Module - 5

Recurrence Relation

I order Recurrence relation

Consider the recurrence relation $a_n = c \cdot a_{n-1} + f(n)$ for $n \geq 1$ where c - known constant & $f(n)$ - known function. This relation is called as linear recurrence relation of I order with constant coefficient.

If $f(n) = 0$, the relation is called homogeneous otherwise it is called non-homogeneous.

If $f(n) = 0$ the solution of this recurrence relation is of the form $a_n = c^n a_0$ for $n \geq 1$

Solve the recurrence relation $a_{n+1} = 4 \cdot a_n$ for $n \geq 0$, given that $a_0 = 3$

The given relation is homogeneous, comparing with general recurrence relation $(a_n = c \cdot a_{n-1})$, we get $c = 4$

$\therefore$ The solution of this recurrence relation is of the form

$a_n = c^n \cdot a_0 \Rightarrow a_n = 4^n \cdot 3$ for $n \geq 1$

Solve the recurrence relation $a_n = 7 \cdot a_{n-1}$ for $n \geq 0$, given that $a_2 = 98$

The given equation is a homogeneous linear recurrence.

Comparing it with general recursion eq^n, we get $c = 7$.

$\therefore$ The soln of this recurrence eqn is given by

$a_n = 7^n \cdot a_0$

To find a_0 using a_2, substitute $n = 2$ in the soln.

we get $a_2 = 7^2 \cdot a_0$

Given that $a_2 = 98$ $\therefore$ $98 = 49 \cdot a_0 \Rightarrow \boxed{a_0 = 2}$

$\therefore$ Soln is $a_n = 7^n \cdot 2$

The number of virus affected files in a system is 1000 to start with, this increases 250% every 2 hrs. Use a recurrence relation to determine the number of virus affected files in a system after one day.

In the beginning the number of virus affected files is 1000.

i.e $a_0 = 1000$

Let a_n denote the number of virus affected files after 2n hrs, then the no. increases by $a_n \times 250/100$ in next 2 hrs.

Thus, after 2n+2 hrs, $a_{n+1} = a_n + a_n \times 250/100$

$= a_n + 2.5 a_n$

$a_{n+1} = 3.5 a_n$

This recurrence relation is for the no. of virus affected files, solving the relation we get soln of the form $a_n = (3.5)^n a_0$

$\rightarrow a_n = (3.5)^n . 1000$

This gives the no. of virus affected files after 2n hrs. from this we get $n = 12$. $\therefore a_{12} = (3.5)^{12} \times 1000 \Rightarrow a_{12} = 3379220508$

'A' invests 10000 for at 10.5% interest per year compounded monthly. Find & solve the recurrence relation for the value of the investment at the end of n months. What is the value of the investment at the end of 1st year. How long will it take to double the investment for A's marriage.

Since annual interest is 10.5%, the monthly interest comes to

$\frac{10.5\%}{12} = 0.875\% = 0.00875$

Let S_0 denote the investment made initially i.e $S_0 = 10000$ and $S_1, S_2, S_3 \ldots S_n$ denote the value of the investment at the end of 'n' months. Then

$S_1 = 10000 + 10{,}000 \times 0.00875$

$S_1 = S_0 + (0.00875) S_0 \Rightarrow (1.00875) S_0$

$S_2 = S_1 + (0.00875) S_1 \Rightarrow (1.00875) S_1$

$S_3 = (1.00875) S_2$

$\vdots$

$S_n = (1.00875)^n S_0$ ... linear homog...
we got 1^{st} order linear ... of the form $S_n = (1.00...)$
the soln of this eqn is given of the form
This relation gives the value at the end of n months.
To find the amount after 1^{st} year i.e $n=12$ we get
$S_{12} = (1.00875)^{12}(10,000) \approx 11102$
To find how long it will take to double the investment,
$S_n = 2S_0$
WKT $S_n = (1.00875)^n S_0$
$\Rightarrow (1.00875)^n S_0 = 2S_0$
$(1.00875)^n = 2$
$\log(1.00875)^n = \log 2$
$n \cdot \log(1.00875) = \log 2$
$n = \dfrac{\log 2}{\log(1.00875)} \approx 79.6$
Thus, the investment will be doubled in 80 months i.e 6 years & 8 months.

