 0.$

$\therefore \boxed{k=2}$ (or) $\boxed{k=-1}$

$$\boxed{a_n^{(h)} = A(2)^n + B(-1)^n}$$

Since $f(n)$ is a polynomial of degree 0 (zero).

$\therefore a_n^{(p)} = A_0$

Substituting in equation.

$\cancel{A_0} - \cancel{A_0} - 2A_0 = 4$

$-2A_0 = 4$

$\cancel{A_0 = -4/3}$

$\boxed{A_0 = -2}$

$\boxed{a_n^{(p)} = -2}$

$a_n = a_n^{(h)} + a_n^{(p)}$

$$\boxed{a_n = A(2)^n + B(-1)^n - 2}$$

$\underline{n=0}$

$a_0 = A + B - 2$

$0 = A + B - 2$

$\boxed{A + B = 2}$

$\underline{n=1}$

$a_1 = A(2) + B(-1) - 2$

$-1 = 2A - B - 2$

$\boxed{2A - B = 1}$

$A + \cancel{B} = 2$

$2A - \cancel{B} = 1$

$\overline{3A = 3}$

$A = \frac{3}{3}$

$\boxed{A=1}$

from

$$2A - B = 1$$
$$2(1) - B = 1$$
$$2 - B = 1$$
$$\boxed{B = 1}$$

$$\therefore \boxed{a_n = 2^n + (-1)^n - 2}$$

(2) Solve the Recurrance Relation, $a_{n+2} + 4a_{n+1} + 4a_n = 7$ given $a_0 = 1$ & $a_1 = 2$.

Soln:-

$$a_n = a_n^{(h)} + a_n^{(P)}$$
$$a_{n+2} + 4a_{n+1} + 4a_n = 0$$

~~$a_{n+2} + 4a_{n+1} + 4a_n = 0$~~

$$k^2 + 4k + 4 = 0$$
$$k^2 + 2k + 2k + 4 = 0$$

$4k^2$: $+2$, $+2$

$$k(k+2) + 2(k+2) = 0$$
$$k+2 = 0, \quad k+2 = 0$$
$$\boxed{k = -2}, \quad \boxed{k = -2}$$

$$a_n = (A + Bn)k_1^n$$
$$a_n = (A + Bn)(-2)^n$$
$$\boxed{a_n^{(h)} = (A + Bn)(-2)^n}$$

$$4a_n + 4a_{n+1} + a_{n+2} = 0$$
$$4k^2 + 4k + 1 = 0$$
$$4k^2 + 2k + 2k + 1 = 0$$
$$2k(2k+1) + 1(2k+1) = 0$$
$$2k+1 = 0, \quad 2k+1 = 0$$
$$2k = -1, \quad 2k = -1$$
$$k = -\frac{1}{2}, \quad k = -\frac{1}{2}$$

$4k^2$: $+2$, $+2$

$$a_n^{(P)} = A_0$$
$$A_0 + 4A_0 + 4A_0 = 7$$
$$9A_0 = 7$$
$$\boxed{A_0 = 7/9}$$
$$\boxed{a_n^{(P)} = 7/9}$$

$$a_n = (A + Bn)(-2)^n + 7/9$$

$$n = 0, \quad a_0 = (A + B(0))(-2)^0 + 7/9$$
$$1 = A(1) + 7/9$$
$$1 = A + 7/9 \Rightarrow A + 7/9 = 1 \Rightarrow A = 1 - 7/9$$
$$\boxed{A = 2/9}$$

$$n = 1, \quad a_1 = (A + B(1))(-2)^1 + 7/9$$
$$2 = (A + B)(-2) + 7/9$$
$$2 = \left(\frac{2}{9} + B\right)(-2) + 7/9$$
$$2 = \frac{2}{9}(-2) + B(-2) + 7/9$$
$$2 = -\frac{4}{9} - 2B + 7/9$$

$$2 = -2B + 7/9 - 4/9$$

$$2 = -2B + 3/9$$

$$-2B + \frac{3}{9} = 2$$

$$-2B = 2 - 3/9$$

$$-2B = \frac{15}{9}$$

$$B = \frac{15}{9 \times -2}$$

$$B = \frac{\cancel{15}\ 5}{-\cancel{18}\ 6}$$

$$\boxed{B = -\frac{5}{6}}$$

$$\therefore \boxed{a_n = \frac{2}{9} + \left(\frac{-5}{6}\right) n (-2)^n + 7/9}$$

