22/07/24

Module – 04

The Principle of Inclusion – Exclusion

Counting

Formula:-

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum A_i - \sum A_i \cap A_j + \sum A_i \cap A_j \cap A_k - \ldots + (-1)^{n-1} \sum A_i \cap A_j \cap \ldots \cap A_n$$

A and B are any two subsets of 'S'

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|\overline{A \cup B}| = |S| - |A \cup B|$$

Examples:-

① Among the students in a hostel, 12 are studying mathematics (A), 20 study physics (B), 20 study chemistry (C) and 8 study biology (D). There are 5 students for A and B, 7 students for A and C, 4 students for A and D. 16 students for B and C, 4 students for B and D and 3 students for C and D. There are 3 students for A, B and C, 2 for A, B and D, 2 for B, C & D, 3 for A, C and D. Finally there are (2) two who study all of these subjects. Furthermore there are 71 students whose do not study any of these students. Find total number of students in the hostel.

Soln:- $|A| = 12$, $|B| = 20$, $|C| = 20$, $|D| = 8$

$|A \cap B| = 5$, $|A \cap C| = 7$, $|A \cap D| = 4$

$|B \cap C| = 16$, $|B \cap D| = 4$, $|C \cap D| = 3$

$|A \cap B \cap C| = 3$, $|A \cap B \cap D| = 2$, $|B \cap C \cap D| = 2$, $|A \cap C \cap D| = 3$,

$|A \cap B \cap C \cap D| = 2$, $|\overline{A \cap B \cap C \cap D}| = 71$

$|A \cup B \cup C \cup D| = \{|A| + |B| + |C| + |D|\} - \{|A \cap B| + |A \cap C| + |A \cap D| + |B \cap C| + |B \cap D| + |C \cap D|\} + \{|A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D|\} - \{|A \cap B \cap C \cap D|$

$= \{12 + 20 + 20 + 8\} - \{5 + 7 + 4 + 16 + 4 + 3\} + \{3 + 2 + 2 + 3\} - 2$

$= 60 - 39 + 10 - 2$

$= 29$

$S - |A \cup B \cup C \cup D| = \overline{A \cap B \cap C \cap D}$

$S - 29 = 71$

$S = 71 + 29$

$\boxed{S = 100}$

② Determine the number of positive integers n such that $1 \le n \le 100$ and n is not divisible by 2, 3 or 5.

Soln:- $|S| = 100$

$|A| = 100/2 = 50$

$|B| = 100/3 = 33$

$|C| = 100/5 = 20$

$|A \cap B| = \frac{100}{2 \times 3} = 16$, $|A \cap C| = \frac{100}{2 \times 5} = 10$

$|B \cap C| = \frac{100}{3 \times 5} = 6$, $|A \cap B \cap C| = \frac{100}{2 \times 3 \times 5} = 3$

$|A \cup B \cup C| = \{|A| + |B| + |C|\} - \{|A \cap B| + |A \cap C| + |B \cap C|\} + |A \cap B \cap C|$

$= \{50 + 33 + 20\} - \{16 + 10 + 6\} + 3$

$= 74$

$|\overline{A \cup B \cup C}| = |S| - |A \cup B \cup C|$

$= 100 - 74$

$= 26$

③ How many integers between 1 and 300 (inclusive) are

(i) Divisible by atleast one of 5, 6, 8

(ii) Divisible by none of 5, 6, 8

Soln:- $|S| = 300$

$|A| = 300/5 = 60$

$|B| = 300/6 = 50$

$|C| = 300/8 = 37$

$|A \cap B| = \frac{300}{5 \times 6} = 10$ gcd(5,6) = 1

$|A \cap C| = \frac{300}{5 \times 8} = 7$ gcd(5,8) = 1

If gcd ≠ 1 then take LCM

$|B \cap C| = \frac{300}{24} = 12$

$|A \cap B \cap C| = \frac{300}{\underset{120}{5 \times 6 \times 8}} = 2$

$$\begin{array}{r|l} 2 & 6, 8 \\ 3 & 3, 4 \\ 2 & 1, 4 \\ 2 & 1, 2 \\ & 1, 1 \end{array}$$

(i) $|A \cup B \cup C| = \{|A|+|B|+|C|\} - \{|A\cap B| + |A\cap C| + |B\cap C|\} + |A\cap B\cap C|$

$= \{60+50+37\} - \{10+7+12\} + 2$

$= 147 - 29 + 1$

$= 1280$

(ii) $|\overline{A\cup B\cup C}| = |S| - |A\cup B\cup C|$

$= 300 - 1280$

$= 180$

(3) In how many ways 5 number of a's, 4 number of b's, and 3 number of c's can be arranged so that all the identical letters are not in a single block.

Soln:-

$5 \rightarrow a's$

$4 \rightarrow b's$

$3 \rightarrow c's$

$$|S| = \frac{12!}{5!\times 4!\times 3!} = 27{,}720.$$

$A = \boxed{aaaaa}\ bbbb\ ccc$

$$= \frac{8!}{1!\times 4!\times 3!} = 280$$

$B = \boxed{bbbb}\ aaaaa\ ccc$

$$= \frac{9!}{1!\times 5!\times 3!} = 504$$

$C = \boxed{ccc}\ aaaaa\ bbbb$

$$= \frac{10!}{1!\times 5!\times 4!} = 1260$$

$A\cap B = \boxed{aaaaa}\boxed{bbbb}\ ccc$

$$= \frac{5!}{1!\times 1!\times 3!} = 20$$

$A\cap C = \boxed{aaaaa}\boxed{ccc}\ bbbb$

$$= \frac{6!}{1!\times 1!\times 4!} = 30$$

$B\cap C = \boxed{bbbb}\boxed{ccc}\ aaaaa$

$$= \frac{7!}{1!\times 1!\times 5!} = 42.$$

$A\cap B\cap C = 3! = 6$

$|A\cup B\cup C| = \{|A|+|B|+|C|\} - \{|A\cap B| + |A\cap C| + |B\cap C|\} + \{A\cap B\cap C\}$

$= \{280+504+1260\} - \{20+30+42\} + 6$

$= 2044 - 92 + 6$

$= 1958$

$|S| - |A\cup B\cup C| = |\overline{A\cup B\cup C}|$

$27{,}720 - 1958 = 25762$

(4) Out of 30 students in a hostel, 15 study history, 8 study economics and 6 study geography it is known that 3 students study all of these subjects. show that 7 or more students study none of subjects.

Soln:-

$|S| = 30$

$|A| = 15$

$|B| = 8$

$|C| = 6$

$|A\cap B\cap C| = 3$

$|A\cup B\cup C| = \{|A|+|B|+|C|\} - \{|A\cap B| + |A\cap C| + |B\cap C|\} + A\cap B\cap C$

$\Rightarrow |A\cup B\cup C| + |\overline{A\cup B\cup C}| = 30$

$|A\cup B\cup C| = |A| + |B| + |C| - |A\cap B| - |A\cap C| - |B\cap C| + |A\cap B\cap C|$

$|A\cap B| + |A\cap C| + |B\cap C| = |A| + |B| + |C| + |A\cap B\cap C| - |A\cup B\cup C|$

$|A\cap B| + |A\cap C| + |B\cap C| = (23) - |A\cup B\cup C|$

$30 - 7 = 23$

But

$|A\cap B| + |A\cap C| + |B\cap C| \geq 0,$

because are non-negative integers.

Then $|A\cup B\cup C| \leq 23$ this implies

$|\overline{A\cup B\cup C}| \geq 7.$

⑤ Find number of permutation of the digits 1 through 9, in which

(a) the block 23, 57 and 468 do not appear.

(b) the block ~~~~ 36, 78 and 672 do not appear.

