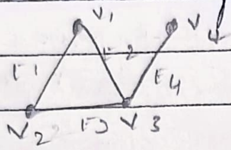
Graph Theory

4th SEM.

Graph - A collection of vertices connected to each other through a set of edges.

formally, A graph is defined as an order pair of set of vertices and a set of edges. i.e., $G = (V, E)$ where 'V' is a set of vertices & 'E' is a set edges.

Eg:



Null graph - A graph whose edge set is empty is called null graph.

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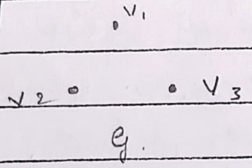

Trivial graph - A graph having only 1 vertex in it is called a trivial graph.

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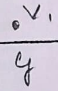

undirected or Non-directed graph - A graph in which all the edges are undirected is called as undirected graph.

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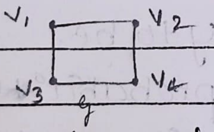

Directed graph - A graph in which all the edges are directed.

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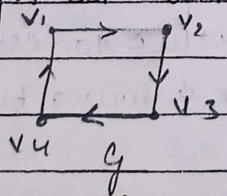
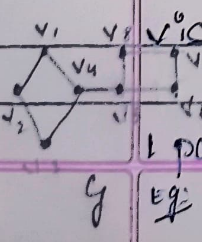


connected graph - A graph in which we can visit from any ~~other~~ vertex to any other vertex. In connected graph there exist atleast 1 path b/n every pair of vertices.



Eg:

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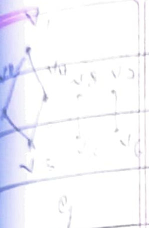
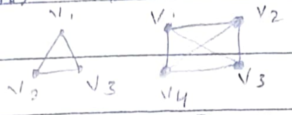


Disconnected graph - A graph in which there does not exist any path b/n atleast 1 pair of vertices

Regular graph - A graph in which all the vertices having same degree is called as regular graph. If all the vertices in a graph are of degree k or deg(k) then it is called as k-regular graph.

eg:

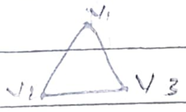


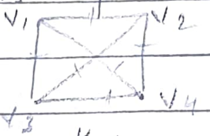
All vertices should have same edges / all should meet same edges.

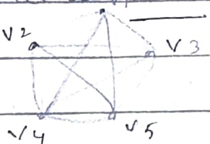
2-regular graph   3-regular graph

Complete graph - A graph in which exactly 1 edge is present b/n every pair of vertices is called as complete graph. A complete graph of 'N' vertices contain exactly N_{C_2} edges. A complete graph of 'n' vertices is represented as, K_n.

Eg:





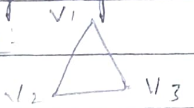


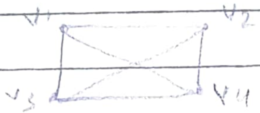
K_3 → no. of vertices K_4 K_5

$4_{C_2} = \frac{4!}{2!\,2!} = \frac{24}{2 \times 2}$

Cycle graph - A simple graph of 'n' vertices ($n \geq 3$) & 'n' edges forming a cycle of length 'n' is called as cycle graph. All the vertices are of degree 2.

Eg:



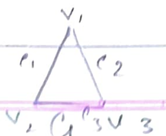


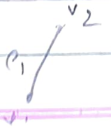
deg(v_1): 3, v_2: 3, v_3: 3
v_4: 3

G_1 deg(v_1) = 2, v_2 = 2, v_3 = 2

Subgraph - Given 2 graphs G_1 ~~say that~~ & G, we say that G_1 is a subgraph of G if the following conditions hold.

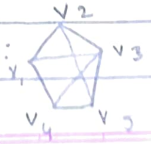
- All the vertices & all the edges of G_1 are in G
- Each edge of G_1 has the same end vertices in G as in G_1



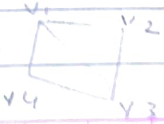


(or)

G:



G_1:

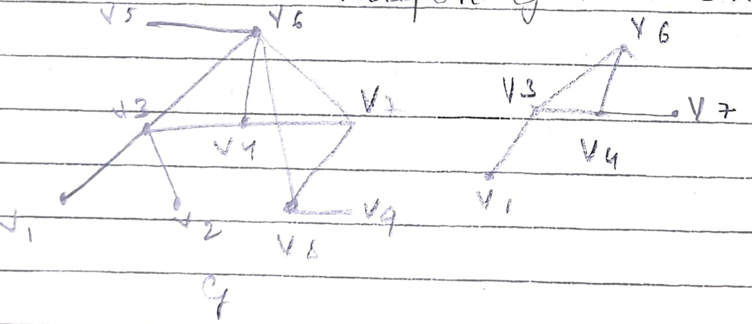


G G_1 ✓ subset

$\boxed{G_1 \subseteq G}$

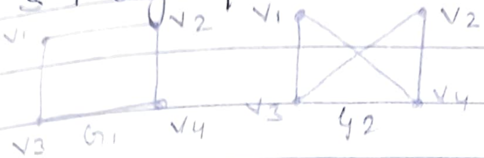
The following results are immediate consequences of the definition of a subgraph:
1. Every graph is a subgraph of itself.
2. Every simple graph of n vertices is a subgraph of the complete graph K_n.
3. If G_1 is a subgraph of a graph G_2 and G_2 is a subgraph of a graph G, then G_1 is a subgraph of G.
4. A single vertex in a graph G is a subgraph of G.
5. A single edge in a graph G, together with its end vertices, is a subgraph of G.

Eg: All the edges of G_1 namely $V_1 = \{v_1, v_3, v_4, v_6, v_7\}$ is a subset of the vertex set $V = \{v_1, v_2, v_3 \ldots v_9\}$ of G. Also all the edges of G_1 are in G. Further each edge in G_1 as the same end vertices in G having G_1 has the same end vertices in G_1. Therefore G_1 is a subgraph of G.



=> Isomorphism of graphs – 2 graphs G_1 & G_2 are said to be isomorphic to each other if the following conditions hold:
1) No. of vertices should be equal.
2) No. of edges should be equal.
3) Degree sequence is same. → Degrees should be in increasing order.
4) Adjacency vertex degree is preserved.

1. S.T 2 graphs are isomers.



vertices	Deg. of vertex
V1	2
V2	2
V3	2
V4	2

No. of vertices of G1 = 4, G2 = 4.

No. of edges of G1 = 4, G2 = 4

Degree sequence of $G_1(V_1) = 2$, $G_1(V_2) = 2$ $G_1(V_3) = 2$
$G_1(V_4) = 2$ & $G_2(V_1) = 2$ $G_2(V_2) = 2$, $G_2(V_3) = 2$ $G_2(V_4) = 2$

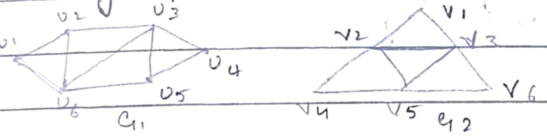
∴ ~~G1 &~~ Degree seq. of $G_1 = (2, 2, 2, 2)$ & $G_2 = (2, 2, 2, 2)$

vertex	Degree	Adj vex Deg.	vertex	Degree	Adj. vert Deg.
U1	2	2, 2	V1	2	2, 2
U2	2	2, 2	V2	2	2, 2
U3	2	2, 2	V3	2	2, 2
U4	2	2, 2	V4	2	2, 2.

∴ $f(U_1) = V_1$, $f(U_2) = V_2$, $f(U_3) = V_3$, $f(U_4) = V_4$

∴ Isomorphism is satisfied.

2. S.T 2 graphs are isomers.



No. of vertices of G1 = V6, G2 = V6.

No. of edges of G1 = 9, G2 = 9.

Degree of $G_1(V_1) = 2$, $G_1(U_2) = 3$. $G_1(V_3) = 4$, $G_1(U_4) = 2$
$G_1(U_5) = 3$. $G_1(U_6) = 4$.

Degree of $G_2(V_1) = 2$, $G_2(U_2) = 4$ $G_2(U_3) = 4$ $G_2(U_4) = 2$
$G_2(U_5) = 4$ $G_2(U_6) = 2$

Degree seq of $G_1 = (2, 2, 3, 3, 4, 4)$. & $G_2 = (2, 2, 2, 4, 4, 4)$

vertex	Degree	Adj. vex Deg.	vertex	Deg	Adj. vex Deg
U1	2	3, 4	V1	2	4, 4
U2	3.	2, 4	V2	4	2, 2
U3	4	2, 4	V3	4	
U4	2	4, 3	V4	2	
U5	3	2, 4	V5	4	
U6	4	2, 3	V6	2	

since

∴ Isomorphism is not satisfied in step 3 only.