The Bank pays a certain % of annual interests on deposits. Compounding the interests once in 3 months. If a deposit doubles in 6 years & 6 months, what is the annual % of interest paid by the bank.

II Order Homogeneous Recurrence Relation.

Consider a recurrence relation of the form $c_n a_n + c_{n-1} a_{n-1} + c_{n-2} a_{n-2} = 0$ for $n \geq 2$, where c_n, c_{n-1}, c_{n-2} are real constants with $c_n \neq 0$. A relation of this type is called 2 order linear Homogenous recurrence relation with constant coefficients.

We seek a solution for the above relation in the form $a_n = c \cdot k^n$ where c & k are non-zero, substituting this we get $c_n k^2 + c_{n-1} k + c_{n-2} = 0$.

This quadratic eqn is called the auxiliary eqn (or) characteristic eqn for the given relation.

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Solution of II order homogeneous recurrence relation

Consider the II order recurrence relation,

$c_n a_n + c_{n-1} a_{n-1} + c_{n-2} a_{n-2} = 0$ for $n \geq 2$ is converted into Quad. eqn of the form $c_n k^2 + c_{n-1} k + c_{n-2} = 0$.

The solution of this Quad. Eqn gives the soln of II order recurrence relation.

The soln of Quad. Eqn has 3 types.

1) The two roots k_1 & k_2 of the Quad. eqn which are real & distinct then we take $a_n = Ak_1^n + Bk_2^n$ where A & B arbitrary real constants, as the general soln of II order recurrence relation.

2) The two roots k_1 & k_2 of the Quad eqn are real & equal then $a_n = (A+Bn)k^n$ where A & B are arbitrary real constants & the general soln of II order recurrence relation

3) The two roots k_1 & k_2 of the Quad. eqn are complex then k_1 & k_2 are complex conjugate of each other. with $k_1 = p+iq$, $k_2 = p-iq$, then $a_n = r^n(A\cos n\theta + B\sin n\theta)$

where $r = |k_1| = |k_2| = \sqrt{p^2+q^2}$

$\theta = \tan^{-1}(q/p)$

$a_n + a_{n-1} - 6a_{n-2} = 0$ for $n \geq 2$, Given that $a_0 = -1$ and $a_1 = 8$

Comparing the given recurrence relation with general recurrence relation (II order)

$c_n a_n + c_{n-1} a_{n-1} + c_{n-2} a_{n-2} = 0$ for $n \geq 2$ we get

$c_n = 1$, $c_{n-1} = 1$, $c_{n-2} = -6$

$\therefore$ The quad. eqn of given recurrence relation is:

$k^2 + k - 6 = 0$

Solving the above eqn, we get $k_1 = 2$, $k_2 = -3$

Here, the roots of the quad eqn are real & distinct.

$\therefore$ The soln of the form $a_n = A k_1^n + B k_2^n$

$a_n = A(2)^n + B(-3)^n$

To find A & B using the initial condition $a_0 = -1$ & $a_1 = 8$

Substitute $n = 0$ in soln, we get

$a_0 = A(2)^0 + B(-3)^0$

$a_0 = A + B$

$\boxed{A + B = -1}$ — ①

Substitute $n = 1$, we get

$a_1 = A(2)^1 + B(-3)^1$

$\boxed{8 = 2A - 3B}$ — ②

Solving ① & ② we get $\boxed{A = 1}$, $\boxed{B = -2}$

$\therefore$ The soln of the given II order recurrence relation is

$a_n = 2^n - 2(-3)^n$

Solve the recurrence relation $a_n - 6a_{n-1} + 9a_{n-2} = 0$ for $n \geq 2$ given that $a_0 = 5$ and $a_1 = 12$

Comparing the given recurrence relation with general recurrence relation (II order) $c_n a_n + c_{n-1} a_{n-1} + c_{n-2} a_{n-2} = 0$ for $n \geq 2$ we get

$c_n = 1$, $c_{n-1} = -6$, $c_{n-2} = 1$

$\therefore$ The quad eqn of given recurrence relation is

$k^2 - 6k + 9 = 0$

$k^2 - 3k - 3k + 9 = 0$

$k(k-3) - 3(k-3) = 0$

$(k-3)(k-3) = 0 \Rightarrow k_1 = 3$ & $k_2 = 3 \Rightarrow \boxed{k = 3}$

9
$-3 \quad -3$
-6

Here, the roots of the quad eqn are real & equal.

