GRAPH THEORY

Module-01 :- Introduction to Graph Theory

Graph:-

Graph G is a finite non-empty set 'V' together with a prescribed set 'E' of unordered pair of distinct elements of V.

→ Each element of vertex V is called vertex or point of graph G.

* The number of vertices in a graph G is called order of the graph.
* The number of edges in a graph G is called as size of the graph.

→ Each pair (u,v) of vertices in E is called edge of G.

→ If e=(u,v) is edge of G, then u and v are adjacent vertices.

→ If e=(u,v) is edge of G, then the vertex u and edge e are incident.

→ If two distinct edges e_1 and e_2 are incident with common vertex then they are called adjacent edges.

→ If both vertex set and edge set of a Graph G are finite then G is called finite graph or else infinite graph.

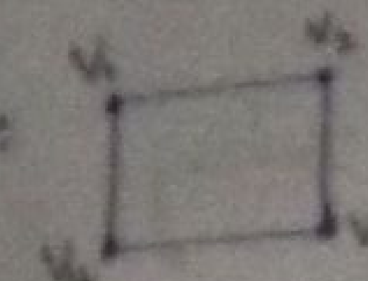
* Loop:- The edge joining vertex to vertex is called loop.

→ If two or more vertices have more than one edge are called multiple edges.

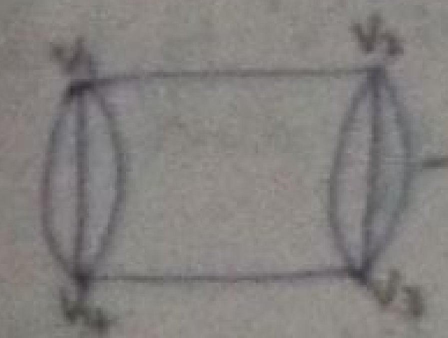
→ If one or more edges join same two vertices in a graph then these edges are called multiple edges

* Multiple Graph:- Graph with multiple edges and no loops is called Multiple Graph.
* PseudoGraph:- A Graph with both multiple edges and loops is called pseudograph.
* Simple Graph:- A Graph without multiple edges and loops is called Simple Graph.

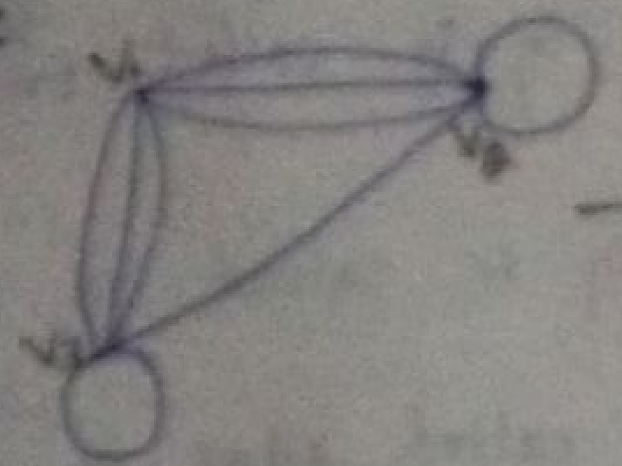
Example:-

① G: 

→ simple graph

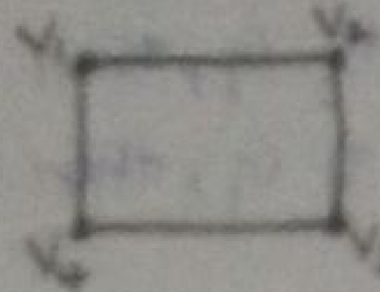
② G: 

→ Multiple Graph

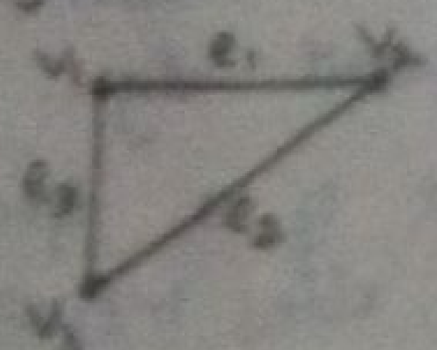
③ G: 

→ Pseudograph

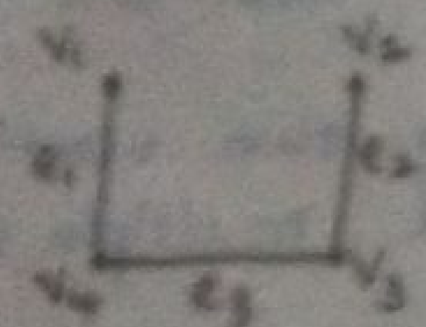
① Graph of order (vertices) 4 :-



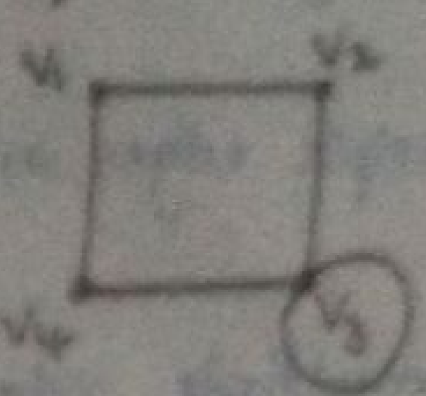
② Graph of size (edges) 3 :-



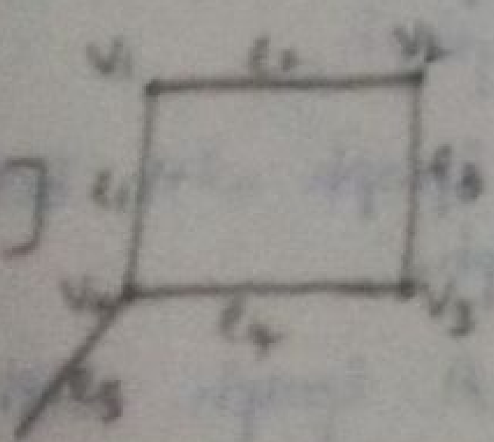
③ Graph of (4, 3) :-



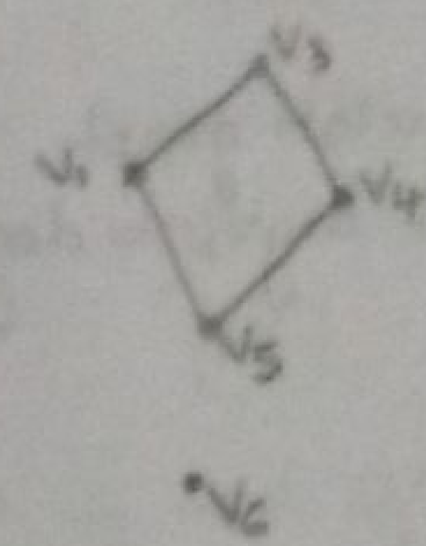
④ Graph of (4, 5) :-



[or]



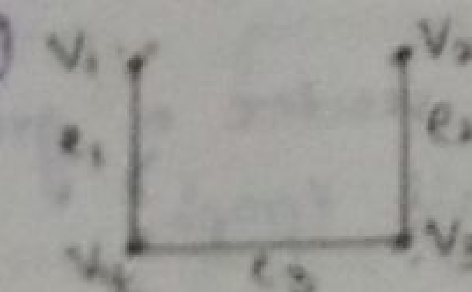
⑤ Graph of (6, 4) :-



⑥ Degree of the Vertex :-

→ The degree of vertex 'V' in a Graph 'G' is the number of edges incident with V. It is denoted by deg V.

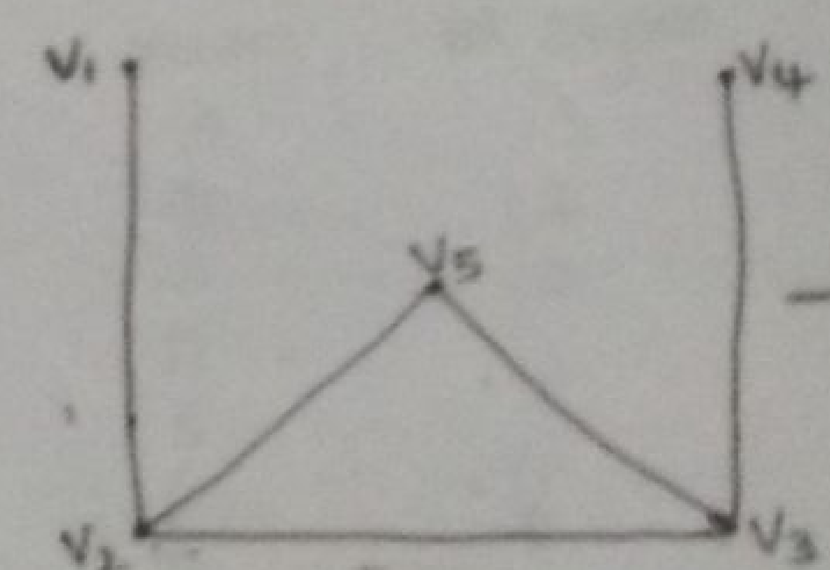
Example :- ①



degree of v_1 ⇒ $\deg v_1 = 1$
$\deg v_2 = 1$
$\deg v_3 = 2$
$\deg v_4 = 2$

* If $\deg V = 1$ ⇒ then it is called end vertex
* If $\deg V = 0$ ⇒ then it is called isolated vertex
* If all the degrees of V is equal, then it is called Regular graph.

②

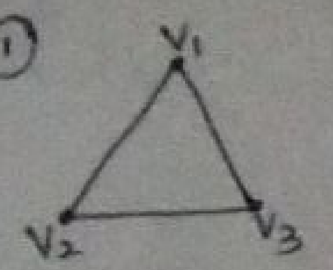


→ $\deg v_1 = 1$
$\deg v_2 = 2$
$\deg v_3 = 3$
$\deg v_4 = 1$
$\deg v_5 = 2$

→ A vertex with degree 0 is called isolated vertex.
→ A vertex with degree 1 is called end vertex.
→ A graph G is said to be regular if every vertex of G has same degree.

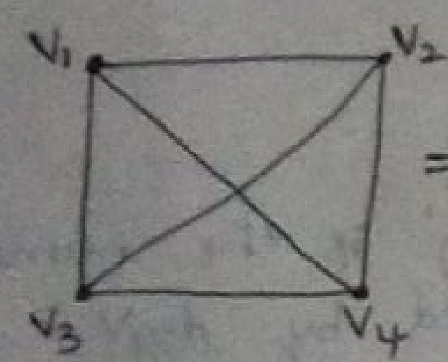
→ A graph G is r-regular if degree of each vertex of G is r.

Example:- ①



⇒ 2-regular graph.
$\deg v_1 = \deg v_2 = \deg v_3 = 2$.

②



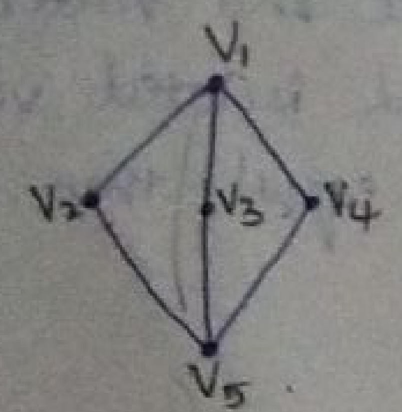
⇒ 3-regular graph.
$\deg v_1 = \deg v_2 = \deg v_3 = \deg v_4 = 3$

→ Null graph:- If degree of each vertex of graph G is 0 [zero] then it is called Null graph.