Sol: let 'S' denote set of permutation of digits 1 through 9 (without repitation) i.e. $|S| = 9!$

(a) let A_1 be the subset of S containing the block 23. Thus A_1 consists of all permutations which contain the block 23 as a single object and the 7 remaining objects 1, 4, 5, 6, 7, 8, 9.

1, 2, 3, 4, 5, 6, 7, 8, 9 $\quad |A_1| = (1+7)! = 8!$ $\quad A_1 = 23 = 2 \Rightarrow 9-2 = 7$

7 $\quad |A_2| = (1+7)! = 8!$ $\quad A_2 = 57 = 2 \Rightarrow 9-2 = 7$

$|A_3| = (1+6)! = 7!$ $\quad A_3 = 468 = 3 \Rightarrow 9-3 = 6$

$A_1 \cap A_2 = (2+5)! = 7!$

$A_1 \cap A_3 = (2+4)! = 6!$

$A_2 \cap A_3 = (2+4)! = 6!$

$A_1 \cap A_2 \cap A_3 = (3+2)! = 5! \Rightarrow$ 23, 57, 468 7 $\quad 9-7 = 2$

$A_1 \cup A_2 \cup A_3 = |A_1| + |A_2| + |A_3| - \{|A_1\cap A_2| + |A_1\cap A_3| + |A_2\cap A_3|\} + \{|A_1\cap A_2\cap A_3|\}$

$= (8! + 8! + 7!) - (7! + 6! + 6!) + 5!$

$= 79320$

$\overline{A_1\cup A_2\cup A_3} = |S| - |A_1\cup A_2\cup A_3|$

$= 9! - 79320$

$= 283560$ //

(b) ~~let A_2 be the subset of S containing~~

$|A_1| = (1+7)! = 8!$ $\quad A_1 = 36 = 2 \Rightarrow 9-2 = 7$

$|A_2| = (1+7)! = 8!$ $\quad A_2 = 78 = 2 \Rightarrow 9-2 = 7$

$|A_3| = (1+6)! = 7!$ $\quad A_3 = 672 = 3 \Rightarrow 9-3 = 6$

$A_1 \cap A_2 = (2+5)! = 7!$ $\quad$ 1, 2, ~~3~~, 4, 5, ~~6~~, ~~7~~, ~~8~~, 9

$A_1 \cap A_3 = (2+5)! = 7!$ $\quad$ 1, ~~2~~, ~~3~~, 4, 5, ~~6~~, ~~7~~, 8, 9

$A_2 \cap A_3 = (2+5)! = 7!$ $\quad$ 1, ~~2~~, 3, 4, 5, ~~6~~, ~~7~~, ~~8~~, 9

$A_1 \cap A_2 \cap A_3 = (3+4)! = 7!$ $\quad$ 1, ~~2~~, ~~3~~, 4, 5, ~~6~~, ~~7~~, ~~8~~, 9

$A_1\cup A_2\cup A_3 = \{|A_1| + |A_2| + |A_3|\} - \{|A_1\cap A_2| + |A_1\cap A_3| + |A_2\cap A_3|\} + \{|A_1\cap A_2\cap A_3|\}$

$= \{8! + 8! + 7!\} - \{7! + 7! + 7!\} + 7!$

$= 85680 - 15120 + 5040$

$= 75600$

$\overline{A_1\cup A_2\cup A_3} = |S| - |A_1\cup A_2\cup A_3|$

$= 9! - 75600$

$= 287280$ //

⑥ In how many ways can the 26 letters of the english alphabet being permuted so that none of the pattern CAR, DOG, PUN or BYTE occurs?

Soln:- Let 'S' denotes set of permutation of digits 1 through 26 (without repitation) i.e. $|S| = 26!$

$A_1 = CAR$
$A_2 = DOG$
$A_3 = PUN$
$A_4 = BYTE$

$|A_1| = (1+23)! = 24!$ $\quad A_1 = \frac{CAR}{3}$ $\quad 26-3=23$

$|A_2| = (1+23)! = 24!$ $\quad A_2 = \frac{DOG}{3}$ $\quad 26-3=23$

$|A_3| = (1+23)! = 24!$ $\quad A_3 = \frac{PUN}{3}$ $\quad 26-3=23$

$|A_4| = (1+22)! = 23!$ $\quad A_4 = \frac{BYTE}{4}$ $\quad 26-4=22$

$A_1 \cap A_2 = (2+20)! = 22!$ $\quad \frac{CARDOG}{6} = 26-6=20$

$A_1 \cap A_3 = (2+20)! = 22!$ $\quad \frac{CARPUN}{6} = 26-6=20$

$A_1 \cap A_4 = (2+19)! = 21!$ $\quad \frac{CARBYTE}{7} = 26-7=19$

$A_2 \cap A_3 = (2+20)! = 22!$

$A_2 \cap A_4 = (2+19)! = 21!$

$A_3 \cap A_4 = (2+19)! = 21!$

$A_1 \cap A_2 \cap A_3 = (3+17)! = 20!$ $\quad \frac{CARDOGPUN}{9} = 26-9=17$

$A_1 \cap A_2 \cap A_4 = (3+16)! = 19!$ $\quad \frac{CARDOGBYTE}{10} = 26-10=16$

$A_1 \cap A_3 \cap A_4 = (3+16)! = 19!$

$A_2 \cap A_3 \cap A_4 = (3+16)! = 19!$

$A_1 \cap A_2 \cap A_3 \cap A_4 = (4+13)! = 17!$ $\quad \frac{CARDOGPUNBYTE}{13} = 26-13=13$

$A_1 \cup A_2 \cup A_3 \cup A_4 = \{|A_1| + |A_2| + |A_3| + |A_4|\} - \{|A_1 \cap A_2| + |A_1 \cap A_3| + |A_1 \cap A_4| + |A_2 \cap A_3| + |A_2 \cap A_4| + |A_3 \cap A_4|\} + \{|A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4|\} - \{|A_1 \cap A_2 \cap A_3 \cap A_4|\}$

$= \{24! + 24! + 24! + 23!\} - \{22! + 22! + 21! + 22! + 21! + 21!\} + \{20! + 19! + 19! + 19!\} - 17!$

$= \{3 \times 24! + 23!\} - \{3 \times 22! + 3 \times 21!\} + \{3 \times 19! + 20!\} - 17!$

$\overline{A_1 \cup A_2 \cup A_3 \cup A_4} = S - |A_1 \cup A_2 \cup A_3 \cup A_4|$

$= 26! - \{3 \times 24! + 23!\} - \{3 \times 22! + 3 \times 21!\} + \{3 \times 19! + 20!\} - 17!$

Formulas:- Generalization of Principle:-

① $E_m = S_m - \binom{m+1}{1} S_{m+1} + \binom{m+2}{2} S_{m+2} + \cdots + (-1)^{n-m} \binom{n}{n-m} S_n$ (Exactly)

② $I_m = S_m - \binom{m}{m-1} S_{m+1} + \binom{m+1}{m-1} S_{m+2} + \cdots + (-1)^{n-m} \binom{n-1}{m-1} S_n$ (Atleast)

$S_0 = |S|$
$S_1 = \sum |A_i|$
$S_2 = \sum |A_i \cap A_j|$
$S_3 = \sum |A_i \cap A_j \cap A_k|$
$S_4 = \sum |A_i \cap A_j \cap A_k \cap A_p|$

Problems:-

① Find number of permutations of english letter which contain
(a) exactly 2
(b) atleast 2
(c) exactly 3
(d) atleast 3 of the patterns CAR, DOG, PUN or BYTE.

Soln:- $A_1 = CAR$, $A_2 = DOG$, $A_3 = PUN$, $A_4 = BYTE$.