$nC_r = \frac{n!}{(n-r)!\, r!}$

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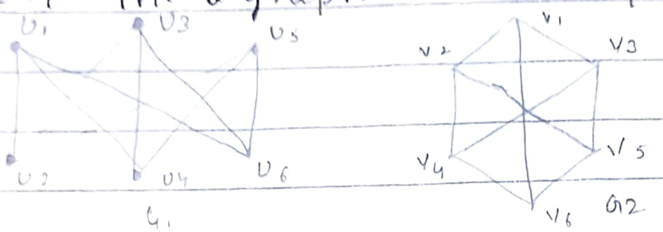
3. S.T the 2 graphs are isomorphism



No of vertices of $G_1 = V_6$, $G_2 = V_6$

No of edges of $G_1 = 9$, $G_2 = 9$

Deg of $G_1(V_1) = 3$, $G_1(U_2) = 3$, $G_1(V_3) = 3$, $G_1(U_4) = 3$.

$G_1(V_5) = 3$, $G_1(U_6) = 3$

Deg of $G_2(V_1) = 3$, $G_2(U_2) = 3$, $G_2(V_3) = 3$, $G_2(U_4) = 3$

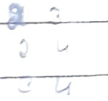
$G_2(V_5) = 3$, $G_2(U_6) = 3$.

Deg seq of $G_1 = (3,3,3,3,3,3)$, $G_2 = (3,3,3,3,3,3)$.

$f(U_1) = V_1$, $f(U_2) = U_2$, $f(U_3) = U_3$, $f(U_4) = V_4$, $f(U_5) = U_5$, $f(U_6) = U_6$

∴ Isomorphism obeyed

4. S.T complete graph with 'n' vertices namely K_n has $\frac{n(n-1)}{2}$ edges.



$K_4 \Rightarrow \frac{4(4-1)}{2} = \frac{4(3)}{2} = \frac{\cancel{12}^{6}}{\cancel{2}_{1}} = 6$ edges

= 6 edges

pair of vertices are:
1 2
1 3
1 4
2 3
2 4
3 4

In a complete graph there exists exactly 1 edge b/n every pair of vertices. As such the no. of edges in a complete graph is equal to no. of pair of vertices. If no. of vertices is 'n' then the no. of pair of vertices is $nC_2 = \frac{n!}{(n-2)! \times 2!}$ $\Rightarrow \frac{n(n-1)(n-2)!}{(n-2)! \times 2}$

$nC_r = \frac{n!}{(n-r)! \times r!}$

$= \frac{n(n-1)}{2}$ //

5. Determine the order of G in the following case.

every vertices should have 3 edges in the graph

graph that has the vertices in which all the vertices have same edges ∴ same degree

i) G is a cubic graph with 9 edges.

ii) G is a regular graph with 16 edges.

i) Suppose the order of G is 'n'. Since G is a cubic graph all vertices of G have degree 3 & therefore the sum of degrees of vertices is 3n. Since G has 9 edges, we should have $3n = 2 \times 9$

i) g is a cubic graph with 9 edges but g n vertices

$\sum_{v \in V} deg(v) = 2|E|$ $\Rightarrow 3+3+3 \ldots + 3 = 2|E|$ $3(n) = 2(9)$ $3n = 18 \Rightarrow n = 6$

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$3n = 2 \times 9$ [handshaking lemma]

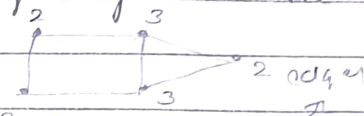
$3n = 18 \Rightarrow n = 6$ $\therefore$ Thus order of g is 6.

2M Handshaking lemma: The sum of degrees of all the vertices in a graph is an even number & this no. is equal to twice the no. of edges in the graph.

$$\sum_{v \in V} deg(v) = 2|E| \rightarrow \text{edges}$$

Hence it is handshaking lemma.

Eg:



$v = 12$ $\therefore 12 = 2(6)$ $12 = 12$

ii) Since g is regular. All the vertices of g must have the same degree, say k. If g has g is of order n, then the sum of degrees of vertices is kn. Since g has 15 edges by handshaking theorem, we will get:

$$\sum_{v \in V} deg(v) = 2|E|$$

$$k + k + k + \ldots + k = 2|E|$$

$$nk = 2(15) \Rightarrow nk = 30 \Rightarrow \boxed{n = 30/k} \text{ or}$$

$\boxed{k = 30/n}$. Since k has to be a '+ve' integer, it follows that 'n' must be a divisor of 30. thus the possible orders of g are 1, 2, 3, 5, 6, 10, 15, 30.