Example:- v_1. $\cdot v_2$ $\cdot v_3$ ⇒ Null graph.

Degree of each vertex of a graph:-

①

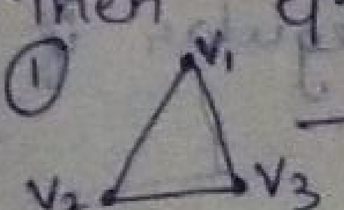


$\deg v_1 = 3$
$\deg v_2 = 2$
$\deg v_3 = 2$
$\deg v_4 = 2$
$\deg v_5 = 3$

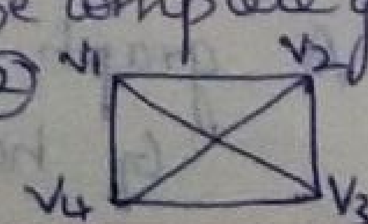
3, 3, 2, 2, 2 ~~3~~ ⟶ Degree Sequence

* Complete graph:-
In a graph G, If each vertex is connected to remaining all vertex. Then Graph 'G' is said to be complete graph.

Eg:- ①



⟶ Complete graph. ②



Isomorphism of graphs:-

① Two graphs G and G' are said to be isomorphic to each other if there is a one-one correspondance or ~~bijenction~~ bijuriction between their vertices and between their edges with that adjacency relationship preserved.

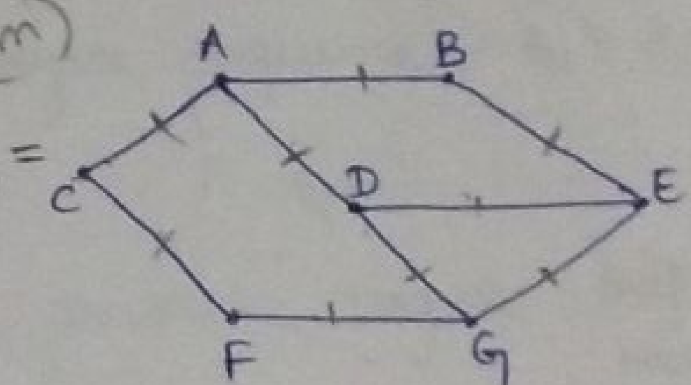
[OR]

Two graphs G and G' are said to be isomorphic,
* Number of vertices are equal.
* Number of edges are equal.
* Degree Sequence is same.
* Adjacency preserved.

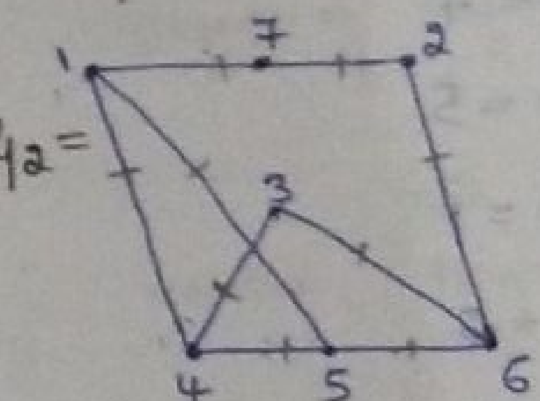
Example:-
Check whether graphs are isomorphic or not.

① (gm)

$G_1 =$



$G_2 =$



Ans:-
① G_1: Vertices (V) = 7 , G_2: V = 7
② Edges :- G1 : 9, G2 : 9
③

G1 Vertex	G1 Degree	G2 Vertex	G2 Degree
A	3	1	3
B	2	2	2
C	2	3	2
D	3	4	3
E	3	5	3
F	2	6	3
G	3	7	2

Degree Sequence :- 3, 3, 3, 3, 2, 2, 2

Degree Sequence :- 3, 3, 3, 3, 2, 2, 2.

④ G_1

Vertex	Degree	Adjacency vertex
A	3	2,2,3
B	2	3,3
C	2	3,3
D	3	3,3,3
E	3	2,3,3
F	2	3,3
G	3	3,3,3

G_2

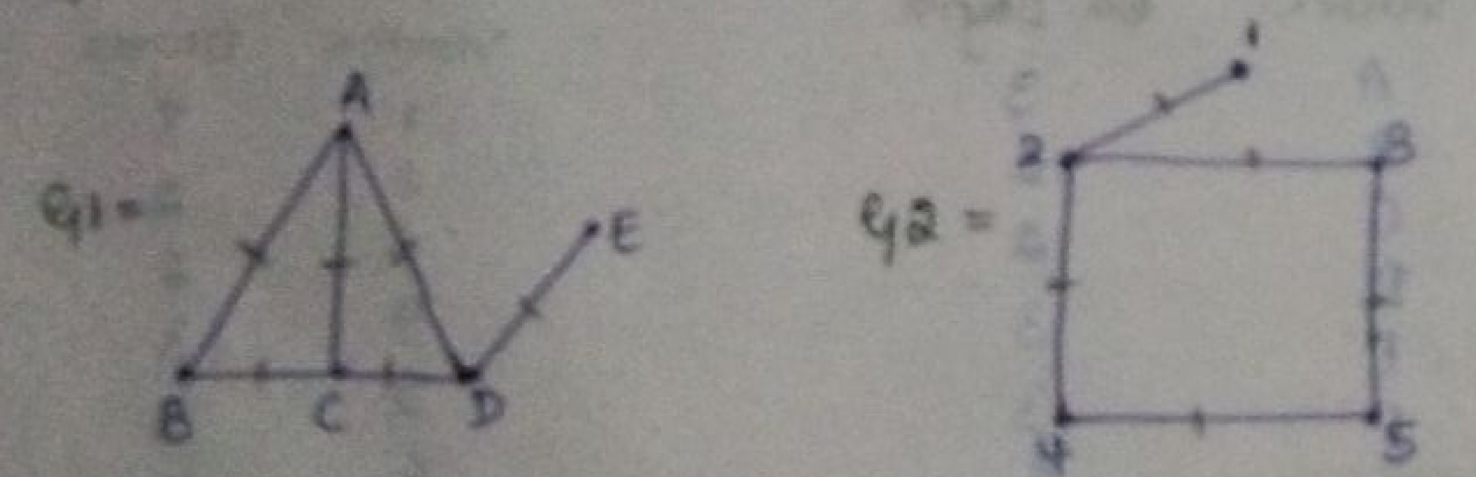
Vertex	Degree	Adjacency vertex
1	3	2,3,3
2	2	3,3
3	2	3,3
4	3	2,3,3
5	3	3,3,3
6	3	2,2,3
7	2	2,3

⑤ Image of vertex :-

$f(A) = 6$
$f(B) = 3$
$f(C) = 7$
$f(D) = 5$
$f(E) = 1$
$f(F) = 2$
$f(G) = 4$

As All the above conditions are satisfied. Hence the Graph is isomorphic to each other.

②



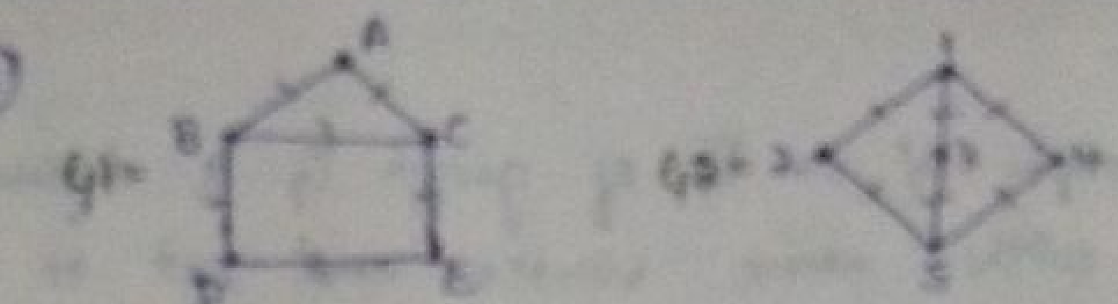
Ans :-

① Vertices :- $G_1 : 5$, $G_2 : 5$

② Edges :- $G_1 : 6$, $G_2 : 5$

∴ As the Edges of G_1 and G_2 are not equal Hence the graph is not isomorphic to each other.

③



Ans :-

① Vertices :- $G_1 : 5$, $G_2 : 5$

② Edges :- $G_1 : 6$, $G_2 : 6$

③ G_1

Vertex	Degree
A	2
B	3
C	3
D	2
E	2

Degree Sequence :- 3,3,2,2,2

G_2

Vertex	Degree
1	3
2	2
3	2
4	2
5	3

Degree of Sequence :- 3,3,2,2,2

④ G_1

Vertex	Degree	Adjacency matrix
A	2	3,3
B	3	2,2,3
C	3	2,2,3
D	2	2,3
E	2	2,3

G_2

Vertex	Degree	Adjacency matrix
1	3	2,2,2
2	2	3,3
3	2	3,3
4	2	3,3
5	3	2,2,2

⑤ Image of vertex :-

$f(A) = 2$
$f(B) =$ No Image

As the above condition is not satisfied, Hence it is not isomorphic to each other.

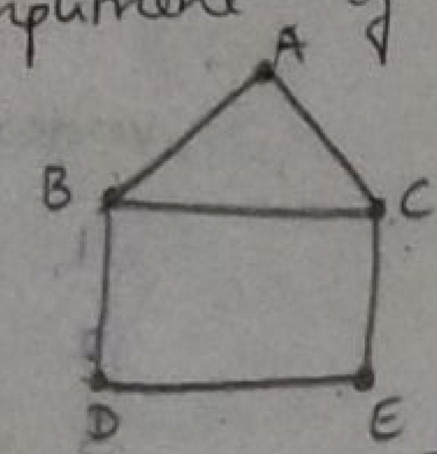
* Compliment Graph:-

The Compliment Graph G' or $\overline{G}$ of graph G is defined as simple graph with same vertex ~~and~~ set as graph G and where two vertices are adjacent only when they are not adjacent in graph G.

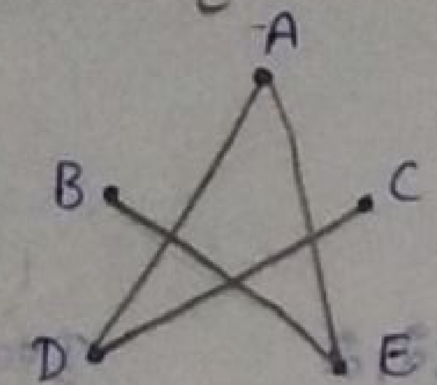
Example:-

* Write the Compliment of graph G where

G :-



$\Rightarrow$ G' (or) $\overline{G}$:-



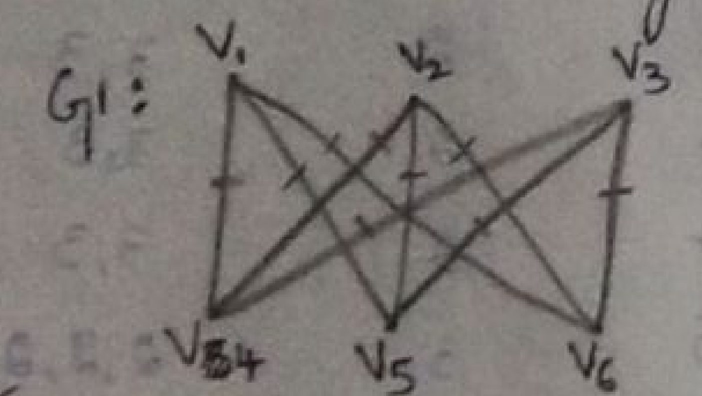
$\Rightarrow$ $G \cup G'$ = regular graph. (Each vertex degree will be same)

Here degree is 4, 4-regular graph.