The soln of the form $a_n = (A + Bn)k^n$

$a_n = (A + Bn)3^n$

To find A & B using initial condition $a_0 = 5$, $a_1 = 12$

Substitute n_0 & n_1 in soln, we get

$a_0 = (A + B0)3^0$, $\quad a_1 = (A + B1)3^1$

$\boxed{5 = A}$

$12 = (5 + B)3 \Rightarrow 12 = 15 + 3B$

$-3 = 3B \Rightarrow \boxed{B = -1}$

we got $\boxed{A=5}$ & $\boxed{B=-1}$

$\therefore$ the soln of the given II order recurrence relation is

$$a_n = (5+(-1)n)3^n$$

Solve the recurrence relation $a_n = 2(a_{n-1} - a_{n-2})$ for $n \geq 2$ given that $a_0 = 1$, $a_1 = 2$

Comparing the given recurrence relation with general recurrence relation (II order)

$C_n a_n + C_{n-1} a_{n-1} + C_{n-2} a_{n-2} = 0$ for $n \geq 2$ we get

$\rightarrow C_n = 1, C_{n-1} = -2, C_{n-2} = 2$

$\rightarrow a_n = 2a_{n-1} - 2a_{n-2}$

$a_n - 2a_{n-1} + 2a_{n-2} = 0$

$k^2 - 2k + 2 = 0$

$a = 1, b = -2, c = 2$

$$k = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$k = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)}$$

$$k = \frac{2 \pm \sqrt{4-8}}{2}$$

$$k = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2\sqrt{-1}}{2} = \frac{2 \pm 2i}{2} = \frac{2(1 \pm i)}{2}$$

$\boxed{k_1 = 1+i}$, $\boxed{k_2 = 1-i}$

The soln of form $a_n = r^n (A \cos n\theta + B \sin n\theta)$

$r = |k_1| = |k_2| = \sqrt{p^2+q^2} = \sqrt{1^2+1^2} = \sqrt{2}$ $\Rightarrow \boxed{r = \sqrt{2}}$

$\theta = \tan^{-1}(q/p) = \tan^{-1}(1/1) = \tan^{-1}(1/1) = \pi/4$ $\boxed{\theta = \pi/4}$

$\therefore a_n = r^n (A \cos n\theta + B \sin n\theta)$

$a_0 = -1$,

$n = 0$

$a_0 = r^0 (A \cos(0)\theta + B \sin(0)\theta) = r^0 (A \cos 0 + B \sin 0)$

$a_0 = r^0 (A(1) + B(0))$

$\boxed{+1 = A}$

$n=1$, $a_1 = 2$

$a_1 = r^1 (A \cos 1(\theta) + B \sin(1)\theta)$

$a_1 = r^1 (A \cos \pi/4 + B \sin \pi/4)$ $\quad \therefore \theta = \pi/4$

$a_1 = r^1 (A \, 1/\sqrt{2} + B \, 1/\sqrt{2})$

$a_1 = \frac{r^1}{\sqrt{2}} (A+B)$

$\boxed{a_1 = \frac{r}{\sqrt{2}} (A+B)} \Rightarrow a_1 = \frac{\sqrt{2}}{\sqrt{2}} (A+B) = A+B$

$\Rightarrow 2 = A+B \Rightarrow 2 = +1 + B \Rightarrow \boxed{1 = B}$

$\therefore a_n = \sqrt{2}^n (+1 \cos n\theta + 1 \sin n\theta)$

$a_n = \sqrt{2}^n (+1 \cos n(\pi/4) + 1 \sin n(\pi/4))$

$a_n = \sqrt{2}^n (\cos n\pi/4 + \sin n\pi/4)$

$F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$, $F_0 = 0$, $F_1 = 1$

→ Non-homogeneous recurrence relation

Consider a Ist & higher order linear non-homogeneous recurrence relation with constant coefficient which of the form $c_n a_n + c_{n-1} a_{n-1} + c_{n-2} a_{n-2} + \dots + c_{n-k} a_{n-k} = f(n)$ —(1) for $n \geq k \geq 2$ when $c_n, c_{n-1}, c_{n-2} \dots c_{n-k}$ are real constants with $c_n \neq 0$ & $f(n)$ is a given real valued function of n.