(a) Exactly two:- $m=2$

$E_m = S_m - \binom{m+1}{1} S_{m+1} + \binom{m+2}{2} S_{m+2} + \cdots + (-1)^{n-m} \binom{n}{n-m} S_n$

$E_2 = S_2 - \binom{2+1}{1} S_{2+1} + \binom{2+2}{2} S_{2+2}$

$E_2 = S_2 - \binom{3}{1} S_3 + \binom{4}{2} S_4$

$S_0 = |S| = 26!$

$S_1 = \sum |A_i| = 24! + 24! + 24! + 23! = 3 \times 24! + 23!$ $\quad \sum |A_i| = |A_1| + |A_2| + |A_3|$

$S_2 = \sum |A_i \cap A_j| = 3 \times 22! + 3 \times 21! = |A_1 \cap A_2| + |A_1 \cap A_3| + |A_1 \cap A_4| + |A_2 \cap A_3| + |A_2 \cap A_4| + |A_3 \cap A_4|$

$S_3 = \sum |A_i \cap A_j \cap A_k| = 3 \times 19! + 20!$

$S_4 = \sum |A_i \cap A_j \cap A_k \cap A_p| = 17!$

$E_2 = S_2 - \binom{3}{1} S_3 + \binom{4}{2} S_4$

$E_2 = (3 \times 22! + 3 \times 21!) - \binom{3}{1}(3 \times 19! + 20!) + \binom{4}{3} 17!$

$E_2 = 3 \times 22! + 3 \times 21! - \frac{3!}{2! \times 1!} * 3 \times 19! + 20! + \frac{4!}{1! \times 3!} * 17!$

(b) Atleast two :-

$m=2$.

$$I_m = S_m - \binom{m}{m-1} S_{m+1} + \binom{m+1}{m-1} S_{m+2} + \dots + (-1)^{n-m}\binom{n-1}{m-1} S_n$$

$$I_2 = S_2 - \binom{2}{2-1} S_{2+1} + \binom{2+1}{2-1} S_{2+2}.$$

$$I_2 = S_2 - \binom{2}{1} S_3 + \binom{3}{1} S_4$$

$S_0 = |S| = 26!$

$S_1 = \Sigma|A_i| = 24! + 24! + 24! + 23!$
$= 3 \times 24! + 23!$

$S_2 = \Sigma|A_i \cap A_j| = 3 \times 22! + 3 \times 21!$

$S_3 = \Sigma|A_i \cap A_j \cap A_k| = 3 \times 19! + 20!$

$S_4 = \Sigma|A_i \cap A_j \cap A_k \cap A_p| = 17!$

$I_2 = S_2 - \binom{2}{1} S_3 + \binom{3}{1} S_4$

$I_2 = 3 \times 21! - \binom{2}{1} 3 \times 19! + 20! + \binom{3}{1} 17!$

$I_2 = 3 \times 21! - \frac{2!}{1! \times 1!} * 3 \times 19! + 20! + \frac{3!}{2! \times 1!} * 17!$

(c) Exactly three :-

$m=3$

$E_3 = S_3 - \binom{3+1}{1} S_{3+1}$

$E_3 = S_3 - \binom{4}{1} S_4$

$S_0 = |S| = 26!$

$S_1 = \Sigma|A_i| = 3 \times 24! + 23!$

$S_2 = \Sigma|A_i \cap A_j| = 3 \times 22! + 3 \times 21!$

$S_3 = \Sigma|A_i \cap A_j \cap A_k| = 3 \times 19! + 20!$

$S_4 = \Sigma|A_i \cap A_j \cap A_k \cap A_p| = 17!$

$E_3 = S_3 - \binom{4}{1} S_4$

$E_3 = 3 \times 19! + 20! - \binom{4}{1} 17!$

$= 3 \times 19! + 20! - \frac{4!}{3! \times 1!} * 17!$

(d) Atleast three :-

$m=3$.

$$I_m = S_m - \binom{m}{m-1} S_{m+1}$$

$$I_3 = S_3 - \binom{3}{3-1} S_{3+1}$$

$$I_3 = S_3 - \binom{3}{2} S_4.$$

$S_0 = 26!$

$S_1 = 3 \times 24! + 23!$

$S_2 = 3 \times 22! + 3 \times 21!$

$S_3 = 3 \times 19! + 20!$

$S_4 = 17!$

$I_3 = S_3 - \binom{3}{2} S_4$
$= 3 \times 19! + 20! - \frac{3!}{1! \times 2!} * 17!$ //.

(2) Determine no. of integers between 1 and 300 (inclusive) which are.

(a) divisible by exactly 2 of 5, 6, 8.
(b) divisible by atleast 2 of 5, 6, 8

Soln:- $n = 300$, $|A_1| = \frac{300}{5} = 60$. $|S| = 300$

$|A_2| = \frac{300}{6} = 50$

$|A_3| = \frac{300}{8} = 37$

$|A_1 \cap A_2| = \frac{300}{5 \times 6} = 10$

$|A_1 \cap A_3| = \frac{300}{5 \times 8} = 7$

$|A_2 \cap A_3| = \frac{300}{24} = 12.$

$$\begin{array}{r|l} 2 & 6,8 \\ 2 & 3,4 \\ 2 & 3,2 \\ 3 & 3,1 \\ & 1,1 \end{array}$$

$A_1 \cap A_2 \cap A_3 = \frac{300}{120} = 2$

$S_0 = N = 300$

$S_1 = \Sigma|A_i| \Rightarrow S_1 = 60 + 50 + 37 = 147$

$S_2 = \Sigma|A_i \cap A_j| \Rightarrow S_2 = 10 + 7 + 12 = 29$

$S_3 = \Sigma|A_i \cap A_j \cap A_k| \Rightarrow S_3 = 2$.

(i) Exactly two:-

$m=2$

$E_m = S_m - \binom{m+1}{1} S_{m+1} + \cancel{\binom{m+2}{2} S_{m+2}}$

$E_2 = S_2 - \binom{2+1}{1} S_{2+1} + \cancel{\binom{2+2}{2} S_{2+2}}$

$E_2 = S_2 - \binom{3}{1} S_3 + \cancel{\binom{4}{2} S_4}$

$= 29 - \frac{3!}{2! \times 1!} * (2)$ $\qquad {}^nC_r = \frac{n!}{(n-r)!\,r!} = \binom{n}{r}$

$= 29 - 6$

$= 23$

(ii) Atleast two :-

$m=2$

$I_2 = S_2 - \binom{2}{1} S_3$

$= 29 - \frac{2!}{1! \times 1!} * (2)$

$= 29 - 4$

$= 25$

③ In how many ways can one arrange the letters in the word CORRESPONDENTS so that

(a) There is no pair of consecutive identical letters

(b) There are exactly 2 pairs of consecutive identical letters.

(c) There are atleast 3 pair of consecutive identical letters.

Soln:-

CORRESPONDENTS.