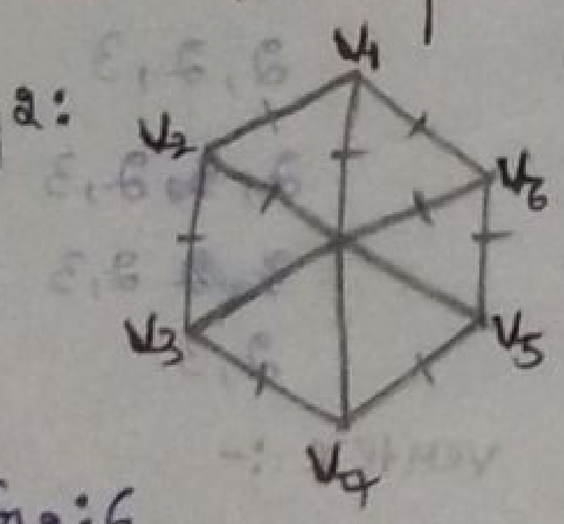
Imp

① Show that two graphs are isomorphic.

G_1:



G_2:



Ans:-

① Vertices :- G_1: 6 , G_2: 6

② Edges :- G_1: 9 G_2: ~~12~~ 9

③

G_1

Vertex	Degree	Adjacency matrix
v_1	3	3,3,3
v_2	3	3,3,3
v_3	3	3,3,3
v_4	3	3,3,3
v_5	3	3,3,3
v_6	3	3,3,3

G_2

Vertex	Degree	Adjacency matrix
u_1	3	3,3,3
u_2	3	3,3,3
u_3	3	3,3,3
u_4	3	3,3,3
u_5	3	3,3,3
u_6	3	3,3,3

Degree Sequence of G1 :- 3,3,3,3,3,3 Degree of Sequence of G2 : 3,3,3,3,3,3.

④ ~~f(x)~~

$f(V_1) = u_2$

$f(V_2) = u_4$

$f(V_3) = u_1$

$f(V_4) = u_3$

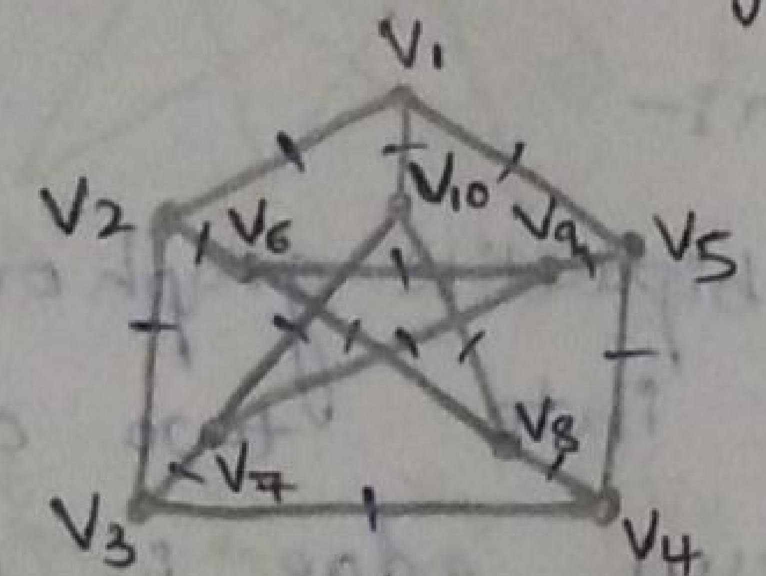
$f(V_5) = u_6$

$f(V_6) = u_5$

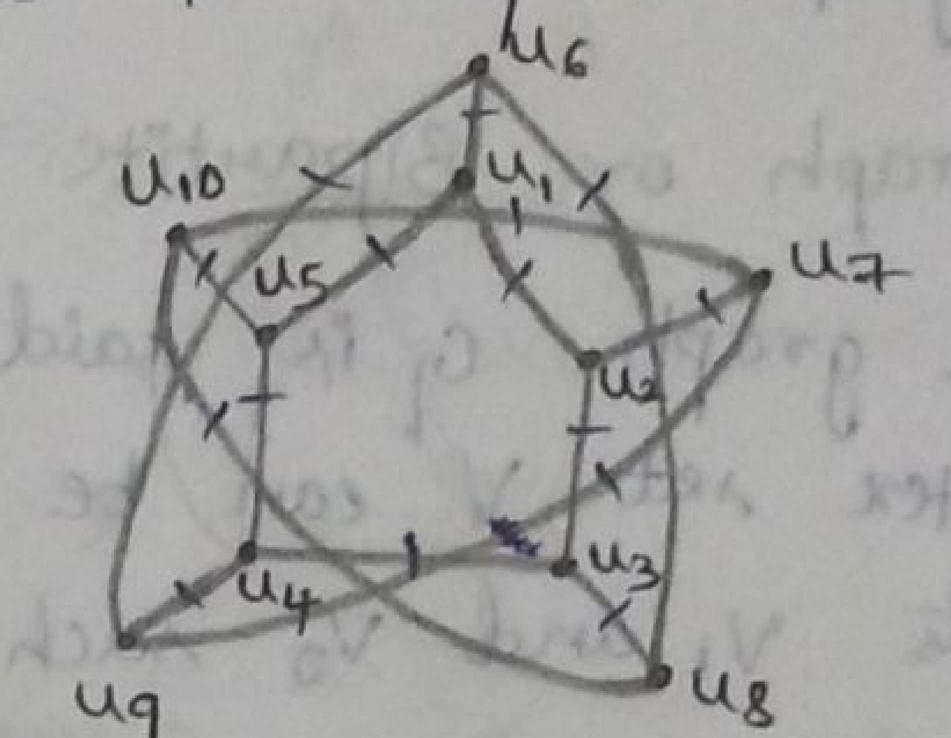
∴ Hence all the conditions are satisfied. Hence the two graphs are isomorphic in nature.

② Show that two graphs are isomorphic.

G1:



G2:



Ans:-

① Vertices :- G1 : 10 , G2 : 10

② Edges :- G1 : 15 , G2 : 15

③

G1 Vertex	Degree	Adjacency matrix	G2 Vertex	Degree	Adjacency matrix
V_1	3	3,3,3	u_1	3	3,3,3
V_2	3	3,3,3	u_2	3	3,3,3
V_3	3	3,3,3	u_3	3	3,3,3
V_4	3	3,3,3	u_4	3	3,3,3
V_5	3	3,3,3	u_5	3	3,3,3
V_6	3	3,3,3	u_6	3	3,3,3
V_7	3	3,3,3	u_7	3	3,3,3
V_8	3	3,3,3	u_8	3	3,3,3
V_9	3	3,3,3	u_9	3	3,3,3
V_{10}	3	3,3,3	u_{10}	3	3,3,3

Degree Sequence :- 3,3,3,3,3,3,3,3,3,3 Degree Sequence :- 3,3,3,3,3,3,3,3,3,3.

Scanned By Camera Scanner

(4) $f(v_1) = u_2$
$f(v_2) = u_3$
$f(v_3) = u_4$
$f(v_4) = u_5$
$f(v_5) = u_6$
$f(v_6) = u_7$
$f(v_7) = u_8$
$f(v_8) = u_9$
$f(v_9) = u_{10}$
$f(v_{10}) = u_1$

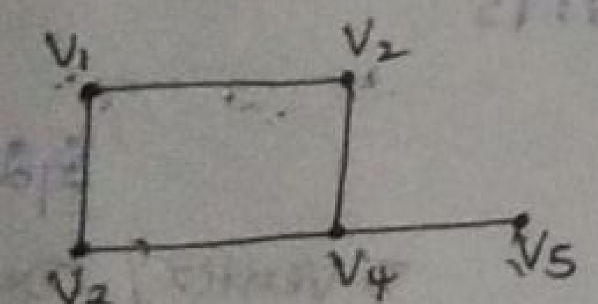
$\therefore$ All the conditions are satisfied Hence the two graphs are isomorphic in nature.

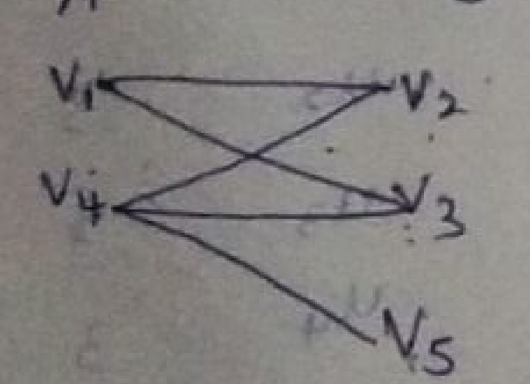
Bigraph or Bipartite ~~grap~~ graph:-

Let a graph G is said to be bipartite graph or bigarph if vertex set V can be partition into two disjoint subsets V_1 and V_2 such that every edge in G. joints vertex of V_1 to ~~V_2~~ vertex of V_2.