Theorem - For a graph with 'n' vertices & 'm' edges if δ (delta) is the minimum & the Δ is the maximum of the degree of vertices show that $\delta \leq \frac{2m}{n} \leq \Delta$

Soln: let $d_1, d_2, d_3, \ldots, d_n$ be the degree of vertices. Then by handshaking property we get $d_1 + d_2 + d_3 + \cdots + d_n = 2m$. Since, $\delta =$

$\delta = min(d_1, d_2, d_3 \ldots d_n)$ we have $d_1 \geq \delta, d_2 \geq \delta,$ $d_3 \geq \delta, \ldots, d_n \geq \delta$ → min. Adding these 'n' equalities we get $d_1 + d_2 + d_3 + \cdots + d_n \geq \delta + \delta + \delta + \cdots + \delta$

$$d_1 + d_2 + d_3 + \cdots + d_n \geq n\delta \rightarrow (1)$$

$$\Delta = max(d_1, d_2, \ldots, d_n)$$

$$d_1 \leq \Delta, d_2 \leq \Delta, d_3 \leq \Delta, \ldots, d_n \leq \Delta$$

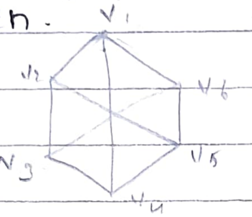
$$d_1 + d_2 + d_3 + \cdots + d_n \leq n\Delta \rightarrow (2)$$

$n\delta \le d_1 + d_2 + d_3 + \cdots + d_n \le n\Delta$

$(n\delta \le 2m \le n\Delta) \div n$

$\boxed{\delta \le \frac{2m}{n} \le \Delta}$ → delta

6. In every graph, the no. of vertices of odd degree is even.



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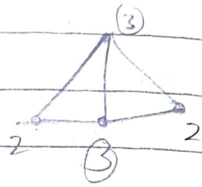


2 vertices have odd degree ∴ 2 is even.

Consider a graph with 'n' vertices suppose 'K' of these vertices are all odd degree so that remaining $(n-K)$ vertices are of even degree. Denote that vertices with odd degree by $v_1, v_2, v_3 \ldots v_K$ & the remaining vertices with even degree by $v_{K+1}, v_{K+2}, v_{K+3} \ldots v_n$ then by handshaking theorem we get

$$\sum_{v \in V} \deg(v) = 2|E| \rightarrow \underbrace{v_1, v_2, v_3 \ldots v_K}_{\text{odd degree}}, \underbrace{v_{K+1}, v_{K+2} \ldots v_n}_{\text{even degree}}$$

$$\sum_{i=1}^{K} \deg(v_i) + \sum_{i=K+1}^{n} \deg(v_i) = 2|E|$$

(odd degree addition) (even degree addition)

$$\sum_{i=1}^{K} \deg(v_i) + \text{even no.} = \text{even no.}$$

$$\sum_{i=1}^{K} \deg(v_i) = \text{even no.}$$

[even no. times odd no = even no]

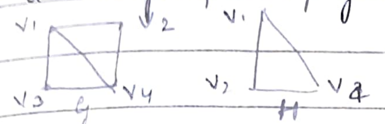
Therefore, the no. of terms in the left hand side must be even. i.e., K is even.

Theorm - Let $G = G(V, E)$ and $G' = G'(V', E')$ be 2 graphs and ~~f(G), G'~~ [f is the func() from $G \to G'$] $f : G \to G'$ be an isomorphism. prove the following.

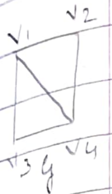
i) $f' : G' \to G$ is also an isomorphism.

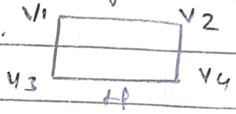
ii) for any vertex 'v' in G, deg(v) in G = deg(f(v)) in G'

Subgraph - Let $G = G(V, E)$ & $H = H(V', E')$ be 2 graphs. H is a subgraph of G if $V' \subseteq V$ & $E' \subseteq E$

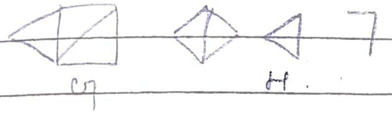


Spanning subgraph - ~~Let $G = G(V, E)$ & $H = H(V', E')$ be 2 graphs. H is a subgraph of G of $V' \subseteq V$~~ & If H is a subgraph of G such that $V(H) = V(G)$ & $E(H) \subseteq E(G)$ then H is called spanning subgraph.