A general solution of the recurrence relation (1) is given by $a_n = a_n^{(h)} + a_n^{(p)}$,

where $a_n^{(h)}$ - general soln of the homogeneous part of relation (1)

$a_n^{(p)}$ - any particular soln of the relation (1)

The part $a_n^{(p)}$ is discussed for some special cases

case 1: Suppose $f(n)$ is a polynomial of deg (q) & 1 is not a root of the characteristic eqn of the homogeneous part of the relation (1), In this case $a_n^{(p)}$ is taken in the form

$$a_n^{(p)} = A_0 + A_1 n + A_2 n^2 + \dots + A_q n^q$$

where $A_0, A_1, A_2 \dots A_q$ are constants to be evaluated by using the fact $a_n = a_n^{(p)}$ satisfies the relation (1) case 2 : Suppose $f(n)$ is a polynomial of deg (q) & 1 is a root of multiplicity m of the characteristic eqn of the homogeneous part of the relation ①. In this case $a_n^{(P)}$ is taken in the form

$$a_n^{(P)} = n^m\left(A_0 + A_1 n + A_2 n^2 + \cdots + A_q n^q\right)$$

where $A_0, A_1, A_2, \ldots A_q$ are constants to be evaluated using the fact that $a_n = a_n^{(P)}$ satisfies the relation ①

case 3 : Suppose $f(n) = \alpha b^n$, where α - constant & b - not root of characteristic eqn of the homogeneous part of the relation ①. Then $a_n^{(P)}$ is taken in the form $\boxed{a_n^{(P)} = A_0 b^n}$ where A_0 - constant to be evaluated using the fact that $a_n = a_n^{(P)}$ satisfies the relation ①

case 4: Suppose $f(n) = \alpha b^n$, where α - constant & b - root of

repetition multiplicity m of the characteristic eqn of the homogeneous part of the relation ①. Then $a_n^{(P)}$ is taken in the form

$$a_n^{(P)} = A_0 n^m b^n$$

where A_0 - constant to be evaluated by using the fact $a_n = a_n^{(P)}$ satisfies the relation ①

Solve the recurrence relation $a_n - a_{n-1} - 2a_{n-2} = 4$ and $a_0 = 0$, $a_1 = -1$

The given eq^n is a non-homogeneous II order recurrence relation. The so/n of this eqn is of the form $a_n = a_n^{(h)} + a_n^{(P)}$.

Comparing the given relation with general II order Non-homogeneous recurrence relation i.e $c_n a_n + c_{n-1} a_{n-1} + c_{n-2} a_{n-2} = f(n), n \geq 2$

We get $c_n = 1$, $c_{n-1} = -1$, $c_{n-2} = -2$ and $f(n) = 4$

∴ The quad. eqn for given recurrence relation is of the form

$c_n k^2 + c_{n-1} k + c_{n-2} = 0$ (calculating $a_n^{(h)}$ i.e homogeneous part)

$$k^2 - k - 2 = 0$$

$$k^2 - 2k + k - 2 = 0$$

Case 2: Suppose $f(n)$ is a polynomial of deg (q) & 1 is a root of multiplicity m of the characteristic eqn of the homogeneous part of the relation ①. In this case $a_n^{(P)}$ is taken in the form

$$a_n^{(P)} = n^m \left(A_0 + A_1 n + A_2 n^2 + \dots + A_q n^q\right)$$

where $A_0, A_1, A_2, \dots A_q$ are constants to be evaluated using the fact that $a_n = a_n^{(P)}$ satisfies the relation ①

Case 3: Suppose $f(n) = \alpha b^n$, where α - constant & b - not root of characteristic eqn of the homogeneous part of the relation ①. Then $a_n^{(P)}$ is taken in the form $\boxed{a_n^{(P)} = A_0 b^n}$ where A_0 - constant to be evaluated using the fact that $a_n = a_n^{(P)}$ satisfies the relation ①

Case 4: Suppose $f(n) = \alpha b^n$ where α - constant & b - root of multiplicity m of the characteristic eqn of the homogeneous part of the relation ①. Then $a_n^{(P)}$ is taken in the form

$$a_n^{(P)} = A_0 n^m b^n$$

where A_0 - constant to be evaluated by using the fact $a_n = a_n^{(P)}$ satisfies the relation ①

Solve the recurrence relation $a_n - a_{n-1} - 2a_{n-2} = 4$ and $a_0 = 0$, $a_1 = -1$

The given eqn is a non-homogeneous II order recurrence relation

The soln of this eqn is of the form $a_n = a_n^{(h)} + a_n^{(P)}$.