$C \to 1$
$O \to 2$
$R \to 2$
$E \to 2$
$S \to 2$
$P \to 1$
$N \to 2$
$D \to 1$
$T \to 1$

14

$\cancel{S_0 = \frac{14!}{2!\,2!}}$

$S_0 = \frac{14!}{2!\,2!\,2!\,2!\,2!}$

$A_1 = O, \quad A_2 = R, \quad A_3 = E, \quad A_4 = S, \quad A_5 = N$

$|A_1| = \frac{13!}{2! \times 2! \times 2! \times 2!}, \quad |A_2| = \frac{13!}{2!\,2!\,2!\,2!}$

$|A_3| = \frac{13!}{2!\,2!\,2!\,2!}, \quad |A_4| = \frac{13!}{2!\,2!\,2!\,2!}$

$|A_5| = \frac{13!}{2!\,2!\,2!\,2!}$

$A_1 = \boxed{O}$ R E S N C P D T

$|A_i \cap A_j| = \frac{12!}{2!\,2!\,2!}$ $\quad \boxed{OR}$ E S N C P P T

$|A_i \cap A_j \cap A_k| = \frac{11!}{(2!)^2}$ $\quad \boxed{ORE}$ S N C P P T

$|A_i \cap A_j \cap A_k \cap A_p| = \frac{10!}{2!}$ $\quad \boxed{ORES}$ N C P P T

$|A_i \cap A_j \cap A_k \cap A_p \cap A_s| = 9!$ $\quad \boxed{ORESN}$ C P P T

$S_0 = N = \frac{14!}{2!\,2!\,2!\,2!\,2!} = \frac{14!}{(2!)^5}$

$S_1 = \Sigma|A_i| = \left(\frac{13!}{2!\,2!\,2!\,2!}\right) * 5 = \frac{13! \times 5}{(2!)^4}$

$S_2 = \Sigma|A_i \cap A_j| = \left(\frac{12!}{2!\,2!\,2!}\right) * 10 = \frac{12!}{(2!)^3} * 10$

$S_3 = \Sigma|A_i \cap A_j \cap A_k| = \left(\frac{11!}{2!\,2!}\right) * 10 = \frac{11!}{(2!)^2} * 10$

$S_4 = \Sigma|A_i \cap A_j \cap A_k \cap A_p| = \frac{10!}{2!} \times 5$

$S_5 = 9!$

(a) No pairs :-

$m=0$

$E_0 = S_0 - \binom{1}{1} S_1 + \binom{2}{2} S_2 - \binom{3}{3} S_3 + \binom{4}{4} S_4 - \binom{5}{5} S_5$

$$= S_0 - S_1 + S_2 - S_3 + S_4 - S_5$$

$$= \frac{14!}{(2!)^5} - \frac{13! \times 5}{(2!)^4} + \frac{12!}{(2!)^3} \times 10 - \frac{11!}{(2!)^2} \times 10 + \frac{10!}{2!} \times 5 - 9!$$

(b) $m = 2$ (2 pairs)

$$E_2 = S_2 - \binom{3}{1} S_3 + \binom{4}{2} S_4 - \binom{5}{3} S_5$$

$$= S_2 - \frac{3!}{2! \times 1!} S_3 + \left(\frac{4!}{2! \times 2!}\right) S_4 - \frac{5!}{2! \times 3!} S_5$$

$$= \frac{12!}{(2!)^3} \times 10 - \frac{3!}{2! \times 1!} * \frac{11!}{(2!)^2} \times 10 + \frac{4!}{2! \times 2!} * \frac{10!}{2!} \times 5 - \frac{5!}{2! \times 3!} * 9!$$

(c) $m = 3$ (3 pairs)

$$I_3 = S_3 - \binom{3}{2} S_4 + \binom{4}{2} S_5 \qquad \binom{n}{r} = \frac{n!}{(n-r)! \times r!}$$

$$= \frac{11! \times 10}{(2!)^2} - \frac{3!}{1! \times 2!} * \frac{10!}{2!} \times 5 + \frac{4!}{2! \times 2!} * 9!$$

Derangements:-

A permutation of n distinct object in which none of the object is in its original place is called the Derangements.

For Example,

A permutation of integers $1, 2, 3, \ldots, n$ in which 1 is not in the first place, 2 is not in the second place, 3 is not in the third place and so on n is not in the n^{th} place is a derangements.

If there is only one object, it continues to be its original place in every arrangement. Therefore

$d_1 = 0$

* $$d_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right)$$

$\boxed{OR}$

This is formula to find derangements for $n \geq 1$.

$$\boxed{d_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}}$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^1 = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

$$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \cdots$$

$$\boxed{e^{-1} = \sum_{k=0}^{n} \frac{(-1)^k}{k!}}$$

$$e^{-1} = \frac{1}{e}$$
$$= 0.36788$$

$$\boxed{\sum_{k=0}^{n} \frac{(-1)^k}{k!} = 0.36788}$$

$$\sum_{k=0}^{n} \frac{(-1)^k}{k!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!}$$
$$= 0.36879$$

$\therefore$ For $\boxed{n \geq 7}$ $d_n \cong (n! \times e^{-1})$

$$\Rightarrow \boxed{d_n \cong (n! \times 0.3679)}$$

---

① Find number of derangements of 1, 2, 3 and 4.

Soln:- $d_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$

$\boxed{n=4}$

$$d_4 = 4!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right)$$

$$= 24\left(\cancel{1} - \cancel{1} + \frac{1}{2} - \frac{1}{6} + \frac{1}{24}\right)$$

$$= 24\left(\frac{3}{8}\right)$$

$$= 9$$

⇒	1	2	3	4
①	2	1	4	3
②	2	4	1	3
③	2	3	4	1
④	3	4	2	1
⑤	3	1	4	2
⑥	4	3	2	1
⑦	4	1	2	3
⑧	4	3	1	2
⑨	3	4	1	2

② Evaluate d_5, d_6, d_7, d_8.

Sol^n:- $\boxed{n=5}$

$$d_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

$$d_5 = 5!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right)$$

$$\boxed{d_5 = 44}$$

$\boxed{n=6}$

$$d_6 = 6!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!}\right)$$

$$\boxed{d_6 = 265}$$

$\boxed{n=7}$

$$d_n = (n! \times e^{-1})$$
$$= (n! \times 0.3679)$$
$$= (7! \times 0.3679)$$
$$d_7 = 1854.216$$
$$\boxed{d_7 = 1854}$$

$\boxed{n=8}$

$$d_8 = (n! \times e^{-1})$$
$$d_8 = (n! \times 0.3679)$$
$$= (8! \times 0.3679)$$
$$\boxed{d_8 = 14833}$$

③ In how many ways can we arrange 1, 2, 3, ---- 10 so that 1 is not in first place, 2 is not in second place and so on 10 is not in 10^{th} place

Sol^n:- $n=10$

$$d_n = (n! \times e^{-1})$$
$$= (10! \times 0.3679)$$
$$\boxed{d_{10} = 1335035}$$

④ There are 8 letters to 8 different people to be placed in 8 different addressed envelop. Find number of ways of doing this so that atleast one letter gets to right person.

Sol^n:- $\boxed{n=8}$ Total number of permutations — Derangements

$$8! - d_8$$

$\boxed{n \geq 7}$

$$d_n = (n! \times e^{-1})$$
$$d_8 = 8! \times 0.3679$$
$$= 8! - d_8$$
$$= 8! - (8! \times 0.3679)$$
$$= \underline{25487}$$

⑤ Find number of dearangements of the integer from 1 to n (inclusive) such that in each dearangement

(i) the elements in the first k places are 1, 2, 3 ---, k in some order

(ii) the elements in the first n−k places are k+1, k+2, ----- n in some order [$0 < k < n$].

Sol^n:-

$$d_k \times d_{n-k}$$

Rook Polynomial :-

Consider a ~~bord~~ board that represent full chess board or a part of it. Let n be number of squares presenting in the board. Two pawns or rooks placed on board said to be in non — capturing position, If they are not in same row or same column.