Non Homogeneous Second order

Case (2):-

(1) $a^2_{n+2} - 5a^2_{n+1} + 6a^2_n = 7n$ for $n \geq 0$, $a_0 = a_1 = 1$

Soln:-

$$a_n = a_n^{(h)} + a_n^{(p)}$$

$$k^2 - 5k + 6 = 0$$

$+6k^2$

$-2 \quad -3$

$$k^2 - 2k - 3k + 6 = 0$$

$$k(k-2) - 3(k-2) = 0$$

$$k-2=0, \; k-3=0$$

$$k=2 \qquad k=3$$

$$a_n = A(k_1)^n + B(k_2)^n$$

$$\boxed{a_n^{(h)} = A(2)^n + B(3)^n}$$

$$a_n^{(p)} \Rightarrow A_0 + A_1(n+2) - 5(A_0 + A_1(n+1)) + 6(A_0 + A_1 n) = 7n$$

$$\Rightarrow A_0 + A_1 n + 2A_1 - 5A_0 - 5A_1 n - 5A_1 + 6A_0 + 6A_1 n = 7n$$

$$\cancel{\Rightarrow 2A_0 + 2A_1 n - 3A_1 = 7n + 0}$$

~~Comparing the coefficients, we get~~

$$(2A_0 - 3A_1) + 2A_1 n = 7n + 0$$

$2A_1 = 7$; $2A_0 - 3A_1 = 0$

$A_1 = \frac{7}{2}$

$$2A_0 - 3 \cdot \frac{7}{2} = 0$$

$$2A_0 - \frac{21}{2} = 0$$

$$2A_0 = \frac{21}{2}$$

$$\boxed{A_0 = 21/4}$$

b_n (or) $a_n^{(P)} = \frac{21}{4} + \frac{7}{2} n$

$$a_n = A(2)^n + B(3)^n + \frac{21}{4} + \frac{7}{2} n$$

$n = 0$

$$a_0 = A(2)^0 + B(3)^0 + \frac{21}{4} + \frac{7}{2}(0)$$

$$1 = A + B + \frac{21}{4}$$

$$A + B + \frac{21}{4} = 1$$

$$A + B = 1 - \frac{21}{4}$$

$$A + B = \frac{-17}{4}$$

$n = 1$

$$a_1 = A(2)^1 + B(3)^1 + \frac{21}{4} + \frac{7}{2}(1)$$

$$1 = 2A + 3B + \frac{21}{4} + \frac{7}{2}$$

$$1 = 2A + 3B + \frac{35}{4}$$

$$2A + 3B + \frac{35}{4} = 1$$

$$2A + 3B = 1 - \frac{35}{4}$$

$$2A + 3B = \frac{-31}{4}$$

$$A + B = \frac{-17}{4}$$

$$2A + 3B = \frac{-31}{4}$$

$$\cancel{A + B = -4.25}$$

$$\cancel{2A + 2B = -7.75}$$

$$A = -5 \qquad B = \frac{3}{4}$$

$$\boxed{a_n = (-5)(2)^n + \frac{3}{4}(3)^n + \frac{21}{4} + \frac{7}{2}n}$$

Method of generating ~~fractions~~ function:-

$$a_n = Ca_{n-1} + F(n) \longrightarrow (1)$$

Equivalently, $a_{n+1} = Ca_n + \phi(n)$ $\quad (\because \phi(n) = F(n+1))$

$\longrightarrow (2)$

$f(x) = \sum_{n=0}^{\infty} a_n x^n$, $\quad f(x) = \dfrac{a_0 + x\,g(x)}{1 - Cx}$

$$g(x) = \sum_{n=0}^{\infty} \phi(n) x^n$$

(1) Find generating function for the recurrance relation, $a_{n+1} - a_n = 3^n$, $n \geq 0$ with $a_0 = 1$. Hence solve the equation.

Soln:

Given:-

$$a_{n+1} - a_n = 3^n$$

$$a_{n+1} = 1a_n + 3^n$$

$$a_{n+1} = Ca_n + \phi(n)$$

$$C = 1, \quad \phi(n) = 3^n, \quad a_0 = 1$$

$$f(x) = \frac{a_0 + x\,g(x)}{1 - Cx}$$

$$\boxed{f(x) = \frac{1 + x\,g(x)}{1 - x}}$$

$$g(x) = \sum_{n=0}^{\infty} \phi(n) x^n$$

$$= \sum_{n=0}^{\infty} 3^n x^n$$

$$= \sum_{n=0}^{\infty} (3x)^n$$

$g(x) = (1-3x)^{-1} = \frac{1}{1-3x}$

Note:

$(1-x)^{-1} = 1 + x + x^2 + \dots + x^n + \dots$

$= \sum_{n=0}^{\infty} x^n$

$f(x) = \frac{1 + \frac{x}{(1-3x)}}{1-x}$

$= \frac{(1-3x)+x}{(1-3x)(1-x)}$

$(1-3x)^{-1} = 1 + 3x + (3x)^2 + \dots + (3x)^n + \dots$

$= \sum_{n=0}^{\infty} (3x)^n$

$$\boxed{f(x) = \frac{1-2x}{(1-3x)(1-x)}}$$

$\frac{1-2x}{(1-3x)(1-x)} = \frac{A}{(1-3x)} + \frac{B}{1-x}$

$\Rightarrow 1-2x = \frac{A(1-3x)(1-x)}{(1-3x)} + \frac{B(1-3x)(1-x)}{(1-x)}$

$1-2x = A(1-x) + B(1-3x)$

To make B as 0 put $x = 1/3$

$1-3x=0$

$x = \frac{1}{3}$

$1 - 2\times\frac{1}{3} = A\left(1-\frac{1}{3}\right) + B\left(1 - 3\times\frac{1}{3}\right)$

$1 - \frac{2}{3} = A\left(2/3\right)$

$\frac{1}{3} = A\left(2/3\right)$

$$\boxed{A = \frac{1}{2}}$$

To make A as 0. put:- $x=1$

$1-x=0$

$\boxed{x=1}$

$1-2(1) = B(1-3(1))$

$-1 = 0 + B(-2) \Rightarrow -1 = -2B$

$\Rightarrow B = \frac{-1}{-2}$

$B = \frac{1}{2}$

$$\boxed{B = \frac{1}{2}}$$

Substitute A & B.

$= \frac{1/2}{1-3x} + \frac{1/2}{1-x}$ $= \frac{1}{2(1-3x)} + \frac{1}{2(1-x)}$

$= \frac{1}{2}\left(\frac{1}{1-3x} + \frac{1}{1-x}\right)$

$$f(x) = \tfrac{1}{2}\left\{(1-3x)^{-1} + (1-x)^{-1}\right\}$$

$$= \tfrac{1}{2}\left\{\sum_{n=0}^{\infty}(3x)^n + \sum_{n=0}^{\infty}x^n\right\} = \tfrac{1}{2}\left\{\sum 3^n x^n + \sum x^n\right\}$$

$$f(x) = \sum_{n=0}^{\infty}\frac{(1+3^n)}{2}x^n \qquad (3^n+1)x^n \qquad \frac{(1+3^n)x^n}{2}$$

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$\therefore \boxed{a_n = \frac{1+3^n}{2}}$$

(2) Find generating function for the reccurence relation, $a_{n+1} - a_n = n^2$, $n \geq 0$ with $a_0 = 1$. Hence solve the equation.

Soln:-

Given,

$$a_{n+1} - a_n = n^2$$

$$a_{n+1} = a_n + n^2$$

$$a_{n+1} = ca_n + \phi(n)$$

$$c = 1, \ \phi(n) = n^2$$

$$f(x) = \frac{a_0 + x\,g(x)}{1 - cx}$$

$$\boxed{f(x) = \frac{1 + x\,g(x)}{1-x}}$$

$$g(x) = \sum_{n=0}^{\infty} n^2 x^n$$

$$= \frac{x(1+x)}{(1-x)^3}$$

Note:- $\displaystyle\sum_{n=0}^{\infty} n^2 x^n = \frac{x(1+x)}{(1-x)^3}$

$$f(x) = \frac{1}{1-x}\left(1 + \frac{x \cdot x(1+x)}{(1-x)^3}\right)$$

$$f(x) = \frac{1}{1-x}\left(1 + \frac{x^2(1+x)}{(1-x)^3}\right)$$

$$= \frac{1}{(1-x)^4}\left((1-x)^3 + x^2(1+x)\right)$$

$$= \frac{1}{(1-x)^4}\left(\cancel{\quad}\right)$$

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$$= \frac{1-3x+4x^2}{(1-x)^4}$$

$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

$$f(x) = (1-3x+4x^2)(1-x)^{-4}$$

$$= (1-3x+4x^2) \sum_{n=0}^{\infty} \binom{3+n}{n} x^n \longrightarrow (1)$$

a_n = coefficient of x^n from (1)

$$a_n = \binom{3+n}{n} - 3\binom{3+n-1}{n-1} + 4\binom{3+n-2}{n-2}$$

$$a_n = \frac{(n+3)(n+2)(n+1)}{3!} - 3 \cdot \frac{(n+2)(n+1)n}{3!} + 4\frac{(n+1)(n)(n-1)}{3!}$$

$$a_n = \frac{n+1}{6}\left(2n^2-5n+6\right)$$

$$\boxed{a_n = 1 + \frac{1}{6} n(n-1)(2n-1)}$$

③ Find generating function for the recurrance relation, $a_n - 3a_{n-1} = n$, $n \geq 1$ with $a_0 = 1$. Hence solve the equation.