Comparing the given relation with general II order Non-homogeneous recurrence relation i.e. $C_n a_n + C_{n-1} a_{n-1} + C_{n-2} a_{n-2} = f(n)$, $n \geq 2$

we get $C_n = 1$, $C_{n-1} = -1$, $C_{n-2} = -2$ and $f(n) = 4$

∴ The quad. eqn for given recurrence relation is of the form

$$C_n k^2 + C_{n-1} k + C_{n-2} = 0 \quad (\text{calculating } a_n^{(h)} \text{ i.e. homogeneous part})$$

$$k^2 - k - 2 = 0$$

$$k^2 - 2k + k - 2 = 0$$

$$k(k-2) + 1(k-2) = 0$$

$$(k+1)(k-2) = 0$$

$$\boxed{k_1 = -1}, \boxed{k_2 = 2}$$

Here, the roots are real & distinct

∴ soln of homogeneous part is written in the form

$$a_n^{(h)} = A k_1^n + B k_2^n$$

$$a_n^{(h)} = A(-1)^n + B(2)^n$$

To find $a_n^{(P)}$,

here $f(n) = 4$

i.e. $f(n) = 4n^0$

here, $f(n)$ is a polynomial of deg '0' & 1 is not a root of the characteristic eqn of the relation. This implies

$$a_n^{(P)} = A_0$$

To find A_0 using the relation $a_n = a_n^{(P)}$,

Substituting in the given relation we get

$$a_n^{(P)} - a_{n-1}^{(P)} - 2a_{n-2}^{(P)} = 4$$

$$A_0 - A_0 - 2A_0 = 4$$

$$-2A_0 = 4$$

$$\boxed{A_0 = -2}$$

$$\Rightarrow \boxed{a_n^{(P)} = -2}$$

∴ The general soln of the given relation is written in the form, $a_n = a_n^{(h)} + a_n^{(P)}$

$$a_n = A(-1)^n + B(2)^n - 2 \quad \text{—} ⊛$$

To find A & B using $a_0 = 0$, $a_1 = -1$

Put $n = 0$ in eqn ⊛

$$a_0 = A(-1)^0 + B(2)^0 - 2$$

$$0 = A + B - 2$$

$$\boxed{A + B = 2} \quad \text{—} ①$$

Put $n = 1$ in eqn ⊛

$$\rightarrow a_1 = A(-1)^1 + B(2)^1 - 2$$

$$-1 = -A + 2B - 2$$

$$\boxed{-A + 2B = 1} \quad \text{—} ②$$

Solving ① & ② we get $A = 1$, $B = 1$

Substitute A & B in eqn ⊛, we get the soln of non-homogeneous II order recurrence relation

$$\boxed{a_n = (-1)^n + (2)^n - 2}$$

Solve the recurrence relation $a_{n+2} + 4a_{n+1} + 4a_n = 7$, $n \geq$
$a_0 = 1$, $a_1 = 2$

Given $a_{n+2} + 4a_{n+1} + 4a_n = 7$
Replace n with $n-2$

$$a_{(n-2)+2} + 4a_{n-2+1} + 4a_{n-2} = 7$$

$$a_n + 4a_{n-1} + 4a_{n-2} = 7 \rightarrow$$ The eqn is a non-homogeneous II order recurrence relation.

→ The soln of this eqn is of the form $a_n = a_n^{(h)} + a_n^{(p)}$.