For $2 \leq k \leq n$, Let r_k denote number of ways in which k rooks can be placed on a board such that no two rooks capture each other. Then the polynomial $1 + r_1x + r_2x^2 + \dots + r_nx^n$ is said to be rook polynomial for the considered board. If C denotes the board then polynomial is given by,

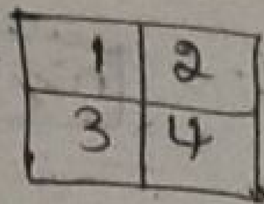
$$r(C, x) = 1 + r_1x + r_2x^2 + \dots + r_nx^n.$$

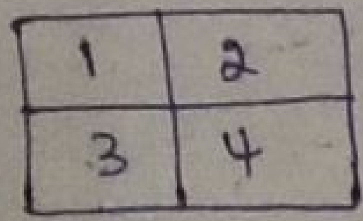
r_1 = number of squares present in the board.

Note:-

If we have only one square box then Rook polynomial is $\boxed{r(C, x) = 1 + x}$

Problems :-

① Find Rook Polynomial for the board 

Soln :- 

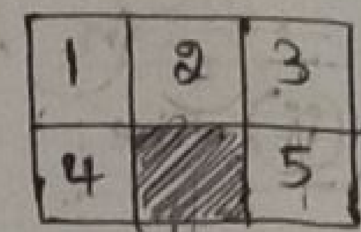
no. of element right
no. of element down

$r(C,x) = 1 + r_1 x + r_2 x^2 + \cdots + r_n x^n$

$r_1 = 4$

$r_2 = (1,4), (2,3) = 2$

$\therefore \boxed{r(C,x) = 1 + 4x + 2x^2}$

② Find Rook polynomial for the board 

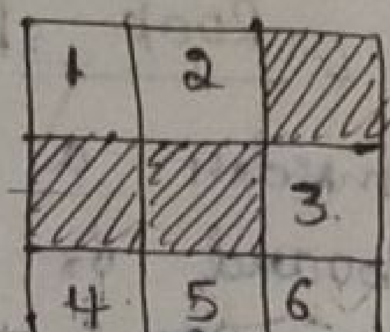
Not there

Soln :-

$r_1 = 5$

$r_2 = (1,5), (2,4), (2,5), (3,4) = 4$

$\therefore \boxed{r(c,x) = 1 + 5x + 4x^2}$

③ Find Rook polynomial for the board 

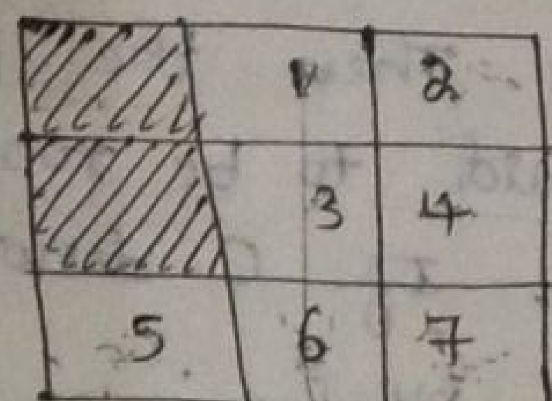
Soln :-

$r_1 = 6$

$r_2 = (1,3), (1,5), (1,6), (2,3), (2,4), (2,6), (3,4), (3,5) = 8$

$r_3 = (1,3,5), (2,3,4) = 2$

$\therefore \boxed{r(C,x) = 1 + 6x + 8x^2 + 2x^3}$

④ 

Soln :-

$r_1 = 7$

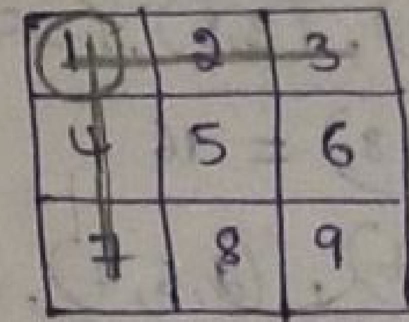
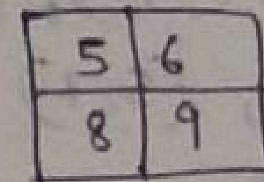
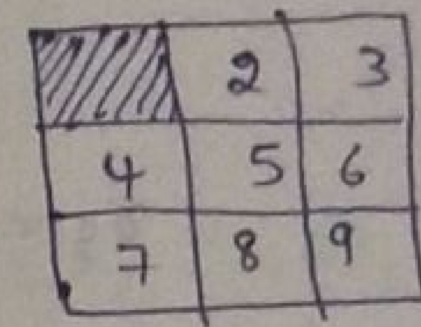
$r_2 = (1,4), (1,5), (2,3), (2,5), (2,6), (1,7), (3,5), (3,7), (4,5), (4,6) = 10$

$r_3 = (2,3,5), (1,4,5) = 2$

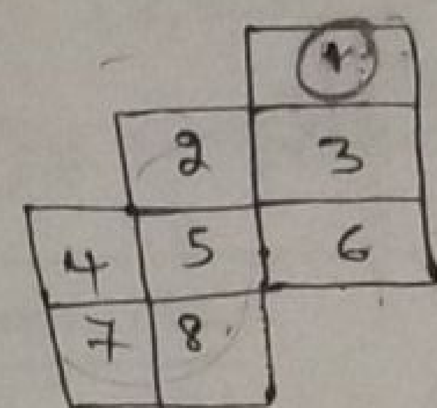
$\therefore \boxed{r(C,x) = 1 + 7x + 10x^2 + 2x^3}$

Expansion formula :-

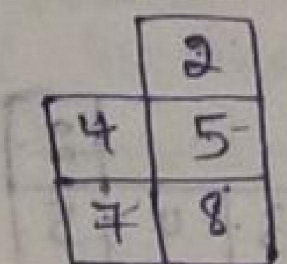
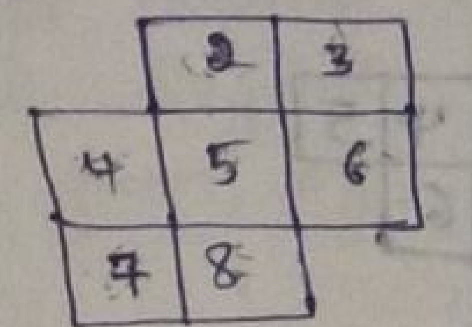
$\boxed{r(C,x) = x\, r(D,x) + r(E,x)}$

Eg :-  D =  E = 

① By using Expansion formula obtain Rook polynomial for the board C.



Soln :-

D =  E = 

$r_1 = 5$

$r_2 = (2,4), (2,7), (4,8), (5,7) = 4$

$\therefore \boxed{r(D,x) = 1 + 5x + 4x^2}$

$r_1 = 7$

$r_2 = (2,4), (2,6), (2,7), (3,4), (3,5), (3,7), (3,8), (4,8), (5,7), (6,7), (6,8) = 11$

$r_3 = (2,6,7), (3,5,7), (3,4,8) = 3$

$\therefore \boxed{r(E,x) = 1 + 7x + 11x^2 + 3x^3}$

$r(C,x) = x\, r(D,x) + r(E,x)$

$r(C,x) = x(1 + 5x + 4x^2) + 1 + 7x + 11x^2 + 3x^3$

$= x + 5x^2 + 4x^3 + 1 + 7x + 11x^2 + 3x^3$

$\boxed{r(C,x) = 1 + 8x + 16x^2 + 7x^3}$

[OR]

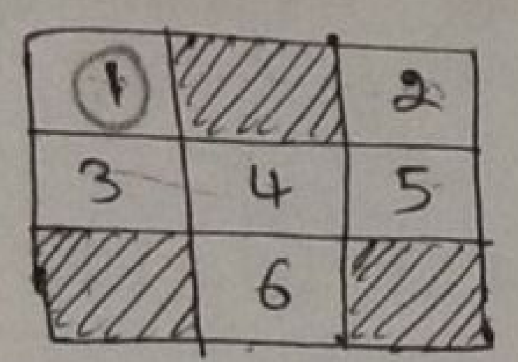
$r_1 = 8$

$r_2 = (1,2), (1,4), (1,5), (1,7), (1,8),$
$(2,4), (2,6), (2,7), (3,4), (3,5), (3,7), (3,8),$
$(4,8), (5,7), (6,7), (6,8) = 16$

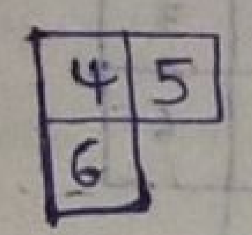
$r_3 = (1,2,4), (1,27), (1,5,7), (2,6,7), (3,5,7),$
$(3,4,8), (1,48) = 7$

$\therefore \boxed{r(C,x) = 1 + 8x + 16x^2 + 7x^3}$

② By using Expansion formula obtain Rook polynomial for the board C.