Induced subgraph - A subgraph $H(G)$ is an induced subgraph if H contains all edges of G that connects the vertices of ~~G~~ H.



proof : i) Since, $f : G \to G'$ is an isomorphism f is a one to one correspondance b/n G & G'.
$\therefore$ f is invertible.
& f^{-1} is a func. from G' to G & is also a 1 to 1 correspondance. further since f is an isomorphism for all vertices a, b in G the set $\{a, b\}$ is an edge in G if & only if the inverse set $\{f(a), f(b)\}$ is an edge in G'. Since $a = f^{-1}(f(a))$ & $b = f^{-1}(f(b))$ this is equivalent to saying the set $\{f(a), f(b)\}$. This follows f^{-1} ~~is a~~ $f^{-1} : G' \to G$ is an isomorphism
f inverse G -- dash

ii) for any ~~$e \in E$~~, $v \in V$. Let 'K' be its degree in G. then there exist K distinct vertices $v_1, v_2 \dots v_K$ such that the set $\{v, v_1\}, \{v, v_2\} \dots \{v, v_K\}$ are edges in G. Consequently $\{f(v), f(v_1)\}, \{f(v), f(v_2)\} \dots \{f(v), f(v_K)\}$ are edges in G'.
Since $v_1, v_2 \dots v_K$ are distinct & f is an 1 to 1 correspondance, $f(v_1), f(v_2) \dots f(v_K)$ are also distinct

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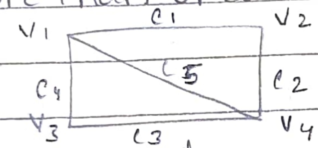
Thus there are 'k' edges in g' which are incident on $f(v)$. Hence $\deg(f(v))$ in g' is k. This proves the result.

<u>walk</u> - Let 'g' be a graph having atleast 1 edge. In g consider a finite alternating sequence of vertices & edge of the form $v_i e_i v_j e_j v_k e_k \ldots v_m e_n$. Such a sequence is called a walk in g. In a walk a vertex or an edge (or both) can appear more than once.

Eg:



$v_1 e_1 v_2 e_2 v_4 e_3 v_3 e_4 v_1 e_5 v_4$

always should end with vertex

The no. of edges present in a walk is called its length.

Here walk having 'u' has an initial vertex & 'v' has the final vertex is called a walk from u to v or it is called ~~u-v~~ u - v walk. ~~yes~~ A walk that begins & ends with same vertex is called a <u>closed walk</u>.

~~yes~~ A walk which is not closed is called <u>open walk</u>.

<u>Trial</u> - If an open walk no edge appears more than once, then walk is called trial.

A closed walk in which no edge appears more than once is called a <u>circuit</u>.

<u>path</u> - A trial in which no vertex appears more than once is called a path.

<u>Cycle</u> - A circuit in which the terminal vertex does not appear as an internal vertex & no internal vertex is repeated is called a cycle.

Euler Circuit (or) Eulerian line (or) Euler tour -

Consider a connected graph 'g'. If there is a circuit in G that contains all the edges of G, then the circuit is called Euler circuit in G.

Euler trial – If there is a trial in G, that contains all the edges of G, then that trial is called Euler trial. or unicursal line.

A connected graph that contains an Euler ~~trial~~ circuit is called Euler graph.

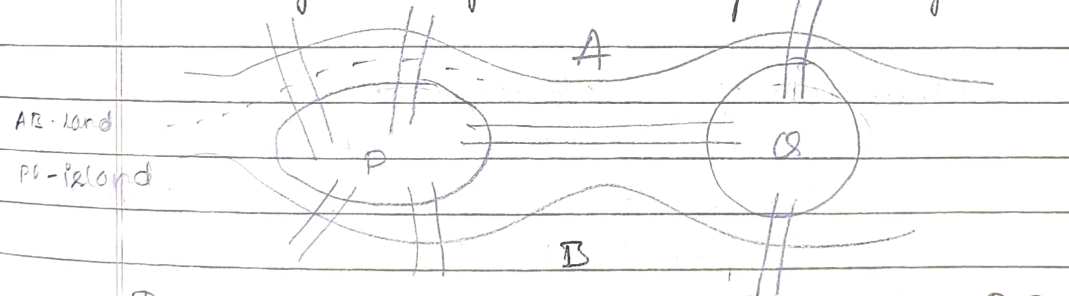
A connected graph that contains an Euler trial is called an semi-~~graph~~ Euler graph.