→ Comparing the given eqn with general II order Non-homogeneous relation

we get $C_n = 1$, $C_{n-1} = 4$, $C_{n-2} = 4$

→ The quad eqn for given recurrence relation is of the form

$$C_n k^2 + C_{n-1} k + C_{n-2} = 0$$

$$k^2 + 4k + 4 = 0$$

$$k^2 + 2k + 2k + 4 = 0$$

$$(k+2)(k+2) = 0$$

$$k_1 = -2 , k_2 = -2 \quad \boxed{\therefore k = -2}$$

$4 = 2 \times 2$, $2 + 2 = 4$

Here, the roots are real & equal

$\therefore$ soln of homogeneous part is written in the form

$$a_n^{(h)} = (A + Bn) k^n$$

$$a_n^{(h)} = (A + Bn)(-2)^n$$

To find $a_n^{(p)}$

here $f(n) = 7$

i.e $f(n) = 7n^0$

Here, $f(n)$ is a polynomial of degree '0' & 1 is not a root of characteristic eqn

$$\boxed{a_n^{(p)} = A_0 b^n}$$

Solve the recurrence relation $a_{n+2} + 3a_{n+1} + 2a_n = 3^n$ for $n \geq 0$ given $a_0 = 0, a_1 = 1$

given $a_{n+2} + 3a_{n+1} + 2a_n = 3^n$

Replace n with n-1 on LHS.

$a_n + 3a_{n-1} + 2a_{n-2} = 3^n$ for $n \geq 2$

General form, $c_n a_n + c_{n-1} a_{n-1} + c_{n-2} a_{n-2} = f(n)$

$c_n = 1$; $c_{n-1} = 3$; $c_{n-2} = 2$; $f(n) = 3^n$

Quad eqn $\quad c_n k^2 + c_{n-1} k + c_{n-2} = 0$

$k^2 + 3k + 2 = 0$

$k^2 + k + 2k + 2 = 0$

$k(k+1) + 2(k+1) = 0$

$(k+1)(k+2) = 0$

$k_1 = -1, \ k_2 = -2$

$\rightarrow a_n^{(h)} = Ak_1^n + Bk_2^n$

$a_n^{(h)} = A(-1)^n + B(-2)^n$

To find $a_n^{(P)}$

$f(n) = 3^n$

$f(n) = 1 \cdot 3^n$

$f(n) = \alpha b^n$

$\alpha = 1$; $b = 3$

$\rightarrow a_n^{(P)} = A_0 b^n$

$a_n^{P} = A_0 3^n$

$a_n = a_n^{(P)}$

$a_{n+1}^{(P)} = A_0 3^{n+1}$

$a_{n+2}^{(P)} = A_0 3^{n+2}$

$A_0 3^n 3^2 + 3(A_0 3^n 3) + 2(A_0 3^n) = 3^n$

$A_0 9 + 3(A_0 3) + 2(A_0) = 1$

$9A_0 + 9A_0 + 2A_0 = 1$

$20 A_0 = 1$

$$\boxed{A_0 = \frac{1}{20}}$$

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$$\boxed{a_n^{(p)} = \frac{1}{20} 3^n}$$

$\rightarrow a_n = a_n^{(h)} + a_n^{(p)}$

$a_n = A(-1)^n + B(-2)^n + \frac{1}{20} 3^n$

$n=0$

$a_0 = A(-1)^0 + B(-2)^0 + \frac{1}{20} 3^0$

$0 = A + B + \frac{1}{20}$

$\boxed{A + B = -\frac{1}{20}}$ — ①

$n=1$

$a_1 = A(-1)^1 + B(-2)^1 + \frac{1}{20} 3^1$

$1 = -A - 2B + \frac{3}{20}$

$\boxed{-A - 2B = \frac{17}{20}}$ — ②

on solving ① & ②

we get $A = \frac{3}{4}$, $B = -\frac{4}{5}$

$\therefore a_n = \frac{3}{4}(-1)^n + \left(-\frac{4}{5}\right)(-2)^n + \frac{1}{20} 3^n$

Solve the recurrence relation $a_n + 4a_{n-1} + 4a_{n-2} = 5(-2)^n$, $n \geq 2$

$a_n + 4a_{n-1} + 4a_{n-2} = 5(-2)^n$.