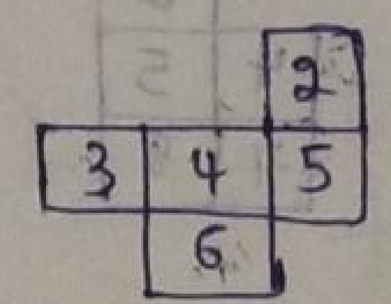
Soln:-

$D =$ 

$r_1 = 3$

$r_2 = (5,6) = 1$

$\therefore \boxed{r(D,x) = 1 + 3x + 1x^2}$

$E =$ 

$r_1 = 5$

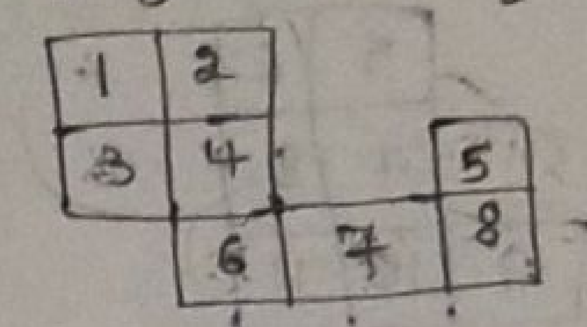
$r_2 = (2,3), (2,4), (2,6), (3,6), (5,6) = 5$

$r_3 = (2,6,3) = 1$

$\therefore \boxed{r(E,x) = 1 + 5x + 5x^2 + 1x^3}$

$$r(C,x) = x \cdot r(D,x) + r(E,x)$$
$$= x(1 + 3x + 1x^2) + (1 + 5x + 5x^2 + 1x^3)$$
$$= x + 3x^2 + 1x^3 + 1 + 5x + 5x^2 + 1x^3$$
$$\boxed{r(C,x) = 1 + 6x + 8x^2 + 2x^3}$$

③ Find rook polynomial for the board shown below.



Soln:- $r_1 = 8$

$r_2 = (1,4), (1,6), (1,7), (1,8), (1,5), (2,3), (2,7), (2,5),$
$(2,8), (3,6), (3,7), (3,8), (4,7), (4,8), (5,6),$
$(5,7) = 16$

$r_3 = (1,4,7), (1,4,8), (1,6,5), (1,5,7), (2,3,7), (2,3,8),$
$(2,5,7) = 7$

$r(C,x) = 1 + 8x + 16x^2 + 7x^3$

Product formula:-

Suppose a board C is made of two parts C_1 and C_2 have no square in the same row or column of 'C', such parts of board C are called disjoint subboards of C. Then Rook polynomial is given by,

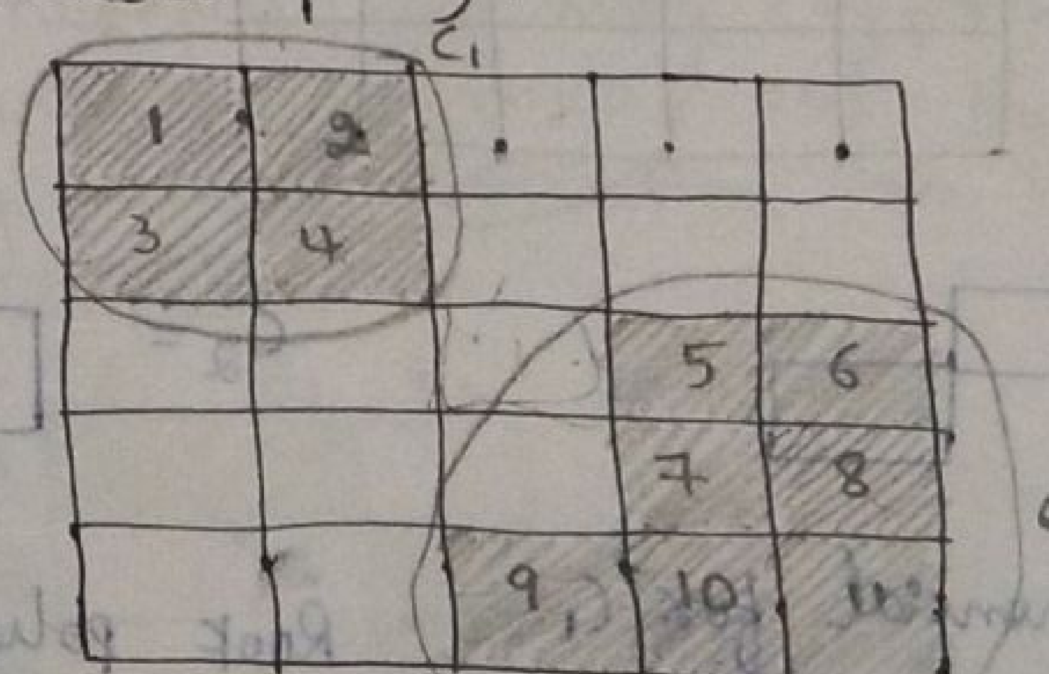
$$r(C,x) = r(C_1,x) * r(C_2,x).$$

Note:-

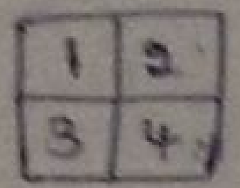
* if board C made of pair wise disjoint subboards $C_1, C_2, \ldots C_n$ then Rook polynomial is given by,

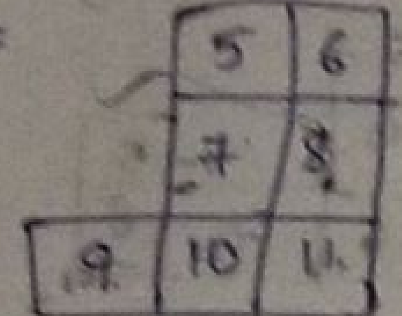
$$\boxed{r(C,x) = r(C_1,x) * r(C_2,x) * \cdots\cdots, r(C_n,x)}$$

① Find Rook polynomial for the below shown board (shaded part).