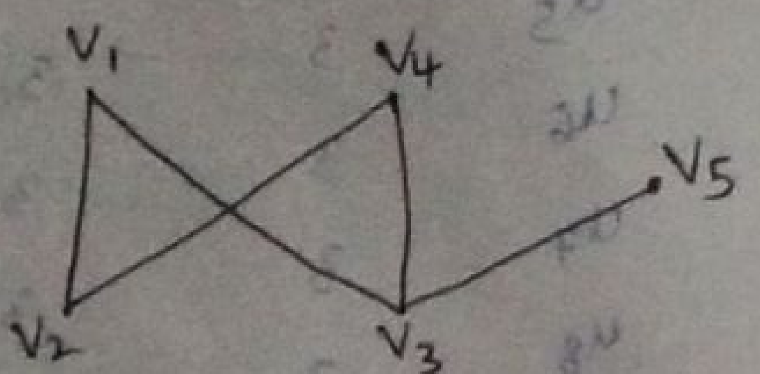
Example:-

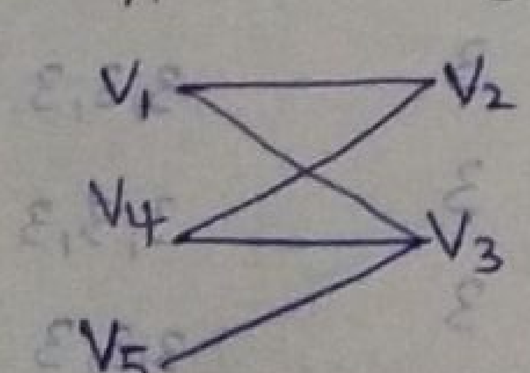
G:



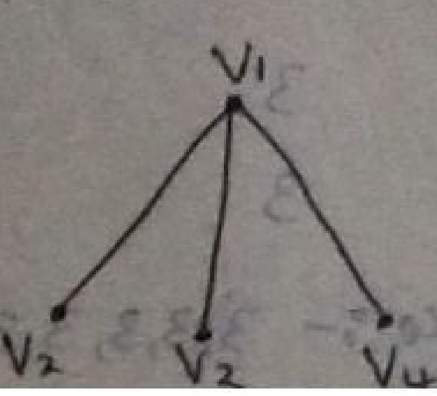


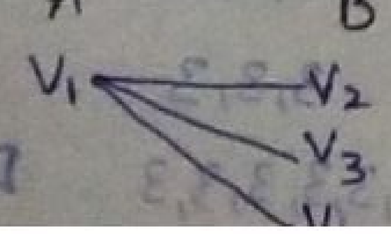
(1)





(2)

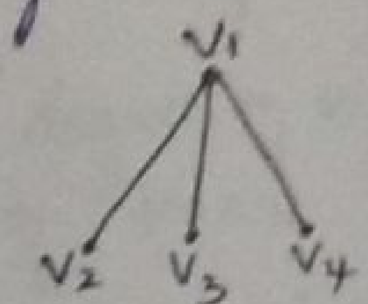




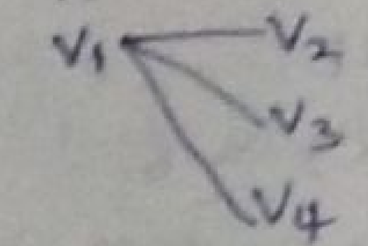
Complete Bigraph:-

A complete bipartite graph is a bipartite graph with vertex partition ~~V_1~~ V_1 and V_2 such that each vertex in V_1 is joined every vertex V_2.

Example:- (1)

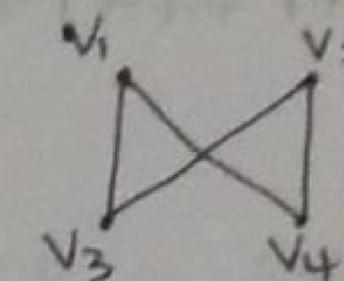


$\Rightarrow$

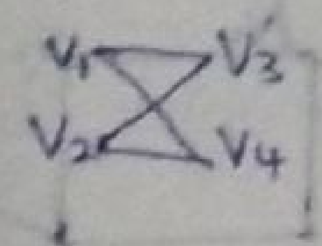


$\Rightarrow K_{1,3}$ graph

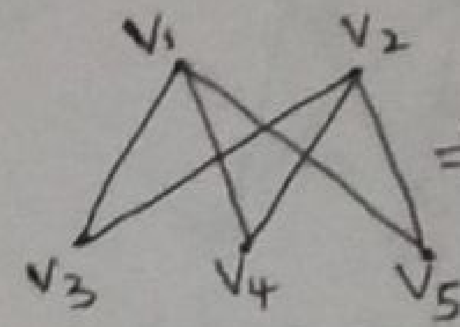
(2)



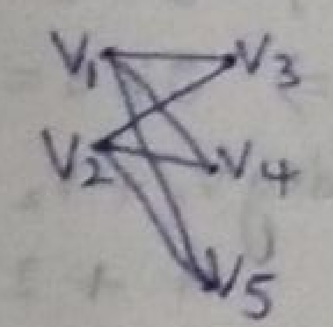
$\Rightarrow K_{2,2}$ graph $\Rightarrow$



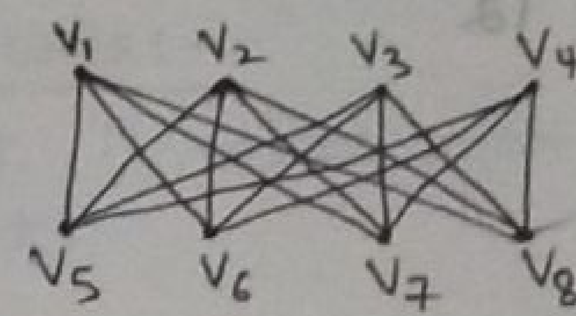
(3)



$\Rightarrow K_{2,3}$ graph $\Rightarrow$

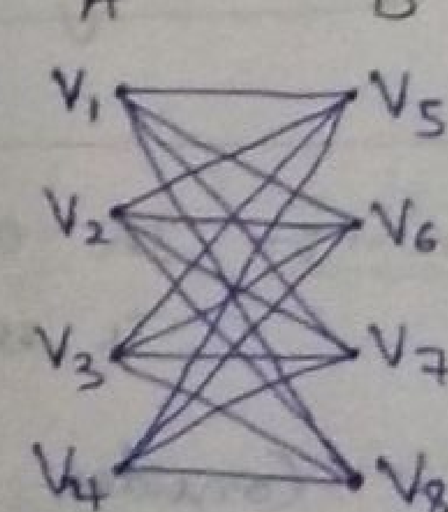


(4)



$\Rightarrow K_{4,4}$ graph

$\Rightarrow$



* **Note:-**

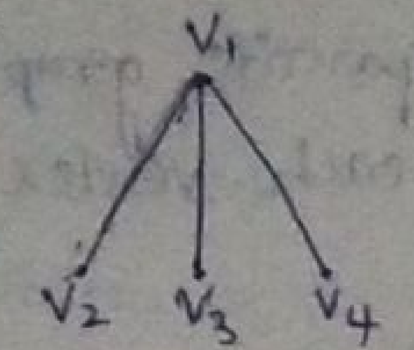
The Sum of degrees of all the vertices in a graph is an ~~g~~ even number and this number is equal to twice the number of edges in a graph.

i.e., $\boxed{\sum_{i=1}^{n} \deg V_i = 2E}$ where E = Edges

$\Downarrow$

Hand shaking property

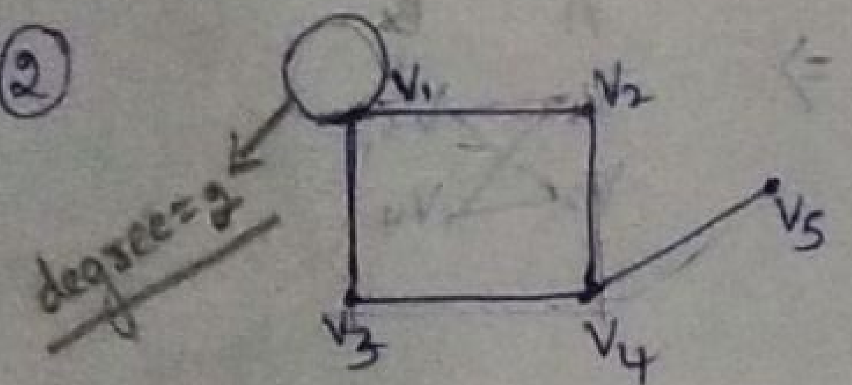
Example :-

① 

$\Rightarrow 2 \times E$

$\Rightarrow 2 \times 3 = 6$

$\deg v_1 + v_2 + v_3 + v_4 = 3 + 1 + 1 + 1 = 6$

② 

$\Rightarrow 2 \times E$

$\Rightarrow 2 \times 6 = 12$

$\deg v_1 + v_2 + v_3 + v_4 + v_5$

$4 + 2 + 2 + 3 + 1 = 12$

$\boxed{\text{The degree of loop} = 2}$

---

v.v. Imp **Theorem :-** In a graph, number of vertices of odd degree is even.

Proof :- Consider a graph with n vertices. Suppose k of these vertices are of odd degree so that remaining $n-k$ vertices are of even degree. Denote the vertices with odd degree by $v_1, v_2, \ldots, v_k$ and vertices of even degree by $v_{k+1}, v_{k+2}, \ldots, v_n$ then the sum of degrees of all the vertices is

$$\sum_{i=1}^{n} \deg v_i = \underbrace{\sum_{i=1}^{k} \deg v_i}_{\text{sum of odd degrees}} + \underbrace{\sum_{i=k+1}^{n} \deg v_i}_{\text{sum of even degrees}} \longrightarrow ①$$

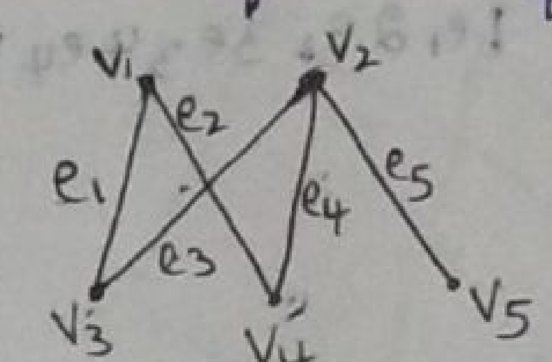
In view of Hand shaking property, the sum eq on left side of above expression is equal to twice the number of edges in the graph. As such this sum is even.

For the the second sum in the right hand side is the sum of degrees of vertices with even degrees. As such sum is even. Therefore first sum in the right hand side must also even. i.e, $\deg v_1 + \deg v_2 + \ldots + \deg v_k = \text{even} \longrightarrow ②$

But each of degree of v_1 *, $\deg v_2, \ldots \deg v_k$ is odd. Therefore the number of terms in the left hand side of equation ② must be even. i.e, k is even.

* **Walk :-**

A walk is a sequence of vertices and edges of a graph.

Example :- 

$v_1 e_1 v_3 e_3 v_2 e_4 v_4 e_4 v_2 e_5 v_5$.

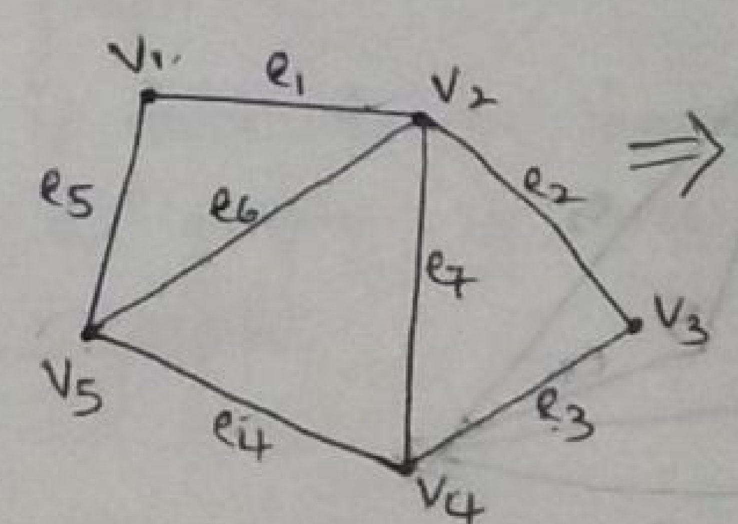
* Open walk :-

A walk is called open if starting and ending vertices are different.

* Closed walk :-

A walk is said to be close if starting and ending vertices are same.