Comparing with general eqn $C_n a_n + C_{n-1} a_{n-1} + C_{n-2} a_{n-2} = f(n)$

$C_n = 1$, $C_{n-1} = 4$, $C_{n-2} = 4$, $f(n) = 5 \times (-2)^n$

Quad eqn $C_n k^2 + C_{n-1} k + C_{n-2} = 0$

$k^2 + 4k + 4 = 0$

$k_1 = -2$, $k_2 = -2$, $k_1 = k_2 = k$.

We get the roots are real & distinct

soln of homogeneous part is written as

$a_n^{(h)} = (A + Bn)(-2)^n$

To find $a_n^{(p)}$;

here $f(n) = 5 \times (-2)^n$

in form, $f(n) = \alpha b^n$

$\alpha = 5,\ b = -2$

as b is root of multiplicity m, $a_n^{(p)}$ can be written

as

$$a_n^{(p)} = A_0 n^m b^n$$
$$a_n^{(p)} = A_0 n^2 (-2)^n$$

$$\rightarrow \quad a_{n-1}^{(p)} = A_0 (n-1)^2 (-2)^{n-1}$$
$$a_{n-2}^{(p)} = A_0 (n-2)^2 (-2)^{n-2}$$
$$a_n = a_n^{(p)}$$

$$\rightarrow A_0 n^2 (-2)^n + 4\left(A_0 (n-1)^2 (-2)^{n-1}\right) + 4\left(A_0 (n-2)^2 (-2)^{n-2}\right) = 5 \times (-2)^n$$

$$A_0 n^2 (-2)^n + 4\left(A_0 (n^2+1-2n)(-2)^n (-2)^{-1}\right) + 4\left(A_0 (n^2+4-4n)(-2)^n (-2)^{-2}\right) = 5 \times (-2)^n$$

$$\rightarrow A_0 n^2 + 4\left(A_0 (n^2+1-2n)\left(\tfrac{1}{-2}\right)\right) + 4\left(A_0 (n^2+4-4n)\left(\tfrac{1}{(-2)^2}\right)\right) = 5$$

$$A_0 n^2 - 2\left(A_0 n^2 + A_0 - 2nA_0\right) + A_0 n^2 + 4A_0 - 4nA_0 = 5$$
$$A_0 n^2 - 2A_0 n^2 - 2A_0 + 4nA_0 + A_0 n^2 + 4A_0 - 4nA_0 = 5$$
$$2A_0 = 5$$
$$\boxed{A_0 = 5/2}$$

$$\therefore \quad a_n^{(p)} = 5/2\, n^2 (-2)^n$$
$$a_n = a_n^{(h)} + a_n^{(p)}$$
$$= (A + Bn)(-2)^n + 5/2\, n^2 (-2)^n$$
$$a_n = \left[A + Bn + \frac{5}{2} n^2\right](-2)^n$$

Method of generating function of I order Recurrence Relation.

Consider the recurrence relation $a_n = Ca_{n-1} + f(n)$, $n \geq$

where C - is a constant & $f(n)$ - is a known function.

If $f(x)$ is a generating function in the form $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Thus $f(x)$ can be expressed in the form $f(x) = \frac{a_0 + x.g(x)}{1 - cx}$.

where $g(x) = \sum_{n=0}^{\infty} f(n) x^n$

Find the generating function for the recurrence relation $a_{n+1} - a_n = 3^n$, $n \geq 0$ with $a_0 = 1$

given $a_{n+1} - a_n = 3^n$

comparing with ~~$k_n a_n + k_n + a_{n-1} = f(n)$~~

we get, → comparing the given eqn with general

→ $\boxed{a_n = c a_{n-1} + f(n)}$ recurrence relation.

we get, $c = 1$, $f(n) = 3^n$

∴ The generating function $f(x)$ is of the form

$f(x) = \frac{a_0 + x.g(x)}{1 - cx}$

where $g(x) = \sum_{n=0}^{\infty} f(n) x^n = \sum_{n=0}^{\infty} 3^n x^n = \sum_{n=0}^{\infty} (3x)^n$