Soln:-

$C_1 =$ 

$C_2 =$ 

Rook polynomial for C_1

$r_1 = 4$

$r_2 = (1,4), (2,3) = 2$

$r(C_1,x) = 1 + 4x + 2x^2$

Rook polynomial for C_2

$r_1 = 7$

$r_2 = (5,8), (5,11), (5,9), (6,7), (6,10), (6,9), \cancel{(7,6)}, (7,11), (7,9), \cancel{(8,5)}\ (8,10), (8,9) = 10$

$r_3 = (5,8,9), (6,7,9) = 2$

$r(C_2,x) = 1 + 7x + 10x^2 + 2x^3$

$r(C,x) = r(C_1,x) * r(C_2,x)$

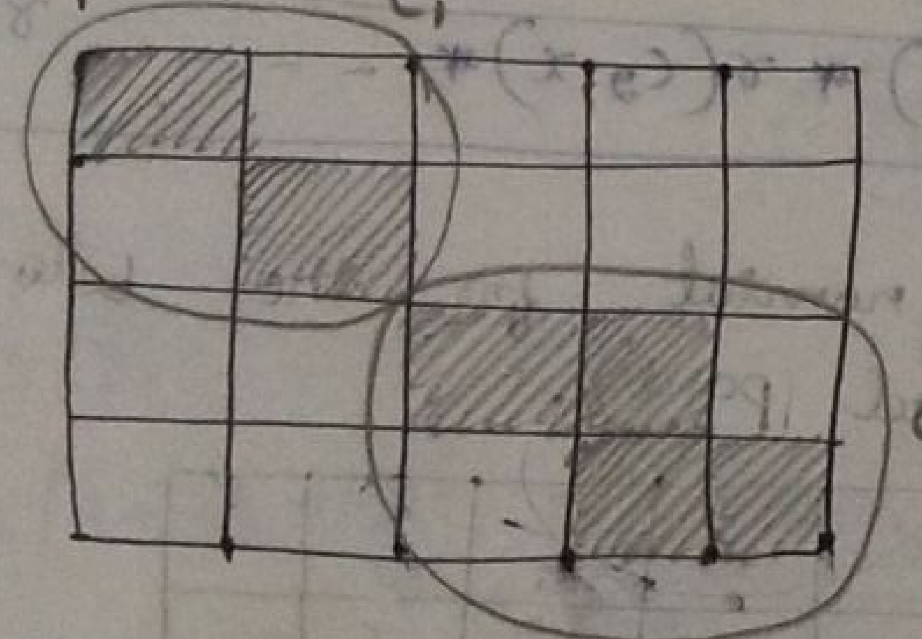
$= (1+4x+2x^2) * (1+7x+10x^2+2x^3)$

$= 1 + 7x + 10x^2 + 2x^3 + 4x + 28x^2 + 40x^3 + 8x^4 + 2x^2 + 14x^3 + 20x^4 + 4x^5$

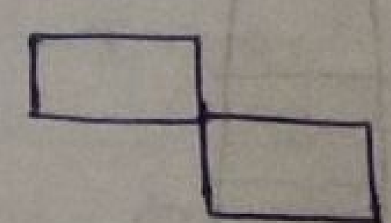
$r(C,x) = 4x^5 + 28x^4 + 56x^3 + 40x^2 + 11x + 1$

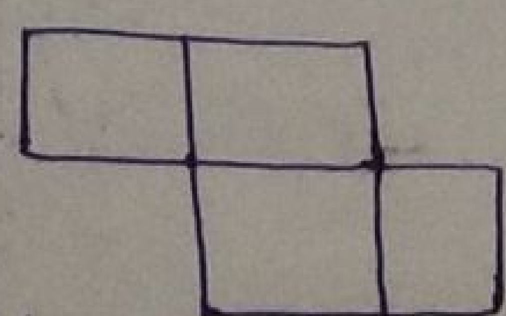
$$\boxed{r(C,x) = 1 + 11x + 40x^2 + 56x^3 + 28x^4 + 4x^5}$$ //

(2) Find rook polynomial for the below shown board (shaded part).



Soln:-

$C_1 =$ 

$C_2 =$ 

Rook polynomial for C_1

$r_1 = 2$

$r_2 = 1$

Rook polynomial for C_2

$r_1 = 4$

$r_2 = 3$

$r(C_1,x) = 1 + 2x + 1x^2$ $\qquad$ $r(C_2,x) = 1 + 4x + 3x^2$

$r(C,x) = r(C_1,x) * r(C_2,x)$

$= (1+2x+1x^2) * (1+4x+3x^2)$

$= 1 + 4x + 3x^2 + 2x + 8x^2 + 6x^3 + 1x^2 + 4x^3 + 3x^4$

$= 1 + 6x + 12x^2 + 10x^3 + 3x^4$ //

Arrangement with Forbidden Positions :-

Suppose m objects are to be arranged in n places where $n \geq m$. Suppose there are constraints ~~and~~ under which some objects cannot occupy certain places, such places are called forbidden positions for the said objects. The Number of ways of carrying out this task is given by following rule.

$\bar{N} = S_0 - S_1 + S_2 - \cdots (-1)^n S_n$

where $S_0 = n!$

and $S_k = (n-k)!\, r_k \quad (k \geq 1)$

where r_k is the coefficient of x^k in the Rook polynomial of the board containing m rows and n columns.

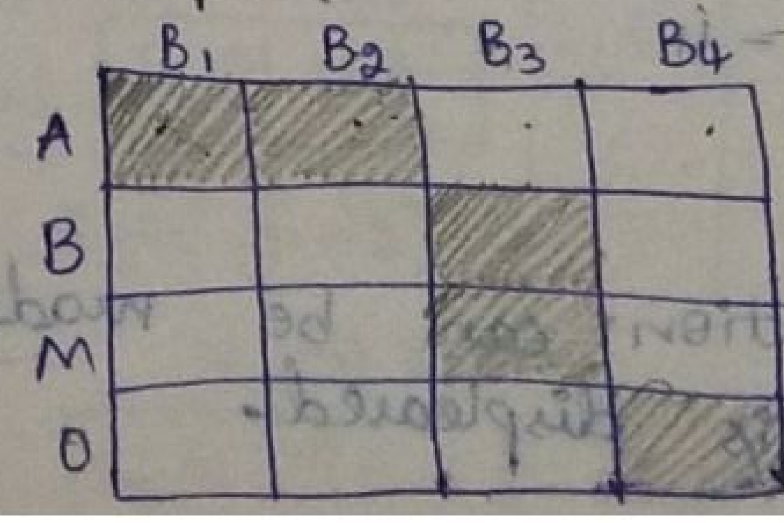
$S_1 = (n-1)!\, r_1$

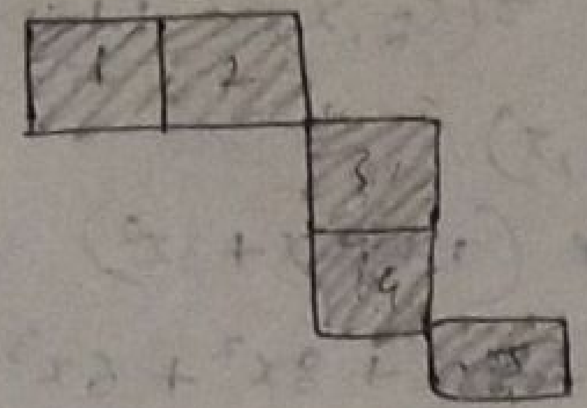
$S_2 = (n-2)!\, r_2$.

Problems :-

(1) An apple, A Banana, A Mango and An orange are to be distributed to four boys B_1, B_2, B_3 and B_4. The Boys B_1 and B_2 do not wish to have apple, The Boy B_3 does not want Banana or Mango, and B_4 refuses orange. In how many ways distribution can be made so that no boy is displeased.