Example :-

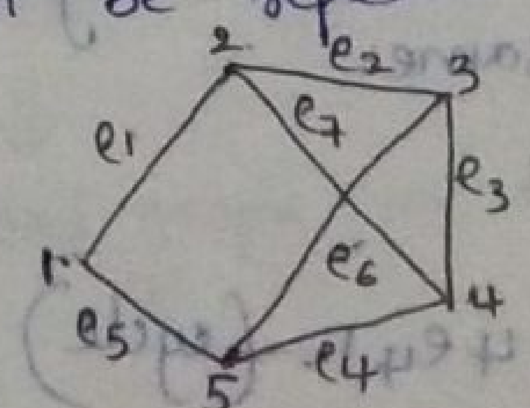


$\Rightarrow$ open walk :- $v_1 e_1 v_2 e_6 v_5 e_4 v_4$

closed walk :- $v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_4 v_5 e_5 v_1$

* Trial :-
* A trial is a open walk in which no edge is repeated.
* Vertex can be repeated.

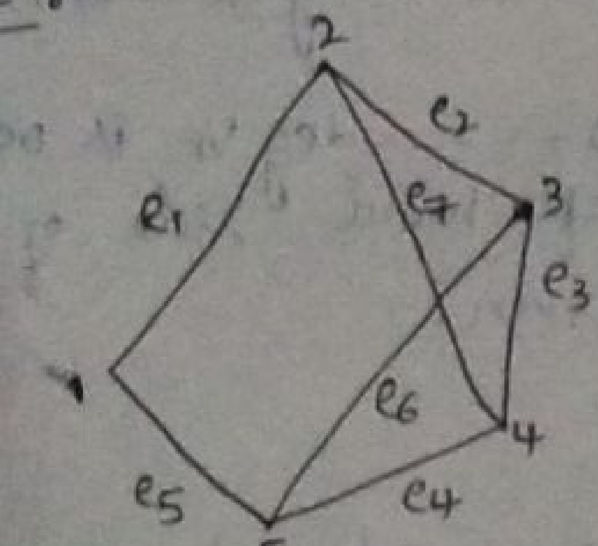
Example :-



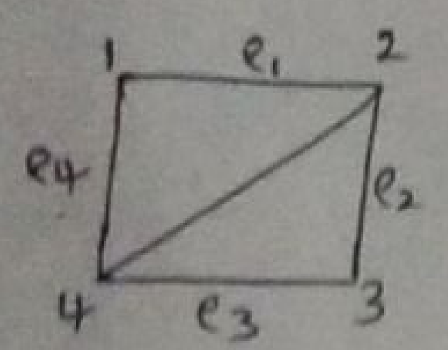
$\boxed{1e_1 2e_2 3e_6 5e_4 4}$ trial (no edge is repeated)

* Closed trail:-
* Closed trail is a closed walk in which no edge is repeated.
* vertex can be repeated.

Example :-

① 

$= 1e_1 2e_2 3e_6 5e_5 1$ (or) $1e_1 2e_7 3e_3 4e_4 5e_5 1$

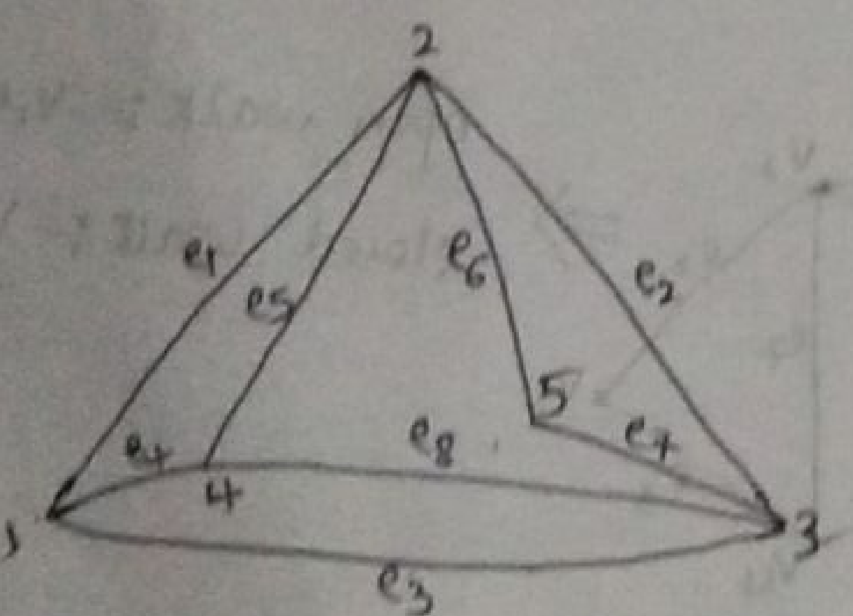
② 

$= 1e_1 2e_2 3e_3 4e_4 1$

→ Closed trail is called circuit.

* Path:-
A trail in which neither vertices nor edges is repeated.

Example :-



$\Rightarrow 1e_1 2e_2 3e_7 5$. (path)

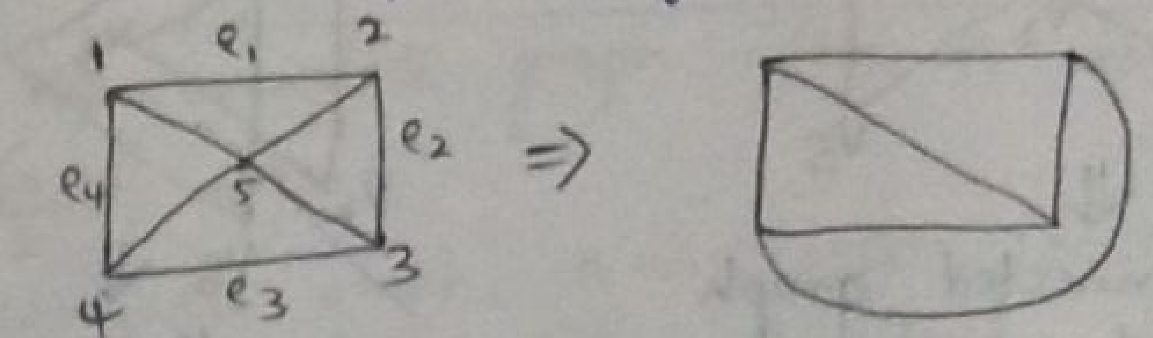
* Cycle:-
A trail in which neither vertices nor edges repeated but starting and ending are same.

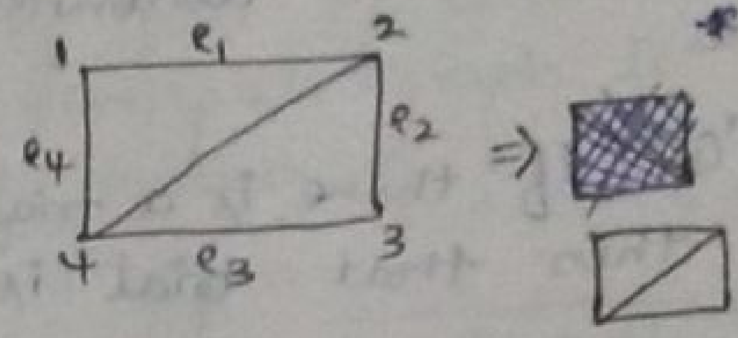
Example:- Above example.

$\Rightarrow 1e_1 2e_2 3e_8 4e_4 1$. (cycle)

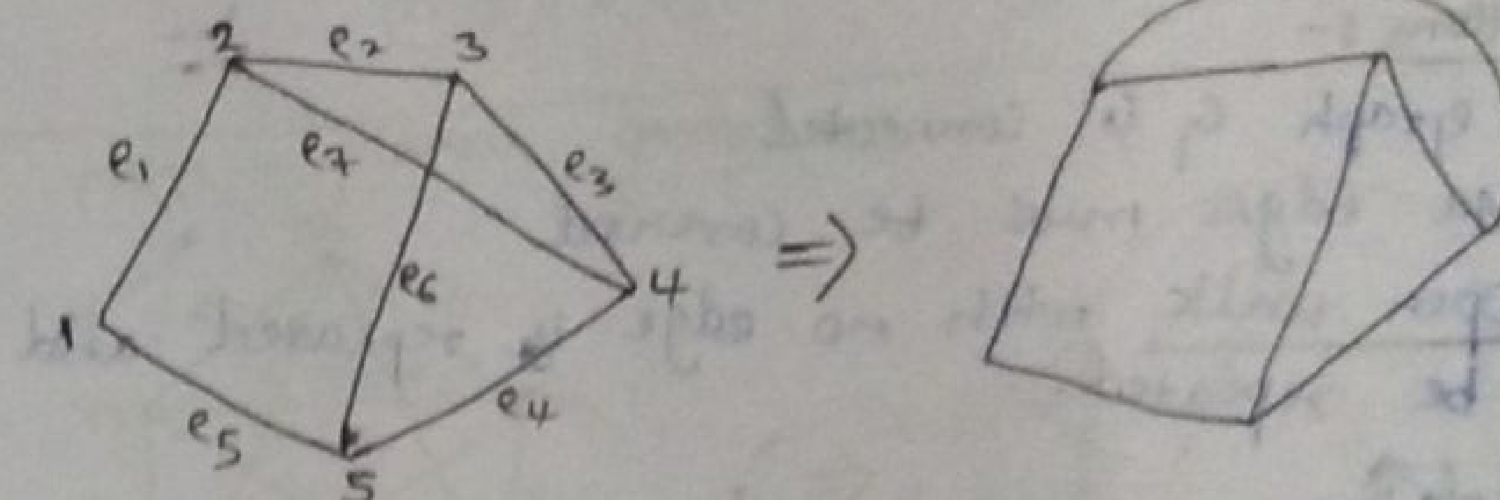
* Planar Graphs:-
A graph that can be drawn on a plane without crossing the edges is called planar graph.