Expanding the Σ, we get $\langle g(x) \rangle : 1, 3x, (3x)^2, (3x)^3, \ldots$

As $g(x)$ is having G.P form $a, ar, ar^2, ar^3, \ldots$ with $a = 1$ & $r = 3x$

∴ summing up to ∞ we get

$S_\infty = \frac{a}{1 - r} = \frac{1}{1 - 3x}$

$g(x) = \sum_{n=0}^{\infty} (3x)^n = \frac{1}{1 - 3x}$

Substituting $g(x)$ in $f(x)$

$$f(x) = \frac{1 + x.\frac{1}{(1-3x)}}{1 + x}$$

$$= \frac{\frac{1 - 3x + x}{(1 - 3x)}}{1 - x}$$

$$f(x) = \frac{1 - 2x}{(1 - 3x)(1 - x)}$$

Solving $f(x)$ by method of partial fraction

$$\frac{1-2x}{(1-3x)(1-x)} = \frac{A}{1-3x} + \frac{B}{1-x}$$

$$\frac{1-2x}{(1-3x)(1-x)} = \frac{A(1-x)+B(1-3x)}{(1-3x)(1-x)}$$

$$1-2x = A(1-x)+B(1-3x)$$

To find A & B

put $x=1$

$1-2(1) = A(1-1)+B(1-3(1))$

$1-2 = B(-2)$

$-1 = -2B$

$\boxed{B = 1/2}$

put $x = 1/3$

$1-2(1/3) = A(1-1/3) + B(1-3(1/3))$

$1/3 = 2/3 A$

$\boxed{A = 1/2}$

$$\therefore f(x) = \frac{1/2}{1-3x} + \frac{1/2}{1-x}$$

$$= \frac{1}{2}\left[\frac{1}{1-3x} + \frac{1}{1-x}\right]$$

$$f(x) = \frac{1}{2}\left[\sum_{n=0}^{\infty}(3x)^n + \sum_{n=0}^{\infty} x^n\right]$$

$$\therefore g(x) = \sum_{n=0}^{\infty}(3x)^n = \frac{1}{1-3x}$$

$1, x, x^2, x^3, \ldots$

$a, ar, ar^2, ar^3, \ldots$ $\sum_{n=0}^{\infty} x^n = 1/1-x$

$S_\infty = 1/1-x$

$$f(x) = \frac{1}{2}\left[\sum_{n=0}^{\infty} 3^n x^n + \sum_{n=0}^{\infty} x^n\right] = \frac{1}{2}\sum_{n=0}^{\infty} x^n\left[3^n+1\right]$$

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$\therefore a_n = 1/2\left[1+3^n\right]$$

Bafna Gold

Date:

Using generating function solve the recurrence relation $a_n - 3a_{n-1} = n$, given $a_0 = 1$

$a_n - 3a_{n-1} + n$

comparing with general recurrence relation. we get

$c = 3, \; f(n) = n.$

the generating function $f(x)$ is given by

$f(x) = \frac{a_0 + x\,g(x)}{1 - cx}$ where $g(x) = \sum_{n=0}^{\infty} f(n)x^n = \sum_{n=0}^{\infty} n x^n$

$\langle g(x) \rangle : 0, x, 2x^2, 3x^3, \ldots$

$$\sum_{n=0}^{\infty} n x^n = \frac{x}{(1-x)^2} = g(x)$$

$$\boxed{\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}}$$

$$\sum_{n=1}^{\infty} n . x^{n-1} = (-1)(1-x)^{-2}(-1) = 1/(1-x)^2$$

$$\sum_{n=0}^{\infty} n x^{n-1} . x = \frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} n . x^n$$

$$\rightarrow f(x) = \frac{1 + x . \frac{x}{(1-x)^2}}{1 - 3x} = \frac{\frac{1 + x^2 - 2x + x^2}{(1-x)^2}}{(1-3x)}$$

$$f(x) = \frac{1 - 2x + 2x^2}{(1-3x)(1-x)^2}$$

$$\rightarrow \frac{1 - 2x + 2x^2}{(1-3x)(1-x)^2} = \frac{A}{1-3x} + \frac{B}{1-x} + \frac{C}{(1-x)^2}$$

$$1 - 2x + 2x^2 = A(1-x)^2 + B(1-3x)(1-x) + C(1-3x)$$

To find A, B & C

put $x = 1$

$$1 - 2(1) + 2(1)^2 = A(0) + B(0) + C(1 - 3(1))$$

$$1 = -2C$$

$$\boxed{C = -1/2}$$