Soln:- So take column as boy





$r_1 = 5$

$r_2 = 8$

$r_3 = 4$

$r(C,x) = 1 + r_1 x + r_2 x^2 + r_3 x^3$

$\boxed{r(C,x) = 1 + 5x + 8x^2 + 4x^3}$

$\bar{N} = S_0 - S_1 + S_2 - S_3$

$n = 4$

$S_0 = 4! = 24$

$n =$ no of columns

$m =$ no of rows

$S_0 = n!$

$S_1 = (n-1)!\, r_1$

$= (4-1)! \times 5$

$= 3! \times 5$

$\boxed{S_1 = 30}$

$S_2 = (n-2)! \times r_2$

$= (4-2)! \times 8$

$= 2! \times 8$

$\boxed{S_2 = 16}$

$S_3 = (n-3)!\, r_3$

$= (4-3)! \times 4$

$= 1! \times 4$

$\boxed{S_3 = 4}$

$\bar{N} = 24 - 30 + 16 - 4$

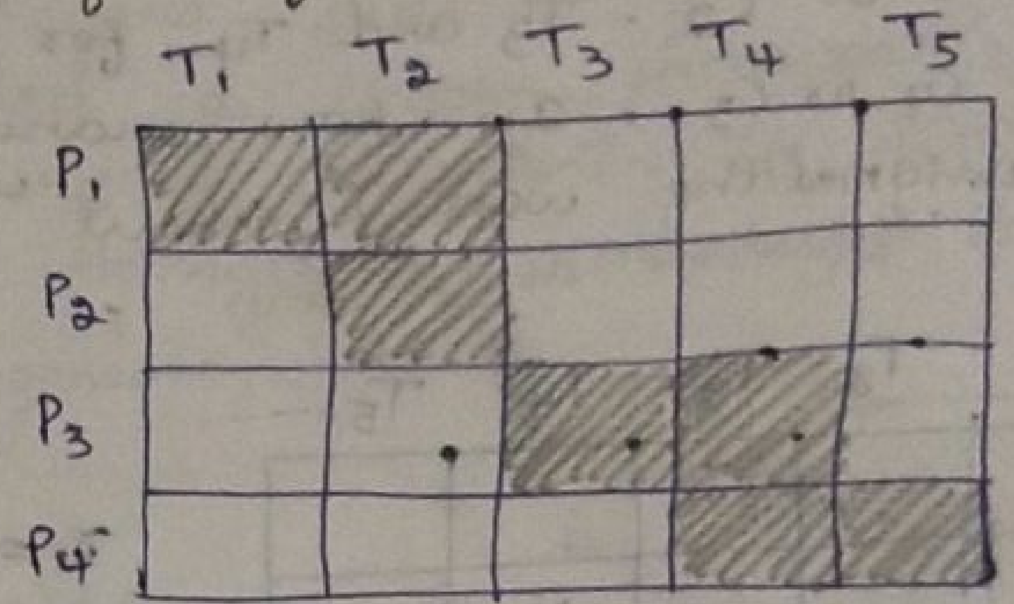
$\boxed{\bar{N} = 6}$

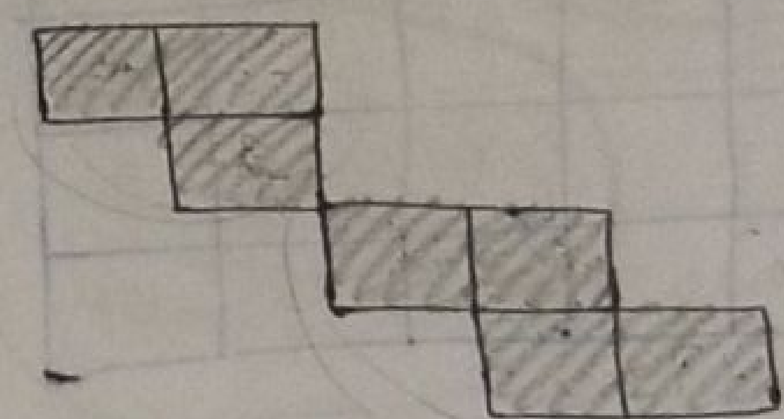
In 6 ways distribution can be made so that no boy is displeased.

② Four persons P_1, P_2, P_3 and P_4 who arrive late for a dinner party. find that only one chair at each of the 5 tables T_1, T_2, T_3, T_4 and T_5. P_1 will not sit at T_1 or T_2, P_2 will not sit at T_2, P_3 will not sit at T_3 or T_4 and P_4 will not sit at T_4 or T_5. Find the number of ways they can occupy the vacant chairs.

So take table as column

Soln:-





$r_1 = 7$

$r_2 = 16$

$r_3 = \cancel{103}\ 13$

$r_4 = 3$

$r(C,x) = 1 + r_1 x + r_2 x^2 + r_3 x^3 + r_4 x^4$

$\boxed{r(C,x) = 1 + 7x + 16x^2 + \underset{13}{\cancel{103}}x^3 + 3x^4}$

$\bar{N} = S_0 - S_1 + S_2 - S_3 + S_4$

$n = 5$

$S_0 = 5! = 120$

$S_1 = (n-1)!\, r_1$

$= 4! \times 7$

$\boxed{S_1 = 168}$

$S_2 = (n-2)!\, r_2$

$= 3! \times 16$

$\boxed{S_2 = 96}$

$S_3 = (n-3)!\, r_3$

$S_3 = 2! \times 103$

$\boxed{S_3 = 26}$

$S_4 = (n-4)!\, r_4$

$= 1! \times 3$

$\boxed{S_4 = 3}$

$\bar{N} = 120 - 168 + 96 - 26 + 3$

$\boxed{\bar{N} = 25}$

(3) Five teachers, T_1, T_2, T_3, T_4, T_5 are to be made class teachers for 5 classes C_1, C_2, C_3, C_4, C_5 one teacher for each class. T_1 and T_2 does not wish to become class teacher C_1 ~~and~~ or C_2. T_3 and T_4 for C_4 or C_5 and T_5 for C_3 or C_4 or C_5.. In how many ways can the teacher be assigned the work [without displeasing any teacher] → take teacher as column

Soln:-

	T_1	T_2	T_3	T_4	T_5
C_1	1	2			
C_2	3	4			
C_3					5
C_4			6	7	8
C_5			9	10	11

$C_1 =$

1	2
3	4

$r_1 = 4$

$r_2 = (1,4), (2,3) = 2$

$r(C_1, x) = 1 + 4x + 2x^2$

$C_2 =$

		5
6	7	8
9	10	11

$r_1 = 7$

$r_2 = (5,6), (5,7), (5,9), (5,10),$
$(6,10), (6,11), (7,9), (7,11),$
$(8,9), (8,10) = 10$

$r_3 = (5,6,10), (5,7,9)$, ~~(6,8,10), (7,5,9), (8,~~
$= 2$

$r(C_2, x) = 1 + 7x + 10x^2 + 2x^3$

$r(C, x) = r(C_1, x) * r(C_2, x)$

$= (1 + 4x + 2x^2) * (1 + 7x + 10x^2 + 2x^3)$

~~$= 1 + 4x + 2x^2$~~

$= 1 + 7x + 10x^2 + 2x^3 + 4x + 28x^2 +$
$40x^3 + 8x^4 + 2x^2 + 14x^3 + 20x^4 + 4x^5$

$\boxed{r(C, x) = 1 + 11x + 40x^2 + 56x^3 + 28x^4 + 4x^5}$

~~$r_1 = 11$~~

~~$r_2 =$~~

$\bar{N} = S_0 - S_1 + S_2 - S_3 + S_4 - S_5$

$\boxed{n = 5}$

$S_0 = 5! = 120$

$S_1 = (n-1)! \, r_1$
$= 4! \times 11$

$\boxed{S_1 = 264}$

$S_2 = (n-2)! \times r_2$
$= 3! \times 40$

$\boxed{S_2 = 240}$

$S_3 = (n-3)! \, r_3$
$= 2! \times 56$

$\boxed{S_3 = 112}$

$S_4 = (n-4)! \, r_4$
$= 1! \times 28$

$\boxed{S_4 = 28}$

$S_5 = (n-5)! \, r_5$
$= 0! \times 4$

$\boxed{S_5 = 4}$

$\bar{N} = 120 - 264 + 240 - 112 + 28 - 4$

$\boxed{\bar{N} = 8}$