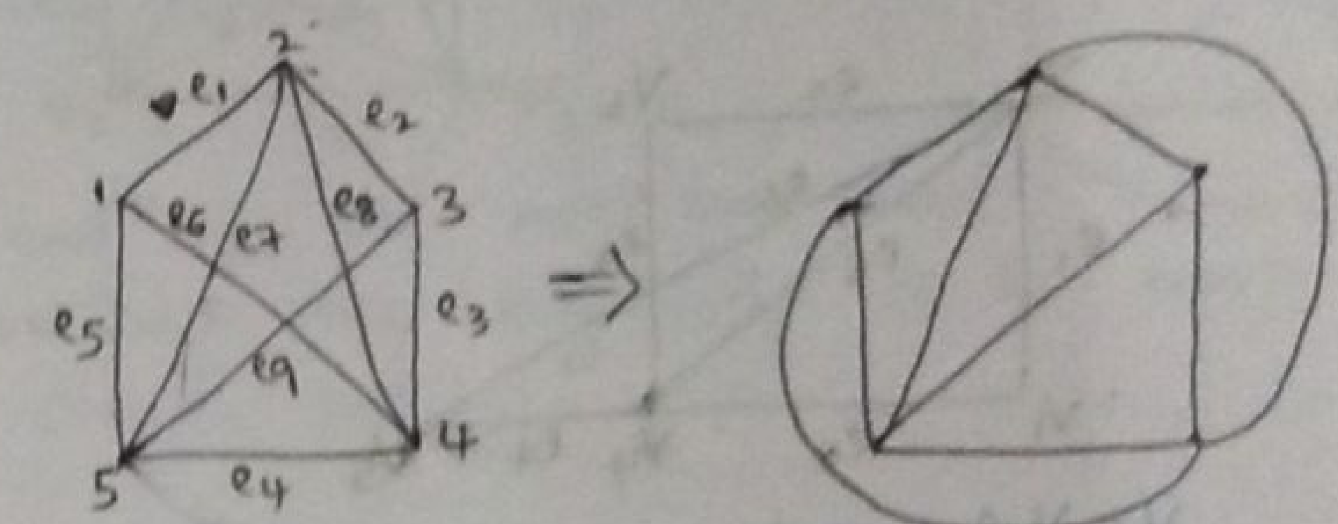
Example :-

① 

② 

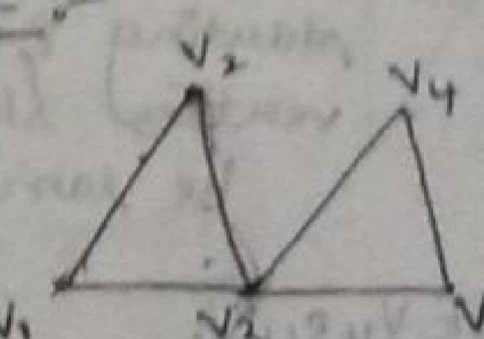
* any one edge should be redrawn without crossing any edge = planar graph

③ 

④ 

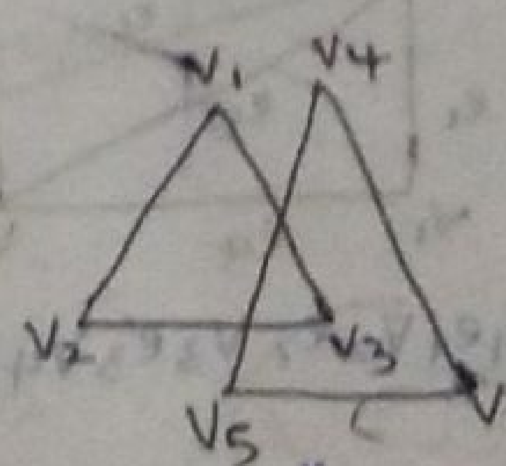
* Connected graph:-
A graph G is said to be connected if every pair of distinct vertices in a graph G are connected. otherwise it is disconnected graph.

Example :-

① 

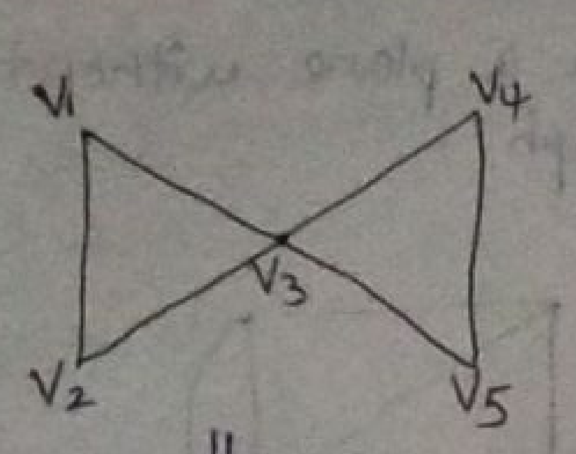
⇓

Connected graph



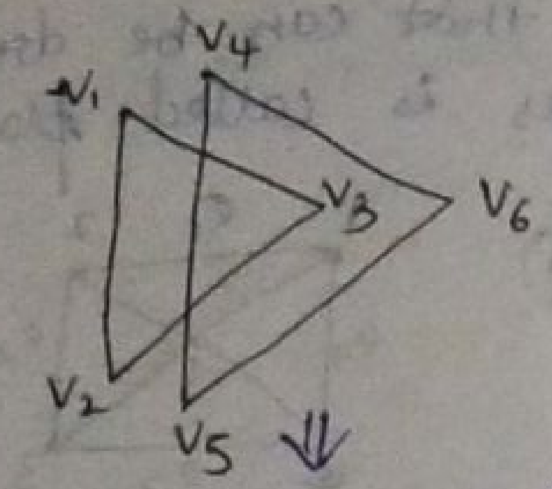
⇓

disconnected graph



⇓

connected graph



⇓

disconnected graph

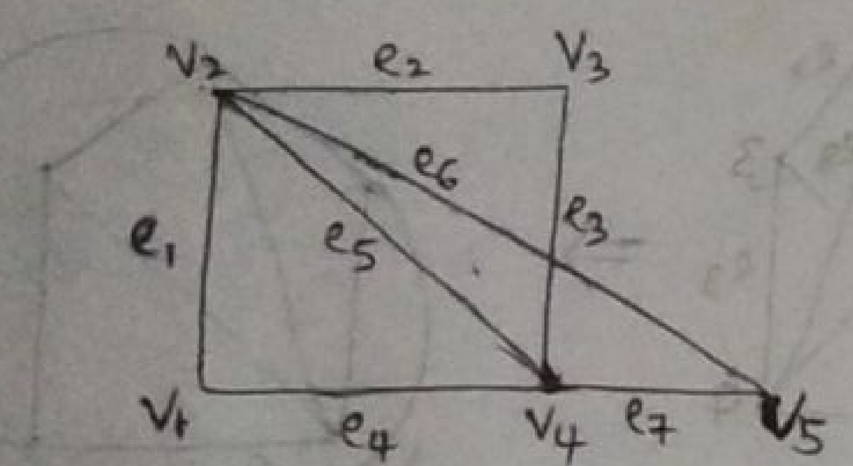
* Euler trial:- /universe line

Consider connected graph 'G', if there is a trial in 'G' contains all the edges of 'G', then that trial is called Euler trial.

Conditions :-

1. graph G is connected
2. All edges must be covered
3. open walk with no edge is repeated and vertex can be repeated.

Example:-



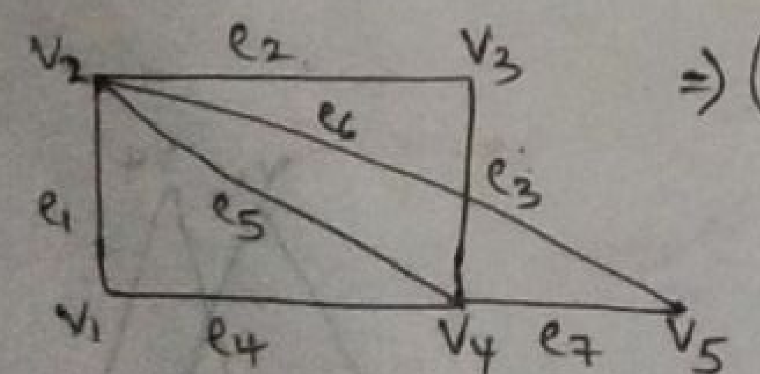
$V_1 e_1 V_2 e_6 V_5 e_7 V_4 e_5 V_2 e_2 V_3 e_3 V_4 e_4 V_1$

Imp

* Euler Circuit:-

Consider a connected graph 'G', if there is a circuit in G that contains all the edges of graph G then that circuit is called Euler circuit.

Example :-



⇒ (closed walk)

starting & ending vertices should be same

$V_1 e_1 V_2 e_2 V_3 e_3 V_4 e_7 V_5 e_6 V_2 e_5 V_4 e_4 V_1$

*① Euler graph:-

A connected graph that contains an Euler circuit is called Euler graph.

*② Semi - Euler graph:-

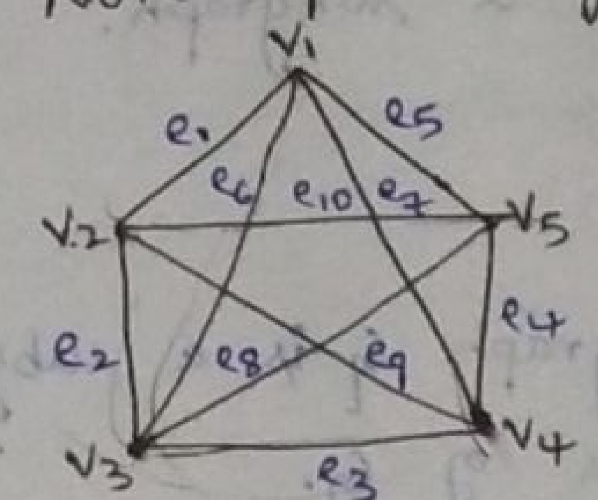
A connected graph that contains Euler trial is called semi - Euler graph.

\# Show that a complete graph with n vertices namely K_n has $\frac{n*(n-1)}{2}$ edges.

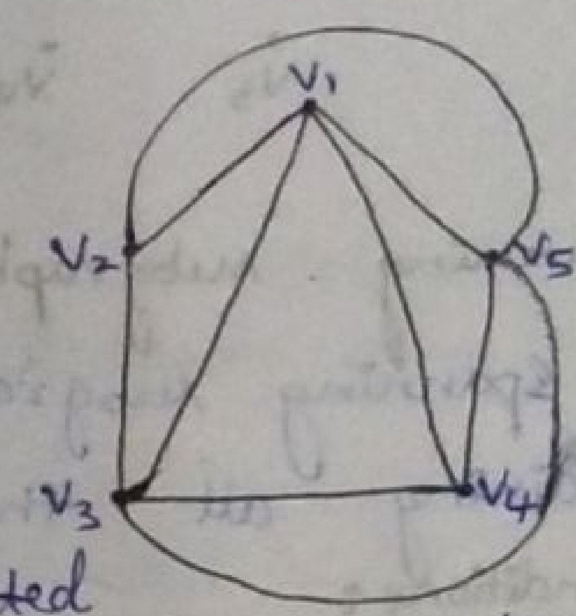
Sol^n:- In a complete graph, there exist exactly one edge between every pair of vertices. As such the number of edges in a complete graph is equal to number of pair of vertices. If number of vertices is n then number of pair of vertices is $^nC_2 = \frac{n!}{(n-2)! * 2!} = \frac{n*(n-1)*\cancel{(n-2)!}}{\cancel{(n-2)!} \times 2} = \frac{n(n-1)}{2}$

Example for Non - planar graph:-

K_5 =



⇒



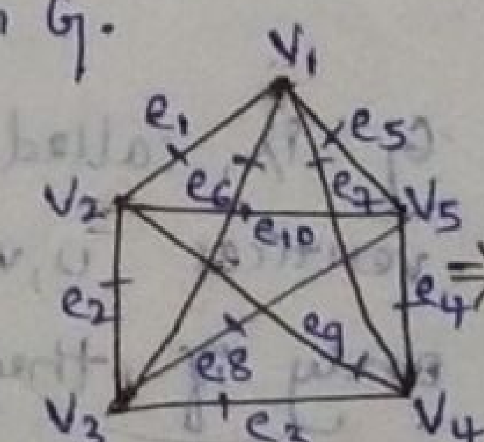
Since V_2 and V_4 are not connected without crossing the edge, so it is non-planar graph.

\# subgraph:-

Let G be a graph then G_1 is said to be subgraph of G if (i) All vertices and edges of graph G_1 are in G.