Soln:- Given:- $a_n = 3a_{n-1} + n$

replace n by $n+1$

$$a_{n+1} = 3a_n + (n+1)$$

$$c = 3, \quad \phi(n) = n+1$$

$$f(x) = \frac{a_0 + x\,g(x)}{1-cx} \qquad \boxed{f(x) = \frac{1+x\,g(x)}{1-3x}}$$

$$g(x) = \sum_{n=0}^{\infty} \phi(n)\, x^n$$

$$= \sum_{n=0}^{\infty} (n+1)x^n = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

$$\boxed{g(x) = (1-x)^{-2}}$$

$$f(x) = \frac{1}{1-3x}\left(1 + x * \frac{1}{(1-x)^2}\right)$$

$$= \frac{x + (1-x)^2}{(1-3x)(1-x)^2} = \frac{x + 1 + x^2 - 2x}{(1-3x)(1-x^2)}$$

$$= \frac{x^2 - x + 1}{(1-3x)(1-x^2)}$$

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$$\frac{x^2-x+1}{(1-3x)(1-x)^2} = \frac{A}{(1-3x)} + \frac{B}{(1-x)} + \frac{C}{(1-x)^2}$$

$$x^2-x+1 = A(1-x)^2 + B(1-3x)(1-x) + C(1-3x)$$

put $x = 1/3$

To make B and C as 0.

$$\frac{1}{9} - \frac{1}{3} + 1 = A\left(\frac{2}{3}\right)^2 + 0 + 0$$

$$\frac{1-3+9}{9} = A * \frac{4}{9}$$

$$\frac{7}{9} = A * \frac{4}{9}$$

$$\boxed{A = \frac{7}{4}}$$

put $x=1$

To make A and B as 0.

$$\cancel{1} - \cancel{1} + 1 = 0 + 0 + C(-2)$$

$$\boxed{C = -\frac{1}{2}}$$

$$x^2-x+1 = A(x^2+1-2x) + B(1-x-3x+3x^2) + C(1-3x)$$

$$x^2-x+1 = (Ax^2 + A - 2Ax) + (B - \cancel{Bx} - 4Bx + 3Bx^2) + (C - 3Cx)$$

$$\cancel{\Rightarrow 4C\ \tfrac{5}{4}\ B}.$$

$$1 = A + B + C$$

$$= \frac{7}{4} + B - \frac{1}{2}.$$

$$\Rightarrow 1 = \frac{5}{4} + B \Rightarrow B = 1 - \frac{5}{4}$$

$$\boxed{B = -\frac{1}{4}}$$

$$f(x) = \frac{7/4}{1-3x} - \frac{1/4}{1-x} \quad \frac{-1/2}{(1-x)^2}$$

$$= \frac{7}{4}(1-3x)^{-1} - \frac{1}{4}(1-x)^{-1} - \frac{1}{2}(1-x)^{-2}$$

$$= \frac{7}{4}\sum(3x)^n - \frac{1}{4}\sum x^n - \frac{1}{2}\sum(n+1)x^n$$

$$\boxed{f(x) = \sum x^n \left(\frac{7}{4}3^n - \frac{1}{4} - \left(\frac{n+1}{2}\right)\right)}$$

$$\therefore \boxed{a_n = \frac{7}{4}3^n - \frac{1}{4} - \left(\frac{n+1}{2}\right)}$$

$$(1-3x)^{-1} = \sum_{n=0}^{\infty} (3x)^n$$

$$(1-x)^{-1} = \sum_{n=0}^{\infty} x^n$$

$$(1-x)^{-2} = \sum_{n=0}^{\infty} (n+1)x^n$$

$$(1-x)^{-4} = \sum_{n=0}^{\infty} \binom{3+r}{r} x^n$$

$$\frac{x(1+x)}{(1-x)^3} = \sum_{n=0}^{\infty} n^2 x^n$$