*NOTE: A connected graph G as an Euler circuit if & only if all the vertices of G are of even degree.

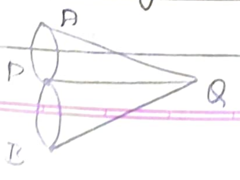
* A connected graph G as an Euler circuit if & only if G can be decomposed into edge disjoint cycles.

Konigsberge bridge problem –

problem: By starting at any of the 4 land areas, can be returned to that area after crossing each of the 7 bridges exactly once.



Denote the land areas of the city by A, B, P, Q where A, B are the banks of river & P, Q are the islands. Construct a graph by treating the 4 Land areas as four vertices & 7 bridges connecting them as 7 edges

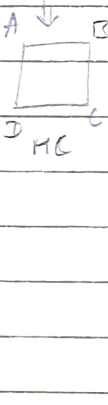


Denote WKT deg(A) = deg(B) = deg(Q) = 3 & deg(P) = 5 which

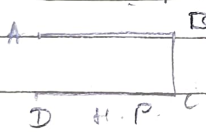
which is not even. By the theorem, "A connected graph 'G' has an Euler circuit if & only if all the vertices of G are of even degree". Therefore, the graph doesnot have an Euler circuit. This means that there doesn't exist a closed walk that contains all the edges exactly once. This amounts to saying i.e. not possible to walk over all 7 bridges exactly once & return to the starting point.

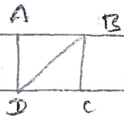
Hamilton cycle - Let 'G' be a connected graph. If there is a cycle in 'G' that contains all the vertices of 'G', then that cycle is called Hamilton cycle in G.

A graph that contains Hamilton cycle is called a 'Hamilton graph' or 'Hamiltonian graph'.



Hamilton path - A path in a connected which include every vertex of the graph is called a Hamilton path or Hamiltonian path.





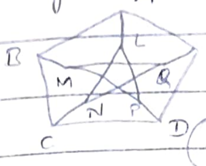
Theorem 1 - If in a simple connected graph with 'n' vertices ($n \geq 3$). the sum of the degrees of every pair of non-adjacent vertices $\geq n$, then the graph is Hamiltonian.

Theorem 2 - If in a simple connected graph with n vertices ($n \geq 3$), the degree of every vertices is greater than or equal to $n/2$. Then the graph is Hamiltonian.

NOTE: Every complete graph K_n where greater than or equal to 3 is a Hamiltonian graph.

Q) Show that a ~~praterson~~ peterson graph has no Hamilton cycle in it. But it has a Hamilton path.

Soln:



All vertices having deg(3)

The peterson graph is a 3-Regular graph with 10 vertices & 15 edges. The vertices are labelled has A, B, C, D, E, M, N, P, Q, L. Since the graph has 10 vertices & 15 edges, a hamilton cycle if any in this graph must pass through all the 10 vertices & must have 15 edges.

it should contain deg(3)

We observe that 3 edges are incident at every vertex of the graph. Of these 3 edges only 2 can be ~~con~~ included in an Hamilton cycle if it exist. Thus, at each of the 10 vertices of the graph one edge has to be excluded. By actual counting, we find that the no. of edges to be excluded is 6. for eg. the edges are AL, BM, CN, DE, PM, QE. Consequently, the no. of edges that remain in the graph is $15 - 6 = 9$. These edges are insufficient to form a Hamilton cycle in the graph. Thus, the peterson graph does not contain a Hamilton cycle in it.

(or)

By the theorem, if in a simple connected graph with 'n' vertices $(n \geq 3)$ the degree of every vertex is greater than or equal to $n/2$. Then the graph is Hamiltonian.

Here, the peterson graph has 10 vertices, by the theorem each vertex should have deg ≥ 5. but here the degree of each vertices is 3. Therefore, peterson graph doesnot have a Hamilton cycle in it.

By seeing peterson graph w.k.T the edges

AB, BC, CD, DE, EQ, QM, MP, PL, LN. forms a ~~these~~ & this path path includes all the vertices. therefore, this path is a Hamilton path.

planar graph - A graph which can be represented atleast 1 plane drawing in which the edges ~~need~~ meet only at the vertices is called a planar graph.

Kuratowski's first graph -