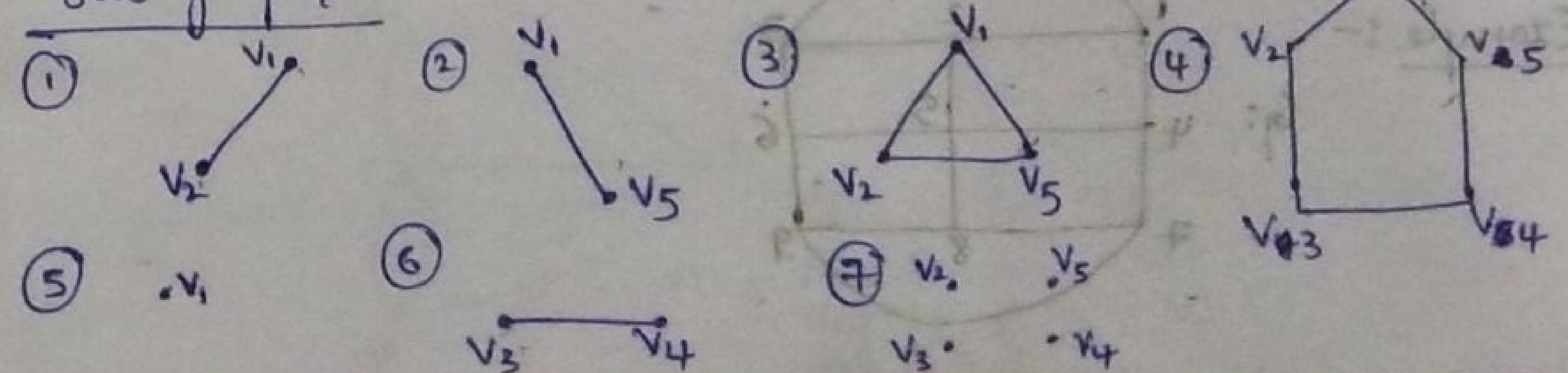
(ii) If $E = (u,v)$ be an edge in G_1 has same end vertex as in G.

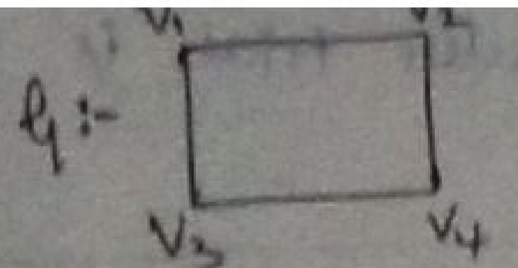
Example:- ①

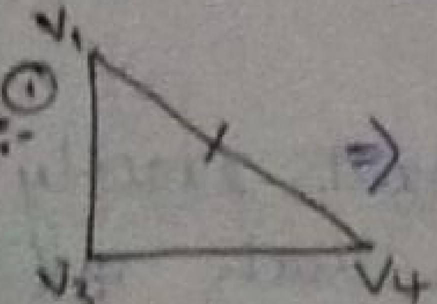


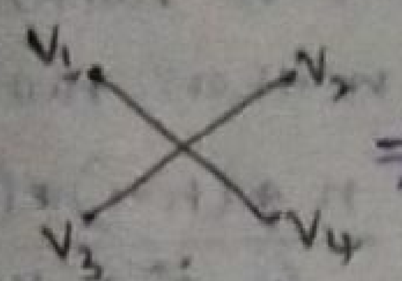
⇒ graph G

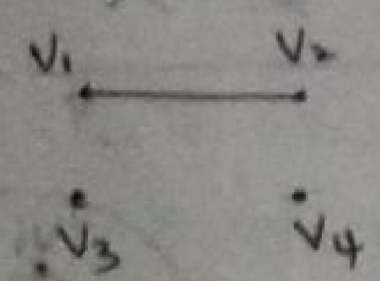
Sub graphs :-



② G:- 

Subgraph:- ①  ⇒ It is not a subgraph because V_1 and V_4 are not connected in graph G.

②  ⇒ It is not a subgraph

③  ⇒ It is a subgraph.

Spanning subgraph:-

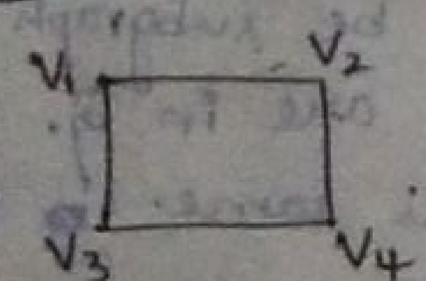
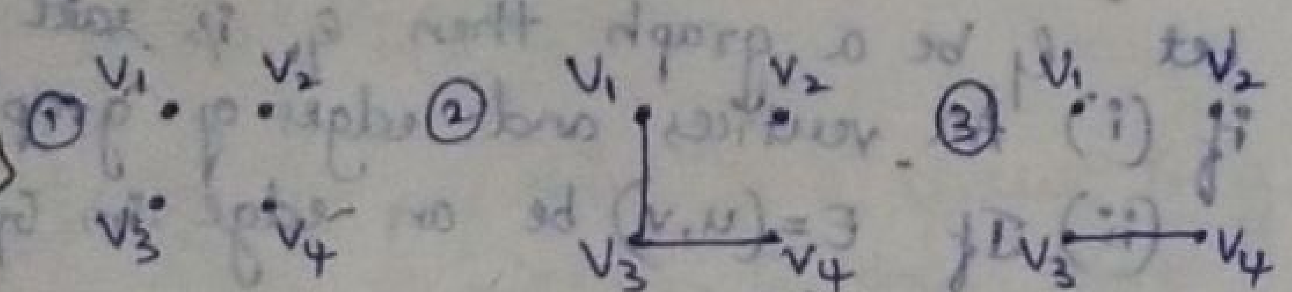
A spanning subgraph of graph G is a subgraph of G containing all the vertices of G.

Conditions:-

(i) It should be a subgraph

(ii) It should contains all the vertices of G.

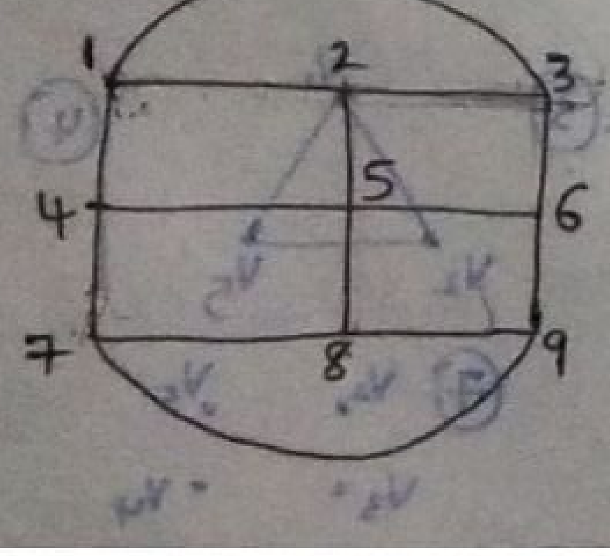
Example:-

 ⇒ 

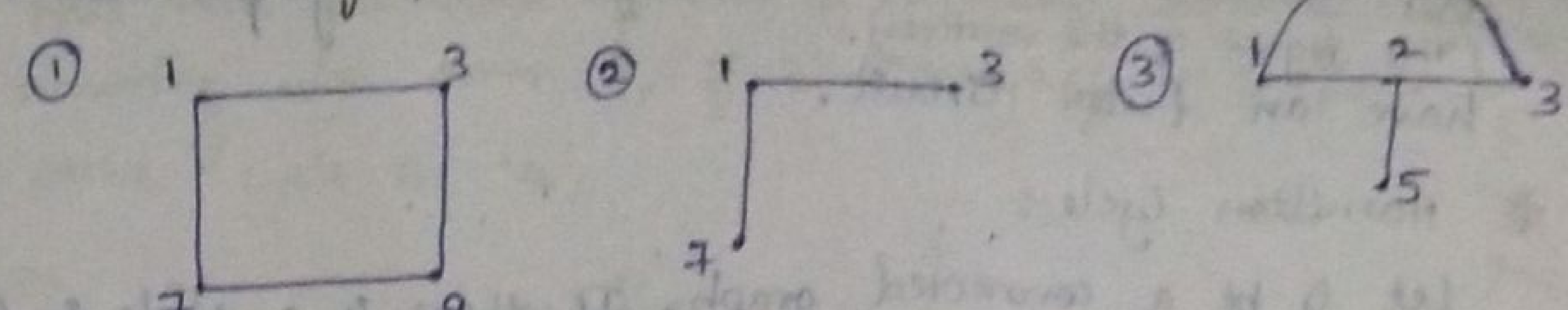
Induced subgraph:-

A subgraph H ~~edge~~ of graph G is called Induced subgraph if for any two vertices u, v in H, u and v are adjacent in H if and only if they are adjacent in G.

Example:-

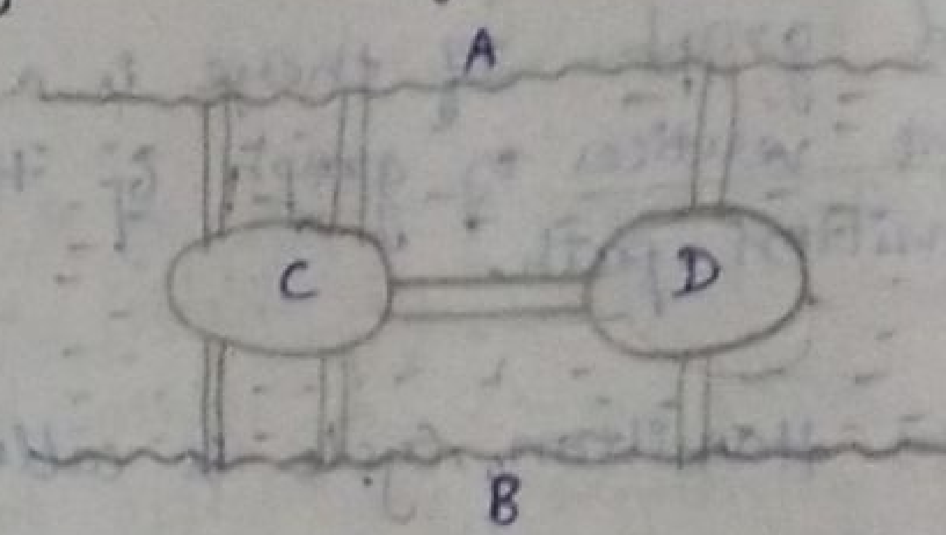


Induced subgraphs:-



Theorem:- (Konigsberg bridge problem).

A connected graph G has an Euler circuit if and only if all vertices of even degree.



In 18th century city name konigsberg in Europe, ~~they~~ their flowed a river named piegel river which divided city into 4 parts. Two of these parts were banks of the river and two were islands. These parts were connected with each other through seven bridges.

The problem was to start from any one of the land area, walk across each bridge exactly once and return to the starting point.

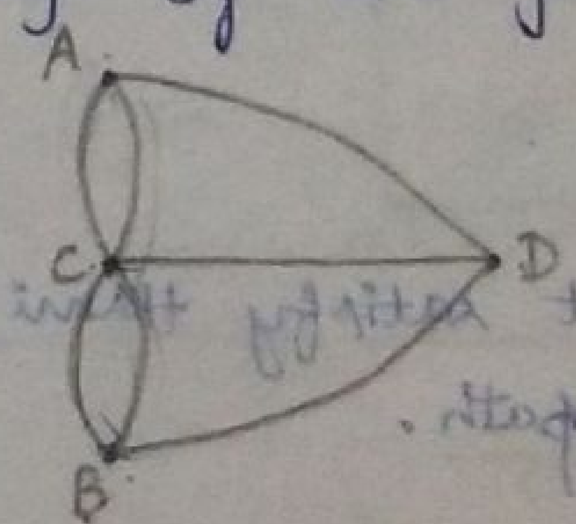
where A, B, C, D are land areas of city. A and B are banks of the river. C and D are islands. This problem now known as Konigsberg bridge problem.

Remained unsolved for several years.

Euler abstracted the problem,

① By replacing each land area by point or vertex.

② Each bridge by line joining the points.



deg A = 3
deg B = 3
deg C = 5
deg D = 3

which are not even (The degree of the vertices). Therefore the graph does not have an Euler Circuit.

Hamilton Cycle :-

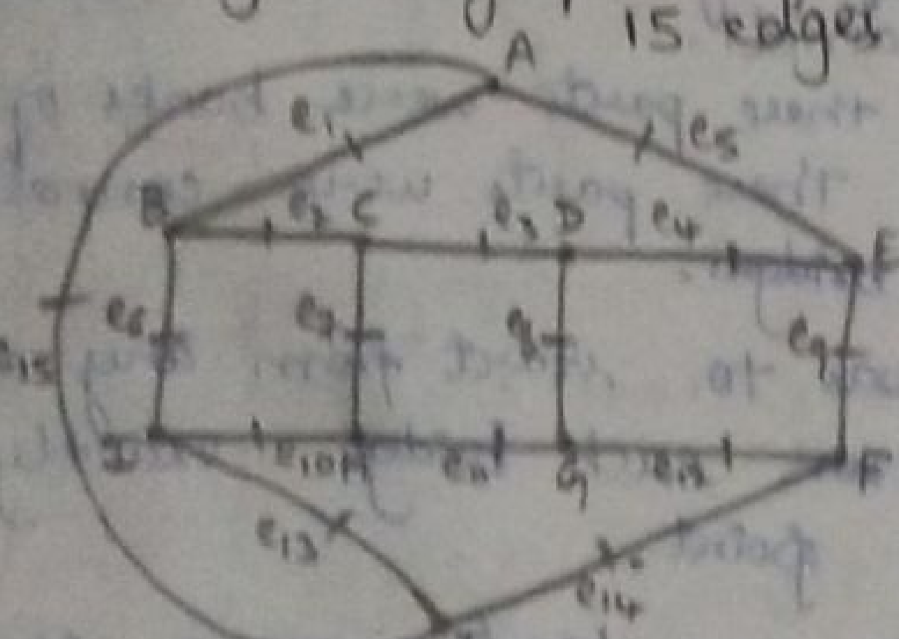
Imp Let G be a connected graph. If there is a cycle in G that contains all the vertices of graph G then that cycle is called Hamilton Cycle.

Hamilton Path :-

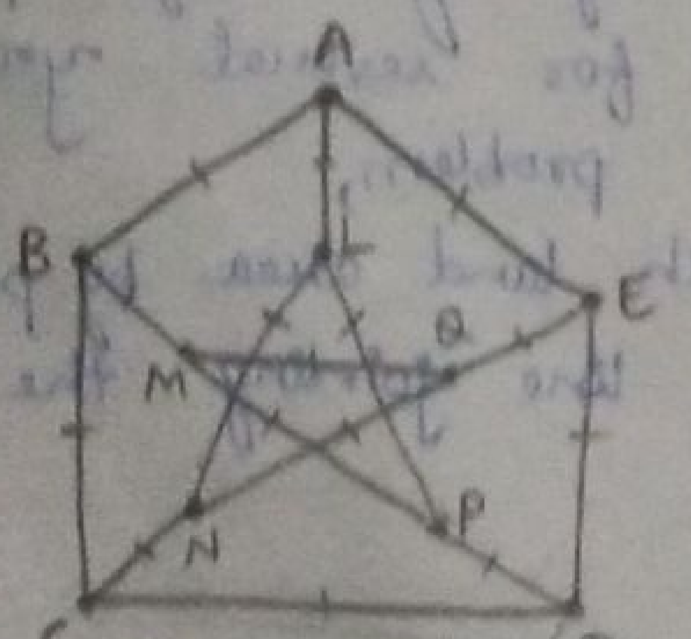
Imp Let G be a connected graph. If there is a path in G that contains all the vertices of graph G then that path is called Hamilton path.

→ A Graph that contains Hamilton Cycle is called Hamilton graph.

(1) Draw a 3-regular graph with 10 vertices and 15 edges / Peterson graph :-



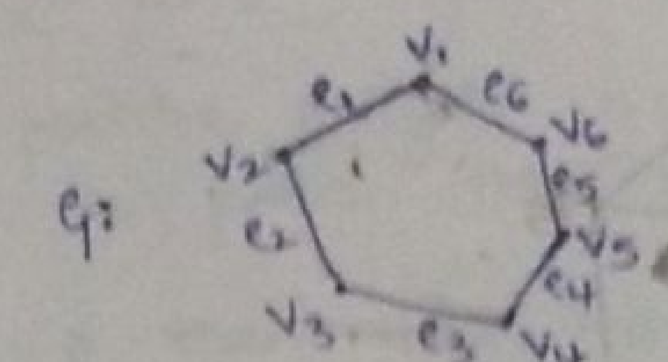
(or)



* Peterson graph does not satisfy Hamilton cycle but it satisfies Hamilton path.

Prove that connected graph 'G' remains connected after removing edge from e_r if and only if 'e' is part of same cycle in 'e_r'.

⇒

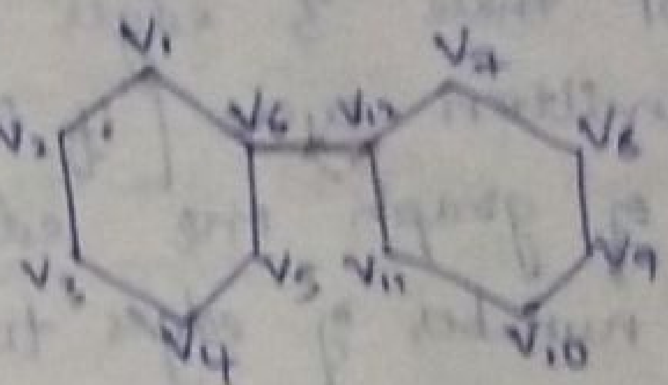


(i) If 'e' is part of some cycle in 'e_r'.

$V_1 e_1 V_2 e_2 V_3 e_3 V_4 e_4 V_5 e_5 V_6 e_6 V_1$

Suppose e is a part of some cycle C in 'G'. Then end vertices of G V are joined by atleast two paths. One of which is e and other is C − e. Hence the removal of e from G will not affect the connectivity of G. Because even after removal of e the end vertices remains connected.

(ii) Conversely,

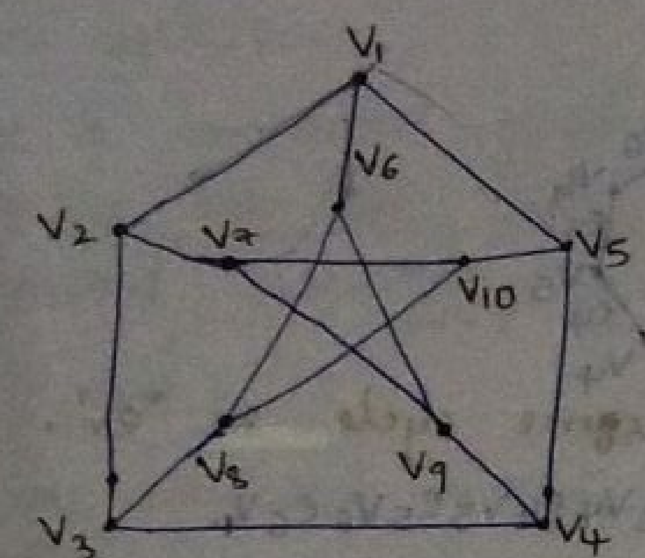


$V_1V_2V_3V_4V_5V_6V_1$ $V_{11}V_1V_7V_8V_9V_{10}V_{11}$

Suppose e is not a part of any cycle in G. then the end vertices of e are connected by atmost one path. Hence removal of e from G disconnects these end points. This means G−e is disconnected graph. Thus if e is not a part of any cycle in G then G−e is disconnected. This is equivalent to saying that G−e is connected if and only if e is part of some cycle C in G.

Show that Peterson graph has no Hamilton cycle in it But it has Hamilton path.

⇒



The Peterson graph is a 3-regular graph with 10 vertices 15 edges. The graph is shown with the vertices labelled as $V_1 V_2 V_3 V_4 V_5 V_6 V_7 V_8 V_9 V_{10}$. Since the graph has 10 vertices and 15 edges, A Hamilton cycle if any in the graph must pass through all the 10 vertices. We observe that 3 edges are incident at every vertex of the graph. Of these 3 edges only two(2) can be included in a Hamilton cycle [if it exists]. Thus at each of 10 vertices of graph one edge as to be excluded. We find the number of edges to be excluded is 6. Consequently the number of edges remain in the graph is 9. These edges are insufficient to form a Hamilton cycle in a graph. Thus peterson graph does not contain Hamilton cycle.

By examining the figure we know the edges [$V_1V_2, V_2V_3, V_3V_4, V_4V_5, V_5V_{10}, V_{10}V_8, V_8V_6, V_6V_9, V_9V_7, V_7V_{10}$] form a path which includes all the vertices.

This path is called Hamilton path. Thus peterson graph has Hamilton path.

~~Show that connected~~ A connected graph with n vertices has atleast $n-1$ edges.

⇒ Since the graph is connected, $n \geq 2$. If m denotes numbers of edges, we have to prove that $m \geq n-1$.

For every positive integer $n \geq 2$ we employ (use) the method of induction to prove this result.

① Suppose $n=2$. then there are exactly two vertices in a graph and since the graph is connected there must be atleast one edge joining these vertices. $\begin{bmatrix} m \geq n-1 \\ 1 \geq 2-1 \\ 1 \geq 1 \end{bmatrix}$.

This verify result is true for $n=2$. Assume that result is true for all connected graphs with $n=k$ number of vertices where k is a positive integer greater than 2. Now consider a connected graph say G_{k+1} with $k+1$ vertices.

Choose a vertex V of this graph and consider graph G_k obtained by deleting an edge from G_{k+1} for which V is the end vertex. Then G_k is a connected graph with k vertices. Let m_k be the number of edges. Then from the assumption made in the preceeding paragraph we have $m_k \geq n-1$

$$m_k+1 \geq (n+1)-1.$$

m_k+1 is the number of edges in G_{k+1} and $k+1$ is the number of vertices in G_{k+1}. Thus the result $m \geq n-1$ is true for $n=k+1$. Hence by induction, the result is true for all integers ≥ 2. Hence it proved.

Prove that in a graph there is a u-v trial if and only if there is a u-v path.